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Index of Applications

Engineering and Physical Sciences

Acceleration, 129, 157, 159, 177, 258, 924
Air pressure, 443
Air traffic control, 155, 762, 868
Aircraft glide path, 196
Airplane separation, 258
Angle of elevation, 152, 156, 157
Angular rate of change, 380, 381
Area, 38, 118, 127, 154, 261, 317, 616, 688
Asteroid Apollo, 754
Atmospheric pressure and altitude, 333, 360, 969
Automobile aerodynamics, 30
Average speed, 40, 89
Average temperature, 1002, 1052
Average velocity, 113
Beam deflection, 197, 707
Beam strength, 35, 225
Billiard balls and normal lines, 945
Boiling temperature, 36
Boyle's Law, 89, 127, 497
Braking load, 791
Breaking strength of a steel cable, 371
Bridge design, 708
Building design, 457, 568, 1026, 1053, 1082
Buoyant force, 513
Cable tension, 774, 782
Capillary action, 1026
Car performance, 35, 36
Carbon dating, 421
Center of mass, of glass, 507
Center of pressure on a sail, 1019
Centripetal acceleration, 868
Centripetal force, 868, 882
Centroid, 506, 507, 516, 531
Chemical mixture problem, 436, 441
Chemical reaction, 223, 399, 432, 562, 980
Circular motion, 857, 858, 866, 882
Comet Hale-Bopp, 757
Construction, 155, 708, 782
Cycloidal motion, 857, 867
Depth
  of gasoline in a tank, 516
  of water in a swimming pool, 154, 155
  of water in a vase, 29
Distance, 245, 967
Distance between two ships, 244
Drag force, 980
Earthquake intensity, 422
Einstein's Special Theory of Relativity and Newton's First Law of Motion, 207
Electric circuits, 414, 438, 441
Electric force, 497
Electric force fields, 1059
Electric potential, 896
Electrical charge, 1122, 1123
Electrical resistance, 188
Electricity, 156, 309
Electromagnetic theory, 589
Emptying a tank of oil, 493
Error
  in area of the end of a log, 240
  in volume of a ball bearing, 237
  in volume and surface area of a cube, 240
Explorer 18, 708, 757
Explorer 55, 709
Falling object, 34, 321, 437, 441
Ferris wheel, 884
Flow rate, 291, 361, 1123
Fluid force, 553
  on a circular plate, 514, 516
  of gasoline, 513, 514
  on a stern of a boat, 514
  in a swimming pool, 516, 518
  on a tank wall, 513, 514
  of water, 513
Force, 294, 513, 788
Free-falling object, 69, 82, 91
Frictional force, 876, 880
Gauss's Law, 1121
Gravitational fields, 1059
Gravitational force, 128, 589
Halley's comet, 709, 753
Harmonic motion, 36, 38, 139, 242, 359, 402
Heat flow, 1141
Heat transfer, 342
Heat-seeking particle, 939
Heat-seeking path, 944
Height
  of a baseball, 29
  of a basketball, 32
  of an oscillating object, 242
Highway design, 171, 196, 882, 884
Honeycomb, 171
Horizontal motion, 159, 361
Hyperbolic detection system, 705
Hyperbolic mirror, 710
Ideal Gas Law, 896, 916, 931
Illumination, 226, 245
Inflating balloon, 151
Kepler's Laws, 753, 754, 880
Kinetic and potential energy, 1089, 1092
Law of Conservation of Energy, 1089
Lawn sprinkler, 171
Length, 616
  of a catenary, 485, 516
  of pursuit, 488
  of a stream, 487
Linear and angular velocity, 160
Linear vs. angular speed, 157
Load supports, 782
Load-supporting cables, 790, 791
Lunar gravity, 257
Magnetic field of Earth, 1142
Map of the ocean floor, 944
Mass, 1073, 1079
  on the surface of Earth, 498
Maximum area, 222, 223, 224, 225, 228, 244, 246, 967
Maximum cross-sectional area of an irrigation canal, 227
Maximum volume, 224, 225, 227
  of a box, 218, 219, 223, 224, 962, 966, 967, 977
  of a can buoy, 977
  of a package, 225, 968, 977
Minimum length, 221, 224, 226, 244
Minimum surface area, 225, 981
Minimum time, 226, 234
Motion
  of a liquid, 1136, 1137
  of a particle, 728
Moving ladder, 155
Moving shadow, 157, 160, 162
Muzzle velocity, 772, 774
Navigation, 710, 762, 774
Newton's Law of Gravitation, 1059
Orbit of Earth, 708
Orbital speed, 868
Parabolic reflector, 698
Particle motion, 129, 292, 296, 841,849, 851, 857, 858, 867, 868, 879, 881
Path
  of a ball, 856
  of a baseball, 855, 856, 857, 877
  of a bomb, 857, 883
  of a football, 857
  of a projectile, 185, 728, 856, 857, 982
Pendulum, 139, 241, 924
Planetary motion, 757
Planetary orbits, 701
Planimeter, 1140
Power, 171, 188, 924
Projectile motion, 158, 159, 241, 553, 689, 720, 774, 854, 856, 857, 865, 867, 868, 877, 882, 931
Radioactive decay, 417, 421, 432, 443
Refraction of light, 977
Refrigeration, 160
Resultant force, 770, 773
Ripples in a pond, 150
Rolling a ball bearing, 188
Satellite antenna, 758
Satellite orbit, 708, 882, 884
Satellites, 128
Sending a space module into orbit, 583
Solar collector, 707

(continued on back inside cover)

Tear out Formula Cards for Homework Success.

DERIVATIVES AND INTEGRALS

Basic Differentiation Rules

1. $\frac{d}{dx}[cu] = cu'$

2. $\frac{d}{dx}[u \pm v] = u' \pm v'$

3. $\frac{d}{dx}[uv] = uv' + vu'$

4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$

5. $\frac{d}{dx}[c] = 0$

6. $\frac{d}{dx}[u^n] = nu^{n-1}u'$

7. $\frac{d}{dx}[x] = 1$

8. $\frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), \quad u \neq 0$

9. $\frac{d}{dx}[\ln u] = \frac{u'}{u}$

10. $\frac{d}{dx}[e^u] = e^u u'$

11. $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$

12. $\frac{d}{dx}[a^u] = (\ln a)a^u u'$

13. $\frac{d}{dx}[\sin u] = (\cos u)u'$

14. $\frac{d}{dx}[\cos u] = -(\sin u)u'$

15. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$

16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$

17. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$

18. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$

19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1 - u^2}}$

20. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1 - u^2}}$

21. $\frac{d}{dx}[\arctan u] = \frac{u'}{1 + u^2}$

22. $\frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1 + u^2}$

23. $\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2 - 1}}$

24. $\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2 - 1}}$

25. $\frac{d}{dx}[\sinh u] = (\cosh u)u'$

26. $\frac{d}{dx}[\cosh u] = (\sinh u)u'$

27. $\frac{d}{dx}[\tanh u] = (\operatorname{sech}^2 u)u'$

28. $\frac{d}{dx}[\coth u] = -(\operatorname{csch}^2 u)u'$

29. $\frac{d}{dx}[\operatorname{sech} u] = -(\operatorname{sech} u \tanh u)u'$

30. $\frac{d}{dx}[\operatorname{csch} u] = -(\operatorname{csch} u \coth u)u'$

31. $\frac{d}{dx}[\sinh^{-1} u] = \frac{u'}{\sqrt{u^2 + 1}}$

32. $\frac{d}{dx}[\cosh^{-1} u] = \frac{u'}{\sqrt{u^2 - 1}}$

33. $\frac{d}{dx}[\tanh^{-1} u] = \frac{u'}{1 - u^2}$

34. $\frac{d}{dx}[\coth^{-1} u] = \frac{u'}{1 - u^2}$

35. $\frac{d}{dx}[\operatorname{sech}^{-1} u] = \frac{-u'}{u\sqrt{1 - u^2}}$

36. $\frac{d}{dx}[\operatorname{csch}^{-1} u] = \frac{-u'}{|u|\sqrt{1 + u^2}}$

Basic Integration Formulas

1. $\int kf(u)\,du = k\int f(u)\,du$

2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$

3. $\int du = u + C$

4. $\int a^u\,du = \left(\frac{1}{\ln a}\right)a^u + C$

5. $\int e^u\,du = e^u + C$

6. $\int \sin u\,du = -\cos u + C$

7. $\int \cos u\,du = \sin u + C$

8. $\int \tan u\,du = -\ln|\cos u| + C$

9. $\int \cot u\,du = \ln|\sin u| + C$

10. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

11. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$

12. $\int \sec^2 u\,du = \tan u + C$

13. $\int \csc^2 u\,du = -\cot u + C$

14. $\int \sec u \tan u\,du = \sec u + C$

15. $\int \csc u \cot u\,du = -\csc u + C$

16. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$

17. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

18. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

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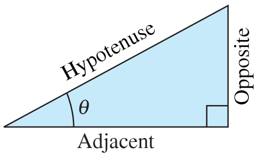
TRIGONOMETRY

Definition of the Six Trigonometric Functions

Right triangle definitions, where $0 < \theta < \pi/2$.

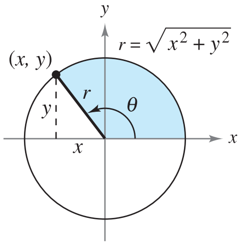

Hypotenuse
Opposite
Adjacent
θ

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

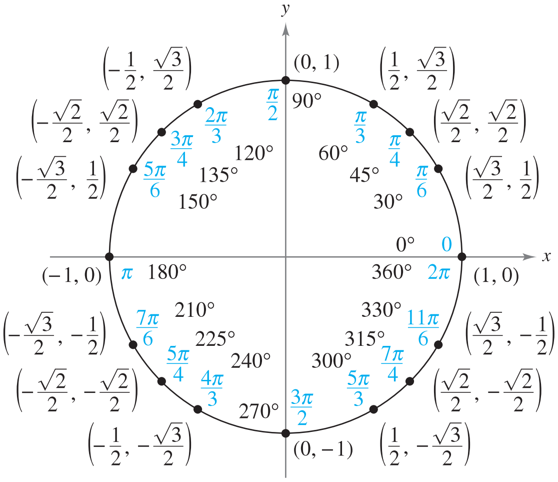
Circular function definitions, where θ *is any angle.*


y
$r = \sqrt{x^2 + y^2}$
(x, y)
r
θ
y
x
x

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$


y
x
(0, 1)
90°
$\frac{\pi}{2}$
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{2\pi}{3}$
120°
$\frac{3\pi}{4}$
135°
$\frac{5\pi}{6}$
150°
$\frac{\pi}{3}$
60°
$\frac{\pi}{4}$
45°
$\frac{\pi}{6}$
30°
0°
0
360°
2π
(1, 0)
(−1, 0)
π
180°
$\frac{7\pi}{6}$
210°
$\frac{5\pi}{4}$
225°
$\frac{4\pi}{3}$
240°
$\frac{3\pi}{2}$
270°
300°
$\frac{5\pi}{3}$
315°
$\frac{7\pi}{4}$
330°
$\frac{11\pi}{6}$
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
(0, −1)
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Reciprocal Identities

$$\sin x = \frac{1}{\csc x} \qquad \sec x = \frac{1}{\cos x} \qquad \tan x = \frac{1}{\cot x}$$

$$\csc x = \frac{1}{\sin x} \qquad \cos x = \frac{1}{\sec x} \qquad \cot x = \frac{1}{\tan x}$$

Tangent and Cotangent Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\csc\left(\frac{\pi}{2} - x\right) = \sec x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x$$

$$\sec\left(\frac{\pi}{2} - x\right) = \csc x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$$

Reduction Formulas

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$

$$\csc(-x) = -\csc x \qquad \tan(-x) = -\tan x$$

$$\sec(-x) = \sec x \qquad \cot(-x) = -\cot x$$

Sum and Difference Formulas

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

Double-Angle Formulas

$$\sin 2u = 2 \sin u \cos u$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$

$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u}$$

Power-Reducing Formulas

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Sum-to-Product Formulas

$$\sin u + \sin v = 2\sin\left(\frac{u + v}{2}\right)\cos\left(\frac{u - v}{2}\right)$$

$$\sin u - \sin v = 2\cos\left(\frac{u + v}{2}\right)\sin\left(\frac{u - v}{2}\right)$$

$$\cos u + \cos v = 2\cos\left(\frac{u + v}{2}\right)\cos\left(\frac{u - v}{2}\right)$$

$$\cos u - \cos v = -2\sin\left(\frac{u + v}{2}\right)\sin\left(\frac{u - v}{2}\right)$$

Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$

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Calculus

Ninth Edition

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University of Florida

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Contents

A Word from the Authors x

Textbook Features xiv

CHAPTER P **Preparation for Calculus** 1

P.1 Graphs and Models 2
P.2 Linear Models and Rates of Change 10
P.3 Functions and Their Graphs 19
P.4 Fitting Models to Data 31
Review Exercises 37
P.S. Problem Solving 39

CHAPTER 1 **Limits and Their Properties** 41

1.1 A Preview of Calculus 42
1.2 Finding Limits Graphically and Numerically 48
1.3 Evaluating Limits Analytically 59
1.4 Continuity and One-Sided Limits 70
1.5 Infinite Limits 83
SECTION PROJECT: **Graphs and Limits of Trigonometric Functions** 90
Review Exercises 91
P.S. Problem Solving 93

CHAPTER 2 **Differentiation** 95

2.1 The Derivative and the Tangent Line Problem 96
2.2 Basic Differentiation Rules and Rates of Change 107
2.3 Product and Quotient Rules and Higher-Order Derivatives 119
2.4 The Chain Rule 130
2.5 Implicit Differentiation 141
SECTION PROJECT: **Optical Illusions** 148
2.6 Related Rates 149
Review Exercises 158
P.S. Problem Solving 161

CHAPTER 3 Applications of Differentiation 163

3.1 Extrema on an Interval 164
3.2 Rolle's Theorem and the Mean Value Theorem 172
3.3 Increasing and Decreasing Functions and the First Derivative Test 179
SECTION PROJECT: **Rainbows** 189
3.4 Concavity and the Second Derivative Test 190
3.5 Limits at Infinity 198
3.6 A Summary of Curve Sketching 209
3.7 Optimization Problems 218
SECTION PROJECT: **Connecticut River** 228
3.8 Newton's Method 229
3.9 Differentials 235
Review Exercises 242
P.S. Problem Solving 245

CHAPTER 4 Integration 247

4.1 Antiderivatives and Indefinite Integration 248
4.2 Area 259
4.3 Riemann Sums and Definite Integrals 271
4.4 The Fundamental Theorem of Calculus 282
SECTION PROJECT: **Demonstrating the Fundamental Theorem** 296
4.5 Integration by Substitution 297
4.6 Numerical Integration 311
Review Exercises 318
P.S. Problem Solving 321

CHAPTER 5 Logarithmic, Exponential, and Other Transcendental Functions 323

5.1 The Natural Logarithmic Function: Differentiation 324
5.2 The Natural Logarithmic Function: Integration 334
5.3 Inverse Functions 343
5.4 Exponential Functions: Differentiation and Integration 352
5.5 Bases Other Than e and Applications 362
SECTION PROJECT: **Using Graphing Utilities to Estimate Slope** 372
5.6 Inverse Trigonometric Functions: Differentiation 373
5.7 Inverse Trigonometric Functions: Integration 382

5.8 Hyperbolic Functions 390
SECTION PROJECT: **St. Louis Arch** 400
Review Exercises 401
P.S. Problem Solving 403

CHAPTER 6 **Differential Equations** **405**

6.1 Slope Fields and Euler's Method 406
6.2 Differential Equations: Growth and Decay 415
6.3 Separation of Variables and the Logistic Equation 423
6.4 First-Order Linear Differential Equations 434
SECTION PROJECT: **Weight Loss** 442
Review Exercises 443
P.S. Problem Solving 445

CHAPTER 7 **Applications of Integration** **447**

7.1 Area of a Region Between Two Curves 448
7.2 Volume: The Disk Method 458
7.3 Volume: The Shell Method 469
SECTION PROJECT: **Saturn** 477
7.4 Arc Length and Surfaces of Revolution 478
7.5 Work 489
SECTION PROJECT: **Tidal Energy** 497
7.6 Moments, Centers of Mass, and Centroids 498
7.7 Fluid Pressure and Fluid Force 509
Review Exercises 515
P.S. Problem Solving 517

CHAPTER 8 **Integration Techniques, L'Hôpital's Rule, and Improper Integrals** **519**

8.1 Basic Integration Rules 520
8.2 Integration by Parts 527
8.3 Trigonometric Integrals 536
SECTION PROJECT: **Power Lines** 544
8.4 Trigonometric Substitution 545
8.5 Partial Fractions 554
8.6 Integration by Tables and Other Integration Techniques 563
8.7 Indeterminate Forms and L'Hôpital's Rule 569
8.8 Improper Integrals 580
Review Exercises 591
P.S. Problem Solving 593

CHAPTER 9 **Infinite Series** 595

9.1 Sequences 596

9.2 Series and Convergence 608

SECTION PROJECT: **Cantor's Disappearing Table** 618

9.3 The Integral Test and *p*-Series 619

SECTION PROJECT: **The Harmonic Series** 625

9.4 Comparisons of Series 626

SECTION PROJECT: **Solera Method** 632

9.5 Alternating Series 633

9.6 The Ratio and Root Tests 641

9.7 Taylor Polynomials and Approximations 650

9.8 Power Series 661

9.9 Representation of Functions by Power Series 671

9.10 Taylor and Maclaurin Series 678

Review Exercises 690

P.S. Problem Solving 693

CHAPTER 10 **Conics, Parametric Equations, and Polar Coordinates** 695

10.1 Conics and Calculus 696

10.2 Plane Curves and Parametric Equations 711

SECTION PROJECT: **Cycloids** 720

10.3 Parametric Equations and Calculus 721

10.4 Polar Coordinates and Polar Graphs 731

SECTION PROJECT: **Anamorphic Art** 740

10.5 Area and Arc Length in Polar Coordinates 741

10.6 Polar Equations of Conics and Kepler's Laws 750

Review Exercises 758

P.S. Problem Solving 761

CHAPTER 11 **Vectors and the Geometry of Space** 763

11.1 Vectors in the Plane 764

11.2 Space Coordinates and Vectors in Space 775

11.3 The Dot Product of Two Vectors 783

11.4 The Cross Product of Two Vectors in Space 792

11.5 Lines and Planes in Space 800

SECTION PROJECT: **Distances in Space** 811

11.6 Surfaces in Space 812

11.7 Cylindrical and Spherical Coordinates 822

Review Exercises 829

P.S. Problem Solving 831

CHAPTER 12 Vector-Valued Functions 833

12.1 Vector-Valued Functions 834

SECTION PROJECT: Witch of Agnesi 841

12.2 Differentiation and Integration of Vector-Valued Functions 842

12.3 Velocity and Acceleration 850

12.4 Tangent Vectors and Normal Vectors 859

12.5 Arc Length and Curvature 869

Review Exercises 881

P.S. Problem Solving 883

CHAPTER 13 Functions of Several Variables 885

13.1 Introduction to Functions of Several Variables 886

13.2 Limits and Continuity 898

13.3 Partial Derivatives 908

SECTION PROJECT: Moiré Fringes 917

13.4 Differentials 918

13.5 Chain Rules for Functions of Several Variables 925

13.6 Directional Derivatives and Gradients 933

13.7 Tangent Planes and Normal Lines 945

SECTION PROJECT: Wildflowers 953

13.8 Extrema of Functions of Two Variables 954

13.9 Applications of Extrema of Functions of Two Variables 962

SECTION PROJECT: Building a Pipeline 969

13.10 Lagrange Multipliers 970

Review Exercises 978

P.S. Problem Solving 981

CHAPTER 14 Multiple Integration 983

14.1 Iterated Integrals and Area in the Plane 984

14.2 Double Integrals and Volume 992

14.3 Change of Variables: Polar Coordinates 1004

14.4 Center of Mass and Moments of Inertia 1012

SECTION PROJECT: Center of Pressure on a Sail 1019

14.5 Surface Area 1020

SECTION PROJECT: Capillary Action 1026

14.6 Triple Integrals and Applications 1027

14.7 Triple Integrals in Cylindrical and Spherical Coordinates 1038

SECTION PROJECT: **Wrinkled and Bumpy Spheres** 1044
14.8 Change of Variables: Jacobians 1045
Review Exercises 1052
P.S. Problem Solving 1055

CHAPTER 15 Vector Analysis 1057

15.1 Vector Fields 1058
15.2 Line Integrals 1069
15.3 Conservative Vector Fields and Independence of Path 1083
15.4 Green's Theorem 1093
SECTION PROJECT: **Hyperbolic and Trigonometric Functions** 1101
15.5 Parametric Surfaces 1102
15.6 Surface Integrals 1112
SECTION PROJECT: **Hyperboloid of One Sheet** 1123
15.7 Divergence Theorem 1124
15.8 Stokes's Theorem 1132
Review Exercises 1138
SECTION PROJECT: **The Planimeter** 1140
P.S. Problem Solving 1141

Appendix A Proofs of Selected Theorems A2

Appendix B Integration Tables A21

Answers to Odd-Numbered Exercises A25

Index A149

ADDITIONAL APPENDICES

Appendix C Precalculus Review (Online)

C.1 Real Numbers and the Real Number Line
C.2 The Cartesian Plane
C.3 Review of Trigonometric Functions

Appendix D Rotation and the General Second-Degree Equation (Online)

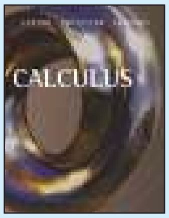
Appendix E Complex Numbers (Online)

Appendix F Business and Economic Applications (Online)

A Word from the Authors

Welcome to the Ninth Edition of *Calculus*! We are proud to offer you a new and revised version of our textbook. Much has changed since we wrote the first edition over 35 years ago. With each edition we have listened to you, our users, and have incorporated many of your suggestions for improvement.

9th



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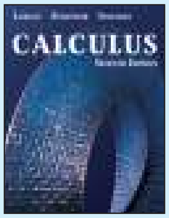

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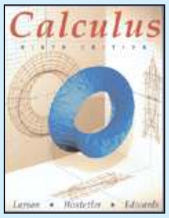

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Throughout the years, our objective has always been to write in a precise, readable manner with the fundamental concepts and rules of calculus clearly defined and demonstrated. When writing for students, we strive to offer features and materials that enable mastery by all types of learners. For the instructors, we aim to provide a comprehensive teaching instrument that employs proven pedagogical techniques, freeing instructors to make the most efficient use of classroom time.

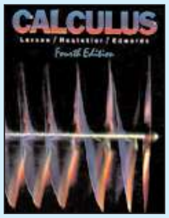
This revision brings us to a new level of change and improvement. For the past several years, we've maintained an independent website**—CalcChat.com—**that provides free solutions to all odd-numbered exercises in the text. Thousands of students using our textbooks have visited the site for practice and help with their homework. With the Ninth Edition, we were able to use information from CalcChat.com, including which solutions students accessed most often, to help guide the revision of the exercises. This edition of *Calculus* will be the first calculus textbook to use actual data from students.

We have also added a new feature called *Capstone* exercises to this edition. These conceptual problems synthesize key topics and provide students with a better understanding of each section's concepts. *Capstone* exercises are excellent for classroom discussion or test prep, and instructors may find value in integrating these problems into their review of the section. These and other new features join our time-tested pedagogy, with the goal of enabling students and instructors to make the most efficient use of this text.

We hope you will enjoy the Ninth Edition of *Calculus*. As always, we welcome comments and suggestions for continued improvements.

Ron Larson　　Bruce H. Edwards

5th



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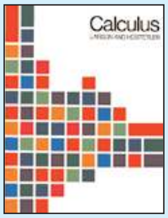

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A special note of thanks goes to the instructors who responded to our survey and to the over 2 million students who have used earlier editions of the text.

We would also like to thank the staff at Larson Texts, Inc., who assisted in preparing the manuscript, rendering the art package, typesetting, and proofreading the pages and supplements.

On a personal level, we are grateful to our wives, Deanna Gilbert Larson and Consuelo Edwards, for their love, patience, and support. Also, a special note of thanks goes out to R. Scott O'Neil.

If you have suggestions for improving this text, please feel free to write to us. Over the years we have received many useful comments from both instructors and students, and we value these very much.

Ron Larson

Bruce H. Edwards

Your Course. Your Way.

Calculus Textbook Options

The Ninth Edition of *Calculus* is available in a variety of textbook configurations to address the different ways instructors teach—and students take—their classes.

It is available in a comprehensive three-semester version or as single-variable and multivariable versions. The book can also be customized to meet your individual needs and is available through iChapters —www.ichapters.com.

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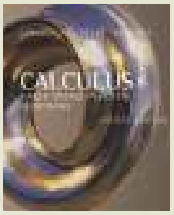
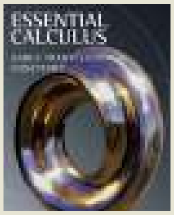
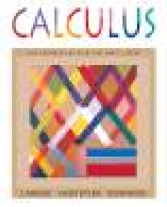
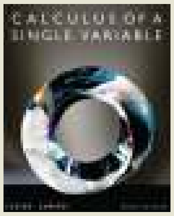
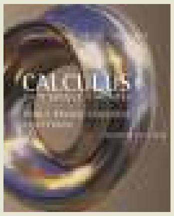
3-semester	Calculus 9e	Calculus: Early Transcendental Functions 4e 	Essential Calculus 	Calculus with Late Trigonometry 
Single Variable Only	Calculus 9e Single Variable 	Calculus: Early Transcendental Functions 4e Single Variable 		
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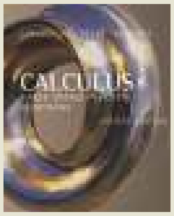
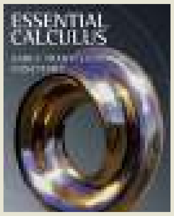
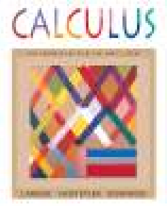
Custom All of these textbook choices can be customized to fit the individual needs of your course.	Calculus 9e	Calculus: Early Transcendental Functions 4e 	Essential Calculus 	Calculus with Late Trigonometry 

Textbook Features

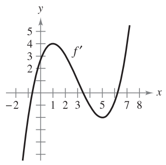
Tools to Build Mastery

CAPSTONES

NEW! Capstone exercises now appear in every section. These exercises synthesize the main concepts of each section and show students how the topics relate. They are often multipart problems that contain conceptual and noncomputational parts, and can be used for classroom discussion or test prep.

CAPSTONE

70. Use the graph of f' shown in the figure to answer the following, given that $f(0) = -4$.



(a) Approximate the slope of f at $x = 4$. Explain.

(b) Is it possible that $f(2) = -1$? Explain.

(c) Is $f(5) - f(4) > 0$? Explain.

(d) Approximate the value of x where f is maximum. Explain.

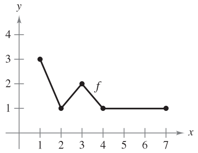
(e) Approximate any intervals in which the graph of f is concave upward and any intervals in which it is concave downward. Approximate the x-coordinates of any points of inflection.

(f) Approximate the x-coordinate of the minimum of $f''(x)$.

(g) Sketch an approximate graph of f. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

WRITING ABOUT CONCEPTS

59. The graph of f is shown in the figure.



(a) Evaluate $\int_1^7 f(x)\,dx$.

(b) Determine the average value of f on the interval $[1, 7]$.

(c) Determine the answers to parts (a) and (b) if the graph is translated two units upward.

60. If $r'(t)$ represents the rate of growth of a dog in pounds per year, what does $r(t)$ represent? What does $\int_2^6 r'(t)\,dt$ represent about the dog?

WRITING ABOUT CONCEPTS

These writing exercises are questions designed to test students' understanding of basic concepts in each section. The exercises encourage students to verbalize and write answers, promoting technical communication skills that will be invaluable in their future careers.

STUDY TIPS

The devil is in the details. Study Tips help point out some of the troublesome common mistakes, indicate special cases that can cause confusion, or expand on important concepts. These tips provide students with valuable information, similar to what an instructor might comment on in class.

STUDY TIP Because integration is usually more difficult than differentiation, you should always check your answer to an integration problem by differentiating. For instance, in Example 4 you should differentiate $\frac{1}{3}(2x -$
that you obtain the

STUDY TIP Later in this chapter, you will learn convenient methods for calculating $\int_a^b f(x)\,dx$ for continuous
or now, you must use the
ion.

STUDY TIP Remember that you can check your answer by differentiating.

EXAMPLE 6 Evaluation of a Definite Integral

Evaluate $\int_1^3 (-x^2 + 4x - 3)\,dx$ using each of the following values.

$$\int_1^3 x^2\,dx = \frac{26}{3}, \qquad \int_1^3 x\,dx = 4, \qquad \int_1^3 dx = 2$$

Solution

$$\begin{aligned}\int_1^3 (-x^2 + 4x - 3)\,dx &= \int_1^3 (-x^2)\,dx + \int_1^3 4x\,dx + \int_1^3 (-3)\,dx \\ &= -\int_1^3 x^2\,dx + 4\int_1^3 x\,dx - 3\int_1^3 dx \\ &= -\left(\frac{26}{3}\right) + 4(4) - 3(2) \\ &= \frac{4}{3}\end{aligned}$$ ■

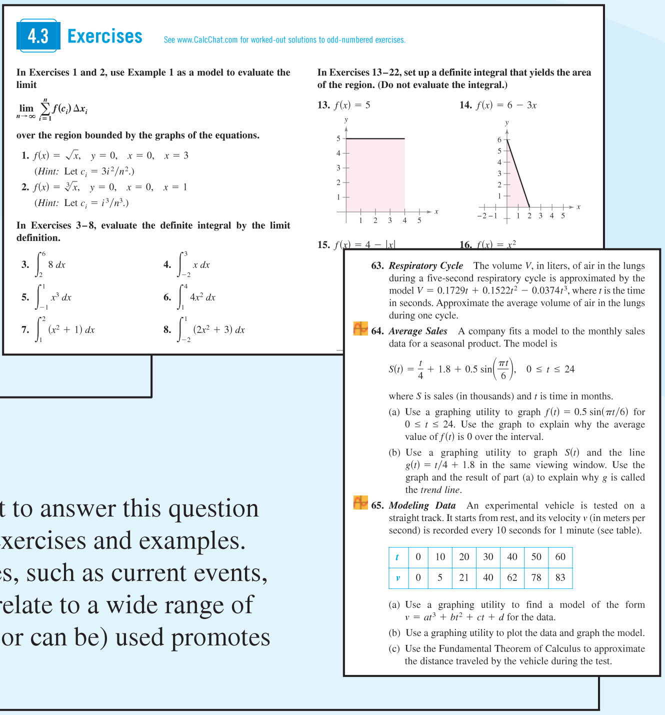
EXAMPLES

Throughout the text, examples are worked out step-by-step. These worked examples demonstrate the procedures and techniques for solving problems, and give students an increased understanding of the concepts of calculus.

EXERCISES

Practice makes perfect. Exercises are often the first place students turn to in a textbook. The authors have spent a great deal of time analyzing and revising the exercises, and the result is a comprehensive and robust set of exercises at the end of every section. A variety of exercise types and levels of difficulty are included to accommodate students with all learning styles.

In addition to the exercises in the book, 3,000 algorithmic exercises appear in the WebAssign® course that accompanies Calculus.



4.3 Exercises See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use Example 1 as a model to evaluate the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

over the region bounded by the graphs of the equations.

1. $f(x) = \sqrt{x}, \quad y = 0, \quad x = 0, \quad x = 3$
(*Hint:* Let $c_i = 3i^2/n^2$.)

2. $f(x) = \sqrt[3]{x}, \quad y = 0, \quad x = 0, \quad x = 1$
(*Hint:* Let $c_i = i^3/n^3$.)

In Exercises 3–8, evaluate the definite integral by the limit definition.

3. $\int_2^6 8\,dx$ **4.** $\int_{-2}^3 x\,dx$

5. $\int_{-1}^1 x^3\,dx$ **6.** $\int_1^4 4x^2\,dx$

7. $\int_1^2 (x^2 + 1)\,dx$ **8.** $\int_{-2}^1 (2x^2 + 3)\,dx$

In Exercises 13–22, set up a definite integral that yields the area of the region. (Do not evaluate the integral.)

13. $f(x) = 5$ **14.** $f(x) = 6 - 3x$

15. $f(x) = 4 - |x|$ **16.** $f(x) = x^2$

63. ***Respiratory Cycle*** The volume V, in liters, of air in the lungs during a five-second respiratory cycle is approximated by the model $V = 0.1729t + 0.1522t^2 - 0.0374t^3$, where t is the time in seconds. Approximate the average volume of air in the lungs during one cycle.

64. ***Average Sales*** A company fits a model to the monthly sales data for a seasonal product. The model is

$$S(t) = \frac{t}{4} + 1.8 + 0.5\sin\left(\frac{\pi t}{6}\right), \quad 0 \le t \le 24$$

where S is sales (in thousands) and t is time in months.

(a) Use a graphing utility to graph $f(t) = 0.5\sin(\pi t/6)$ for $0 \le t \le 24$. Use the graph to explain why the average value of $f(t)$ is 0 over the interval.

(b) Use a graphing utility to graph $S(t)$ and the line $g(t) = t/4 + 1.8$ in the same viewing window. Use the graph and the result of part (a) to explain why g is called the *trend line*.

65. ***Modeling Data*** An experimental vehicle is tested on a straight track. It starts from rest, and its velocity v (in meters per second) is recorded every 10 seconds for 1 minute (see table).

t	0	10	20	30	40	50	60
v	0	5	21	40	62	78	83

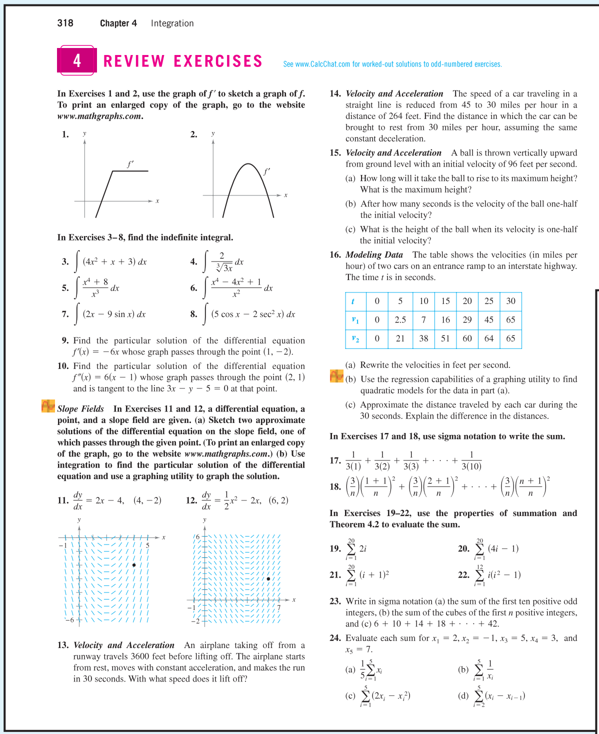
(a) Use a graphing utility to find a model of the form $v = at^3 + bt^2 + ct + d$ for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the Fundamental Theorem of Calculus to approximate the distance traveled by the vehicle during the test.

APPLICATIONS

"When will I use this?" The authors attempt to answer this question for students with carefully chosen applied exercises and examples. Applications are pulled from diverse sources, such as current events, world data, industry trends, and more, and relate to a wide range of interests. Understanding where calculus is (or can be) used promotes fuller understanding of the material.



318 Chapter 4 Integration

4 REVIEW EXERCISES See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use the graph of f' to sketch a graph of f. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

1. **2.**

In Exercises 3–8, find the indefinite integral.

3. $\int (4x^2 + x + 3)\,dx$ **4.** $\int \frac{2}{\sqrt[3]{3x}}\,dx$

5. $\int \frac{x^4 + 8}{x^3}\,dx$ **6.** $\int \frac{x^4 - 4x^2 + 1}{x^2}\,dx$

7. $\int (2x - 9\sin x)\,dx$ **8.** $\int (5\cos x - 2\sec^2 x)\,dx$

9. Find the particular solution of the differential equation $f'(x) = -6x$ whose graph passes through the point $(1, -2)$.

10. Find the particular solution of the differential equation $f''(x) = 6(x - 1)$ whose graph passes through the point $(2, 1)$ and is tangent to the line $3x - y - 5 = 0$ at that point.

Slope Fields **In Exercises 11 and 12, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution.**

11. $\frac{dy}{dx} = 2x - 4, \quad (4, -2)$ **12.** $\frac{dy}{dx} = \frac{1}{2}x^2 - 2x, \quad (6, 2)$

13. ***Velocity and Acceleration*** An airplane taking off from a runway travels 3600 feet before lifting off. The airplane starts from rest, moves with constant acceleration, and makes the run in 30 seconds. With what speed does it lift off?

14. ***Velocity and Acceleration*** The speed of a car traveling in a straight line is reduced from 45 to 30 miles per hour in a distance of 264 feet. Find the distance in which the car can be brought to rest from 30 miles per hour, assuming the same constant deceleration.

15. ***Velocity and Acceleration*** A ball is thrown vertically upward from ground level with an initial velocity of 96 feet per second.

(a) How long will it take the ball to rise to its maximum height? What is the maximum height?

(b) After how many seconds is the velocity of the ball one-half the initial velocity?

(c) What is the height of the ball when its velocity is one-half the initial velocity?

16. ***Modeling Data*** The table shows the velocities (in miles per hour) of two cars on an entrance ramp to an interstate highway. The time t is in seconds.

t	0	5	10	15	20	25	30
v_1	0	2.5	7	16	29	45	65
v_2	0	21	38	51	60	64	65

(a) Rewrite the velocities in feet per second.

(b) Use the regression capabilities of a graphing utility to find quadratic models for the data in part (a).

(c) Approximate the distance traveled by each car during the 30 seconds. Explain the difference in the distances.

In Exercises 17 and 18, use sigma notation to write the sum.

17. $\frac{1}{3(1)} + \frac{1}{3(2)} + \frac{1}{3(3)} + \cdots + \frac{1}{3(10)}$

18. $\left(\frac{3}{n}\right)\left(\frac{1+1}{n}\right)^2 + \left(\frac{3}{n}\right)\left(\frac{2+1}{n}\right)^2 + \cdots + \left(\frac{3}{n}\right)\left(\frac{n+1}{n}\right)^2$

In Exercises 19–22, use the properties of summation and Theorem 4.2 to evaluate the sum.

19. $\sum_{i=1}^{20} 2i$ **20.** $\sum_{i=1}^{20} (4i - 1)$

21. $\sum_{i=1}^{20} (i + 1)^2$ **22.** $\sum_{i=1}^{12} i(i^2 - 1)$

23. Write in sigma notation (a) the sum of the first ten positive odd integers, (b) the sum of the cubes of the first n positive integers, and (c) $6 + 10 + 14 + 18 + \cdots + 42$.

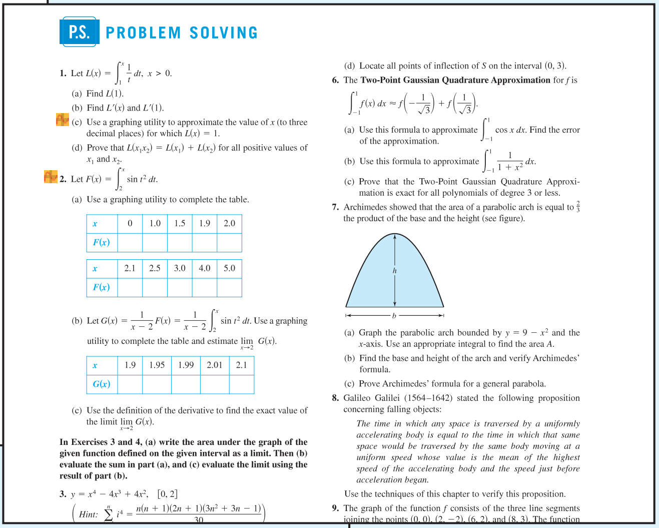
24. Evaluate each sum for $x_1 = 2, x_2 = -1, x_3 = 5, x_4 = 3,$ and $x_5 = 7$.

(a) $\frac{1}{5}\sum_{i=1}^{5} x_i$ (b) $\sum_{i=1}^{5} \frac{1}{x_i}$

(c) $\sum_{i=1}^{5} (2x_i - x_i^2)$ (d) $\sum_{i=2}^{5} (x_i - x_{i-1})$

REVIEW EXERCISES

Review Exercises at the end of each chapter provide more practice for students. These exercise sets provide a comprehensive review of the chapter's concepts and are an excellent way for students to prepare for an exam.



P.S. PROBLEM SOLVING

1. Let $L(x) = \int_1^x \frac{1}{t}\,dt, \; x > 0$.

(a) Find $L(1)$.

(b) Find $L'(x)$ and $L'(1)$.

(c) Use a graphing utility to approximate the value of x (to three decimal places) for which $L(x) = 1$.

(d) Prove that $L(x_1x_2) = L(x_1) + L(x_2)$ for all positive values of x_1 and x_2.

2. Let $F(x) = \int_2^x \sin t^2\,dt$.

(a) Use a graphing utility to complete the table.

x	0	1.0	1.5	1.9	2.0
$F(x)$					

x	2.1	2.5	3.0	4.0	5.0
$F(x)$					

(b) Let $G(x) = \frac{1}{x-2}F(x) = \frac{1}{x-2}\int_2^x \sin t^2\,dt$. Use a graphing utility to complete the table and estimate $\lim_{x\to 2} G(x)$.

x	1.9	1.95	1.99	2.01	2.1
$G(x)$					

(c) Use the definition of the derivative to find the exact value of the limit $\lim_{x\to 2} G(x)$.

In Exercises 3 and 4, (a) write the area under the graph of the given function defined on the given interval as a limit. Then (b) evaluate the sum in part (a), and (c) evaluate the limit using the result of part (b).

3. $y = x^4 - 4x^3 + 4x^2, \quad [0, 2]$

$\left(\textit{Hint: } \sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\right)$

(d) Locate all points of inflection of S on the interval $(0, 3)$.

6. The **Two-Point Gaussian Quadrature Approximation** for f is

$$\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

(a) Use this formula to approximate $\int_{-1}^{1} \cos x\,dx$. Find the error of the approximation.

(b) Use this formula to approximate $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$.

(c) Prove that the Two-Point Gaussian Quadrature Approximation is exact for all polynomials of degree 3 or less.

7. Archimedes showed that the area of a parabolic arch is equal to $\frac{2}{3}$ the product of the base and the height (see figure).

(a) Graph the parabolic arch bounded by $y = 9 - x^2$ and the x-axis. Use an appropriate integral to find the area A.

(b) Find the base and height of the arch and verify Archimedes' formula.

(c) Prove Archimedes' formula for a general parabola.

8. Galileo Galilei (1564–1642) stated the following proposition concerning falling objects:

> *The time in which any space is traversed by a uniformly accelerating body is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed of the accelerating body and the speed just before acceleration began.*

Use the techniques of this chapter to verify this proposition.

9. The graph of the function f consists of the three line segments joining the points $(0, 0)$, $(2, -2)$, $(6, 2)$, and $(8, 3)$. The function

P.S. PROBLEM SOLVING

These sets of exercises at the end of each chapter test students' abilities with challenging, thought-provoking questions.

Classic Calculus with Contemporary Relevance

THEOREMS

Theorems provide the conceptual framework for calculus. Theorems are clearly stated and separated from the rest of the text by boxes for quick visual reference. Key proofs often follow the theorem, and other proofs are provided in an in-text appendix.

THEOREM 4.9 THE FUNDAMENTAL THEOREM OF CALCULUS

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

DEFINITIONS

As with the theorems, definitions are clearly stated using precise, formal wording and are separated from the text by boxes for quick visual reference.

DEFINITION OF DEFINITE INTEGRAL

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The limit is called the **definite integral** of f from a to b. **lower limit** of integration, and the number b is the **upper**

PROCEDURES

Formal procedures are set apart from the text for easy reference. The procedures provide students with step-by-step instructions that will help them solve problems quickly and efficiently.

To complete the change of variables in Example 5, you solved for x in terms of u. Sometimes this is very difficult. Fortunately it is not always necessary, as shown in the next example.

EXAMPLE 6 Change of Variables

Find $\int \sin^2 3x \cos 3x\,dx$.

Solution Because $\sin^2 3x = (\sin 3x)^2$, you can let $u = \sin 3x$. Then

$$du = (\cos 3x)(3)\,dx.$$

Now, because $\cos 3x\,dx$ is part of the original integral, you can write

$$\frac{du}{3} = \cos 3x\,dx.$$

Substituting u and $du/3$ in the original integral yields

$$\begin{aligned}\int \sin^2 3x \cos 3x\,dx &= \int u^2\,\frac{du}{3}\\ &= \frac{1}{3}\int u^2\,du\\ &= \frac{1}{3}\left(\frac{u^3}{3}\right) + C\\ &= \frac{1}{9}\sin^3 3x + C.\end{aligned}$$

You can check this by differentiating.

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{9}\sin^3 3x\right] &= \left(\frac{1}{9}\right)(3)(\sin 3x)^2(\cos 3x)(3)\\ &= \sin^2 3x \cos 3x\end{aligned}$$

that you have ■

NOTES

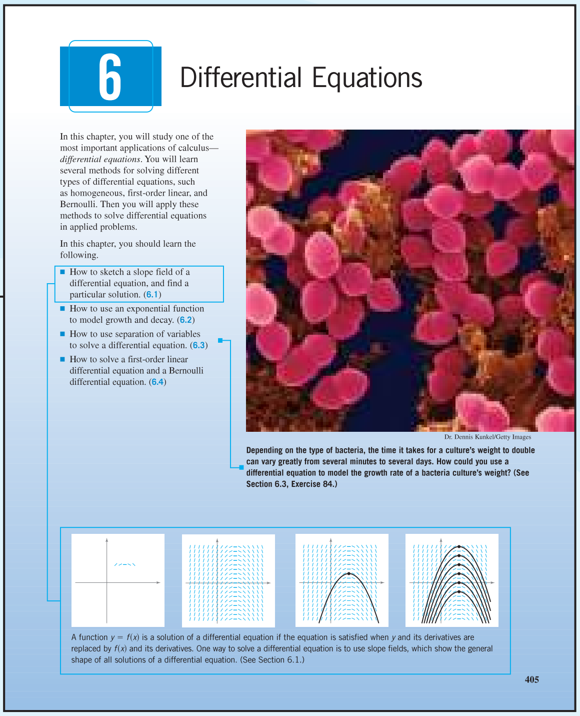
Notes provide additional details about theorems, definitions, and examples. They offer additional insight, or important generalizations that students might not immediately see. Like the study tips, notes can be invaluable to students.

NOTE There are two important points that should be made concerning the Trapezoidal Rule (or the Midpoint Rule). First, the approximation tends to become more accurate as n increases. For instance, in Example 1, if $n = 16$, the Trapezoidal Rule yields an approximation of 1.994. Second, although you could have used the Fundamental Theorem to evaluate the integral in Example 1, this theorem cannot be used to evaluate an integral as simple as $\int_0^{\pi} \sin x^2\,dx$ because $\sin x^2$ has no elementary antiderivative. Yet, the Trapezoidal Rule can be applied easily to estimate this integral. ■

Expanding the Experience of Calculus

CHAPTER OPENERS

Chapter Openers provide initial motivation for the upcoming chapter material. Along with a map of the chapter objectives, an important concept in the chapter is related to an application of the topic in the real world. Students are encouraged to see the real-life relevance of calculus.



6 Differential Equations

In this chapter, you will study one of the most important applications of calculus—*differential equations*. You will learn several methods for solving different types of differential equations, such as homogeneous, first-order linear, and Bernoulli. Then you will apply these methods to solve differential equations in applied problems.

In this chapter, you should learn the following.

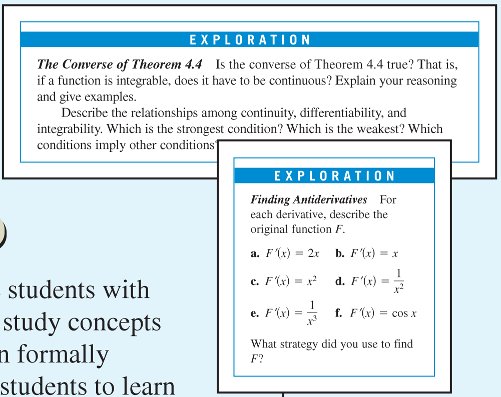
- How to sketch a slope field of a differential equation, and find a particular solution. (6.1)
- How to use an exponential function to model growth and decay. (6.2)
- How to use separation of variables to solve a differential equation. (6.3)
- How to solve a first-order linear differential equation and a Bernoulli differential equation. (6.4)

Dr. Dennis Kunkel/Getty Images

Depending on the type of bacteria, the time it takes for a culture's weight to double can vary greatly from several minutes to several days. How could you use a differential equation to model the growth rate of a bacteria culture's weight? (See Section 6.3, Exercise 84.)

A function $y = f(x)$ is a solution of a differential equation if the equation is satisfied when y and its derivatives are replaced by $f(x)$ and its derivatives. One way to solve a differential equation is to use slope fields, which show the general shape of all solutions of a differential equation. (See Section 6.1.)

405



EXPLORATION

The Converse of Theorem 4.4 Is the converse of Theorem 4.4 true? That is, if a function is integrable, does it have to be continuous? Explain your reasoning and give examples.

Describe the relationships among continuity, differentiability, and integrability. Which is the strongest condition? Which is the weakest? Which conditions imply other conditions

EXPLORATION

Finding Antiderivatives For each derivative, describe the original function F.

a. $F'(x) = 2x$ **b.** $F'(x) = x$

c. $F'(x) = x^2$ **d.** $F'(x) = \frac{1}{x^2}$

e. $F'(x) = \frac{1}{x^3}$ **f.** $F'(x) = \cos x$

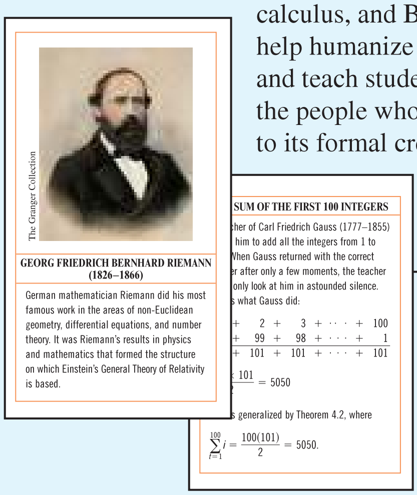
What strategy did you use to find F?

EXPLORATIONS

Explorations provide students with unique challenges to study concepts that have not yet been formally covered. They allow students to learn by discovery and introduce topics related to ones they are presently studying. By exploring topics in this way, students are encouraged to think outside the box.

HISTORICAL NOTES AND BIOGRAPHIES

Historical Notes provide students with background information on the foundations of calculus, and Biographies help humanize calculus and teach students about the people who contributed to its formal creation.



The Granger Collection

GEORG FRIEDRICH BERNHARD RIEMANN (1826–1866)

German mathematician Riemann did his most famous work in the areas of non-Euclidean geometry, differential equations, and number theory. It was Riemann's results in physics and mathematics that formed the structure on which Einstein's General Theory of Relativity is based.

SUM OF THE FIRST 100 INTEGERS

cher of Carl Friedrich Gauss (1777–1855) him to add all the integers from 1 to When Gauss returned with the correct er after only a few moments, the teacher only look at him in astounded silence. s what Gauss did:

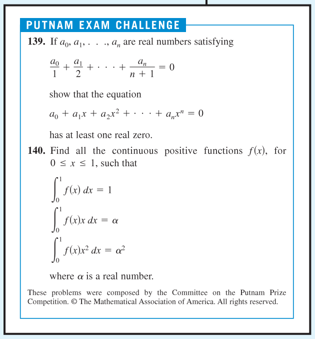
$$\begin{array}{rrrrrrrr} + & 2 & + & 3 & + & \cdots & + & 100 \\ + & 99 & + & 98 & + & \cdots & + & 1 \\ \hline + & 101 & + & 101 & + & \cdots & + & 101 \end{array}$$

$$101 = 5050$$

s generalized by Theorem 4.2, where

$$\sum_{i=1}^{100} i = \frac{100(101)}{2} = 5050.$$

PUTNAM EXAM CHALLENGES



PUTNAM EXAM CHALLENGE

139. If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$$

show that the equation

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

has at least one real zero.

140. Find all the continuous positive functions $f(x)$, for $0 \le x \le 1$, such that

$$\int_0^1 f(x)\,dx = 1$$

$$\int_0^1 f(x)x\,dx = \alpha$$

$$\int_0^1 f(x)x^2\,dx = \alpha^2$$

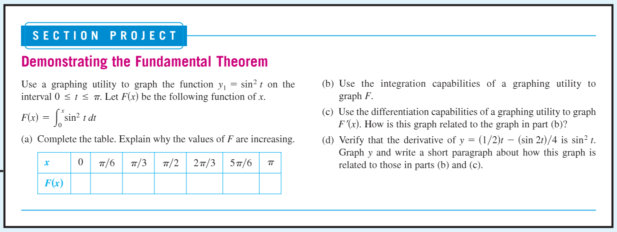
where α is a real number.

These problems were composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.

Putnam Exam questions appear in selected sections and are drawn from actual Putnam Exams. These exercises will push the limits of students' understanding of calculus and provide extra challenges for motivated students.

SECTION PROJECTS

Projects appear in selected sections and more deeply explore applications related to the topics being studied. They provide an interesting and engaging way for students to work and investigate ideas collaboratively.



SECTION PROJECT

Demonstrating the Fundamental Theorem

Use a graphing utility to graph the function $y_1 = \sin^2 t$ on the interval $0 \le t \le \pi$. Let $F(x)$ be the following function of x.

$$F(x) = \int_0^x \sin^2 t\,dt$$

(a) Complete the table. Explain why the values of F are increasing.

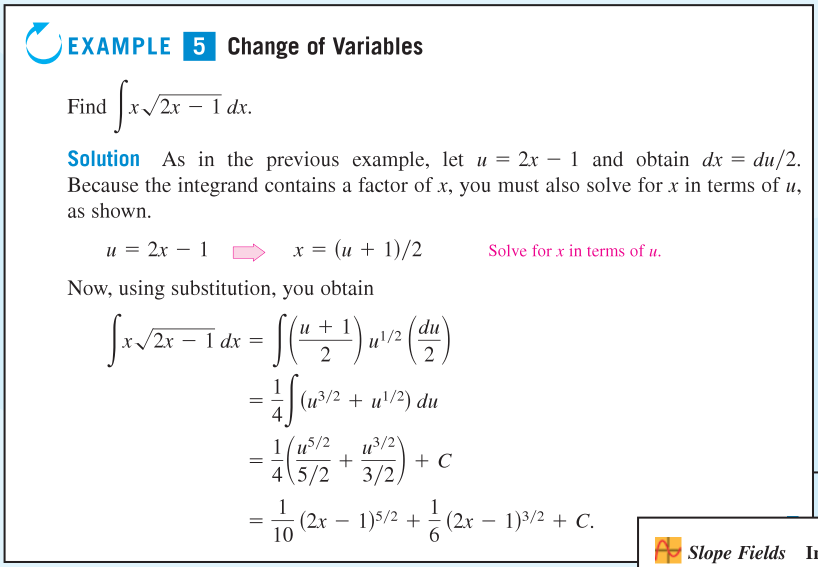
x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$F(x)$							

(b) Use the integration capabilities of a graphing utility to graph F.

(c) Use the differentiation capabilities of a graphing utility to graph $F'(x)$. How is this graph related to the graph in part (b)?

(d) Verify that the derivative of $y = (1/2)t - (\sin 2t)/4$ is $\sin^2 t$. Graph y and write a short paragraph about how this graph is related to those in parts (b) and (c).

Integrated Technology for Today's World



EXAMPLE 5 Change of Variables

Find $\int x\sqrt{2x-1}\,dx$.

Solution As in the previous example, let $u = 2x - 1$ and obtain $dx = du/2$. Because the integrand contains a factor of x, you must also solve for x in terms of u, as shown.

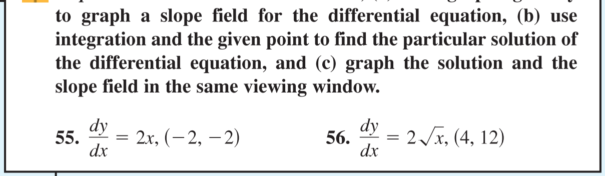
$$u = 2x - 1 \quad \Rightarrow \quad x = (u+1)/2 \qquad \text{Solve for } x \text{ in terms of } u.$$

Now, using substitution, you obtain

$$\begin{aligned}\int x\sqrt{2x-1}\,dx &= \int \left(\frac{u+1}{2}\right) u^{1/2} \left(\frac{du}{2}\right)\\ &= \frac{1}{4}\int (u^{3/2} + u^{1/2})\,du\\ &= \frac{1}{4}\left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C\\ &= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C.\end{aligned}$$

CAS INVESTIGATIONS

Examples throughout the book are accompanied by CAS Investigations. These investigations are linked explorations that use a computer algebra system (e.g., Maple®) to further explore a related example in the book. They allow students to explore calculus by manipulating functions, graphs, etc. and observing the results. (Formerly called Open Explorations)



Slope Fields **In Exercises 55 and 56, (a) use a graphing utility to graph a slope field for the differential equation, (b) use integration and the given point to find the particular solution of the differential equation, and (c) graph the solution and the slope field in the same viewing window.**

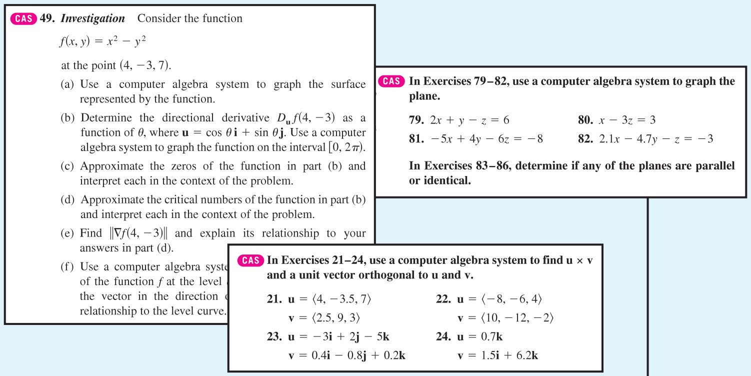
55. $\dfrac{dy}{dx} = 2x,\ (-2, -2)$ **56.** $\dfrac{dy}{dx} = 2\sqrt{x},\ (4, 12)$

GRAPHING TECH EXERCISES

Understanding is often enhanced by using a graph or visualization. Graphing Tech Exercises are exercises that ask students to make use of a graphing utility to help find a solution. These exercises are marked with a special icon.

TECHNOLOGY

Throughout the book, technology boxes give students a glimpse of how technology may be used to help solve problems and explore the concepts of calculus. They provide discussions of not only where technology succeeds, but also where it may fail.



CAS **49.** ***Investigation*** Consider the function

$$f(x, y) = x^2 - y^2$$

at the point $(4, -3, 7)$.

(a) Use a computer algebra system to graph the surface represented by the function.

(b) Determine the directional derivative $D_{\mathbf{u}}f(4, -3)$ as a function of θ, where $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$. Use a computer algebra system to graph the function on the interval $[0, 2\pi)$.

(c) Approximate the zeros of the function in part (b) and interpret each in the context of the problem.

(d) Approximate the critical numbers of the function in part (b) and interpret each in the context of the problem.

(e) Find $\|\nabla f(4, -3)\|$ and explain its relationship to your answers in part (d).

(f) Use a computer algebra syste
of the function f at the level
the vector in the direction
relationship to the level curve.

CAS **In Exercises 79–82, use a computer algebra system to graph the plane.**

79. $2x + y - z = 6$ **80.** $x - 3z = 3$

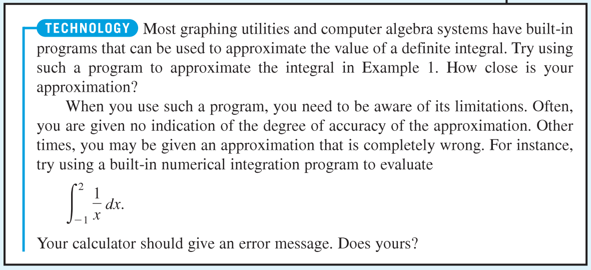
81. $-5x + 4y - 6z = -8$ **82.** $2.1x - 4.7y - z = -3$

In Exercises 83–86, determine if any of the planes are parallel or identical.

CAS **In Exercises 21–24, use a computer algebra system to find u × v and a unit vector orthogonal to u and v.**

21. $\mathbf{u} = \langle 4, -3.5, 7\rangle$, $\mathbf{v} = \langle 2.5, 9, 3\rangle$ **22.** $\mathbf{u} = \langle -8, -6, 4\rangle$, $\mathbf{v} = \langle 10, -12, -2\rangle$

23. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{v} = 0.4\mathbf{i} - 0.8\mathbf{j} + 0.2\mathbf{k}$ **24.** $\mathbf{u} = 0.7\mathbf{k}$, $\mathbf{v} = 1.5\mathbf{i} + 6.2\mathbf{k}$



TECHNOLOGY Most graphing utilities and computer algebra systems have built-in programs that can be used to approximate the value of a definite integral. Try using such a program to approximate the integral in Example 1. How close is your approximation?

When you use such a program, you need to be aware of its limitations. Often, you are given no indication of the degree of accuracy of the approximation. Other times, you may be given an approximation that is completely wrong. For instance, try using a built-in numerical integration program to evaluate

$$\int_{-1}^{2} \frac{1}{x}\,dx.$$

Your calculator should give an error message. Does yours?

CAS EXERCISES

NEW! Like the Graphing Tech Exercises, some exercises may best be solved using a computer algebra system. These CAS Exercises are new to this edition and are denoted by a special icon.

Additional Resources

Student Resources

- **Student Solutions Manual**—Need a leg up on your homework or help to prepare for an exam? The *Student Solutions Manual* contains worked-out solutions for all odd-numbered exercises in the text. It is a great resource to help you understand how to solve those tough problems.
- **Notetaking Guide**—This notebook organizer is designed to help you organize your notes, and provides section-by-section summaries of key topics and other helpful study tools. The Notetaking Guide is available for download on the book's website.
- **WebAssign®**—The most widely used homework system in higher education, WebAssign offers instant feedback and repeatable problems, everything you could ask for in an online homework system. WebAssign's homework system lets you practice and submit homework via the web. It is easy to use and loaded with extra resources. With this edition of Larson's *Calculus*, there are over 3,000 algorithmic homework exercises to use for practice and review.
- **DVD Lecture Series**—Comprehensive, instructional lecture presentations serve a number of uses. They are great if you need to catch up after missing a class, need to supplement online or hybrid instruction, or need material for self-study or review.
- **CalcLabs with Maple® and Mathematica®**—Working with *Maple* or *Mathematica* in class? Be sure to pick up one of these comprehensive manuals that will help you use each program efficiently.

Instructor Resources

- **WebAssign®**—Instant feedback, grading precision, and ease of use are just three reasons why WebAssign is the most widely used homework system in higher education. WebAssign's homework delivery system lets instructors deliver, collect, grade, and record assignments via the web. With this edition of Larson's *Calculus,* there are over 3,000 algorithmic homework exercises to choose from. These algorithmic exercises are based on the section exercises from the textbook to ensure alignment with course goals.
- **Instructor's Complete Solutions Manual**—This manual contains worked-out solutions for all exercises in the text. It also contains solutions for the special features in the text such as Explorations, Section Projects, etc. It is available on the *Instructor's Resource Center* at the book's website.
- **Instructor's Resource Manual**—This robust manual contains an abundance of resources keyed to the textbook by chapter and section, including chapter summaries and teaching strategies. New to this edition's manual are the authors' findings from CalcChat.com (*see* A Word from the Authors). They offer suggestions for exercises to cover in class, identify tricky exercises with tips on how best to use them, and explain what changes were made in the exercise set based on the research.
- **Power Lecture**—This comprehensive CD-ROM includes the *Instructor's Complete Solutions Manual*, PowerPoint® slides, and the computerized test bank featuring algorithmically created questions that can be used to create, deliver, and customize tests.
- **Computerized Test Bank**—Create, deliver, and customize tests and study guides in minutes with this easy to use assessment software on CD. The thousands of algorithmic questions in the test bank are derived from the textbook exercises, ensuring consistency between exams and the book.
- **JoinIn on TurningPoint**—Enhance your students' interactions with you, your lectures, and each other. Cengage Learning is now pleased to offer you book-specific content for Response Systems tailored to Larson's *Calculus*, allowing you to transform your classroom and assess your students' progress with instant in-class quizzes and polls.

P

Preparation for Calculus

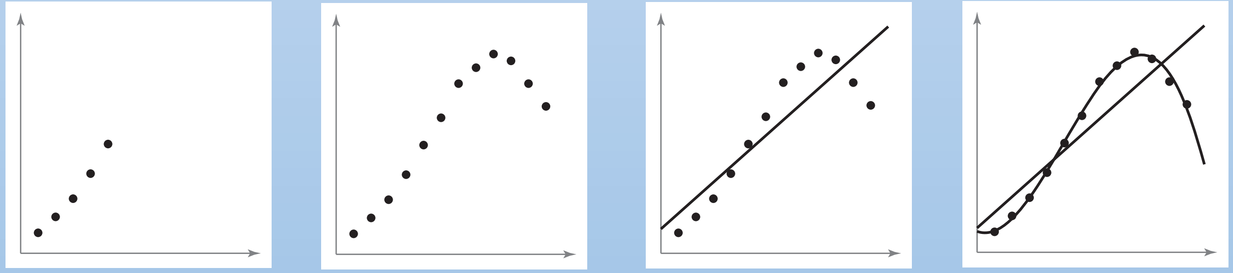
This chapter reviews several concepts that will help you prepare for your study of calculus. These concepts include sketching the graphs of equations and functions, and fitting mathematical models to data. It is important to review these concepts before moving on to calculus.

In this chapter, you should learn the following.

- How to identify the characteristics of equations and sketch their graphs. (**P.1**)
- How to find and graph equations of lines, including parallel and perpendicular lines, using the concept of slope. (**P.2**)
- How to evaluate and graph functions and their transformations. (**P.3**)
- How to fit mathematical models to real-life data sets. (**P.4**)

Jeremy Walker/Getty Images

In 2006, China surpassed the United States as the world's biggest emitter of carbon dioxide, the main greenhouse gas. Given the carbon dioxide concentrations in the atmosphere for several years, can older mathematical models still accurately predict future atmospheric concentrations compared with more recent models? (See Section P.1, Example 6.)



Mathematical models are commonly used to describe data sets. These models can be represented by many different types of functions, such as linear, quadratic, cubic, rational, and trigonometric functions. (See Section P.4.)

P.1 Graphs and Models

- Sketch the graph of an equation.
- Find the intercepts of a graph.
- Test a graph for symmetry with respect to an axis and the origin.
- Find the points of intersection of two graphs.
- Interpret mathematical models for real-life data.

Archive Photos

RENÉ DESCARTES (1596–1650)

Descartes made many contributions to philosophy, science, and mathematics. The idea of representing points in the plane by pairs of real numbers and representing curves in the plane by equations was described by Descartes in his book *La Géométrie*, published in 1637.

The Graph of an Equation

In 1637 the French mathematician René Descartes revolutionized the study of mathematics by joining its two major fields—algebra and geometry. With Descartes's coordinate plane, geometric concepts could be formulated analytically and algebraic concepts could be viewed graphically. The power of this approach was such that within a century of its introduction, much of calculus had been developed.

The same approach can be followed in your study of calculus. That is, by viewing calculus from multiple perspectives—*graphically*, *analytically*, and *numerically*—you will increase your understanding of core concepts.

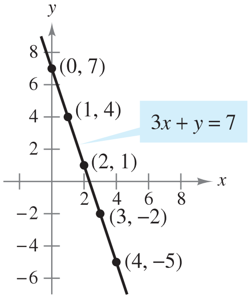
Consider the equation $3x + y = 7$. The point $(2, 1)$ is a **solution point** of the equation because the equation is satisfied (is true) when 2 is substituted for x and 1 is substituted for y. This equation has many other solutions, such as $(1, 4)$ and $(0, 7)$. To find other solutions systematically, solve the original equation for y.

$y = 7 - 3x$ Analytic approach

Then construct a **table of values** by substituting several values of x.

x	0	1	2	3	4
y	7	4	1	-2	-5

Numerical approach



Graphical approach: $3x + y = 7$
Figure P.1

From the table, you can see that $(0, 7)$, $(1, 4)$, $(2, 1)$, $(3, -2)$, and $(4, -5)$ are solutions of the original equation $3x + y = 7$. Like many equations, this equation has an infinite number of solutions. The set of all solution points is the **graph** of the equation, as shown in Figure P.1.

NOTE Even though we refer to the sketch shown in Figure P.1 as the graph of $3x + y = 7$, it really represents only a *portion* of the graph. The entire graph would extend beyond the page. ■

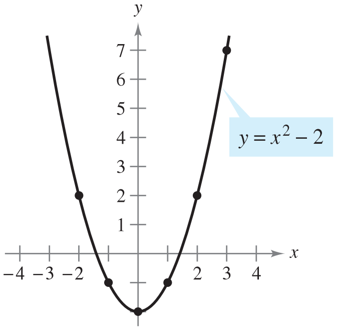
In this course, you will study many sketching techniques. The simplest is point plotting—that is, you plot points until the basic shape of the graph seems apparent.

EXAMPLE 1 Sketching a Graph by Point Plotting

Sketch the graph of $y = x^2 - 2$.

Solution First construct a table of values. Then plot the points shown in the table.

x	-2	-1	0	1	2	3
y	2	-1	-2	-1	2	7



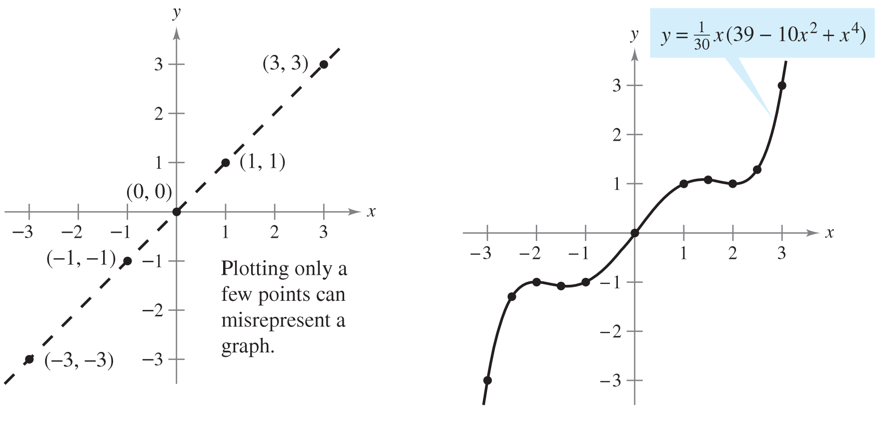
The parabola $y = x^2 - 2$
Figure P.2

Finally, connect the points with a *smooth curve*, as shown in Figure P.2. This graph is a **parabola.** It is one of the conics you will study in Chapter 10. ■

One disadvantage of point plotting is that to get a good idea about the shape of a graph, you may need to plot many points. With only a few points, you could badly misrepresent the graph. For instance, suppose that to sketch the graph of

$$y = \frac{1}{30}x(39 - 10x^2 + x^4)$$

you plotted only five points: $(-3, -3)$, $(-1, -1)$, $(0, 0)$, $(1, 1)$, and $(3, 3)$, as shown in Figure P.3(a). From these five points, you might conclude that the graph is a line. This, however, is not correct. By plotting several more points, you can see that the graph is more complicated, as shown in Figure P.3(b).



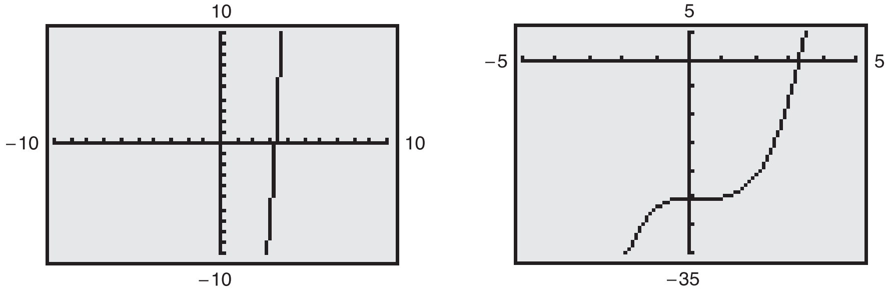
(a) **(b)**

Figure P.3

TECHNOLOGY Technology has made sketching of graphs easier. Even with technology, however, it is possible to misrepresent a graph badly. For instance, each of the graphing utility screens in Figure P.4 shows a portion of the graph of

$$y = x^3 - x^2 - 25.$$

From the screen on the left, you might assume that the graph is a line. From the screen on the right, however, you can see that the graph is not a line. So, whether you are sketching a graph by hand or using a graphing utility, you must realize that different "viewing windows" can produce very different views of a graph. In choosing a viewing window, your goal is to show a view of the graph that fits well in the context of the problem.



Graphing utility screens of $y = x^3 - x^2 - 25$

Figure P.4

NOTE In this text, the term *graphing utility* means either a graphing calculator or computer graphing software such as *Maple*, *Mathematica*, or the *TI-89*. ■

EXPLORATION

Comparing Graphical and Analytic Approaches Use a graphing utility to graph each equation. In each case, find a viewing window that shows the important characteristics of the graph.

a. $y = x^3 - 3x^2 + 2x + 5$

b. $y = x^3 - 3x^2 + 2x + 25$

c. $y = -x^3 - 3x^2 + 20x + 5$

d. $y = 3x^3 - 40x^2 + 50x - 45$

e. $y = -(x + 12)^3$

f. $y = (x - 2)(x - 4)(x - 6)$

A purely graphical approach to this problem would involve a simple "guess, check, and revise" strategy. What types of things do you think an analytic approach might involve? For instance, does the graph have symmetry? Does the graph have turns? If so, where are they?

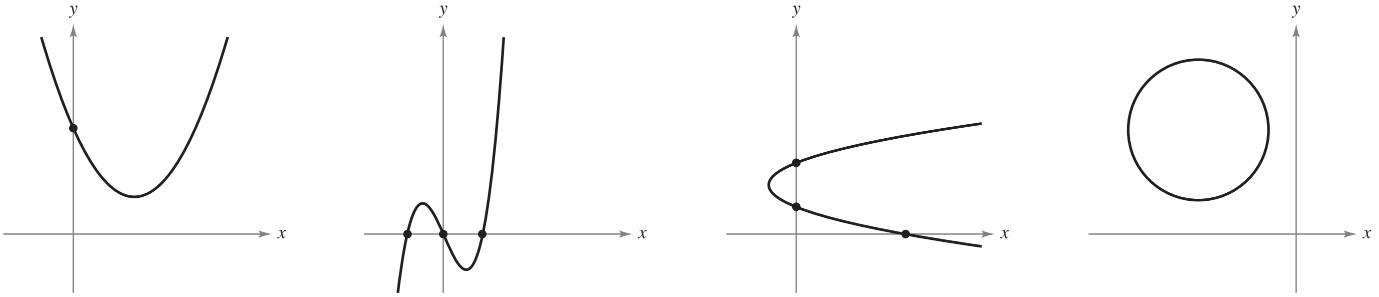
As you proceed through Chapters 1, 2, and 3 of this text, you will study many new analytic tools that will help you analyze graphs of equations such as these.

Intercepts of a Graph

Two types of solution points that are especially useful in graphing an equation are those having zero as their x- or y-coordinate. Such points are called **intercepts** because they are the points at which the graph intersects the x- or y-axis. The point $(a, 0)$ is an ***x*-intercept** of the graph of an equation if it is a solution point of the equation. To find the x-intercepts of a graph, let y be zero and solve the equation for x. The point $(0, b)$ is a ***y*-intercept** of the graph of an equation if it is a solution point of the equation. To find the y-intercepts of a graph, let x be zero and solve the equation for y.

NOTE Some texts denote the x-intercept as the x-coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, we will use the term *intercept* to mean either the point or the coordinate. ■

It is possible for a graph to have no intercepts, or it might have several. For instance, consider the four graphs shown in Figure P.5.



No x-intercepts
One y-intercept

Three x-intercepts
One y-intercept

One x-intercept
Two y-intercepts

No intercepts

Figure P.5

EXAMPLE 2 Finding x- and y-intercepts

Find the x- and y-intercepts of the graph of $y = x^3 - 4x$.

Solution To find the x-intercepts, let y be zero and solve for x.

$$\begin{aligned} x^3 - 4x &= 0 && \text{Let } y \text{ be zero.} \\ x(x-2)(x+2) &= 0 && \text{Factor.} \\ x &= 0, 2, \text{ or } -2 && \text{Solve for } x. \end{aligned}$$

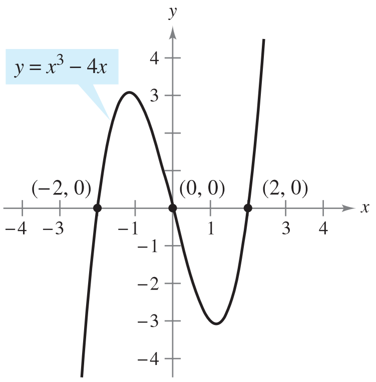
Because this equation has three solutions, you can conclude that the graph has three x-intercepts:

$$(0, 0), (2, 0), \text{ and } (-2, 0). \qquad x\text{-intercepts}$$

To find the y-intercepts, let x be zero. Doing this produces $y = 0$. So, the y-intercept is

$$(0, 0). \qquad y\text{-intercept}$$

(See Figure P.6.) ■



Intercepts of a graph
Figure P.6

TECHNOLOGY Example 2 uses an analytic approach to finding intercepts. When an analytic approach is not possible, you can use a graphical approach by finding the points at which the graph intersects the axes. Use a graphing utility to approximate the intercepts.

Symmetry of a Graph

Knowing the symmetry of a graph *before* attempting to sketch it is useful because you need only half as many points to sketch the graph. The following three types of symmetry can be used to help sketch the graphs of equations (see Figure P.7).

1. A graph is **symmetric with respect to the y-axis** if, whenever (x, y) is a point on the graph, $(-x, y)$ is also a point on the graph. This means that the portion of the graph to the left of the y-axis is a mirror image of the portion to the right of the y-axis.
2. A graph is **symmetric with respect to the x-axis** if, whenever (x, y) is a point on the graph, $(x, -y)$ is also a point on the graph. This means that the portion of the graph above the x-axis is a mirror image of the portion below the x-axis.
3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is a point on the graph, $(-x, -y)$ is also a point on the graph. This means that the graph is unchanged by a rotation of 180° about the origin.

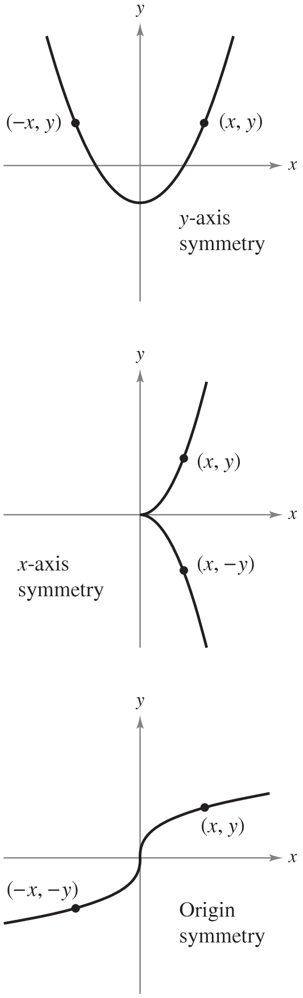


Figure P.7

TESTS FOR SYMMETRY

1. The graph of an equation in x and y is symmetric with respect to the y-axis if replacing x by $-x$ yields an equivalent equation.
2. The graph of an equation in x and y is symmetric with respect to the x-axis if replacing y by $-y$ yields an equivalent equation.
3. The graph of an equation in x and y is symmetric with respect to the origin if replacing x by $-x$ and y by $-y$ yields an equivalent equation.

The graph of a polynomial has symmetry with respect to the y-axis if each term has an even exponent (or is a constant). For instance, the graph of $y = 2x^4 - x^2 + 2$ has symmetry with respect to the y-axis. Similarly, the graph of a polynomial has symmetry with respect to the origin if each term has an odd exponent, as illustrated in Example 3.

EXAMPLE 3 Testing for Symmetry

Test the graph of $y = 2x^3 - x$ for symmetry with respect to the y-axis and to the origin.

Solution

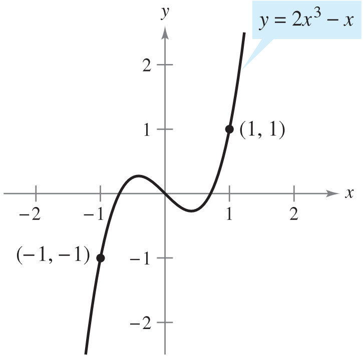
y-axis Symmetry:

$$\begin{aligned} y &= 2x^3 - x && \text{Write original equation.} \\ y &= 2(-x)^3 - (-x) && \text{Replace } x \text{ by } -x. \\ y &= -2x^3 + x && \text{Simplify. It is not an equivalent equation.} \end{aligned}$$

Origin Symmetry:

$$\begin{aligned} y &= 2x^3 - x && \text{Write original equation.} \\ -y &= 2(-x)^3 - (-x) && \text{Replace } x \text{ by } -x \text{ and } y \text{ by } -y. \\ -y &= -2x^3 + x && \text{Simplify.} \\ y &= 2x^3 - x && \text{Equivalent equation} \end{aligned}$$

Because replacing both x by $-x$ and y by $-y$ yields an equivalent equation, you can conclude that the graph of $y = 2x^3 - x$ is symmetric with respect to the origin, as shown in Figure P.8. ■



Origin symmetry
Figure P.8

EXAMPLE 4 Using Intercepts and Symmetry to Sketch a Graph

Sketch the graph of $x - y^2 = 1$.

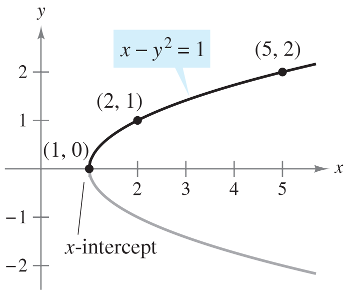


Figure P.9

Solution The graph is symmetric with respect to the x-axis because replacing y by $-y$ yields an equivalent equation.

$$\begin{aligned} x - y^2 &= 1 && \text{Write original equation.} \\ x - (-y)^2 &= 1 && \text{Replace } y \text{ by } -y. \\ x - y^2 &= 1 && \text{Equivalent equation} \end{aligned}$$

This means that the portion of the graph below the x-axis is a mirror image of the portion above the x-axis. To sketch the graph, first plot the x-intercept and the points above the x-axis. Then reflect in the x-axis to obtain the entire graph, as shown in Figure P.9. ■

TECHNOLOGY Graphing utilities are designed so that they most easily graph equations in which y is a function of x (see Section P.3 for a definition of **function**). To graph other types of equations, you need to split the graph into two or more parts *or* you need to use a different graphing mode. For instance, to graph the equation in Example 4, you can split it into two parts.

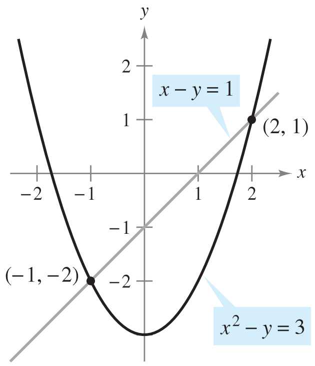
$$\begin{aligned} y_1 &= \sqrt{x - 1} && \text{Top portion of graph} \\ y_2 &= -\sqrt{x - 1} && \text{Bottom portion of graph} \end{aligned}$$

Points of Intersection

A **point of intersection** of the graphs of two equations is a point that satisfies both equations. You can find the point(s) of intersection of two graphs by solving their equations simultaneously.

EXAMPLE 5 Finding Points of Intersection

Find all points of intersection of the graphs of $x^2 - y = 3$ and $x - y = 1$.



Two points of intersection
Figure P.10

Solution Begin by sketching the graphs of both equations on the *same* rectangular coordinate system, as shown in Figure P.10. Having done this, it appears that the graphs have two points of intersection. You can find these two points, as follows.

$$\begin{aligned} y &= x^2 - 3 && \text{Solve first equation for } y. \\ y &= x - 1 && \text{Solve second equation for } y. \\ x^2 - 3 &= x - 1 && \text{Equate } y\text{-values.} \\ x^2 - x - 2 &= 0 && \text{Write in general form.} \\ (x - 2)(x + 1) &= 0 && \text{Factor.} \\ x &= 2 \text{ or } -1 && \text{Solve for } x. \end{aligned}$$

The corresponding values of y are obtained by substituting $x = 2$ and $x = -1$ into either of the original equations. Doing this produces two points of intersection:

$$(2, 1) \quad \text{and} \quad (-1, -2). \qquad \text{Points of intersection}$$ ■

STUDY TIP You can check the points of intersection in Example 5 by substituting into *both* of the original equations or by using the *intersect* feature of a graphing utility.

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

Mathematical Models

Real-life applications of mathematics often use equations as **mathematical models.** In developing a mathematical model to represent actual data, you should strive for two (often conflicting) goals: accuracy and simplicity. That is, you want the model to be simple enough to be workable, yet accurate enough to produce meaningful results. Section P.4 explores these goals more completely.

EXAMPLE 6 Comparing Two Mathematical Models

The Mauna Loa Observatory in Hawaii records the carbon dioxide concentration y (in parts per million) in Earth's atmosphere. The January readings for various years are shown in Figure P.11. In the July 1990 issue of *Scientific American*, these data were used to predict the carbon dioxide level in Earth's atmosphere in the year 2035, using the quadratic model

$$y = 316.2 + 0.70t + 0.018t^2$$

Quadratic model for 1960–1990 data

where $t = 0$ represents 1960, as shown in Figure P.11(a).

The data shown in Figure P.11(b) represent the years 1980 through 2007 and can be modeled by

$$y = 304.1 + 1.64t$$

Linear model for 1980–2007 data

where $t = 0$ represents 1960. What was the prediction given in the *Scientific American* article in 1990? Given the new data for 1990 through 2007, does this prediction for the year 2035 seem accurate?

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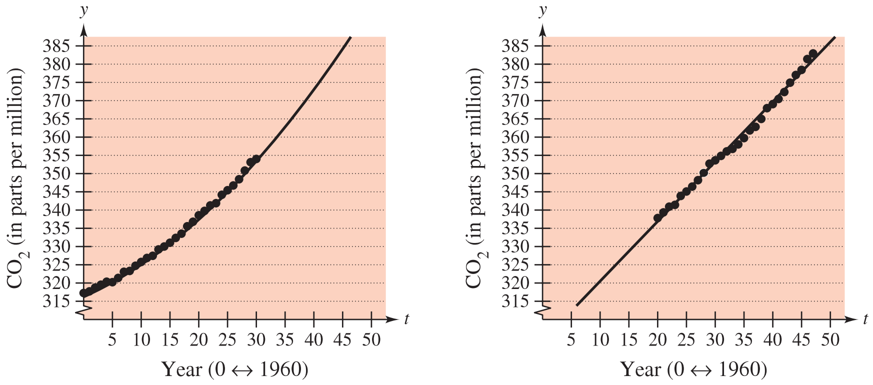
The Mauna Loa Observatory in Hawaii has been measuring the increasing concentration of carbon dioxide in Earth's atmosphere since 1958. Carbon dioxide is the main greenhouse gas responsible for global climate warming.



(a) **(b)**

Figure P.11

Solution To answer the first question, substitute $t = 75$ (for 2035) into the quadratic model.

$$y = 316.2 + 0.70(75) + 0.018(75)^2 = 469.95$$

Quadratic model

So, the prediction in the *Scientific American* article was that the carbon dioxide concentration in Earth's atmosphere would reach about 470 parts per million in the year 2035. Using the linear model for the 1980–2007 data, the prediction for the year 2035 is

$$y = 304.1 + 1.64(75) = 427.1.$$

Linear model

So, based on the linear model for 1980–2007, it appears that the 1990 prediction was too high. ■

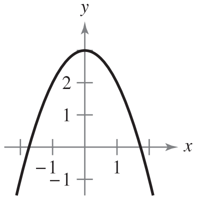
NOTE The models in Example 6 were developed using a procedure called *least squares regression* (see Section 13.9). The quadratic and linear models have correlations given by $r^2 = 0.997$ and $r^2 = 0.994$, respectively. The closer r^2 is to 1, the "better" the model.

P.1 Exercises

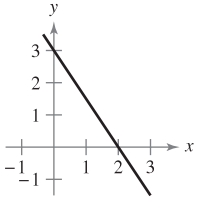
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]

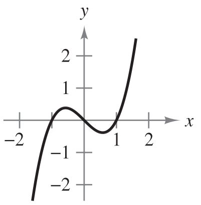
(a)



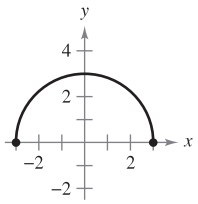
(b)



(c)



(d)



1. $y = -\frac{3}{2}x + 3$

2. $y = \sqrt{9 - x^2}$

3. $y = 3 - x^2$

4. $y = x^3 - x$

In Exercises 5–14, sketch the graph of the equation by point plotting.

5. $y = \frac{1}{2}x + 2$

6. $y = 5 - 2x$

7. $y = 4 - x^2$

8. $y = (x - 3)^2$

9. $y = |x + 2|$

10. $y = |x| - 1$

11. $y = \sqrt{x} - 6$

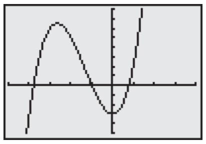
12. $y = \sqrt{x + 2}$

13. $y = \dfrac{3}{x}$

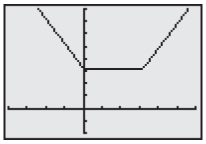
14. $y = \dfrac{1}{x + 2}$

In Exercises 15 and 16, describe the viewing window that yields the figure.

15. $y = x^3 + 4x^2 - 3$



16. $y = |x| + |x - 16|$



In Exercises 17 and 18, use a graphing utility to graph the equation. Move the cursor along the curve to approximate the unknown coordinate of each solution point accurate to two decimal places.

17. $y = \sqrt{5 - x}$ (a) $(2, y)$ (b) $(x, 3)$

18. $y = x^5 - 5x$ (a) $(-0.5, y)$ (b) $(x, -4)$

In Exercises 19–28, find any intercepts.

19. $y = 2x - 5$

20. $y = 4x^2 + 3$

21. $y = x^2 + x - 2$

22. $y^2 = x^3 - 4x$

23. $y = x\sqrt{16 - x^2}$

24. $y = (x - 1)\sqrt{x^2 + 1}$

25. $y = \dfrac{2 - \sqrt{x}}{5x}$

26. $y = \dfrac{x^2 + 3x}{(3x + 1)^2}$

27. $x^2y - x^2 + 4y = 0$

28. $y = 2x - \sqrt{x^2 + 1}$

In Exercises 29–40, test for symmetry with respect to each axis and to the origin.

29. $y = x^2 - 6$

30. $y = x^2 - x$

31. $y^2 = x^3 - 8x$

32. $y = x^3 + x$

33. $xy = 4$

34. $xy^2 = -10$

35. $y = 4 - \sqrt{x + 3}$

36. $xy - \sqrt{4 - x^2} = 0$

37. $y = \dfrac{x}{x^2 + 1}$

38. $y = \dfrac{x^2}{x^2 + 1}$

39. $y = |x^3 + x|$

40. $|y| - x = 3$

In Exercises 41–58, sketch the graph of the equation. Identify any intercepts and test for symmetry.

41. $y = 2 - 3x$

42. $y = -\frac{3}{2}x + 6$

43. $y = \frac{1}{2}x - 4$

44. $y = \frac{2}{3}x + 1$

45. $y = 9 - x^2$

46. $y = x^2 + 3$

47. $y = (x + 3)^2$

48. $y = 2x^2 + x$

49. $y = x^3 + 2$

50. $y = x^3 - 4x$

51. $y = x\sqrt{x + 5}$

52. $y = \sqrt{25 - x^2}$

53. $x = y^3$

54. $x = y^2 - 4$

55. $y = \dfrac{8}{x}$

56. $y = \dfrac{10}{x^2 + 1}$

57. $y = 6 - |x|$

58. $y = |6 - x|$

In Exercises 59–62, use a graphing utility to graph the equation. Identify any intercepts and test for symmetry.

59. $y^2 - x = 9$

60. $x^2 + 4y^2 = 4$

61. $x + 3y^2 = 6$

62. $3x - 4y^2 = 8$

In Exercises 63–70, find the points of intersection of the graphs of the equations.

63. $x + y = 8$
$4x - y = 7$

64. $3x - 2y = -4$
$4x + 2y = -10$

65. $x^2 + y = 6$
$x + y = 4$

66. $x = 3 - y^2$
$y = x - 1$

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by use of appropriate technology.

67. $x^2 + y^2 = 5$, $x - y = 1$

68. $x^2 + y^2 = 25$, $-3x + y = 15$

69. $y = x^3$, $y = x$

70. $y = x^3 - 4x$, $y = -(x + 2)$

In Exercises 71–74, use a graphing utility to find the points of intersection of the graphs. Check your results analytically.

71. $y = x^3 - 2x^2 + x - 1$, $y = -x^2 + 3x - 1$

72. $y = x^4 - 2x^2 + 1$, $y = 1 - x^2$

73. $y = \sqrt{x + 6}$, $y = \sqrt{-x^2 - 4x}$

74. $y = -|2x - 3| + 6$, $y = 6 - x$

75. ***Modeling Data*** The table shows the Consumer Price Index (CPI) for selected years. *(Source: Bureau of Labor Statistics)*

Year	1975	1980	1985	1990	1995	2000	2005
CPI	53.8	82.4	107.6	130.7	152.4	172.2	195.3

(a) Use the regression capabilities of a graphing utility to find a mathematical model of the form $y = at^2 + bt + c$ for the data. In the model, y represents the CPI and t represents the year, with $t = 5$ corresponding to 1975.

(b) Use a graphing utility to plot the data and graph the model. Compare the data with the model.

(c) Use the model to predict the CPI for the year 2010.

76. ***Modeling Data*** The table shows the numbers of cellular phone subscribers (in millions) in the United States for selected years. *(Source: Cellular Telecommunications and Internet Association)*

Year	1990	1993	1996	1999	2002	2005
Number	5	16	44	86	141	208

(a) Use the regression capabilities of a graphing utility to find a mathematical model of the form $y = at^2 + bt + c$ for the data. In the model, y represents the number of subscribers and t represents the year, with $t = 0$ corresponding to 1990.

(b) Use a graphing utility to plot the data and graph the model. Compare the data with the model.

(c) Use the model to predict the number of cellular phone subscribers in the United States in the year 2015.

77. ***Break-Even Point*** Find the sales necessary to break even $(R = C)$ if the cost C of producing x units is

$$C = 5.5\sqrt{x} + 10{,}000 \qquad \text{Cost equation}$$

and the revenue R from selling x units is

$$R = 3.29x. \qquad \text{Revenue equation}$$

78. ***Copper Wire*** The resistance y in ohms of 1000 feet of solid copper wire at 77°F can be approximated by the model

$$y = \frac{10{,}770}{x^2} - 0.37, \quad 5 \le x \le 100$$

where x is the diameter of the wire in mils (0.001 in.). Use a graphing utility to graph the model. If the diameter of the wire is doubled, the resistance is changed by about what factor?

WRITING ABOUT CONCEPTS

In Exercises 79 and 80, write an equation whose graph has the indicated property. (There may be more than one correct answer.)

79. The graph has intercepts at $x = -4$, $x = 3$, and $x = 8$.

80. The graph has intercepts at $x = -\frac{3}{2}$, $x = 4$, and $x = \frac{5}{2}$.

81. (a) Prove that if a graph is symmetric with respect to the x-axis and to the y-axis, then it is symmetric with respect to the origin. Give an example to show that the converse is not true.

(b) Prove that if a graph is symmetric with respect to one axis and to the origin, then it is symmetric with respect to the other axis.

CAPSTONE

82. Match the equation or equations with the given characteristic.

(i) $y = 3x^3 - 3x$ (ii) $y = (x + 3)^2$ (iii) $y = 3x - 3$

(iv) $y = \sqrt[3]{x}$ (v) $y = 3x^2 + 3$ (vi) $y = \sqrt{x + 3}$

(a) Symmetric with respect to the y-axis

(b) Three x-intercepts

(c) Symmetric with respect to the x-axis

(d) $(-2, 1)$ is a point on the graph

(e) Symmetric with respect to the origin

(f) Graph passes through the origin

True or False? **In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

83. If $(-4, -5)$ is a point on a graph that is symmetric with respect to the x-axis, then $(4, -5)$ is also a point on the graph.

84. If $(-4, -5)$ is a point on a graph that is symmetric with respect to the y-axis, then $(4, -5)$ is also a point on the graph.

85. If $b^2 - 4ac > 0$ and $a \neq 0$, then the graph of $y = ax^2 + bx + c$ has two x-intercepts.

86. If $b^2 - 4ac = 0$ and $a \neq 0$, then the graph of $y = ax^2 + bx + c$ has only one x-intercept.

In Exercises 87 and 88, find an equation of the graph that consists of all points (x, y) having the given distance from the origin. (For a review of the Distance Formula, see Appendix C.)

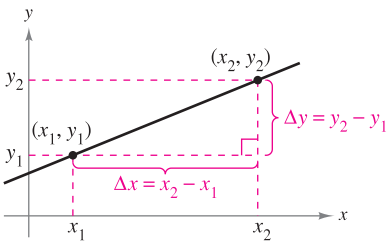
87. The distance from the origin is twice the distance from $(0, 3)$.

88. The distance from the origin is $K\,(K \neq 1)$ times the distance from $(2, 0)$.

P.2 Linear Models and Rates of Change

- **Find the slope of a line passing through two points.**
- **Write the equation of a line with a given point and slope.**
- **Interpret slope as a ratio or as a rate in a real-life application.**
- **Sketch the graph of a linear equation in slope-intercept form.**
- **Write equations of lines that are parallel or perpendicular to a given line.**

The Slope of a Line



$\Delta y = y_2 - y_1 =$ change in y
$\Delta x = x_2 - x_1 =$ change in x
Figure P.12

The **slope** of a nonvertical line is a measure of the number of units the line rises (or falls) vertically for each unit of horizontal change from left to right. Consider the two points (x_1, y_1) and (x_2, y_2) on the line in Figure P.12. As you move from left to right along this line, a vertical change of

$$\Delta y = y_2 - y_1 \qquad \text{Change in } y$$

units corresponds to a horizontal change of

$$\Delta x = x_2 - x_1 \qquad \text{Change in } x$$

units. (Δ is the Greek uppercase letter *delta*, and the symbols Δy and Δx are read "delta y" and "delta x.")

DEFINITION OF THE SLOPE OF A LINE

The **slope** m of the nonvertical line passing through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2.$$

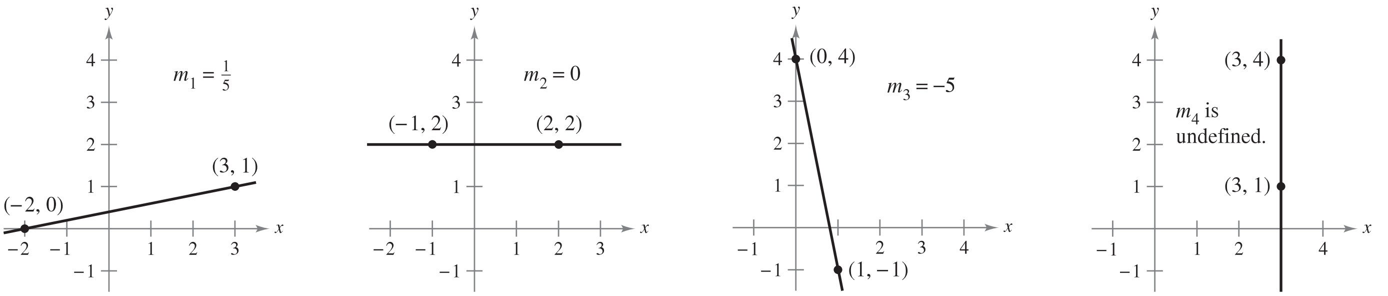
Slope is not defined for vertical lines.

NOTE When using the formula for slope, note that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{-(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2}.$$

So, it does not matter in which order you subtract *as long as* you are consistent and both "subtracted coordinates" come from the same point. ■

Figure P.13 shows four lines: one has a positive slope, one has a slope of zero, one has a negative slope, and one has an "undefined" slope. In general, the greater the absolute value of the slope of a line, the steeper the line is. For instance, in Figure P.13, the line with a slope of -5 is steeper than the line with a slope of $\frac{1}{5}$.



If m is positive, then the line rises from left to right.
Figure P.13

If m is zero, then the line is horizontal.

If m is negative, then the line falls from left to right.

If m is undefined, then the line is vertical.

EXPLORATION

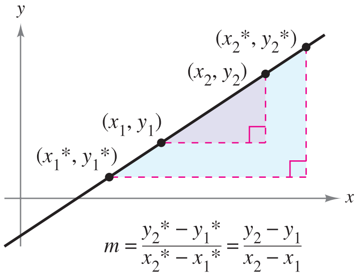
Investigating Equations of Lines Use a graphing utility to graph each of the linear equations. Which point is common to all seven lines? Which value in the equation determines the slope of each line?

a. $y - 4 = -2(x + 1)$
b. $y - 4 = -1(x + 1)$
c. $y - 4 = -\frac{1}{2}(x + 1)$
d. $y - 4 = 0(x + 1)$
e. $y - 4 = \frac{1}{2}(x + 1)$
f. $y - 4 = 1(x + 1)$
g. $y - 4 = 2(x + 1)$

Use your results to write an equation of a line passing through $(-1, 4)$ with a slope of m.

Equations of Lines

Any two points on a nonvertical line can be used to calculate its slope. This can be verified from the similar triangles shown in Figure P.14. (Recall that the ratios of corresponding sides of similar triangles are equal.)


y
x
(x_2^*, y_2^*)
(x_2, y_2)
(x_1, y_1)
(x_1^*, y_1^*)
$m = \frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{y_2 - y_1}{x_2 - x_1}$

Any two points on a nonvertical line can be used to determine its slope.
Figure P.14

You can write an equation of a nonvertical line if you know the slope of the line and the coordinates of one point on the line. Suppose the slope is m and the point is (x_1, y_1). If (x, y) is any other point on the line, then

$$\frac{y - y_1}{x - x_1} = m.$$

This equation, involving the two variables x and y, can be rewritten in the form $y - y_1 = m(x - x_1)$, which is called the **point-slope equation of a line.**

POINT-SLOPE EQUATION OF A LINE

An equation of the line with slope m passing through the point (x_1, y_1) is given by

$$y - y_1 = m(x - x_1).$$

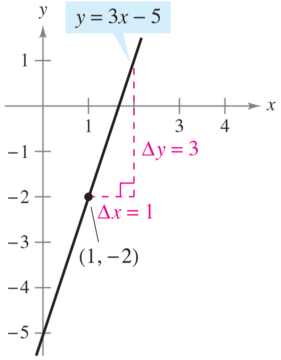
EXAMPLE 1 Finding an Equation of a Line

Find an equation of the line that has a slope of 3 and passes through the point $(1, -2)$.

Solution

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-2) = 3(x - 1)$	Substitute -2 for y_1, 1 for x_1, and 3 for m.
$y + 2 = 3x - 3$	Simplify.
$y = 3x - 5$	Solve for y.

(See Figure P.15.) ■



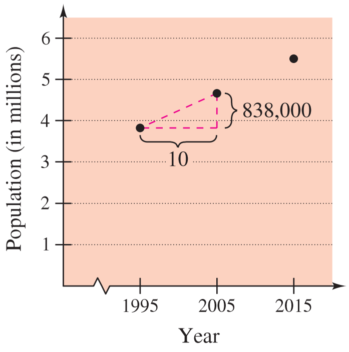
The line with a slope of 3 passing through the point $(1, -2)$
Figure P.15

NOTE Remember that only nonvertical lines have a slope. Consequently, vertical lines cannot be written in point-slope form. For instance, the equation of the vertical line passing through the point $(1, -2)$ is $x = 1$. ■

Ratios and Rates of Change

The slope of a line can be interpreted as either a *ratio* or a *rate*. If the x- and y-axes have the same unit of measure, the slope has no units and is a **ratio.** If the x- and y-axes have different units of measure, the slope is a rate or **rate of change.** In your study of calculus, you will encounter applications involving both interpretations of slope.

EXAMPLE 2 Population Growth and Engineering Design



Population of Colorado
Figure P.16

a. The population of Colorado was 3,827,000 in 1995 and 4,665,000 in 2005. Over this 10-year period, the average rate of change of the population was

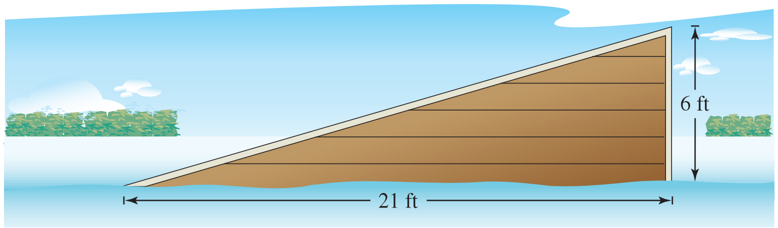
$$\begin{aligned}\text{Rate of change} &= \frac{\text{change in population}}{\text{change in years}}\\ &= \frac{4{,}665{,}000 - 3{,}827{,}000}{2005 - 1995}\\ &= 83{,}800 \text{ people per year.}\end{aligned}$$

If Colorado's population continues to increase at this same rate for the next 10 years, it will have a 2015 population of 5,503,000 (see Figure P.16). *(Source: U.S. Census Bureau)*

b. In tournament water-ski jumping, the ramp rises to a height of 6 feet on a raft that is 21 feet long, as shown in Figure P.17. The slope of the ski ramp is the ratio of its height (the rise) to the length of its base (the run).

$$\begin{aligned}\text{Slope of ramp} &= \frac{\text{rise}}{\text{run}} && \text{Rise is vertical change, run is horizontal change.}\\ &= \frac{6 \text{ feet}}{21 \text{ feet}}\\ &= \frac{2}{7}\end{aligned}$$

In this case, note that the slope is a ratio and has no units.



Dimensions of a water-ski ramp
Figure P.17

The rate of change found in Example 2(a) is an **average rate of change.** An average rate of change is always calculated over an interval. In this case, the interval is $[1995, 2005]$. In Chapter 2 you will study another type of rate of change called an *instantaneous rate of change*.

Graphing Linear Models

Many problems in analytic geometry can be classified in two basic categories: (1) Given a graph, what is its equation? and (2) Given an equation, what is its graph? The point-slope equation of a line can be used to solve problems in the first category. However, this form is not especially useful for solving problems in the second category. The form that is better suited to sketching the graph of a line is the **slope-intercept** form of the equation of a line.

THE SLOPE-INTERCEPT EQUATION OF A LINE

The graph of the linear equation

$$y = mx + b$$

is a line having a *slope* of m and a y-*intercept* at $(0, b)$.

EXAMPLE 3 Sketching Lines in the Plane

Sketch the graph of each equation.

a. $y = 2x + 1$ **b.** $y = 2$ **c.** $3y + x - 6 = 0$

Solution

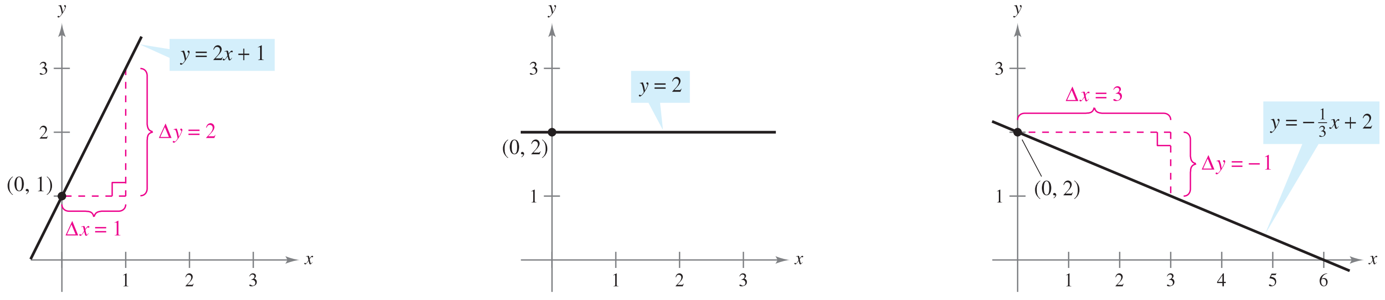
a. Because $b = 1$, the y-intercept is $(0, 1)$. Because the slope is $m = 2$, you know that the line rises two units for each unit it moves to the right, as shown in Figure P.18(a).

b. Because $b = 2$, the y-intercept is $(0, 2)$. Because the slope is $m = 0$, you know that the line is horizontal, as shown in Figure P.18(b).

c. Begin by writing the equation in slope-intercept form.

$$\begin{aligned} 3y + x - 6 &= 0 && \text{Write original equation.} \\ 3y &= -x + 6 && \text{Isolate } y\text{-term on the left.} \\ y &= -\frac{1}{3}x + 2 && \text{Slope-intercept form} \end{aligned}$$

In this form, you can see that the y-intercept is $(0, 2)$ and the slope is $m = -\frac{1}{3}$. This means that the line falls one unit for every three units it moves to the right, as shown in Figure P.18(c).



(a) $m = 2$; line rises

(b) $m = 0$; line is horizontal

(c) $m = -\frac{1}{3}$; line falls

Figure P.18

■

Because the slope of a vertical line is not defined, its equation cannot be written in the slope-intercept form. However, the equation of any line can be written in the **general form**

$$Ax + By + C = 0$$ General form of the equation of a line

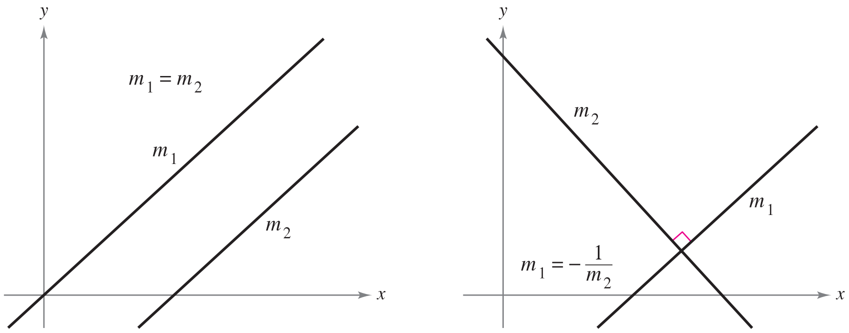
where A and B are not *both* zero. For instance, the vertical line given by $x = a$ can be represented by the general form $x - a = 0$.

SUMMARY OF EQUATIONS OF LINES

1. General form: $Ax + By + C = 0, \quad (A, B \neq 0)$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Point-slope form: $y - y_1 = m(x - x_1)$
5. Slope-intercept form: $y = mx + b$

Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular, as shown in Figure P.19. Specifically, nonvertical lines with the same slope are parallel and nonvertical lines whose slopes are negative reciprocals are perpendicular.



Parallel lines   Perpendicular lines

Figure P.19

PARALLEL AND PERPENDICULAR LINES

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal—that is, if and only if $m_1 = m_2$.

2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other—that is, if and only if

$$m_1 = -\frac{1}{m_2}.$$

STUDY TIP In mathematics, the phrase "if and only if" is a way of stating two implications in one statement. For instance, the first statement at the right could be rewritten as the following two implications.

a. If two distinct nonvertical lines are parallel, then their slopes are equal.

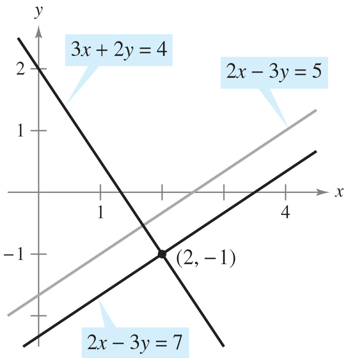
b. If two distinct nonvertical lines have equal slopes, then they are parallel.

EXAMPLE 4 Finding Parallel and Perpendicular Lines

Find the general forms of the equations of the lines that pass through the point $(2, -1)$ and are

a. parallel to the line $2x - 3y = 5$ **b.** perpendicular to the line $2x - 3y = 5$.

(See Figure P.20.)



Lines parallel and perpendicular to $2x - 3y = 5$
Figure P.20

Solution By writing the linear equation $2x - 3y = 5$ in slope-intercept form, $y = \frac{2}{3}x - \frac{5}{3}$, you can see that the given line has a slope of $m = \frac{2}{3}$.

a. The line through $(2, -1)$ that is parallel to the given line also has a slope of $\frac{2}{3}$.

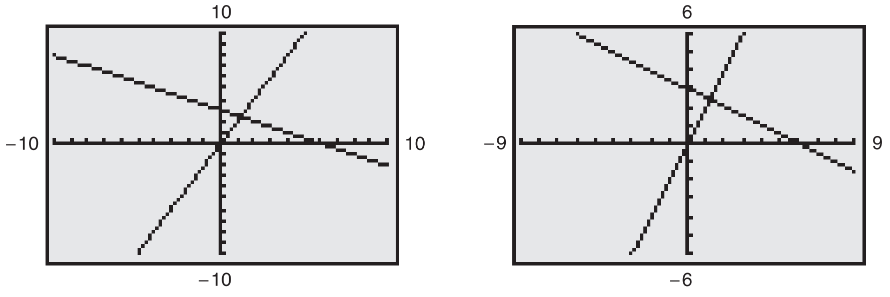
$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form} \\ y - (-1) &= \tfrac{2}{3}(x - 2) && \text{Substitute.} \\ 3(y + 1) &= 2(x - 2) && \text{Simplify.} \\ 2x - 3y - 7 &= 0 && \text{General form} \end{aligned}$$

Note the similarity to the original equation.

b. Using the negative reciprocal of the slope of the given line, you can determine that the slope of a line perpendicular to the given line is $-\frac{3}{2}$. So, the line through the point $(2, -1)$ that is perpendicular to the given line has the following equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form} \\ y - (-1) &= -\tfrac{3}{2}(x - 2) && \text{Substitute.} \\ 2(y + 1) &= -3(x - 2) && \text{Simplify.} \\ 3x + 2y - 4 &= 0 && \text{General form} \end{aligned}$$

TECHNOLOGY PITFALL The slope of a line will appear distorted if you use different tick-mark spacing on the x- and y-axes. For instance, the graphing calculator screens in Figures P.21(a) and P.21(b) both show the lines given by $y = 2x$ and $y = -\frac{1}{2}x + 3$. Because these lines have slopes that are negative reciprocals, they must be perpendicular. In Figure P.21(a), however, the lines don't appear to be perpendicular because the tick-mark spacing on the x-axis is not the same as that on the y-axis. In Figure P.21(b), the lines appear perpendicular because the tick-mark spacing on the x-axis is the same as on the y-axis. This type of viewing window is said to have a *square setting*.



(a) Tick-mark spacing on the x-axis is not the same as tick-mark spacing on the y-axis.

(b) Tick-mark spacing on the x-axis is the same as tick-mark spacing on the y-axis.

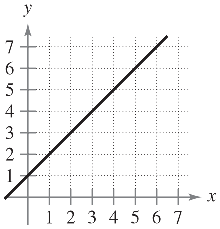
Figure P.21

P.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, estimate the slope of the line from its graph. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

1.



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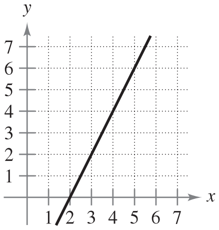

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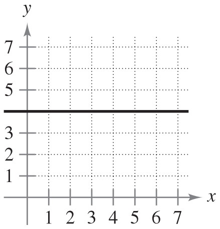

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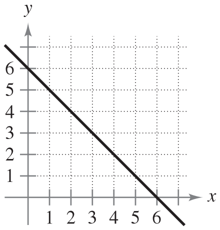

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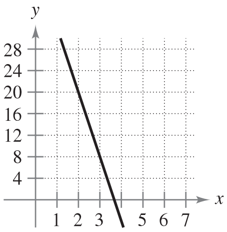

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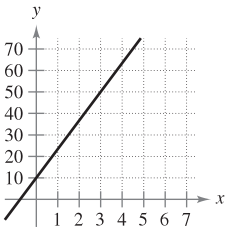


In Exercises 7 and 8, sketch the lines through the point with the indicated slopes. Make the sketches on the same set of coordinate axes.

	Point	*Slopes*			
7.	(3, 4)	(a) 1	(b) −2	(c) $-\frac{3}{2}$	(d) Undefined
8.	(−2, 5)	(a) 3	(b) −3	(c) $\frac{1}{3}$	(d) 0

In Exercises 9–14, plot the pair of points and find the slope of the line passing through them.

9. $(3, -4), (5, 2)$

10. $(1, 1), (-2, 7)$

11. $(4, 6), (4, 1)$

12. $(3, -5), (5, -5)$

13. $\left(-\frac{1}{2}, \frac{2}{3}\right), \left(-\frac{3}{4}, \frac{1}{6}\right)$

14. $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

In Exercises 15–18, use the point on the line and the slope of the line to find three additional points that the line passes through. (There is more than one correct answer.)

	Point	*Slope*		*Point*	*Slope*
15.	(6, 2)	$m = 0$	**16.**	(−4, 3)	m is undefined.
17.	(1, 7)	$m = -3$	**18.**	(−2, −2)	$m = 2$

19. ***Conveyor Design*** A moving conveyor is built to rise 1 meter for each 3 meters of horizontal change.

(a) Find the slope of the conveyor.

(b) Suppose the conveyor runs between two floors in a factory. Find the length of the conveyor if the vertical distance between floors is 10 feet.

20. ***Rate of Change*** Each of the following is the slope of a line representing daily revenue y in terms of time x in days. Use the slope to interpret any change in daily revenue for a one-day increase in time.

(a) $m = 800$ (b) $m = 250$ (c) $m = 0$

21. ***Modeling Data*** The table shows the populations y (in millions) of the United States for 2000 through 2005. The variable t represents the time in years, with $t = 0$ corresponding to 2000. *(Source: U.S. Bureau of the Census)*

t	0	1	2	3	4	5
y	282.4	285.3	288.2	291.1	293.9	296.6

(a) Plot the data by hand and connect adjacent points with a line segment.

(b) Use the slope of each line segment to determine the year when the population increased least rapidly.

22. ***Modeling Data*** The table shows the rate r (in miles per hour) that a vehicle is traveling after t seconds.

t	5	10	15	20	25	30
r	57	74	85	84	61	43

(a) Plot the data by hand and connect adjacent points with a line segment.

(b) Use the slope of each line segment to determine the interval when the vehicle's rate changed most rapidly. How did the rate change?

In Exercises 23–28, find the slope and the *y*-intercept (if possible) of the line.

23. $y = 4x - 3$

24. $-x + y = 1$

25. $x + 5y = 20$

26. $6x - 5y = 15$

27. $x = 4$

28. $y = -1$

In Exercises 29–34, find an equation of the line that passes through the point and has the indicated slope. Sketch the line.

	Point	*Slope*		*Point*	*Slope*
29.	(0, 3)	$m = \frac{3}{4}$	**30.**	(−5, −2)	m is undefined.
31.	(0, 0)	$m = \frac{2}{3}$	**32.**	(0, 4)	$m = 0$
33.	(3, −2)	$m = 3$	**34.**	(−2, 4)	$m = -\frac{3}{5}$

In Exercises 35–44, find an equation of the line that passes through the points, and sketch the line.

35. $(0, 0), (4, 8)$

36. $(0, 0), (-1, 5)$

37. $(2, 1), (0, -3)$

38. $(-2, -2), (1, 7)$

39. $(2, 8), (5, 0)$

40. $(-3, 6), (1, 2)$

41. $(6, 3), (6, 8)$

42. $(1, -2), (3, -2)$

43. $\left(\frac{1}{2}, \frac{7}{2}\right), \left(0, \frac{3}{4}\right)$

44. $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

45. Find an equation of the vertical line with x-intercept at 3.

46. Show that the line with intercepts $(a, 0)$ and $(0, b)$ has the following equation.

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a \neq 0, b \neq 0$$

In Exercises 47–50, use the result of Exercise 46 to write an equation of the line in general form.

47. x-intercept: $(2, 0)$
y-intercept: $(0, 3)$

48. x-intercept: $\left(-\frac{2}{3}, 0\right)$
y-intercept: $(0, -2)$

49. Point on line: $(1, 2)$
x-intercept: $(a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

50. Point on line: $(-3, 4)$
x-intercept: $(a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

In Exercises 51–58, sketch a graph of the equation.

51. $y = -3$

52. $x = 4$

53. $y = -2x + 1$

54. $y = \frac{1}{3}x - 1$

55. $y - 2 = \frac{3}{2}(x - 1)$

56. $y - 1 = 3(x + 4)$

57. $2x - y - 3 = 0$

58. $x + 2y + 6 = 0$

59. ***Square Setting*** Use a graphing utility to graph the lines $y = 2x - 3$ and $y = -\frac{1}{2}x + 1$ in each viewing window. Compare the graphs. Do the lines appear perpendicular? Are the lines perpendicular? Explain.

(a)
Xmin = -5
Xmax = 5
Xscl = 1
Ymin = -5
Ymax = 5
Yscl = 1

(b)
Xmin = -6
Xmax = 6
Xscl = 1
Ymin = -4
Ymax = 4
Yscl = 1

CAPSTONE

60. A line is represented by the equation $ax + by = 4$.

(a) When is the line parallel to the x-axis?

(b) When is the line parallel to the y-axis?

(c) Give values for a and b such that the line has a slope of $\frac{5}{8}$.

(d) Give values for a and b such that the line is perpendicular to $y = \frac{2}{5}x + 3$.

(e) Give values for a and b such that the line coincides with the graph of $5x + 6y = 8$.

In Exercises 61–66, write the general forms of the equations of the lines through the point (a) parallel to the given line and (b) perpendicular to the given line.

	Point	*Line*		*Point*	*Line*
61.	$(-7, -2)$	$x = 1$	**62.**	$(-1, 0)$	$y = -3$
63.	$(2, 1)$	$4x - 2y = 3$	**64.**	$(-3, 2)$	$x + y = 7$
65.	$\left(\frac{3}{4}, \frac{7}{8}\right)$	$5x - 3y = 0$	**66.**	$(4, -5)$	$3x + 4y = 7$

Rate of Change **In Exercises 67–70, you are given the dollar value of a product in 2008 *and* the rate at which the value of the product is expected to change during the next 5 years. Write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 0$ represent 2000.)**

	2008 Value	*Rate*
67.	\$1850	\$250 increase per year
68.	\$156	\$4.50 increase per year
69.	\$17,200	\$1600 decrease per year
70.	\$245,000	\$5600 decrease per year

In Exercises 71 and 72, use a graphing utility to graph the parabolas and find their points of intersection. Find an equation of the line through the points of intersection and graph the line in the same viewing window.

71. $y = x^2$
$y = 4x - x^2$

72. $y = x^2 - 4x + 3$
$y = -x^2 + 2x + 3$

In Exercises 73 and 74, determine whether the points are collinear. (Three points are *collinear* if they lie on the same line.)

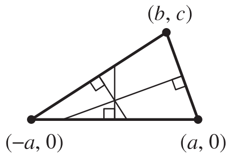
73. $(-2, 1), (-1, 0), (2, -2)$

74. $(0, 4), (7, -6), (-5, 11)$

WRITING ABOUT CONCEPTS

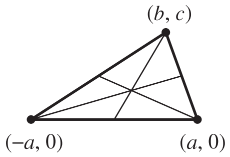
In Exercises 75–77, find the coordinates of the point of intersection of the given segments. Explain your reasoning.

75.



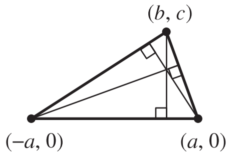
Perpendicular bisectors

76.



Medians

77.



Altitudes

78. Show that the points of intersection in Exercises 75, 76, and 77 are collinear.

79. ***Temperature Conversion*** Find a linear equation that expresses the relationship between the temperature in degrees Celsius C and degrees Fahrenheit F. Use the fact that water freezes at 0°C (32°F) and boils at 100°C (212°F). Use the equation to convert 72°F to degrees Celsius.

80. ***Reimbursed Expenses*** A company reimburses its sales representatives \$175 per day for lodging and meals plus 48¢ per mile driven. Write a linear equation giving the daily cost C to the company in terms of x, the number of miles driven. How much does it cost the company if a sales representative drives 137 miles on a given day?

81. ***Career Choice*** An employee has two options for positions in a large corporation. One position pays \$14.50 per hour *plus* an additional unit rate of \$0.75 per unit produced. The other pays \$11.20 per hour *plus* a unit rate of \$1.30.

(a) Find linear equations for the hourly wages W in terms of x, the number of units produced per hour, for each option.

(b) Use a graphing utility to graph the linear equations and find the point of intersection.

(c) Interpret the meaning of the point of intersection of the graphs in part (b). How would you use this information to select the correct option if the goal were to obtain the highest hourly wage?

82. ***Straight-Line Depreciation*** A small business purchases a piece of equipment for \$875. After 5 years the equipment will be outdated, having no value.

(a) Write a linear equation giving the value y of the equipment in terms of the time x, $0 \le x \le 5$.

(b) Find the value of the equipment when $x = 2$.

(c) Estimate (to two-decimal-place accuracy) the time when the value of the equipment is \$200.

83. ***Apartment Rental*** A real estate office manages an apartment complex with 50 units. When the rent is \$780 per month, all 50 units are occupied. However, when the rent is \$825, the average number of occupied units drops to 47. Assume that the relationship between the monthly rent p and the demand x is linear. (*Note:* The term *demand* refers to the number of occupied units.)

(a) Write a linear equation giving the demand x in terms of the rent p.

(b) *Linear extrapolation* Use a graphing utility to graph the demand equation and use the *trace* feature to predict the number of units occupied if the rent is raised to \$855.

(c) *Linear interpolation* Predict the number of units occupied if the rent is lowered to \$795. Verify graphically.

84. ***Modeling Data*** An instructor gives regular 20-point quizzes and 100-point exams in a mathematics course. Average scores for six students, given as ordered pairs (x, y), where x is the average quiz score and y is the average test score, are (18, 87), (10, 55), (19, 96), (16, 79), (13, 76), and (15, 82).

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use a graphing utility to plot the points and graph the regression line in the same viewing window.

(c) Use the regression line to predict the average exam score for a student with an average quiz score of 17.

(d) Interpret the meaning of the slope of the regression line.

(e) The instructor adds 4 points to the average test score of everyone in the class. Describe the changes in the positions of the plotted points and the change in the equation of the line.

85. ***Tangent Line*** Find an equation of the line tangent to the circle $x^2 + y^2 = 169$ at the point (5, 12).

86. ***Tangent Line*** Find an equation of the line tangent to the circle $(x - 1)^2 + (y - 1)^2 = 25$ at the point $(4, -3)$.

Distance **In Exercises 87–92, find the distance between the point and line, or between the lines, using the formula for the distance between the point (x_1, y_1) and the line $Ax + By + C = 0$.**

$$\textbf{Distance} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

87. Point: (0, 0)
Line: $4x + 3y = 10$

88. Point: (2, 3)
Line: $4x + 3y = 10$

89. Point: $(-2, 1)$
Line: $x - y - 2 = 0$

90. Point: (6, 2)
Line: $x = -1$

91. Line: $x + y = 1$
Line: $x + y = 5$

92. Line: $3x - 4y = 1$
Line: $3x - 4y = 10$

93. Show that the distance between the point (x_1, y_1) and the line $Ax + By + C = 0$ is

$$\text{Distance} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

94. Write the distance d between the point (3, 1) and the line $y = mx + 4$ in terms of m. Use a graphing utility to graph the equation. When is the distance 0? Explain the result geometrically.

95. Prove that the diagonals of a rhombus intersect at right angles. (A rhombus is a quadrilateral with sides of equal lengths.)

96. Prove that the figure formed by connecting consecutive midpoints of the sides of any quadrilateral is a parallelogram.

97. Prove that if the points (x_1, y_1) and (x_2, y_2) lie on the same line as (x_1^*, y_1^*) and (x_2^*, y_2^*), then

$$\frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Assume $x_1 \neq x_2$ and $x_1^* \neq x_2^*$.

98. Prove that if the slopes of two nonvertical lines are negative reciprocals of each other, then the lines are perpendicular.

True or False? **In Exercises 99 and 100, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

99. The lines represented by $ax + by = c_1$ and $bx - ay = c_2$ are perpendicular. Assume $a \neq 0$ and $b \neq 0$.

100. It is possible for two lines with positive slopes to be perpendicular to each other.

P.3 Functions and Their Graphs

- **Use function notation to represent and evaluate a function.**
- **Find the domain and range of a function.**
- **Sketch the graph of a function.**
- **Identify different types of transformations of functions.**
- **Classify functions and recognize combinations of functions.**

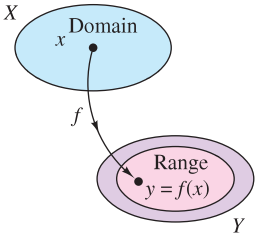
Functions and Function Notation

A **relation** between two sets X and Y is a set of ordered pairs, each of the form (x, y), where x is a member of X and y is a member of Y. A **function** from X to Y is a relation between X and Y that has the property that any two ordered pairs with the same x-value also have the same y-value. The variable x is the **independent variable,** and the variable y is the **dependent variable.**

Many real-life situations can be modeled by functions. For instance, the area A of a circle is a function of the circle's radius r.

$$A = \pi r^2 \qquad A \text{ is a function of } r.$$

In this case r is the independent variable and A is the dependent variable.



A real-valued function f of a real variable
Figure P.22

DEFINITION OF A REAL-VALUED FUNCTION OF A REAL VARIABLE

Let X and Y be sets of real numbers. A **real-valued function *f* of a real variable *x*** from X to Y is a correspondence that assigns to each number x in X exactly one number y in Y.

The **domain** of f is the set X. The number y is the **image** of x under f and is denoted by $f(x)$, which is called the **value of *f* at *x*.** The **range** of f is a subset of Y and consists of all images of numbers in X (see Figure P.22).

Functions can be specified in a variety of ways. In this text, however, we will concentrate primarily on functions that are given by equations involving the dependent and independent variables. For instance, the equation

$$x^2 + 2y = 1 \qquad \text{Equation in implicit form}$$

defines y, the dependent variable, as a function of x, the independent variable. To **evaluate** this function (that is, to find the y-value that corresponds to a given x-value), it is convenient to isolate y on the left side of the equation.

$$y = \frac{1}{2}(1 - x^2) \qquad \text{Equation in explicit form}$$

Using f as the name of the function, you can write this equation as

$$f(x) = \frac{1}{2}(1 - x^2). \qquad \text{Function notation}$$

The original equation, $x^2 + 2y = 1$, **implicitly** defines y as a function of x. When you solve the equation for y, you are writing the equation in **explicit** form.

Function notation has the advantage of clearly identifying the dependent variable as $f(x)$ while at the same time telling you that x is the independent variable and that the function itself is "f." The symbol $f(x)$ is read "f of x." Function notation allows you to be less wordy. Instead of asking "What is the value of y that corresponds to $x = 3$?" you can ask "What is $f(3)$?"

FUNCTION NOTATION

The word *function* was first used by Gottfried Wilhelm Leibniz in 1694 as a term to denote any quantity connected with a curve, such as the coordinates of a point on a curve or the slope of a curve. Forty years later, Leonhard Euler used the word "function" to describe any expression made up of a variable and some constants. He introduced the notation $y = f(x)$.

In an equation that defines a function, the role of the variable x is simply that of a placeholder. For instance, the function given by

$$f(x) = 2x^2 - 4x + 1$$

can be described by the form

$$f(\square) = 2(\square)^2 - 4(\square) + 1$$

where parentheses are used instead of x. To evaluate $f(-2)$, simply place -2 in each set of parentheses.

$$\begin{aligned} f(-2) &= 2(-2)^2 - 4(-2) + 1 && \text{Substitute } -2 \text{ for } x. \\ &= 2(4) + 8 + 1 && \text{Simplify.} \\ &= 17 && \text{Simplify.} \end{aligned}$$

NOTE Although f is often used as a convenient function name and x as the independent variable, you can use other symbols. For instance, the following equations all define the same function.

$$\begin{aligned} f(x) &= x^2 - 4x + 7 && \text{Function name is } f\text{, independent variable is } x. \\ f(t) &= t^2 - 4t + 7 && \text{Function name is } f\text{, independent variable is } t. \\ g(s) &= s^2 - 4s + 7 && \text{Function name is } g\text{, independent variable is } s. \end{aligned}$$

■

EXAMPLE 1 Evaluating a Function

For the function f defined by $f(x) = x^2 + 7$, evaluate each expression.

a. $f(3a)$ **b.** $f(b - 1)$ **c.** $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0$

Solution

a.

$$\begin{aligned} f(3a) &= (3a)^2 + 7 && \text{Substitute } 3a \text{ for } x. \\ &= 9a^2 + 7 && \text{Simplify.} \end{aligned}$$

b.

$$\begin{aligned} f(b - 1) &= (b - 1)^2 + 7 && \text{Substitute } b - 1 \text{ for } x. \\ &= b^2 - 2b + 1 + 7 && \text{Expand binomial.} \\ &= b^2 - 2b + 8 && \text{Simplify.} \end{aligned}$$

c.

$$\begin{aligned} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \frac{[(x + \Delta x)^2 + 7] - (x^2 + 7)}{\Delta x} \\ &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 7 - x^2 - 7}{\Delta x} \\ &= \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \frac{\cancel{\Delta x}(2x + \Delta x)}{\cancel{\Delta x}} \\ &= 2x + \Delta x, \quad \Delta x \neq 0 \end{aligned}$$

■

STUDY TIP In calculus, it is important to specify clearly the domain of a function or expression. For instance, in Example 1(c) the two expressions

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{and} \quad 2x + \Delta x, \quad \Delta x \neq 0$$

are equivalent because $\Delta x = 0$ is excluded from the domain of each expression. Without a stated domain restriction, the two expressions would not be equivalent.

NOTE The expression in Example 1(c) is called a *difference quotient* and has a special significance in calculus. You will learn more about this in Chapter 2. ■

The Domain and Range of a Function

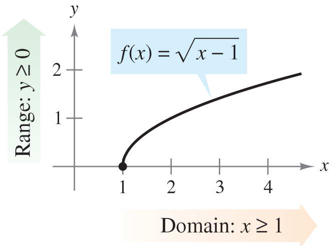
The domain of a function can be described explicitly, or it may be described *implicitly* by an equation used to define the function. The implied domain is the set of all real numbers for which the equation is defined, whereas an explicitly defined domain is one that is given along with the function. For example, the function given by

$$f(x) = \frac{1}{x^2 - 4}, \quad 4 \le x \le 5$$

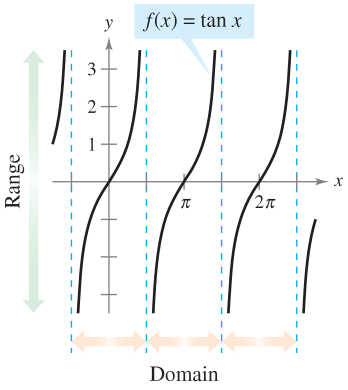
has an explicitly defined domain given by $\{x\colon\ 4 \le x \le 5\}$. On the other hand, the function given by

$$g(x) = \frac{1}{x^2 - 4}$$

has an implied domain that is the set $\{x\colon\ x \ne \pm 2\}$.



(a) The domain of f is $[1, \infty)$ and the range is $[0, \infty)$.



(b) The domain of f is all x-values such that $x \ne \frac{\pi}{2} + n\pi$ and the range is $(-\infty, \infty)$.

Figure P.23

EXAMPLE 2 Finding the Domain and Range of a Function

a. The domain of the function

$$f(x) = \sqrt{x - 1}$$

is the set of all x-values for which $x - 1 \ge 0$, which is the interval $[1, \infty)$. To find the range, observe that $f(x) = \sqrt{x-1}$ is never negative. So, the range is the interval $[0, \infty)$, as indicated in Figure P.23(a).

b. The domain of the tangent function, as shown in Figure P.23(b),

$$f(x) = \tan x$$

is the set of all x-values such that

$$x \ne \frac{\pi}{2} + n\pi, \quad n \text{ is an integer.}$$

Domain of tangent function

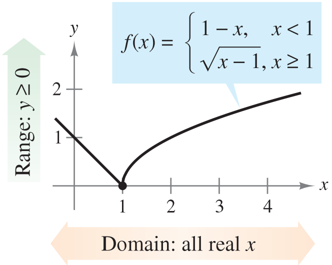
The range of this function is the set of all real numbers. For a review of the characteristics of this and other trigonometric functions, see Appendix C.

EXAMPLE 3 A Function Defined by More than One Equation

Determine the domain and range of the function.

$$f(x) = \begin{cases} 1 - x, & \text{if } x < 1 \\ \sqrt{x - 1}, & \text{if } x \ge 1 \end{cases}$$

Solution Because f is defined for $x < 1$ and $x \ge 1$, the domain is the entire set of real numbers. On the portion of the domain for which $x \ge 1$, the function behaves as in Example 2(a). For $x < 1$, the values of $1 - x$ are positive. So, the range of the function is the interval $[0, \infty)$. (See Figure P.24.) ■



The domain of f is $(-\infty, \infty)$ and the range is $[0, \infty)$.

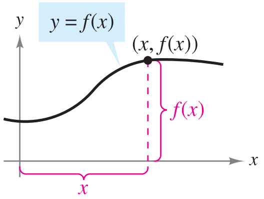
Figure P.24

A function from X to Y is **one-to-one** if to each y-value in the range there corresponds exactly one x-value in the domain. For instance, the function given in Example 2(a) is one-to-one, whereas the functions given in Examples 2(b) and 3 are not one-to-one. A function from X to Y is **onto** if its range consists of all of Y.

The Graph of a Function

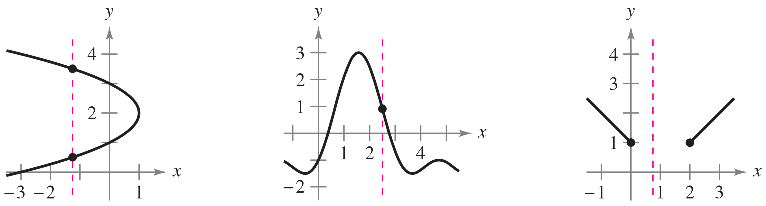
The graph of the function $y = f(x)$ consists of all points $(x, f(x))$, where x is in the domain of f. In Figure P.25, note that

$$x = \text{the directed distance from the } y\text{-axis}$$
$$f(x) = \text{the directed distance from the } x\text{-axis.}$$



The graph of a function
Figure P.25

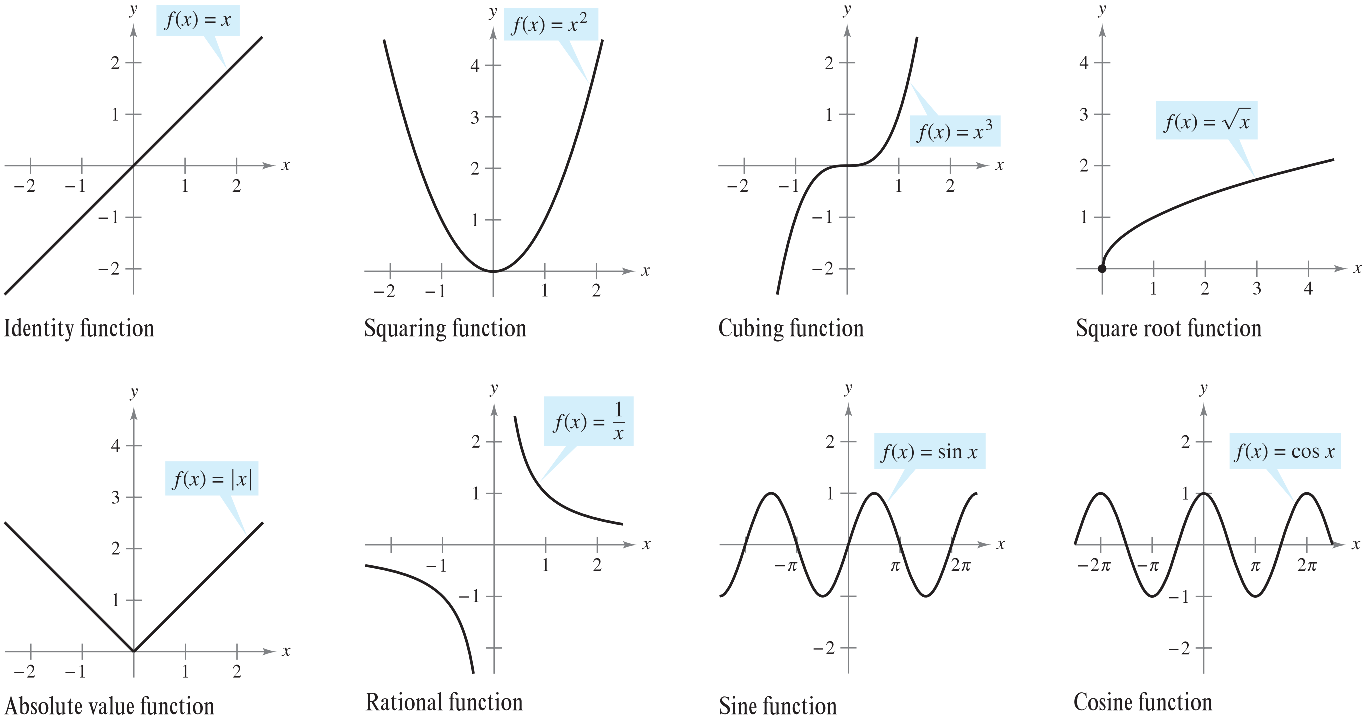
A vertical line can intersect the graph of a function of x at most *once*. This observation provides a convenient visual test, called the **Vertical Line Test,** for functions of x. That is, a graph in the coordinate plane is the graph of a function of x if and only if no vertical line intersects the graph at more than one point. For example, in Figure P.26(a), you can see that the graph does not define y as a function of x because a vertical line intersects the graph twice, whereas in Figures P.26(b) and (c), the graphs do define y as a function of x.



(a) Not a function of x **(b)** A function of x **(c)** A function of x

Figure P.26

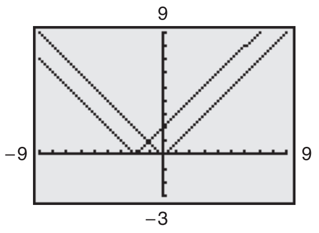
Figure P.27 shows the graphs of eight basic functions. You should be able to recognize these graphs. (Graphs of the other four basic trigonometric functions are shown in Appendix C.)



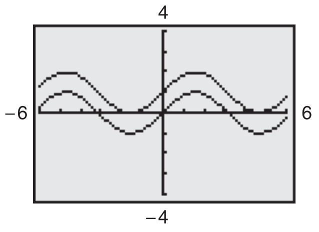
The graphs of eight basic functions
Figure P.27

EXPLORATION

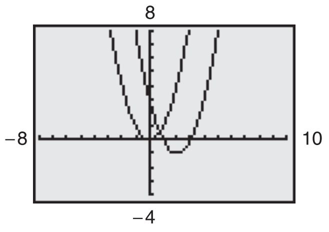
Writing Equations for Functions Each of the graphing utility screens below shows the graph of one of the eight basic functions shown on page 22. Each screen also shows a transformation of the graph. Describe the transformation. Then use your description to write an equation for the transformation.



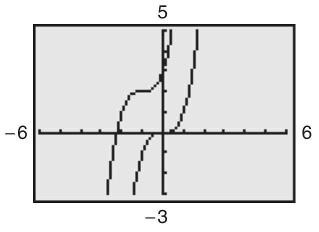
a.



b.



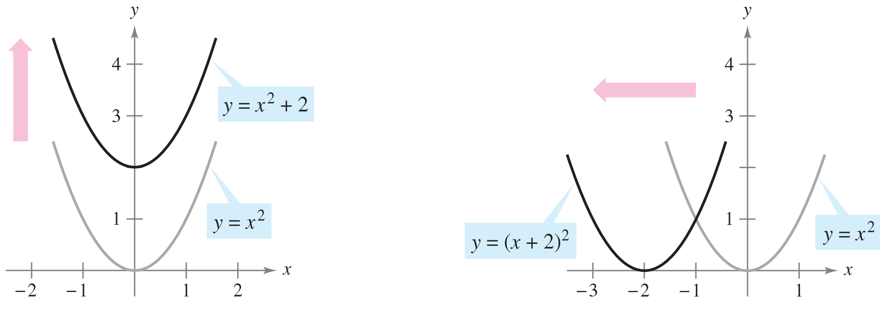
c.



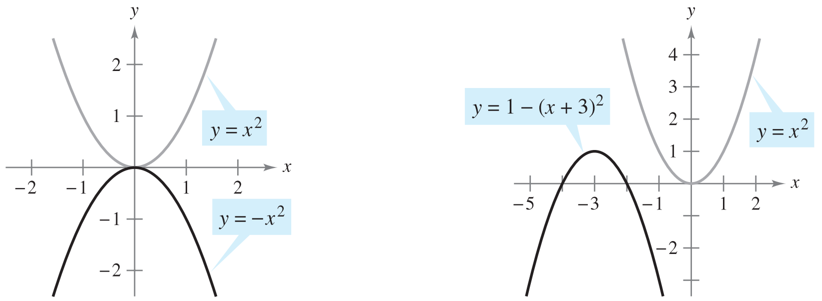
d.

Transformations of Functions

Some families of graphs have the same basic shape. For example, compare the graph of $y = x^2$ with the graphs of the four other quadratic functions shown in Figure P.28.



(a) Vertical shift upward **(b)** Horizontal shift to the left



(c) Reflection **(d)** Shift left, reflect, and shift upward

Figure P.28

Each of the graphs in Figure P.28 is a **transformation** of the graph of $y = x^2$. The three basic types of transformations illustrated by these graphs are vertical shifts, horizontal shifts, and reflections. Function notation lends itself well to describing transformations of graphs in the plane. For instance, if $f(x) = x^2$ is considered to be the original function in Figure P.28, the transformations shown can be represented by the following equations.

$y = f(x) + 2$	Vertical shift up 2 units
$y = f(x + 2)$	Horizontal shift to the left 2 units
$y = -f(x)$	Reflection about the x-axis
$y = -f(x + 3) + 1$	Shift left 3 units, reflect about x-axis, and shift up 1 unit

BASIC TYPES OF TRANSFORMATIONS ($c > 0$)

Original graph:	$y = f(x)$
Horizontal shift c units to the **right:**	$y = f(x - c)$
Horizontal shift c units to the **left:**	$y = f(x + c)$
Vertical shift c units **downward:**	$y = f(x) - c$
Vertical shift c units **upward:**	$y = f(x) + c$
Reflection (about the x-axis):	$y = -f(x)$
Reflection (about the y-axis):	$y = f(-x)$
Reflection (about the origin):	$y = -f(-x)$

Bettmann/Corbis

LEONHARD EULER (1707–1783)

In addition to making major contributions to almost every branch of mathematics, Euler was one of the first to apply calculus to real-life problems in physics. His extensive published writings include such topics as shipbuilding, acoustics, optics, astronomy, mechanics, and magnetism.

Classifications and Combinations of Functions

The modern notion of a function is derived from the efforts of many seventeenth- and eighteenth-century mathematicians. Of particular note was Leonhard Euler, to whom we are indebted for the function notation $y = f(x)$. By the end of the eighteenth century, mathematicians and scientists had concluded that many real-world phenomena could be represented by mathematical models taken from a collection of functions called **elementary functions.** Elementary functions fall into three categories.

1. Algebraic functions (polynomial, radical, rational)
2. Trigonometric functions (sine, cosine, tangent, and so on)
3. Exponential and logarithmic functions

You can review the trigonometric functions in Appendix C. The other nonalgebraic functions, such as the inverse trigonometric functions and the exponential and logarithmic functions, are introduced in Chapter 5.

The most common type of algebraic function is a **polynomial function**

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

where n is a nonnegative integer. The numbers a_i are **coefficients,** with a_n the **leading coefficient** and a_0 the **constant term** of the polynomial function. If $a_n \neq 0$, then n is the **degree** of the polynomial function. The zero polynomial $f(x) = 0$ is not assigned a degree. It is common practice to use subscript notation for coefficients of general polynomial functions, but for polynomial functions of low degree, the following simpler forms are often used. (Note that $a \neq 0$.)

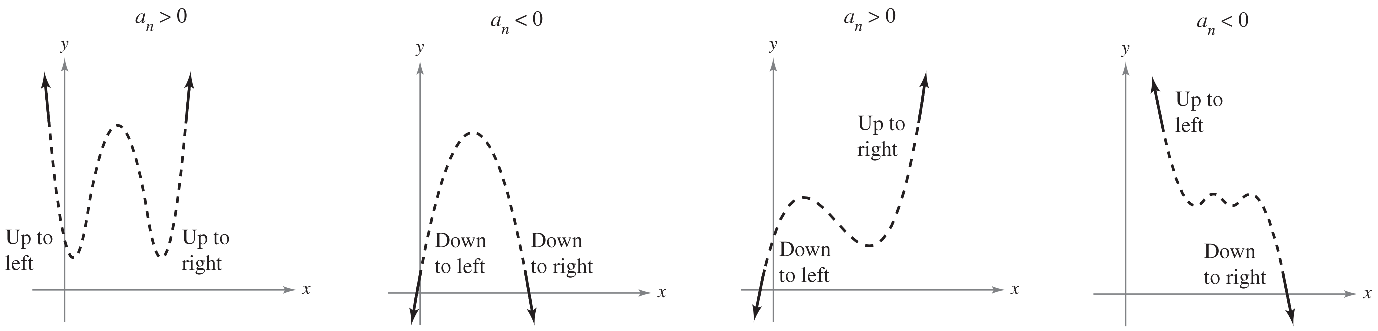
Zeroth degree: $f(x) = a$ Constant function
First degree: $f(x) = ax + b$ Linear function
Second degree: $f(x) = ax^2 + bx + c$ Quadratic function
Third degree: $f(x) = ax^3 + bx^2 + cx + d$ Cubic function

■ **FOR FURTHER INFORMATION** For more on the history of the concept of a function, see the article "Evolution of the Function Concept: A Brief Survey" by Israel Kleiner in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

Although the graph of a nonconstant polynomial function can have several turns, eventually the graph will rise or fall without bound as x moves to the right or left. Whether the graph of

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

eventually rises or falls can be determined by the function's degree (odd or even) and by the leading coefficient a_n, as indicated in Figure P.29. Note that the dashed portions of the graphs indicate that the **Leading Coefficient Test** determines *only* the right and left behavior of the graph.



Graphs of polynomial functions of even degree

Graphs of polynomial functions of odd degree

The Leading Coefficient Test for polynomial functions
Figure P.29

Just as a rational number can be written as the quotient of two integers, a **rational function** can be written as the quotient of two polynomials. Specifically, a function f is rational if it has the form

$$f(x) = \frac{p(x)}{q(x)}, \qquad q(x) \neq 0$$

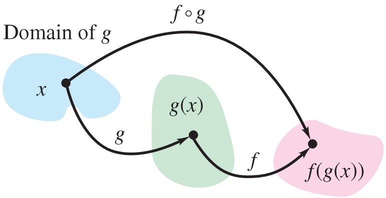
where $p(x)$ and $q(x)$ are polynomials.

Polynomial functions and rational functions are examples of **algebraic functions.** An algebraic function of x is one that can be expressed as a finite number of sums, differences, multiples, quotients, and radicals involving x^n. For example, $f(x) = \sqrt{x+1}$ is algebraic. Functions that are not algebraic are **transcendental.** For instance, the trigonometric functions are transcendental.

Two functions can be combined in various ways to create new functions. For example, given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, you can form the functions shown.

$$(f+g)(x) = f(x) + g(x) = (2x-3) + (x^2+1) \qquad \text{Sum}$$
$$(f-g)(x) = f(x) - g(x) = (2x-3) - (x^2+1) \qquad \text{Difference}$$
$$(fg)(x) = f(x)g(x) = (2x-3)(x^2+1) \qquad \text{Product}$$
$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{2x-3}{x^2+1} \qquad \text{Quotient}$$

You can combine two functions in yet another way, called **composition.** The resulting function is called a **composite function.**



The domain of the composite function $f \circ g$
Figure P.30

DEFINITION OF COMPOSITE FUNCTION

Let f and g be functions. The function given by $(f \circ g)(x) = f(g(x))$ is called the **composite** of f with g. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f (see Figure P.30).

The composite of f with g may not be equal to the composite of g with f.

EXAMPLE 4 Finding Composite Functions

Given $f(x) = 2x - 3$ and $g(x) = \cos x$, find each composite function.

a. $f \circ g$ **b.** $g \circ f$

Solution

a.
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g\\ &= f(\cos x) && \text{Substitute } \cos x \text{ for } g(x).\\ &= 2(\cos x) - 3 && \text{Definition of } f(x)\\ &= 2\cos x - 3 && \text{Simplify.}\end{aligned}$$

b.
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f\\ &= g(2x-3) && \text{Substitute } 2x-3 \text{ for } f(x).\\ &= \cos(2x-3) && \text{Definition of } g(x)\end{aligned}$$

Note that $(f \circ g)(x) \neq (g \circ f)(x)$. ■

EXPLORATION

Use a graphing utility to graph each function. Determine whether the function is *even*, *odd*, or *neither*.

$f(x) = x^2 - x^4$

$g(x) = 2x^3 + 1$

$h(x) = x^5 - 2x^3 + x$

$j(x) = 2 - x^6 - x^8$

$k(x) = x^5 - 2x^4 + x - 2$

$p(x) = x^9 + 3x^5 - x^3 + x$

Describe a way to identify a function as odd or even by inspecting the equation.

In Section P.1, an x-intercept of a graph was defined to be a point $(a, 0)$ at which the graph crosses the x-axis. If the graph represents a function f, the number a is a **zero** of f. In other words, *the zeros of a function f are the solutions of the equation $f(x) = 0$.* For example, the function $f(x) = x - 4$ has a zero at $x = 4$ because $f(4) = 0$.

In Section P.1 you also studied different types of symmetry. In the terminology of functions, a function is **even** if its graph is symmetric with respect to the y-axis, and is **odd** if its graph is symmetric with respect to the origin. The symmetry tests in Section P.1 yield the following test for even and odd functions.

TEST FOR EVEN AND ODD FUNCTIONS

The function $y = f(x)$ is **even** if $f(-x) = f(x)$.

The function $y = f(x)$ is **odd** if $f(-x) = -f(x)$.

NOTE Except for the constant function $f(x) = 0$, the graph of a function of x cannot have symmetry with respect to the x-axis because it then would fail the Vertical Line Test for the graph of the function. ■

EXAMPLE 5 Even and Odd Functions and Zeros of Functions

Determine whether each function is even, odd, or neither. Then find the zeros of the function.

a. $f(x) = x^3 - x$ **b.** $g(x) = 1 + \cos x$

Solution

a. This function is odd because

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x).$$

The zeros of f are found as shown.

$$x^3 - x = 0 \qquad \text{Let } f(x) = 0.$$

$$x(x^2 - 1) = x(x - 1)(x + 1) = 0 \qquad \text{Factor.}$$

$$x = 0, 1, -1 \qquad \text{Zeros of } f$$

See Figure P.31(a).

b. This function is even because

$$g(-x) = 1 + \cos(-x) = 1 + \cos x = g(x). \qquad \cos(-x) = \cos(x)$$

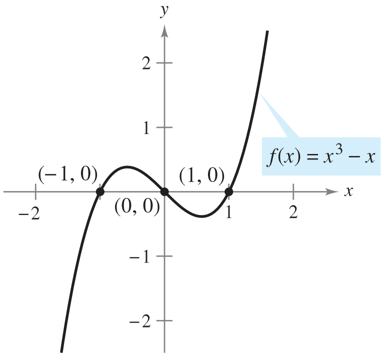
The zeros of g are found as shown.

$$1 + \cos x = 0 \qquad \text{Let } g(x) = 0.$$

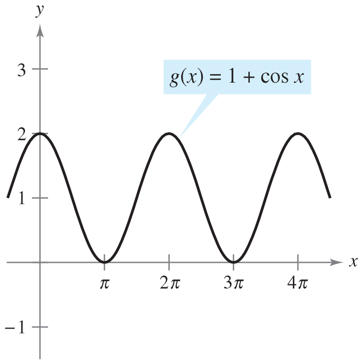
$$\cos x = -1 \qquad \text{Subtract 1 from each side.}$$

$$x = (2n + 1)\pi,\ n \text{ is an integer.} \qquad \text{Zeros of } g$$

See Figure P.31(b). ■



(a) Odd function



(b) Even function

Figure P.31

NOTE Each of the functions in Example 5 is either even or odd. However, some functions, such as $f(x) = x^2 + x + 1$, are neither even nor odd. ■

P.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use the graphs of f and g to answer the following.

(a) Identify the domains and ranges of f and g.

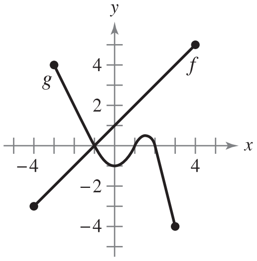
(b) Identify $f(-2)$ and $g(3)$.

(c) For what value(s) of x is $f(x) = g(x)$?

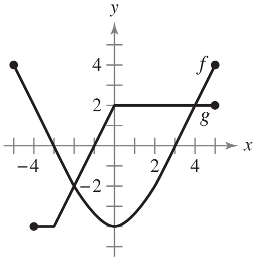
(d) Estimate the solution(s) of $f(x) = 2$.

(e) Estimate the solutions of $g(x) = 0$.

1.



2.



In Exercises 3–12, evaluate (if possible) the function at the given value(s) of the independent variable. Simplify the results.

3. $f(x) = 7x - 4$
- (a) $f(0)$
- (b) $f(-3)$
- (c) $f(b)$
- (d) $f(x - 1)$

4. $f(x) = \sqrt{x + 5}$
- (a) $f(-4)$
- (b) $f(11)$
- (c) $f(-8)$
- (d) $f(x + \Delta x)$

5. $g(x) = 5 - x^2$
- (a) $g(0)$
- (b) $g(\sqrt{5})$
- (c) $g(-2)$
- (d) $g(t - 1)$

6. $g(x) = x^2(x - 4)$
- (a) $g(4)$
- (b) $g\left(\frac{3}{2}\right)$
- (c) $g(c)$
- (d) $g(t + 4)$

7. $f(x) = \cos 2x$
- (a) $f(0)$
- (b) $f(-\pi/4)$
- (c) $f(\pi/3)$

8. $f(x) = \sin x$
- (a) $f(\pi)$
- (b) $f(5\pi/4)$
- (c) $f(2\pi/3)$

9. $f(x) = x^3$

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

10. $f(x) = 3x - 1$

$$\frac{f(x) - f(1)}{x - 1}$$

11. $f(x) = \dfrac{1}{\sqrt{x - 1}}$

$$\frac{f(x) - f(2)}{x - 2}$$

12. $f(x) = x^3 - x$

$$\frac{f(x) - f(1)}{x - 1}$$

In Exercises 13–20, find the domain and range of the function.

13. $f(x) = 4x^2$

14. $g(x) = x^2 - 5$

15. $g(x) = \sqrt{6x}$

16. $h(x) = -\sqrt{x + 3}$

17. $f(t) = \sec \dfrac{\pi t}{4}$

18. $h(t) = \cot t$

19. $f(x) = \dfrac{3}{x}$

20. $g(x) = \dfrac{2}{x - 1}$

In Exercises 21–26, find the domain of the function.

21. $f(x) = \sqrt{x} + \sqrt{1 - x}$

22. $f(x) = \sqrt{x^2 - 3x + 2}$

23. $g(x) = \dfrac{2}{1 - \cos x}$

24. $h(x) = \dfrac{1}{\sin x - \frac{1}{2}}$

25. $f(x) = \dfrac{1}{|x + 3|}$

26. $g(x) = \dfrac{1}{|x^2 - 4|}$

In Exercises 27–30, evaluate the function as indicated. Determine its domain and range.

27. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$

(a) $f(-1)$ (b) $f(0)$ (c) $f(2)$ (d) $f(t^2 + 1)$

28. $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$

(a) $f(-2)$ (b) $f(0)$ (c) $f(1)$ (d) $f(s^2 + 2)$

29. $f(x) = \begin{cases} |x| + 1, & x < 1 \\ -x + 1, & x \geq 1 \end{cases}$

(a) $f(-3)$ (b) $f(1)$ (c) $f(3)$ (d) $f(b^2 + 1)$

30. $f(x) = \begin{cases} \sqrt{x + 4}, & x \leq 5 \\ (x - 5)^2, & x > 5 \end{cases}$

(a) $f(-3)$ (b) $f(0)$ (c) $f(5)$ (d) $f(10)$

In Exercises 31–38, sketch a graph of the function and find its domain and range. Use a graphing utility to verify your graph.

31. $f(x) = 4 - x$

32. $g(x) = \dfrac{4}{x}$

33. $h(x) = \sqrt{x - 6}$

34. $f(x) = \frac{1}{4}x^3 + 3$

35. $f(x) = \sqrt{9 - x^2}$

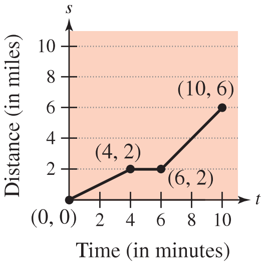
36. $f(x) = x + \sqrt{4 - x^2}$

37. $g(t) = 3 \sin \pi t$

38. $h(\theta) = -5 \cos \dfrac{\theta}{2}$

WRITING ABOUT CONCEPTS

39. The graph of the distance that a student drives in a 10-minute trip to school is shown in the figure. Give a verbal description of characteristics of the student's drive to school.

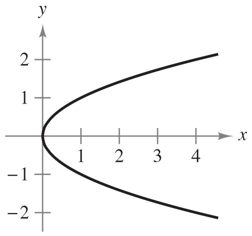


WRITING ABOUT CONCEPTS (continued)

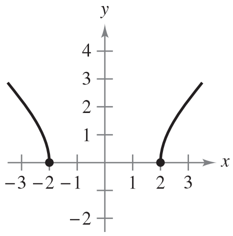
40. A student who commutes 27 miles to attend college remembers, after driving a few minutes, that a term paper that is due has been forgotten. Driving faster than usual, the student returns home, picks up the paper, and once again starts toward school. Sketch a possible graph of the student's distance from home as a function of time.

In Exercises 41–44, use the Vertical Line Test to determine whether y is a function of x. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

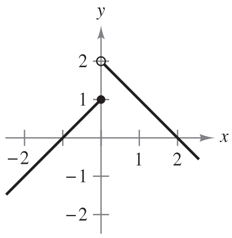
41. $x - y^2 = 0$



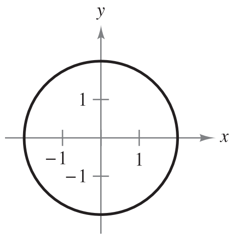
42. $\sqrt{x^2 - 4} - y = 0$



43. $y = \begin{cases} x + 1, & x \le 0 \\ -x + 2, & x > 0 \end{cases}$



44. $x^2 + y^2 = 4$



In Exercises 45–48, determine whether y is a function of x.

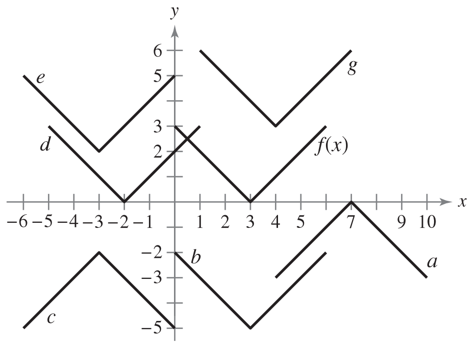
45. $x^2 + y^2 = 16$

46. $x^2 + y = 16$

47. $y^2 = x^2 - 1$

48. $x^2y - x^2 + 4y = 0$

In Exercises 49–54, use the graph of $y = f(x)$ to match the function with its graph.



49. $y = f(x + 5)$

50. $y = f(x) - 5$

51. $y = -f(-x) - 2$

52. $y = -f(x - 4)$

53. $y = f(x + 6) + 2$

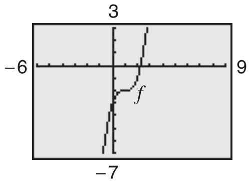
54. $y = f(x - 1) + 3$

55. Use the graph of f shown in the figure to sketch the graph of each function. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(a) $f(x + 3)$ (b) $f(x - 1)$

(c) $f(x) + 2$ (d) $f(x) - 4$

(e) $3f(x)$ (f) $\frac{1}{4}f(x)$

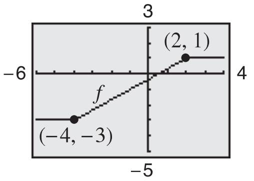


56. Use the graph of f shown in the figure to sketch the graph of each function. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(a) $f(x - 4)$ (b) $f(x + 2)$

(c) $f(x) + 4$ (d) $f(x) - 1$

(e) $2f(x)$ (f) $\frac{1}{2}f(x)$



57. Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of each function. In each case, describe the transformation.

(a) $y = \sqrt{x} + 2$ (b) $y = -\sqrt{x}$ (c) $y = \sqrt{x - 2}$

58. Specify a sequence of transformations that will yield each graph of h from the graph of the function $f(x) = \sin x$.

(a) $h(x) = \sin\left(x + \frac{\pi}{2}\right) + 1$ (b) $h(x) = -\sin(x - 1)$

59. Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, evaluate each expression.

(a) $f(g(1))$ (b) $g(f(1))$ (c) $g(f(0))$

(d) $f(g(-4))$ (e) $f(g(x))$ (f) $g(f(x))$

60. Given $f(x) = \sin x$ and $g(x) = \pi x$, evaluate each expression.

(a) $f(g(2))$ (b) $f\left(g\left(\frac{1}{2}\right)\right)$ (c) $g(f(0))$

(d) $g\left(f\left(\frac{\pi}{4}\right)\right)$ (e) $f(g(x))$ (f) $g(f(x))$

In Exercises 61–64, find the composite functions $(f \circ g)$ and $(g \circ f)$. What is the domain of each composite function? Are the two composite functions equal?

61. $f(x) = x^2$
$g(x) = \sqrt{x}$

62. $f(x) = x^2 - 1$
$g(x) = \cos x$

63. $f(x) = \dfrac{3}{x}$
$g(x) = x^2 - 1$

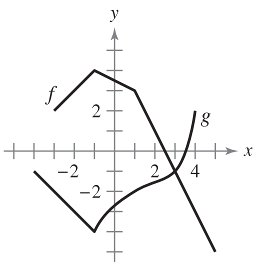
64. $f(x) = \dfrac{1}{x}$
$g(x) = \sqrt{x + 2}$

65. Use the graphs of f and g to evaluate each expression. If the result is undefined, explain why.

(a) $(f \circ g)(3)$ (b) $g(f(2))$

(c) $g(f(5))$ (d) $(f \circ g)(-3)$

(e) $(g \circ f)(-1)$ (f) $f(g(-1))$



66. ***Ripples*** A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius (in feet) of the outer ripple is given by $r(t) = 0.6t$, where t is the time in seconds after the pebble strikes the water. The area of the circle is given by the function $A(r) = \pi r^2$. Find and interpret $(A \circ r)(t)$.

Think About It **In Exercises 67 and 68, $F(x) = f \circ g \circ h$. Identify functions for f, g, and h. (There are many correct answers.)**

67. $F(x) = \sqrt{2x - 2}$

68. $F(x) = -4 \sin(1 - x)$

In Exercises 69–72, determine whether the function is even, odd, or neither. Use a graphing utility to verify your result.

69. $f(x) = x^2(4 - x^2)$

70. $f(x) = \sqrt[3]{x}$

71. $f(x) = x \cos x$

72. $f(x) = \sin^2 x$

Think About It **In Exercises 73 and 74, find the coordinates of a second point on the graph of a function f if the given point is on the graph and the function is (a) even and (b) odd.**

73. $\left(-\frac{3}{2}, 4\right)$

74. $(4, 9)$

75. The graphs of f, g, and h are shown in the figure. Decide whether each function is even, odd, or neither.

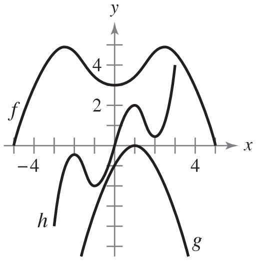


Figure for 75

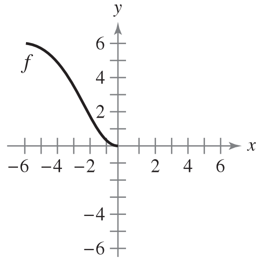


Figure for 76

76. The domain of the function f shown in the figure is $-6 \le x \le 6$.

(a) Complete the graph of f given that f is even.

(b) Complete the graph of f given that f is odd.

Writing Functions **In Exercises 77–80, write an equation for a function that has the given graph.**

77. Line segment connecting $(-2, 4)$ and $(0, -6)$

78. Line segment connecting $(3, 1)$ and $(5, 8)$

79. The bottom half of the parabola $x + y^2 = 0$

80. The bottom half of the circle $x^2 + y^2 = 36$

In Exercises 81–84, sketch a possible graph of the situation.

81. The speed of an airplane as a function of time during a 5-hour flight

82. The height of a baseball as a function of horizontal distance during a home run

83. The amount of a certain brand of sneaker sold by a sporting goods store as a function of the price of the sneaker

84. The value of a new car as a function of time over a period of 8 years

85. Find the value of c such that the domain of

$$f(x) = \sqrt{c - x^2}$$

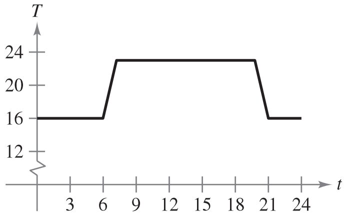
is $[-5, 5]$.

86. Find all values of c such that the domain of

$$f(x) = \frac{x + 3}{x^2 + 3cx + 6}$$

is the set of all real numbers.

87. ***Graphical Reasoning*** An electronically controlled thermostat is programmed to lower the temperature during the night automatically (see figure). The temperature T in degrees Celsius is given in terms of t, the time in hours on a 24-hour clock.



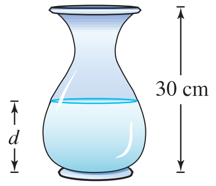
(a) Approximate $T(4)$ and $T(15)$.

(b) The thermostat is reprogrammed to produce a temperature $H(t) = T(t - 1)$. How does this change the temperature? Explain.

(c) The thermostat is reprogrammed to produce a temperature $H(t) = T(t) - 1$. How does this change the temperature? Explain.

CAPSTONE

88. Water runs into a vase of height 30 centimeters at a constant rate. The vase is full after 5 seconds. Use this information and the shape of the vase shown to answer the questions if d is the depth of the water in centimeters and t is the time in seconds (see figure).



(a) Explain why d is a function of t.

(b) Determine the domain and range of the function.

(c) Sketch a possible graph of the function.

(d) Use the graph in part (c) to approximate $d(4)$. What does this represent?

89. ***Modeling Data*** The table shows the average numbers of acres per farm in the United States for selected years. *(Source: U.S. Department of Agriculture)*

Year	1955	1965	1975	1985	1995	2005
Acreage	258	340	420	441	438	444

(a) Plot the data, where A is the acreage and t is the time in years, with $t = 5$ corresponding to 1955. Sketch a freehand curve that approximates the data.

(b) Use the curve in part (a) to approximate $A(20)$.

90. ***Automobile Aerodynamics*** The horsepower H required to overcome wind drag on a certain automobile is approximated by

$$H(x) = 0.002x^2 + 0.005x - 0.029, \quad 10 \le x \le 100$$

where x is the speed of the car in miles per hour.

(a) Use a graphing utility to graph H.

(b) Rewrite the power function so that x represents the speed in kilometers per hour. [Find $H(x/1.6)$.]

91. ***Think About It*** Write the function

$$f(x) = |x| + |x - 2|$$

without using absolute value signs. (For a review of absolute value, see Appendix C.)

92. ***Writing*** Use a graphing utility to graph the polynomial functions $p_1(x) = x^3 - x + 1$ and $p_2(x) = x^3 - x$. How many zeros does each function have? Is there a cubic polynomial that has no zeros? Explain.

93. Prove that the function is odd.

$$f(x) = a_{2n+1}x^{2n+1} + \cdot \cdot \cdot + a_3x^3 + a_1x$$

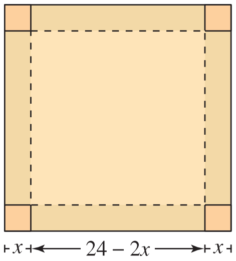
94. Prove that the function is even.

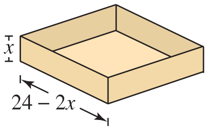
$$f(x) = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdot \cdot \cdot + a_2x^2 + a_0$$

95. Prove that the product of two even (or two odd) functions is even.

96. Prove that the product of an odd function and an even function is odd.

97. ***Volume*** An open box of maximum volume is to be made from a square piece of material 24 centimeters on a side by cutting equal squares from the corners and turning up the sides (see figure).





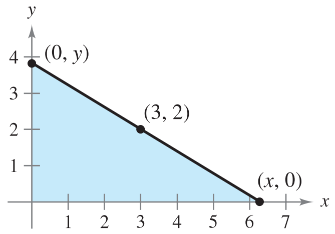
(a) Write the volume V as a function of x, the length of the corner squares. What is the domain of the function?

(b) Use a graphing utility to graph the volume function and approximate the dimensions of the box that yield a maximum volume.

(c) Use the *table* feature of a graphing utility to verify your answer in part (b). (The first two rows of the table are shown.)

Height, x	Length and Width	Volume, V
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$

98. ***Length*** A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(3, 2)$ (see figure). Write the length L of the hypotenuse as a function of x.



True or False? **In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

99. If $f(a) = f(b)$, then $a = b$.

100. A vertical line can intersect the graph of a function at most once.

101. If $f(x) = f(-x)$ for all x in the domain of f, then the graph of f is symmetric with respect to the y-axis.

102. If f is a function, then $f(ax) = af(x)$.

PUTNAM EXAM CHALLENGE

103. Let R be the region consisting of the points (x, y) of the Cartesian plane satisfying both $|x| - |y| \le 1$ and $|y| \le 1$. Sketch the region R and find its area.

104. Consider a polynomial $f(x)$ with real coefficients having the property $f(g(x)) = g(f(x))$ for every polynomial $g(x)$ with real coefficients. Determine and prove the nature of $f(x)$.

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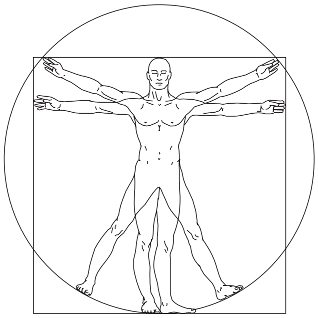
P.4 Fitting Models to Data

- **Fit a linear model to a real-life data set.**
- **Fit a quadratic model to a real-life data set.**
- **Fit a trigonometric model to a real-life data set.**

Fitting a Linear Model to Data

A basic premise of science is that much of the physical world can be described mathematically and that many physical phenomena are predictable. This scientific outlook was part of the scientific revolution that took place in Europe during the late 1500s. Two early publications connected with this revolution were *On the Revolutions of the Heavenly Spheres* by the Polish astronomer Nicolaus Copernicus and *On the Structure of the Human Body* by the Belgian anatomist Andreas Vesalius. Each of these books was published in 1543, and each broke with prior tradition by suggesting the use of a scientific method rather than unquestioned reliance on authority.

One basic technique of modern science is gathering data and then describing the data with a mathematical model. For instance, the data given in Example 1 are inspired by Leonardo da Vinci's famous drawing that indicates that a person's height and arm span are equal.



A computer graphics drawing based on the pen and ink drawing of Leonardo da Vinci's famous study of human proportions, called *Vitruvian Man*

EXAMPLE 1 Fitting a Linear Model to Data

A class of 28 people collected the following data, which represent their heights x and arm spans y (rounded to the nearest inch).

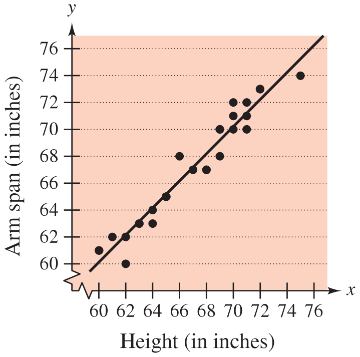
(60, 61), (65, 65), (68, 67), (72, 73), (61, 62), (63, 63), (70, 71),
(75, 74), (71, 72), (62, 60), (65, 65), (66, 68), (62, 62), (72, 73),
(70, 70), (69, 68), (69, 70), (60, 61), (63, 63), (64, 64), (71, 71),
(68, 67), (69, 70), (70, 72), (65, 65), (64, 63), (71, 70), (67, 67)

Find a linear model to represent these data.

Solution There are different ways to model these data with an equation. The simplest would be to observe that x and y are about the same and list the model as simply $y = x$. A more careful analysis would be to use a procedure from statistics called linear regression. (You will study this procedure in Section 13.9.) The least squares regression line for these data is

$$y = 1.006x - 0.23. \qquad \text{Least squares regression line}$$

The graph of the model and the data are shown in Figure P.32. From this model, you can see that a person's arm span tends to be about the same as his or her height. ■



Linear model and data
Figure P.32

TECHNOLOGY Many scientific and graphing calculators have built-in least squares regression programs. Typically, you enter the data into the calculator and then run the linear regression program. The program usually displays the slope and y-intercept of the best-fitting line and the *correlation coefficient* r. The correlation coefficient gives a measure of how well the model fits the data. The closer $|r|$ is to 1, the better the model fits the data. For instance, the correlation coefficient for the model in Example 1 is $r \approx 0.97$, which indicates that the model is a good fit for the data. If the r-value is positive, the variables have a positive correlation, as in Example 1. If the r-value is negative, the variables have a negative correlation.

Fitting a Quadratic Model to Data

A function that gives the height s of a falling object in terms of the time t is called a position function. If air resistance is not considered, the position of a falling object can be modeled by

$$s(t) = \tfrac{1}{2}gt^2 + v_0 t + s_0$$

where g is the acceleration due to gravity, v_0 is the initial velocity, and s_0 is the initial height. The value of g depends on where the object is dropped. On Earth, g is approximately -32 feet per second per second, or -9.8 meters per second per second.

To discover the value of g experimentally, you could record the heights of a falling object at several increments, as shown in Example 2.

EXAMPLE 2 Fitting a Quadratic Model to Data

A basketball is dropped from a height of about $5\frac{1}{4}$ feet. The height of the basketball is recorded 23 times at intervals of about 0.02 second.* The results are shown in the table.

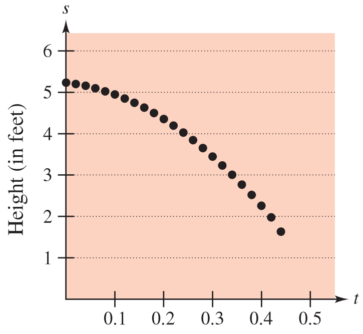
Time	0.0	0.02	0.04	0.06	0.08	0.099996
Height	5.23594	5.20353	5.16031	5.0991	5.02707	4.95146

Time	0.119996	0.139992	0.159988	0.179988	0.199984	0.219984
Height	4.85062	4.74979	4.63096	4.50132	4.35728	4.19523

Time	0.23998	0.25993	0.27998	0.299976	0.319972	0.339961
Height	4.02958	3.84593	3.65507	3.44981	3.23375	3.01048

Time	0.359961	0.379951	0.399941	0.419941	0.439941
Height	2.76921	2.52074	2.25786	1.98058	1.63488

Find a model to fit these data. Then use the model to predict the time when the basketball will hit the ground.



Scatter plot of data
Figure P.33

Solution Begin by drawing a scatter plot of the data, as shown in Figure P.33. From the scatter plot, you can see that the data do not appear to be linear. It does appear, however, that they might be quadratic. To check this, enter the data into a calculator or computer that has a quadratic regression program. You should obtain the model

$$s = -15.45t^2 - 1.302t + 5.2340. \qquad \text{Least squares regression quadratic}$$

Using this model, you can predict the time when the basketball hits the ground by substituting 0 for s and solving the resulting equation for t.

$$0 = -15.45t^2 - 1.302t + 5.2340 \qquad \text{Let } s = 0.$$

$$t = \frac{1.302 \pm \sqrt{(-1.302)^2 - 4(-15.45)(5.2340)}}{2(-15.45)} \qquad \text{Quadratic Formula}$$

$$t \approx 0.54 \qquad \text{Choose positive solution.}$$

The solution is about 0.54 second. In other words, the basketball will continue to fall for about 0.1 second more before hitting the ground. ■

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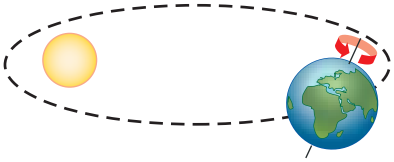
* *Data were collected with a Texas Instruments CBL (Calculator-Based Laboratory) System.*



The plane of Earth's orbit about the sun and its axis of rotation are not perpendicular. Instead, Earth's axis is tilted with respect to its orbit. The result is that the amount of daylight received by locations on Earth varies with the time of year. That is, it varies with the position of Earth in its orbit.

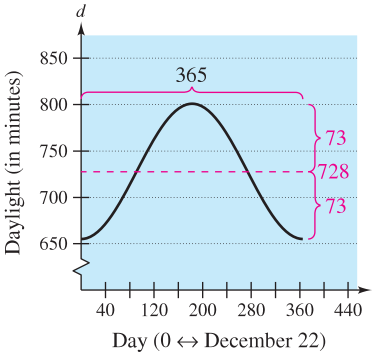
Fitting a Trigonometric Model to Data

What is mathematical modeling? This is one of the questions that is asked in the book *Guide to Mathematical Modelling*. Here is part of the answer.*

1. Mathematical modeling consists of applying your mathematical skills to obtain useful answers to real problems.
2. Learning to apply mathematical skills is very different from learning mathematics itself.
3. Models are used in a very wide range of applications, some of which do not appear initially to be mathematical in nature.
4. Models often allow quick and cheap evaluation of alternatives, leading to optimal solutions that are not otherwise obvious.
5. There are no precise rules in mathematical modeling and no "correct" answers.
6. Modeling can be learned only by *doing*.

EXAMPLE 3 Fitting a Trigonometric Model to Data

The number of hours of daylight on a given day on Earth depends on the latitude and the time of year. Here are the numbers of minutes of daylight at a location of 20°N latitude on the longest and shortest days of the year: June 21, 801 minutes; December 22, 655 minutes. Use these data to write a model for the amount of daylight d (in minutes) on each day of the year at a location of 20°N latitude. How could you check the accuracy of your model?



Graph of model
Figure P.34

Solution Here is one way to create a model. You can hypothesize that the model is a sine function whose period is 365 days. Using the given data, you can conclude that the amplitude of the graph is $(801 - 655)/2$, or 73. So, one possible model is

$$d = 728 - 73 \sin\left(\frac{2\pi t}{365} + \frac{\pi}{2}\right).$$

In this model, t represents the number of each day of the year, with December 22 represented by $t = 0$. A graph of this model is shown in Figure P.34. To check the accuracy of this model, a weather almanac was used to find the numbers of minutes of daylight on different days of the year at the location of 20°N latitude.

Date	*Value of t*	*Actual Daylight*	*Daylight Given by Model*
Dec 22	0	655 min	655 min
Jan 1	10	657 min	656 min
Feb 1	41	676 min	672 min
Mar 1	69	705 min	701 min
Apr 1	100	740 min	739 min
May 1	130	772 min	773 min
Jun 1	161	796 min	796 min
Jun 21	181	801 min	801 min
Jul 1	191	799 min	800 min
Aug 1	222	782 min	785 min
Sep 1	253	752 min	754 min
Oct 1	283	718 min	716 min
Nov 1	314	685 min	681 min
Dec 1	344	661 min	660 min

You can see that the model is fairly accurate. ■

NOTE For a review of trigonometric functions, see Appendix C.

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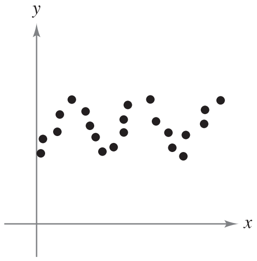
* *Text from Dilwyn Edwards and Mike Hamson,* Guide to Mathematical Modelling *(Boca Raton: CRC Press, 1990), p. 4. Used by permission of the authors.*

P.4 Exercises

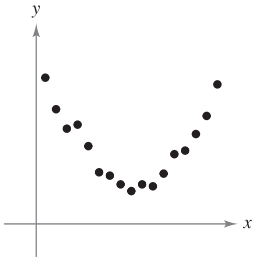
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, a scatter plot of data is given. Determine whether the data can be modeled by a linear function, a quadratic function, or a trigonometric function, or that there appears to be no relationship between x and y. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

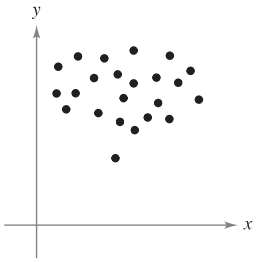
1.



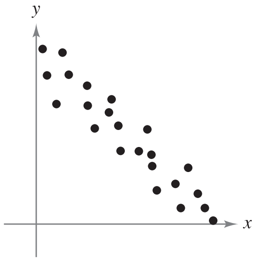
2.



3.



4.



5. ***Carcinogens*** Each ordered pair gives the exposure index x of a carcinogenic substance and the cancer mortality y per 100,000 people in the population.

(3.50, 150.1), (3.58, 133.1), (4.42, 132.9),
(2.26, 116.7), (2.63, 140.7), (4.85, 165.5),
(12.65, 210.7), (7.42, 181.0), (9.35, 213.4)

(a) Plot the data. From the graph, do the data appear to be approximately linear?

(b) Visually find a linear model for the data. Graph the model.

(c) Use the model to approximate y if $x = 3$.

6. ***Quiz Scores*** The ordered pairs represent the scores on two consecutive 15-point quizzes for a class of 18 students.

(7, 13), (9, 7), (14, 14), (15, 15), (10, 15), (9, 7),
(14, 11), (14, 15), (8, 10), (15, 9), (10, 11), (9, 10),
(11, 14), (7, 14), (11, 10), (14, 11), (10, 15), (9, 6)

(a) Plot the data. From the graph, does the relationship between consecutive scores appear to be approximately linear?

(b) If the data appear to be approximately linear, find a linear model for the data. If not, give some possible explanations.

7. ***Hooke's Law*** Hooke's Law states that the force F required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed or stretched from its original length. That is, $F = kd$, where k is a measure of the stiffness of the spring and is called the *spring constant*. The table shows the elongation d in centimeters of a spring when a force of F newtons is applied.

F	20	40	60	80	100
d	1.4	2.5	4.0	5.3	6.6

(a) Use the regression capabilities of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model. How well does the model fit the data? Explain your reasoning.

(c) Use the model to estimate the elongation of the spring when a force of 55 newtons is applied.

8. ***Falling Object*** In an experiment, students measured the speed s (in meters per second) of a falling object t seconds after it was released. The results are shown in the table.

t	0	1	2	3	4
s	0	11.0	19.4	29.2	39.4

(a) Use the regression capabilities of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model. How well does the model fit the data? Explain your reasoning.

(c) Use the model to estimate the speed of the object after 2.5 seconds.

9. ***Energy Consumption and Gross National Product*** The data show the per capita energy consumptions (in millions of Btu) and the per capita gross national products (in thousands of U.S. dollars) for several countries in 2004. *(Source: U.S. Census Bureau)*

Argentina	(71, 12.53)	**Bangladesh**	(5, 1.97)
Chile	(75, 10.61)	**Ecuador**	(29, 3.77)
Greece	(136, 22.23)	**Hong Kong**	(159, 31.56)
Hungary	(106, 15.8)	**India**	(15, 3.12)
Mexico	(63, 9.64)	**Poland**	(95, 12.73)
Portugal	(106, 19.24)	**South Korea**	(186, 20.53)
Spain	(159, 24.75)	**Turkey**	(51, 7.72)
United Kingdom	(167, 31.43)	**Venezuela**	(115, 5.83)

(a) Use the regression capabilities of a graphing utility to find a linear model for the data. What is the correlation coefficient?

(b) Use a graphing utility to plot the data and graph the model.

(c) Interpret the graph in part (b). Use the graph to identify the four countries that differ most from the linear model.

(d) Delete the data for the four countries identified in part (c). Fit a linear model to the remaining data and give the correlation coefficient.

10. *Brinell Hardness* The data in the table show the Brinell hardness H of 0.35 carbon steel when hardened and tempered at temperature t (degrees Fahrenheit). *(Source: Standard Handbook for Mechanical Engineers)*

t	200	400	600	800	1000	1200
H	534	495	415	352	269	217

(a) Use the regression capabilities of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model. How well does the model fit the data? Explain your reasoning.

(c) Use the model to estimate the hardness when t is 500°F.

11. *Automobile Costs* The data in the table show the variable costs of operating an automobile in the United States for several recent years. The functions y_1, y_2, and y_3 represent the costs in cents per mile for gas, maintenance, and tires, respectively. *(Source: Bureau of Transportation Statistics)*

Year	y_1	y_2	y_3
0	5.60	3.30	1.70
1	6.90	3.60	1.70
2	7.90	3.90	1.80
3	5.90	4.10	1.80
4	7.20	4.10	1.80
5	6.50	5.40	0.70
6	9.50	4.90	0.70
7	8.90	4.90	0.70

(a) Use the regression capabilities of a graphing utility to find cubic models for y_1 and y_3 and a linear model for y_2.

(b) Use a graphing utility to graph y_1, y_2, y_3, and $y_1 + y_2 + y_3$ in the same viewing window. Use the model to estimate the total variable cost per mile in year 12.

12. *Beam Strength* Students in a lab measured the breaking strength S (in pounds) of wood 2 inches thick, x inches high, and 12 inches long. The results are shown in the table.

x	4	6	8	10	12
S	2370	5460	10,310	16,250	23,860

(a) Use the regression capabilities of a graphing utility to fit a quadratic model to the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model to approximate the breaking strength when $x = 2$.

13. *Car Performance* The time t (in seconds) required to attain a speed of s miles per hour from a standing start for a Honda Accord Hybrid is shown in the table. *(Source: Car & Driver)*

s	30	40	50	60	70	80	90
t	2.5	3.5	5.0	6.7	8.7	11.5	14.4

(a) Use the regression capabilities of a graphing utility to find a quadratic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

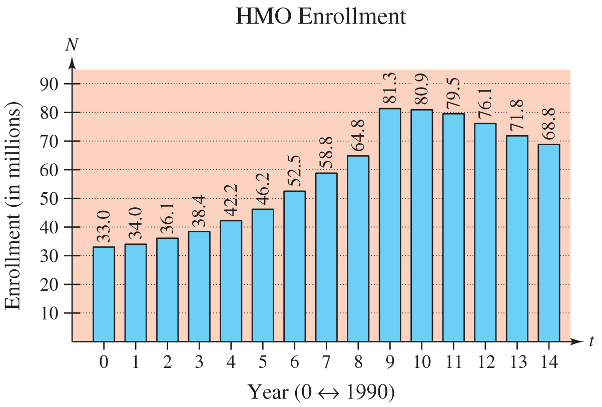
(c) Use the graph in part (b) to state why the model is not appropriate for determining the times required to attain speeds of less than 20 miles per hour.

(d) Because the test began from a standing start, add the point (0, 0) to the data. Fit a quadratic model to the revised data and graph the new model.

(e) Does the quadratic model in part (d) more accurately model the behavior of the car? Explain.

CAPSTONE

14. *Health Maintenance Organizations* The bar graph shows the numbers of people N (in millions) receiving care in HMOs for the years 1990 through 2004. *(Source: HealthLeaders-InterStudy)*



(a) Let t be the time in years, with $t = 0$ corresponding to 1990. Use the regression capabilities of a graphing utility to find linear and cubic models for the data.

(b) Use a graphing utility to graph the data and the linear and cubic models.

(c) Use the graphs in part (b) to determine which is the better model.

(d) Use a graphing utility to find and graph a quadratic model for the data. How well does the model fit the data? Explain your reasoning.

(e) Use the linear and cubic models to estimate the number of people receiving care in HMOs in the year 2007. What do you notice?

(f) Use a graphing utility to find other models for the data. Which models do you think best represent the data? Explain.

15. ***Car Performance*** A V8 car engine is coupled to a dynamometer, and the horsepower y is measured at different engine speeds x (in thousands of revolutions per minute). The results are shown in the table.

x	1	2	3	4	5	6
y	40	85	140	200	225	245

(a) Use the regression capabilities of a graphing utility to find a cubic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model to approximate the horsepower when the engine is running at 4500 revolutions per minute.

16. ***Boiling Temperature*** The table shows the temperatures T (°F) at which water boils at selected pressures p (pounds per square inch). *(Source: Standard Handbook for Mechanical Engineers)*

p	5	10	14.696 (1 atmosphere)	20
T	162.24°	193.21°	212.00°	227.96°

p	30	40	60	80	100
T	250.33°	267.25°	292.71°	312.03°	327.81°

(a) Use the regression capabilities of a graphing utility to find a cubic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the graph to estimate the pressure required for the boiling point of water to exceed 300°F.

(d) Explain why the model would not be accurate for pressures exceeding 100 pounds per square inch.

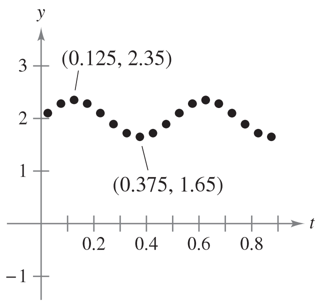
17. ***Harmonic Motion*** The motion of an oscillating weight suspended by a spring was measured by a motion detector. The data collected and the approximate maximum (positive and negative) displacements from equilibrium are shown in the figure. The displacement y is measured in centimeters and the time t is measured in seconds.

(a) Is y a function of t? Explain.

(b) Approximate the amplitude and period of the oscillations.

(c) Find a model for the data.

(d) Use a graphing utility to graph the model in part (c). Compare the result with the data in the figure.



18. ***Temperature*** The table shows the normal daily high temperatures for Miami M and Syracuse S (in degrees Fahrenheit) for month t, with $t = 1$ corresponding to January. *(Source: NOAA)*

t	1	2	3	4	5	6
M	76.5	77.7	80.7	83.8	87.2	89.5
S	31.4	33.5	43.1	55.7	68.5	77.0

t	7	8	9	10	11	12
M	90.9	90.6	89.0	85.4	81.2	77.5
S	81.7	79.6	71.4	59.8	47.4	36.3

(a) A model for Miami is

$$M(t) = 83.70 + 7.46 \sin(0.4912t - 1.95).$$

Find a model for Syracuse.

(b) Use a graphing utility to graph the data and the model for the temperatures in Miami. How well does the model fit?

(c) Use a graphing utility to graph the data and the model for the temperatures in Syracuse. How well does the model fit?

(d) Use the models to estimate the average annual temperature in each city. Which term of the model did you use? Explain.

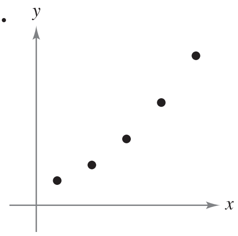
(e) What is the period of each model? Is it what you expected? Explain.

(f) Which city has a greater variability in temperature throughout the year? Which factor of the models determines this variability? Explain.

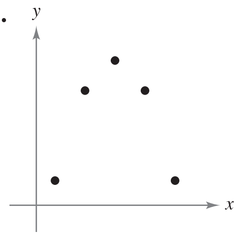
WRITING ABOUT CONCEPTS

In Exercises 19 and 20, describe a possible real-life situation for each data set. Then describe how a model could be used in the real-life setting.

19.



20.



PUTNAM EXAM CHALLENGE

21. For $i = 1, 2$ let T_i be a triangle with side lengths $a_i, b_i, c_i,$ and area A_i. Suppose that $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \leq A_2$?

This problem was composed by the Committee on the Putnam Prize Competition.
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P REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find the intercepts (if any).

1. $y = 5x - 8$

2. $y = (x - 2)(x - 6)$

3. $y = \dfrac{x - 3}{x - 4}$

4. $xy = 4$

In Exercises 5 and 6, check for symmetry with respect to both axes and to the origin.

5. $x^2y - x^2 + 4y = 0$

6. $y = x^4 - x^2 + 3$

In Exercises 7–14, sketch the graph of the equation.

7. $y = \frac{1}{2}(-x + 3)$

8. $6x - 3y = 12$

9. $-\frac{1}{3}x + \frac{5}{6}y = 1$

10. $0.02x + 0.15y = 0.25$

11. $y = 9 - 8x - x^2$

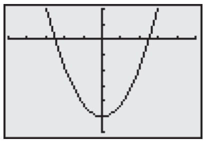
12. $y = 6x - x^2$

13. $y = 2\sqrt{4 - x}$

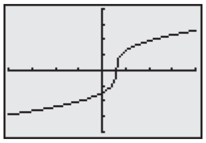
14. $y = |x - 4| - 4$

In Exercises 15 and 16, describe the viewing window of a graphing utility that yields the figure.

15. $y = 4x^2 - 25$



16. $y = 8\sqrt[3]{x - 6}$



In Exercises 17 and 18, use a graphing utility to find the point(s) of intersection of the graphs of the equations.

17. $5x + 3y = -1$
$x - y = -5$

18. $x - y + 1 = 0$
$y - x^2 = 7$

19. ***Think About It*** Write an equation whose graph has intercepts at $x = -4$ and $x = 4$ and is symmetric with respect to the origin.

20. ***Think About It*** For what value of k does the graph of $y = kx^3$ pass through the point?

(a) $(1, 4)$ (b) $(-2, 1)$ (c) $(0, 0)$ (d) $(-1, -1)$

In Exercises 21 and 22, plot the points and find the slope of the line passing through the points.

21. $\left(\frac{3}{2}, 1\right), \left(5, \frac{5}{2}\right)$

22. $(-7, 8), (-1, 8)$

In Exercises 23 and 24, use the concept of slope to find t such that the three points are collinear.

23. $(-8, 5), (0, t), (2, -1)$

24. $(-3, 3), (t, -1), (8, 6)$

In Exercises 25–28, find an equation of the line that passes through the point with the indicated slope. Sketch the line.

25. $(3, -5),\quad m = \frac{7}{4}$

26. $(-8, 1),\quad m$ is undefined.

27. $(-3, 0),\quad m = -\frac{2}{3}$

28. $(5, 4),\quad m = 0$

29. Find equations of the lines passing through $(-3, 5)$ and having the following characteristics.

(a) Slope of $\frac{7}{16}$

(b) Parallel to the line $5x - 3y = 3$

(c) Passing through the origin

(d) Parallel to the y-axis

30. Find equations of the lines passing through $(2, 4)$ and having the following characteristics.

(a) Slope of $-\frac{2}{3}$

(b) Perpendicular to the line $x + y = 0$

(c) Passing through the point $(6, 1)$

(d) Parallel to the x-axis

31. ***Rate of Change*** The purchase price of a new machine is \$12,500, and its value will decrease by \$850 per year. Use this information to write a linear equation that gives the value V of the machine t years after it is purchased. Find its value at the end of 3 years.

32. ***Break-Even Analysis*** A contractor purchases a piece of equipment for \$36,500 that costs an average of \$9.25 per hour for fuel and maintenance. The equipment operator is paid \$13.50 per hour, and customers are charged \$30 per hour.

(a) Write an equation for the cost C of operating this equipment for t hours.

(b) Write an equation for the revenue R derived from t hours of use.

(c) Find the break-even point for this equipment by finding the time at which $R = C$.

In Exercises 33–36, sketch the graph of the equation and use the Vertical Line Test to determine whether the equation expresses y as a function of x.

33. $x - y^2 = 6$

34. $x^2 - y = 0$

35. $y = \dfrac{|x - 2|}{x - 2}$

36. $x = 9 - y^2$

37. Evaluate (if possible) the function $f(x) = 1/x$ at the specified values of the independent variable, and simplify the results.

(a) $f(0)$ (b) $\dfrac{f(1 + \Delta x) - f(1)}{\Delta x}$

38. Evaluate (if possible) the function at each value of the independent variable.

$$f(x) = \begin{cases} x^2 + 2, & x < 0 \\ |x - 2|, & x \geq 0 \end{cases}$$

(a) $f(-4)$ (b) $f(0)$ (c) $f(1)$

39. Find the domain and range of each function.

(a) $y = \sqrt{36 - x^2}$ (b) $y = \dfrac{7}{2x - 10}$ (c) $y = \begin{cases} x^2, & x < 0 \\ 2 - x, & x \geq 0 \end{cases}$

40. Given $f(x) = 1 - x^2$ and $g(x) = 2x + 1$, evaluate each expression.

(a) $f(x) - g(x)$ (b) $f(x)g(x)$ (c) $g(f(x))$

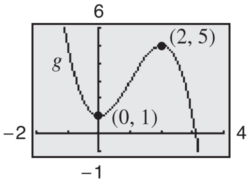
41. Sketch (on the same set of coordinate axes) graphs of f for $c = -2$, 0, and 2.

(a) $f(x) = x^3 + c$ (b) $f(x) = (x - c)^3$

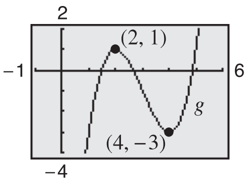
(c) $f(x) = (x - 2)^3 + c$ (d) $f(x) = cx^3$

42. Use a graphing utility to graph $f(x) = x^3 - 3x^2$. Use the graph to write a formula for the function g shown in the figure. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(a)



(b)



43. ***Conjecture***

(a) Use a graphing utility to graph the functions f, g, and h in the same viewing window. Write a description of any similarities and differences you observe among the graphs.

Odd powers: $f(x) = x$, $g(x) = x^3$, $h(x) = x^5$

Even powers: $f(x) = x^2$, $g(x) = x^4$, $h(x) = x^6$

(b) Use the result in part (a) to make a conjecture about the graphs of the functions $y = x^7$ and $y = x^8$. Use a graphing utility to verify your conjecture.

44. ***Think About It*** Use the results of Exercise 43 to guess the shapes of the graphs of the functions f, g, and h. Then use a graphing utility to graph each function and compare the result with your guess.

(a) $f(x) = x^2(x - 6)^2$ (b) $g(x) = x^3(x - 6)^2$

(c) $h(x) = x^3(x - 6)^3$

45. ***Area*** A wire 24 inches long is to be cut into four pieces to form a rectangle whose shortest side has a length of x.

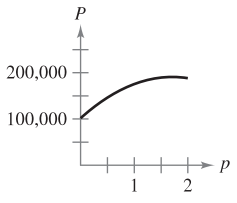
(a) Write the area A of the rectangle as a function of x.

(b) Determine the domain of the function and use a graphing utility to graph the function over that domain.

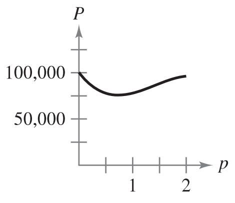
(c) Use the graph of the function to approximate the maximum area of the rectangle. Make a conjecture about the dimensions that yield a maximum area.

46. ***Writing*** The following graphs give the profits P for two small companies over a period p of 2 years. Create a story to describe the behavior of each profit function for some hypothetical product the company produces.

(a)

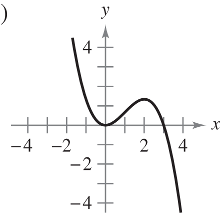


(b)

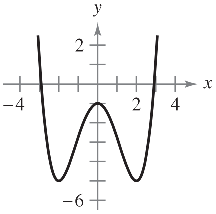


47. ***Think About It*** What is the minimum degree of the polynomial function whose graph approximates the given graph? What sign must the leading coefficient have?

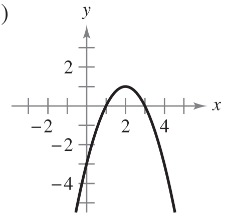
(a)



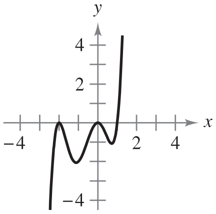
(b)



(c)



(d)



48. ***Stress Test*** A machine part was tested by bending it x centimeters 10 times per minute until the time y (in hours) of failure. The results are recorded in the table.

x	3	6	9	12	15	18	21	24	27	30
y	61	56	53	55	48	35	36	33	44	23

(a) Use the regression capabilities of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the graph to determine whether there may have been an error made in conducting one of the tests or in recording the results. If so, eliminate the erroneous point and find the model for the remaining data.

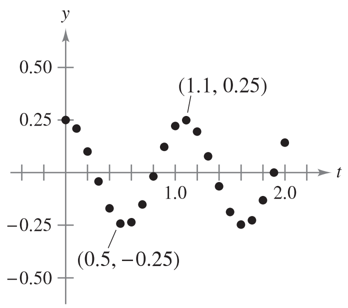
49. ***Harmonic Motion*** The motion of an oscillating weight suspended by a spring was measured by a motion detector. The data collected and the approximate maximum (positive and negative) displacements from equilibrium are shown in the figure. The displacement y is measured in feet and the time t is measured in seconds.

(a) Is y a function of t? Explain.

(b) Approximate the amplitude and period of the oscillations.

(c) Find a model for the data.

(d) Use a graphing utility to graph the model in part (c). Compare the result with the data in the figure.



P.S. PROBLEM SOLVING

1. Consider the circle $x^2 + y^2 - 6x - 8y = 0$, as shown in the figure.

(a) Find the center and radius of the circle.

(b) Find an equation of the tangent line to the circle at the point $(0, 0)$.

(c) Find an equation of the tangent line to the circle at the point $(6, 0)$.

(d) Where do the two tangent lines intersect?

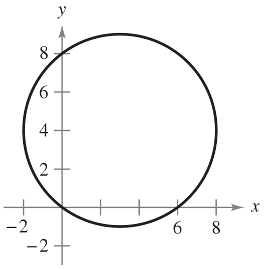


Figure for 1

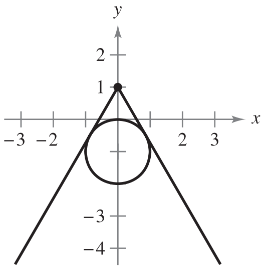


Figure for 2

2. There are two tangent lines from the point $(0, 1)$ to the circle $x^2 + (y + 1)^2 = 1$ (see figure). Find equations of these two lines by using the fact that each tangent line intersects the circle at *exactly* one point.

3. The Heaviside function $H(x)$ is widely used in engineering applications.

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Sketch the graph of the Heaviside function and the graphs of the following functions by hand.

(a) $H(x) - 2$	(b) $H(x - 2)$	(c) $-H(x)$
(d) $H(-x)$	(e) $\frac{1}{2}H(x)$	(f) $-H(x - 2) + 2$

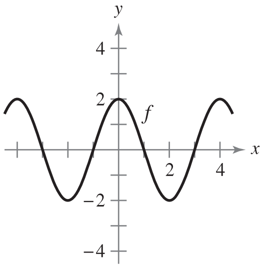
Institute of Electrical Engineers, London

OLIVER HEAVISIDE (1850–1925)

Heaviside was a British mathematician and physicist who contributed to the field of applied mathematics, especially applications of mathematics to electrical engineering. The *Heaviside function* is a classic type of "on-off" function that has applications to electricity and computer science.

4. Consider the graph of the function f shown below. Use this graph to sketch the graphs of the following functions. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(a) $f(x + 1)$	(b) $f(x) + 1$
(c) $2f(x)$	(d) $f(-x)$
(e) $-f(x)$	(f) $\lvert f(x) \rvert$
(g) $f(\lvert x \rvert)$	



5. A rancher plans to fence a rectangular pasture adjacent to a river. The rancher has 100 meters of fencing, and no fencing is needed along the river (see figure).

(a) Write the area A of the pasture as a function of x, the length of the side parallel to the river. What is the domain of A?

(b) Graph the area function $A(x)$ and estimate the dimensions that yield the maximum amount of area for the pasture.

(c) Find the dimensions that yield the maximum amount of area for the pasture by completing the square.

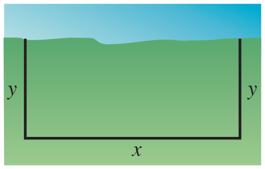


Figure for 5

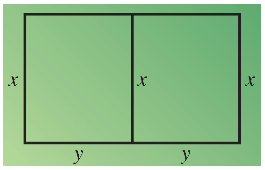


Figure for 6

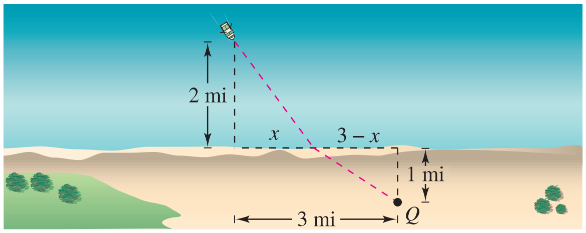
6. A rancher has 300 feet of fencing to enclose two adjacent pastures.

(a) Write the total area A of the two pastures as a function of x (see figure). What is the domain of A?

(b) Graph the area function and estimate the dimensions that yield the maximum amount of area for the pastures.

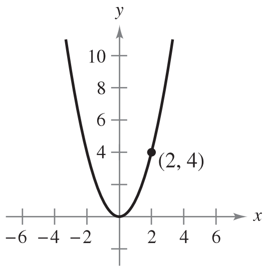
(c) Find the dimensions that yield the maximum amount of area for the pastures by completing the square.

7. You are in a boat 2 miles from the nearest point on the coast. You are to go to a point Q located 3 miles down the coast and 1 mile inland (see figure). You can row at 2 miles per hour and walk at 4 miles per hour. Write the total time T of the trip as a function of x.



8. You drive to the beach at a rate of 120 kilometers per hour. On the return trip, you drive at a rate of 60 kilometers per hour. What is your average speed for the entire trip? Explain your reasoning.

9. One of the fundamental themes of calculus is to find the slope of the tangent line to a curve at a point. To see how this can be done, consider the point $(2, 4)$ on the graph of $f(x) = x^2$ (see figure).



(a) Find the slope of the line joining $(2, 4)$ and $(3, 9)$. Is the slope of the tangent line at $(2, 4)$ greater than or less than this number?

(b) Find the slope of the line joining $(2, 4)$ and $(1, 1)$. Is the slope of the tangent line at $(2, 4)$ greater than or less than this number?

(c) Find the slope of the line joining $(2, 4)$ and $(2.1, 4.41)$. Is the slope of the tangent line at $(2, 4)$ greater than or less than this number?

(d) Find the slope of the line joining $(2, 4)$ and $(2 + h, f(2 + h))$ in terms of the nonzero number h. Verify that $h = 1, -1$, and 0.1 yield the solutions to parts (a)–(c) above.

(e) What is the slope of the tangent line at $(2, 4)$? Explain how you arrived at your answer.

10. Sketch the graph of the function $f(x) = \sqrt{x}$ and label the point $(4, 2)$ on the graph.

(a) Find the slope of the line joining $(4, 2)$ and $(9, 3)$. Is the slope of the tangent line at $(4, 2)$ greater than or less than this number?

(b) Find the slope of the line joining $(4, 2)$ and $(1, 1)$. Is the slope of the tangent line at $(4, 2)$ greater than or less than this number?

(c) Find the slope of the line joining $(4, 2)$ and $(4.41, 2.1)$. Is the slope of the tangent line at $(4, 2)$ greater than or less than this number?

(d) Find the slope of the line joining $(4, 2)$ and $(4 + h, f(4 + h))$ in terms of the nonzero number h.

(e) What is the slope of the tangent line at the point $(4, 2)$? Explain how you arrived at your answer.

11. Explain how you would graph the equation

$$y + |y| = x + |x|.$$

Then sketch the graph.

12. A large room contains two speakers that are 3 meters apart. The sound intensity I of one speaker is twice that of the other, as shown in the figure. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.) Suppose the listener is free to move about the room to find those positions that receive equal amounts of sound from both speakers. Such a location satisfies two conditions: (1) the sound intensity at the listener's position is directly proportional to the sound level of a source, and (2) the sound intensity is inversely proportional to the square of the distance from the source.

(a) Find the points on the x-axis that receive equal amounts of sound from both speakers.

(b) Find and graph the equation of all locations (x, y) where one could stand and receive equal amounts of sound from both speakers.

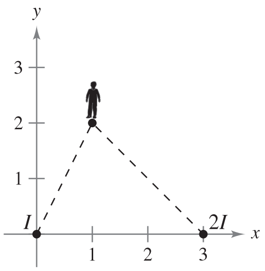


Figure for 12

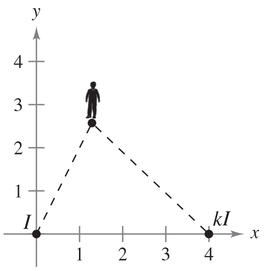


Figure for 13

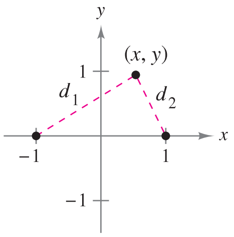
13. Suppose the speakers in Exercise 12 are 4 meters apart and the sound intensity of one speaker is k times that of the other, as shown in the figure. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(a) Find the equation of all locations (x, y) where one could stand and receive equal amounts of sound from both speakers.

(b) Graph the equation for the case $k = 3$.

(c) Describe the set of locations of equal sound as k becomes very large.

14. Let d_1 and d_2 be the distances from the point (x, y) to the points $(-1, 0)$ and $(1, 0)$, respectively, as shown in the figure. Show that the equation of the graph of all points (x, y) satisfying $d_1 d_2 = 1$ is $(x^2 + y^2)^2 = 2(x^2 - y^2)$. This curve is called a **lemniscate.** Graph the lemniscate and identify three points on the graph.



15. Let $f(x) = \dfrac{1}{1 - x}$.

(a) What are the domain and range of f?

(b) Find the composition $f(f(x))$. What is the domain of this function?

(c) Find $f(f(f(x)))$. What is the domain of this function?

(d) Graph $f(f(f(x)))$. Is the graph a line? Why or why not?

1 Limits and Their Properties

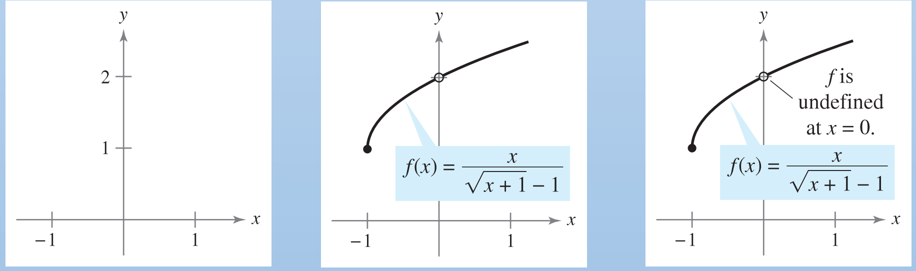
The limit of a function is the primary concept that distinguishes calculus from algebra and analytic geometry. The notion of a limit is fundamental to the study of calculus. Thus, it is important to acquire a good working knowledge of limits before moving on to other topics in calculus.

In this chapter, you should learn the following.

- How calculus compares with precalculus. (**1.1**)
- How to find limits graphically and numerically. (**1.2**)
- How to evaluate limits analytically. (**1.3**)
- How to determine continuity at a point and on an open interval, and how to determine one-sided limits. (**1.4**)
- How to determine infinite limits and find vertical asymptotes. (**1.5**)

European Space Agency/NASA

According to NASA, the coldest place in the known universe is the Boomerang nebula. The nebula is five thousand light years from Earth and has a temperature of −272°C. That is only 1° warmer than absolute zero, the coldest possible temperature. How did scientists determine that absolute zero is the "lower limit" of the temperature of matter? (See Section 1.4, Example 5.)



The limit process is a fundamental concept of calculus. One technique you can use to estimate a limit is to graph the function and then determine the behavior of the graph as the independent variable approaches a specific value. (See Section 1.2.)

1.1 A Preview of Calculus

- **Understand what calculus is and how it compares with precalculus.**
- **Understand that the tangent line problem is basic to calculus.**
- **Understand that the area problem is also basic to calculus.**

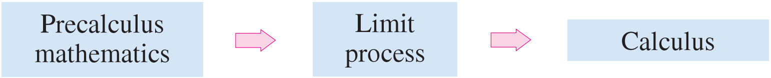
What Is Calculus?

Calculus is the mathematics of change. For instance, calculus is the mathematics of velocities, accelerations, tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures, and a variety of other concepts that have enabled scientists, engineers, and economists to model real-life situations.

Although precalculus mathematics also deals with velocities, accelerations, tangent lines, slopes, and so on, there is a fundamental difference between precalculus mathematics and calculus. Precalculus mathematics is more static, whereas calculus is more dynamic. Here are some examples.

- An object traveling at a constant velocity can be analyzed with precalculus mathematics. To analyze the velocity of an accelerating object, you need calculus.
- The slope of a line can be analyzed with precalculus mathematics. To analyze the slope of a curve, you need calculus.
- The curvature of a circle is constant and can be analyzed with precalculus mathematics. To analyze the variable curvature of a general curve, you need calculus.
- The area of a rectangle can be analyzed with precalculus mathematics. To analyze the area under a general curve, you need calculus.

Each of these situations involves the same general strategy—the reformulation of precalculus mathematics through the use of a limit process. So, one way to answer the question "What is calculus?" is to say that calculus is a "limit machine" that involves three stages. The first stage is precalculus mathematics, such as the slope of a line or the area of a rectangle. The second stage is the limit process, and the third stage is a new calculus formulation, such as a derivative or integral.



Some students try to learn calculus as if it were simply a collection of new formulas. This is unfortunate. If you reduce calculus to the memorization of differentiation and integration formulas, you will miss a great deal of understanding, self-confidence, and satisfaction.

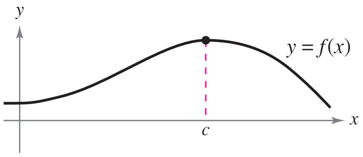
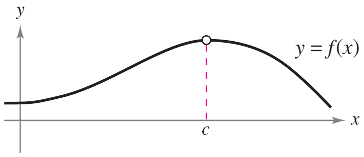
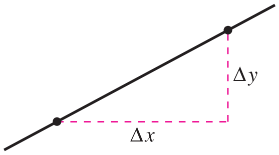
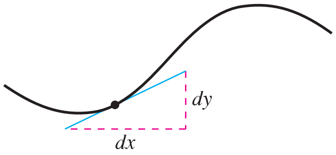
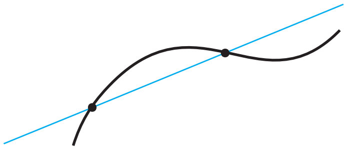
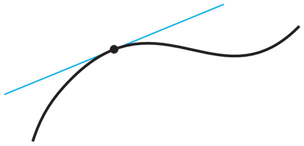
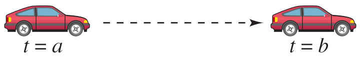
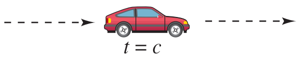
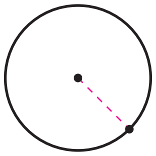
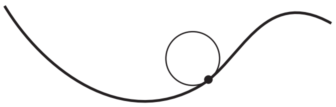
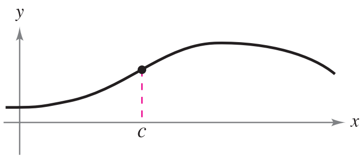
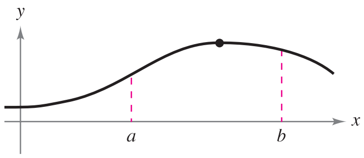
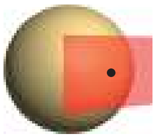
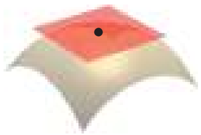
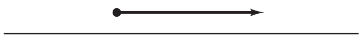
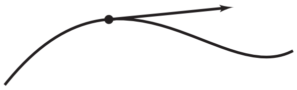
On the following two pages are listed some familiar precalculus concepts coupled with their calculus counterparts. Throughout the text, your goal should be to learn how precalculus formulas and techniques are used as building blocks to produce the more general calculus formulas and techniques. Don't worry if you are unfamiliar with some of the "old formulas" listed on the following two pages—you will be reviewing all of them.

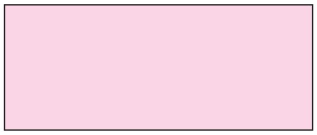
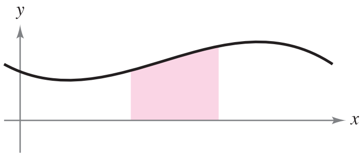
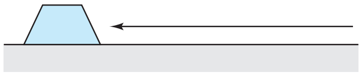
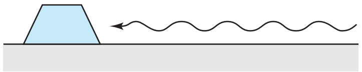
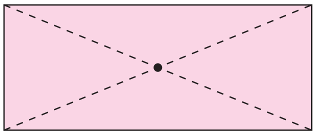
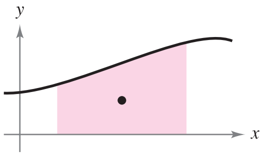
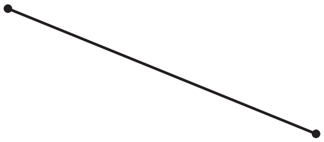
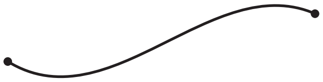
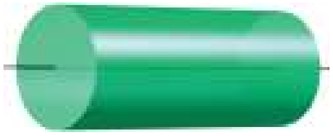
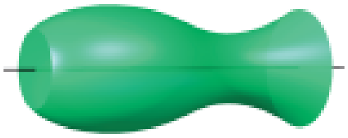
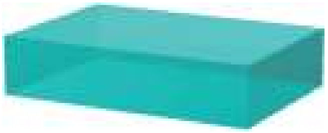
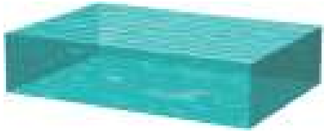
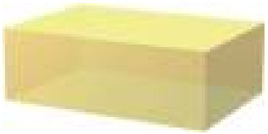
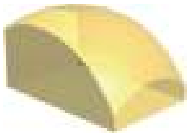
As you proceed through this text, come back to this discussion repeatedly. Try to keep track of where you are relative to the three stages involved in the study of calculus. For example, the first three chapters break down as follows.

Chapter P: Preparation for Calculus	Precalculus
Chapter 1: Limits and Their Properties	Limit process
Chapter 2: Differentiation	Calculus

STUDY TIP As you progress through this course, remember that learning calculus is just one of your goals. Your most important goal is to learn how to use calculus to model and solve real-life problems. Here are a few problem-solving strategies that may help you.

- Be sure you understand the question. What is given? What are you asked to find?
- Outline a plan. There are many approaches you could use: look for a pattern, solve a simpler problem, work backwards, draw a diagram, use technology, or any of many other approaches.
- Complete your plan. Be sure to answer the question. Verbalize your answer. For example, rather than writing the answer as $x = 4.6$, it would be better to write the answer as "The area of the region is 4.6 square meters."
- Look back at your work. Does your answer make sense? Is there a way you can check the reasonableness of your answer?

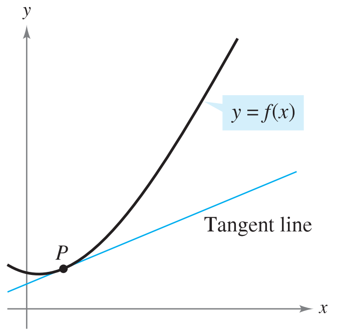
Without Calculus	With Differential Calculus
Value of $f(x)$ when $x = c$ 	Limit of $f(x)$ as x approaches c 
Slope of a line 	Slope of a curve 
Secant line to a curve 	Tangent line to a curve 
Average rate of change between $t = a$ and $t = b$ 	Instantaneous rate of change at $t = c$ 
Curvature of a circle 	Curvature of a curve 
Height of a curve when $x = c$ 	Maximum height of a curve on an interval 
Tangent plane to a sphere 	Tangent plane to a surface 
Direction of motion along a line 	Direction of motion along a curve 

Without Calculus	With Integral Calculus
Area of a rectangle 	Area under a curve 
Work done by a constant force 	Work done by a variable force 
Center of a rectangle 	Centroid of a region 
Length of a line segment 	Length of an arc 
Surface area of a cylinder 	Surface area of a solid of revolution 
Mass of a solid of constant density 	Mass of a solid of variable density 
Volume of a rectangular solid 	Volume of a region under a surface 
Sum of a finite number of terms $a_1 + a_2 + \cdot \cdot \cdot + a_n = S$	Sum of an infinite number of terms $a_1 + a_2 + a_3 + \cdot \cdot \cdot = S$

The Tangent Line Problem

The notion of a limit is fundamental to the study of calculus. The following brief descriptions of two classic problems in calculus—*the tangent line problem* and *the area problem*—should give you some idea of the way limits are used in calculus.

In the tangent line problem, you are given a function f and a point P on its graph and are asked to find an equation of the tangent line to the graph at point P, as shown in Figure 1.1.



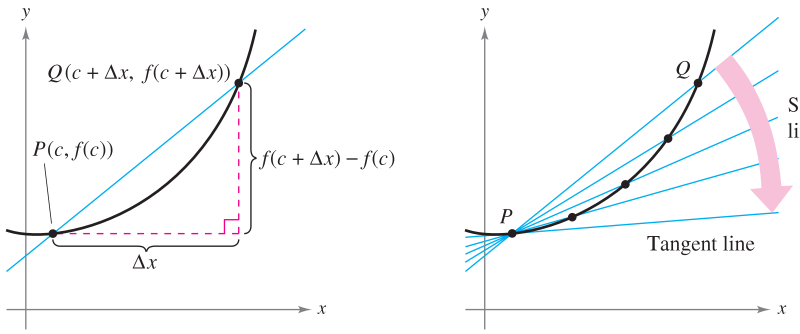
The tangent line to the graph of f at P
Figure 1.1

Except for cases involving a vertical tangent line, the problem of finding the **tangent line** at a point P is equivalent to finding the *slope* of the tangent line at P. You can approximate this slope by using a line through the point of tangency and a second point on the curve, as shown in Figure 1.2(a). Such a line is called a **secant line.** If $P(c, f(c))$ is the point of tangency and

$$Q(c + \Delta x, f(c + \Delta x))$$

is a second point on the graph of f, the slope of the secant line through these two points can be found using precalculus and is given by

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{c + \Delta x - c} = \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$



(a) The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$

(b) As Q approaches P, the secant lines approach the tangent line.

Figure 1.2

As point Q approaches point P, the slopes of the secant lines approach the slope of the tangent line, as shown in Figure 1.2(b). When such a "limiting position" exists, the slope of the tangent line is said to be the **limit** of the slopes of the secant lines. (Much more will be said about this important calculus concept in Chapter 2.)

The Mistress Fellows, Girton College, Cambridge

GRACE CHISHOLM YOUNG (1868–1944)

Grace Chisholm Young received her degree in mathematics from Girton College in Cambridge, England. Her early work was published under the name of William Young, her husband. Between 1914 and 1916, Grace Young published work on the foundations of calculus that won her the Gamble Prize from Girton College.

EXPLORATION

The following points lie on the graph of $f(x) = x^2$.

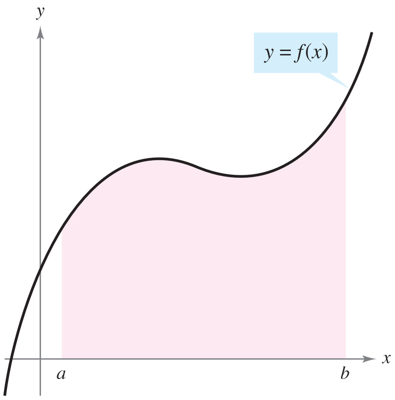
$$Q_1(1.5, f(1.5)),\quad Q_2(1.1, f(1.1)),\quad Q_3(1.01, f(1.01)),$$
$$Q_4(1.001, f(1.001)),\quad Q_5(1.0001, f(1.0001))$$

Each successive point gets closer to the point $P(1, 1)$. Find the slopes of the secant lines through Q_1 and P, Q_2 and P, and so on. Graph these secant lines on a graphing utility. Then use your results to estimate the slope of the tangent line to the graph of f at the point P.

The Area Problem

In the tangent line problem, you saw how the limit process can be applied to the slope of a line to find the slope of a general curve. A second classic problem in calculus is finding the area of a plane region that is bounded by the graphs of functions. This problem can also be solved with a limit process. In this case, the limit process is applied to the area of a rectangle to find the area of a general region.

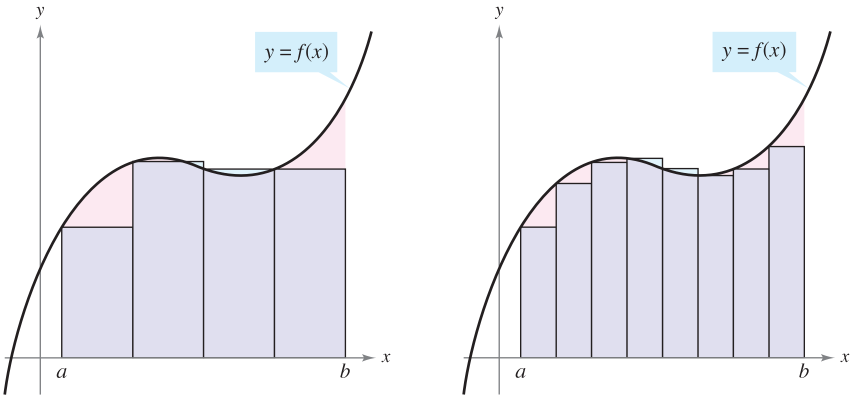
As a simple example, consider the region bounded by the graph of the function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 1.3. You can approximate the area of the region with several rectangular regions, as shown in Figure 1.4. As you increase the number of rectangles, the approximation tends to become better and better because the amount of area missed by the rectangles decreases. Your goal is to determine the limit of the sum of the areas of the rectangles as the number of rectangles increases without bound.



Area under a curve
Figure 1.3

HISTORICAL NOTE

In one of the most astounding events ever to occur in mathematics, it was discovered that the tangent line problem and the area problem are closely related. This discovery led to the birth of calculus. You will learn about the relationship between these two problems when you study the Fundamental Theorem of Calculus in Chapter 4.

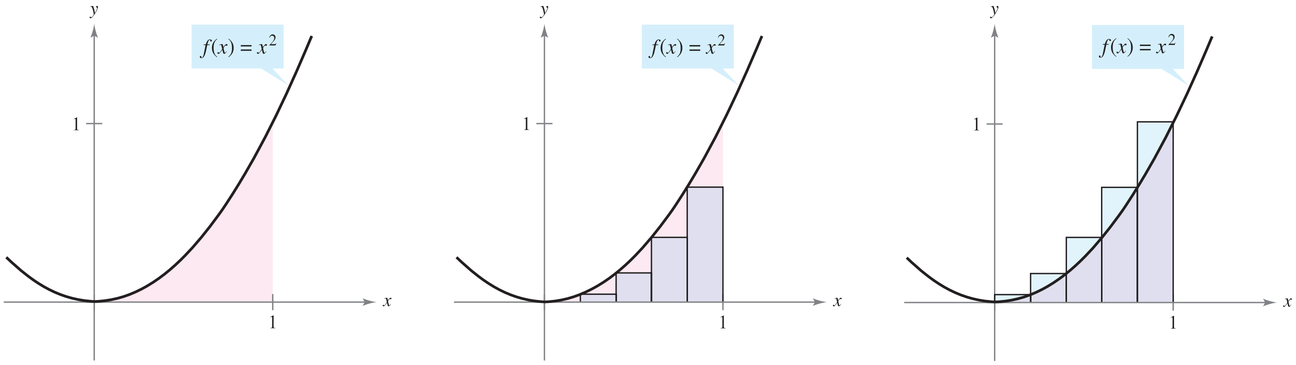


Approximation using four rectangles
Figure 1.4

Approximation using eight rectangles

EXPLORATION

Consider the region bounded by the graphs of $f(x) = x^2$, $y = 0$, and $x = 1$, as shown in part (a) of the figure. The area of the region can be approximated by two sets of rectangles—one set inscribed within the region and the other set circumscribed over the region, as shown in parts (b) and (c). Find the sum of the areas of each set of rectangles. Then use your results to approximate the area of the region.



(a) Bounded region **(b)** Inscribed rectangles **(c)** Circumscribed rectangles

1.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–5, decide whether the problem can be solved using precalculus or whether calculus is required. If the problem can be solved using precalculus, solve it. If the problem seems to require calculus, explain your reasoning and use a graphical or numerical approach to estimate the solution.

1. Find the distance traveled in 15 seconds by an object traveling at a constant velocity of 20 feet per second.

2. Find the distance traveled in 15 seconds by an object moving with a velocity of $v(t) = 20 + 7 \cos t$ feet per second.

3. A bicyclist is riding on a path modeled by the function $f(x) = 0.04(8x - x^2)$, where x and $f(x)$ are measured in miles. Find the rate of change of elevation at $x = 2$.

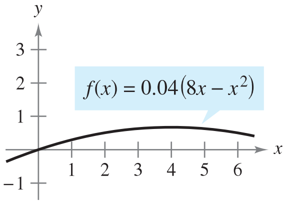


Figure for 3

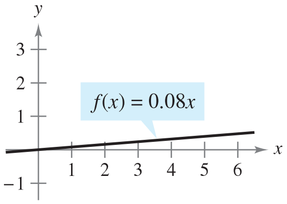
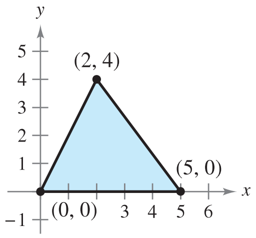


Figure for 4

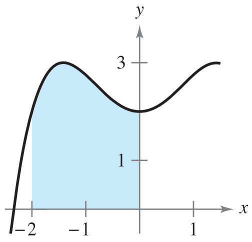
4. A bicyclist is riding on a path modeled by the function $f(x) = 0.08x$, where x and $f(x)$ are measured in miles. Find the rate of change of elevation at $x = 2$.

5. Find the area of the shaded region.

(a)



(b)



6. ***Secant Lines*** Consider the function $f(x) = \sqrt{x}$ and the point $P(4, 2)$ on the graph of f.

(a) Graph f and the secant lines passing through $P(4, 2)$ and $Q(x, f(x))$ for x-values of 1, 3, and 5.

(b) Find the slope of each secant line.

(c) Use the results of part (b) to estimate the slope of the tangent line to the graph of f at $P(4, 2)$. Describe how to improve your approximation of the slope.

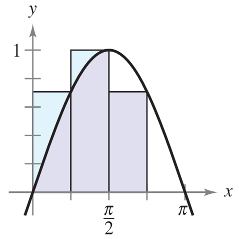
7. ***Secant Lines*** Consider the function $f(x) = 6x - x^2$ and the point $P(2, 8)$ on the graph of f.

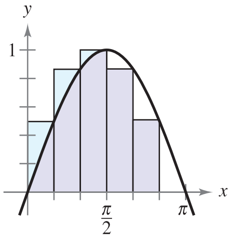
(a) Graph f and the secant lines passing through $P(2, 8)$ and $Q(x, f(x))$ for x-values of 3, 2.5, and 1.5.

(b) Find the slope of each secant line.

(c) Use the results of part (b) to estimate the slope of the tangent line to the graph of f at $P(2, 8)$. Describe how to improve your approximation of the slope.

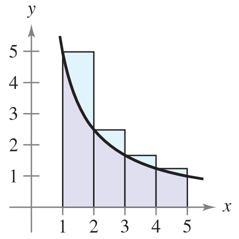
8. (a) Use the rectangles in each graph to approximate the area of the region bounded by $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$.

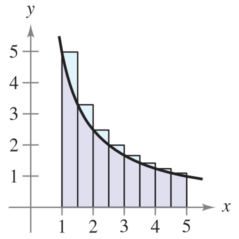




(b) Describe how you could continue this process to obtain a more accurate approximation of the area.

9. (a) Use the rectangles in each graph to approximate the area of the region bounded by $y = 5/x$, $y = 0$, $x = 1$, and $x = 5$.





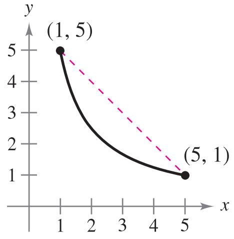
(b) Describe how you could continue this process to obtain a more accurate approximation of the area.

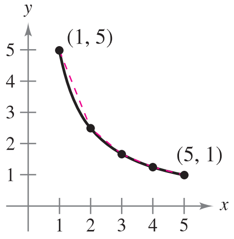
CAPSTONE

10. How would you describe the instantaneous rate of change of an automobile's position on the highway?

WRITING ABOUT CONCEPTS

11. Consider the length of the graph of $f(x) = 5/x$ from $(1, 5)$ to $(5, 1)$.





(a) Approximate the length of the curve by finding the distance between its two endpoints, as shown in the first figure.

(b) Approximate the length of the curve by finding the sum of the lengths of four line segments, as shown in the second figure.

(c) Describe how you could continue this process to obtain a more accurate approximation of the length of the curve.

1.2 Finding Limits Graphically and Numerically

- **Estimate a limit using a numerical or graphical approach.**
- **Learn different ways that a limit can fail to exist.**
- **Study and use a formal definition of limit.**

An Introduction to Limits

Suppose you are asked to sketch the graph of the function f given by

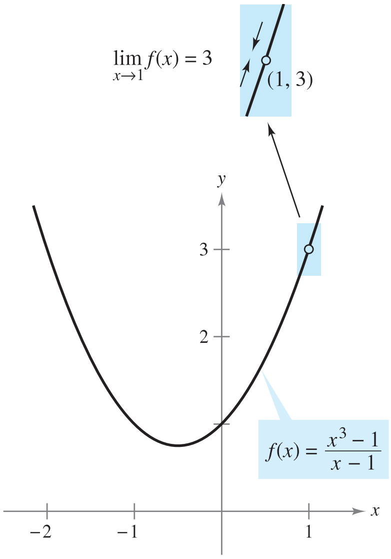
$$f(x) = \frac{x^3 - 1}{x - 1}, \quad x \neq 1.$$

For all values other than $x = 1$, you can use standard curve-sketching techniques. However, at $x = 1$, it is not clear what to expect. To get an idea of the behavior of the graph of f near $x = 1$, you can use two sets of x-values—one set that approaches 1 from the left and one set that approaches 1 from the right, as shown in the table.

x approaches 1 from the left. → ← x approaches 1 from the right.

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

$f(x)$ approaches 3. → ← $f(x)$ approaches 3.



The limit of $f(x)$ as x approaches 1 is 3.
Figure 1.5

The graph of f is a parabola that has a gap at the point $(1, 3)$, as shown in Figure 1.5. Although x cannot equal 1, you can move arbitrarily close to 1, and as a result $f(x)$ moves arbitrarily close to 3. Using limit notation, you can write

$$\lim_{x \to 1} f(x) = 3. \qquad \text{This is read as "the limit of } f(x) \text{ as } x \text{ approaches 1 is 3."}$$

This discussion leads to an informal definition of limit. If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, the **limit** of $f(x)$, as x approaches c, is L. This limit is written as

$$\lim_{x \to c} f(x) = L.$$

EXPLORATION

The discussion above gives an example of how you can estimate a limit *numerically* by constructing a table and *graphically* by drawing a graph. Estimate the following limit numerically by completing the table.

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$$

x	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25
$f(x)$	?	?	?	?	?	?	?	?	?

Then use a graphing utility to estimate the limit graphically.

EXAMPLE 1 Estimating a Limit Numerically

Evaluate the function $f(x) = x/(\sqrt{x+1} - 1)$ at several points near $x = 0$ and use the results to estimate the limit

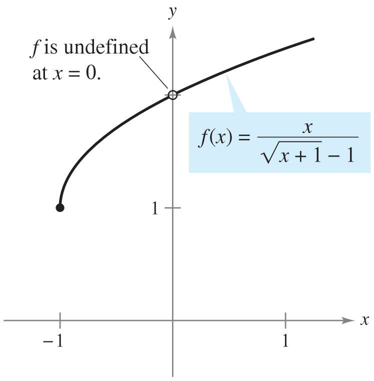
$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}.$$

Solution The table lists the values of $f(x)$ for several x-values near 0.

x approaches 0 from the left. *x* approaches 0 from the right.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.99995	?	2.00005	2.00050	2.00499

$f(x)$ approaches 2. $f(x)$ approaches 2.



The limit of $f(x)$ as x approaches 0 is 2.
Figure 1.6

From the results shown in the table, you can estimate the limit to be 2. This limit is reinforced by the graph of f (see Figure 1.6). ■

In Example 1, note that the function is undefined at $x = 0$ and yet $f(x)$ appears to be approaching a limit as x approaches 0. This often happens, and it is important to realize that *the existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c.*

EXAMPLE 2 Finding a Limit

Find the limit of $f(x)$ as x approaches 2, where f is defined as

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}.$$

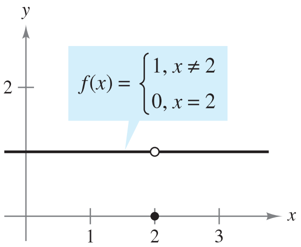
Solution Because $f(x) = 1$ for all x other than $x = 2$, you can conclude that the limit is 1, as shown in Figure 1.7. So, you can write

$$\lim_{x \to 2} f(x) = 1.$$

The fact that $f(2) = 0$ has no bearing on the existence or value of the limit as x approaches 2. For instance, if the function were defined as

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 2, & x = 2 \end{cases}$$

the limit would be the same. ■



The limit of $f(x)$ as x approaches 2 is 1.
Figure 1.7

So far in this section, you have been estimating limits numerically and graphically. Each of these approaches produces an estimate of the limit. In Section 1.3, you will study analytic techniques for evaluating limits. Throughout the course, try to develop a habit of using this three-pronged approach to problem solving.

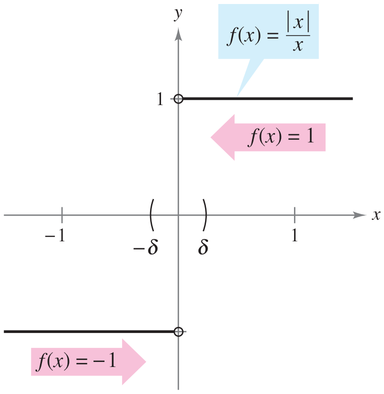
1. Numerical approach	Construct a table of values.
2. Graphical approach	Draw a graph by hand or using technology.
3. Analytic approach	Use algebra or calculus.

Limits That Fail to Exist

In the next three examples you will examine some limits that fail to exist.

EXAMPLE 3 Behavior That Differs from the Right and from the Left

Show that the limit $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.



$\lim_{x\to 0} f(x)$ does not exist.

Figure 1.8

Solution Consider the graph of the function $f(x) = |x|/x$. From Figure 1.8 and the definition of absolute value

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Definition of absolute value

you can see that

$$\frac{|x|}{x} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}.$$

This means that no matter how close x gets to 0, there will be both positive and negative x-values that yield $f(x) = 1$ or $f(x) = -1$. Specifically, if δ (the lowercase Greek letter *delta*) is a positive number, then for x-values satisfying the inequality $0 < |x| < \delta$, you can classify the values of $|x|/x$ as follows.

$$(-\delta, 0) \qquad\qquad (0, \delta)$$

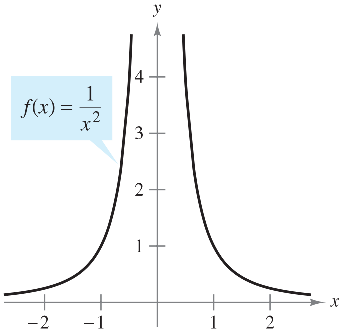
Negative x-values yield $|x|/x = -1$.

Positive x-values yield $|x|/x = 1$.

Because $|x|/x$ approaches a different number from the right side of 0 than it approaches from the left side, the limit $\lim_{x\to 0} (|x|/x)$ does not exist.

EXAMPLE 4 Unbounded Behavior

Discuss the existence of the limit $\lim_{x\to 0} \frac{1}{x^2}$.



$\lim_{x\to 0} f(x)$ does not exist.

Figure 1.9

Solution Let $f(x) = 1/x^2$. In Figure 1.9, you can see that as x approaches 0 from either the right or the left, $f(x)$ increases without bound. This means that by choosing x close enough to 0, you can force $f(x)$ to be as large as you want. For instance, $f(x)$ will be larger than 100 if you choose x that is within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 100.$$

Similarly, you can force $f(x)$ to be larger than 1,000,000, as follows.

$$0 < |x| < \frac{1}{1000} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 1{,}000{,}000$$

Because $f(x)$ is not approaching a real number L as x approaches 0, you can conclude that the limit does not exist. ■

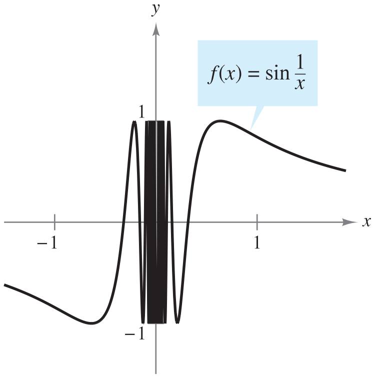
EXAMPLE 5 Oscillating Behavior

Discuss the existence of the limit $\lim_{x\to 0} \sin \frac{1}{x}$.

Solution Let $f(x) = \sin(1/x)$. In Figure 1.10, you can see that as x approaches 0, $f(x)$ oscillates between -1 and 1. So, the limit does not exist because no matter how small you choose δ, it is possible to choose x_1 and x_2 within δ units of 0 such that $\sin(1/x_1) = 1$ and $\sin(1/x_2) = -1$, as shown in the table.

x	$2/\pi$	$2/3\pi$	$2/5\pi$	$2/7\pi$	$2/9\pi$	$2/11\pi$	$x \to 0$
$\sin(1/x)$	1	-1	1	-1	1	-1	Limit does not exist.

■



$\lim_{x\to 0} f(x)$ does not exist.

Figure 1.10

COMMON TYPES OF BEHAVIOR ASSOCIATED WITH NONEXISTENCE OF A LIMIT

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

There are many other interesting functions that have unusual limit behavior. An often cited one is the *Dirichlet function*

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational.} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

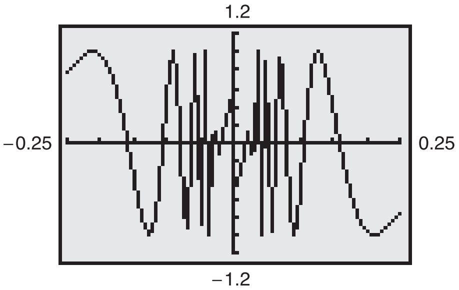
Because this function has *no limit* at any real number c, it is *not continuous* at any real number c. You will study continuity more closely in Section 1.4.

The Granger Collection

PETER GUSTAV DIRICHLET (1805–1859)

In the early development of calculus, the definition of a function was much more restricted than it is today, and "functions" such as the Dirichlet function would not have been considered. The modern definition of function is attributed to the German mathematician Peter Gustav Dirichlet.

TECHNOLOGY PITFALL When you use a graphing utility to investigate the behavior of a function near the x-value at which you are trying to evaluate a limit, remember that you can't always trust the pictures that graphing utilities draw. If you use a graphing utility to graph the function in Example 5 over an interval containing 0, you will most likely obtain an incorrect graph such as that shown in Figure 1.11. The reason that a graphing utility can't show the correct graph is that the graph has infinitely many oscillations over any interval that contains 0.



Incorrect graph of $f(x) = \sin(1/x)$.

Figure 1.11

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

A Formal Definition of Limit

Let's take another look at the informal definition of limit. If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L, written as

$$\lim_{x\to c} f(x) = L.$$

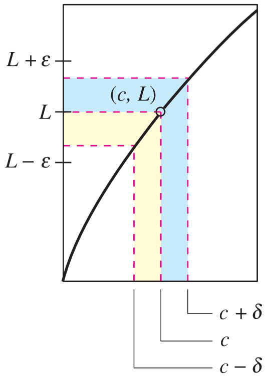
At first glance, this definition looks fairly technical. Even so, it is informal because exact meanings have not yet been given to the two phrases

"$f(x)$ becomes arbitrarily close to L"

and

"x approaches c."

The first person to assign mathematically rigorous meanings to these two phrases was Augustin-Louis Cauchy. His ε-δ **definition of limit** is the standard used today.



The ε-δ definition of the limit of $f(x)$ as x approaches c
Figure 1.12

In Figure 1.12, let ε (the lowercase Greek letter *epsilon*) represent a (small) positive number. Then the phrase "$f(x)$ becomes arbitrarily close to L" means that $f(x)$ lies in the interval $(L - \varepsilon, L + \varepsilon)$. Using absolute value, you can write this as

$$|f(x) - L| < \varepsilon.$$

Similarly, the phrase "x approaches c" means that there exists a positive number δ such that x lies in either the interval $(c - \delta, c)$ or the interval $(c, c + \delta)$. This fact can be concisely expressed by the double inequality

$$0 < |x - c| < \delta.$$

The first inequality

$$0 < |x - c|$$

The distance between x and c is more than 0.

expresses the fact that $x \neq c$. The second inequality

$$|x - c| < \delta$$

x is within δ units of c.

says that x is within a distance δ of c.

DEFINITION OF LIMIT

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement

$$\lim_{x\to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

NOTE Throughout this text, the expression

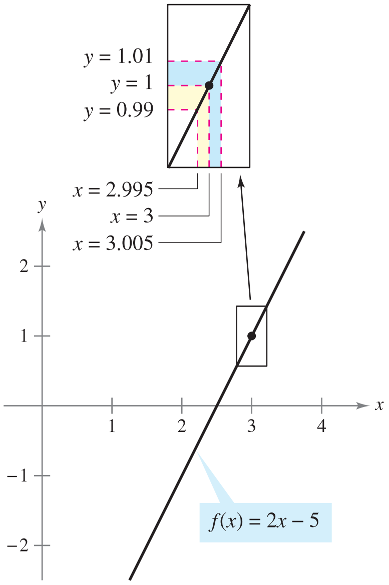
$$\lim_{x\to c} f(x) = L$$

implies two statements—the limit exists *and* the limit is L. ■

Some functions do not have limits as $x \to c$, but those that do cannot have two different limits as $x \to c$. That is, *if the limit of a function exists, it is unique* (see Exercise 79).

■ **FOR FURTHER INFORMATION** For more on the introduction of rigor to calculus, see "Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus" by Judith V. Grabiner in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com.*

The next three examples should help you develop a better understanding of the ε-δ definition of limit.



The limit of $f(x)$ as x approaches 3 is 1.
Figure 1.13

EXAMPLE 6 Finding a δ for a Given ε

Given the limit

$$\lim_{x \to 3} (2x - 5) = 1$$

find δ such that $|(2x - 5) - 1| < 0.01$ whenever $0 < |x - 3| < \delta$.

Solution In this problem, you are working with a given value of ε—namely, $\varepsilon = 0.01$. To find an appropriate δ, notice that

$$|(2x - 5) - 1| = |2x - 6| = 2|x - 3|.$$

Because the inequality $|(2x - 5) - 1| < 0.01$ is equivalent to $2|x - 3| < 0.01$, you can choose $\delta = \frac{1}{2}(0.01) = 0.005$. This choice works because

$$0 < |x - 3| < 0.005$$

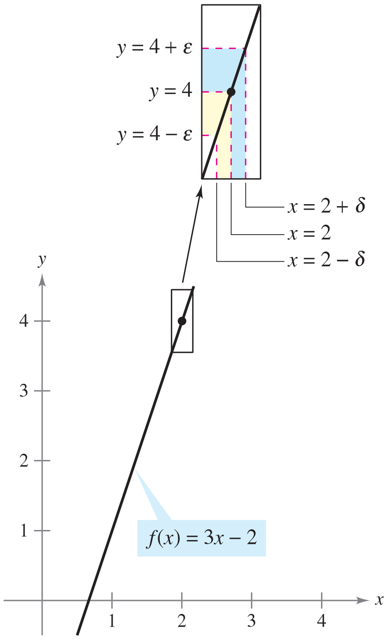
implies that

$$|(2x - 5) - 1| = 2|x - 3| < 2(0.005) = 0.01$$

as shown in Figure 1.13. ■

NOTE In Example 6, note that 0.005 is the *largest* value of δ that will guarantee $|(2x - 5) - 1| < 0.01$ whenever $0 < |x - 3| < \delta$. Any *smaller* positive value of δ would also work. ■

In Example 6, you found a δ-value for a *given* ε. This does not prove the existence of the limit. To do that, you must prove that you can find a δ for *any* ε, as shown in the next example.



The limit of $f(x)$ as x approaches 2 is 4.
Figure 1.14

EXAMPLE 7 Using the ε-δ Definition of Limit

Use the ε-δ definition of limit to prove that

$$\lim_{x \to 2} (3x - 2) = 4.$$

Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|(3x - 2) - 4| < \varepsilon$ whenever $0 < |x - 2| < \delta$. Because your choice of δ depends on ε, you need to establish a connection between the absolute values $|(3x - 2) - 4|$ and $|x - 2|$.

$$|(3x - 2) - 4| = |3x - 6| = 3|x - 2|$$

So, for a given $\varepsilon > 0$ you can choose $\delta = \varepsilon/3$. This choice works because

$$0 < |x - 2| < \delta = \frac{\varepsilon}{3}$$

implies that

$$|(3x - 2) - 4| = 3|x - 2| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

as shown in Figure 1.14. ■

EXAMPLE 8 Using the ε-δ Definition of Limit

Use the ε-δ definition of limit to prove that

$$\lim_{x\to 2} x^2 = 4.$$

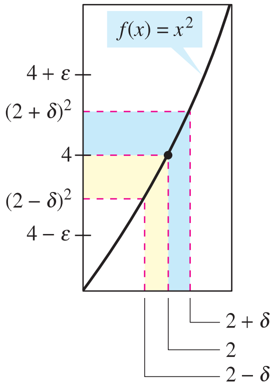
Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x^2 - 4| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

To find an appropriate δ, begin by writing $|x^2 - 4| = |x - 2||x + 2|$. For all x in the interval $(1, 3)$, $x + 2 < 5$ and thus $|x + 2| < 5$. So, letting δ be the minimum of $\varepsilon/5$ and 1, it follows that, whenever $0 < |x - 2| < \delta$, you have

$$|x^2 - 4| = |x - 2||x + 2| < \left(\frac{\varepsilon}{5}\right)(5) = \varepsilon$$

as shown in Figure 1.15. ■



The limit of $f(x)$ as x approaches 2 is 4.
Figure 1.15

Throughout this chapter you will use the ε-δ definition of limit primarily to prove theorems about limits and to establish the existence or nonexistence of particular types of limits. For *finding* limits, you will learn techniques that are easier to use than the ε-δ definition of limit.

1.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

1. $\displaystyle\lim_{x\to 4} \frac{x - 4}{x^2 - 3x - 4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

2. $\displaystyle\lim_{x\to 2} \frac{x - 2}{x^2 - 4}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

3. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x + 6} - \sqrt{6}}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

4. $\displaystyle\lim_{x\to -5} \frac{\sqrt{4 - x} - 3}{x + 5}$

x	-5.1	-5.01	-5.001	-4.999	-4.99	-4.9
$f(x)$						

5. $\displaystyle\lim_{x\to 3} \frac{[1/(x + 1)] - (1/4)}{x - 3}$

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$						

6. $\displaystyle\lim_{x\to 4} \frac{[x/(x + 1)] - (4/5)}{x - 4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

7. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

8. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

In Exercises 9–14, create a table of values for the function and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

9. $\lim_{x\to 1}\dfrac{x-2}{x^2+x-6}$

10. $\lim_{x\to -3}\dfrac{x+3}{x^2+7x+12}$

11. $\lim_{x\to 1}\dfrac{x^4-1}{x^6-1}$

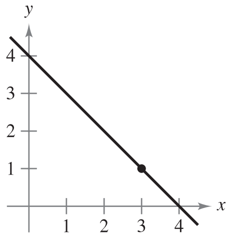
12. $\lim_{x\to -2}\dfrac{x^3+8}{x+2}$

13. $\lim_{x\to 0}\dfrac{\sin 2x}{x}$

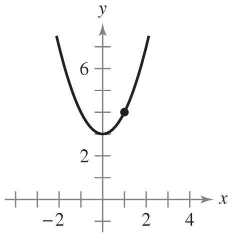
14. $\lim_{x\to 0}\dfrac{\tan x}{\tan 2x}$

In Exercises 15–24, use the graph to find the limit (if it exists). If the limit does not exist, explain why.

15. $\lim_{x\to 3}(4-x)$

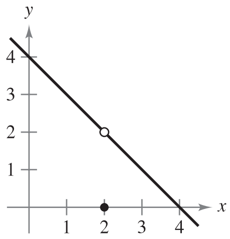


16. $\lim_{x\to 1}(x^2+3)$



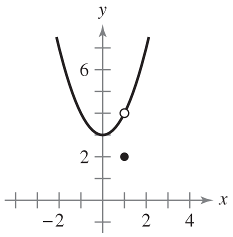
17. $\lim_{x\to 2} f(x)$

$$f(x)=\begin{cases}4-x, & x\neq 2\\ 0, & x=2\end{cases}$$

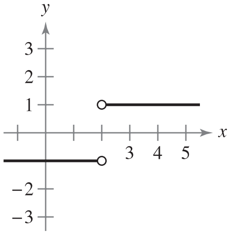


18. $\lim_{x\to 1} f(x)$

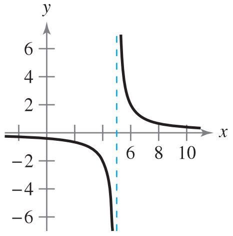
$$f(x)=\begin{cases}x^2+3, & x\neq 1\\ 2, & x=1\end{cases}$$



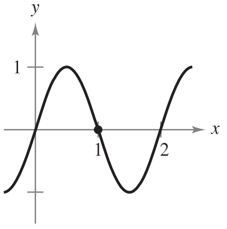
19. $\lim_{x\to 2}\dfrac{|x-2|}{x-2}$



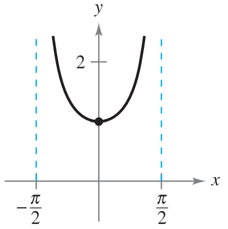
20. $\lim_{x\to 5}\dfrac{2}{x-5}$



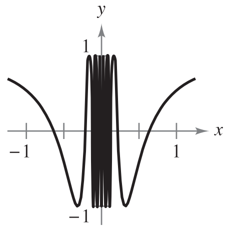
21. $\lim_{x\to 1}\sin \pi x$



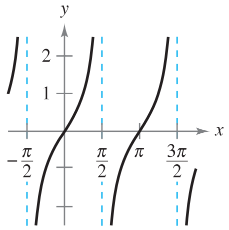
22. $\lim_{x\to 0}\sec x$



23. $\lim_{x\to 0}\cos\dfrac{1}{x}$



24. $\lim_{x\to \pi/2}\tan x$



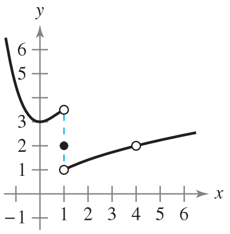
In Exercises 25 and 26, use the graph of the function *f* to decide whether the value of the given quantity exists. If it does, find it. If not, explain why.

25. (a) $f(1)$

(b) $\lim_{x\to 1} f(x)$

(c) $f(4)$

(d) $\lim_{x\to 4} f(x)$



26. (a) $f(-2)$

(b) $\lim_{x\to -2} f(x)$

(c) $f(0)$

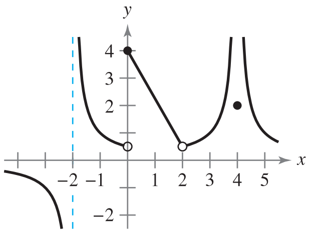
(d) $\lim_{x\to 0} f(x)$

(e) $f(2)$

(f) $\lim_{x\to 2} f(x)$

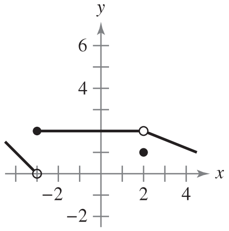
(g) $f(4)$

(h) $\lim_{x\to 4} f(x)$

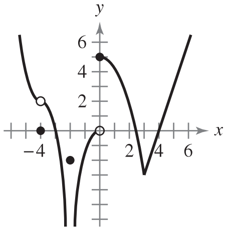


In Exercises 27 and 28, use the graph of *f* to identify the values of *c* for which $\lim_{x\to c} f(x)$ exists.

27.



28.



In Exercises 29 and 30, sketch the graph of *f*. Then identify the values of *c* for which $\lim_{x\to c} f(x)$ exists.

29. $f(x)=\begin{cases}x^2, & x\le 2\\ 8-2x, & 2<x<4\\ 4, & x\ge 4\end{cases}$

30. $f(x)=\begin{cases}\sin x, & x<0\\ 1-\cos x, & 0\le x\le \pi\\ \cos x, & x>\pi\end{cases}$

In Exercises 31 and 32, sketch a graph of a function f that satisfies the given values. (There are many correct answers.)

31. $f(0)$ is undefined.

$\lim_{x\to 0} f(x) = 4$

$f(2) = 6$

$\lim_{x\to 2} f(x) = 3$

32. $f(-2) = 0$

$f(2) = 0$

$\lim_{x\to -2} f(x) = 0$

$\lim_{x\to 2} f(x)$ does not exist.

33. ***Modeling Data*** For a long distance phone call, a hotel charges \$9.99 for the first minute and \$0.79 for each additional minute or fraction thereof. A formula for the cost is given by

$$C(t) = 9.99 - 0.79[\![-(t-1)]\!]$$

where t is the time in minutes.

(*Note:* $[\![x]\!]$ = greatest integer n such that $n \le x$. For example, $[\![3.2]\!] = 3$ and $[\![-1.6]\!] = -2$.)

(a) Use a graphing utility to graph the cost function for $0 < t \le 6$.

(b) Use the graph to complete the table and observe the behavior of the function as t approaches 3.5. Use the graph and the table to find

$$\lim_{t\to 3.5} C(t).$$

t	3	3.3	3.4	3.5	3.6	3.7	4
C				?			

(c) Use the graph to complete the table and observe the behavior of the function as t approaches 3.

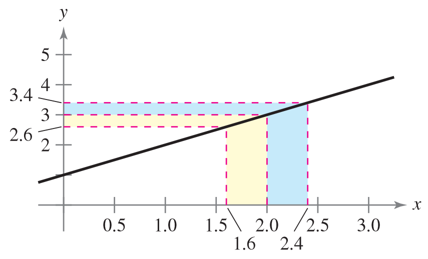
t	2	2.5	2.9	3	3.1	3.5	4
C				?			

Does the limit of $C(t)$ as t approaches 3 exist? Explain.

34. Repeat Exercise 33 for

$$C(t) = 5.79 - 0.99[\![-(t-1)]\!].$$

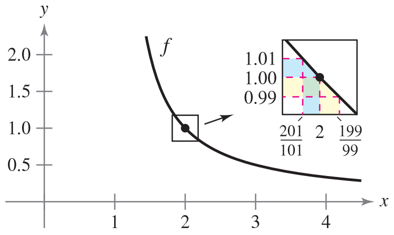
35. The graph of $f(x) = x + 1$ is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$ then $|f(x) - 3| < 0.4$.



36. The graph of

$$f(x) = \frac{1}{x-1}$$

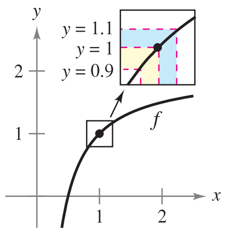
is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$ then $|f(x) - 1| < 0.01$.



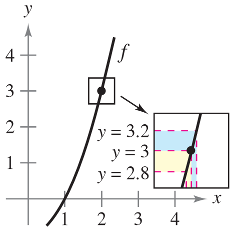
37. The graph of

$$f(x) = 2 - \frac{1}{x}$$

is shown in the figure. Find δ such that if $0 < |x - 1| < \delta$ then $|f(x) - 1| < 0.1$.



38. The graph of $f(x) = x^2 - 1$ is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$ then $|f(x) - 3| < 0.2$.



In Exercises 39–42, find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

39. $\lim_{x\to 2} (3x + 2)$

40. $\lim_{x\to 4} \left(4 - \frac{x}{2}\right)$

41. $\lim_{x\to 2} (x^2 - 3)$

42. $\lim_{x\to 5} (x^2 + 4)$

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by use of appropriate technology.

In Exercises 43–54, find the limit L. Then use the ε-δ definition to prove that the limit is L.

43. $\lim\limits_{x \to 4} (x + 2)$

44. $\lim\limits_{x \to -3} (2x + 5)$

45. $\lim\limits_{x \to -4} \left(\frac{1}{2}x - 1\right)$

46. $\lim\limits_{x \to 1} \left(\frac{2}{5}x + 7\right)$

47. $\lim\limits_{x \to 6} 3$

48. $\lim\limits_{x \to 2} (-1)$

49. $\lim\limits_{x \to 0} \sqrt[3]{x}$

50. $\lim\limits_{x \to 4} \sqrt{x}$

51. $\lim\limits_{x \to -5} |x - 5|$

52. $\lim\limits_{x \to 6} |x - 6|$

53. $\lim\limits_{x \to 1} (x^2 + 1)$

54. $\lim\limits_{x \to -3} (x^2 + 3x)$

55. What is the limit of $f(x) = 4$ as x approaches π?

56. What is the limit of $g(x) = x$ as x approaches π?

Writing **In Exercises 57–60, use a graphing utility to graph the function and estimate the limit (if it exists). What is the domain of the function? Can you detect a possible error in determining the domain of a function solely by analyzing the graph generated by a graphing utility? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

57. $f(x) = \dfrac{\sqrt{x + 5} - 3}{x - 4}$

$\lim\limits_{x \to 4} f(x)$

58. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

$\lim\limits_{x \to 3} f(x)$

59. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

$\lim\limits_{x \to 9} f(x)$

60. $f(x) = \dfrac{x - 3}{x^2 - 9}$

$\lim\limits_{x \to 3} f(x)$

WRITING ABOUT CONCEPTS

61. Write a brief description of the meaning of the notation

$\lim\limits_{x \to 8} f(x) = 25.$

62. The definition of limit on page 52 requires that f is a function defined on an open interval containing c, except possibly at c. Why is this requirement necessary?

63. Identify three types of behavior associated with the nonexistence of a limit. Illustrate each type with a graph of a function.

CAPSTONE

64. (a) If $f(2) = 4$, can you conclude anything about the limit of $f(x)$ as x approaches 2? Explain your reasoning.

(b) If the limit of $f(x)$ as x approaches 2 is 4, can you conclude anything about $f(2)$? Explain your reasoning.

65. ***Jewelry*** A jeweler resizes a ring so that its inner circumference is 6 centimeters.

(a) What is the radius of the ring?

(b) If the ring's inner circumference can vary between 5.5 centimeters and 6.5 centimeters, how can the radius vary?

(c) Use the ε-δ definition of limit to describe this situation. Identify ε and δ.

66. ***Sports*** A sporting goods manufacturer designs a golf ball having a volume of 2.48 cubic inches.

(a) What is the radius of the golf ball?

(b) If the ball's volume can vary between 2.45 cubic inches and 2.51 cubic inches, how can the radius vary?

(c) Use the ε-δ definition of limit to describe this situation. Identify ε and δ.

67. Consider the function $f(x) = (1 + x)^{1/x}$. Estimate the limit

$$\lim_{x \to 0} (1 + x)^{1/x}$$

by evaluating f at x-values near 0. Sketch the graph of f.

68. Consider the function

$$f(x) = \frac{|x + 1| - |x - 1|}{x}.$$

Estimate

$$\lim_{x \to 0} \frac{|x + 1| - |x - 1|}{x}$$

by evaluating f at x-values near 0. Sketch the graph of f.

69. ***Graphical Analysis*** The statement

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$$

means that for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that if $0 < |x - 2| < \delta$, then

$$\left| \frac{x^2 - 4}{x - 2} - 4 \right| < \varepsilon.$$

If $\varepsilon = 0.001$, then

$$\left| \frac{x^2 - 4}{x - 2} - 4 \right| < 0.001.$$

Use a graphing utility to graph each side of this inequality. Use the *zoom* feature to find an interval $(2 - \delta, 2 + \delta)$ such that the graph of the left side is below the graph of the right side of the inequality.

70. ***Graphical Analysis*** The statement

$$\lim_{x\to 3} \frac{x^2 - 3x}{x - 3}$$

means that for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that if $0 < |x - 3| < \delta$, then

$$\left|\frac{x^2 - 3x}{x - 3} - 3\right| < \varepsilon.$$

If $\varepsilon = 0.001$, then

$$\left|\frac{x^2 - 3x}{x - 3} - 3\right| < 0.001.$$

Use a graphing utility to graph each side of this inequality. Use the *zoom* feature to find an interval $(3 - \delta, 3 + \delta)$ such that the graph of the left side is below the graph of the right side of the inequality.

True or False? **In Exercises 71–74, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

71. If f is undefined at $x = c$, then the limit of $f(x)$ as x approaches c does not exist.

72. If the limit of $f(x)$ as x approaches c is 0, then there must exist a number k such that $f(k) < 0.001$.

73. If $f(c) = L$, then $\lim_{x\to c} f(x) = L$.

74. If $\lim_{x\to c} f(x) = L$, then $f(c) = L$.

In Exercises 75 and 76, consider the function $f(x) = \sqrt{x}$.

75. Is $\lim_{x\to 0.25} \sqrt{x} = 0.5$ a true statement? Explain.

76. Is $\lim_{x\to 0} \sqrt{x} = 0$ a true statement? Explain.

77. Use a graphing utility to evaluate the limit $\lim_{x\to 0} \frac{\sin nx}{x}$ for several values of n. What do you notice?

78. Use a graphing utility to evaluate the limit $\lim_{x\to 0} \frac{\tan nx}{x}$ for several values of n. What do you notice?

79. Prove that if the limit of $f(x)$ as $x \to c$ exists, then the limit must be unique. [*Hint:* Let

$$\lim_{x\to c} f(x) = L_1 \quad \text{and} \quad \lim_{x\to c} f(x) = L_2$$

and prove that $L_1 = L_2$.]

80. Consider the line $f(x) = mx + b$, where $m \neq 0$. Use the ε-δ definition of limit to prove that $\lim_{x\to c} f(x) = mc + b$.

81. Prove that $\lim_{x\to c} f(x) = L$ is equivalent to $\lim_{x\to c} [f(x) - L] = 0$.

82. (a) Given that

$$\lim_{x\to 0} (3x + 1)(3x - 1)x^2 + 0.01 = 0.01$$

prove that there exists an open interval (a, b) containing 0 such that $(3x + 1)(3x - 1)x^2 + 0.01 > 0$ for all $x \neq 0$ in (a, b).

(b) Given that $\lim_{x\to c} g(x) = L$, where $L > 0$, prove that there exists an open interval (a, b) containing c such that $g(x) > 0$ for all $x \neq c$ in (a, b).

83. ***Programming*** Use the programming capabilities of a graphing utility to write a program for approximating $\lim_{x\to c} f(x)$.

Assume the program will be applied only to functions whose limits exist as x approaches c. Let $y_1 = f(x)$ and generate two lists whose entries form the ordered pairs

$$(c \pm [0.1]^n,\ f(c \pm [0.1]^n))$$

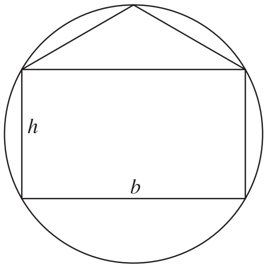
for $n = 0, 1, 2, 3,$ and 4.

84. ***Programming*** Use the program you created in Exercise 83 to approximate the limit

$$\lim_{x\to 4} \frac{x^2 - x - 12}{x - 4}.$$

PUTNAM EXAM CHALLENGE

85. Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown. For what value of h do the rectangle and triangle have the same area?



86. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

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1.3 Evaluating Limits Analytically

- **Evaluate a limit using properties of limits.**
- **Develop and use a strategy for finding limits.**
- **Evaluate a limit using dividing out and rationalizing techniques.**
- **Evaluate a limit using the Squeeze Theorem.**

Properties of Limits

In Section 1.2, you learned that the limit of $f(x)$ as x approaches c does not depend on the value of f at $x = c$. It may happen, however, that the limit is precisely $f(c)$. In such cases, the limit can be evaluated by **direct substitution.** That is,

$$\lim_{x \to c} f(x) = f(c). \qquad \text{Substitute } c \text{ for } x.$$

Such *well-behaved* functions are **continuous at *c*.** You will examine this concept more closely in Section 1.4.

THEOREM 1.1 SOME BASIC LIMITS

Let b and c be real numbers and let n be a positive integer.

1. $\lim\limits_{x \to c} b = b$ **2.** $\lim\limits_{x \to c} x = c$ **3.** $\lim\limits_{x \to c} x^n = c^n$

PROOF To prove Property 2 of Theorem 1.1, you need to show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - c| < \varepsilon$ whenever $0 < |x - c| < \delta$. To do this, choose $\delta = \varepsilon$. The second inequality then implies the first, as shown in Figure 1.16. This completes the proof. (Proofs of the other properties of limits in this section are listed in Appendix A or are discussed in the exercises.) ■

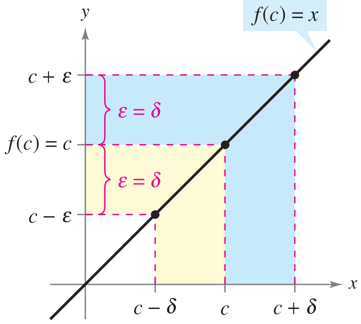


Figure 1.16

EXAMPLE 1 Evaluating Basic Limits

a. $\lim\limits_{x \to 2} 3 = 3$ **b.** $\lim\limits_{x \to -4} x = -4$ **c.** $\lim\limits_{x \to 2} x^2 = 2^2 = 4$ ■

NOTE When you encounter new notations or symbols in mathematics, be sure you know how the notations are read. For instance, the limit in Example 1(c) is read as "the limit of x^2 as x approaches 2 is 4."

THEOREM 1.2 PROPERTIES OF LIMITS

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar multiple: $\lim\limits_{x \to c} [b f(x)] = bL$

2. Sum or difference: $\lim\limits_{x \to c} [f(x) \pm g(x)] = L \pm K$

3. Product: $\lim\limits_{x \to c} [f(x)g(x)] = LK$

4. Quotient: $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$

5. Power: $\lim\limits_{x \to c} [f(x)]^n = L^n$

EXAMPLE 2 The Limit of a Polynomial

$$\begin{aligned}
\lim_{x\to 2}(4x^2+3) &= \lim_{x\to 2}4x^2 + \lim_{x\to 2}3 && \text{Property 2}\\
&= 4\left(\lim_{x\to 2}x^2\right) + \lim_{x\to 2}3 && \text{Property 1}\\
&= 4(2^2)+3 && \text{Example 1}\\
&= 19 && \text{Simplify.}
\end{aligned}$$

■

In Example 2, note that the limit (as $x \to 2$) of the *polynomial function* $p(x) = 4x^2 + 3$ is simply the value of p at $x = 2$.

$$\lim_{x\to 2} p(x) = p(2) = 4(2^2) + 3 = 19$$

This *direct substitution* property is valid for all polynomial and rational functions with nonzero denominators.

THEOREM 1.3 LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS

If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

EXAMPLE 3 The Limit of a Rational Function

Find the limit: $\displaystyle\lim_{x\to 1}\frac{x^2+x+2}{x+1}$.

Solution Because the denominator is not 0 when $x = 1$, you can apply Theorem 1.3 to obtain

$$\lim_{x\to 1}\frac{x^2+x+2}{x+1} = \frac{1^2+1+2}{1+1} = \frac{4}{2} = 2.$$

■

Polynomial functions and rational functions are two of the three basic types of algebraic functions. The following theorem deals with the limit of the third type of algebraic function—one that involves a radical. See Appendix A for a proof of this theorem.

THE SQUARE ROOT SYMBOL

The first use of a symbol to denote the square root can be traced to the sixteenth century. Mathematicians first used the symbol $\surd$, which had only two strokes. This symbol was chosen because it resembled a lowercase r, to stand for the Latin word *radix*, meaning root.

THEOREM 1.4 THE LIMIT OF A FUNCTION INVOLVING A RADICAL

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x\to c}\sqrt[n]{x} = \sqrt[n]{c}$$

The following theorem greatly expands your ability to evaluate limits because it shows how to analyze the limit of a composite function. See Appendix A for a proof of this theorem.

THEOREM 1.5 THE LIMIT OF A COMPOSITE FUNCTION

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

EXAMPLE 4 The Limit of a Composite Function

a. Because

$$\lim_{x \to 0} (x^2 + 4) = 0^2 + 4 = 4 \quad \text{and} \quad \lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$$

it follows that

$$\lim_{x \to 0} \sqrt{x^2 + 4} = \sqrt{4} = 2.$$

b. Because

$$\lim_{x \to 3} (2x^2 - 10) = 2(3^2) - 10 = 8 \quad \text{and} \quad \lim_{x \to 8} \sqrt[3]{x} = \sqrt[3]{8} = 2$$

it follows that

$$\lim_{x \to 3} \sqrt[3]{2x^2 - 10} = \sqrt[3]{8} = 2.$$

■

You have seen that the limits of many algebraic functions can be evaluated by direct substitution. The six basic trigonometric functions also exhibit this desirable quality, as shown in the next theorem (presented without proof).

THEOREM 1.6 LIMITS OF TRIGONOMETRIC FUNCTIONS

Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x \to c} \sin x = \sin c$

2. $\lim_{x \to c} \cos x = \cos c$

3. $\lim_{x \to c} \tan x = \tan c$

4. $\lim_{x \to c} \cot x = \cot c$

5. $\lim_{x \to c} \sec x = \sec c$

6. $\lim_{x \to c} \csc x = \csc c$

EXAMPLE 5 Limits of Trigonometric Functions

a. $\lim_{x \to 0} \tan x = \tan(0) = 0$

b. $\lim_{x \to \pi} (x \cos x) = \left(\lim_{x \to \pi} x\right)\left(\lim_{x \to \pi} \cos x\right) = \pi \cos(\pi) = -\pi$

c. $\lim_{x \to 0} \sin^2 x = \lim_{x \to 0} (\sin x)^2 = 0^2 = 0$

■

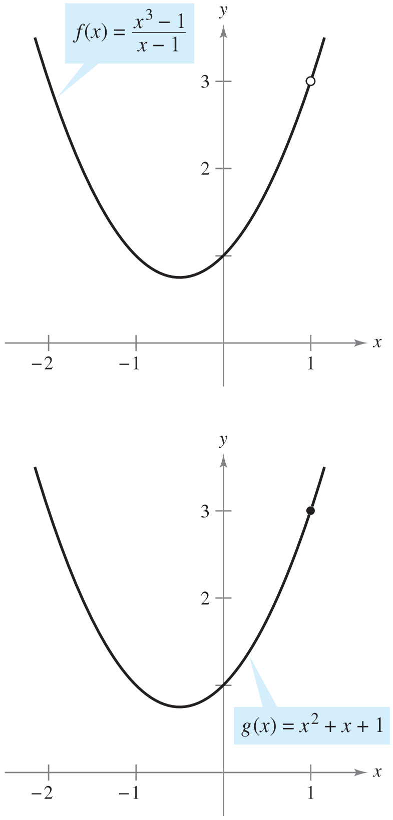
A Strategy for Finding Limits

On the previous three pages, you studied several types of functions whose limits can be evaluated by direct substitution. This knowledge, together with the following theorem, can be used to develop a strategy for finding limits. A proof of this theorem is given in Appendix A.

THEOREM 1.7 FUNCTIONS THAT AGREE AT ALL BUT ONE POINT

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x).$$



f and *g* agree at all but one point.
Figure 1.17

EXAMPLE 6 Finding the Limit of a Function

Find the limit: $\displaystyle\lim_{x\to 1} \frac{x^3-1}{x-1}$.

Solution Let $f(x) = (x^3 - 1)/(x - 1)$. By factoring and dividing out like factors, you can rewrite f as

$$f(x) = \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = x^2 + x + 1 = g(x), \quad x \neq 1.$$

So, for all x-values other than $x = 1$, the functions f and g agree, as shown in Figure 1.17. Because $\displaystyle\lim_{x\to 1} g(x)$ exists, you can apply Theorem 1.7 to conclude that f and g have the same limit at $x = 1$.

$$\begin{aligned}
\lim_{x\to 1} \frac{x^3-1}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1} && \text{Factor.}\\
&= \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} && \text{Divide out like factors.}\\
&= \lim_{x\to 1} (x^2+x+1) && \text{Apply Theorem 1.7.}\\
&= 1^2 + 1 + 1 && \text{Use direct substitution.}\\
&= 3 && \text{Simplify.}
\end{aligned}$$

■

STUDY TIP When applying this strategy for finding a limit, remember that some functions do not have a limit (as x approaches c). For instance, the following limit does not exist.

$$\lim_{x\to 1} \frac{x^3+1}{x-1}$$

A STRATEGY FOR FINDING LIMITS

1. Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)
2. If the limit of $f(x)$ as x approaches c *cannot* be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$. [Choose g such that the limit of $g(x)$ *can* be evaluated by direct substitution.]
3. Apply Theorem 1.7 to conclude *analytically* that
$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = g(c).$$
4. Use a *graph* or *table* to reinforce your conclusion.

Dividing Out and Rationalizing Techniques

Two techniques for finding limits analytically are shown in Examples 7 and 8. The dividing out technique involves dividing out common factors, and the rationalizing technique involves rationalizing the numerator of a fractional expression.

EXAMPLE 7 Dividing Out Technique

Find the limit: $\lim_{x\to -3} \dfrac{x^2+x-6}{x+3}$.

Solution Although you are taking the limit of a rational function, you *cannot* apply Theorem 1.3 because the limit of the denominator is 0.

$$\begin{tikzcd} & \lim_{x\to -3}(x^2+x-6)=0 \\ \lim_{x\to -3}\dfrac{x^2+x-6}{x+3} \arrow[ur] \arrow[dr] & \\ & \lim_{x\to -3}(x+3)=0 \end{tikzcd}$$

Direct substitution fails.

Because the limit of the numerator is also 0, the numerator and denominator have a *common factor* of $(x+3)$. So, for all $x \neq -3$, you can divide out this factor to obtain

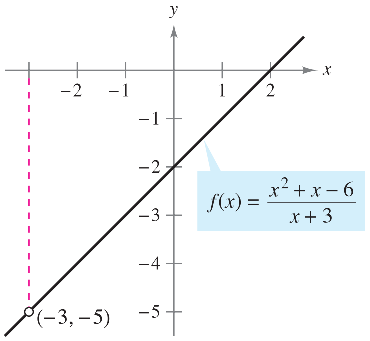
$$f(x) = \frac{x^2+x-6}{x+3} = \frac{\cancel{(x+3)}(x-2)}{\cancel{x+3}} = x-2 = g(x), \quad x \neq -3.$$

Using Theorem 1.7, it follows that

$$\lim_{x\to -3}\frac{x^2+x-6}{x+3} = \lim_{x\to -3}(x-2)$$ Apply Theorem 1.7.

$$= -5.$$ Use direct substitution.

This result is shown graphically in Figure 1.18. Note that the graph of the function f coincides with the graph of the function $g(x) = x - 2$, except that the graph of f has a gap at the point $(-3, -5)$. ■



f is undefined when $x = -3$.
Figure 1.18

NOTE In the solution of Example 7, be sure you see the usefulness of the Factor Theorem of Algebra. This theorem states that if c is a zero of a polynomial function, $(x - c)$ is a factor of the polynomial. So, if you apply direct substitution to a rational function and obtain

$$r(c) = \frac{p(c)}{q(c)} = \frac{0}{0}$$

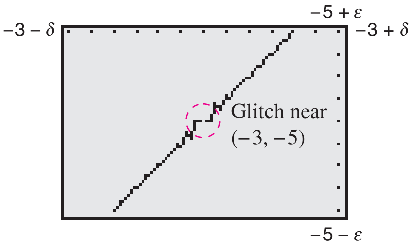
you can conclude that $(x - c)$ must be a common factor of both $p(x)$ and $q(x)$.

In Example 7, direct substitution produced the meaningless fractional form $0/0$. An expression such as $0/0$ is called an **indeterminate form** because you cannot (from the form alone) determine the limit. When you try to evaluate a limit and encounter this form, remember that you must rewrite the fraction so that the new denominator does not have 0 as its limit. One way to do this is to *divide out like factors*, as shown in Example 7. A second way is to *rationalize the numerator*, as shown in Example 8.

TECHNOLOGY PITFALL Because the graphs of

$$f(x) = \frac{x^2+x-6}{x+3} \quad \text{and} \quad g(x) = x-2$$

differ only at the point $(-3, -5)$, a standard graphing utility setting may not distinguish clearly between these graphs. However, because of the pixel configuration and rounding error of a graphing utility, it may be possible to find screen settings that distinguish between the graphs. Specifically, by repeatedly zooming in near the point $(-3, -5)$ on the graph of f, your graphing utility may show glitches or irregularities that do not exist on the actual graph. (See Figure 1.19.) By changing the screen settings on your graphing utility you may obtain the correct graph of f.



Incorrect graph of f
Figure 1.19

EXAMPLE 8 Rationalizing Technique

Find the limit: $\lim_{x\to 0} \dfrac{\sqrt{x+1}-1}{x}$.

Solution By direct substitution, you obtain the indeterminate form $0/0$.

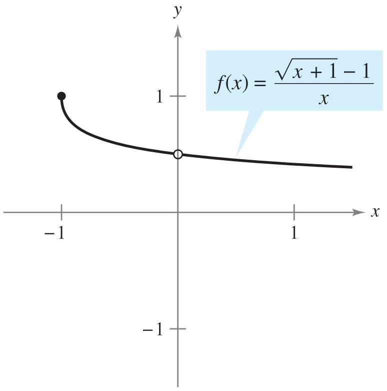
$$\begin{tikzcd} & \lim_{x\to 0}\left(\sqrt{x+1}-1\right)=0 \\ \lim_{x\to 0}\dfrac{\sqrt{x+1}-1}{x} \arrow[ur] \arrow[dr] & \text{Direct substitution fails.} \\ & \lim_{x\to 0} x = 0 \end{tikzcd}$$

In this case, you can rewrite the fraction by rationalizing the numerator.

$$\begin{aligned} \frac{\sqrt{x+1}-1}{x} &= \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) \\ &= \frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)} \\ &= \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)} \\ &= \frac{1}{\sqrt{x+1}+1}, \quad x \neq 0 \end{aligned}$$

Now, using Theorem 1.7, you can evaluate the limit as shown.

$$\begin{aligned} \lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1} \\ &= \frac{1}{1+1} \\ &= \frac{1}{2} \end{aligned}$$



The limit of $f(x)$ as x approaches 0 is $\frac{1}{2}$.
Figure 1.20

A table or a graph can reinforce your conclusion that the limit is $\frac{1}{2}$. (See Figure 1.20.)

x approaches 0 from the left. → ← x approaches 0 from the right.

x	−0.25	−0.1	−0.01	−0.001	0	0.001	0.01	0.1	0.25
$f(x)$	0.5359	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881	0.4721

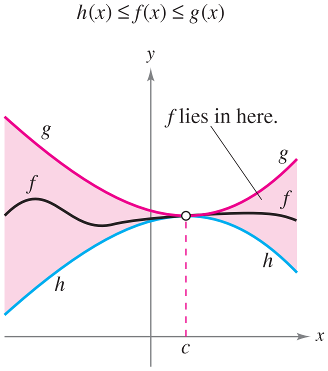
$f(x)$ approaches 0.5. → ← $f(x)$ approaches 0.5.

NOTE The rationalizing technique for evaluating limits is based on multiplication by a convenient form of 1. In Example 8, the convenient form is

$$1 = \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}.$$

The Squeeze Theorem

The next theorem concerns the limit of a function that is squeezed between two other functions, each of which has the same limit at a given x-value, as shown in Figure 1.21. (The proof of this theorem is given in Appendix A.)



The Squeeze Theorem
Figure 1.21

THEOREM 1.8 THE SQUEEZE THEOREM

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

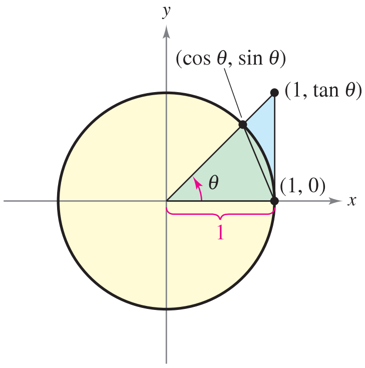
then $\lim_{x\to c} f(x)$ exists and is equal to L.

You can see the usefulness of the Squeeze Theorem (also called the Sandwich Theorem or the Pinching Theorem) in the proof of Theorem 1.9.

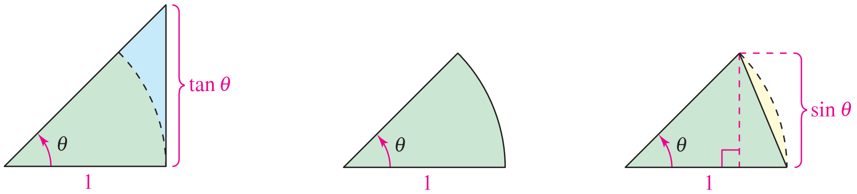
THEOREM 1.9 TWO SPECIAL TRIGONOMETRIC LIMITS

1. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$ **2.** $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

PROOF To avoid the confusion of two different uses of x, the proof is presented using the variable θ, where θ is an acute positive angle *measured in radians*. Figure 1.22 shows a circular sector that is squeezed between two triangles.



A circular sector is used to prove Theorem 1.9.
Figure 1.22



Area of triangle ≥ Area of sector ≥ Area of triangle

$$\frac{\tan\theta}{2} \ge \frac{\theta}{2} \ge \frac{\sin\theta}{2}$$

Multiplying each expression by $2/\sin\theta$ produces

$$\frac{1}{\cos\theta} \ge \frac{\theta}{\sin\theta} \ge 1$$

and taking reciprocals and reversing the inequalities yields

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1.$$

Because $\cos\theta = \cos(-\theta)$ and $(\sin\theta)/\theta = [\sin(-\theta)]/(-\theta)$, you can conclude that this inequality is valid for *all* nonzero θ in the open interval $(-\pi/2, \pi/2)$. Finally, because $\lim_{\theta\to 0} \cos\theta = 1$ and $\lim_{\theta\to 0} 1 = 1$, you can apply the Squeeze Theorem to conclude that $\lim_{\theta\to 0} (\sin\theta)/\theta = 1$. The proof of the second limit is left as an exercise (see Exercise 123). ■

EXAMPLE 9 A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x\to 0} \frac{\tan x}{x}$.

Solution Direct substitution yields the indeterminate form 0/0. To solve this problem, you can write $\tan x$ as $(\sin x)/(\cos x)$ and obtain

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right).$$

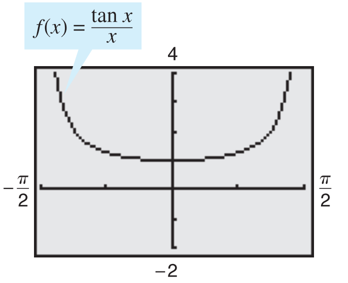
Now, because

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{1}{\cos x} = 1$$

you can obtain

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x}{x} &= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{\cos x}\right)\\ &= (1)(1)\\ &= 1.\end{aligned}$$

(See Figure 1.23.)



The limit of $f(x)$ as x approaches 0 is 1.
Figure 1.23

EXAMPLE 10 A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x\to 0} \frac{\sin 4x}{x}$.

Solution Direct substitution yields the indeterminate form 0/0. To solve this problem, you can rewrite the limit as

$$\lim_{x\to 0} \frac{\sin 4x}{x} = 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right).$$

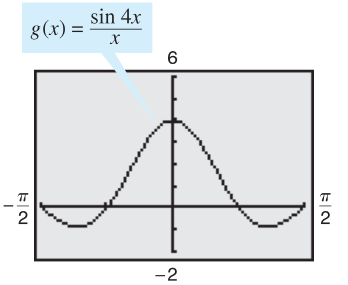
Multiply and divide by 4.

Now, by letting $y = 4x$ and observing that $x \to 0$ if and only if $y \to 0$, you can write

$$\begin{aligned}\lim_{x\to 0} \frac{\sin 4x}{x} &= 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right)\\ &= 4\left(\lim_{y\to 0} \frac{\sin y}{y}\right)\\ &= 4(1)\\ &= 4.\end{aligned}$$

Apply Theorem 1.9(1).

(See Figure 1.24.) ■



The limit of $g(x)$ as x approaches 0 is 4.
Figure 1.24

TECHNOLOGY Use a graphing utility to confirm the limits in the examples and in the exercise set. For instance, Figures 1.23 and 1.24 show the graphs of

$$f(x) = \frac{\tan x}{x} \qquad \text{and} \qquad g(x) = \frac{\sin 4x}{x}.$$

Note that the first graph appears to contain the point (0, 1) and the second graph appears to contain the point (0, 4), which lends support to the conclusions obtained in Examples 9 and 10.

1.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, use a graphing utility to graph the function and visually estimate the limits.

1. $h(x) = -x^2 + 4x$

(a) $\lim_{x\to 4} h(x)$

(b) $\lim_{x\to -1} h(x)$

2. $g(x) = \dfrac{12(\sqrt{x} - 3)}{x - 9}$

(a) $\lim_{x\to 4} g(x)$

(b) $\lim_{x\to 0} g(x)$

3. $f(x) = x \cos x$

(a) $\lim_{x\to 0} f(x)$

(b) $\lim_{x\to \pi/3} f(x)$

4. $f(t) = t|t - 4|$

(a) $\lim_{t\to 4} f(t)$

(b) $\lim_{t\to -1} f(t)$

In Exercises 5–22, find the limit.

5. $\lim_{x\to 2} x^3$

6. $\lim_{x\to -2} x^4$

7. $\lim_{x\to 0} (2x - 1)$

8. $\lim_{x\to -3} (3x + 2)$

9. $\lim_{x\to -3} (x^2 + 3x)$

10. $\lim_{x\to 1} (-x^2 + 1)$

11. $\lim_{x\to -3} (2x^2 + 4x + 1)$

12. $\lim_{x\to 1} (3x^3 - 2x^2 + 4)$

13. $\lim_{x\to 3} \sqrt{x + 1}$

14. $\lim_{x\to 4} \sqrt[3]{x + 4}$

15. $\lim_{x\to -4} (x + 3)^2$

16. $\lim_{x\to 0} (2x - 1)^3$

17. $\lim_{x\to 2} \dfrac{1}{x}$

18. $\lim_{x\to -3} \dfrac{2}{x + 2}$

19. $\lim_{x\to 1} \dfrac{x}{x^2 + 4}$

20. $\lim_{x\to 1} \dfrac{2x - 3}{x + 5}$

21. $\lim_{x\to 7} \dfrac{3x}{\sqrt{x + 2}}$

22. $\lim_{x\to 2} \dfrac{\sqrt{x + 2}}{x - 4}$

In Exercises 23–26, find the limits.

23. $f(x) = 5 - x,\ g(x) = x^3$

(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

24. $f(x) = x + 7,\ g(x) = x^2$

(a) $\lim_{x\to -3} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to -3} g(f(x))$

25. $f(x) = 4 - x^2,\ g(x) = \sqrt{x + 1}$

(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 3} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

26. $f(x) = 2x^2 - 3x + 1,\ g(x) = \sqrt[3]{x + 6}$

(a) $\lim_{x\to 4} f(x)$ (b) $\lim_{x\to 21} g(x)$ (c) $\lim_{x\to 4} g(f(x))$

In Exercises 27–36, find the limit of the trigonometric function.

27. $\lim_{x\to \pi/2} \sin x$

28. $\lim_{x\to \pi} \tan x$

29. $\lim_{x\to 1} \cos \dfrac{\pi x}{3}$

30. $\lim_{x\to 2} \sin \dfrac{\pi x}{2}$

31. $\lim_{x\to 0} \sec 2x$

32. $\lim_{x\to \pi} \cos 3x$

33. $\lim_{x\to 5\pi/6} \sin x$

34. $\lim_{x\to 5\pi/3} \cos x$

35. $\lim_{x\to 3} \tan\left(\dfrac{\pi x}{4}\right)$

36. $\lim_{x\to 7} \sec\left(\dfrac{\pi x}{6}\right)$

In Exercises 37–40, use the information to evaluate the limits.

37. $\lim_{x\to c} f(x) = 3$

$\lim_{x\to c} g(x) = 2$

(a) $\lim_{x\to c} [5g(x)]$

(b) $\lim_{x\to c} [f(x) + g(x)]$

(c) $\lim_{x\to c} [f(x)g(x)]$

(d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

38. $\lim_{x\to c} f(x) = \frac{3}{2}$

$\lim_{x\to c} g(x) = \frac{1}{2}$

(a) $\lim_{x\to c} [4f(x)]$

(b) $\lim_{x\to c} [f(x) + g(x)]$

(c) $\lim_{x\to c} [f(x)g(x)]$

(d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

39. $\lim_{x\to c} f(x) = 4$

(a) $\lim_{x\to c} [f(x)]^3$

(b) $\lim_{x\to c} \sqrt{f(x)}$

(c) $\lim_{x\to c} [3f(x)]$

(d) $\lim_{x\to c} [f(x)]^{3/2}$

40. $\lim_{x\to c} f(x) = 27$

(a) $\lim_{x\to c} \sqrt[3]{f(x)}$

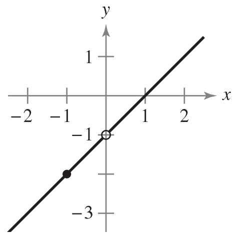
(b) $\lim_{x\to c} \dfrac{f(x)}{18}$

(c) $\lim_{x\to c} [f(x)]^2$

(d) $\lim_{x\to c} [f(x)]^{2/3}$

In Exercises 41–44, use the graph to determine the limit visually (if it exists). Write a simpler function that agrees with the given function at all but one point.

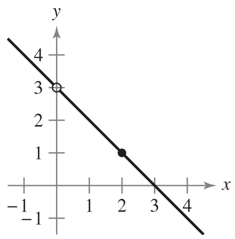
41. $g(x) = \dfrac{x^2 - x}{x}$



(a) $\lim_{x\to 0} g(x)$

(b) $\lim_{x\to -1} g(x)$

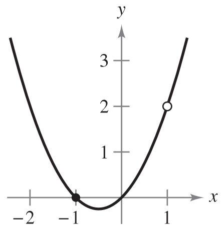
42. $h(x) = \dfrac{-x^2 + 3x}{x}$



(a) $\lim_{x\to 2} h(x)$

(b) $\lim_{x\to 0} h(x)$

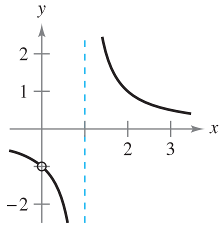
43. $g(x) = \dfrac{x^3 - x}{x - 1}$



(a) $\lim_{x\to 1} g(x)$

(b) $\lim_{x\to -1} g(x)$

44. $f(x) = \dfrac{x}{x^2 - x}$



(a) $\lim_{x\to 1} f(x)$

(b) $\lim_{x\to 0} f(x)$

In Exercises 45–48, find the limit of the function (if it exists). Write a simpler function that agrees with the given function at all but one point. Use a graphing utility to confirm your result.

45. $\lim_{x\to -1}\frac{x^2-1}{x+1}$ **46.** $\lim_{x\to -1}\frac{2x^2-x-3}{x+1}$

47. $\lim_{x\to 2}\frac{x^3-8}{x-2}$ **48.** $\lim_{x\to -1}\frac{x^3+1}{x+1}$

In Exercises 49–64, find the limit (if it exists).

49. $\lim_{x\to 0}\frac{x}{x^2-x}$ **50.** $\lim_{x\to 0}\frac{3x}{x^2+2x}$

51. $\lim_{x\to 4}\frac{x-4}{x^2-16}$ **52.** $\lim_{x\to 3}\frac{3-x}{x^2-9}$

53. $\lim_{x\to -3}\frac{x^2+x-6}{x^2-9}$ **54.** $\lim_{x\to 4}\frac{x^2-5x+4}{x^2-2x-8}$

55. $\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}$ **56.** $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}$

57. $\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}$ **58.** $\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$

59. $\lim_{x\to 0}\frac{[1/(3+x)]-(1/3)}{x}$ **60.** $\lim_{x\to 0}\frac{[1/(x+4)]-(1/4)}{x}$

61. $\lim_{\Delta x\to 0}\frac{2(x+\Delta x)-2x}{\Delta x}$ **62.** $\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}$

63. $\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x}$

64. $\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x}$

In Exercises 65–76, determine the limit of the trigonometric function (if it exists).

65. $\lim_{x\to 0}\frac{\sin x}{5x}$ **66.** $\lim_{x\to 0}\frac{3(1-\cos x)}{x}$

67. $\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^2}$ **68.** $\lim_{\theta\to 0}\frac{\cos\theta\tan\theta}{\theta}$

69. $\lim_{x\to 0}\frac{\sin^2 x}{x}$ **70.** $\lim_{x\to 0}\frac{\tan^2 x}{x}$

71. $\lim_{h\to 0}\frac{(1-\cos h)^2}{h}$ **72.** $\lim_{\phi\to\pi}\phi\sec\phi$

73. $\lim_{x\to\pi/2}\frac{\cos x}{\cot x}$ **74.** $\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x}$

75. $\lim_{t\to 0}\frac{\sin 3t}{2t}$

76. $\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}$ $\left[\textit{Hint}\text{: Find } \lim_{x\to 0}\left(\frac{2\sin 2x}{2x}\right)\left(\frac{3x}{3\sin 3x}\right).\right]$

Graphical, Numerical, and Analytic Analysis **In Exercises 77–84, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.**

77. $\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}$ **78.** $\lim_{x\to 16}\frac{4-\sqrt{x}}{x-16}$

79. $\lim_{x\to 0}\frac{[1/(2+x)]-(1/2)}{x}$ **80.** $\lim_{x\to 2}\frac{x^5-32}{x-2}$

81. $\lim_{t\to 0}\frac{\sin 3t}{t}$ **82.** $\lim_{x\to 0}\frac{\cos x-1}{2x^2}$

83. $\lim_{x\to 0}\frac{\sin x^2}{x}$ **84.** $\lim_{x\to 0}\frac{\sin x}{\sqrt[3]{x}}$

In Exercises 85–88, find $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$.

85. $f(x)=3x-2$

86. $f(x)=\sqrt{x}$

87. $f(x)=\frac{1}{x+3}$

88. $f(x)=x^2-4x$

In Exercises 89 and 90, use the Squeeze Theorem to find $\lim_{x\to c} f(x)$.

89. $c=0$

$4-x^2\le f(x)\le 4+x^2$

90. $c=a$

$b-|x-a|\le f(x)\le b+|x-a|$

In Exercises 91–96, use a graphing utility to graph the given function and the equations $y=|x|$ **and** $y=-|x|$ **in the same viewing window. Using the graphs to observe the Squeeze Theorem visually, find** $\lim_{x\to 0} f(x)$.

91. $f(x)=x\cos x$ **92.** $f(x)=|x\sin x|$

93. $f(x)=|x|\sin x$ **94.** $f(x)=|x|\cos x$

95. $f(x)=x\sin\frac{1}{x}$ **96.** $h(x)=x\cos\frac{1}{x}$

WRITING ABOUT CONCEPTS

97. In the context of finding limits, discuss what is meant by two functions that agree at all but one point.

98. Give an example of two functions that agree at all but one point.

99. What is meant by an indeterminate form?

100. In your own words, explain the Squeeze Theorem.

101. ***Writing*** Use a graphing utility to graph

$$f(x)=x,\quad g(x)=\sin x,\quad \text{and}\quad h(x)=\frac{\sin x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why

$$\lim_{x\to 0} h(x)=1.$$

102. ***Writing*** Use a graphing utility to graph

$$f(x) = x, \; g(x) = \sin^2 x, \text{ and } h(x) = \frac{\sin^2 x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why $\lim_{x\to 0} h(x) = 0$.

Free-Falling Object **In Exercises 103 and 104, use the position function $s(t) = -16t^2 + 500$, which gives the height (in feet) of an object that has fallen for t seconds from a height of 500 feet. The velocity at time $t = a$ seconds is given by**

$$\lim_{t\to a} \frac{s(a) - s(t)}{a - t}.$$

103. If a construction worker drops a wrench from a height of 500 feet, how fast will the wrench be falling after 2 seconds?

104. If a construction worker drops a wrench from a height of 500 feet, when will the wrench hit the ground? At what velocity will the wrench impact the ground?

Free-Falling Object **In Exercises 105 and 106, use the position function $s(t) = -4.9t^2 + 200$, which gives the height (in meters) of an object that has fallen from a height of 200 meters. The velocity at time $t = a$ seconds is given by**

$$\lim_{t\to a} \frac{s(a) - s(t)}{a - t}.$$

105. Find the velocity of the object when $t = 3$.

106. At what velocity will the object impact the ground?

107. Find two functions f and g such that $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist, but $\lim_{x\to 0} [f(x) + g(x)]$ does exist.

108. Prove that if $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} [f(x) + g(x)]$ does not exist, then $\lim_{x\to c} g(x)$ does not exist.

109. Prove Property 1 of Theorem 1.1.

110. Prove Property 3 of Theorem 1.1. (You may use Property 3 of Theorem 1.2.)

111. Prove Property 1 of Theorem 1.2.

112. Prove that if $\lim_{x\to c} f(x) = 0$, then $\lim_{x\to c} |f(x)| = 0$.

113. Prove that if $\lim_{x\to c} f(x) = 0$ and $|g(x)| \le M$ for a fixed number M and all $x \ne c$, then $\lim_{x\to c} f(x)g(x) = 0$.

114. (a) Prove that if $\lim_{x\to c} |f(x)| = 0$, then $\lim_{x\to c} f(x) = 0$.

(*Note:* This is the converse of Exercise 112.)

(b) Prove that if $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c} |f(x)| = |L|$.

[*Hint:* Use the inequality $\big||f(x)| - |L|\big| \le |f(x) - L|$.]

115. ***Think About It*** Find a function f to show that the converse of Exercise 114(b) is not true. [*Hint:* Find a function f such that $\lim_{x\to c} |f(x)| = |L|$ but $\lim_{x\to c} f(x)$ does not exist.]

CAPSTONE

116. Let $f(x) = \begin{cases} 3, & x \ne 2 \\ 5, & x = 2 \end{cases}$. Find $\lim_{x\to 2} f(x)$.

True or False? **In Exercises 117–122, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

117. $\lim_{x\to 0} \frac{|x|}{x} = 1$

118. $\lim_{x\to \pi} \frac{\sin x}{x} = 1$

119. If $f(x) = g(x)$ for all real numbers other than $x = 0$, and $\lim_{x\to 0} f(x) = L$, then $\lim_{x\to 0} g(x) = L$.

120. If $\lim_{x\to c} f(x) = L$, then $f(c) = L$.

121. $\lim_{x\to 2} f(x) = 3$, where $f(x) = \begin{cases} 3, & x \le 2 \\ 0, & x > 2 \end{cases}$

122. If $f(x) < g(x)$ for all $x \ne a$, then

$\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$.

123. Prove the second part of Theorem 1.9.

$$\lim_{x\to 0} \frac{1 - \cos x}{x} = 0$$

124. Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

and

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational.} \end{cases}$$

Find (if possible) $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$.

125. ***Graphical Reasoning*** Consider $f(x) = \dfrac{\sec x - 1}{x^2}$.

(a) Find the domain of f.

(b) Use a graphing utility to graph f. Is the domain of f obvious from the graph? If not, explain.

(c) Use the graph of f to approximate $\lim_{x\to 0} f(x)$.

(d) Confirm your answer to part (c) analytically.

126. ***Approximation***

(a) Find $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2}$.

(b) Use your answer to part (a) to derive the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ for x near 0.

(c) Use your answer to part (b) to approximate $\cos(0.1)$.

(d) Use a calculator to approximate $\cos(0.1)$ to four decimal places. Compare the result with part (c).

127. ***Think About It*** When using a graphing utility to generate a table to approximate $\lim_{x\to 0} [(\sin x)/x]$, a student concluded that the limit was 0.01745 rather than 1. Determine the probable cause of the error.

1.4 Continuity and One-Sided Limits

- **Determine continuity at a point and continuity on an open interval.**
- **Determine one-sided limits and continuity on a closed interval.**
- **Use properties of continuity.**
- **Understand and use the Intermediate Value Theorem.**

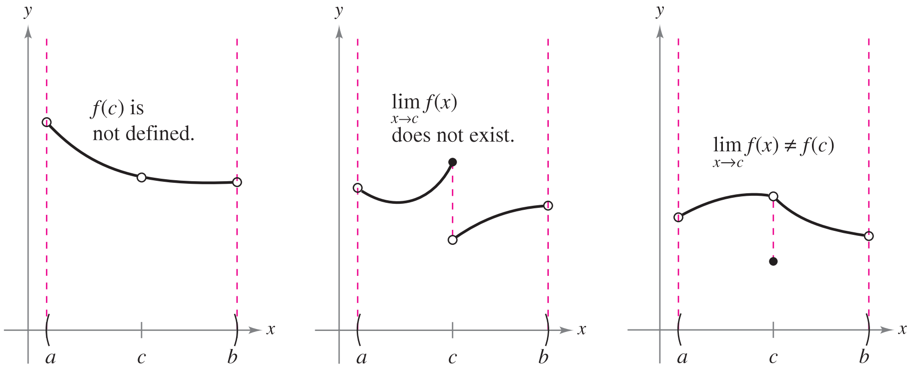
Continuity at a Point and on an Open Interval

In mathematics, the term *continuous* has much the same meaning as it has in everyday usage. Informally, to say that a function f is continuous at $x = c$ means that there is no interruption in the graph of f at c. That is, its graph is unbroken at c and there are no holes, jumps, or gaps. Figure 1.25 identifies three values of x at which the graph of f is *not* continuous. At all other points in the interval (a, b), the graph of f is uninterrupted and **continuous.**

EXPLORATION

Informally, you might say that a function is *continuous* on an open interval if its graph can be drawn with a pencil without lifting the pencil from the paper. Use a graphing utility to graph each function on the given interval. From the graphs, which functions would you say are continuous on the interval? Do you think you can trust the results you obtained graphically? Explain your reasoning.

Function	*Interval*
a. $y = x^2 + 1$	$(-3, 3)$
b. $y = \dfrac{1}{x - 2}$	$(-3, 3)$
c. $y = \dfrac{\sin x}{x}$	$(-\pi, \pi)$
d. $y = \dfrac{x^2 - 4}{x + 2}$	$(-3, 3)$
e. $y = \begin{cases} 2x - 4, & x \le 0 \\ x + 1, & x > 0 \end{cases}$	$(-3, 3)$


y
$f(c)$ is not defined.
x
a
c
b
y
$\lim\limits_{x\to c} f(x)$ does not exist.
x
a
c
b
y
$\lim\limits_{x\to c} f(x) \neq f(c)$
x
a
c
b

Three conditions exist for which the graph of f is not continuous at $x = c$.
Figure 1.25

In Figure 1.25, it appears that continuity at $x = c$ can be destroyed by any one of the following conditions.

1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

If *none* of the three conditions above is true, the function f is called **continuous at c,** as indicated in the following important definition.

DEFINITION OF CONTINUITY

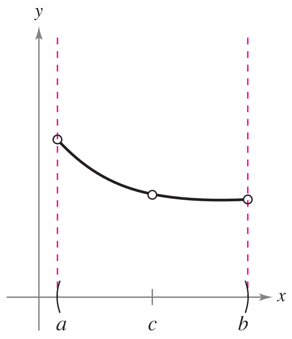
Continuity at a Point: A function f is **continuous at c** if the following three conditions are met.

1. $f(c)$ is defined.
2. $\lim\limits_{x\to c} f(x)$ exists.
3. $\lim\limits_{x\to c} f(x) = f(c)$

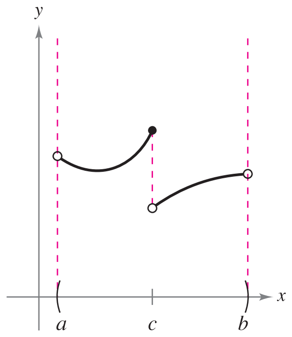
Continuity on an Open Interval: A function is **continuous on an open interval (a, b)** if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$ is **everywhere continuous.**

FOR FURTHER INFORMATION For more information on the concept of continuity, see the article "Leibniz and the Spell of the Continuous" by Hardy Grant in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com.*

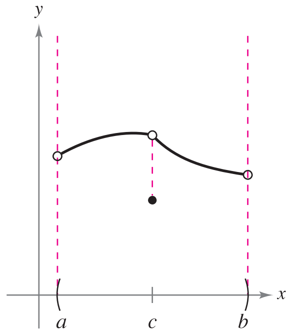
Consider an open interval I that contains a real number c. If a function f is defined on I (except possibly at c), and f is not continuous at c, then f is said to have a **discontinuity** at c. Discontinuities fall into two categories: **removable** and **nonremovable.** A discontinuity at c is called removable if f can be made continuous by appropriately defining (or redefining) $f(c)$. For instance, the functions shown in Figures 1.26(a) and (c) have removable discontinuities at c and the function shown in Figure 1.26(b) has a nonremovable discontinuity at c.



(a) Removable discontinuity



(b) Nonremovable discontinuity



(c) Removable discontinuity

Figure 1.26

EXAMPLE 1 Continuity of a Function

Discuss the continuity of each function.

a. $f(x) = \dfrac{1}{x}$ **b.** $g(x) = \dfrac{x^2 - 1}{x - 1}$ **c.** $h(x) = \begin{cases} x + 1, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$ **d.** $y = \sin x$

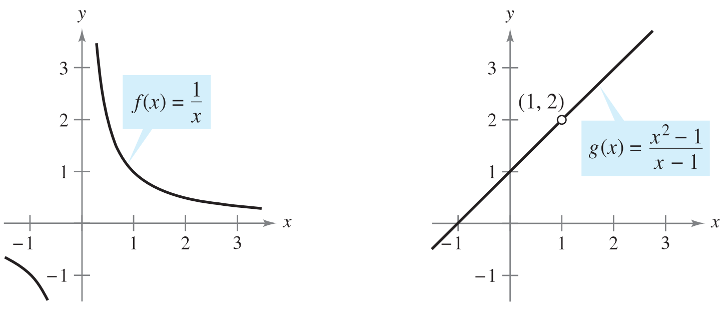
Solution

a. The domain of f is all nonzero real numbers. From Theorem 1.3, you can conclude that f is continuous at every x-value in its domain. At $x = 0$, f has a nonremovable discontinuity, as shown in Figure 1.27(a). In other words, there is no way to define $f(0)$ so as to make the function continuous at $x = 0$.

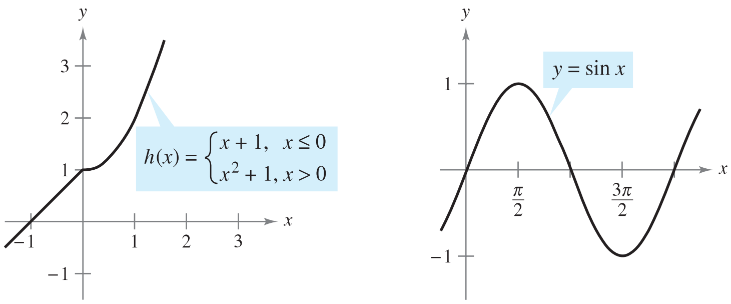
b. The domain of g is all real numbers except $x = 1$. From Theorem 1.3, you can conclude that g is continuous at every x-value in its domain. At $x = 1$, the function has a removable discontinuity, as shown in Figure 1.27(b). If $g(1)$ is defined as 2, the "newly defined" function is continuous for all real numbers.

c. The domain of h is all real numbers. The function h is continuous on $(-\infty, 0)$ and $(0, \infty)$, and, because $\lim\limits_{x \to 0} h(x) = 1$, h is continuous on the entire real line, as shown in Figure 1.27(c).

d. The domain of y is all real numbers. From Theorem 1.6, you can conclude that the function is continuous on its entire domain, $(-\infty, \infty)$, as shown in Figure 1.27(d).



(a) Nonremovable discontinuity at $x = 0$ **(b)** Removable discontinuity at $x = 1$



(c) Continuous on entire real line **(d)** Continuous on entire real line

Figure 1.27

■

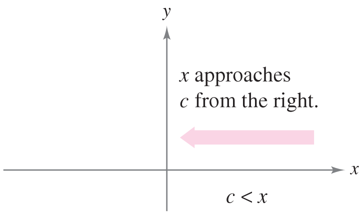
STUDY TIP Some people may refer to the function in Example 1(a) as "discontinuous." We have found that this terminology can be confusing. Rather than saying that the function is discontinuous, we prefer to say that it has a discontinuity at $x = 0$.

One-Sided Limits and Continuity on a Closed Interval

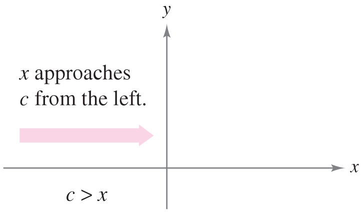
To understand continuity on a closed interval, you first need to look at a different type of limit called a **one-sided limit.** For example, the **limit from the right** (or right-hand limit) means that x approaches c from values greater than c [see Figure 1.28(a)]. This limit is denoted as

$$\lim_{x\to c^+} f(x) = L.$$

Limit from the right



(a) Limit from right



(b) Limit from left
Figure 1.28

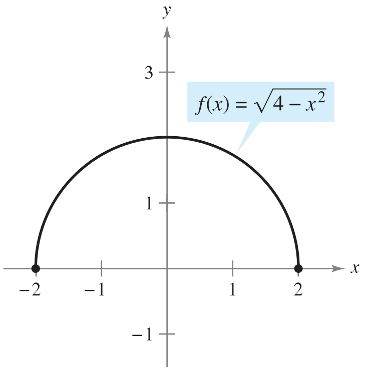
Similarly, the **limit from the left** (or left-hand limit) means that x approaches c from values less than c [see Figure 1.28(b)]. This limit is denoted as

$$\lim_{x\to c^-} f(x) = L.$$

Limit from the left

One-sided limits are useful in taking limits of functions involving radicals. For instance, if n is an even integer,

$$\lim_{x\to 0^+} \sqrt[n]{x} = 0.$$



The limit of $f(x)$ as x approaches -2 from the right is 0.
Figure 1.29

EXAMPLE 2 A One-Sided Limit

Find the limit of $f(x) = \sqrt{4 - x^2}$ as x approaches -2 from the right.

Solution As shown in Figure 1.29, the limit as x approaches -2 from the right is

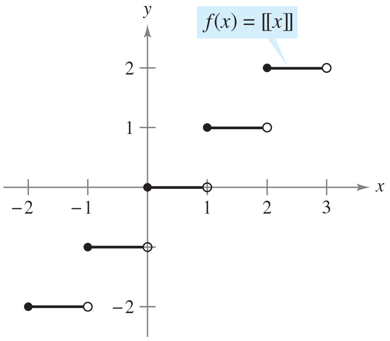
$$\lim_{x\to -2^+} \sqrt{4 - x^2} = 0.$$ ■

One-sided limits can be used to investigate the behavior of **step functions.** One common type of step function is the **greatest integer function** $[\![x]\!]$, defined by

$$[\![x]\!] = \text{greatest integer } n \text{ such that } n \le x.$$

Greatest integer function

For instance, $[\![2.5]\!] = 2$ and $[\![-2.5]\!] = -3$.



Greatest integer function
Figure 1.30

EXAMPLE 3 The Greatest Integer Function

Find the limit of the greatest integer function $f(x) = [\![x]\!]$ as x approaches 0 from the left and from the right.

Solution As shown in Figure 1.30, the limit as x approaches 0 *from the left* is given by

$$\lim_{x\to 0^-} [\![x]\!] = -1$$

and the limit as x approaches 0 *from the right* is given by

$$\lim_{x\to 0^+} [\![x]\!] = 0.$$

The greatest integer function has a discontinuity at zero because the left and right limits at zero are different. By similar reasoning, you can see that the greatest integer function has a discontinuity at any integer n. ■

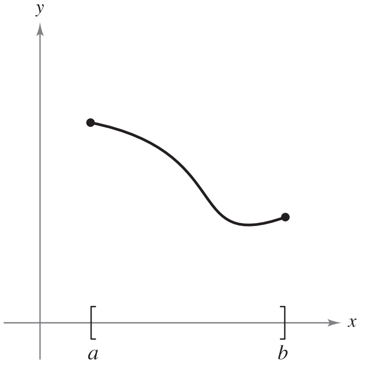
When the limit from the left is not equal to the limit from the right, the (two-sided) limit *does not exist*. The next theorem makes this more explicit. The proof of this theorem follows directly from the definition of a one-sided limit.

THEOREM 1.10 THE EXISTENCE OF A LIMIT

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

The concept of a one-sided limit allows you to extend the definition of continuity to closed intervals. Basically, a function is continuous on a closed interval if it is continuous in the interior of the interval and exhibits one-sided continuity at the endpoints. This is stated formally as follows.



Continuous function on a closed interval
Figure 1.31

DEFINITION OF CONTINUITY ON A CLOSED INTERVAL

A function f is **continuous on the closed interval $[a, b]$** if it is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b).$$

The function f is **continuous from the right** at a and **continuous from the left** at b (see Figure 1.31).

Similar definitions can be made to cover continuity on intervals of the form $(a, b]$ and $[a, b)$ that are neither open nor closed, or on infinite intervals. For example, the function

$$f(x) = \sqrt{x}$$

is continuous on the infinite interval $[0, \infty)$, and the function

$$g(x) = \sqrt{2 - x}$$

is continuous on the infinite interval $(-\infty, 2]$.

EXAMPLE 4 Continuity on a Closed Interval

Discuss the continuity of $f(x) = \sqrt{1 - x^2}$.

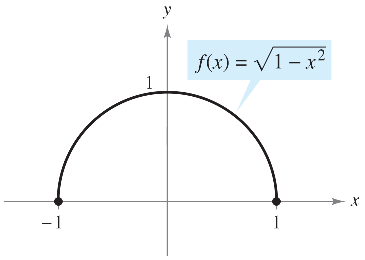
Solution The domain of f is the closed interval $[-1, 1]$. At all points in the open interval $(-1, 1)$, the continuity of f follows from Theorems 1.4 and 1.5. Moreover, because

$$\lim_{x\to -1^+} \sqrt{1 - x^2} = 0 = f(-1) \qquad \text{Continuous from the right}$$

and

$$\lim_{x\to 1^-} \sqrt{1 - x^2} = 0 = f(1) \qquad \text{Continuous from the left}$$

you can conclude that f is continuous on the closed interval $[-1, 1]$, as shown in Figure 1.32. ■

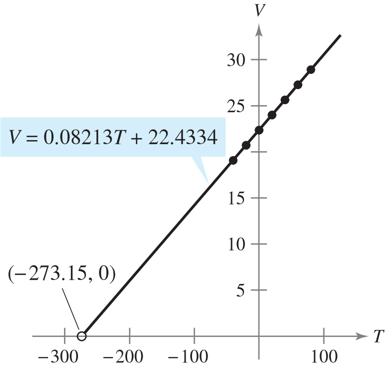


f is continuous on $[-1, 1]$.
Figure 1.32

The next example shows how a one-sided limit can be used to determine the value of absolute zero on the Kelvin scale.

EXAMPLE 5 Charles's Law and Absolute Zero

On the Kelvin scale, *absolute zero* is the temperature 0 K. Although temperatures very close to 0 K have been produced in laboratories, absolute zero has never been attained. In fact, evidence suggests that absolute zero *cannot* be attained. How did scientists determine that 0 K is the "lower limit" of the temperature of matter? What is absolute zero on the Celsius scale?



The volume of hydrogen gas depends on its temperature.
Figure 1.33

Solution The determination of absolute zero stems from the work of the French physicist Jacques Charles (1746–1823). Charles discovered that the volume of gas at a constant pressure increases linearly with the temperature of the gas. The table illustrates this relationship between volume and temperature. To generate the values in the table, one mole of hydrogen is held at a constant pressure of one atmosphere. The volume V is approximated and is measured in liters, and the temperature T is measured in degrees Celsius.

T	-40	-20	0	20	40	60	80
V	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038

The points represented by the table are shown in Figure 1.33. Moreover, by using the points in the table, you can determine that T and V are related by the linear equation

$$V = 0.08213T + 22.4334 \qquad \text{or} \qquad T = \frac{V - 22.4334}{0.08213}.$$

By reasoning that the volume of the gas can approach 0 (but can never equal or go below 0), you can determine that the "least possible temperature" is given by

$$\begin{aligned}\lim_{V\to 0^+} T &= \lim_{V\to 0^+} \frac{V - 22.4334}{0.08213}\\ &= \frac{0 - 22.4334}{0.08213} \qquad \text{Use direct substitution.}\\ &\approx -273.15.\end{aligned}$$

So, absolute zero on the Kelvin scale (0 K) is approximately −273.15° on the Celsius scale. ■

Photo courtesy of W. Ketterle, MIT

In 2003, researchers at the Massachusetts Institute of Technology used lasers and evaporation to produce a supercold gas in which atoms overlap. This gas is called a Bose-Einstein condensate. They measured a temperature of about 450 pK (picokelvin), or approximately −273.14999999955°C. (*Source:* Science *magazine, September 12, 2003*)

The following table shows the temperatures in Example 5 converted to the Fahrenheit scale. Try repeating the solution shown in Example 5 using these temperatures and volumes. Use the result to find the value of absolute zero on the Fahrenheit scale.

T	-40	-4	32	68	104	140	176
V	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038

NOTE Charles's Law for gases (assuming constant pressure) can be stated as

$$V = RT \qquad \text{Charles's Law}$$

where V is volume, R is a constant, and T is temperature. In the statement of this law, what property must the temperature scale have? ■

Bettmann/Corbis

AUGUSTIN-LOUIS CAUCHY (1789–1857)

The concept of a continuous function was first introduced by Augustin-Louis Cauchy in 1821. The definition given in his text *Cours d'Analyse* stated that indefinite small changes in y were the result of indefinite small changes in x. "… $f(x)$ will be called a *continuous* function if … the numerical values of the difference $f(x + \alpha) - f(x)$ decrease indefinitely with those of α …."

Properties of Continuity

In Section 1.3, you studied several properties of limits. Each of those properties yields a corresponding property pertaining to the continuity of a function. For instance, Theorem 1.11 follows directly from Theorem 1.2. (A proof of Theorem 1.11 is given in Appendix A.)

THEOREM 1.11 PROPERTIES OF CONTINUITY

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\dfrac{f}{g}$, if $g(c) \neq 0$

The following types of functions are continuous at every point in their domains.

1. Polynomial: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0$
2. Rational: $r(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$
3. Radical: $f(x) = \sqrt[n]{x}$
4. Trigonometric: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$

By combining Theorem 1.11 with this summary, you can conclude that a wide variety of elementary functions are continuous at every point in their domains.

EXAMPLE 6 Applying Properties of Continuity

By Theorem 1.11, it follows that each of the functions below is continuous at every point in its domain.

$$f(x) = x + \sin x, \qquad f(x) = 3\tan x, \qquad f(x) = \frac{x^2 + 1}{\cos x}$$ ■

The next theorem, which is a consequence of Theorem 1.5, allows you to determine the continuity of *composite* functions such as

$$f(x) = \sin 3x, \qquad f(x) = \sqrt{x^2 + 1}, \qquad f(x) = \tan \frac{1}{x}.$$

THEOREM 1.12 CONTINUITY OF A COMPOSITE FUNCTION

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

PROOF By the definition of continuity, $\lim\limits_{x \to c} g(x) = g(c)$ and $\lim\limits_{x \to g(c)} f(x) = f(g(c))$. Apply Theorem 1.5 with $L = g(c)$ to obtain $\lim\limits_{x \to c} f(g(x)) = f\left(\lim\limits_{x \to c} g(x)\right) = f(g(c))$. So, $(f \circ g) = f(g(x))$ is continuous at c. ■

NOTE One consequence of Theorem 1.12 is that if f and g satisfy the given conditions, you can determine the limit of $f(g(x))$ as x approaches c to be

$$\lim_{x \to c} f(g(x)) = f(g(c)).$$

EXAMPLE 7 Testing for Continuity

Describe the interval(s) on which each function is continuous.

a. $f(x) = \tan x$ **b.** $g(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ **c.** $h(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution

a. The tangent function $f(x) = \tan x$ is undefined at

$$x = \frac{\pi}{2} + n\pi, \qquad n \text{ is an integer.}$$

At all other points it is continuous. So, $f(x) = \tan x$ is continuous on the open intervals

$$\ldots, \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$

as shown in Figure 1.34(a).

b. Because $y = 1/x$ is continuous except at $x = 0$ and the sine function is continuous for all real values of x, it follows that $y = \sin(1/x)$ is continuous at all real values except $x = 0$. At $x = 0$, the limit of $g(x)$ does not exist (see Example 5, Section 1.2). So, g is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$, as shown in Figure 1.34(b).

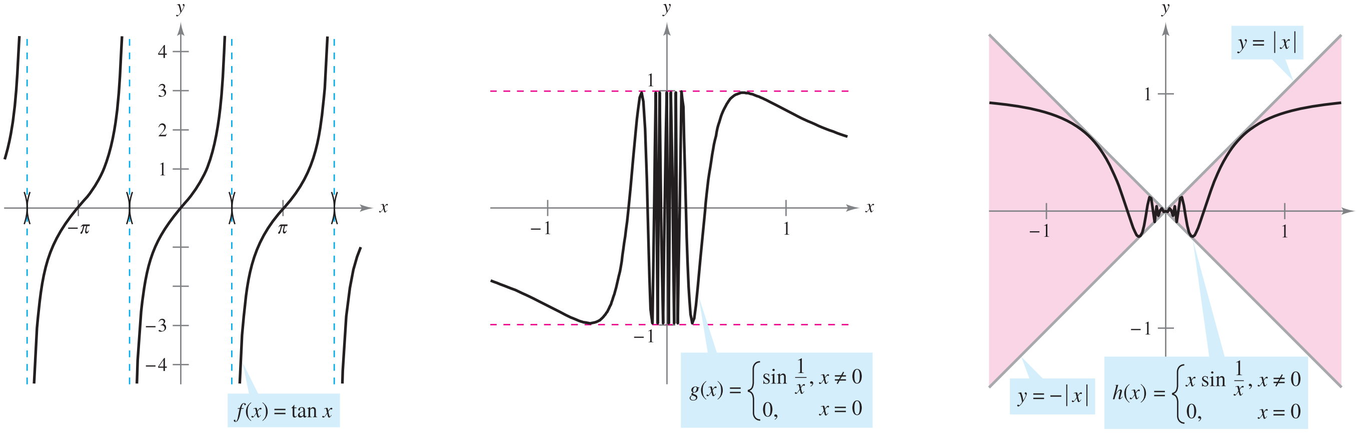
c. This function is similar to the function in part (b) except that the oscillations are damped by the factor x. Using the Squeeze Theorem, you obtain

$$-|x| \leq x \sin \frac{1}{x} \leq |x|, \quad x \neq 0$$

and you can conclude that

$$\lim_{x \to 0} h(x) = 0.$$

So, h is continuous on the entire real line, as shown in Figure 1.34(c).



(a) f is continuous on each open interval in its domain.

(b) g is continuous on $(-\infty, 0)$ and $(0, \infty)$.

(c) h is continuous on the entire real line

Figure 1.34

The Intermediate Value Theorem

Theorem 1.13 is an important theorem concerning the behavior of functions that are continuous on a closed interval.

THEOREM 1.13 INTERMEDIATE VALUE THEOREM

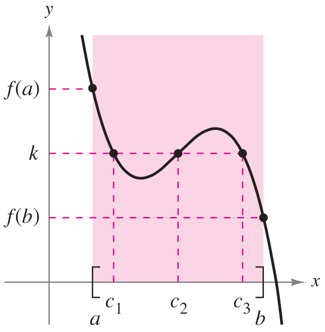
If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

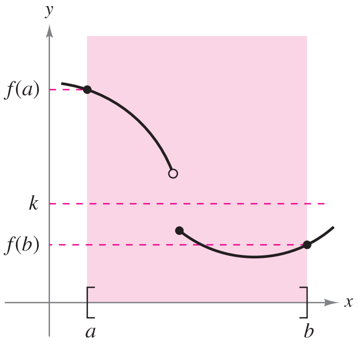
NOTE The Intermediate Value Theorem tells you that at least one number c exists, but it does not provide a method for finding c. Such theorems are called **existence theorems.** By referring to a text on advanced calculus, you will find that a proof of this theorem is based on a property of real numbers called *completeness*. The Intermediate Value Theorem states that for a continuous function f, if x takes on all values between a and b, $f(x)$ must take on all values between $f(a)$ and $f(b)$. ■

As a simple example of the application of this theorem, consider a person's height. Suppose that a girl is 5 feet tall on her thirteenth birthday and 5 feet 7 inches tall on her fourteenth birthday. Then, for any height h between 5 feet and 5 feet 7 inches, there must have been a time t when her height was exactly h. This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.

The Intermediate Value Theorem guarantees the existence of *at least one* number c in the closed interval $[a, b]$. There may, of course, be more than one number c such that $f(c) = k$, as shown in Figure 1.35. A function that is not continuous does not necessarily exhibit the intermediate value property. For example, the graph of the function shown in Figure 1.36 jumps over the horizontal line given by $y = k$, and for this function there is no value of c in $[a, b]$ such that $f(c) = k$.

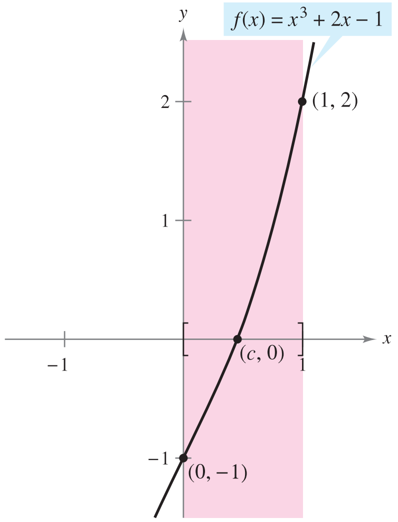


f is continuous on $[a, b]$.
[There exist three c's such that $f(c) = k$.]
Figure 1.35



f is not continuous on $[a, b]$.
[There are no c's such that $f(c) = k$.]
Figure 1.36

The Intermediate Value Theorem often can be used to locate the zeros of a function that is continuous on a closed interval. Specifically, if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, the Intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval $[a, b]$.



f is continuous on $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$.

Figure 1.37

EXAMPLE 8 An Application of the Intermediate Value Theorem

Use the Intermediate Value Theorem to show that the polynomial function $f(x) = x^3 + 2x - 1$ has a zero in the interval $[0, 1]$.

Solution Note that f is continuous on the closed interval $[0, 1]$. Because

$$f(0) = 0^3 + 2(0) - 1 = -1 \quad \text{and} \quad f(1) = 1^3 + 2(1) - 1 = 2$$

it follows that $f(0) < 0$ and $f(1) > 0$. You can therefore apply the Intermediate Value Theorem to conclude that there must be some c in $[0, 1]$ such that

$$f(c) = 0 \qquad f \text{ has a zero in the closed interval } [0, 1].$$

as shown in Figure 1.37. ■

The **bisection method** for approximating the real zeros of a continuous function is similar to the method used in Example 8. If you know that a zero exists in the closed interval $[a, b]$, the zero must lie in the interval $[a, (a + b)/2]$ or $[(a + b)/2, b]$. From the sign of $f([a + b]/2)$, you can determine which interval contains the zero. By repeatedly bisecting the interval, you can "close in" on the zero of the function.

TECHNOLOGY You can also use the *zoom* feature of a graphing utility to approximate the real zeros of a continuous function. By repeatedly zooming in on the point where the graph crosses the x-axis, and adjusting the x-axis scale, you can approximate the zero of the function to any desired accuracy. The zero of $x^3 + 2x - 1$ is approximately 0.453, as shown in Figure 1.38.

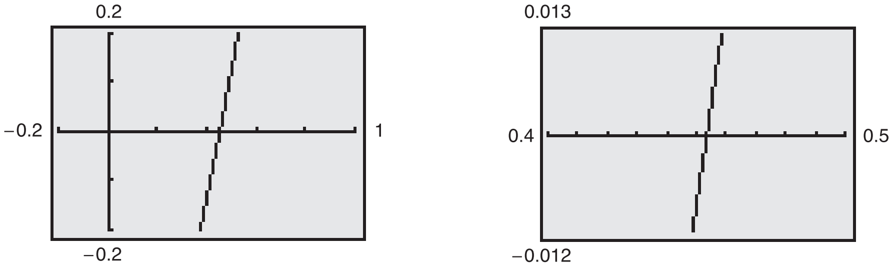


Figure 1.38 Zooming in on the zero of $f(x) = x^3 + 2x - 1$

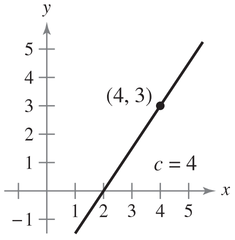
1.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, use the graph to determine the limit, and discuss the continuity of the function.

(a) $\lim\limits_{x\to c^+} f(x)$ **(b)** $\lim\limits_{x\to c^-} f(x)$ **(c)** $\lim\limits_{x\to c} f(x)$

1.



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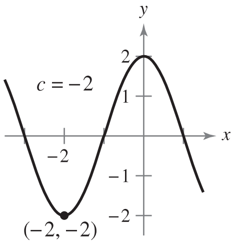

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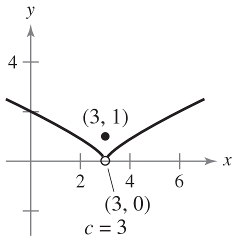

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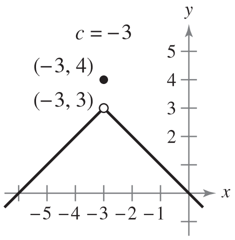

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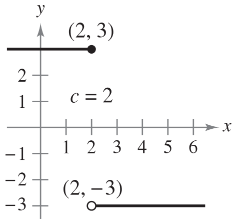
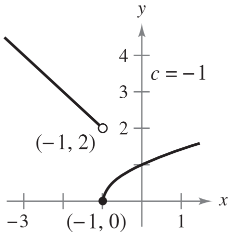


6.



In Exercises 7–26, find the limit (if it exists). If it does not exist, explain why.

7. $\lim\limits_{x\to 8^+} \dfrac{1}{x+8}$

8. $\lim\limits_{x\to 5^-} -\dfrac{3}{x+5}$

9. $\lim\limits_{x\to 5^+} \dfrac{x-5}{x^2-25}$

10. $\lim\limits_{x\to 2^+} \dfrac{2-x}{x^2-4}$

11. $\lim\limits_{x\to -3^-} \dfrac{x}{\sqrt{x^2-9}}$

12. $\lim\limits_{x\to 9^-} \dfrac{\sqrt{x}-3}{x-9}$

13. $\lim\limits_{x\to 0^-} \dfrac{|x|}{x}$

14. $\lim\limits_{x\to 10^+} \dfrac{|x-10|}{x-10}$

15. $\lim\limits_{\Delta x\to 0^-} \dfrac{\dfrac{1}{x+\Delta x}-\dfrac{1}{x}}{\Delta x}$

16. $\lim\limits_{\Delta x\to 0^+} \dfrac{(x+\Delta x)^2+x+\Delta x-(x^2+x)}{\Delta x}$

17. $\lim\limits_{x\to 3^-} f(x)$, where $f(x)=\begin{cases}\dfrac{x+2}{2}, & x\le 3\\[4pt] \dfrac{12-2x}{3}, & x>3\end{cases}$

18. $\lim\limits_{x\to 2} f(x)$, where $f(x)=\begin{cases}x^2-4x+6, & x<2\\ -x^2+4x-2, & x\ge 2\end{cases}$

19. $\lim\limits_{x\to 1} f(x)$, where $f(x)=\begin{cases}x^3+1, & x<1\\ x+1, & x\ge 1\end{cases}$

20. $\lim\limits_{x\to 1^+} f(x)$, where $f(x)=\begin{cases}x, & x\le 1\\ 1-x, & x>1\end{cases}$

21. $\lim\limits_{x\to \pi} \cot x$

22. $\lim\limits_{x\to \pi/2} \sec x$

23. $\lim\limits_{x\to 4^-} (5[\![x]\!]-7)$

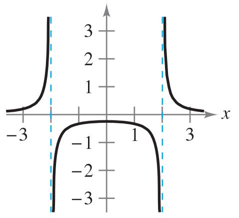
24. $\lim\limits_{x\to 2^+} (2x-[\![x]\!])$

25. $\lim\limits_{x\to 3} (2-[\![-x]\!])$

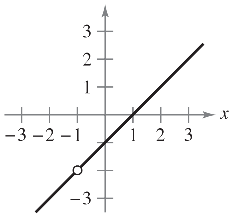
26. $\lim\limits_{x\to 1} \left(1-\left[\!\!\left[-\dfrac{x}{2}\right]\!\!\right]\right)$

In Exercises 27–30, discuss the continuity of each function.

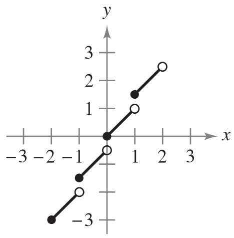
27. $f(x)=\dfrac{1}{x^2-4}$



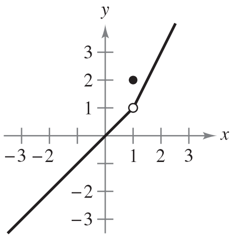
28. $f(x)=\dfrac{x^2-1}{x+1}$



29. $f(x)=\frac{1}{2}[\![x]\!]+x$



30. $f(x)=\begin{cases}x, & x<1\\ 2, & x=1\\ 2x-1, & x>1\end{cases}$



In Exercises 31–34, discuss the continuity of the function on the closed interval.

	Function	*Interval*
31.	$g(x)=\sqrt{49-x^2}$	$[-7, 7]$
32.	$f(t)=3-\sqrt{9-t^2}$	$[-3, 3]$
33.	$f(x)=\begin{cases}3-x, & x\le 0\\ 3+\frac{1}{2}x, & x>0\end{cases}$	$[-1, 4]$
34.	$g(x)=\dfrac{1}{x^2-4}$	$[-1, 2]$

In Exercises 35–60, find the *x*-values (if any) at which *f* is not continuous. Which of the discontinuities are removable?

35. $f(x)=\dfrac{6}{x}$

36. $f(x)=\dfrac{3}{x-2}$

37. $f(x)=x^2-9$

38. $f(x)=x^2-2x+1$

39. $f(x)=\dfrac{1}{4-x^2}$

40. $f(x)=\dfrac{1}{x^2+1}$

41. $f(x)=3x-\cos x$

42. $f(x)=\cos\dfrac{\pi x}{2}$

43. $f(x)=\dfrac{x}{x^2-x}$

44. $f(x)=\dfrac{x}{x^2-1}$

45. $f(x)=\dfrac{x}{x^2+1}$

46. $f(x)=\dfrac{x-6}{x^2-36}$

47. $f(x)=\dfrac{x+2}{x^2-3x-10}$

48. $f(x)=\dfrac{x-1}{x^2+x-2}$

49. $f(x)=\dfrac{|x+7|}{x+7}$

50. $f(x)=\dfrac{|x-8|}{x-8}$

51. $f(x)=\begin{cases}x, & x\le 1\\ x^2, & x>1\end{cases}$

52. $f(x)=\begin{cases}-2x+3, & x<1\\ x^2, & x\ge 1\end{cases}$

53. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

54. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

55. $f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

56. $f(x) = \begin{cases} \csc \frac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

57. $f(x) = \csc 2x$

58. $f(x) = \tan \frac{\pi x}{2}$

59. $f(x) = [\![x - 8]\!]$

60. $f(x) = 5 - [\![x]\!]$

In Exercises 61 and 62, use a graphing utility to graph the function. From the graph, estimate

$$\lim_{x\to 0^+} f(x) \quad \text{and} \quad \lim_{x\to 0^-} f(x).$$

Is the function continuous on the entire real line? Explain.

61. $f(x) = \dfrac{|x^2 - 4|x}{x + 2}$

62. $f(x) = \dfrac{|x^2 + 4x|(x + 2)}{x + 4}$

In Exercises 63–68, find the constant a, or the constants a and b, such that the function is continuous on the entire real line.

63. $f(x) = \begin{cases} 3x^2, & x \ge 1 \\ ax - 4, & x < 1 \end{cases}$

64. $f(x) = \begin{cases} 3x^3, & x \le 1 \\ ax + 5, & x > 1 \end{cases}$

65. $f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$

66. $g(x) = \begin{cases} \dfrac{4 \sin x}{x}, & x < 0 \\ a - 2x, & x \ge 0 \end{cases}$

67. $f(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

68. $g(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, & x \ne a \\ 8, & x = a \end{cases}$

In Exercises 69–72, discuss the continuity of the composite function $h(x) = f(g(x))$.

69. $f(x) = x^2$

$g(x) = x - 1$

70. $f(x) = \dfrac{1}{\sqrt{x}}$

$g(x) = x - 1$

71. $f(x) = \dfrac{1}{x - 6}$

$g(x) = x^2 + 5$

72. $f(x) = \sin x$

$g(x) = x^2$

In Exercises 73–76, use a graphing utility to graph the function. Use the graph to determine any x-values at which the function is not continuous.

73. $f(x) = [\![x]\!] - x$

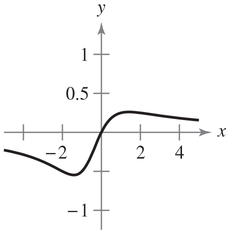
74. $h(x) = \dfrac{1}{x^2 - x - 2}$

75. $g(x) = \begin{cases} x^2 - 3x, & x > 4 \\ 2x - 5, & x \le 4 \end{cases}$

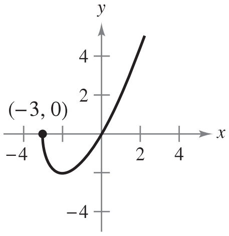
76. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

In Exercises 77–80, describe the interval(s) on which the function is continuous.

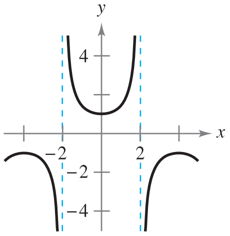
77. $f(x) = \dfrac{x}{x^2 + x + 2}$



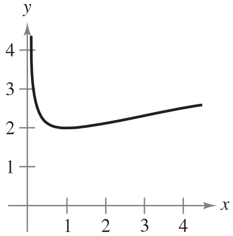
78. $f(x) = x\sqrt{x + 3}$



79. $f(x) = \sec \dfrac{\pi x}{4}$



80. $f(x) = \dfrac{x + 1}{\sqrt{x}}$



Writing **In Exercises 81 and 82, use a graphing utility to graph the function on the interval $[-4, 4]$. Does the graph of the function appear to be continuous on this interval? Is the function continuous on $[-4, 4]$? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

81. $f(x) = \dfrac{\sin x}{x}$

82. $f(x) = \dfrac{x^3 - 8}{x - 2}$

Writing **In Exercises 83–86, explain why the function has a zero in the given interval.**

	Function	*Interval*
83.	$f(x) = \frac{1}{12}x^4 - x^3 + 4$	$[1, 2]$
84.	$f(x) = x^3 + 5x - 3$	$[0, 1]$
85.	$f(x) = x^2 - 2 - \cos x$	$[0, \pi]$
86.	$f(x) = -\dfrac{5}{x} + \tan\left(\dfrac{\pi x}{10}\right)$	$[1, 4]$

In Exercises 87–90, use the Intermediate Value Theorem and a graphing utility to approximate the zero of the function in the interval [0, 1]. Repeatedly “zoom in” on the graph of the function to approximate the zero accurate to two decimal places. Use the *zero* or *root* feature of the graphing utility to approximate the zero accurate to four decimal places.

87. $f(x) = x^3 + x - 1$

88. $f(x) = x^3 + 5x - 3$

89. $g(t) = 2\cos t - 3t$

90. $h(\theta) = 1 + \theta - 3\tan\theta$

In Exercises 91–94, verify that the Intermediate Value Theorem applies to the indicated interval and find the value of *c* guaranteed by the theorem.

91. $f(x) = x^2 + x - 1, \quad [0, 5], \quad f(c) = 11$

92. $f(x) = x^2 - 6x + 8, \quad [0, 3], \quad f(c) = 0$

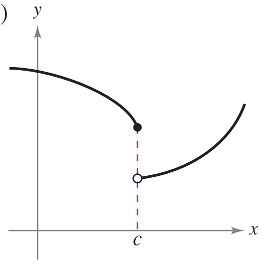
93. $f(x) = x^3 - x^2 + x - 2, \quad [0, 3], \quad f(c) = 4$

94. $f(x) = \dfrac{x^2 + x}{x - 1}, \quad \left[\dfrac{5}{2}, 4\right], \quad f(c) = 6$

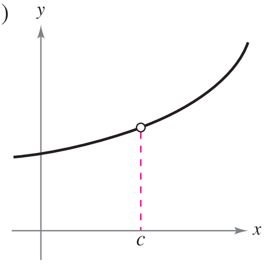
WRITING ABOUT CONCEPTS

95. State how continuity is destroyed at $x = c$ for each of the following graphs.

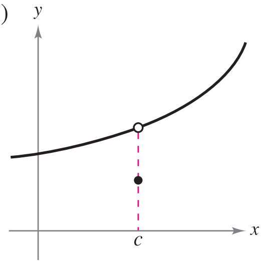
(a)



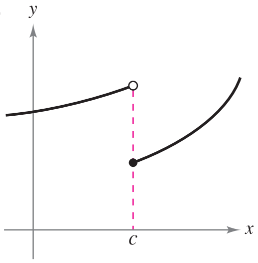
(b)



(c)



(d)



96. Sketch the graph of any function f such that

$$\lim_{x\to 3^+} f(x) = 1 \quad \text{and} \quad \lim_{x\to 3^-} f(x) = 0.$$

Is the function continuous at $x = 3$? Explain.

97. If the functions f and g are continuous for all real x, is $f + g$ always continuous for all real x? Is f/g always continuous for all real x? If either is not continuous, give an example to verify your conclusion.

CAPSTONE

98. Describe the difference between a discontinuity that is removable and one that is nonremovable. In your explanation, give examples of the following descriptions.

(a) A function with a nonremovable discontinuity at $x = 4$

(b) A function with a removable discontinuity at $x = -4$

(c) A function that has both of the characteristics described in parts (a) and (b)

True or False? **In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

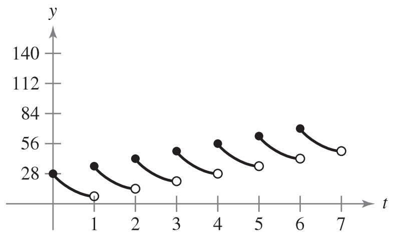
99. If $\lim_{x\to c} f(x) = L$ and $f(c) = L$, then f is continuous at c.

100. If $f(x) = g(x)$ for $x \neq c$ and $f(c) \neq g(c)$, then either f or g is not continuous at c.

101. A rational function can have infinitely many x-values at which it is not continuous.

102. The function $f(x) = |x - 1|/(x - 1)$ is continuous on $(-\infty, \infty)$.

103. ***Swimming Pool*** Every day you dissolve 28 ounces of chlorine in a swimming pool. The graph shows the amount of chlorine $f(t)$ in the pool after t days.



Estimate and interpret $\lim_{t\to 4^-} f(t)$ and $\lim_{t\to 4^+} f(t)$.

104. ***Think About It*** Describe how the functions

$$f(x) = 3 + [\![x]\!]$$

and

$$g(x) = 3 - [\![-x]\!]$$

differ.

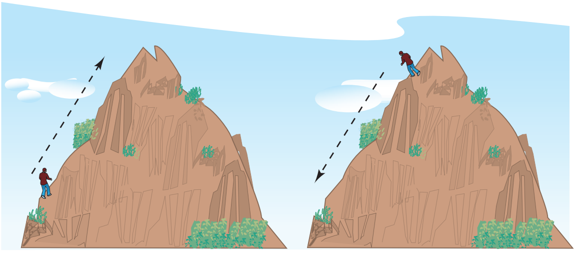
105. ***Telephone Charges*** A long distance phone service charges \$0.40 for the first 10 minutes and \$0.05 for each additional minute or fraction thereof. Use the greatest integer function to write the cost C of a call in terms of time t (in minutes). Sketch the graph of this function and discuss its continuity.

106. ***Inventory Management*** The number of units in inventory in a small company is given by

$$N(t) = 25\left(2\left[\!\left[\frac{t+2}{2}\right]\!\right] - t\right)$$

where t is the time in months. Sketch the graph of this function and discuss its continuity. How often must this company replenish its inventory?

107. ***Déjà Vu*** At 8:00 A.M. on Saturday a man begins running up the side of a mountain to his weekend campsite (see figure). On Sunday morning at 8:00 A.M. he runs back down the mountain. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes that he passed the same place at exactly the same time on Saturday. Prove that he is correct. [*Hint:* Let $s(t)$ and $r(t)$ be the position functions for the runs up and down, and apply the Intermediate Value Theorem to the function $f(t) = s(t) - r(t)$.]



Not drawn to scale

Saturday 8:00 A.M. Sunday 8:00 A.M.

108. ***Volume*** Use the Intermediate Value Theorem to show that for all spheres with radii in the interval $[5, 8]$, there is one with a volume of 1500 cubic centimeters.

109. Prove that if f is continuous and has no zeros on $[a, b]$, then either

$f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

110. Show that the Dirichlet function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

is not continuous at any real number.

111. Show that the function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ kx, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$. (Assume that k is any nonzero real number.)

112. The **signum function** is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

Sketch a graph of $\operatorname{sgn}(x)$ and find the following (if possible).

(a) $\lim_{x\to 0^-} \operatorname{sgn}(x)$ (b) $\lim_{x\to 0^+} \operatorname{sgn}(x)$ (c) $\lim_{x\to 0} \operatorname{sgn}(x)$

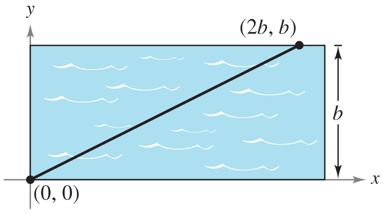
113. ***Modeling Data*** The table lists the speeds S (in feet per second) of a falling object at various times t (in seconds).

t	0	5	10	15	20	25	30
S	0	48.2	53.5	55.2	55.9	56.2	56.3

(a) Create a line graph of the data.

(b) Does there appear to be a limiting speed of the object? If there is a limiting speed, identify a possible cause.

114. ***Creating Models*** A swimmer crosses a pool of width b by swimming in a straight line from $(0, 0)$ to $(2b, b)$. (See figure.)



(a) Let f be a function defined as the y-coordinate of the point on the long side of the pool that is nearest the swimmer at any given time during the swimmer's crossing of the pool. Determine the function f and sketch its graph. Is f continuous? Explain.

(b) Let g be the minimum distance between the swimmer and the long sides of the pool. Determine the function g and sketch its graph. Is g continuous? Explain.

115. Find all values of c such that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$$

116. Prove that for any real number y there exists x in $(-\pi/2, \pi/2)$ such that $\tan x = y$.

117. Let $f(x) = \left(\sqrt{x + c^2} - c\right)/x$, $c > 0$. What is the domain of f? How can you define f at $x = 0$ in order for f to be continuous there?

118. Prove that if $\lim_{\Delta x\to 0} f(c + \Delta x) = f(c)$, then f is continuous at c.

119. Discuss the continuity of the function $h(x) = x[\![x]\!]$.

120. (a) Let $f_1(x)$ and $f_2(x)$ be continuous on the closed interval $[a, b]$. If $f_1(a) < f_2(a)$ and $f_1(b) > f_2(b)$, prove that there exists c between a and b such that $f_1(c) = f_2(c)$.

(b) Show that there exists c in $\left[0, \frac{\pi}{2}\right]$ such that $\cos x = x$. Use a graphing utility to approximate c to three decimal places.

PUTNAM EXAM CHALLENGE

121. Prove or disprove: if x and y are real numbers with $y \ge 0$ and $y(y + 1) \le (x + 1)^2$, then $y(y - 1) \le x^2$.

122. Determine all polynomials $P(x)$ such that

$P(x^2 + 1) = (P(x))^2 + 1$ and $P(0) = 0$.

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1.5 Infinite Limits

- **Determine infinite limits from the left and from the right.**
- **Find and sketch the vertical asymptotes of the graph of a function.**

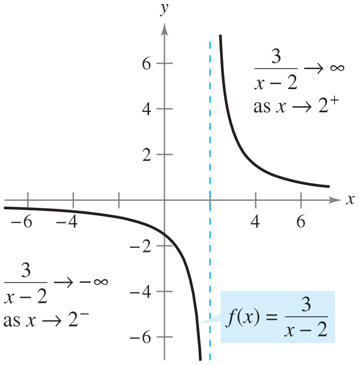
Infinite Limits

Let f be the function given by $3/(x-2)$. From Figure 1.39 and the table, you can see that $f(x)$ *decreases without bound* as x approaches 2 from the left, and $f(x)$ *increases without bound* as x approaches 2 from the right. This behavior is denoted as

$$\lim_{x \to 2^-} \frac{3}{x-2} = -\infty \qquad f(x) \text{ decreases without bound as } x \text{ approaches 2 from the left.}$$

and

$$\lim_{x \to 2^+} \frac{3}{x-2} = \infty \qquad f(x) \text{ increases without bound as } x \text{ approaches 2 from the right.}$$



$f(x)$ increases and decreases without bound as x approaches 2.
Figure 1.39

x approaches 2 from the left. → ← x approaches 2 from the right.

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	−6	−30	−300	−3000	?	3000	300	30	6

$f(x)$ decreases without bound. → ← $f(x)$ increases without bound.

A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an **infinite limit.**

DEFINITION OF INFINITE LIMITS

Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

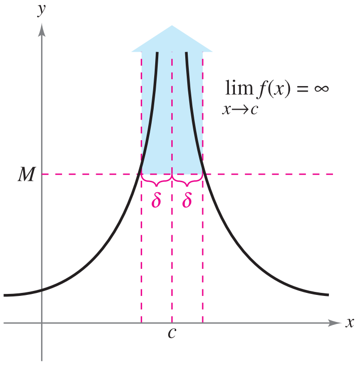
$$\lim_{x \to c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$ (see Figure 1.40). Similarly, the statement

$$\lim_{x \to c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - c| < \delta$.

To define the **infinite limit from the left,** replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right,** replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.



Infinite limits
Figure 1.40

Be sure you see that the equal sign in the statement $\lim f(x) = \infty$ does not mean that the limit exists! On the contrary, it tells you how the limit *fails to exist* by denoting the unbounded behavior of $f(x)$ as x approaches c.

EXPLORATION

Use a graphing utility to graph each function. For each function, analytically find the single real number c that is not in the domain. Then graphically find the limit (if it exists) of $f(x)$ as x approaches c from the left and from the right.

a. $f(x) = \dfrac{3}{x - 4}$

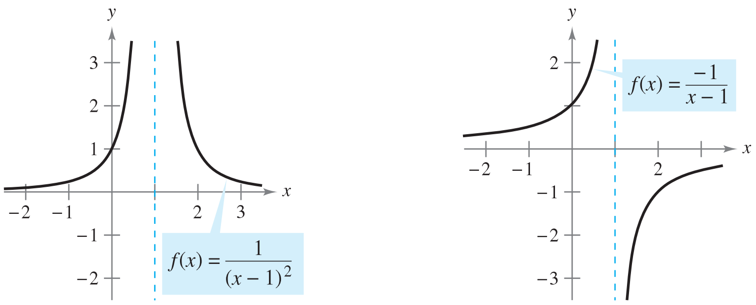
b. $f(x) = \dfrac{1}{2 - x}$

c. $f(x) = \dfrac{2}{(x - 3)^2}$

d. $f(x) = \dfrac{-3}{(x + 2)^2}$

EXAMPLE 1 Determining Infinite Limits from a Graph

Determine the limit of each function shown in Figure 1.41 as x approaches 1 from the left and from the right.



(a) **(b)**

Each graph has an asymptote at $x = 1$.

Figure 1.41

Solution

a. When x approaches 1 from the left or the right, $(x - 1)^2$ is a small positive number. Thus, the quotient $1/(x - 1)^2$ is a large positive number and $f(x)$ approaches infinity from each side of $x = 1$. So, you can conclude that

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty. \qquad \text{Limit from each side is infinity.}$$

Figure 1.41(a) confirms this analysis.

b. When x approaches 1 from the left, $x - 1$ is a small negative number. Thus, the quotient $-1/(x - 1)$ is a large positive number and $f(x)$ approaches infinity from the left of $x = 1$. So, you can conclude that

$$\lim_{x\to 1^-} \frac{-1}{x-1} = \infty. \qquad \text{Limit from the left side is infinity.}$$

When x approaches 1 from the right, $x - 1$ is a small positive number. Thus, the quotient $-1/(x - 1)$ is a large negative number and $f(x)$ approaches negative infinity from the right of $x = 1$. So, you can conclude that

$$\lim_{x\to 1^+} \frac{-1}{x-1} = -\infty. \qquad \text{Limit from the right side is negative infinity.}$$

Figure 1.41(b) confirms this analysis. ■

Vertical Asymptotes

If it were possible to extend the graphs in Figure 1.41 toward positive and negative infinity, you would see that each graph becomes arbitrarily close to the vertical line $x = 1$. This line is a **vertical asymptote** of the graph of f. (You will study other types of asymptotes in Sections 3.5 and 3.6.)

NOTE If the graph of a function f has a vertical asymptote at $x = c$, then f is *not continuous* at c.

DEFINITION OF VERTICAL ASYMPTOTE

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f.

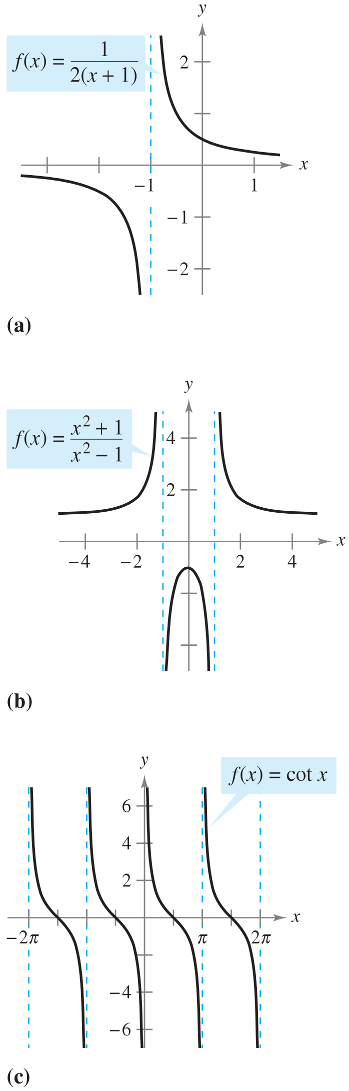
In Example 1, note that each of the functions is a *quotient* and that the vertical asymptote occurs at a number at which the denominator is 0 (and the numerator is not 0). The next theorem generalizes this observation. (A proof of this theorem is given in Appendix A.)

THEOREM 1.14 VERTICAL ASYMPTOTES

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.



Functions with vertical asymptotes

Figure 1.42

EXAMPLE 2 Finding Vertical Asymptotes

Determine all vertical asymptotes of the graph of each function.

a. $f(x) = \dfrac{1}{2(x+1)}$ **b.** $f(x) = \dfrac{x^2+1}{x^2-1}$ **c.** $f(x) = \cot x$

Solution

a. When $x = -1$, the denominator of

$$f(x) = \frac{1}{2(x+1)}$$

is 0 and the numerator is not 0. So, by Theorem 1.14, you can conclude that $x = -1$ is a vertical asymptote, as shown in Figure 1.42(a).

b. By factoring the denominator as

$$f(x) = \frac{x^2+1}{x^2-1} = \frac{x^2+1}{(x-1)(x+1)}$$

you can see that the denominator is 0 at $x = -1$ and $x = 1$. Moreover, because the numerator is not 0 at these two points, you can apply Theorem 1.14 to conclude that the graph of f has two vertical asymptotes, as shown in Figure 1.42(b).

c. By writing the cotangent function in the form

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

you can apply Theorem 1.14 to conclude that vertical asymptotes occur at all values of x such that $\sin x = 0$ and $\cos x \neq 0$, as shown in Figure 1.42(c). So, the graph of this function has infinitely many vertical asymptotes. These asymptotes occur at $x = n\pi$, where n is an integer. ■

Theorem 1.14 requires that the value of the numerator at $x = c$ be nonzero. If both the numerator and the denominator are 0 at $x = c$, you obtain the *indeterminate form* $0/0$, and you cannot determine the limit behavior at $x = c$ without further investigation, as illustrated in Example 3.

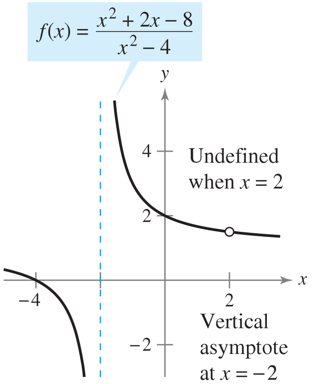
EXAMPLE 3 A Rational Function with Common Factors

Determine all vertical asymptotes of the graph of

$$f(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$

Solution Begin by simplifying the expression, as shown.

$$\begin{aligned} f(x) &= \frac{x^2 + 2x - 8}{x^2 - 4} \\ &= \frac{(x+4)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} \\ &= \frac{x+4}{x+2}, \quad x \neq 2 \end{aligned}$$



$f(x)$ increases and decreases without bound as x approaches -2.
Figure 1.43

At all x-values other than $x = 2$, the graph of f coincides with the graph of $g(x) = (x + 4)/(x + 2)$. So, you can apply Theorem 1.14 to g to conclude that there is a vertical asymptote at $x = -2$, as shown in Figure 1.43. From the graph, you can see that

$$\lim_{x \to -2^-} \frac{x^2 + 2x - 8}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x \to -2^+} \frac{x^2 + 2x - 8}{x^2 - 4} = \infty.$$

Note that $x = 2$ is *not* a vertical asymptote.

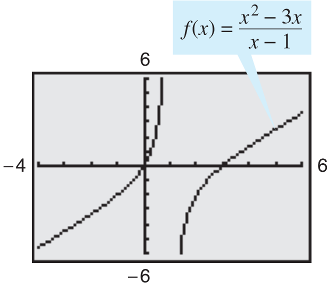
EXAMPLE 4 Determining Infinite Limits

Find each limit.

$$\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1} \quad \text{and} \quad \lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1}$$

Solution Because the denominator is 0 when $x = 1$ (and the numerator is not zero), you know that the graph of

$$f(x) = \frac{x^2 - 3x}{x - 1}$$



f has a vertical asymptote at $x = 1$.
Figure 1.44

has a vertical asymptote at $x = 1$. This means that each of the given limits is either ∞ or $-\infty$. You can determine the result by analyzing f at values of x close to 1, or by using a graphing utility. From the graph of f shown in Figure 1.44, you can see that the graph approaches ∞ from the left of $x = 1$ and approaches $-\infty$ from the right of $x = 1$. So, you can conclude that

$$\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1} = \infty$$ The limit from the left is infinity.

and

$$\lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1} = -\infty.$$ The limit from the right is negative infinity. ■

TECHNOLOGY PITFALL When using a graphing calculator or graphing software, be careful to interpret correctly the graph of a function with a vertical asymptote—graphing utilities often have difficulty drawing this type of graph.

THEOREM 1.15 PROPERTIES OF INFINITE LIMITS

Let c and L be real numbers and let f and g be functions such that

$$\lim_{x\to c} f(x) = \infty \quad \text{and} \quad \lim_{x\to c} g(x) = L.$$

1. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = \infty$
2. Product: $\lim_{x\to c} [f(x)g(x)] = \infty, \quad L > 0$
   $\lim_{x\to c} [f(x)g(x)] = -\infty, \quad L < 0$
3. Quotient: $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$.

PROOF To show that the limit of $f(x) + g(x)$ is infinite, choose $M > 0$. You then need to find $\delta > 0$ such that

$$[f(x) + g(x)] > M$$

whenever $0 < |x - c| < \delta$. For simplicity's sake, you can assume L is positive. Let $M_1 = M + 1$. Because the limit of $f(x)$ is infinite, there exists δ_1 such that $f(x) > M_1$ whenever $0 < |x - c| < \delta_1$. Also, because the limit of $g(x)$ is L, there exists δ_2 such that $|g(x) - L| < 1$ whenever $0 < |x - c| < \delta_2$. By letting δ be the smaller of δ_1 and δ_2, you can conclude that $0 < |x - c| < \delta$ implies $f(x) > M + 1$ and $|g(x) - L| < 1$. The second of these two inequalities implies that $g(x) > L - 1$, and, adding this to the first inequality, you can write

$$f(x) + g(x) > (M + 1) + (L - 1) = M + L > M.$$

So, you can conclude that

$$\lim_{x\to c} [f(x) + g(x)] = \infty.$$

The proofs of the remaining properties are left as exercises (see Exercise 78). ■

EXAMPLE 5 Determining Limits

a. Because $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$, you can write

$$\lim_{x\to 0}\left(1 + \frac{1}{x^2}\right) = \infty.$$

Property 1, Theorem 1.15

b. Because $\lim_{x\to 1^-} (x^2 + 1) = 2$ and $\lim_{x\to 1^-} (\cot \pi x) = -\infty$, you can write

$$\lim_{x\to 1^-} \frac{x^2 + 1}{\cot \pi x} = 0.$$

Property 3, Theorem 1.15

c. Because $\lim_{x\to 0^+} 3 = 3$ and $\lim_{x\to 0^+} \cot x = \infty$, you can write

$$\lim_{x\to 0^+} 3 \cot x = \infty.$$

Property 2, Theorem 1.15 ■

1.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches 4 from the left and from the right.

1. $f(x) = \dfrac{1}{x-4}$

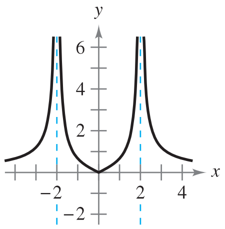
2. $f(x) = \dfrac{-1}{x-4}$

3. $f(x) = \dfrac{1}{(x-4)^2}$

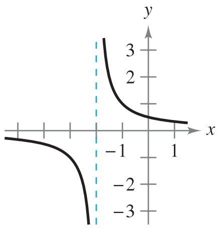
4. $f(x) = \dfrac{-1}{(x-4)^2}$

In Exercises 5–8, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -2 from the left and from the right.

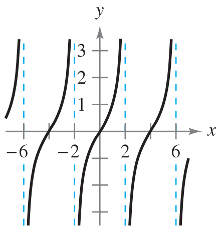
5. $f(x) = 2\left|\dfrac{x}{x^2-4}\right|$



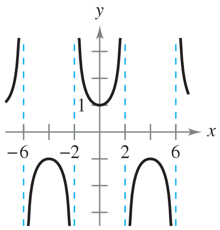
6. $f(x) = \dfrac{1}{x+2}$



7. $f(x) = \tan\dfrac{\pi x}{4}$



8. $f(x) = \sec\dfrac{\pi x}{4}$



Numerical and Graphical Analysis **In Exercises 9–12, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -3 from the left and from the right by completing the table. Use a graphing utility to graph the function to confirm your answer.**

x	-3.5	-3.1	-3.01	-3.001
$f(x)$				

x	-2.999	-2.99	-2.9	-2.5
$f(x)$				

9. $f(x) = \dfrac{1}{x^2-9}$

10. $f(x) = \dfrac{x}{x^2-9}$

11. $f(x) = \dfrac{x^2}{x^2-9}$

12. $f(x) = \sec\dfrac{\pi x}{6}$

In Exercises 13–32, find the vertical asymptotes (if any) of the graph of the function.

13. $f(x) = \dfrac{1}{x^2}$

14. $f(x) = \dfrac{4}{(x-2)^3}$

15. $f(x) = \dfrac{x^2}{x^2-4}$

16. $f(x) = \dfrac{-4x}{x^2+4}$

17. $g(t) = \dfrac{t-1}{t^2+1}$

18. $h(s) = \dfrac{2s-3}{s^2-25}$

19. $h(x) = \dfrac{x^2-2}{x^2-x-2}$

20. $g(x) = \dfrac{2+x}{x^2(1-x)}$

21. $T(t) = 1 - \dfrac{4}{t^2}$

22. $g(x) = \dfrac{\frac{1}{2}x^3 - x^2 - 4x}{3x^2-6x-24}$

23. $f(x) = \dfrac{3}{x^2+x-2}$

24. $f(x) = \dfrac{4x^2+4x-24}{x^4-2x^3-9x^2+18x}$

25. $g(x) = \dfrac{x^3+1}{x+1}$

26. $h(x) = \dfrac{x^2-4}{x^3+2x^2+x+2}$

27. $f(x) = \dfrac{x^2-2x-15}{x^3-5x^2+x-5}$

28. $h(t) = \dfrac{t^2-2t}{t^4-16}$

29. $f(x) = \tan \pi x$

30. $f(x) = \sec \pi x$

31. $s(t) = \dfrac{t}{\sin t}$

32. $g(\theta) = \dfrac{\tan\theta}{\theta}$

In Exercises 33–36, determine whether the graph of the function has a vertical asymptote or a removable discontinuity at $x = -1$. Graph the function using a graphing utility to confirm your answer.

33. $f(x) = \dfrac{x^2-1}{x+1}$

34. $f(x) = \dfrac{x^2-6x-7}{x+1}$

35. $f(x) = \dfrac{x^2+1}{x+1}$

36. $f(x) = \dfrac{\sin(x+1)}{x+1}$

In Exercises 37–54, find the limit (if it exists).

37. $\displaystyle\lim_{x\to -1^+} \frac{1}{x+1}$

38. $\displaystyle\lim_{x\to 1^-} \frac{-1}{(x-1)^2}$

39. $\displaystyle\lim_{x\to 2^+} \frac{x}{x-2}$

40. $\displaystyle\lim_{x\to 1^+} \frac{2+x}{1-x}$

41. $\displaystyle\lim_{x\to 1^+} \frac{x^2}{(x-1)^2}$

42. $\displaystyle\lim_{x\to 4^-} \frac{x^2}{x^2+16}$

43. $\displaystyle\lim_{x\to -3^-} \frac{x+3}{x^2+x-6}$

44. $\displaystyle\lim_{x\to (-1/2)^+} \frac{6x^2+x-1}{4x^2-4x-3}$

45. $\displaystyle\lim_{x\to 1} \frac{x-1}{(x^2+1)(x-1)}$

46. $\displaystyle\lim_{x\to 3} \frac{x-2}{x^2}$

47. $\displaystyle\lim_{x\to 0^-} \left(1 + \frac{1}{x}\right)$

48. $\displaystyle\lim_{x\to 0^-} \left(x^2 - \frac{1}{x}\right)$

49. $\displaystyle\lim_{x\to 0^+} \frac{2}{\sin x}$

50. $\displaystyle\lim_{x\to (\pi/2)^+} \frac{-2}{\cos x}$

51. $\displaystyle\lim_{x\to \pi} \frac{\sqrt{x}}{\csc x}$

52. $\displaystyle\lim_{x\to 0} \frac{x+2}{\cot x}$

53. $\displaystyle\lim_{x\to 1/2} x \sec \pi x$

54. $\displaystyle\lim_{x\to 1/2} x^2 \tan \pi x$

In Exercises 55–58, use a graphing utility to graph the function and determine the one-sided limit.

55. $f(x) = \dfrac{x^2 + x + 1}{x^3 - 1}$

$\lim\limits_{x \to 1^+} f(x)$

56. $f(x) = \dfrac{x^3 - 1}{x^2 + x + 1}$

$\lim\limits_{x \to 1^-} f(x)$

57. $f(x) = \dfrac{1}{x^2 - 25}$

$\lim\limits_{x \to 5^-} f(x)$

58. $f(x) = \sec \dfrac{\pi x}{8}$

$\lim\limits_{x \to 4^+} f(x)$

WRITING ABOUT CONCEPTS

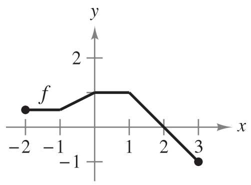
59. In your own words, describe the meaning of an infinite limit. Is ∞ a real number?

60. In your own words, describe what is meant by an asymptote of a graph.

61. Write a rational function with vertical asymptotes at $x = 6$ and $x = -2$, and with a zero at $x = 3$.

62. Does the graph of every rational function have a vertical asymptote? Explain.

63. Use the graph of the function f (see figure) to sketch the graph of $g(x) = 1/f(x)$ on the interval $[-2, 3]$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



CAPSTONE

64. Given a polynomial $p(x)$, is it true that the graph of the function given by $f(x) = \dfrac{p(x)}{x - 1}$ has a vertical asymptote at $x = 1$? Why or why not?

65. *Relativity* According to the theory of relativity, the mass m of a particle depends on its velocity v. That is,

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$$

where m_0 is the mass when the particle is at rest and c is the speed of light. Find the limit of the mass as v approaches c^-.

66. *Boyle's Law* For a quantity of gas at a constant temperature, the pressure P is inversely proportional to the volume V. Find the limit of P as $V \to 0^+$.

67. *Rate of Change* A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of $\frac{1}{2}$ revolution per second. The rate at which the light beam moves along the wall is $r = 50\pi \sec^2 \theta$ ft/sec.

(a) Find the rate r when θ is $\pi/6$.

(b) Find the rate r when θ is $\pi/3$.

(c) Find the limit of r as $\theta \to (\pi/2)^-$.

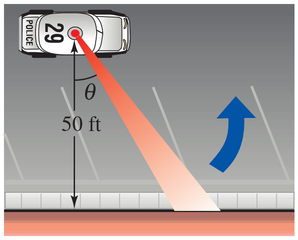


Figure for 67

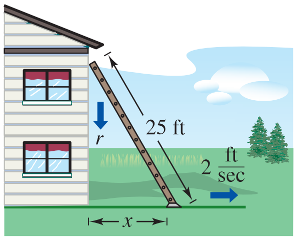


Figure for 68

68. *Rate of Change* A 25-foot ladder is leaning against a house (see figure). If the base of the ladder is pulled away from the house at a rate of 2 feet per second, the top will move down the wall at a rate of

$$r = \frac{2x}{\sqrt{625 - x^2}} \text{ ft/sec}$$

where x is the distance between the base of the ladder and the house.

(a) Find the rate r when x is 7 feet.

(b) Find the rate r when x is 15 feet.

(c) Find the limit of r as $x \to 25^-$.

69. *Average Speed* On a trip of d miles to another city, a truck driver's average speed was x miles per hour. On the return trip the average speed was y miles per hour. The average speed for the round trip was 50 miles per hour.

(a) Verify that $y = \dfrac{25x}{x - 25}$. What is the domain?

(b) Complete the table.

x	30	40	50	60
y				

Are the values of y different than you expected? Explain.

(c) Find the limit of y as $x \to 25^+$ and interpret its meaning.

70. *Numerical and Graphical Analysis* Use a graphing utility to complete the table for each function and graph each function to estimate the limit. What is the value of the limit when the power of x in the denominator is greater than 3?

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$							

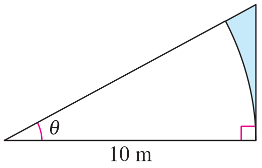
(a) $\lim\limits_{x \to 0^+} \dfrac{x - \sin x}{x}$

(b) $\lim\limits_{x \to 0^+} \dfrac{x - \sin x}{x^2}$

(c) $\lim\limits_{x \to 0^+} \dfrac{x - \sin x}{x^3}$

(d) $\lim\limits_{x \to 0^+} \dfrac{x - \sin x}{x^4}$

71. ***Numerical and Graphical Analysis*** Consider the shaded region outside the sector of a circle of radius 10 meters and inside a right triangle (see figure).



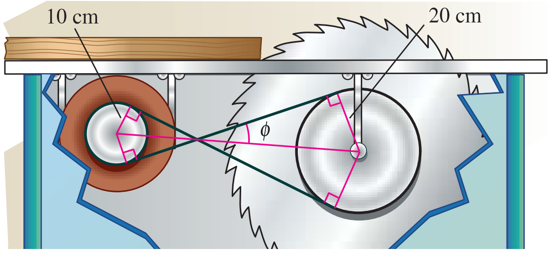
(a) Write the area $A = f(\theta)$ of the region as a function of θ. Determine the domain of the function.

(b) Use a graphing utility to complete the table and graph the function over the appropriate domain.

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$					

(c) Find the limit of A as $\theta \to (\pi/2)^-$.

72. ***Numerical and Graphical Reasoning*** A crossed belt connects a 20-centimeter pulley (10-cm radius) on an electric motor with a 40-centimeter pulley (20-cm radius) on a saw arbor (see figure). The electric motor runs at 1700 revolutions per minute.



(a) Determine the number of revolutions per minute of the saw.

(b) How does crossing the belt affect the saw in relation to the motor?

(c) Let L be the total length of the belt. Write L as a function of ϕ, where ϕ is measured in radians. What is the domain of the function? (*Hint:* Add the lengths of the straight sections of the belt and the length of the belt around each pulley.)

(d) Use a graphing utility to complete the table.

ϕ	0.3	0.6	0.9	1.2	1.5
L					

(e) Use a graphing utility to graph the function over the appropriate domain.

(f) Find $\displaystyle\lim_{\phi\to(\pi/2)^-} L$. Use a geometric argument as the basis of a second method of finding this limit.

(g) Find $\displaystyle\lim_{\phi\to 0^+} L$.

True or False? **In Exercises 73–76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

73. The graph of a rational function has at least one vertical asymptote.

74. The graphs of polynomial functions have no vertical asymptotes.

75. The graphs of trigonometric functions have no vertical asymptotes.

76. If f has a vertical asymptote at $x = 0$, then f is undefined at $x = 0$.

77. Find functions f and g such that $\displaystyle\lim_{x\to c} f(x) = \infty$ and $\displaystyle\lim_{x\to c} g(x) = \infty$ but $\displaystyle\lim_{x\to c} [f(x) - g(x)] \neq 0$.

78. Prove the difference, product, and quotient properties in Theorem 1.15.

79. Prove that if $\displaystyle\lim_{x\to c} f(x) = \infty$, then $\displaystyle\lim_{x\to c} \frac{1}{f(x)} = 0$.

80. Prove that if $\displaystyle\lim_{x\to c} \frac{1}{f(x)} = 0$, then $\displaystyle\lim_{x\to c} f(x)$ does not exist.

Infinite Limits **In Exercises 81 and 82, use the ε-δ definition of infinite limits to prove the statement.**

81. $\displaystyle\lim_{x\to 3^+} \frac{1}{x-3} = \infty$

82. $\displaystyle\lim_{x\to 5^-} \frac{1}{x-5} = -\infty$

SECTION PROJECT

Graphs and Limits of Trigonometric Functions

Recall from Theorem 1.9 that the limit of $f(x) = (\sin x)/x$ as x approaches 0 is 1.

(a) Use a graphing utility to graph the function f on the interval $-\pi \le x \le \pi$. Explain how the graph helps confirm this theorem.

(b) Explain how you could use a table of values to confirm the value of this limit numerically.

(c) Graph $g(x) = \sin x$ by hand. Sketch a tangent line at the point $(0, 0)$ and visually estimate the slope of this tangent line.

(d) Let $(x, \sin x)$ be a point on the graph of g near $(0, 0)$, and write a formula for the slope of the secant line joining $(x, \sin x)$ and $(0, 0)$. Evaluate this formula at $x = 0.1$ and $x = 0.01$. Then find the exact slope of the tangent line to g at the point $(0, 0)$.

(e) Sketch the graph of the cosine function $h(x) = \cos x$. What is the slope of the tangent line at the point $(0, 1)$? Use limits to find this slope analytically.

(f) Find the slope of the tangent line to $k(x) = \tan x$ at $(0, 0)$.

1 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, determine whether the problem can be solved using precalculus or if calculus is required. If the problem can be solved using precalculus, solve it. If the problem seems to require calculus, explain your reasoning. Use a graphical or numerical approach to estimate the solution.

1. Find the distance between the points $(1, 1)$ and $(3, 9)$ along the curve $y = x^2$.
2. Find the distance between the points $(1, 1)$ and $(3, 9)$ along the line $y = 4x - 3$.

In Exercises 3 and 4, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

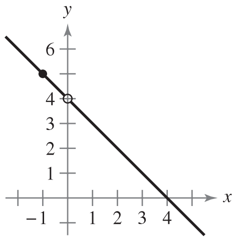
3. $\displaystyle\lim_{x\to0}\frac{[4/(x+2)]-2}{x}$
4. $\displaystyle\lim_{x\to0}\frac{4\left(\sqrt{x+2}-\sqrt{2}\right)}{x}$

In Exercises 5–8, find the limit L. Then use the ε-δ definition to prove that the limit is L.

5. $\displaystyle\lim_{x\to1}(x+4)$
6. $\displaystyle\lim_{x\to9}\sqrt{x}$
7. $\displaystyle\lim_{x\to2}(1-x^2)$
8. $\displaystyle\lim_{x\to5}9$

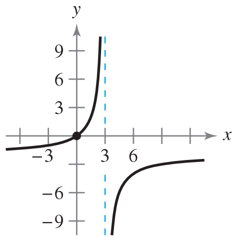
In Exercises 9 and 10, use the graph to determine each limit.

9. $h(x) = \dfrac{4x - x^2}{x}$



(a) $\displaystyle\lim_{x\to0}h(x)$ (b) $\displaystyle\lim_{x\to-1}h(x)$

10. $g(x) = \dfrac{-2x}{x-3}$



(a) $\displaystyle\lim_{x\to3}g(x)$ (b) $\displaystyle\lim_{x\to0}g(x)$

In Exercises 11–26, find the limit (if it exists).

11. $\displaystyle\lim_{x\to6}(x-2)^2$
12. $\displaystyle\lim_{x\to7}(10-x)^4$
13. $\displaystyle\lim_{t\to4}\sqrt{t+2}$
14. $\displaystyle\lim_{y\to4}3|y-1|$
15. $\displaystyle\lim_{t\to-2}\frac{t+2}{t^2-4}$
16. $\displaystyle\lim_{t\to3}\frac{t^2-9}{t-3}$
17. $\displaystyle\lim_{x\to4}\frac{\sqrt{x-3}-1}{x-4}$
18. $\displaystyle\lim_{x\to0}\frac{\sqrt{4+x}-2}{x}$
19. $\displaystyle\lim_{x\to0}\frac{[1/(x+1)]-1}{x}$
20. $\displaystyle\lim_{s\to0}\frac{\left(1/\sqrt{1+s}\right)-1}{s}$
21. $\displaystyle\lim_{x\to-5}\frac{x^3+125}{x+5}$
22. $\displaystyle\lim_{x\to-2}\frac{x^2-4}{x^3+8}$
23. $\displaystyle\lim_{x\to0}\frac{1-\cos x}{\sin x}$
24. $\displaystyle\lim_{x\to\pi/4}\frac{4x}{\tan x}$
25. $\displaystyle\lim_{\Delta x\to0}\frac{\sin[(\pi/6)+\Delta x]-(1/2)}{\Delta x}$

    [*Hint:* $\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi$]
26. $\displaystyle\lim_{\Delta x\to0}\frac{\cos(\pi+\Delta x)+1}{\Delta x}$

    [*Hint:* $\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$]

In Exercises 27–30, evaluate the limit given $\lim_{x\to c}f(x) = -\frac{3}{4}$ and $\lim_{x\to c}g(x) = \frac{2}{3}$.

27. $\displaystyle\lim_{x\to c}[f(x)g(x)]$
28. $\displaystyle\lim_{x\to c}\frac{f(x)}{g(x)}$
29. $\displaystyle\lim_{x\to c}[f(x)+2g(x)]$
30. $\displaystyle\lim_{x\to c}[f(x)]^2$

Numerical, Graphical, and Analytic Analysis **In Exercises 31 and 32, consider**

$$\lim_{x\to1^+}f(x).$$

(a) Complete the table to estimate the limit.

(b) Use a graphing utility to graph the function and use the graph to estimate the limit.

(c) Rationalize the numerator to find the exact value of the limit analytically.

x	1.1	1.01	1.001	1.0001
$f(x)$				

31. $f(x) = \dfrac{\sqrt{2x+1}-\sqrt{3}}{x-1}$
32. $f(x) = \dfrac{1-\sqrt[3]{x}}{x-1}$

    [*Hint:* $a^3-b^3=(a-b)(a^2+ab+b^2)$]

Free-Falling Object **In Exercises 33 and 34, use the position function $s(t) = -4.9t^2 + 250$, which gives the height (in meters) of an object that has fallen from a height of 250 meters. The velocity at time $t = a$ seconds is given by**

$$\lim_{t\to a}\frac{s(a)-s(t)}{a-t}.$$

33. Find the velocity of the object when $t = 4$.
34. At what velocity will the object impact the ground?

In Exercises 35–40, find the limit (if it exists). If the limit does not exist, explain why.

35. $\lim\limits_{x\to 3^-} \dfrac{|x-3|}{x-3}$

36. $\lim\limits_{x\to 4} [\![x-1]\!]$

37. $\lim\limits_{x\to 2} f(x)$, where $f(x) = \begin{cases} (x-2)^2, & x \le 2 \\ 2-x, & x > 2 \end{cases}$

38. $\lim\limits_{x\to 1^+} g(x)$, where $g(x) = \begin{cases} \sqrt{1-x}, & x \le 1 \\ x+1, & x > 1 \end{cases}$

39. $\lim\limits_{t\to 1} h(t)$, where $h(t) = \begin{cases} t^3+1, & t < 1 \\ \frac{1}{2}(t+1), & t \ge 1 \end{cases}$

40. $\lim\limits_{s\to -2} f(s)$, where $f(s) = \begin{cases} -s^2-4s-2, & s \le -2 \\ s^2+4s+6, & s > -2 \end{cases}$

In Exercises 41–52, determine the intervals on which the function is continuous.

41. $f(x) = -3x^2 + 7$

42. $f(x) = x^2 - \dfrac{2}{x}$

43. $f(x) = [\![x+3]\!]$

44. $f(x) = \dfrac{3x^2-x-2}{x-1}$

45. $f(x) = \begin{cases} \dfrac{3x^2-x-2}{x-1}, & x \ne 1 \\ 0, & x = 1 \end{cases}$

46. $f(x) = \begin{cases} 5-x, & x \le 2 \\ 2x-3, & x > 2 \end{cases}$

47. $f(x) = \dfrac{1}{(x-2)^2}$

48. $f(x) = \sqrt{\dfrac{x+1}{x}}$

49. $f(x) = \dfrac{3}{x+1}$

50. $f(x) = \dfrac{x+1}{2x+2}$

51. $f(x) = \csc \dfrac{\pi x}{2}$

52. $f(x) = \tan 2x$

53. Determine the value of c such that the function is continuous on the entire real line.

$$f(x) = \begin{cases} x+3, & x \le 2 \\ cx+6, & x > 2 \end{cases}$$

54. Determine the values of b and c such that the function is continuous on the entire real line.

$$f(x) = \begin{cases} x+1, & 1 < x < 3 \\ x^2+bx+c, & |x-2| \ge 1 \end{cases}$$

55. Use the Intermediate Value Theorem to show that $f(x) = 2x^3 - 3$ has a zero in the interval $[1, 2]$.

56. ***Delivery Charges*** The cost of sending an overnight package from New York to Atlanta is \$12.80 for the first pound and \$2.50 for each additional pound or fraction thereof. Use the greatest integer function to create a model for the cost C of overnight delivery of a package weighing x pounds. Use a graphing utility to graph the function and discuss its continuity.

57. Let $f(x) = \dfrac{x^2-4}{|x-2|}$. Find each limit (if possible).

(a) $\lim\limits_{x\to 2^-} f(x)$

(b) $\lim\limits_{x\to 2^+} f(x)$

(c) $\lim\limits_{x\to 2} f(x)$

58. Let $f(x) = \sqrt{x(x-1)}$.

(a) Find the domain of f.

(b) Find $\lim\limits_{x\to 0^-} f(x)$.

(c) Find $\lim\limits_{x\to 1^+} f(x)$.

In Exercises 59–62, find the vertical asymptotes (if any) of the graph of the function.

59. $g(x) = 1 + \dfrac{2}{x}$

60. $h(x) = \dfrac{4x}{4-x^2}$

61. $f(x) = \dfrac{8}{(x-10)^2}$

62. $f(x) = \csc \pi x$

In Exercises 63–74, find the one-sided limit (if it exists).

63. $\lim\limits_{x\to -2^-} \dfrac{2x^2+x+1}{x+2}$

64. $\lim\limits_{x\to (1/2)^+} \dfrac{x}{2x-1}$

65. $\lim\limits_{x\to -1^+} \dfrac{x+1}{x^3+1}$

66. $\lim\limits_{x\to -1^-} \dfrac{x+1}{x^4-1}$

67. $\lim\limits_{x\to 1^-} \dfrac{x^2+2x+1}{x-1}$

68. $\lim\limits_{x\to -1^+} \dfrac{x^2-2x+1}{x+1}$

69. $\lim\limits_{x\to 0^+} \left(x - \dfrac{1}{x^3}\right)$

70. $\lim\limits_{x\to 2^-} \dfrac{1}{\sqrt[3]{x^2-4}}$

71. $\lim\limits_{x\to 0^+} \dfrac{\sin 4x}{5x}$

72. $\lim\limits_{x\to 0^+} \dfrac{\sec x}{x}$

73. $\lim\limits_{x\to 0^+} \dfrac{\csc 2x}{x}$

74. $\lim\limits_{x\to 0^-} \dfrac{\cos^2 x}{x}$

75. ***Environment*** A utility company burns coal to generate electricity. The cost C in dollars of removing $p\%$ of the air pollutants in the stack emissions is

$$C = \frac{80{,}000p}{100-p}, \quad 0 \le p < 100.$$

Find the costs of removing (a) 15%, (b) 50%, and (c) 90% of the pollutants. (d) Find the limit of C as $p \to 100^-$.

76. The function f is defined as shown.

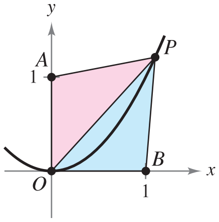
$$f(x) = \frac{\tan 2x}{x}, \quad x \ne 0$$

(a) Find $\lim\limits_{x\to 0} \dfrac{\tan 2x}{x}$ (if it exists).

(b) Can the function f be defined at $x = 0$ such that it is continuous at $x = 0$?

P.S. PROBLEM SOLVING

1. Let $P(x, y)$ be a point on the parabola $y = x^2$ in the first quadrant. Consider the triangle $\triangle PAO$ formed by P, $A(0, 1)$, and the origin $O(0, 0)$, and the triangle $\triangle PBO$ formed by P, $B(1, 0)$, and the origin.



(a) Write the perimeter of each triangle in terms of x.

(b) Let $r(x)$ be the ratio of the perimeters of the two triangles,

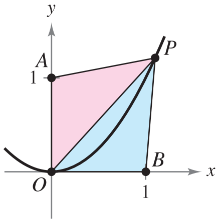
$$r(x) = \frac{\text{Perimeter } \triangle PAO}{\text{Perimeter } \triangle PBO}.$$

Complete the table.

x	4	2	1	0.1	0.01
Perimeter $\triangle PAO$					
Perimeter $\triangle PBO$					
$r(x)$					

(c) Calculate $\lim_{x \to 0^+} r(x)$.

2. Let $P(x, y)$ be a point on the parabola $y = x^2$ in the first quadrant. Consider the triangle $\triangle PAO$ formed by P, $A(0, 1)$, and the origin $O(0, 0)$, and the triangle $\triangle PBO$ formed by P, $B(1, 0)$, and the origin.



(a) Write the area of each triangle in terms of x.

(b) Let $a(x)$ be the ratio of the areas of the two triangles,

$$a(x) = \frac{\text{Area } \triangle PBO}{\text{Area } \triangle PAO}.$$

Complete the table.

x	4	2	1	0.1	0.01
Area $\triangle PAO$					
Area $\triangle PBO$					
$a(x)$					

(c) Calculate $\lim_{x \to 0^+} a(x)$.

3. (a) Find the area of a regular hexagon inscribed in a circle of radius 1. How close is this area to that of the circle?

(b) Find the area A_n of an n-sided regular polygon inscribed in a circle of radius 1. Write your answer as a function of n.

(c) Complete the table.

n	6	12	24	48	96
A_n					

(d) What number does A_n approach as n gets larger and larger?

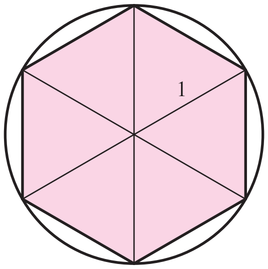


Figure for 3

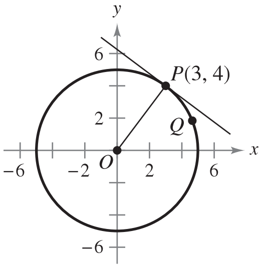


Figure for 4

4. Let $P(3, 4)$ be a point on the circle $x^2 + y^2 = 25$.

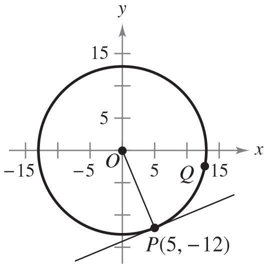
(a) What is the slope of the line joining P and $O(0, 0)$?

(b) Find an equation of the tangent line to the circle at P.

(c) Let $Q(x, y)$ be another point on the circle in the first quadrant. Find the slope m_x of the line joining P and Q in terms of x.

(d) Calculate $\lim_{x \to 3} m_x$. How does this number relate to your answer in part (b)?

5. Let $P(5, -12)$ be a point on the circle $x^2 + y^2 = 169$.



(a) What is the slope of the line joining P and $O(0, 0)$?

(b) Find an equation of the tangent line to the circle at P.

(c) Let $Q(x, y)$ be another point on the circle in the fourth quadrant. Find the slope m_x of the line joining P and Q in terms of x.

(d) Calculate $\lim_{x \to 5} m_x$. How does this number relate to your answer in part (b)?

6. Find the values of the constants a and b such that

$$\lim_{x \to 0} \frac{\sqrt{a + bx} - \sqrt{3}}{x} = \sqrt{3}.$$

7. Consider the function $f(x) = \dfrac{\sqrt{3 + x^{1/3}} - 2}{x - 1}$.

(a) Find the domain of f.

(b) Use a graphing utility to graph the function.

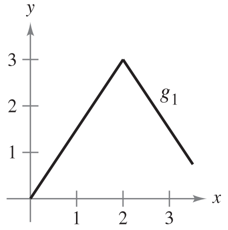
(c) Calculate $\lim\limits_{x \to -27^+} f(x)$.

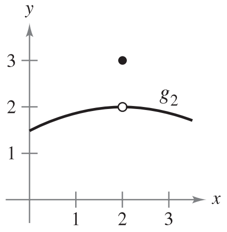
(d) Calculate $\lim\limits_{x \to 1} f(x)$.

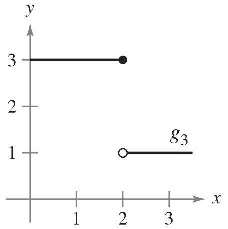
8. Determine all values of the constant a such that the following function is continuous for all real numbers.

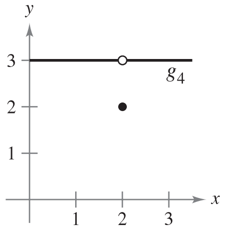
$$f(x) = \begin{cases} \dfrac{ax}{\tan x}, & x \geq 0 \\ a^2 - 2, & x < 0 \end{cases}$$

9. Consider the graphs of the four functions g_1, g_2, g_3, and g_4.









For each given condition of the function f, which of the graphs could be the graph of f?

(a) $\lim\limits_{x \to 2} f(x) = 3$

(b) f is continuous at 2.

(c) $\lim\limits_{x \to 2^-} f(x) = 3$

10. Sketch the graph of the function $f(x) = \left\llbracket \dfrac{1}{x} \right\rrbracket$.

(a) Evaluate $f\left(\frac{1}{4}\right)$, $f(3)$, and $f(1)$.

(b) Evaluate the limits $\lim\limits_{x \to 1^-} f(x)$, $\lim\limits_{x \to 1^+} f(x)$, $\lim\limits_{x \to 0^-} f(x)$, and $\lim\limits_{x \to 0^+} f(x)$.

(c) Discuss the continuity of the function.

11. Sketch the graph of the function $f(x) = \llbracket x \rrbracket + \llbracket -x \rrbracket$.

(a) Evaluate $f(1)$, $f(0)$, $f\left(\frac{1}{2}\right)$, and $f(-2.7)$.

(b) Evaluate the limits $\lim\limits_{x \to 1^-} f(x)$, $\lim\limits_{x \to 1^+} f(x)$, and $\lim\limits_{x \to \frac{1}{2}} f(x)$.

(c) Discuss the continuity of the function.

12. To escape Earth's gravitational field, a rocket must be launched with an initial velocity called the **escape velocity.** A rocket launched from the surface of Earth has velocity v (in miles per second) given by

$$v = \sqrt{\frac{2GM}{r} + v_0^2 - \frac{2GM}{R}} \approx \sqrt{\frac{192{,}000}{r} + v_0^2 - 48}$$

where v_0 is the initial velocity, r is the distance from the rocket to the center of Earth, G is the gravitational constant, M is the mass of Earth, and R is the radius of Earth (approximately 4000 miles).

(a) Find the value of v_0 for which you obtain an infinite limit for r as v approaches zero. This value of v_0 is the escape velocity for Earth.

(b) A rocket launched from the surface of the moon has velocity v (in miles per second) given by

$$v = \sqrt{\frac{1920}{r} + v_0^2 - 2.17}.$$

Find the escape velocity for the moon.

(c) A rocket launched from the surface of a planet has velocity v (in miles per second) given by

$$v = \sqrt{\frac{10{,}600}{r} + v_0^2 - 6.99}.$$

Find the escape velocity for this planet. Is the mass of this planet larger or smaller than that of Earth? (Assume that the mean density of this planet is the same as that of Earth.)

13. For positive numbers $a < b$, the **pulse function** is defined as

$$P_{a,b}(x) = H(x - a) - H(x - b) = \begin{cases} 0, & x < a \\ 1, & a \leq x < b \\ 0, & x \geq b \end{cases}$$

where $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is the Heaviside function.

(a) Sketch the graph of the pulse function.

(b) Find the following limits:

(i) $\lim\limits_{x \to a^+} P_{a,b}(x)$ (ii) $\lim\limits_{x \to a^-} P_{a,b}(x)$

(iii) $\lim\limits_{x \to b^+} P_{a,b}(x)$ (iv) $\lim\limits_{x \to b^-} P_{a,b}(x)$

(c) Discuss the continuity of the pulse function.

(d) Why is

$$U(x) = \frac{1}{b - a} P_{a,b}(x)$$

called the **unit** pulse function?

14. Let a be a nonzero constant. Prove that if $\lim\limits_{x \to 0} f(x) = L$, then $\lim\limits_{x \to 0} f(ax) = L$. Show by means of an example that a must be nonzero.

2 Differentiation

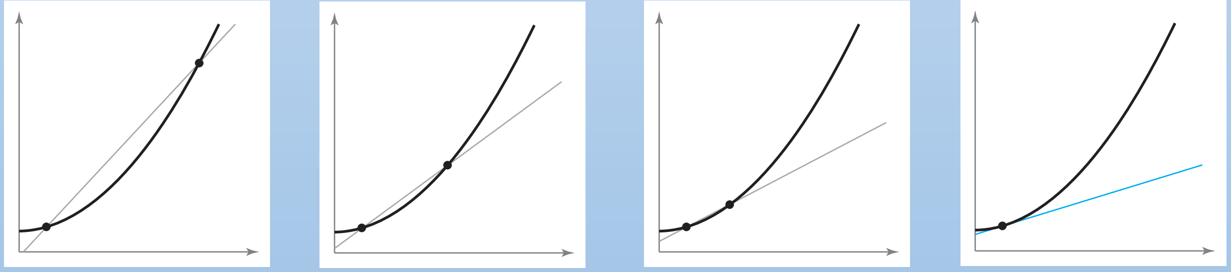
In this chapter you will study one of the most important processes of calculus–*differentiation*. In each section, you will learn new methods and rules for finding derivatives of functions. Then you will apply these rules to find such things as velocity, acceleration, and the rates of change of two or more related variables.

In this chapter, you should learn the following.

- How to find the derivative of a function using the limit definition and understand the relationship between differentiability and continuity. (2.1)
- How to find the derivative of a function using basic differentiation rules. (2.2)
- How to find the derivative of a function using the Product Rule and the Quotient Rule. (2.3)
- How to find the derivative of a function using the Chain Rule and the General Power Rule. (2.4)
- How to find the derivative of a function using implicit differentiation. (2.5)
- How to find a related rate. (2.6)

Al Bello/Getty Images

When jumping from a platform, a diver's velocity is briefly positive because of the upward movement, but then becomes negative when falling. How can you use calculus to determine the velocity of a diver at impact? (See Section 2.2, Example 10.)



To approximate the slope of a tangent line to a graph at a given point, find the slope of the secant line through the given point and a second point on the graph. As the second point approaches the given point, the approximation tends to become more accurate. (See Section 2.1.)

2.1 The Derivative and the Tangent Line Problem

- **Find the slope of the tangent line to a curve at a point.**
- **Use the limit definition to find the derivative of a function.**
- **Understand the relationship between differentiability and continuity.**

The Tangent Line Problem

Calculus grew out of four major problems that European mathematicians were working on during the seventeenth century.

1. The tangent line problem (Section 1.1 and this section)
2. The velocity and acceleration problem (Sections 2.2 and 2.3)
3. The minimum and maximum problem (Section 3.1)
4. The area problem (Sections 1.1 and 4.2)

Each problem involves the notion of a limit, and calculus can be introduced with any of the four problems.

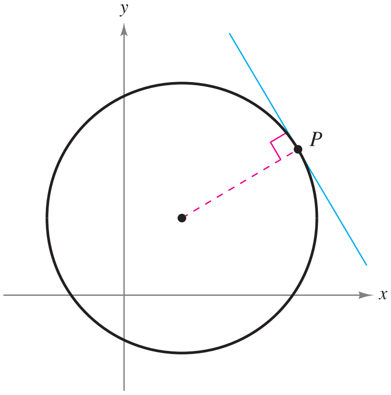
A brief introduction to the tangent line problem is given in Section 1.1. Although partial solutions to this problem were given by Pierre de Fermat (1601–1665), René Descartes (1596–1650), Christian Huygens (1629–1695), and Isaac Barrow (1630–1677), credit for the first general solution is usually given to Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716). Newton's work on this problem stemmed from his interest in optics and light refraction.

Mary Evans Picture Library

ISAAC NEWTON (1642–1727)

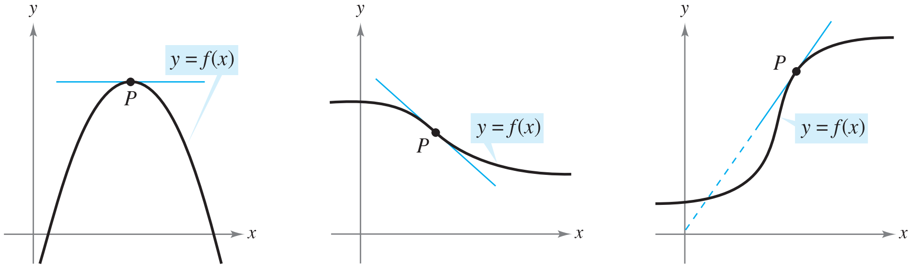
In addition to his work in calculus, Newton made revolutionary contributions to physics, including the Law of Universal Gravitation and his three laws of motion.

What does it mean to say that a line is tangent to a curve at a point? For a circle, the tangent line at a point P is the line that is perpendicular to the radial line at point P, as shown in Figure 2.1.



Tangent line to a circle
Figure 2.1

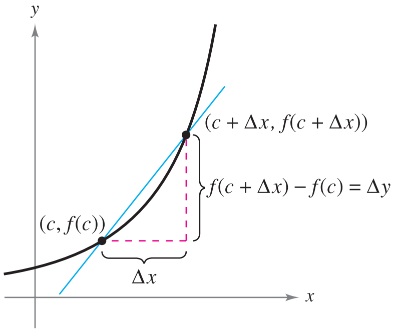
For a general curve, however, the problem is more difficult. For example, how would you define the tangent lines shown in Figure 2.2? You might say that a line is tangent to a curve at a point P if it touches, but does not cross, the curve at point P. This definition would work for the first curve shown in Figure 2.2, but not for the second. *Or* you might say that a line is tangent to a curve if the line touches or intersects the curve at exactly one point. This definition would work for a circle but not for more general curves, as the third curve in Figure 2.2 shows.



Tangent line to a curve at a point
Figure 2.2

EXPLORATION

Identifying a Tangent Line Use a graphing utility to graph the function $f(x) = 2x^3 - 4x^2 + 3x - 5$. On the same screen, graph $y = x - 5$, $y = 2x - 5$, and $y = 3x - 5$. Which of these lines, if any, appears to be tangent to the graph of f at the point $(0, -5)$? Explain your reasoning.



The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$
Figure 2.3

Essentially, the problem of finding the tangent line at a point P boils down to the problem of finding the *slope* of the tangent line at point P. You can approximate this slope using a **secant line*** through the point of tangency and a second point on the curve, as shown in Figure 2.3. If $(c, f(c))$ is the point of tangency and $(c + \Delta x, f(c + \Delta x))$ is a second point on the graph of f, the slope of the secant line through the two points is given by substitution into the slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c}$$

Change in y / Change in x

$$\boxed{m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{\Delta x}.}$$

Slope of secant line

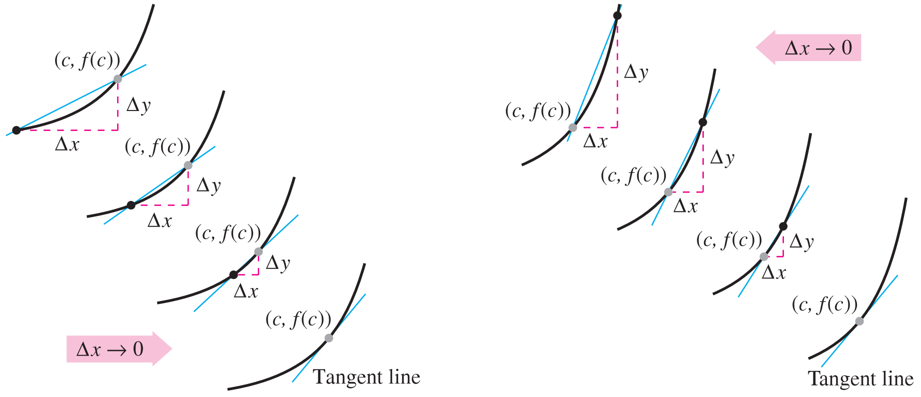
The right-hand side of this equation is a **difference quotient.** The denominator Δx is the **change in x,** and the numerator $\Delta y = f(c + \Delta x) - f(c)$ is the **change in y.**

The beauty of this procedure is that you can obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in Figure 2.4.

THE TANGENT LINE PROBLEM

In 1637, mathematician René Descartes stated this about the tangent line problem:

"And I dare say that this is not only the most useful and general problem in geometry that I know, but even that I ever desire to know."



Tangent line approximations
Figure 2.4

DEFINITION OF TANGENT LINE WITH SLOPE *m*

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of f at $x = c$.**

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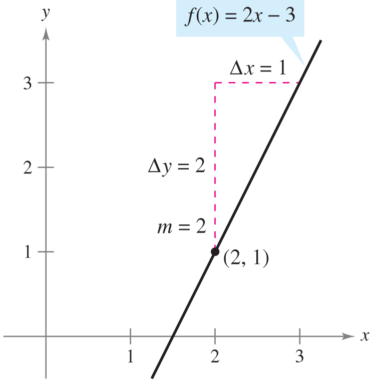
* *This use of the word* secant *comes from the Latin* secare, *meaning to cut, and is not a reference to the trigonometric function of the same name.*



The slope of f at $(2, 1)$ is $m = 2$.
Figure 2.5

EXAMPLE 1 The Slope of the Graph of a Linear Function

Find the slope of the graph of

$$f(x) = 2x - 3$$

at the point $(2, 1)$.

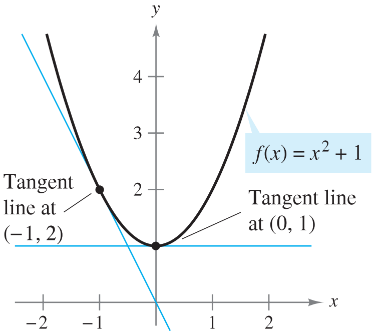
Solution To find the slope of the graph of f when $c = 2$, you can apply the definition of the slope of a tangent line, as shown.

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[2(2 + \Delta x) - 3] - [2(2) - 3]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{4 + 2\Delta x - 3 - 4 + 3}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2\cancel{\Delta x}}{\cancel{\Delta x}} \\
&= \lim_{\Delta x \to 0} 2 \\
&= 2
\end{aligned}$$

The slope of f at $(c, f(c)) = (2, 1)$ is $m = 2$, as shown in Figure 2.5. ■

NOTE In Example 1, the limit definition of the slope of f agrees with the definition of the slope of a line as discussed in Section P.2. ■

The graph of a linear function has the same slope at any point. This is not true of nonlinear functions, as shown in the following example.



The slope of f at any point $(c, f(c))$ is $m = 2c$.
Figure 2.6

EXAMPLE 2 Tangent Lines to the Graph of a Nonlinear Function

Find the slopes of the tangent lines to the graph of

$$f(x) = x^2 + 1$$

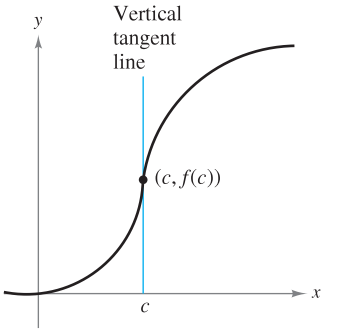
at the points $(0, 1)$ and $(-1, 2)$, as shown in Figure 2.6.

Solution Let $(c, f(c))$ represent an arbitrary point on the graph of f. Then the slope of the tangent line at $(c, f(c))$ is given by

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[(c + \Delta x)^2 + 1] - (c^2 + 1)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{c^2 + 2c(\Delta x) + (\Delta x)^2 + 1 - c^2 - 1}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2c(\Delta x) + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2c + \Delta x) \\
&= 2c.
\end{aligned}$$

So, the slope at *any* point $(c, f(c))$ on the graph of f is $m = 2c$. At the point $(0, 1)$, the slope is $m = 2(0) = 0$, and at $(-1, 2)$, the slope is $m = 2(-1) = -2$. ■

NOTE In Example 2, note that c is held constant in the limit process (as $\Delta x \to 0$). ■



The graph of f has a vertical tangent line at $(c, f(c))$.
Figure 2.7

The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line. For vertical tangent lines, you can use the following definition. If f is continuous at c and

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = -\infty$$

the vertical line $x = c$ passing through $(c, f(c))$ is a **vertical tangent line** to the graph of f. For example, the function shown in Figure 2.7 has a vertical tangent line at $(c, f(c))$. If the domain of f is the closed interval $[a, b]$, you can extend the definition of a vertical tangent line to include the endpoints by considering continuity and limits from the right (for $x = a$) and from the left (for $x = b$).

The Derivative of a Function

You have now arrived at a crucial point in the study of calculus. The limit used to define the slope of a tangent line is also used to define one of the two fundamental operations of calculus—**differentiation.**

DEFINITION OF THE DERIVATIVE OF A FUNCTION

The **derivative** of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

Be sure you see that the derivative of a function of x is also a function of x. This "new" function gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided that the graph has a tangent line at this point.

The process of finding the derivative of a function is called **differentiation.** A function is **differentiable** at x if its derivative exists at x and is **differentiable on an open interval (a, b)** if it is differentiable at every point in the interval.

In addition to $f'(x)$, which is read as "f prime of x," other notations are used to denote the derivative of $y = f(x)$. The most common are

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y].$$

Notation for derivatives

The notation dy/dx is read as "the derivative of y *with respect to* x" or simply "dy, dx." Using limit notation, you can write

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= f'(x). \end{aligned}$$

■ **FOR FURTHER INFORMATION** For more information on the crediting of mathematical discoveries to the first "discoverers," see the article "Mathematical Firsts—Who Done It?" by Richard H. Williams and Roy D. Mazzagatti in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

EXAMPLE 3 Finding the Derivative by the Limit Process

Find the derivative of $f(x) = x^3 + 2x$.

Solution

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{Definition of derivative}\\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 + 2(x + \Delta x) - (x^3 + 2x)}{\Delta x}\\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x}\\
&= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2\Delta x}{\Delta x}\\
&= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}\left[3x^2 + 3x\Delta x + (\Delta x)^2 + 2\right]}{\cancel{\Delta x}}\\
&= \lim_{\Delta x \to 0} \left[3x^2 + 3x\Delta x + (\Delta x)^2 + 2\right]\\
&= 3x^2 + 2
\end{aligned}$$

■

STUDY TIP When using the definition to find a derivative of a function, the key is to rewrite the difference quotient so that Δx does not occur as a factor of the denominator.

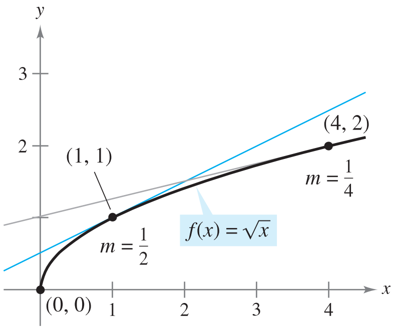
Remember that the derivative of a function f is itself a function, which can be used to find the slope of the tangent line at the point $(x, f(x))$ on the graph of f.

EXAMPLE 4 Using the Derivative to Find the Slope at a Point

Find $f'(x)$ for $f(x) = \sqrt{x}$. Then find the slopes of the graph of f at the points $(1, 1)$ and $(4, 2)$. Discuss the behavior of f at $(0, 0)$.

Solution Use the procedure for rationalizing numerators, as discussed in Section 1.3.

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{Definition of derivative}\\
&= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}\\
&= \lim_{\Delta x \to 0} \left(\frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}\right)\left(\frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}\right)\\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}\\
&= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}\\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}\\
&= \frac{1}{2\sqrt{x}}, \quad x > 0
\end{aligned}$$



The slope of f at $(x, f(x))$, $x > 0$, is $m = 1/\left(2\sqrt{x}\right)$.
Figure 2.8

At the point $(1, 1)$, the slope is $f'(1) = \frac{1}{2}$. At the point $(4, 2)$, the slope is $f'(4) = \frac{1}{4}$. See Figure 2.8. At the point $(0, 0)$, the slope is undefined. Moreover, the graph of f has a vertical tangent line at $(0, 0)$. ■

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

In many applications, it is convenient to use a variable other than x as the independent variable, as shown in Example 5.

EXAMPLE 5 Finding the Derivative of a Function

Find the derivative with respect to t for the function $y = 2/t$.

Solution Considering $y = f(t)$, you obtain

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \qquad \text{Definition of derivative}$$

$$= \lim_{\Delta t \to 0} \frac{\dfrac{2}{t + \Delta t} - \dfrac{2}{t}}{\Delta t} \qquad f(t + \Delta t) = 2/(t + \Delta t) \text{ and } f(t) = 2/t$$

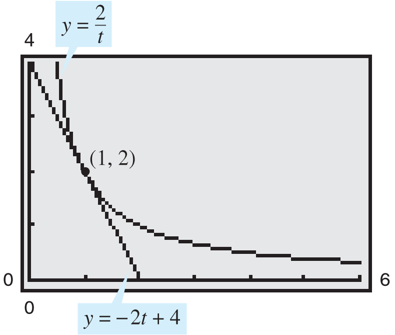
$$= \lim_{\Delta t \to 0} \frac{\dfrac{2t - 2(t + \Delta t)}{t(t + \Delta t)}}{\Delta t} \qquad \text{Combine fractions in numerator.}$$

$$= \lim_{\Delta t \to 0} \frac{-2\cancel{\Delta t}}{\cancel{\Delta t}(t)(t + \Delta t)} \qquad \text{Divide out common factor of } \Delta t.$$

$$= \lim_{\Delta t \to 0} \frac{-2}{t(t + \Delta t)} \qquad \text{Simplify.}$$

$$= -\frac{2}{t^2}. \qquad \text{Evaluate limit as } \Delta t \to 0.$$

■



At the point $(1, 2)$, the line $y = -2t + 4$ is tangent to the graph of $y = 2/t$.
Figure 2.9

TECHNOLOGY A graphing utility can be used to reinforce the result given in Example 5. For instance, using the formula $dy/dt = -2/t^2$, you know that the slope of the graph of $y = 2/t$ at the point $(1, 2)$ is $m = -2$. Using the point-slope form, you can find that the equation of the tangent line to the graph at $(1, 2)$ is

$$y - 2 = -2(t - 1) \quad \text{or} \quad y = -2t + 4$$

as shown in Figure 2.9.

Differentiability and Continuity

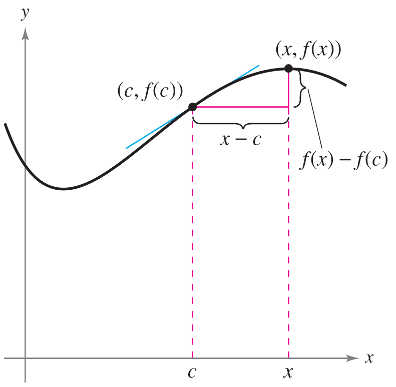
The following alternative limit form of the derivative is useful in investigating the relationship between differentiability and continuity. The derivative of f at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{Alternative form of derivative}$$

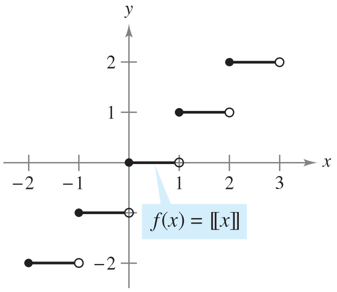
provided this limit exists (see Figure 2.10). (A proof of the equivalence of this form is given in Appendix A.) Note that the existence of the limit in this alternative form requires that the one-sided limits

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

exist and are equal. These one-sided limits are called the **derivatives from the left and from the right,** respectively. It follows that f is **differentiable on the closed interval $[a, b]$** if it is differentiable on (a, b) and if the derivative from the right at a and the derivative from the left at b both exist.



As x approaches c, the secant line approaches the tangent line.
Figure 2.10



The greatest integer function is not differentiable at $x = 0$, because it is not continuous at $x = 0$.
Figure 2.11

If a function is not continuous at $x = c$, it is also not differentiable at $x = c$. For instance, the greatest integer function

$$f(x) = \llbracket x \rrbracket$$

is not continuous at $x = 0$, and so it is not differentiable at $x = 0$ (see Figure 2.11). You can verify this by observing that

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{\llbracket x \rrbracket - 0}{x} = \infty$$

Derivative from the left

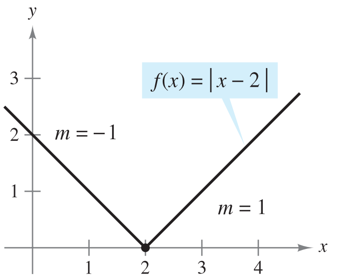
and

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{\llbracket x \rrbracket - 0}{x} = 0.$$

Derivative from the right

Although it is true that differentiability implies continuity (as shown in Theorem 2.1 on the next page), the converse is not true. That is, it is possible for a function to be continuous at $x = c$ and *not* differentiable at $x = c$. Examples 6 and 7 illustrate this possibility.

EXAMPLE 6 A Graph with a Sharp Turn



f is not differentiable at $x = 2$, because the derivatives from the left and from the right are not equal.
Figure 2.12

The function

$$f(x) = |x - 2|$$

shown in Figure 2.12 is continuous at $x = 2$. However, the one-sided limits

$$\lim_{x\to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^-} \frac{|x - 2| - 0}{x - 2} = -1$$

Derivative from the left

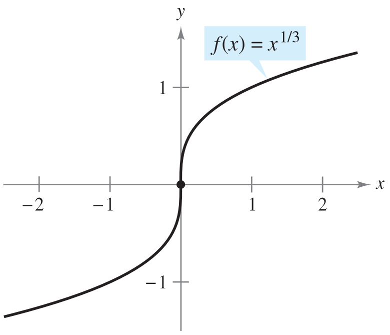
and

$$\lim_{x\to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^+} \frac{|x - 2| - 0}{x - 2} = 1$$

Derivative from the right

are not equal. So, *f* is not differentiable at $x = 2$ and the graph of *f* does not have a tangent line at the point $(2, 0)$.

EXAMPLE 7 A Graph with a Vertical Tangent Line



f is not differentiable at $x = 0$, because *f* has a vertical tangent line at $x = 0$.
Figure 2.13

The function

$$f(x) = x^{1/3}$$

is continuous at $x = 0$, as shown in Figure 2.13. However, because the limit

$$\begin{aligned}\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x\to 0} \frac{x^{1/3} - 0}{x}\\ &= \lim_{x\to 0} \frac{1}{x^{2/3}}\\ &= \infty\end{aligned}$$

is infinite, you can conclude that the tangent line is vertical at $x = 0$. So, *f* is not differentiable at $x = 0$. ■

From Examples 6 and 7, you can see that a function is not differentiable at a point at which its graph has a sharp turn *or* a vertical tangent line.

TECHNOLOGY Some graphing utilities, such as *Maple*, *Mathematica*, and the *TI-89*, perform symbolic differentiation. Others perform *numerical differentiation* by finding values of derivatives using the formula

$$f'(x) \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

where Δx is a small number such as 0.001. Can you see any problems with this definition? For instance, using this definition, what is the value of the derivative of $f(x) = |x|$ when $x = 0$?

THEOREM 2.1 DIFFERENTIABILITY IMPLIES CONTINUITY

If f is differentiable at $x = c$, then f is continuous at $x = c$.

PROOF You can prove that f is continuous at $x = c$ by showing that $f(x)$ approaches $f(c)$ as $x \to c$. To do this, use the differentiability of f at $x = c$ and consider the following limit.

$$\begin{aligned}\lim_{x \to c} [f(x) - f(c)] &= \lim_{x \to c} \left[(x - c)\left(\frac{f(x) - f(c)}{x - c}\right)\right] \\ &= \left[\lim_{x \to c} (x - c)\right]\left[\lim_{x \to c} \frac{f(x) - f(c)}{x - c}\right] \\ &= (0)[f'(c)] \\ &= 0\end{aligned}$$

Because the difference $f(x) - f(c)$ approaches zero as $x \to c$, you can conclude that $\lim_{x \to c} f(x) = f(c)$. So, f is continuous at $x = c$. ■

The following statements summarize the relationship between continuity and differentiability.

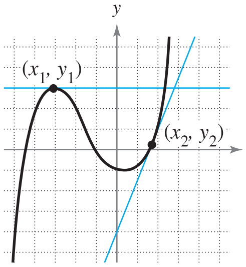
1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. So, differentiability implies continuity.
2. It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, continuity does not imply differentiability (see Example 6).

2.1 Exercises

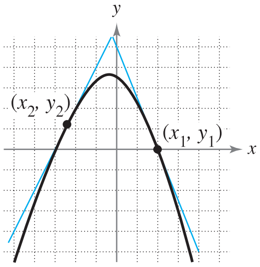
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, estimate the slope of the graph at the points (x_1, y_1) and (x_2, y_2).

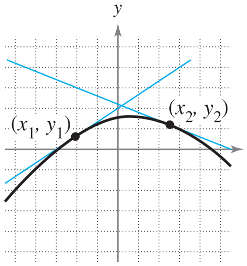
1. (a)



(b)



2. (a)



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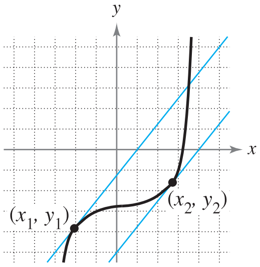
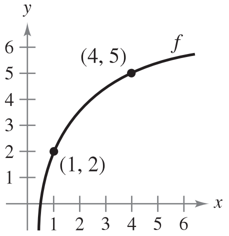


In Exercises 3 and 4, use the graph shown in the figure. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



3. Identify or sketch each of the quantities on the figure.

(a) $f(1)$ and $f(4)$ (b) $f(4) - f(1)$

(c) $y = \dfrac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$

4. Insert the proper inequality symbol ($<$ or $>$) between the given quantities.

(a) $\dfrac{f(4) - f(1)}{4 - 1}$ ☐ $\dfrac{f(4) - f(3)}{4 - 3}$

(b) $\dfrac{f(4) - f(1)}{4 - 1}$ ☐ $f'(1)$

In Exercises 5–10, find the slope of the tangent line to the graph of the function at the given point.

5. $f(x) = 3 - 5x, \quad (-1, 8)$ **6.** $g(x) = \frac{3}{2}x + 1, \quad (-2, -2)$

7. $g(x) = x^2 - 9, \quad (2, -5)$ **8.** $g(x) = 6 - x^2, \quad (1, 5)$

9. $f(t) = 3t - t^2, \quad (0, 0)$ **10.** $h(t) = t^2 + 3, \quad (-2, 7)$

In Exercises 11–24, find the derivative by the limit process.

11. $f(x) = 7$ **12.** $g(x) = -3$

13. $f(x) = -10x$ **14.** $f(x) = 3x + 2$

15. $h(s) = 3 + \frac{2}{3}s$ **16.** $f(x) = 8 - \frac{1}{5}x$

17. $f(x) = x^2 + x - 3$ **18.** $f(x) = 2 - x^2$

19. $f(x) = x^3 - 12x$ **20.** $f(x) = x^3 + x^2$

21. $f(x) = \dfrac{1}{x - 1}$ **22.** $f(x) = \dfrac{1}{x^2}$

23. $f(x) = \sqrt{x + 4}$ **24.** $f(x) = \dfrac{4}{\sqrt{x}}$

In Exercises 25–32, (a) find an equation of the tangent line to the graph of *f* at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

25. $f(x) = x^2 + 3, \quad (1, 4)$

26. $f(x) = x^2 + 3x + 4, \quad (-2, 2)$

27. $f(x) = x^3, \quad (2, 8)$ **28.** $f(x) = x^3 + 1, \quad (1, 2)$

29. $f(x) = \sqrt{x}, \quad (1, 1)$ **30.** $f(x) = \sqrt{x - 1}, \quad (5, 2)$

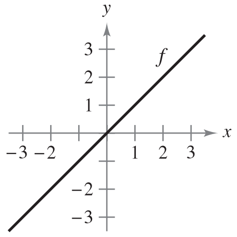
31. $f(x) = x + \dfrac{4}{x}, \quad (4, 5)$ **32.** $f(x) = \dfrac{1}{x + 1}, \quad (0, 1)$

In Exercises 33–38, find an equation of the line that is tangent to the graph of *f and* parallel to the given line.

	Function	*Line*
33.	$f(x) = x^2$	$2x - y + 1 = 0$
34.	$f(x) = 2x^2$	$4x + y + 3 = 0$
35.	$f(x) = x^3$	$3x - y + 1 = 0$
36.	$f(x) = x^3 + 2$	$3x - y - 4 = 0$
37.	$f(x) = \dfrac{1}{\sqrt{x}}$	$x + 2y - 6 = 0$
38.	$f(x) = \dfrac{1}{\sqrt{x - 1}}$	$x + 2y + 7 = 0$

In Exercises 39–42, the graph of *f* is given. Select the graph of *f′*.

39.



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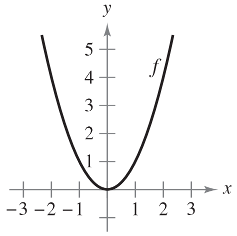

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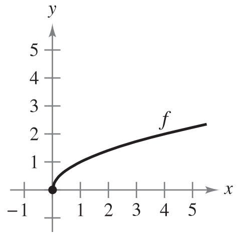
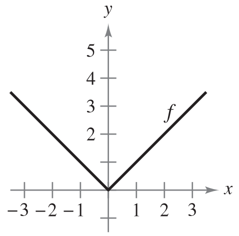
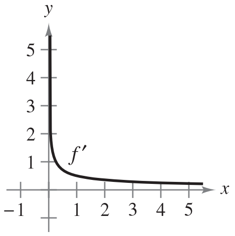


42.



(a)



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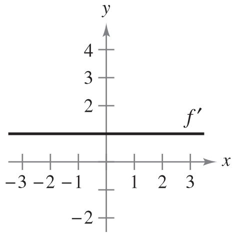

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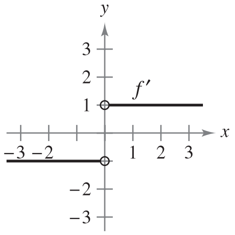

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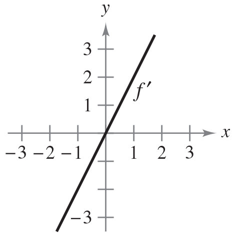


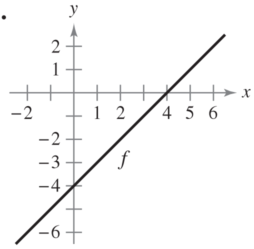
43. The tangent line to the graph of $y = g(x)$ at the point $(4, 5)$ passes through the point $(7, 0)$. Find $g(4)$ and $g'(4)$.

44. The tangent line to the graph of $y = h(x)$ at the point $(-1, 4)$ passes through the point $(3, 6)$. Find $h(-1)$ and $h'(-1)$.

WRITING ABOUT CONCEPTS

In Exercises 45–50, sketch the graph of *f′*. Explain how you found your answer.

45.



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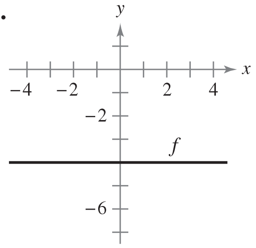

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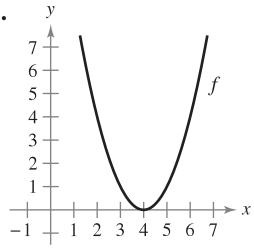

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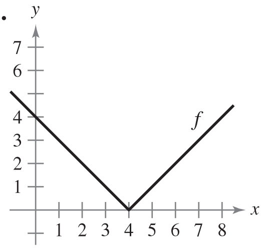
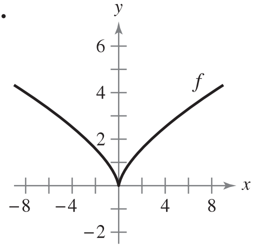


WRITING ABOUT CONCEPTS (continued)

49.



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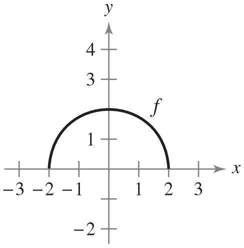


51. Sketch a graph of a function whose derivative is always negative. Explain how you found your answer.

52. Sketch a graph of a function whose derivative is always positive. Explain how you found your answer.

In Exercises 53–56, the limit represents $f'(c)$ for a function f and a number c. Find f and c.

53. $\lim\limits_{\Delta x \to 0} \dfrac{[5 - 3(1 + \Delta x)] - 2}{\Delta x}$

54. $\lim\limits_{\Delta x \to 0} \dfrac{(-2 + \Delta x)^3 + 8}{\Delta x}$

55. $\lim\limits_{x \to 6} \dfrac{-x^2 + 36}{x - 6}$

56. $\lim\limits_{x \to 9} \dfrac{2\sqrt{x} - 6}{x - 9}$

In Exercises 57–59, identify a function f that has the given characteristics. Then sketch the function.

57. $f(0) = 2;$
$f'(x) = -3, -\infty < x < \infty$

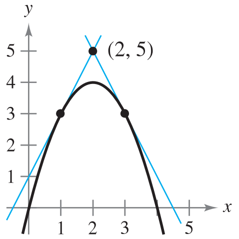
58. $f(0) = 4; f'(0) = 0;$
$f'(x) < 0$ for $x < 0;$
$f'(x) > 0$ for $x > 0$

59. $f(0) = 0; f'(0) = 0; f'(x) > 0$ for $x \neq 0$

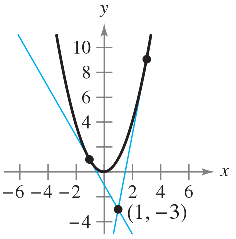
60. Assume that $f'(c) = 3$. Find $f'(-c)$ if (a) f is an odd function and if (b) f is an even function.

In Exercises 61 and 62, find equations of the two tangent lines to the graph of f that pass through the indicated point.

61. $f(x) = 4x - x^2$



62. $f(x) = x^2$

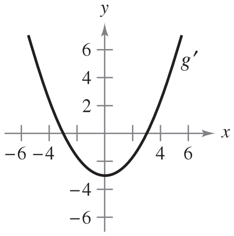


63. ***Graphical Reasoning*** Use a graphing utility to graph each function and its tangent lines at $x = -1$, $x = 0$, and $x = 1$. Based on the results, determine whether the slopes of tangent lines to the graph of a function at different values of x are always distinct.

(a) $f(x) = x^2$ (b) $g(x) = x^3$

CAPSTONE

64. The figure shows the graph of g'.



(a) $g'(0) = \square$ (b) $g'(3) = \square$

(c) What can you conclude about the graph of g knowing that $g'(1) = -\frac{8}{3}$?

(d) What can you conclude about the graph of g knowing that $g'(-4) = \frac{7}{3}$?

(e) Is $g(6) - g(4)$ positive or negative? Explain.

(f) Is it possible to find $g(2)$ from the graph? Explain.

65. ***Graphical Analysis*** Consider the function $f(x) = \frac{1}{2}x^2$.

(a) Use a graphing utility to graph the function and estimate the values of $f'(0)$, $f'\left(\frac{1}{2}\right)$, $f'(1)$, and $f'(2)$.

(b) Use your results from part (a) to determine the values of $f'\left(-\frac{1}{2}\right)$, $f'(-1)$, and $f'(-2)$.

(c) Sketch a possible graph of f'.

(d) Use the definition of derivative to find $f'(x)$.

66. ***Graphical Analysis*** Consider the function $f(x) = \frac{1}{3}x^3$.

(a) Use a graphing utility to graph the function and estimate the values of $f'(0)$, $f'\left(\frac{1}{2}\right)$, $f'(1)$, $f'(2)$, and $f'(3)$.

(b) Use your results from part (a) to determine the values of $f'\left(-\frac{1}{2}\right)$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.

(c) Sketch a possible graph of f'.

(d) Use the definition of derivative to find $f'(x)$.

Graphical Reasoning **In Exercises 67 and 68, use a graphing utility to graph the functions f and g in the same viewing window where**

$$g(x) = \frac{f(x + 0.01) - f(x)}{0.01}.$$

Label the graphs and describe the relationship between them.

67. $f(x) = 2x - x^2$

68. $f(x) = 3\sqrt{x}$

In Exercises 69 and 70, evaluate $f(2)$ and $f(2.1)$ and use the results to approximate $f'(2)$.

69. $f(x) = x(4 - x)$

70. $f(x) = \frac{1}{4}x^3$

Graphical Reasoning **In Exercises 71 and 72, use a graphing utility to graph the function and its derivative in the same viewing window. Label the graphs and describe the relationship between them.**

71. $f(x) = \dfrac{1}{\sqrt{x}}$

72. $f(x) = \dfrac{x^3}{4} - 3x$

In Exercises 73–82, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).

73. $f(x) = x^2 - 5, \quad c = 3$

74. $g(x) = x(x - 1), \quad c = 1$

75. $f(x) = x^3 + 2x^2 + 1, \quad c = -2$

76. $f(x) = x^3 + 6x, \quad c = 2$

77. $g(x) = \sqrt{|x|}, \quad c = 0$

78. $f(x) = 2/x, \quad c = 5$

79. $f(x) = (x - 6)^{2/3}, \quad c = 6$

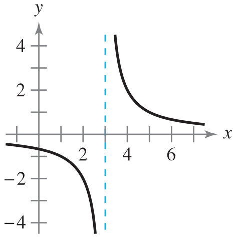
80. $g(x) = (x + 3)^{1/3}, \quad c = -3$

81. $h(x) = |x + 7|, \quad c = -7$

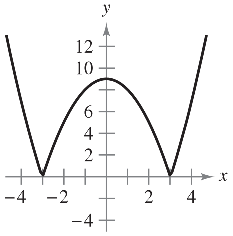
82. $f(x) = |x - 6|, \quad c = 6$

In Exercises 83–88, describe the x-values at which f is differentiable.

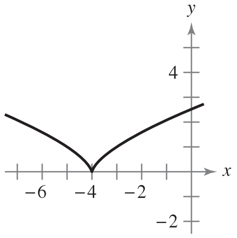
83. $f(x) = \dfrac{2}{x - 3}$



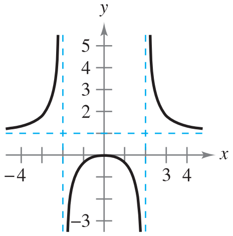
84. $f(x) = |x^2 - 9|$



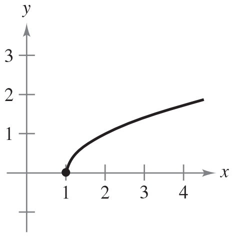
85. $f(x) = (x + 4)^{2/3}$



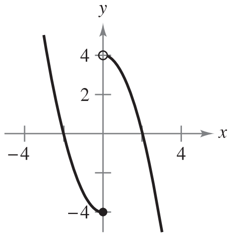
86. $f(x) = \dfrac{x^2}{x^2 - 4}$



87. $f(x) = \sqrt{x - 1}$



88. $f(x) = \begin{cases} x^2 - 4, & x \le 0 \\ 4 - x^2, & x > 0 \end{cases}$



Graphical Analysis **In Exercises 89–92, use a graphing utility to graph the function and find the x-values at which f is differentiable.**

89. $f(x) = |x - 5|$

90. $f(x) = \dfrac{4x}{x - 3}$

91. $f(x) = x^{2/5}$

92. $f(x) = \begin{cases} x^3 - 3x^2 + 3x, & x \le 1 \\ x^2 - 2x, & x > 1 \end{cases}$

In Exercises 93–96, find the derivatives from the left and from the right at $x = 1$ (if they exist). Is the function differentiable at $x = 1$?

93. $f(x) = |x - 1|$

94. $f(x) = \sqrt{1 - x^2}$

95. $f(x) = \begin{cases} (x - 1)^3, & x \le 1 \\ (x - 1)^2, & x > 1 \end{cases}$

96. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

In Exercises 97 and 98, determine whether the function is differentiable at $x = 2$.

97. $f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$

98. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \ge 2 \end{cases}$

99. ***Graphical Reasoning*** A line with slope m passes through the point $(0, 4)$ and has the equation $y = mx + 4$.

(a) Write the distance d between the line and the point $(3, 1)$ as a function of m.

(b) Use a graphing utility to graph the function d in part (a). Based on the graph, is the function differentiable at every value of m? If not, where is it not differentiable?

100. ***Conjecture*** Consider the functions $f(x) = x^2$ and $g(x) = x^3$.

(a) Graph f and f' on the same set of axes.

(b) Graph g and g' on the same set of axes.

(c) Identify a pattern between f and g and their respective derivatives. Use the pattern to make a conjecture about $h'(x)$ if $h(x) = x^n$, where n is an integer and $n \ge 2$.

(d) Find $f'(x)$ if $f(x) = x^4$. Compare the result with the conjecture in part (c). Is this a proof of your conjecture? Explain.

True or False? **In Exercises 101–104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

101. The slope of the tangent line to the differentiable function f at the point $(2, f(2))$ is $\dfrac{f(2 + \Delta x) - f(2)}{\Delta x}$.

102. If a function is continuous at a point, then it is differentiable at that point.

103. If a function has derivatives from both the right and the left at a point, then it is differentiable at that point.

104. If a function is differentiable at a point, then it is continuous at that point.

105. Let $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ and $g(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$.

Show that f is continuous, but not differentiable, at $x = 0$. Show that g is differentiable at 0, and find $g'(0)$.

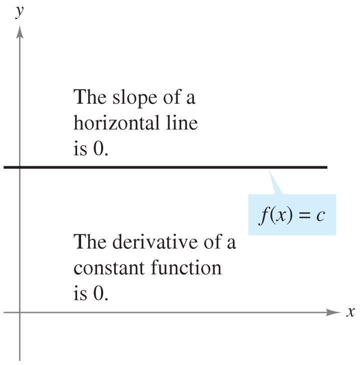
106. ***Writing*** Use a graphing utility to graph the two functions $f(x) = x^2 + 1$ and $g(x) = |x| + 1$ in the same viewing window. Use the *zoom* and *trace* features to analyze the graphs near the point $(0, 1)$. What do you observe? Which function is differentiable at this point? Write a short paragraph describing the geometric significance of differentiability at a point.

2.2 Basic Differentiation Rules and Rates of Change

- **Find the derivative of a function using the Constant Rule.**
- **Find the derivative of a function using the Power Rule.**
- **Find the derivative of a function using the Constant Multiple Rule.**
- **Find the derivative of a function using the Sum and Difference Rules.**
- **Find the derivatives of the sine function and of the cosine function.**
- **Use derivatives to find rates of change.**

The Constant Rule

In Section 2.1 you used the limit definition to find derivatives. In this and the next two sections you will be introduced to several "differentiation rules" that allow you to find derivatives without the *direct* use of the limit definition.



Notice that the Constant Rule is equivalent to saying that the slope of a horizontal line is 0. This demonstrates the relationship between slope and derivative.
Figure 2.14

THEOREM 2.2 THE CONSTANT RULE

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$

(See Figure 2.14.)

PROOF Let $f(x) = c$. Then, by the limit definition of the derivative,

$$\begin{aligned}\frac{d}{dx}[c] &= f'(x)\\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x}\\ &= \lim_{\Delta x \to 0} 0 = 0.\end{aligned}$$

■

EXAMPLE 1 Using the Constant Rule

	Function	*Derivative*
a.	$y = 7$	$dy/dx = 0$
b.	$f(x) = 0$	$f'(x) = 0$
c.	$s(t) = -3$	$s'(t) = 0$
d.	$y = k\pi^2$, k is constant	$y' = 0$

■

EXPLORATION

Writing a Conjecture Use the definition of the derivative given in Section 2.1 to find the derivative of each function. What patterns do you see? Use your results to write a conjecture about the derivative of $f(x) = x^n$.

a. $f(x) = x^1$ **b.** $f(x) = x^2$ **c.** $f(x) = x^3$
d. $f(x) = x^4$ **e.** $f(x) = x^{1/2}$ **f.** $f(x) = x^{-1}$

The Power Rule

Before proving the next rule, it is important to review the procedure for expanding a binomial.

$$(x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$
$$(x + \Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

The general binomial expansion for a positive integer n is

$$(x + \Delta x)^n = x^n + nx^{n-1}(\Delta x) + \underbrace{\frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdot \cdot \cdot + (\Delta x)^n}_{(\Delta x)^2 \text{ is a factor of these terms.}}.$$

This binomial expansion is used in proving a special case of the Power Rule.

NOTE From Example 7 in Section 2.1, you know that the function $f(x) = x^{1/3}$ is defined at $x = 0$, but is not differentiable at $x = 0$. This is because $x^{-2/3}$ is not defined on an interval containing 0.

THEOREM 2.3 THE POWER RULE

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.

PROOF If n is a positive integer greater than 1, then the binomial expansion produces

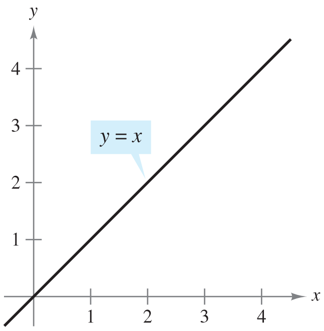
$$\begin{aligned}
\frac{d}{dx}[x^n] &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}(\Delta x) + \frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdot \cdot \cdot + (\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)x^{n-2}}{2}(\Delta x) + \cdot \cdot \cdot + (\Delta x)^{n-1} \right] \\
&= nx^{n-1} + 0 + \cdot \cdot \cdot + 0 \\
&= nx^{n-1}.
\end{aligned}$$

This proves the case for which n is a positive integer greater than 1. You will prove the case for $n = 1$. Example 7 in Section 2.3 proves the case for which n is a negative integer. In Exercise 76 in Section 2.5 you are asked to prove the case for which n is rational. (In Section 5.5, the Power Rule will be extended to cover irrational values of n.) ■

When using the Power Rule, the case for which $n = 1$ is best thought of as a separate differentiation rule. That is,

$$\boxed{\frac{d}{dx}[x] = 1.} \qquad \text{Power Rule when } n = 1$$

This rule is consistent with the fact that the slope of the line $y = x$ is 1, as shown in Figure 2.15.



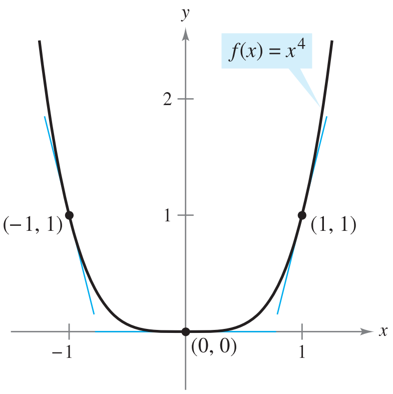
The slope of the line $y = x$ is 1.
Figure 2.15

EXAMPLE 2 Using the Power Rule

Function	*Derivative*
a. $f(x) = x^3$	$f'(x) = 3x^2$
b. $g(x) = \sqrt[3]{x}$	$g'(x) = \dfrac{d}{dx}[x^{1/3}] = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$
c. $y = \dfrac{1}{x^2}$	$\dfrac{dy}{dx} = \dfrac{d}{dx}[x^{-2}] = (-2)x^{-3} = -\dfrac{2}{x^3}$ ■

In Example 2(c), note that *before* differentiating, $1/x^2$ was rewritten as x^{-2}. Rewriting is the first step in *many* differentiation problems.

Given: $y = \dfrac{1}{x^2}$ ⇨ Rewrite: $y = x^{-2}$ ⇨ Differentiate: $\dfrac{dy}{dx} = (-2)x^{-3}$ ⇨ Simplify: $\dfrac{dy}{dx} = -\dfrac{2}{x^3}$



Note that the slope of the graph is negative at the point $(-1, 1)$, the slope is zero at the point $(0, 0)$, and the slope is positive at the point $(1, 1)$.
Figure 2.16

EXAMPLE 3 Finding the Slope of a Graph

Find the slope of the graph of $f(x) = x^4$ when

a. $x = -1$ **b.** $x = 0$ **c.** $x = 1$.

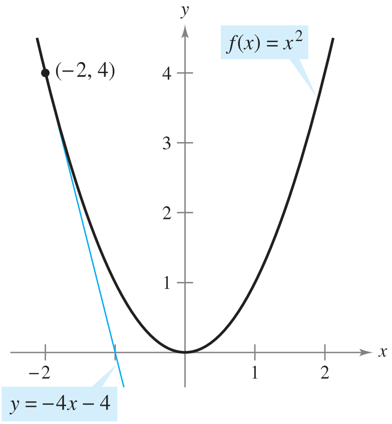
Solution The slope of a graph at a point is the value of the derivative at that point. The derivative of f is $f'(x) = 4x^3$.

a. When $x = -1$, the slope is $f'(-1) = 4(-1)^3 = -4$. Slope is negative.

b. When $x = 0$, the slope is $f'(0) = 4(0)^3 = 0$. Slope is zero.

c. When $x = 1$, the slope is $f'(1) = 4(1)^3 = 4$. Slope is positive.

See Figure 2.16.



The line $y = -4x - 4$ is tangent to the graph of $f(x) = x^2$ at the point $(-2, 4)$.
Figure 2.17

EXAMPLE 4 Finding an Equation of a Tangent Line

Find an equation of the tangent line to the graph of $f(x) = x^2$ when $x = -2$.

Solution To find the *point* on the graph of f, evaluate the original function at $x = -2$.

$(-2, f(-2)) = (-2, 4)$ Point on graph

To find the *slope* of the graph when $x = -2$, evaluate the derivative, $f'(x) = 2x$, at $x = -2$.

$m = f'(-2) = -4$ Slope of graph at $(-2, 4)$

Now, using the point-slope form of the equation of a line, you can write

$y - y_1 = m(x - x_1)$ Point-slope form

$y - 4 = -4[x - (-2)]$ Substitute for y_1, m, and x_1.

$y = -4x - 4$. Simplify.

See Figure 2.17. ■

The Constant Multiple Rule

THEOREM 2.4 THE CONSTANT MULTIPLE RULE

If f is a differentiable function and c is a real number, then cf is also differentiable and $\frac{d}{dx}[cf(x)] = cf'(x)$.

PROOF

$$\begin{aligned}
\frac{d}{dx}[cf(x)] &= \lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} && \text{Definition of derivative}\\
&= \lim_{\Delta x \to 0} c\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]\\
&= c\left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right] && \text{Apply Theorem 1.2.}\\
&= cf'(x)
\end{aligned}$$

■

Informally, the Constant Multiple Rule states that constants can be factored out of the differentiation process, even if the constants appear in the denominator.

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[\;f(x)] = cf'(x)$$

$$\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{c}\right] &= \frac{d}{dx}\left[\left(\frac{1}{c}\right)f(x)\right]\\
&= \left(\frac{1}{c}\right)\frac{d}{dx}[\;f(x)] = \left(\frac{1}{c}\right)f'(x)
\end{aligned}$$

EXAMPLE 5 Using the Constant Multiple Rule

	Function	*Derivative*
a.	$y = \frac{2}{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{-1}] = 2\frac{d}{dx}[x^{-1}] = 2(-1)x^{-2} = -\frac{2}{x^2}$
b.	$f(t) = \frac{4t^2}{5}$	$f'(t) = \frac{d}{dt}\left[\frac{4}{5}t^2\right] = \frac{4}{5}\frac{d}{dt}[t^2] = \frac{4}{5}(2t) = \frac{8}{5}t$
c.	$y = 2\sqrt{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{1/2}] = 2\left(\frac{1}{2}x^{-1/2}\right) = x^{-1/2} = \frac{1}{\sqrt{x}}$
d.	$y = \frac{1}{2\sqrt[3]{x^2}}$	$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}x^{-2/3}\right] = \frac{1}{2}\left(-\frac{2}{3}\right)x^{-5/3} = -\frac{1}{3x^{5/3}}$
e.	$y = -\frac{3x}{2}$	$y' = \frac{d}{dx}\left[-\frac{3}{2}x\right] = -\frac{3}{2}(1) = -\frac{3}{2}$

■

The Constant Multiple Rule and the Power Rule can be combined into one rule. The combination rule is

$$\frac{d}{dx}[cx^n] = cnx^{n-1}.$$

EXAMPLE 6 Using Parentheses When Differentiating

Original Function	*Rewrite*	*Differentiate*	*Simplify*
a. $y = \dfrac{5}{2x^3}$	$y = \dfrac{5}{2}(x^{-3})$	$y' = \dfrac{5}{2}(-3x^{-4})$	$y' = -\dfrac{15}{2x^4}$
b. $y = \dfrac{5}{(2x)^3}$	$y = \dfrac{5}{8}(x^{-3})$	$y' = \dfrac{5}{8}(-3x^{-4})$	$y' = -\dfrac{15}{8x^4}$
c. $y = \dfrac{7}{3x^{-2}}$	$y = \dfrac{7}{3}(x^2)$	$y' = \dfrac{7}{3}(2x)$	$y' = \dfrac{14x}{3}$
d. $y = \dfrac{7}{(3x)^{-2}}$	$y = 63(x^2)$	$y' = 63(2x)$	$y' = 126x$

■

The Sum and Difference Rules

THEOREM 2.5 THE SUM AND DIFFERENCE RULES

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$ Sum Rule

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$ Difference Rule

PROOF A proof of the Sum Rule follows from Theorem 1.2. (The Difference Rule can be proved in a similar way.)

$$\begin{aligned}
\frac{d}{dx}[f(x) + g(x)] &= \lim_{\Delta x \to 0} \frac{[f(x + \Delta x) + g(x + \Delta x)] - [f(x) + g(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x)
\end{aligned}$$

■

The Sum and Difference Rules can be extended to any finite number of functions. For instance, if $F(x) = f(x) + g(x) - h(x)$, then $F'(x) = f'(x) + g'(x) - h'(x)$.

EXAMPLE 7 Using the Sum and Difference Rules

Function	*Derivative*
a. $f(x) = x^3 - 4x + 5$	$f'(x) = 3x^2 - 4$
b. $g(x) = -\dfrac{x^4}{2} + 3x^3 - 2x$	$g'(x) = -2x^3 + 9x^2 - 2$

■

Derivatives of the Sine and Cosine Functions

In Section 1.3, you studied the following limits.

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0$$

These two limits can be used to prove differentiation rules for the sine and cosine functions. (The derivatives of the other four trigonometric functions are discussed in Section 2.3.)

FOR FURTHER INFORMATION For the outline of a geometric proof of the derivatives of the sine and cosine functions, see the article "The Spider's Spacewalk Derivation of sin′ and cos′" by Tim Hesterberg in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

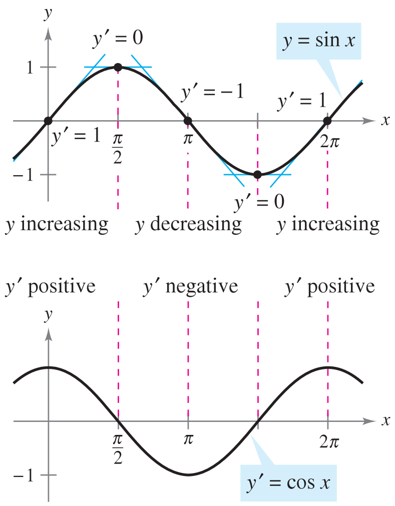
THEOREM 2.6 DERIVATIVES OF SINE AND COSINE FUNCTIONS

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

PROOF

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} & \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x - (\sin x)(1 - \cos \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[(\cos x)\left(\frac{\sin \Delta x}{\Delta x}\right) - (\sin x)\left(\frac{1 - \cos \Delta x}{\Delta x}\right) \right] \\
&= \cos x \left(\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \right) - \sin x \left(\lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} \right) \\
&= (\cos x)(1) - (\sin x)(0) \\
&= \cos x
\end{aligned}$$

This differentiation rule is shown graphically in Figure 2.18. Note that for each x, the *slope* of the sine curve is equal to the value of the cosine. The proof of the second rule is left as an exercise (see Exercise 120). ■



The derivative of the sine function is the cosine function.

Figure 2.18

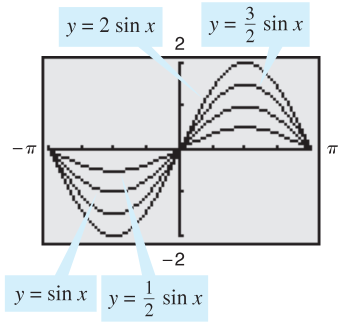
EXAMPLE 8 Derivatives Involving Sines and Cosines

Function	*Derivative*
a. $y = 2 \sin x$	$y' = 2 \cos x$
b. $y = \dfrac{\sin x}{2} = \dfrac{1}{2} \sin x$	$y' = \dfrac{1}{2} \cos x = \dfrac{\cos x}{2}$
c. $y = x + \cos x$	$y' = 1 - \sin x$ ■

TECHNOLOGY A graphing utility can provide insight into the interpretation of a derivative. For instance, Figure 2.19 shows the graphs of

$$y = a \sin x$$

for $a = \frac{1}{2}, 1, \frac{3}{2}$, and 2. Estimate the slope of each graph at the point $(0, 0)$. Then verify your estimates analytically by evaluating the derivative of each function when $x = 0$.



$$\frac{d}{dx}[a \sin x] = a \cos x$$

Figure 2.19

Rates of Change

You have seen how the derivative is used to determine slope. The derivative can also be used to determine the rate of change of one variable with respect to another. Applications involving rates of change occur in a wide variety of fields. A few examples are population growth rates, production rates, water flow rates, velocity, and acceleration.

A common use for rate of change is to describe the motion of an object moving in a straight line. In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion. On such lines, movement to the right (or upward) is considered to be in the positive direction, and movement to the left (or downward) is considered to be in the negative direction.

The function s that gives the position (relative to the origin) of an object as a function of time t is called a **position function.** If, over a period of time Δt, the object changes its position by the amount $\Delta s = s(t + \Delta t) - s(t)$, then, by the familiar formula

$$\text{Rate} = \frac{\text{distance}}{\text{time}}$$

the **average velocity** is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t}. \qquad \text{Average velocity}$$

EXAMPLE 9 Finding Average Velocity of a Falling Object

If a billiard ball is dropped from a height of 100 feet, its height s at time t is given by the position function

$$s = -16t^2 + 100 \qquad \text{Position function}$$

where s is measured in feet and t is measured in seconds. Find the average velocity over each of the following time intervals.

a. $[1, 2]$ **b.** $[1, 1.5]$ **c.** $[1, 1.1]$

Solution

a. For the interval $[1, 2]$, the object falls from a height of $s(1) = -16(1)^2 + 100 = 84$ feet to a height of $s(2) = -16(2)^2 + 100 = 36$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{36 - 84}{2 - 1} = \frac{-48}{1} = -48 \text{ feet per second.}$$

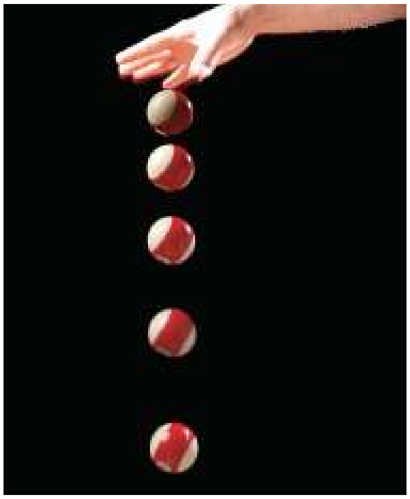
b. For the interval $[1, 1.5]$, the object falls from a height of 84 feet to a height of 64 feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{64 - 84}{1.5 - 1} = \frac{-20}{0.5} = -40 \text{ feet per second.}$$

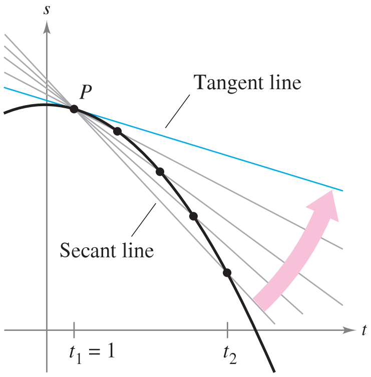
c. For the interval $[1, 1.1]$, the object falls from a height of 84 feet to a height of 80.64 feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{80.64 - 84}{1.1 - 1} = \frac{-3.36}{0.1} = -33.6 \text{ feet per second.}$$

Note that the average velocities are *negative*, indicating that the object is moving downward. ■


Richard Megna/Fundamental Photographs

Time-lapse photograph of a free-falling billiard ball



The average velocity between t_1 and t_2 is the slope of the secant line, and the instantaneous velocity at t_1 is the slope of the tangent line.
Figure 2.20

Suppose that in Example 9 you wanted to find the *instantaneous* velocity (or simply the velocity) of the object when $t = 1$. Just as you can approximate the slope of the tangent line by calculating the slope of the secant line, you can approximate the velocity at $t = 1$ by calculating the average velocity over a small interval $[1, 1 + \Delta t]$ (see Figure 2.20). By taking the limit as Δt approaches zero, you obtain the velocity when $t = 1$. Try doing this—you will find that the velocity when $t = 1$ is -32 feet per second.

In general, if $s = s(t)$ is the position function for an object moving along a straight line, the **velocity** of the object at time t is

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t).$$

Velocity function

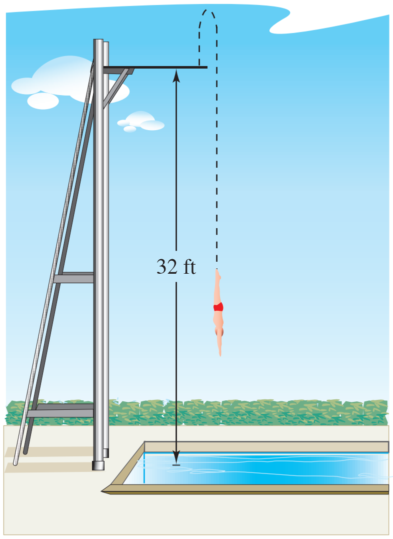
In other words, the velocity function is the derivative of the position function. Velocity can be negative, zero, or positive. The **speed** of an object is the absolute value of its velocity. Speed cannot be negative.

The position of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation

$$s(t) = \frac{1}{2}gt^2 + v_0 t + s_0$$

Position function

where s_0 is the initial height of the object, v_0 is the initial velocity of the object, and g is the acceleration due to gravity. On Earth, the value of g is approximately -32 feet per second per second or -9.8 meters per second per second.



Velocity is positive when an object is rising, and is negative when an object is falling. Notice that the diver moves upward for the first half-second because the velocity is positive for $0 < t < \frac{1}{2}$. When the velocity is 0, the diver has reached the maximum height of the dive.
Figure 2.21

EXAMPLE 10 Using the Derivative to Find Velocity

At time $t = 0$, a diver jumps from a platform diving board that is 32 feet above the water (see Figure 2.21). The position of the diver is given by

$$s(t) = -16t^2 + 16t + 32$$

Position function

where s is measured in feet and t is measured in seconds.

a. When does the diver hit the water?

b. What is the diver's velocity at impact?

Solution

a. To find the time t when the diver hits the water, let $s = 0$ and solve for t.

$$\begin{aligned} -16t^2 + 16t + 32 &= 0 && \text{Set position function equal to 0.} \\ -16(t + 1)(t - 2) &= 0 && \text{Factor.} \\ t &= -1 \text{ or } 2 && \text{Solve for } t. \end{aligned}$$

Because $t \geq 0$, choose the positive value to conclude that the diver hits the water at $t = 2$ seconds.

b. The velocity at time t is given by the derivative $s'(t) = -32t + 16$. So, the velocity at time $t = 2$ is

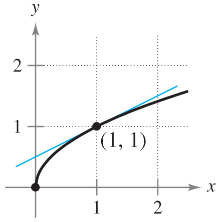
$$s'(2) = -32(2) + 16 = -48 \text{ feet per second.} \quad ■$$

2.2 Exercises

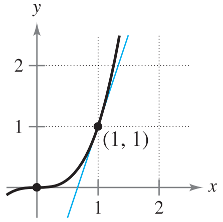
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use the graph to estimate the slope of the tangent line to $y = x^n$ at the point $(1, 1)$. Verify your answer analytically. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

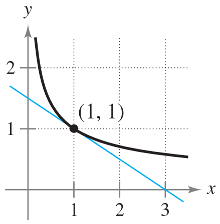
1. (a) $y = x^{1/2}$



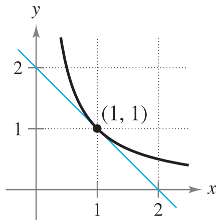
(b) $y = x^3$



2. (a) $y = x^{-1/2}$



(b) $y = x^{-1}$



In Exercises 3–24, use the rules of differentiation to find the derivative of the function.

3. $y = 12$

4. $f(x) = -9$

5. $y = x^7$

6. $y = x^{16}$

7. $y = \dfrac{1}{x^5}$

8. $y = \dfrac{1}{x^8}$

9. $f(x) = \sqrt[5]{x}$

10. $g(x) = \sqrt[4]{x}$

11. $f(x) = x + 11$

12. $g(x) = 3x - 1$

13. $f(t) = -2t^2 + 3t - 6$

14. $y = t^2 + 2t - 3$

15. $g(x) = x^2 + 4x^3$

16. $y = 8 - x^3$

17. $s(t) = t^3 + 5t^2 - 3t + 8$

18. $f(x) = 2x^3 - x^2 + 3x$

19. $y = \dfrac{\pi}{2}\sin\theta - \cos\theta$

20. $g(t) = \pi\cos t$

21. $y = x^2 - \frac{1}{2}\cos x$

22. $y = 7 + \sin x$

23. $y = \dfrac{1}{x} - 3\sin x$

24. $y = \dfrac{5}{(2x)^3} + 2\cos x$

In Exercises 25–30, complete the table.

	Original Function	*Rewrite*	*Differentiate*	*Simplify*
25.	$y = \dfrac{5}{2x^2}$			
26.	$y = \dfrac{2}{3x^2}$			
27.	$y = \dfrac{6}{(5x)^3}$			
28.	$y = \dfrac{\pi}{(3x)^2}$			
29.	$y = \dfrac{\sqrt{x}}{x}$			
30.	$y = \dfrac{4}{x^{-3}}$			

In Exercises 31–38, find the slope of the graph of the function at the given point. Use the *derivative* feature of a graphing utility to confirm your results.

	Function	*Point*
31.	$f(x) = \dfrac{8}{x^2}$	$(2, 2)$
32.	$f(t) = 3 - \dfrac{3}{5t}$	$\left(\frac{3}{5}, 2\right)$
33.	$f(x) = -\frac{1}{2} + \frac{7}{5}x^3$	$\left(0, -\frac{1}{2}\right)$
34.	$y = 3x^3 - 10$	$(2, 14)$
35.	$y = (4x + 1)^2$	$(0, 1)$
36.	$f(x) = 3(5 - x)^2$	$(5, 0)$
37.	$f(\theta) = 4\sin\theta - \theta$	$(0, 0)$
38.	$g(t) = -2\cos t + 5$	$(\pi, 7)$

In Exercises 39–54, find the derivative of the function.

39. $f(x) = x^2 + 5 - 3x^{-2}$

40. $f(x) = x^2 - 3x - 3x^{-2}$

41. $g(t) = t^2 - \dfrac{4}{t^3}$

42. $f(x) = x + \dfrac{1}{x^2}$

43. $f(x) = \dfrac{4x^3 + 3x^2}{x}$

44. $f(x) = \dfrac{x^3 - 6}{x^2}$

45. $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2}$

46. $h(x) = \dfrac{2x^2 - 3x + 1}{x}$

47. $y = x(x^2 + 1)$

48. $y = 3x(6x - 5x^2)$

49. $f(x) = \sqrt{x} - 6\sqrt[3]{x}$

50. $f(x) = \sqrt[3]{x} + \sqrt[5]{x}$

51. $h(s) = s^{4/5} - s^{2/3}$

52. $f(t) = t^{2/3} - t^{1/3} + 4$

53. $f(x) = 6\sqrt{x} + 5\cos x$

54. $f(x) = \dfrac{2}{\sqrt[3]{x}} + 3\cos x$

In Exercises 55–58, (a) find an equation of the tangent line to the graph of *f* at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

	Function	*Point*
55.	$y = x^4 - 3x^2 + 2$	$(1, 0)$
56.	$y = x^3 + x$	$(-1, -2)$
57.	$f(x) = \dfrac{2}{\sqrt[4]{x^3}}$	$(1, 2)$
58.	$y = (x^2 + 2x)(x + 1)$	$(1, 6)$

In Exercises 59–64, determine the point(s) (if any) at which the graph of the function has a horizontal tangent line.

59. $y = x^4 - 2x^2 + 3$

60. $y = x^3 + x$

61. $y = \dfrac{1}{x^2}$

62. $y = x^2 + 9$

63. $y = x + \sin x, \quad 0 \le x < 2\pi$

64. $y = \sqrt{3}x + 2\cos x, \quad 0 \le x < 2\pi$

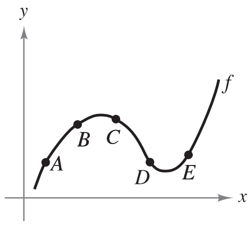
In Exercises 65–70, find k such that the line is tangent to the graph of the function.

Function	*Line*
65. $f(x) = x^2 - kx$	$y = 5x - 4$
66. $f(x) = k - x^2$	$y = -6x + 1$
67. $f(x) = \dfrac{k}{x}$	$y = -\dfrac{3}{4}x + 3$
68. $f(x) = k\sqrt{x}$	$y = x + 4$
69. $f(x) = kx^3$	$y = x + 1$
70. $f(x) = kx^4$	$y = 4x - 1$

71. Sketch the graph of a function f such that $f' > 0$ for all x and the rate of change of the function is decreasing.

CAPSTONE

72. Use the graph of f to answer each question. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) Between which two consecutive points is the average rate of change of the function greatest?

(b) Is the average rate of change of the function between A and B greater than or less than the instantaneous rate of change at B?

(c) Sketch a tangent line to the graph between C and D such that the slope of the tangent line is the same as the average rate of change of the function between C and D.

WRITING ABOUT CONCEPTS

In Exercises 73 and 74, the relationship between f and g is given. Explain the relationship between f' and g'.

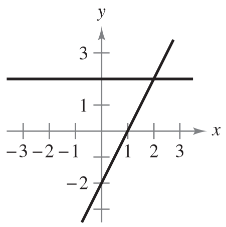
73. $g(x) = f(x) + 6$

74. $g(x) = -5f(x)$

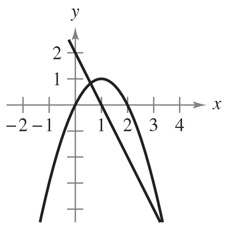
WRITING ABOUT CONCEPTS (continued)

In Exercises 75 and 76, the graphs of a function f and its derivative f' are shown on the same set of coordinate axes. Label the graphs as f or f' and write a short paragraph stating the criteria you used in making your selection. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

75.



76.



77. Sketch the graphs of $y = x^2$ and $y = -x^2 + 6x - 5$, and sketch the two lines that are tangent to both graphs. Find equations of these lines.

78. Show that the graphs of the two equations $y = x$ and $y = 1/x$ have tangent lines that are perpendicular to each other at their point of intersection.

79. Show that the graph of the function

$$f(x) = 3x + \sin x + 2$$

does not have a horizontal tangent line.

80. Show that the graph of the function

$$f(x) = x^5 + 3x^3 + 5x$$

does not have a tangent line with a slope of 3.

In Exercises 81 and 82, find an equation of the tangent line to the graph of the function f through the point (x_0, y_0) not on the graph. To find the point of tangency (x, y) on the graph of f, solve the equation

$$f'(x) = \frac{y_0 - y}{x_0 - x}.$$

81. $f(x) = \sqrt{x}$

$(x_0, y_0) = (-4, 0)$

82. $f(x) = \dfrac{2}{x}$

$(x_0, y_0) = (5, 0)$

83. ***Linear Approximation*** Use a graphing utility, with a square window setting, to zoom in on the graph of

$$f(x) = 4 - \tfrac{1}{2}x^2$$

to approximate $f'(1)$. Use the derivative to find $f'(1)$.

84. ***Linear Approximation*** Use a graphing utility, with a square window setting, to zoom in on the graph of

$$f(x) = 4\sqrt{x} + 1$$

to approximate $f'(4)$. Use the derivative to find $f'(4)$.

85. ***Linear Approximation*** Consider the function $f(x) = x^{3/2}$ with the solution point $(4, 8)$.

(a) Use a graphing utility to graph f. Use the *zoom* feature to obtain successive magnifications of the graph in the neighborhood of the point $(4, 8)$. After zooming in a few times, the graph should appear nearly linear. Use the *trace* feature to determine the coordinates of a point near $(4, 8)$. Find an equation of the secant line $S(x)$ through the two points.

(b) Find the equation of the line

$$T(x) = f'(4)(x - 4) + f(4)$$

tangent to the graph of f passing through the given point. Why are the linear functions S and T nearly the same?

(c) Use a graphing utility to graph f and T on the same set of coordinate axes. Note that T is a good approximation of f when x is close to 4. What happens to the accuracy of the approximation as you move farther away from the point of tangency?

(d) Demonstrate the conclusion in part (c) by completing the table.

Δx	-3	-2	-1	-0.5	-0.1	0
$f(4 + \Delta x)$						
$T(4 + \Delta x)$						

Δx	0.1	0.5	1	2	3
$f(4 + \Delta x)$					
$T(4 + \Delta x)$					

86. ***Linear Approximation*** Repeat Exercise 85 for the function $f(x) = x^3$ where $T(x)$ is the line tangent to the graph at the point $(1, 1)$. Explain why the accuracy of the linear approximation decreases more rapidly than in Exercise 85.

True or False? **In Exercises 87–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

87. If $f'(x) = g'(x)$, then $f(x) = g(x)$.

88. If $f(x) = g(x) + c$, then $f'(x) = g'(x)$.

89. If $y = \pi^2$, then $dy/dx = 2\pi$.

90. If $y = x/\pi$, then $dy/dx = 1/\pi$.

91. If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.

92. If $f(x) = 1/x^n$, then $f'(x) = 1/(nx^{n-1})$.

In Exercises 93–96, find the average rate of change of the function over the given interval. Compare this average rate of change with the instantaneous rates of change at the endpoints of the interval.

93. $f(t) = 4t + 5, \quad [1, 2]$

94. $f(t) = t^2 - 7, \quad [3, 3.1]$

95. $f(x) = \dfrac{-1}{x}, \quad [1, 2]$

96. $f(x) = \sin x, \quad \left[0, \dfrac{\pi}{6}\right]$

Vertical Motion **In Exercises 97 and 98, use the position function $s(t) = -16t^2 + v_0 t + s_0$ for free-falling objects.**

97. A silver dollar is dropped from the top of a building that is 1362 feet tall.

(a) Determine the position and velocity functions for the coin.

(b) Determine the average velocity on the interval $[1, 2]$.

(c) Find the instantaneous velocities when $t = 1$ and $t = 2$.

(d) Find the time required for the coin to reach ground level.

(e) Find the velocity of the coin at impact.

98. A ball is thrown straight down from the top of a 220-foot building with an initial velocity of -22 feet per second. What is its velocity after 3 seconds? What is its velocity after falling 108 feet?

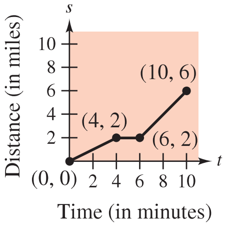
Vertical Motion **In Exercises 99 and 100, use the position function $s(t) = -4.9t^2 + v_0 t + s_0$ for free-falling objects.**

99. A projectile is shot upward from the surface of Earth with an initial velocity of 120 meters per second. What is its velocity after 5 seconds? After 10 seconds?

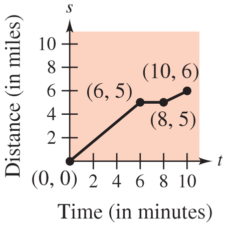
100. To estimate the height of a building, a stone is dropped from the top of the building into a pool of water at ground level. How high is the building if the splash is seen 5.6 seconds after the stone is dropped?

Think About It **In Exercises 101 and 102, the graph of a position function is shown. It represents the distance in miles that a person drives during a 10-minute trip to work. Make a sketch of the corresponding velocity function.**

101.

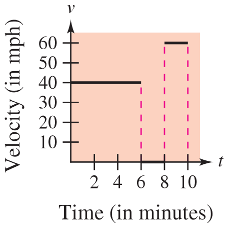


102.

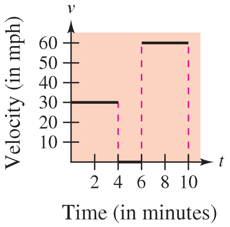


Think About It **In Exercises 103 and 104, the graph of a velocity function is shown. It represents the velocity in miles per hour during a 10-minute drive to work. Make a sketch of the corresponding position function.**

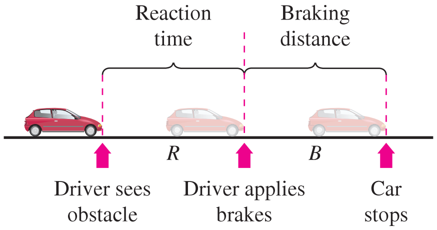
103.



104.



105. ***Modeling Data*** The stopping distance of an automobile, on dry, level pavement, traveling at a speed v (kilometers per hour) is the distance R (meters) the car travels during the reaction time of the driver plus the distance B (meters) the car travels after the brakes are applied (see figure). The table shows the results of an experiment.



Speed, v	20	40	60	80	100
Reaction Time Distance, R	8.3	16.7	25.0	33.3	41.7
Braking Time Distance, B	2.3	9.0	20.2	35.8	55.9

(a) Use the regression capabilities of a graphing utility to find a linear model for reaction time distance.

(b) Use the regression capabilities of a graphing utility to find a quadratic model for braking distance.

(c) Determine the polynomial giving the total stopping distance T.

(d) Use a graphing utility to graph the functions R, B, and T in the same viewing window.

(e) Find the derivative of T and the rates of change of the total stopping distance for $v = 40$, $v = 80$, and $v = 100$.

(f) Use the results of this exercise to draw conclusions about the total stopping distance as speed increases.

106. ***Fuel Cost*** A car is driven 15,000 miles a year and gets x miles per gallon. Assume that the average fuel cost is \$2.76 per gallon. Find the annual cost of fuel C as a function of x and use this function to complete the table.

x	10	15	20	25	30	35	40
C							
dC/dx							

Who would benefit more from a one-mile-per-gallon increase in fuel efficiency—the driver of a car that gets 15 miles per gallon or the driver of a car that gets 35 miles per gallon? Explain.

107. ***Volume*** The volume of a cube with sides of length s is given by $V = s^3$. Find the rate of change of the volume with respect to s when $s = 6$ centimeters.

108. ***Area*** The area of a square with sides of length s is given by $A = s^2$. Find the rate of change of the area with respect to s when $s = 6$ meters.

109. ***Velocity*** Verify that the average velocity over the time interval $[t_0 - \Delta t, t_0 + \Delta t]$ is the same as the instantaneous velocity at $t = t_0$ for the position function

$$s(t) = -\tfrac{1}{2}at^2 + c.$$

110. ***Inventory Management*** The annual inventory cost C for a manufacturer is

$$C = \frac{1{,}008{,}000}{Q} + 6.3Q$$

where Q is the order size when the inventory is replenished. Find the change in annual cost when Q is increased from 350 to 351, and compare this with the instantaneous rate of change when $Q = 350$.

111. ***Writing*** The number of gallons N of regular unleaded gasoline sold by a gasoline station at a price of p dollars per gallon is given by $N = f(p)$.

(a) Describe the meaning of $f'(2.979)$.

(b) Is $f'(2.979)$ usually positive or negative? Explain.

112. ***Newton's Law of Cooling*** This law states that the rate of change of the temperature of an object is proportional to the difference between the object's temperature T and the temperature T_a of the surrounding medium. Write an equation for this law.

113. Find an equation of the parabola $y = ax^2 + bx + c$ that passes through $(0, 1)$ and is tangent to the line $y = x - 1$ at $(1, 0)$.

114. Let (a, b) be an arbitrary point on the graph of $y = 1/x$, $x > 0$. Prove that the area of the triangle formed by the tangent line through (a, b) and the coordinate axes is 2.

115. Find the tangent line(s) to the curve $y = x^3 - 9x$ through the point $(1, -9)$.

116. Find the equation(s) of the tangent line(s) to the parabola $y = x^2$ through the given point.

(a) $(0, a)$ (b) $(a, 0)$

Are there any restrictions on the constant a?

In Exercises 117 and 118, find a and b such that f is differentiable everywhere.

117. $f(x) = \begin{cases} ax^3, & x \le 2 \\ x^2 + b, & x > 2 \end{cases}$

118. $f(x) = \begin{cases} \cos x, & x < 0 \\ ax + b, & x \ge 0 \end{cases}$

119. Where are the functions $f_1(x) = |\sin x|$ and $f_2(x) = \sin |x|$ differentiable?

120. Prove that $\dfrac{d}{dx}[\cos x] = -\sin x$.

■ **FOR FURTHER INFORMATION** For a geometric interpretation of the derivatives of trigonometric functions, see the article "Sines and Cosines of the Times" by Victor J. Katz in *Math Horizons*. To view this article, go to the website *www.matharticles.com*.

2.3 Product and Quotient Rules and Higher-Order Derivatives

- **Find the derivative of a function using the Product Rule.**
- **Find the derivative of a function using the Quotient Rule.**
- **Find the derivative of a trigonometric function.**
- **Find a higher-order derivative of a function.**

The Product Rule

In Section 2.2 you learned that the derivative of the sum of two functions is simply the sum of their derivatives. The rules for the derivatives of the product and quotient of two functions are not as simple.

NOTE A version of the Product Rule that some people prefer is

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

The advantage of this form is that it generalizes easily to products of three or more factors.

THEOREM 2.7 THE PRODUCT RULE

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

PROOF Some mathematical proofs, such as the proof of the Sum Rule, are straightforward. Others involve clever steps that may appear unmotivated to a reader. This proof involves such a step—subtracting and adding the same quantity—which is shown in color.

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left[f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x} + g(x)\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]\\
&= \lim_{\Delta x\to 0}\left[f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x}\right] + \lim_{\Delta x\to 0}\left[g(x)\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]\\
&= \lim_{\Delta x\to 0} f(x+\Delta x)\cdot\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x} + \lim_{\Delta x\to 0} g(x)\cdot\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= f(x)g'(x) + g(x)f'(x) \qquad \blacksquare
\end{aligned}$$

Note that $\lim_{\Delta x\to 0} f(x+\Delta x) = f(x)$ because f is given to be differentiable and therefore is continuous.

The Product Rule can be extended to cover products involving more than two factors. For example, if f, g, and h are differentiable functions of x, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

For instance, the derivative of $y = x^2 \sin x \cos x$ is

$$\begin{aligned}
\frac{dy}{dx} &= 2x\sin x\cos x + x^2\cos x\cos x + x^2\sin x(-\sin x)\\
&= 2x\sin x\cos x + x^2(\cos^2 x - \sin^2 x).
\end{aligned}$$

NOTE The proof of the Product Rule for products of more than two factors is left as an exercise (see Exercise 141).

THE PRODUCT RULE

When Leibniz originally wrote a formula for the Product Rule, he was motivated by the expression

$$(x + dx)(y + dy) - xy$$

from which he subtracted $dx\,dy$ (as being negligible) and obtained the differential form $x\,dy + y\,dx$. This derivation resulted in the traditional form of the Product Rule. (*Source: The History of Mathematics* by David M. Burton)

The derivative of a product of two functions is not (in general) given by the product of the derivatives of the two functions. To see this, try comparing the product of the derivatives of $f(x) = 3x - 2x^2$ and $g(x) = 5 + 4x$ with the derivative in Example 1.

EXAMPLE 1 Using the Product Rule

Find the derivative of $h(x) = (3x - 2x^2)(5 + 4x)$.

Solution

$$\begin{aligned} h'(x) &= \overbrace{(3x - 2x^2)}^{\text{First}}\,\overbrace{\frac{d}{dx}[5 + 4x]}^{\text{Derivative of second}} + \overbrace{(5 + 4x)}^{\text{Second}}\,\overbrace{\frac{d}{dx}[3x - 2x^2]}^{\text{Derivative of first}} && \text{Apply Product Rule.}\\ &= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x)\\ &= (12x - 8x^2) + (15 - 8x - 16x^2)\\ &= -24x^2 + 4x + 15 \end{aligned}$$

■

In Example 1, you have the option of finding the derivative with or without the Product Rule. To find the derivative without the Product Rule, you can write

$$\begin{aligned} D_x[(3x - 2x^2)(5 + 4x)] &= D_x[-8x^3 + 2x^2 + 15x]\\ &= -24x^2 + 4x + 15. \end{aligned}$$

In the next example, you must use the Product Rule.

EXAMPLE 2 Using the Product Rule

Find the derivative of $y = 3x^2 \sin x$.

Solution

$$\begin{aligned} \frac{d}{dx}[3x^2 \sin x] &= 3x^2 \frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[3x^2] && \text{Apply Product Rule.}\\ &= 3x^2 \cos x + (\sin x)(6x)\\ &= 3x^2 \cos x + 6x \sin x\\ &= 3x(x \cos x + 2 \sin x) \end{aligned}$$

EXAMPLE 3 Using the Product Rule

Find the derivative of $y = 2x \cos x - 2 \sin x$.

Solution

$$\begin{aligned} \frac{dy}{dx} &= \overbrace{(2x)\left(\frac{d}{dx}[\cos x]\right) + (\cos x)\left(\frac{d}{dx}[2x]\right)}^{\text{Product Rule}} - \overbrace{2\frac{d}{dx}[\sin x]}^{\text{Constant Multiple Rule}}\\ &= (2x)(-\sin x) + (\cos x)(2) - 2(\cos x)\\ &= -2x \sin x \end{aligned}$$

■

NOTE In Example 3, notice that you use the Product Rule when both factors of the product are variable, and you use the Constant Multiple Rule when one of the factors is a constant.

The Quotient Rule

THEOREM 2.8 THE QUOTIENT RULE

The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

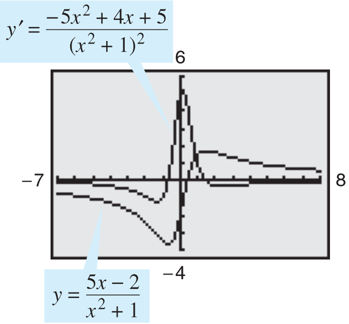
PROOF As with the proof of Theorem 2.7, the key to this proof is subtracting and adding the same quantity.

$$\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{\Delta x \to 0} \frac{\dfrac{f(x+\Delta x)}{g(x+\Delta x)} - \dfrac{f(x)}{g(x)}}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x+\Delta x)}{\Delta x g(x)g(x+\Delta x)} \\
&= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x) + f(x)g(x) - f(x)g(x+\Delta x)}{\Delta x g(x)g(x+\Delta x)} \\
&= \frac{\displaystyle\lim_{\Delta x \to 0} \frac{g(x)[f(x+\Delta x) - f(x)]}{\Delta x} - \lim_{\Delta x \to 0} \frac{f(x)[g(x+\Delta x) - g(x)]}{\Delta x}}{\displaystyle\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]} \\
&= \frac{g(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right] - f(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}\right]}{\displaystyle\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]} \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\end{aligned}$$

■

Note that $\lim_{\Delta x \to 0} g(x+\Delta x) = g(x)$ because g is given to be differentiable and therefore is continuous.

TECHNOLOGY A graphing utility can be used to compare the graph of a function with the graph of its derivative. For instance, in Figure 2.22, the graph of the function in Example 4 appears to have two points that have horizontal tangent lines. What are the values of y' at these two points?



Graphical comparison of a function and its derivative

Figure 2.22

EXAMPLE 4 Using the Quotient Rule

Find the derivative of $y = \dfrac{5x - 2}{x^2 + 1}$.

Solution

$$\begin{aligned}
\frac{d}{dx}\left[\frac{5x-2}{x^2+1}\right] &= \frac{(x^2+1)\dfrac{d}{dx}[5x-2] - (5x-2)\dfrac{d}{dx}[x^2+1]}{(x^2+1)^2} && \text{Apply Quotient Rule.} \\
&= \frac{(x^2+1)(5) - (5x-2)(2x)}{(x^2+1)^2} \\
&= \frac{(5x^2+5) - (10x^2-4x)}{(x^2+1)^2} \\
&= \frac{-5x^2+4x+5}{(x^2+1)^2}
\end{aligned}$$

■

Note the use of parentheses in Example 4. A liberal use of parentheses is recommended for *all* types of differentiation problems. For instance, with the Quotient Rule, it is a good idea to enclose all factors and derivatives in parentheses, and to pay special attention to the subtraction required in the numerator.

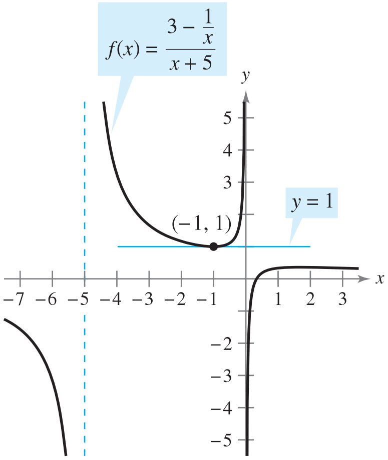
When differentiation rules were introduced in the preceding section, the need for rewriting *before* differentiating was emphasized. The next example illustrates this point with the Quotient Rule.

EXAMPLE 5 Rewriting Before Differentiating

Find an equation of the tangent line to the graph of $f(x) = \dfrac{3 - (1/x)}{x + 5}$ at $(-1, 1)$.

Solution Begin by rewriting the function.

$$\begin{aligned}
f(x) &= \frac{3 - (1/x)}{x + 5} && \text{Write original function.}\\
&= \frac{x\left(3 - \dfrac{1}{x}\right)}{x(x + 5)} && \text{Multiply numerator and denominator by } x.\\
&= \frac{3x - 1}{x^2 + 5x} && \text{Rewrite.}\\
f'(x) &= \frac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2} && \text{Quotient Rule}\\
&= \frac{(3x^2 + 15x) - (6x^2 + 13x - 5)}{(x^2 + 5x)^2}\\
&= \frac{-3x^2 + 2x + 5}{(x^2 + 5x)^2} && \text{Simplify.}
\end{aligned}$$



The line $y = 1$ is tangent to the graph of $f(x)$ at the point $(-1, 1)$.
Figure 2.23

To find the slope at $(-1, 1)$, evaluate $f'(-1)$.

$$f'(-1) = 0 \qquad \text{Slope of graph at } (-1, 1)$$

Then, using the point-slope form of the equation of a line, you can determine that the equation of the tangent line at $(-1, 1)$ is $y = 1$. See Figure 2.23. ■

Not every quotient needs to be differentiated by the Quotient Rule. For example, each quotient in the next example can be considered as the product of a constant times a function of x. In such cases it is more convenient to use the Constant Multiple Rule.

EXAMPLE 6 Using the Constant Multiple Rule

	Original Function	*Rewrite*	*Differentiate*	*Simplify*
a.	$y = \dfrac{x^2 + 3x}{6}$	$y = \dfrac{1}{6}(x^2 + 3x)$	$y' = \dfrac{1}{6}(2x + 3)$	$y' = \dfrac{2x + 3}{6}$
b.	$y = \dfrac{5x^4}{8}$	$y = \dfrac{5}{8}x^4$	$y' = \dfrac{5}{8}(4x^3)$	$y' = \dfrac{5}{2}x^3$
c.	$y = \dfrac{-3(3x - 2x^2)}{7x}$	$y = -\dfrac{3}{7}(3 - 2x)$	$y' = -\dfrac{3}{7}(-2)$	$y' = \dfrac{6}{7}$
d.	$y = \dfrac{9}{5x^2}$	$y = \dfrac{9}{5}(x^{-2})$	$y' = \dfrac{9}{5}(-2x^{-3})$	$y' = -\dfrac{18}{5x^3}$

■

NOTE To see the benefit of using the Constant Multiple Rule for some quotients, try using the Quotient Rule to differentiate the functions in Example 6—you should obtain the same results, but with more work.

In Section 2.2, the Power Rule was proved only for the case in which the exponent n is a positive integer greater than 1. The next example extends the proof to include negative integer exponents.

EXAMPLE 7 Proof of the Power Rule (Negative Integer Exponents)

If n is a negative integer, there exists a positive integer k such that $n = -k$. So, by the Quotient Rule, you can write

$$\begin{aligned}\frac{d}{dx}[x^n] &= \frac{d}{dx}\left[\frac{1}{x^k}\right]\\ &= \frac{x^k(0) - (1)(kx^{k-1})}{(x^k)^2} && \text{Quotient Rule and Power Rule}\\ &= \frac{0 - kx^{k-1}}{x^{2k}}\\ &= -kx^{-k-1}\\ &= nx^{n-1}. && n = -k\end{aligned}$$

So, the Power Rule

$$\frac{d}{dx}[x^n] = nx^{n-1} \qquad \text{Power Rule}$$

is valid for any integer. In Exercise 76 in Section 2.5, you are asked to prove the case for which n is any rational number. ■

Derivatives of Trigonometric Functions

Knowing the derivatives of the sine and cosine functions, you can use the Quotient Rule to find the derivatives of the four remaining trigonometric functions.

THEOREM 2.9 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

PROOF Considering $\tan x = (\sin x)/(\cos x)$ and applying the Quotient Rule, you obtain

$$\begin{aligned}\frac{d}{dx}[\tan x] &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} && \text{Apply Quotient Rule.}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x.\end{aligned}$$

The proofs of the other three parts of the theorem are left as an exercise (see Exercise 89). ■

EXAMPLE 8 Differentiating Trigonometric Functions

NOTE Because of trigonometric identities, the derivative of a trigonometric function can take many forms. This presents a challenge when you are trying to match your answers to those given in the back of the text.

	Function	*Derivative*
a.	$y = x - \tan x$	$\dfrac{dy}{dx} = 1 - \sec^2 x$
b.	$y = x \sec x$	$y' = x(\sec x \tan x) + (\sec x)(1)$ $= (\sec x)(1 + x \tan x)$

EXAMPLE 9 Different Forms of a Derivative

Differentiate both forms of $y = \dfrac{1 - \cos x}{\sin x} = \csc x - \cot x.$

Solution

First form: $y = \dfrac{1 - \cos x}{\sin x}$

$$\begin{aligned} y' &= \frac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x} \\ &= \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x} \\ &= \frac{1 - \cos x}{\sin^2 x} \end{aligned}$$

Second form: $y = \csc x - \cot x$

$$y' = -\csc x \cot x + \csc^2 x$$

To show that the two derivatives are equal, you can write

$$\begin{aligned} \frac{1 - \cos x}{\sin^2 x} &= \frac{1}{\sin^2 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) \\ &= \csc^2 x - \csc x \cot x. \end{aligned}$$

■

The summary below shows that much of the work in obtaining a simplified form of a derivative occurs *after* differentiating. Note that two characteristics of a simplified form are the absence of negative exponents and the combining of like terms.

	$f'(x)$ After Differentiating	$f'(x)$ After Simplifying
Example 1	$(3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$	$-24x^2 + 4x + 15$
Example 3	$(2x)(-\sin x) + (\cos x)(2) - 2(\cos x)$	$-2x \sin x$
Example 4	$\dfrac{(x^2 + 1)(5) - (5x - 2)(2x)}{(x^2 + 1)^2}$	$\dfrac{-5x^2 + 4x + 5}{(x^2 + 1)^2}$
Example 5	$\dfrac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2}$	$\dfrac{-3x^2 + 2x + 5}{(x^2 + 5x)^2}$
Example 9	$\dfrac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$	$\dfrac{1 - \cos x}{\sin^2 x}$

Higher-Order Derivatives

Just as you can obtain a velocity function by differentiating a position function, you can obtain an **acceleration** function by differentiating a velocity function. Another way of looking at this is that you can obtain an acceleration function by differentiating a position function *twice*.

$$\begin{aligned} s(t) & & \text{Position function} \\ v(t) &= s'(t) & \text{Velocity function} \\ a(t) = v'(t) &= s''(t) & \text{Acceleration function} \end{aligned}$$

NOTE The second derivative of f is the derivative of the first derivative of f.

The function given by $a(t)$ is the **second derivative** of $s(t)$ and is denoted by $s''(t)$.

The second derivative is an example of a **higher-order derivative.** You can define derivatives of any positive integer order. For instance, the **third derivative** is the derivative of the second derivative. Higher-order derivatives are denoted as follows.

First derivative:	y',	$f'(x)$,	$\dfrac{dy}{dx}$,	$\dfrac{d}{dx}[f(x)]$,	$D_x[y]$
Second derivative:	y'',	$f''(x)$,	$\dfrac{d^2y}{dx^2}$,	$\dfrac{d^2}{dx^2}[f(x)]$,	$D_x^2[y]$
Third derivative:	y''',	$f'''(x)$,	$\dfrac{d^3y}{dx^3}$,	$\dfrac{d^3}{dx^3}[f(x)]$,	$D_x^3[y]$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\dfrac{d^4y}{dx^4}$,	$\dfrac{d^4}{dx^4}[f(x)]$,	$D_x^4[y]$
$\vdots$					
nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\dfrac{d^ny}{dx^n}$,	$\dfrac{d^n}{dx^n}[f(x)]$,	$D_x^n[y]$

EXAMPLE 10 Finding the Acceleration Due to Gravity

Because the moon has no atmosphere, a falling object on the moon encounters no air resistance. In 1971, astronaut David Scott demonstrated that a feather and a hammer fall at the same rate on the moon. The position function for each of these falling objects is given by

$$s(t) = -0.81t^2 + 2$$

where $s(t)$ is the height in meters and t is the time in seconds. What is the ratio of Earth's gravitational force to the moon's?

Solution To find the acceleration, differentiate the position function twice.

$$\begin{aligned} s(t) &= -0.81t^2 + 2 & \text{Position function} \\ s'(t) &= -1.62t & \text{Velocity function} \\ s''(t) &= -1.62 & \text{Acceleration function} \end{aligned}$$

So, the acceleration due to gravity on the moon is -1.62 meters per second per second. Because the acceleration due to gravity on Earth is -9.8 meters per second per second, the ratio of Earth's gravitational force to the moon's is

$$\begin{aligned} \frac{\text{Earth's gravitational force}}{\text{Moon's gravitational force}} &= \frac{-9.8}{-1.62} \\ &\approx 6.0. \end{aligned}$$

■

Seth Resnick/Getty Images

THE MOON

The moon's mass is 7.349×10^{22} kilograms, and Earth's mass is 5.976×10^{24} kilograms. The moon's radius is 1737 kilometers, and Earth's radius is 6378 kilometers. Because the gravitational force on the surface of a planet is directly proportional to its mass and inversely proportional to the square of its radius, the ratio of the gravitational force on Earth to the gravitational force on the moon is

$$\frac{(5.976 \times 10^{24})/6378^2}{(7.349 \times 10^{22})/1737^2} \approx 6.0.$$

2.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, use the Product Rule to differentiate the function.

1. $g(x) = (x^2 + 3)(x^2 - 4x)$

2. $f(x) = (6x + 5)(x^3 - 2)$

3. $h(t) = \sqrt{t}(1 - t^2)$

4. $g(s) = \sqrt{s}(s^2 + 8)$

5. $f(x) = x^3 \cos x$

6. $g(x) = \sqrt{x} \sin x$

In Exercises 7–12, use the Quotient Rule to differentiate the function.

7. $f(x) = \dfrac{x}{x^2 + 1}$

8. $g(t) = \dfrac{t^2 + 4}{5t - 3}$

9. $h(x) = \dfrac{\sqrt{x}}{x^3 + 1}$

10. $h(s) = \dfrac{s}{\sqrt{s} - 1}$

11. $g(x) = \dfrac{\sin x}{x^2}$

12. $f(t) = \dfrac{\cos t}{t^3}$

In Exercises 13–18, find $f'(x)$ and $f'(c)$.

	Function	*Value of c*
13.	$f(x) = (x^3 + 4x)(3x^2 + 2x - 5)$	$c = 0$
14.	$f(x) = (x^2 - 2x + 1)(x^3 - 1)$	$c = 1$
15.	$f(x) = \dfrac{x^2 - 4}{x - 3}$	$c = 1$
16.	$f(x) = \dfrac{x + 5}{x - 5}$	$c = 4$
17.	$f(x) = x \cos x$	$c = \dfrac{\pi}{4}$
18.	$f(x) = \dfrac{\sin x}{x}$	$c = \dfrac{\pi}{6}$

In Exercises 19–24, complete the table without using the Quotient Rule.

	Function	*Rewrite*	*Differentiate*	*Simplify*
19.	$y = \dfrac{x^2 + 3x}{7}$			
20.	$y = \dfrac{5x^2 - 3}{4}$			
21.	$y = \dfrac{6}{7x^2}$			
22.	$y = \dfrac{10}{3x^3}$			
23.	$y = \dfrac{4x^{3/2}}{x}$			
24.	$y = \dfrac{5x^2 - 8}{11}$			

In Exercises 25–38, find the derivative of the algebraic function.

25. $f(x) = \dfrac{4 - 3x - x^2}{x^2 - 1}$

26. $f(x) = \dfrac{x^3 + 5x + 3}{x^2 - 1}$

27. $f(x) = x\left(1 - \dfrac{4}{x + 3}\right)$

28. $f(x) = x^4\left(1 - \dfrac{2}{x + 1}\right)$

29. $f(x) = \dfrac{3x - 1}{\sqrt{x}}$

30. $f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$

31. $h(s) = (s^3 - 2)^2$

32. $h(x) = (x^2 - 1)^2$

33. $f(x) = \dfrac{2 - \dfrac{1}{x}}{x - 3}$

34. $g(x) = x^2\left(\dfrac{2}{x} - \dfrac{1}{x + 1}\right)$

35. $f(x) = (2x^3 + 5x)(x - 3)(x + 2)$

36. $f(x) = (x^3 - x)(x^2 + 2)(x^2 + x - 1)$

37. $f(x) = \dfrac{x^2 + c^2}{x^2 - c^2}$, c is a constant

38. $f(x) = \dfrac{c^2 - x^2}{c^2 + x^2}$, c is a constant

In Exercises 39–54, find the derivative of the trigonometric function.

39. $f(t) = t^2 \sin t$

40. $f(\theta) = (\theta + 1) \cos \theta$

41. $f(t) = \dfrac{\cos t}{t}$

42. $f(x) = \dfrac{\sin x}{x^3}$

43. $f(x) = -x + \tan x$

44. $y = x + \cot x$

45. $g(t) = \sqrt[4]{t} + 6 \csc t$

46. $h(x) = \dfrac{1}{x} - 12 \sec x$

47. $y = \dfrac{3(1 - \sin x)}{2 \cos x}$

48. $y = \dfrac{\sec x}{x}$

49. $y = -\csc x - \sin x$

50. $y = x \sin x + \cos x$

51. $f(x) = x^2 \tan x$

52. $f(x) = \sin x \cos x$

53. $y = 2x \sin x + x^2 \cos x$

54. $h(\theta) = 5\theta \sec \theta + \theta \tan \theta$

CAS **In Exercises 55–58, use a computer algebra system to differentiate the function.**

55. $g(x) = \left(\dfrac{x + 1}{x + 2}\right)(2x - 5)$

56. $f(x) = \left(\dfrac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$

57. $g(\theta) = \dfrac{\theta}{1 - \sin \theta}$

58. $f(\theta) = \dfrac{\sin \theta}{1 - \cos \theta}$

The symbol CAS *indicates an exercise in which you are instructed to specifically use a computer algebra system.*

In Exercises 59–62, evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.

	Function	*Point*
59.	$y = \dfrac{1 + \csc x}{1 - \csc x}$	$\left(\dfrac{\pi}{6}, -3\right)$
60.	$f(x) = \tan x \cot x$	$(1, 1)$
61.	$h(t) = \dfrac{\sec t}{t}$	$\left(\pi, -\dfrac{1}{\pi}\right)$
62.	$f(x) = \sin x(\sin x + \cos x)$	$\left(\dfrac{\pi}{4}, 1\right)$

In Exercises 63–68, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

63. $f(x) = (x^3 + 4x - 1)(x - 2), \quad (1, -4)$

64. $f(x) = (x + 3)(x^2 - 2), \quad (-2, 2)$

65. $f(x) = \dfrac{x}{x + 4}, \quad (-5, 5)$

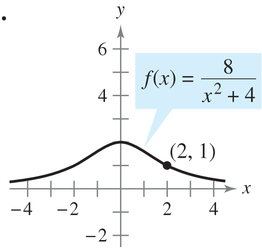
66. $f(x) = \dfrac{(x - 1)}{(x + 1)}, \quad \left(2, \dfrac{1}{3}\right)$

67. $f(x) = \tan x, \quad \left(\dfrac{\pi}{4}, 1\right)$

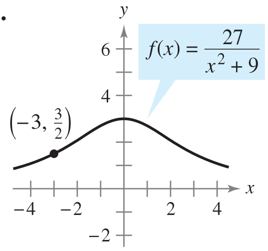
68. $f(x) = \sec x, \quad \left(\dfrac{\pi}{3}, 2\right)$

Famous Curves **In Exercises 69–72, find an equation of the tangent line to the graph at the given point. (The graphs in Exercises 69 and 70 are called *Witches of Agnesi*. The graphs in Exercises 71 and 72 are called *serpentines*.)**

69.



70.



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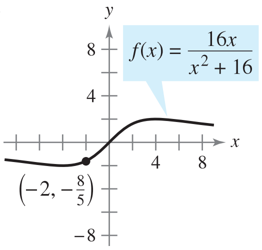
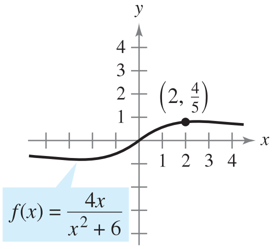


72.



In Exercises 73–76, determine the point(s) at which the graph of the function has a horizontal tangent line.

73. $f(x) = \dfrac{2x - 1}{x^2}$

74. $f(x) = \dfrac{x^2}{x^2 + 1}$

75. $f(x) = \dfrac{x^2}{x - 1}$

76. $f(x) = \dfrac{x - 4}{x^2 - 7}$

77. ***Tangent Lines*** Find equations of the tangent lines to the graph of $f(x) = (x + 1)/(x - 1)$ that are parallel to the line $2y + x = 6$. Then graph the function and the tangent lines.

78. ***Tangent Lines*** Find equations of the tangent lines to the graph of $f(x) = x/(x - 1)$ that pass through the point $(-1, 5)$. Then graph the function and the tangent lines.

In Exercises 79 and 80, verify that $f'(x) = g'(x)$, and explain the relationship between f and g.

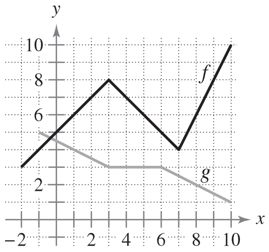
79. $f(x) = \dfrac{3x}{x + 2}, \quad g(x) = \dfrac{5x + 4}{x + 2}$

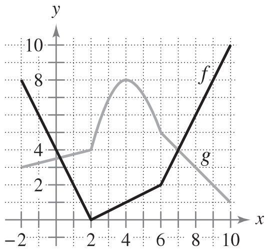
80. $f(x) = \dfrac{\sin x - 3x}{x}, \quad g(x) = \dfrac{\sin x + 2x}{x}$

In Exercises 81 and 82, use the graphs of f and g. Let $p(x) = f(x)g(x)$ and $q(x) = f(x)/g(x)$.

81. (a) Find $p'(1)$.
(b) Find $q'(4)$.

82. (a) Find $p'(4)$.
(b) Find $q'(7)$.





83. ***Area*** The length of a rectangle is given by $6t + 5$ and its height is $\sqrt{t}$, where t is time in seconds and the dimensions are in centimeters. Find the rate of change of the area with respect to time.

84. ***Volume*** The radius of a right circular cylinder is given by $\sqrt{t + 2}$ and its height is $\frac{1}{2}\sqrt{t}$, where t is time in seconds and the dimensions are in inches. Find the rate of change of the volume with respect to time.

85. ***Inventory Replenishment*** The ordering and transportation cost C for the components used in manufacturing a product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$$

where C is measured in thousands of dollars and x is the order size in hundreds. Find the rate of change of C with respect to x when (a) $x = 10$, (b) $x = 15$, and (c) $x = 20$. What do these rates of change imply about increasing order size?

86. ***Boyle's Law*** This law states that if the temperature of a gas remains constant, its pressure is inversely proportional to its volume. Use the derivative to show that the rate of change of the pressure is inversely proportional to the square of the volume.

87. ***Population Growth*** A population of 500 bacteria is introduced into a culture and grows in number according to the equation

$$P(t) = 500\left(1 + \frac{4t}{50 + t^2}\right)$$

where t is measured in hours. Find the rate at which the population is growing when $t = 2$.

88. ***Gravitational Force*** Newton's Law of Universal Gravitation states that the force F between two masses, m_1 and m_2, is

$$F = \frac{Gm_1m_2}{d^2}$$

where G is a constant and d is the distance between the masses. Find an equation that gives an instantaneous rate of change of F with respect to d. (Assume that m_1 and m_2 represent moving points.)

89. Prove the following differentiation rules.

(a) $\frac{d}{dx}[\sec x] = \sec x \tan x$ (b) $\frac{d}{dx}[\csc x] = -\csc x \cot x$

(c) $\frac{d}{dx}[\cot x] = -\csc^2 x$

90. ***Rate of Change*** Determine whether there exist any values of x in the interval $[0, 2\pi)$ such that the rate of change of $f(x) = \sec x$ and the rate of change of $g(x) = \csc x$ are equal.

91. ***Modeling Data*** The table shows the quantities q (in millions) of personal computers shipped in the United States and the values v (in billions of dollars) of these shipments for the years 1999 through 2004. The year is represented by t, with $t = 9$ corresponding to 1999. (*Source: U.S. Census Bureau*)

Year, t	9	10	11	12	13	14
q	19.6	15.9	14.6	12.9	15.0	15.8
v	26.8	22.6	18.9	16.2	14.7	15.3

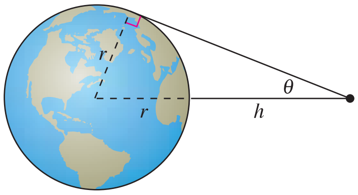
(a) Use a graphing utility to find cubic models for the quantity of personal computers shipped $q(t)$ and the value $v(t)$ of the personal computers.

(b) Graph each model found in part (a).

(c) Find $A = v(t)/q(t)$, then graph A. What does this function represent?

(d) Interpret $A'(t)$ in the context of these data.

92. ***Satellites*** When satellites observe Earth, they can scan only part of Earth's surface. Some satellites have sensors that can measure the angle θ shown in the figure. Let h represent the satellite's distance from Earth's surface and let r represent Earth's radius.



(a) Show that $h = r(\csc \theta - 1)$.

(b) Find the rate at which h is changing with respect to θ when $\theta = 30°$. (Assume $r = 3960$ miles.)

In Exercises 93–100, find the second derivative of the function.

93. $f(x) = x^4 + 2x^3 - 3x^2 - x$ **94.** $f(x) = 8x^6 - 10x^5 + 5x^3$

95. $f(x) = 4x^{3/2}$ **96.** $f(x) = x + 32x^{-2}$

97. $f(x) = \frac{x}{x - 1}$ **98.** $f(x) = \frac{x^2 + 2x - 1}{x}$

99. $f(x) = x \sin x$ **100.** $f(x) = \sec x$

In Exercises 101–104, find the given higher-order derivative.

101. $f'(x) = x^2$, $f''(x)$ **102.** $f''(x) = 2 - \frac{2}{x}$, $f'''(x)$

103. $f'''(x) = 2\sqrt{x}$, $f^{(4)}(x)$ **104.** $f^{(4)}(x) = 2x + 1$, $f^{(6)}(x)$

In Exercises 105–108, use the given information to find $f'(2)$.

$g(2) = 3$ **and** $g'(2) = -2$

$h(2) = -1$ **and** $h'(2) = 4$

105. $f(x) = 2g(x) + h(x)$ **106.** $f(x) = 4 - h(x)$

107. $f(x) = \frac{g(x)}{h(x)}$ **108.** $f(x) = g(x)h(x)$

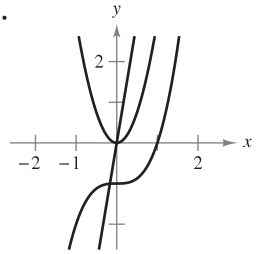
WRITING ABOUT CONCEPTS

109. Sketch the graph of a differentiable function f such that $f(2) = 0$, $f' < 0$ for $-\infty < x < 2$, and $f' > 0$ for $2 < x < \infty$. Explain how you found your answer.

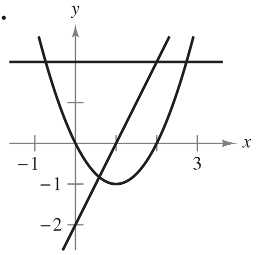
110. Sketch the graph of a differentiable function f such that $f > 0$ and $f' < 0$ for all real numbers x. Explain how you found your answer.

In Exercises 111 and 112, the graphs of f, f', and f'' are shown on the same set of coordinate axes. Identify each graph. Explain your reasoning. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

111.

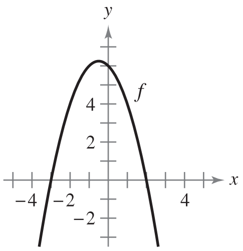


112.

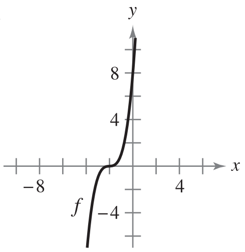


In Exercises 113–116, the graph of f is shown. Sketch the graphs of f' and f''. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*

113.



114.



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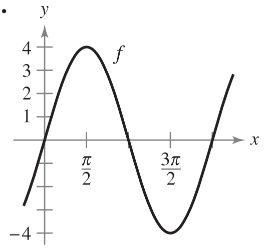
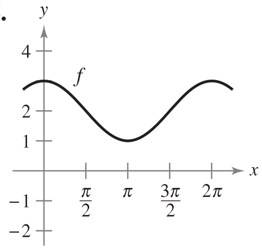


116.



117. ***Acceleration*** The velocity of an object in meters per second is $v(t) = 36 - t^2$, $0 \le t \le 6$. Find the velocity and acceleration of the object when $t = 3$. What can be said about the speed of the object when the velocity and acceleration have opposite signs?

118. ***Acceleration*** An automobile's velocity starting from rest is

$$v(t) = \frac{100t}{2t + 15}$$

where v is measured in feet per second. Find the acceleration at (a) 5 seconds, (b) 10 seconds, and (c) 20 seconds.

119. ***Stopping Distance*** A car is traveling at a rate of 66 feet per second (45 miles per hour) when the brakes are applied. The position function for the car is $s(t) = -8.25t^2 + 66t$, where s is measured in feet and t is measured in seconds. Use this function to complete the table, and find the average velocity during each time interval.

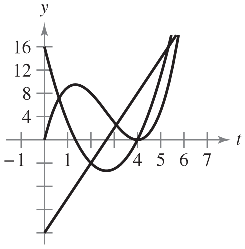
t	0	1	2	3	4
$s(t)$					
$v(t)$					
$a(t)$					

CAPSTONE

120. ***Particle Motion*** The figure shows the graphs of the position, velocity, and acceleration functions of a particle.

(a) Copy the graphs of the functions shown. Identify each graph. Explain your reasoning. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(b) On your sketch, identify when the particle speeds up and when it slows down. Explain your reasoning.



Finding a Pattern **In Exercises 121 and 122, develop a general rule for $f^{(n)}(x)$ given $f(x)$.**

121. $f(x) = x^n$

122. $f(x) = \dfrac{1}{x}$

123. ***Finding a Pattern*** Consider the function $f(x) = g(x)h(x)$.

(a) Use the Product Rule to generate rules for finding $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$.

(b) Use the results of part (a) to write a general rule for $f^{(n)}(x)$.

124. ***Finding a Pattern*** Develop a general rule for $[xf(x)]^{(n)}$ where f is a differentiable function of x.

In Exercises 125 and 126, find the derivatives of the function f for $n = 1, 2, 3,$ and 4. Use the results to write a general rule for $f'(x)$ in terms of n.

125. $f(x) = x^n \sin x$

126. $f(x) = \dfrac{\cos x}{x^n}$

Differential Equations **In Exercises 127–130, verify that the function satisfies the differential equation.**

	Function	*Differential Equation*
127.	$y = \dfrac{1}{x}, x > 0$	$x^3 y'' + 2x^2 y' = 0$
128.	$y = 2x^3 - 6x + 10$	$-y''' - xy'' - 2y' = -24x^2$
129.	$y = 2 \sin x + 3$	$y'' + y = 3$
130.	$y = 3 \cos x + \sin x$	$y'' + y = 0$

True or False? **In Exercises 131–136, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

131. If $y = f(x)g(x)$, then $dy/dx = f'(x)g'(x)$.

132. If $y = (x + 1)(x + 2)(x + 3)(x + 4)$, then $d^5y/dx^5 = 0$.

133. If $f'(c)$ and $g'(c)$ are zero and $h(x) = f(x)g(x)$, then $h'(c) = 0$.

134. If $f(x)$ is an nth-degree polynomial, then $f^{(n+1)}(x) = 0$.

135. The second derivative represents the rate of change of the first derivative.

136. If the velocity of an object is constant, then its acceleration is zero.

137. Find a second-degree polynomial $f(x) = ax^2 + bx + c$ such that its graph has a tangent line with slope 10 at the point $(2, 7)$ and an x-intercept at $(1, 0)$.

138. Consider the third-degree polynomial

$$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0.$$

Determine conditions for a, b, c, and d if the graph of f has (a) no horizontal tangents, (b) exactly one horizontal tangent, and (c) exactly two horizontal tangents. Give an example for each case.

139. Find the derivative of $f(x) = x|x|$. Does $f''(0)$ exist?

140. ***Think About It*** Let f and g be functions whose first and second derivatives exist on an interval I. Which of the following formulas is (are) true?

(a) $fg'' - f''g = (fg' - f'g)'$ (b) $fg'' + f''g = (fg)''$

141. Use the Product Rule twice to prove that if f, g, and h are differentiable functions of x, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

2.4 The Chain Rule

- **Find the derivative of a composite function using the Chain Rule.**
- **Find the derivative of a function using the General Power Rule.**
- **Simplify the derivative of a function using algebra.**
- **Find the derivative of a trigonometric function using the Chain Rule.**

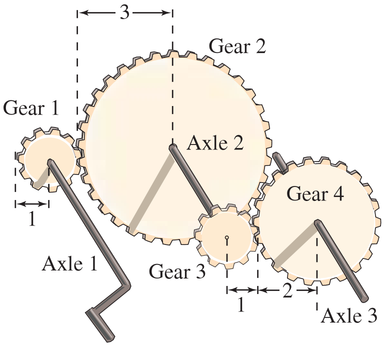
The Chain Rule

This text has yet to discuss one of the most powerful differentiation rules—the **Chain Rule.** This rule deals with composite functions and adds a surprising versatility to the rules discussed in the two previous sections. For example, compare the functions shown below. Those on the left can be differentiated without the Chain Rule, and those on the right are best differentiated with the Chain Rule.

Without the Chain Rule	*With the Chain Rule*
$y = x^2 + 1$	$y = \sqrt{x^2 + 1}$
$y = \sin x$	$y = \sin 6x$
$y = 3x + 2$	$y = (3x + 2)^5$
$y = x + \tan x$	$y = x + \tan x^2$

Basically, the Chain Rule states that if y changes dy/du times as fast as u, and u changes du/dx times as fast as x, then y changes $(dy/du)(du/dx)$ times as fast as x.

EXAMPLE 1 The Derivative of a Composite Function



Axle 1: y revolutions per minute
Axle 2: u revolutions per minute
Axle 3: x revolutions per minute
Figure 2.24

A set of gears is constructed, as shown in Figure 2.24, such that the second and third gears are on the same axle. As the first axle revolves, it drives the second axle, which in turn drives the third axle. Let y, u, and x represent the numbers of revolutions per minute of the first, second, and third axles, respectively. Find dy/du, du/dx, and dy/dx, and show that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Solution Because the circumference of the second gear is three times that of the first, the first axle must make three revolutions to turn the second axle once. Similarly, the second axle must make two revolutions to turn the third axle once, and you can write

$$\frac{dy}{du} = 3 \quad \text{and} \quad \frac{du}{dx} = 2.$$

Combining these two results, you know that the first axle must make six revolutions to turn the third axle once. So, you can write

$$\begin{aligned}\frac{dy}{dx} &= \boxed{\begin{array}{l}\text{Rate of change of first axle}\\ \text{with respect to second axle}\end{array}} \cdot \boxed{\begin{array}{l}\text{Rate of change of second axle}\\ \text{with respect to third axle}\end{array}}\\ &= \frac{dy}{du} \cdot \frac{du}{dx} = 3 \cdot 2 = 6\\ &= \boxed{\begin{array}{l}\text{Rate of change of first axle}\\ \text{with respect to third axle}\end{array}}.\end{aligned}$$

In other words, the rate of change of y with respect to x is the product of the rate of change of y with respect to u and the rate of change of u with respect to x. ■

EXPLORATION

Using the Chain Rule Each of the following functions can be differentiated using rules that you studied in Sections 2.2 and 2.3. For each function, find the derivative using those rules. Then find the derivative using the Chain Rule. Compare your results. Which method is simpler?

a. $\dfrac{2}{3x + 1}$

b. $(x + 2)^3$

c. $\sin 2x$

Example 1 illustrates a simple case of the Chain Rule. The general rule is stated below.

THEOREM 2.10 THE CHAIN RULE

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

PROOF Let $h(x) = f(g(x))$. Then, using the alternative form of the derivative, you need to show that, for $x = c$,

$$h'(c) = f'(g(c))g'(c).$$

An important consideration in this proof is the behavior of g as x approaches c. A problem occurs if there are values of x, other than c, such that $g(x) = g(c)$. Appendix A shows how to use the differentiability of f and g to overcome this problem. For now, assume that $g(x) \neq g(c)$ for values of x other than c. In the proofs of the Product Rule and the Quotient Rule, the same quantity was added and subtracted to obtain the desired form. This proof uses a similar technique—multiplying and dividing by the same (nonzero) quantity. Note that because g is differentiable, it is also continuous, and it follows that $g(x) \to g(c)$ as $x \to c$.

$$\begin{aligned}
h'(c) &= \lim_{x\to c} \frac{f(g(x)) - f(g(c))}{x - c} \\
&= \lim_{x\to c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c}\right], \quad g(x) \neq g(c) \\
&= \left[\lim_{x\to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)}\right]\left[\lim_{x\to c} \frac{g(x) - g(c)}{x - c}\right] \\
&= f'(g(c))g'(c)
\end{aligned}$$

■

When applying the Chain Rule, it is helpful to think of the composite function $f \circ g$ as having two parts—an inner part and an outer part.

Outer function

$$y = f(g(x)) = f(u)$$

Inner function

The derivative of $y = f(u)$ is the derivative of the outer function (at the inner function u) *times* the derivative of the inner function.

$$y' = f'(u) \cdot u'$$

EXAMPLE 2 Decomposition of a Composite Function

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
a. $y = \dfrac{1}{x+1}$	$u = x + 1$	$y = \dfrac{1}{u}$
b. $y = \sin 2x$	$u = 2x$	$y = \sin u$
c. $y = \sqrt{3x^2 - x + 1}$	$u = 3x^2 - x + 1$	$y = \sqrt{u}$
d. $y = \tan^2 x$	$u = \tan x$	$y = u^2$

EXAMPLE 3 Using the Chain Rule

Find dy/dx for $y = (x^2 + 1)^3$.

Solution For this function, you can consider the inside function to be $u = x^2 + 1$. By the Chain Rule, you obtain

$$\frac{dy}{dx} = \underbrace{3(x^2+1)^2}_{\frac{dy}{du}}\underbrace{(2x)}_{\frac{du}{dx}} = 6x(x^2+1)^2.$$

■

STUDY TIP You could also solve the problem in Example 3 without using the Chain Rule by observing that

$$y = x^6 + 3x^4 + 3x^2 + 1$$

and

$$y' = 6x^5 + 12x^3 + 6x.$$

Verify that this is the same as the derivative in Example 3. Which method would you use to find

$$\frac{d}{dx}(x^2 + 1)^{50}?$$

The General Power Rule

The function in Example 3 is an example of one of the most common types of composite functions, $y = [u(x)]^n$. The rule for differentiating such functions is called the **General Power Rule,** and it is a special case of the Chain Rule.

THEOREM 2.11 THE GENERAL POWER RULE

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = n u^{n-1} u'.$$

PROOF Because $y = u^n$, you apply the Chain Rule to obtain

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

$$= \frac{d}{du}[u^n]\frac{du}{dx}.$$

By the (Simple) Power Rule in Section 2.2, you have $D_u[u^n] = n u^{n-1}$, and it follows that

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}.$$

■

EXAMPLE 4 Applying the General Power Rule

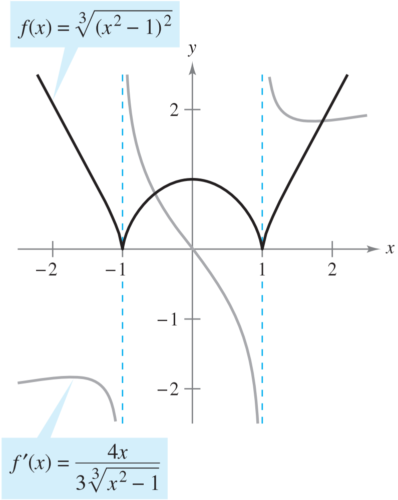
Find the derivative of $f(x) = (3x - 2x^2)^3$.

Solution Let $u = 3x - 2x^2$. Then

$$f(x) = (3x - 2x^2)^3 = u^3$$

and, by the General Power Rule, the derivative is

$$\begin{aligned} f'(x) &= \overset{n}{3}\overbrace{(3x - 2x^2)^2}^{u^{n-1}}\overbrace{\frac{d}{dx}[3x - 2x^2]}^{u'} && \text{Apply General Power Rule.} \\ &= 3(3x - 2x^2)^2(3 - 4x). && \text{Differentiate } 3x - 2x^2. \end{aligned}$$



The derivative of f is 0 at $x = 0$ and is undefined at $x = \pm 1$.
Figure 2.25

EXAMPLE 5 Differentiating Functions Involving Radicals

Find all points on the graph of $f(x) = \sqrt[3]{(x^2 - 1)^2}$ for which $f'(x) = 0$ and those for which $f'(x)$ does not exist.

Solution Begin by rewriting the function as

$$f(x) = (x^2 - 1)^{2/3}.$$

Then, applying the General Power Rule (with $u = x^2 - 1$) produces

$$\begin{aligned} f'(x) &= \overset{n}{\frac{2}{3}}\overbrace{(x^2 - 1)^{-1/3}}^{u^{n-1}}\overbrace{(2x)}^{u'} && \text{Apply General Power Rule.} \\ &= \frac{4x}{3\sqrt[3]{x^2 - 1}}. && \text{Write in radical form.} \end{aligned}$$

So, $f'(x) = 0$ when $x = 0$ and $f'(x)$ does not exist when $x = \pm 1$, as shown in Figure 2.25.

EXAMPLE 6 Differentiating Quotients with Constant Numerators

Differentiate $g(t) = \dfrac{-7}{(2t - 3)^2}$.

Solution Begin by rewriting the function as

$$g(t) = -7(2t - 3)^{-2}.$$

Then, applying the General Power Rule produces

$$\begin{aligned} g'(t) &= \underbrace{(-7)}_{\substack{\text{Constant}\\ \text{Multiple Rule}}}\overbrace{(-2)}^{n}\overbrace{(2t - 3)^{-3}}^{u^{n-1}}\overbrace{(2)}^{u'} && \text{Apply General Power Rule.} \\ &= 28(2t - 3)^{-3} && \text{Simplify.} \\ &= \frac{28}{(2t - 3)^3}. && \text{Write with positive exponent.} \end{aligned}$$

■

NOTE Try differentiating the function in Example 6 using the Quotient Rule. You should obtain the same result, but using the Quotient Rule is less efficient than using the General Power Rule.

Simplifying Derivatives

The next three examples illustrate some techniques for simplifying the "raw derivatives" of functions involving products, quotients, and composites.

EXAMPLE 7 Simplifying by Factoring Out the Least Powers

$$
\begin{aligned}
f(x) &= x^2\sqrt{1-x^2} && \text{Original function}\\
&= x^2(1-x^2)^{1/2} && \text{Rewrite.}\\
f'(x) &= x^2\frac{d}{dx}\left[(1-x^2)^{1/2}\right] + (1-x^2)^{1/2}\frac{d}{dx}\left[x^2\right] && \text{Product Rule}\\
&= x^2\left[\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right] + (1-x^2)^{1/2}(2x) && \text{General Power Rule}\\
&= -x^3(1-x^2)^{-1/2} + 2x(1-x^2)^{1/2} && \text{Simplify.}\\
&= x(1-x^2)^{-1/2}\left[-x^2(1) + 2(1-x^2)\right] && \text{Factor.}\\
&= \frac{x(2-3x^2)}{\sqrt{1-x^2}} && \text{Simplify.}
\end{aligned}
$$

TECHNOLOGY Symbolic differentiation utilities are capable of differentiating very complicated functions. Often, however, the result is given in unsimplified form. If you have access to such a utility, use it to find the derivatives of the functions given in Examples 7, 8, and 9. Then compare the results with those given in these examples.

EXAMPLE 8 Simplifying the Derivative of a Quotient

$$
\begin{aligned}
f(x) &= \frac{x}{\sqrt[3]{x^2+4}} && \text{Original function}\\
&= \frac{x}{(x^2+4)^{1/3}} && \text{Rewrite.}\\
f'(x) &= \frac{(x^2+4)^{1/3}(1) - x(1/3)(x^2+4)^{-2/3}(2x)}{(x^2+4)^{2/3}} && \text{Quotient Rule}\\
&= \frac{1}{3}(x^2+4)^{-2/3}\left[\frac{3(x^2+4)-(2x^2)(1)}{(x^2+4)^{2/3}}\right] && \text{Factor.}\\
&= \frac{x^2+12}{3(x^2+4)^{4/3}} && \text{Simplify.}
\end{aligned}
$$

EXAMPLE 9 Simplifying the Derivative of a Power

$$
\begin{aligned}
y &= \left(\frac{3x-1}{x^2+3}\right)^2 && \text{Original function}\\
y' &= \overset{n}{2}\overbrace{\left(\frac{3x-1}{x^2+3}\right)}^{u^{n-1}}\overbrace{\frac{d}{dx}\left[\frac{3x-1}{x^2+3}\right]}^{u'} && \text{General Power Rule}\\
&= \left[\frac{2(3x-1)}{x^2+3}\right]\left[\frac{(x^2+3)(3)-(3x-1)(2x)}{(x^2+3)^2}\right] && \text{Quotient Rule}\\
&= \frac{2(3x-1)(3x^2+9-6x^2+2x)}{(x^2+3)^3} && \text{Multiply.}\\
&= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3} && \text{Simplify.}
\end{aligned}
$$

■

Trigonometric Functions and the Chain Rule

The “Chain Rule versions” of the derivatives of the six trigonometric functions are as follows.

$$\frac{d}{dx}[\sin u] = (\cos u)\,u' \qquad \frac{d}{dx}[\cos u] = -(\sin u)\,u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)\,u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)\,u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)\,u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)\,u'$$

EXAMPLE 10 Applying the Chain Rule to Trigonometric Functions

a. $y = \sin \overbrace{2x}^{u}$ $\qquad y' = \overbrace{\cos 2x}^{\cos u}\;\overbrace{\frac{d}{dx}[2x]}^{u'} = (\cos 2x)(2) = 2\cos 2x$

b. $y = \cos(x - 1)$ $\qquad y' = -\sin(x - 1)$

c. $y = \tan 3x$ $\qquad y' = 3\sec^2 3x$ ■

Be sure that you understand the mathematical conventions regarding parentheses and trigonometric functions. For instance, in Example 10(a), $\sin 2x$ is written to mean $\sin(2x)$.

EXAMPLE 11 Parentheses and Trigonometric Functions

a. $y = \cos 3x^2 = \cos(3x^2)$ $\qquad y' = (-\sin 3x^2)(6x) = -6x \sin 3x^2$

b. $y = (\cos 3)x^2$ $\qquad y' = (\cos 3)(2x) = 2x \cos 3$

c. $y = \cos(3x)^2 = \cos(9x^2)$ $\qquad y' = (-\sin 9x^2)(18x) = -18x \sin 9x^2$

d. $y = \cos^2 x = (\cos x)^2$ $\qquad y' = 2(\cos x)(-\sin x) = -2\cos x \sin x$

e. $y = \sqrt{\cos x} = (\cos x)^{1/2}$ $\qquad y' = \dfrac{1}{2}(\cos x)^{-1/2}(-\sin x) = -\dfrac{\sin x}{2\sqrt{\cos x}}$ ■

To find the derivative of a function of the form $k(x) = f(g(h(x)))$, you need to apply the Chain Rule twice, as shown in Example 12.

EXAMPLE 12 Repeated Application of the Chain Rule

$$\begin{aligned}
f(t) &= \sin^3 4t && \text{Original function} \\
&= (\sin 4t)^3 && \text{Rewrite.} \\
f'(t) &= 3(\sin 4t)^2 \frac{d}{dt}[\sin 4t] && \text{Apply Chain Rule once.} \\
&= 3(\sin 4t)^2(\cos 4t)\frac{d}{dt}[4t] && \text{Apply Chain Rule a second time.} \\
&= 3(\sin 4t)^2(\cos 4t)(4) && \\
&= 12\sin^2 4t \cos 4t && \text{Simplify.}
\end{aligned}$$

■

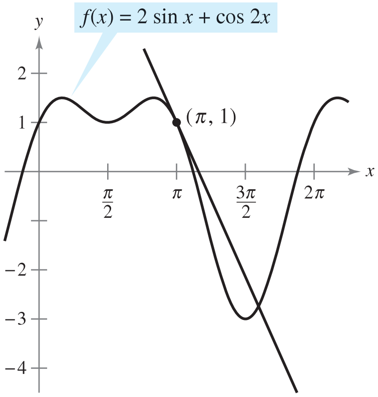


Figure 2.26

EXAMPLE 13 Tangent Line of a Trigonometric Function

Find an equation of the tangent line to the graph of

$$f(x) = 2\sin x + \cos 2x$$

at the point $(\pi, 1)$, as shown in Figure 2.26. Then determine all values of x in the interval $(0, 2\pi)$ at which the graph of f has a horizontal tangent.

Solution Begin by finding $f'(x)$.

$$\begin{aligned} f(x) &= 2\sin x + \cos 2x && \text{Write original function.}\\ f'(x) &= 2\cos x + (-\sin 2x)(2) && \text{Apply Chain Rule to } \cos 2x.\\ &= 2\cos x - 2\sin 2x && \text{Simplify.} \end{aligned}$$

To find the equation of the tangent line at $(\pi, 1)$, evaluate $f'(\pi)$.

$$\begin{aligned} f'(\pi) &= 2\cos \pi - 2\sin 2\pi && \text{Substitute.}\\ &= -2 && \text{Slope of graph at } (\pi, 1) \end{aligned}$$

Now, using the point-slope form of the equation of a line, you can write

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form}\\ y - 1 &= -2(x - \pi) && \text{Substitute for } y_1, m, \text{ and } x_1.\\ y &= 1 - 2x + 2\pi. && \text{Equation of tangent line at } (\pi, 1) \end{aligned}$$

You can then determine that $f'(x) = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$. So, f has horizontal tangents at $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$. ■

STUDY TIP To become skilled at differentiation, you should memorize each rule in words, not symbols. As an aid to memorization, note that the cofunctions (cosine, cotangent, and cosecant) require a negative sign as part of their derivatives.

This section concludes with a summary of the differentiation rules studied so far.

SUMMARY OF DIFFERENTIATION RULES

General Differentiation Rules

Let f, g, and u be differentiable functions of x.

Constant Multiple Rule: $\frac{d}{dx}[cf] = cf'$

Sum or Difference Rule: $\frac{d}{dx}[f \pm g] = f' \pm g'$

Product Rule: $\frac{d}{dx}[fg] = fg' + gf'$

Quotient Rule: $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$

Derivatives of Algebraic Functions

Constant Rule: $\frac{d}{dx}[c] = 0$

(Simple) Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$

Derivatives of Trigonometric Functions

$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$

$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$

Chain Rule

Chain Rule: $\frac{d}{dx}[f(u)] = f'(u)\,u'$

General Power Rule: $\frac{d}{dx}[u^n] = nu^{n-1}u'$

2.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, complete the table.

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1. $y = (5x - 8)^4$		
2. $y = \dfrac{1}{\sqrt{x+1}}$		
3. $y = \sqrt{x^3 - 7}$		
4. $y = 3\tan(\pi x^2)$		
5. $y = \csc^3 x$		
6. $y = \sin\dfrac{5x}{2}$		

In Exercises 7–36, find the derivative of the function.

7. $y = (4x - 1)^3$

8. $y = 2(6 - x^2)^5$

9. $g(x) = 3(4 - 9x)^4$

10. $f(t) = (9t + 2)^{2/3}$

11. $f(t) = \sqrt{5 - t}$

12. $g(x) = \sqrt{9 - 4x}$

13. $y = \sqrt[3]{6x^2 + 1}$

14. $g(x) = \sqrt{x^2 - 2x + 1}$

15. $y = 2\sqrt[4]{9 - x^2}$

16. $f(x) = -3\sqrt[4]{2 - 9x}$

17. $y = \dfrac{1}{x - 2}$

18. $s(t) = \dfrac{1}{t^2 + 3t - 1}$

19. $f(t) = \left(\dfrac{1}{t - 3}\right)^2$

20. $y = -\dfrac{5}{(t + 3)^3}$

21. $y = \dfrac{1}{\sqrt{x + 2}}$

22. $g(t) = \sqrt{\dfrac{1}{t^2 - 2}}$

23. $f(x) = x^2(x - 2)^4$

24. $f(x) = x(3x - 9)^3$

25. $y = x\sqrt{1 - x^2}$

26. $y = \frac{1}{2}x^2\sqrt{16 - x^2}$

27. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

28. $y = \dfrac{x}{\sqrt{x^4 + 4}}$

29. $g(x) = \left(\dfrac{x + 5}{x^2 + 2}\right)^2$

30. $h(t) = \left(\dfrac{t^2}{t^3 + 2}\right)^2$

31. $f(v) = \left(\dfrac{1 - 2v}{1 + v}\right)^3$

32. $g(x) = \left(\dfrac{3x^2 - 2}{2x + 3}\right)^3$

33. $f(x) = ((x^2 + 3)^5 + x)^2$

34. $g(x) = (2 + (x^2 + 1)^4)^3$

35. $f(x) = \sqrt{2 + \sqrt{2 + \sqrt{x}}}$

36. $g(t) = \sqrt{\sqrt{t + 1} + 1}$

CAS **In Exercises 37–42, use a computer algebra system to find the derivative of the function. Then use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.**

37. $y = \dfrac{\sqrt{x} + 1}{x^2 + 1}$

38. $y = \sqrt{\dfrac{2x}{x + 1}}$

39. $y = \sqrt{\dfrac{x + 1}{x}}$

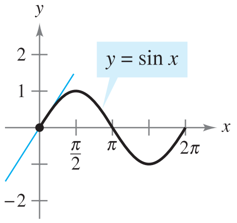
40. $g(x) = \sqrt{x - 1} + \sqrt{x + 1}$

41. $y = \dfrac{\cos \pi x + 1}{x}$

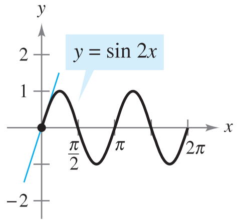
42. $y = x^2 \tan\dfrac{1}{x}$

In Exercises 43 and 44, find the slope of the tangent line to the sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$. What can you conclude about the slope of the sine function $\sin ax$ at the origin?

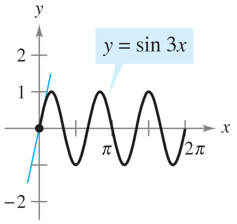
43. (a)



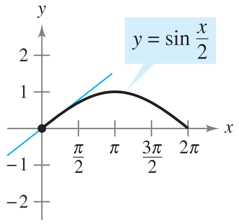
(b)



44. (a)



(b)



In Exercises 45–66, find the derivative of the function.

45. $y = \cos 4x$

46. $y = \sin \pi x$

47. $g(x) = 5\tan 3x$

48. $h(x) = \sec x^2$

49. $y = \sin(\pi x)^2$

50. $y = \cos(1 - 2x)^2$

51. $h(x) = \sin 2x \cos 2x$

52. $g(\theta) = \sec\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{2}\theta\right)$

53. $f(x) = \dfrac{\cot x}{\sin x}$

54. $g(v) = \dfrac{\cos v}{\csc v}$

55. $y = 4\sec^2 x$

56. $g(t) = 5\cos^2 \pi t$

57. $f(\theta) = \tan^2 5\theta$

58. $g(\theta) = \cos^2 8\theta$

59. $f(\theta) = \frac{1}{4}\sin^2 2\theta$

60. $h(t) = 2\cot^2(\pi t + 2)$

61. $f(t) = 3\sec^2(\pi t - 1)$

62. $y = 3x - 5\cos(\pi x)^2$

63. $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2$

64. $y = \sin\sqrt[3]{x} + \sqrt[3]{\sin x}$

65. $y = \sin(\tan 2x)$

66. $y = \cos\sqrt{\sin(\tan \pi x)}$

In Exercises 67–74, evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.

67. $s(t) = \sqrt{t^2 + 6t - 2}$, $(3, 5)$

68. $y = \sqrt[5]{3x^3 + 4x}$, $(2, 2)$

69. $f(x) = \dfrac{5}{x^3 - 2}$, $\left(-2, -\dfrac{1}{2}\right)$

70. $f(x) = \dfrac{1}{(x^2 - 3x)^2}$, $\left(4, \dfrac{1}{16}\right)$

71. $f(t) = \dfrac{3t + 2}{t - 1}$, $(0, -2)$

72. $f(x) = \dfrac{x + 1}{2x - 3}$, $(2, 3)$

73. $y = 26 - \sec^3 4x$, $(0, 25)$

74. $y = \dfrac{1}{x} + \sqrt{\cos x}$, $\left(\dfrac{\pi}{2}, \dfrac{2}{\pi}\right)$

In Exercises 75–82, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of the graphing utility to confirm your results.

Function	*Point*
75. $f(x) = \sqrt{2x^2 - 7}$	$(4, 5)$
76. $f(x) = \frac{1}{3}x\sqrt{x^2 + 5}$	$(2, 2)$
77. $y = (4x^3 + 3)^2$	$(-1, 1)$
78. $f(x) = (9 - x^2)^{2/3}$	$(1, 4)$
79. $f(x) = \sin 2x$	$(\pi, 0)$
80. $y = \cos 3x$	$\left(\frac{\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
81. $f(x) = \tan^2 x$	$\left(\frac{\pi}{4}, 1\right)$
82. $y = 2\tan^3 x$	$\left(\frac{\pi}{4}, 2\right)$

In Exercises 83–86, (a) use a graphing utility to find the derivative of the function at the given point, (b) find an equation of the tangent line to the graph of the function at the given point, and (c) use the utility to graph the function and its tangent line in the same viewing window.

83. $g(t) = \dfrac{3t^2}{\sqrt{t^2 + 2t - 1}}, \quad \left(\frac{1}{2}, \frac{3}{2}\right)$

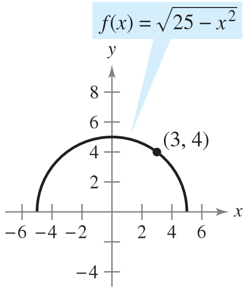
84. $f(x) = \sqrt{x}(2 - x)^2, \quad (4, 8)$

85. $s(t) = \dfrac{(4 - 2t)\sqrt{1 + t}}{3}, \quad \left(0, \frac{4}{3}\right)$

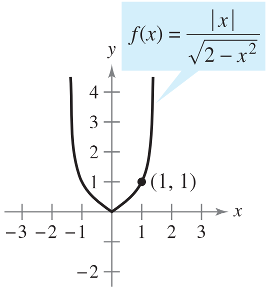
86. $y = (t^2 - 9)\sqrt{t + 2}, \quad (2, -10)$

Famous Curves **In Exercises 87 and 88, find an equation of the tangent line to the graph at the given point. Then use a graphing utility to graph the function and its tangent line in the same viewing window.**

87. Top half of circle



88. Bullet-nose curve



89. ***Horizontal Tangent Line*** Determine the point(s) in the interval $(0, 2\pi)$ at which the graph of $f(x) = 2\cos x + \sin 2x$ has a horizontal tangent.

90. ***Horizontal Tangent Line*** Determine the point(s) at which the graph of $f(x) = \dfrac{x}{\sqrt{2x - 1}}$ has a horizontal tangent.

In Exercises 91–96, find the second derivative of the function.

91. $f(x) = 5(2 - 7x)^4$

92. $f(x) = 4(x^2 - 2)^3$

93. $f(x) = \dfrac{1}{x - 6}$

94. $f(x) = \dfrac{4}{(x + 2)^3}$

95. $f(x) = \sin x^2$

96. $f(x) = \sec^2 \pi x$

In Exercises 97–100, evaluate the second derivative of the function at the given point. Use a computer algebra system to verify your result.

97. $h(x) = \frac{1}{9}(3x + 1)^3, \quad \left(1, \frac{64}{9}\right)$

98. $f(x) = \dfrac{1}{\sqrt{x + 4}}, \quad \left(0, \frac{1}{2}\right)$

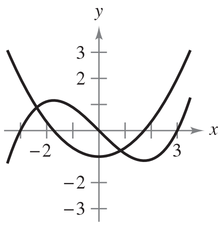
99. $f(x) = \cos x^2, \quad (0, 1)$

100. $g(t) = \tan 2t, \quad \left(\frac{\pi}{6}, \sqrt{3}\right)$

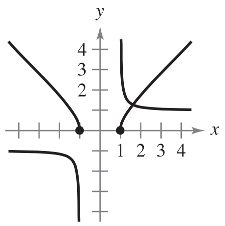
WRITING ABOUT CONCEPTS

In Exercises 101–104, the graphs of a function f and its derivative f' are shown. Label the graphs as f or f' and write a short paragraph stating the criteria you used in making your selection. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

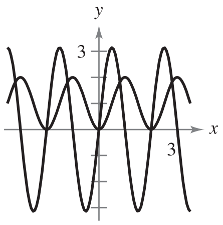
101.



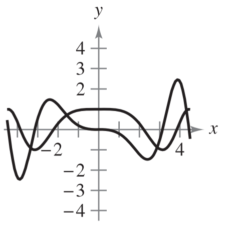
102.



103.



104.



In Exercises 105 and 106, the relationship between f and g is given. Explain the relationship between f' and g'.

105. $g(x) = f(3x)$

106. $g(x) = f(x^2)$

107. ***Think About It*** The table shows some values of the derivative of an unknown function f. Complete the table by finding (if possible) the derivative of each transformation of f.

(a) $g(x) = f(x) - 2$

(b) $h(x) = 2f(x)$

(c) $r(x) = f(-3x)$

(d) $s(x) = f(x + 2)$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

Table for 107

CAPSTONE

108. Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ (if possible) for each of the following. If it is not possible, state what additional information is required.

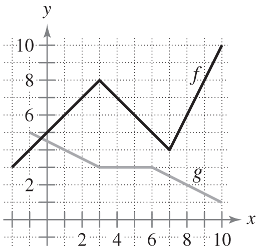
(a) $f(x) = g(x)h(x)$ (b) $f(x) = g(h(x))$

(c) $f(x) = \dfrac{g(x)}{h(x)}$ (d) $f(x) = [g(x)]^3$

In Exercises 109 and 110, the graphs of *f* and *g* are shown. Let $h(x) = f(g(x))$ and $s(x) = g(f(x))$. Find each derivative, if it exists. If the derivative does not exist, explain why.

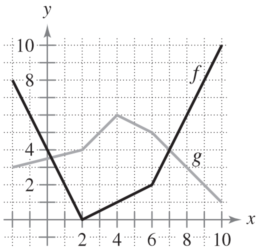
109. (a) Find $h'(1)$.

(b) Find $s'(5)$.



110. (a) Find $h'(3)$.

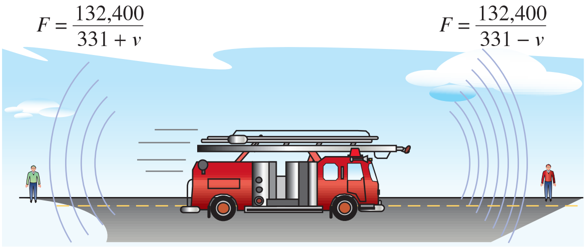
(b) Find $s'(9)$.



111. ***Doppler Effect*** The frequency F of a fire truck siren heard by a stationary observer is $F = 132{,}400/(331 \pm v)$, where $\pm v$ represents the velocity of the accelerating fire truck in meters per second (see figure). Find the rate of change of F with respect to v when

(a) the fire truck is approaching at a velocity of 30 meters per second (use $-v$).

(b) the fire truck is moving away at a velocity of 30 meters per second (use $+v$).


$F = \dfrac{132{,}400}{331 + v}$ $F = \dfrac{132{,}400}{331 - v}$

112. ***Harmonic Motion*** The displacement from equilibrium of an object in harmonic motion on the end of a spring is

$$y = \tfrac{1}{3}\cos 12t - \tfrac{1}{4}\sin 12t$$

where y is measured in feet and t is the time in seconds. Determine the position and velocity of the object when $t = \pi/8$.

113. ***Pendulum*** A 15-centimeter pendulum moves according to the equation $\theta = 0.2 \cos 8t$, where θ is the angular displacement from the vertical in radians and t is the time in seconds. Determine the maximum angular displacement and the rate of change of θ when $t = 3$ seconds.

114. ***Wave Motion*** A buoy oscillates in simple harmonic motion $y = A \cos \omega t$ as waves move past it. The buoy moves a total of 3.5 feet (vertically) from its low point to its high point. It returns to its high point every 10 seconds.

(a) Write an equation describing the motion of the buoy if it is at its high point at $t = 0$.

(b) Determine the velocity of the buoy as a function of t.

115. ***Circulatory System*** The speed S of blood that is r centimeters from the center of an artery is

$$S = C(R^2 - r^2)$$

where C is a constant, R is the radius of the artery, and S is measured in centimeters per second. Suppose a drug is administered and the artery begins to dilate at a rate of dR/dt. At a constant distance r, find the rate at which S changes with respect to t for $C = 1.76 \times 10^5$, $R = 1.2 \times 10^{-2}$, and $dR/dt = 10^{-5}$.

116. ***Modeling Data*** The normal daily maximum temperatures T (in degrees Fahrenheit) for Chicago, Illinois are shown in the table. *(Source: National Oceanic and Atmospheric Administration)*

Month	Jan	Feb	Mar	Apr	May	Jun
Temperature	29.6	34.7	46.1	58.0	69.9	79.2

Month	Jul	Aug	Sep	Oct	Nov	Dec
Temperature	83.5	81.2	73.9	62.1	47.1	34.4

(a) Use a graphing utility to plot the data and find a model for the data of the form

$$T(t) = a + b \sin(ct - d)$$

where T is the temperature and t is the time in months, with $t = 1$ corresponding to January.

(b) Use a graphing utility to graph the model. How well does the model fit the data?

(c) Find T' and use a graphing utility to graph the derivative.

(d) Based on the graph of the derivative, during what times does the temperature change most rapidly? Most slowly? Do your answers agree with your observations of the temperature changes? Explain.

117. ***Modeling Data*** The cost of producing x units of a product is $C = 60x + 1350$. For one week management determined the number of units produced at the end of t hours during an eight-hour shift. The average values of x for the week are shown in the table.

t	0	1	2	3	4	5	6	7	8
x	0	16	60	130	205	271	336	384	392

(a) Use a graphing utility to fit a cubic model to the data.

(b) Use the Chain Rule to find dC/dt.

(c) Explain why the cost function is not increasing at a constant rate during the eight-hour shift.

118. ***Finding a Pattern*** Consider the function $f(x) = \sin \beta x$, where β is a constant.

(a) Find the first-, second-, third-, and fourth-order derivatives of the function.

(b) Verify that the function and its second derivative satisfy the equation $f''(x) + \beta^2 f(x) = 0$.

(c) Use the results of part (a) to write general rules for the even- and odd-order derivatives

$$f^{(2k)}(x) \quad \text{and} \quad f^{(2k-1)}(x).$$

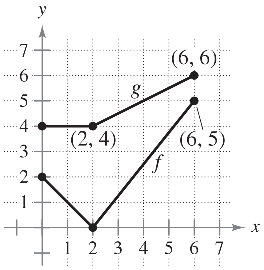
[*Hint:* $(-1)^k$ is positive if k is even and negative if k is odd.]

119. ***Conjecture*** Let f be a differentiable function of period p.

(a) Is the function f' periodic? Verify your answer.

(b) Consider the function $g(x) = f(2x)$. Is the function $g'(x)$ periodic? Verify your answer.

120. ***Think About It*** Let $r(x) = f(g(x))$ and $s(x) = g(f(x))$, where f and g are shown in the figure. Find (a) $r'(1)$ and (b) $s'(4)$.



121. (a) Find the derivative of the function $g(x) = \sin^2 x + \cos^2 x$ in two ways.

(b) For $f(x) = \sec^2 x$ and $g(x) = \tan^2 x$, show that

$$f'(x) = g'(x).$$

122. (a) Show that the derivative of an odd function is even. That is, if $f(-x) = -f(x)$, then $f'(-x) = f'(x)$.

(b) Show that the derivative of an even function is odd. That is, if $f(-x) = f(x)$, then $f'(-x) = -f'(x)$.

123. Let u be a differentiable function of x. Use the fact that $|u| = \sqrt{u^2}$ to prove that

$$\frac{d}{dx}[|u|] = u'\frac{u}{|u|}, \quad u \neq 0.$$

In Exercises 124–127, use the result of Exercise 123 to find the derivative of the function.

124. $g(x) = |3x - 5|$

125. $f(x) = |x^2 - 9|$

126. $h(x) = |x| \cos x$

127. $f(x) = |\sin x|$

Linear and Quadratic Approximations **The linear and quadratic approximations of a function f at $x = a$ are**

$$P_1(x) = f'(a)(x - a) + f(a) \text{ and}$$

$$P_2(x) = \tfrac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a).$$

In Exercises 128 and 129, (a) find the specified linear and quadratic approximations of f, (b) use a graphing utility to graph f and the approximations, (c) determine whether P_1 or P_2 is the better approximation, and (d) state how the accuracy changes as you move farther from $x = a$.

128. $f(x) = \tan x$

$$a = \frac{\pi}{4}$$

129. $f(x) = \sec x$

$$a = \frac{\pi}{6}$$

True or False? **In Exercises 130–132, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

130. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}$.

131. If $f(x) = \sin^2(2x)$, then $f'(x) = 2(\sin 2x)(\cos 2x)$.

132. If y is a differentiable function of u, u is a differentiable function of v, and v is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

PUTNAM EXAM CHALLENGE

133. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, where $a_1, a_2, \ldots, a_n$ are real numbers and where n is a positive integer. Given that $|f(x)| \leq |\sin x|$ for all real x, prove that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$.

134. Let k be a fixed positive integer. The nth derivative of $\dfrac{1}{x^k - 1}$ has the form

$$\frac{P_n(x)}{(x^k - 1)^{n+1}}$$

where $P_n(x)$ is a polynomial. Find $P_n(1)$.

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2.5 Implicit Differentiation

- **Distinguish between functions written in implicit form and explicit form.**
- **Use implicit differentiation to find the derivative of a function.**

EXPLORATION

Graphing an Implicit Equation
How could you use a graphing utility to sketch the graph of the equation

$$x^2 - 2y^3 + 4y = 2?$$

Here are two possible approaches.

a. Solve the equation for x. Switch the roles of x and y and graph the two resulting equations. The combined graphs will show a 90° rotation of the graph of the original equation.

b. Set the graphing utility to *parametric* mode and graph the equations

$$x = -\sqrt{2t^3 - 4t + 2}$$
$$y = t$$

and

$$x = \sqrt{2t^3 - 4t + 2}$$
$$y = t.$$

From either of these two approaches, can you decide whether the graph has a tangent line at the point $(0, 1)$? Explain your reasoning.

Implicit and Explicit Functions

Up to this point in the text, most functions have been expressed in **explicit form.** For example, in the equation

$$y = 3x^2 - 5 \qquad \text{Explicit form}$$

the variable y is explicitly written as a function of x. Some functions, however, are only implied by an equation. For instance, the function $y = 1/x$ is defined **implicitly** by the equation $xy = 1$. Suppose you were asked to find dy/dx for this equation. You could begin by writing y explicitly as a function of x and then differentiating.

Implicit Form	*Explicit Form*	*Derivative*
$xy = 1$	$y = \frac{1}{x} = x^{-1}$	$\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$

This strategy works whenever you can solve for the function explicitly. You cannot, however, use this procedure when you are unable to solve for y as a function of x. For instance, how would you find dy/dx for the equation

$$x^2 - 2y^3 + 4y = 2$$

where it is very difficult to express y as a function of x explicitly? To do this, you can use **implicit differentiation.**

To understand how to find dy/dx implicitly, you must realize that the differentiation is taking place *with respect to* x. This means that when you differentiate terms involving x alone, you can differentiate as usual. However, when you differentiate terms involving y, you must apply the Chain Rule, because you are assuming that y is defined implicitly as a differentiable function of x.

EXAMPLE 1 Differentiating with Respect to *x*

a. $\frac{d}{dx}[x^3] = 3x^2$ Variables agree: use Simple Power Rule.

Variables agree

b. $\frac{d}{dx}\overbrace{[y^3]}^{u^n} = \overbrace{3y^2}^{nu^{n-1}}\overbrace{\frac{dy}{dx}}^{u'}$ Variables disagree: use Chain Rule.

Variables disagree

c. $\frac{d}{dx}[x + 3y] = 1 + 3\frac{dy}{dx}$ Chain Rule: $\frac{d}{dx}[3y] = 3y'$

d. $\frac{d}{dx}[xy^2] = x\frac{d}{dx}[y^2] + y^2\frac{d}{dx}[x]$ Product Rule

$= x\left(2y\frac{dy}{dx}\right) + y^2(1)$ Chain Rule

$= 2xy\frac{dy}{dx} + y^2$ Simplify. ■

Implicit Differentiation

GUIDELINES FOR IMPLICIT DIFFERENTIATION

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx.

In Example 2, note that implicit differentiation can produce an expression for dy/dx that contains both x and y.

EXAMPLE 2 Implicit Differentiation

Find dy/dx given that $y^3 + y^2 - 5y - x^2 = -4$.

Solution

1. Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}[y^3 + y^2 - 5y - x^2] = \frac{d}{dx}[-4]$$

$$\frac{d}{dx}[y^3] + \frac{d}{dx}[y^2] - \frac{d}{dx}[5y] - \frac{d}{dx}[x^2] = \frac{d}{dx}[-4]$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

2. Collect the dy/dx terms on the left side of the equation and move all other terms to the right side of the equation.

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} = 2x$$

3. Factor dy/dx out of the left side of the equation.

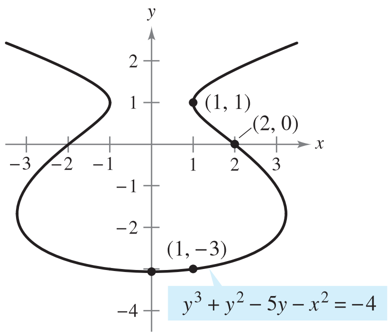
$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

4. Solve for dy/dx by dividing by $(3y^2 + 2y - 5)$.

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$

■

To see how you can use an *implicit derivative*, consider the graph shown in Figure 2.27. From the graph, you can see that y is not a function of x. Even so, the derivative found in Example 2 gives a formula for the slope of the tangent line at a point on this graph. The slopes at several points on the graph are shown below the graph.



Point on Graph	*Slope of Graph*
(2, 0)	$-\frac{4}{5}$
(1, −3)	$\frac{1}{8}$
$x = 0$	0
(1, 1)	Undefined

The implicit equation

$$y^3 + y^2 - 5y - x^2 = -4$$

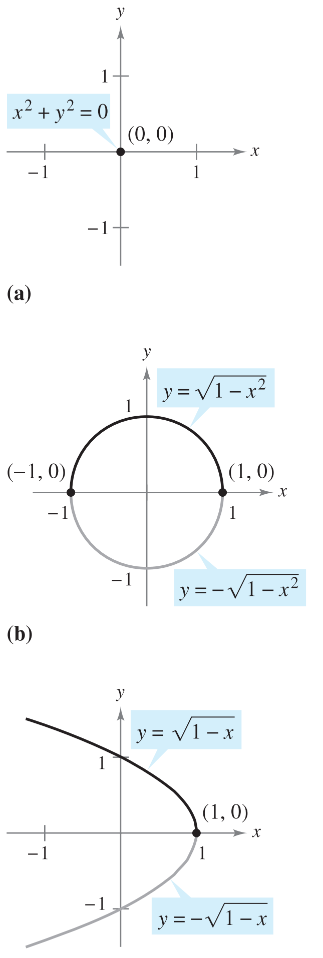
has the derivative

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}.$$

Figure 2.27

TECHNOLOGY With most graphing utilities, it is easy to graph an equation that explicitly represents y as a function of x. Graphing other equations, however, can require some ingenuity. For instance, to graph the equation given in Example 2, use a graphing utility, set in *parametric* mode, to graph the parametric representations $x = \sqrt{t^3 + t^2 - 5t + 4}$, $y = t$, and $x = -\sqrt{t^3 + t^2 - 5t + 4}$, $y = t$, for $-5 \le t \le 5$. How does the result compare with the graph shown in Figure 2.27?

It is meaningless to solve for dy/dx in an equation that has no solution points. (For example, $x^2 + y^2 = -4$ has no solution points.) If, however, a segment of a graph can be represented by a differentiable function, dy/dx will have meaning as the slope at each point on the segment. Recall that a function is not differentiable at (a) points with vertical tangents and (b) points at which the function is not continuous.



Some graph segments can be represented by differentiable functions.

Figure 2.28

EXAMPLE 3 Representing a Graph by Differentiable Functions

If possible, represent y as a differentiable function of x.

a. $x^2 + y^2 = 0$ **b.** $x^2 + y^2 = 1$ **c.** $x + y^2 = 1$

Solution

a. The graph of this equation is a single point. So, it does not define y as a differentiable function of x. See Figure 2.28(a).

b. The graph of this equation is the unit circle, centered at $(0, 0)$. The upper semicircle is given by the differentiable function

$$y = \sqrt{1 - x^2}, \quad -1 < x < 1$$

and the lower semicircle is given by the differentiable function

$$y = -\sqrt{1 - x^2}, \quad -1 < x < 1.$$

At the points $(-1, 0)$ and $(1, 0)$, the slope of the graph is undefined. See Figure 2.28(b).

c. The upper half of this parabola is given by the differentiable function

$$y = \sqrt{1 - x}, \quad x < 1$$

and the lower half of this parabola is given by the differentiable function

$$y = -\sqrt{1 - x}, \quad x < 1.$$

At the point $(1, 0)$, the slope of the graph is undefined. See Figure 2.28(c).

EXAMPLE 4 Finding the Slope of a Graph Implicitly

Determine the slope of the tangent line to the graph of

$$x^2 + 4y^2 = 4$$

at the point $\left(\sqrt{2}, -1/\sqrt{2}\right)$. See Figure 2.29.

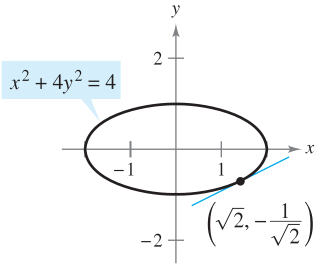


Figure 2.29

Solution

$$x^2 + 4y^2 = 4 \qquad \text{Write original equation.}$$

$$2x + 8y\frac{dy}{dx} = 0 \qquad \text{Differentiate with respect to } x.$$

$$\frac{dy}{dx} = \frac{-2x}{8y} = \frac{-x}{4y} \qquad \text{Solve for } \frac{dy}{dx}.$$

So, at $\left(\sqrt{2}, -1/\sqrt{2}\right)$, the slope is

$$\frac{dy}{dx} = \frac{-\sqrt{2}}{-4/\sqrt{2}} = \frac{1}{2}. \qquad \text{Evaluate } \frac{dy}{dx} \text{ when } x = \sqrt{2} \text{ and } y = -\frac{1}{\sqrt{2}}.$$

■

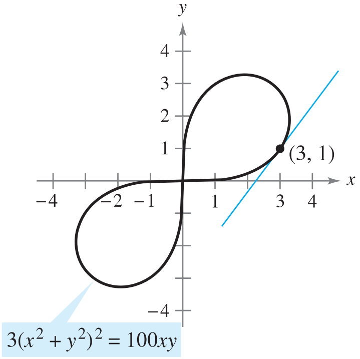
NOTE To see the benefit of implicit differentiation, try doing Example 4 using the explicit function $y = -\frac{1}{2}\sqrt{4 - x^2}$. ■

EXAMPLE 5 Finding the Slope of a Graph Implicitly

Determine the slope of the graph of $3(x^2 + y^2)^2 = 100xy$ at the point $(3, 1)$.

Solution

$$\begin{aligned}
\frac{d}{dx}[3(x^2 + y^2)^2] &= \frac{d}{dx}[100xy] \\
3(2)(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) &= 100\left[x\frac{dy}{dx} + y(1)\right] \\
12y(x^2 + y^2)\frac{dy}{dx} - 100x\frac{dy}{dx} &= 100y - 12x(x^2 + y^2) \\
[12y(x^2 + y^2) - 100x]\frac{dy}{dx} &= 100y - 12x(x^2 + y^2) \\
\frac{dy}{dx} &= \frac{100y - 12x(x^2 + y^2)}{-100x + 12y(x^2 + y^2)} \\
&= \frac{25y - 3x(x^2 + y^2)}{-25x + 3y(x^2 + y^2)}
\end{aligned}$$



Leminscate
Figure 2.30

At the point $(3, 1)$, the slope of the graph is

$$\frac{dy}{dx} = \frac{25(1) - 3(3)(3^2 + 1^2)}{-25(3) + 3(1)(3^2 + 1^2)} = \frac{25 - 90}{-75 + 30} = \frac{-65}{-45} = \frac{13}{9}$$

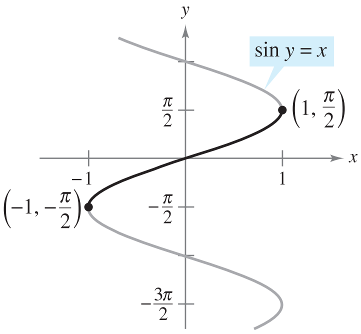
as shown in Figure 2.30. This graph is called a **lemniscate.**

EXAMPLE 6 Determining a Differentiable Function

Find dy/dx implicitly for the equation $\sin y = x$. Then find the largest interval of the form $-a < y < a$ on which y is a differentiable function of x (see Figure 2.31).

Solution

$$\begin{aligned}
\frac{d}{dx}[\sin y] &= \frac{d}{dx}[x] \\
\cos y\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos y}
\end{aligned}$$



The derivative is $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$.

Figure 2.31

The largest interval about the origin for which y is a differentiable function of x is $-\pi/2 < y < \pi/2$. To see this, note that $\cos y$ is positive for all y in this interval and is 0 at the endpoints. If you restrict y to the interval $-\pi/2 < y < \pi/2$, you should be able to write dy/dx explicitly as a function of x. To do this, you can use

$$\begin{aligned}
\cos y &= \sqrt{1 - \sin^2 y} \\
&= \sqrt{1 - x^2}, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}
\end{aligned}$$

and conclude that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

■

You will study this example further when inverse trigonometric functions are defined in Section 5.6.

The Granger Collection

ISAAC BARROW (1630–1677)

The graph in Figure 2.32 is called the **kappa curve** because it resembles the Greek letter kappa, κ. The general solution for the tangent line to this curve was discovered by the English mathematician Isaac Barrow. Newton was Barrow's student, and they corresponded frequently regarding their work in the early development of calculus.

With implicit differentiation, the form of the derivative often can be simplified (as in Example 6) by an appropriate use of the *original* equation. A similar technique can be used to find and simplify higher-order derivatives obtained implicitly.

EXAMPLE 7 Finding the Second Derivative Implicitly

Given $x^2 + y^2 = 25$, find $\dfrac{d^2y}{dx^2}\cdot$ Evaluate the first and second derivatives at the point $(-3, 4)$.

Solution Differentiating each term with respect to x produces

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} = -\frac{x}{y}. \end{aligned}$$

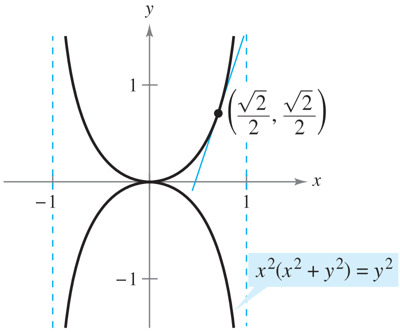
At $(-3, 4)$: $\dfrac{dy}{dx} = -\dfrac{(-3)}{4} = \dfrac{3}{4}.$

Differentiating a second time with respect to x yields

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{(y)(1) - (x)(dy/dx)}{y^2} && \text{Quotient Rule} \\ &= -\frac{y - (x)(-x/y)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{25}{y^3}. \end{aligned}$$

At $(-3, 4)$: $\dfrac{d^2y}{dx^2} = -\dfrac{25}{4^3} = -\dfrac{25}{64}.$

EXAMPLE 8 Finding a Tangent Line to a Graph



The kappa curve
Figure 2.32

Find the tangent line to the graph given by $x^2(x^2 + y^2) = y^2$ at the point $\left(\sqrt{2}/2, \sqrt{2}/2\right)$, as shown in Figure 2.32.

Solution By rewriting and differentiating implicitly, you obtain

$$\begin{aligned} x^4 + x^2y^2 - y^2 &= 0 \\ 4x^3 + x^2\left(2y\frac{dy}{dx}\right) + 2xy^2 - 2y\frac{dy}{dx} &= 0 \\ 2y(x^2 - 1)\frac{dy}{dx} &= -2x(2x^2 + y^2) \\ \frac{dy}{dx} &= \frac{x(2x^2 + y^2)}{y(1 - x^2)}. \end{aligned}$$

At the point $\left(\sqrt{2}/2, \sqrt{2}/2\right)$, the slope is

$$\frac{dy}{dx} = \frac{\left(\sqrt{2}/2\right)[2(1/2) + (1/2)]}{\left(\sqrt{2}/2\right)[1 - (1/2)]} = \frac{3/2}{1/2} = 3$$

and the equation of the tangent line at this point is

$$\begin{aligned} y - \frac{\sqrt{2}}{2} &= 3\left(x - \frac{\sqrt{2}}{2}\right) \\ y &= 3x - \sqrt{2}. \end{aligned}$$

■

2.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–16, find dy/dx by implicit differentiation.

1. $x^2 + y^2 = 9$

2. $x^2 - y^2 = 25$

3. $x^{1/2} + y^{1/2} = 16$

4. $x^3 + y^3 = 64$

5. $x^3 - xy + y^2 = 7$

6. $x^2y + y^2x = -2$

7. $x^3y^3 - y = x$

8. $\sqrt{xy} = x^2y + 1$

9. $x^3 - 3x^2y + 2xy^2 = 12$

10. $4 \cos x \sin y = 1$

11. $\sin x + 2 \cos 2y = 1$

12. $(\sin \pi x + \cos \pi y)^2 = 2$

13. $\sin x = x(1 + \tan y)$

14. $\cot y = x - y$

15. $y = \sin xy$

16. $x = \sec \dfrac{1}{y}$

In Exercises 17–20, (a) find two explicit functions by solving the equation for y in terms of x, (b) sketch the graph of the equation and label the parts given by the corresponding explicit functions, (c) differentiate the explicit functions, and (d) find dy/dx and show that the result is equivalent to that of part (c).

17. $x^2 + y^2 = 64$

18. $x^2 + y^2 - 4x + 6y + 9 = 0$

19. $16x^2 + 25y^2 = 400$

20. $16y^2 - x^2 = 16$

In Exercises 21–28, find dy/dx by implicit differentiation and evaluate the derivative at the given point.

21. $xy = 6$, $(-6, -1)$

22. $x^2 - y^3 = 0$, $(1, 1)$

23. $y^2 = \dfrac{x^2 - 49}{x^2 + 49}$, $(7, 0)$

24. $(x + y)^3 = x^3 + y^3$, $(-1, 1)$

25. $x^{2/3} + y^{2/3} = 5$, $(8, 1)$

26. $x^3 + y^3 = 6xy - 1$, $(2, 3)$

27. $\tan(x + y) = x$, $(0, 0)$

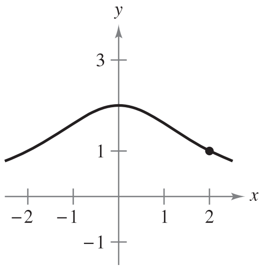
28. $x \cos y = 1$, $\left(2, \dfrac{\pi}{3}\right)$

Famous Curves **In Exercises 29–32, find the slope of the tangent line to the graph at the given point.**

29. Witch of Agnesi:

$(x^2 + 4)y = 8$

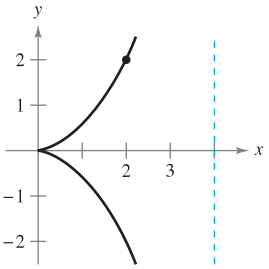
Point: $(2, 1)$



30. Cissoid:

$(4 - x)y^2 = x^3$

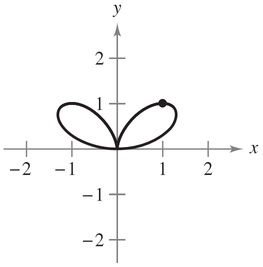
Point: $(2, 2)$



31. Bifolium:

$(x^2 + y^2)^2 = 4x^2y$

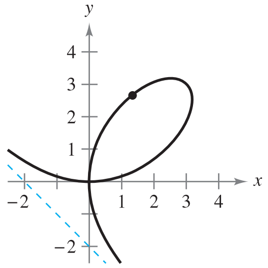
Point: $(1, 1)$



32. Folium of Descartes:

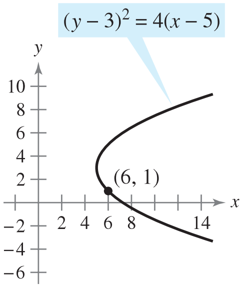
$x^3 + y^3 - 6xy = 0$

Point: $\left(\frac{4}{3}, \frac{8}{3}\right)$

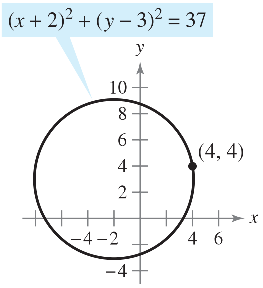


Famous Curves **In Exercises 33–40, find an equation of the tangent line to the graph at the given point. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

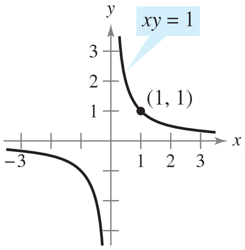
33. Parabola



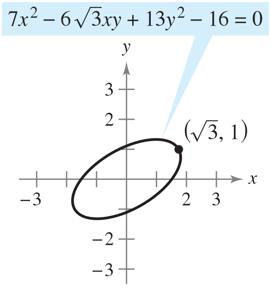
34. Circle



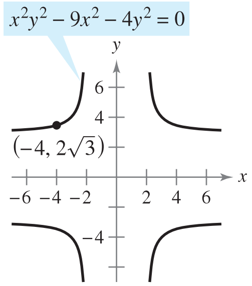
35. Rotated hyperbola



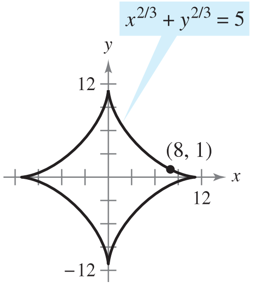
36. Rotated ellipse



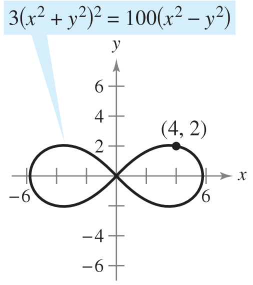
37. Cruciform



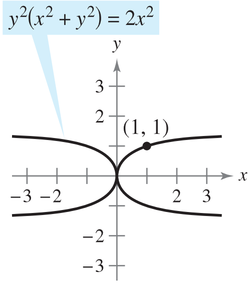
38. Astroid



39. Lemniscate



40. Kappa curve



41. (a) Use implicit differentiation to find an equation of the tangent line to the ellipse $\frac{x^2}{2} + \frac{y^2}{8} = 1$ at $(1, 2)$.

(b) Show that the equation of the tangent line to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_0, y_0) is $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$.

42. (a) Use implicit differentiation to find an equation of the tangent line to the hyperbola $\frac{x^2}{6} - \frac{y^2}{8} = 1$ at $(3, -2)$.

(b) Show that the equation of the tangent line to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_0, y_0) is $\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$.

In Exercises 43 and 44, find *dy/dx* implicitly and find the largest interval of the form $-a < y < a$ or $0 < y < a$ such that *y* is a differentiable function of *x*. Write *dy/dx* as a function of *x*.

43. $\tan y = x$

44. $\cos y = x$

In Exercises 45–50, find d^2y/dx^2 in terms of *x* and *y*.

45. $x^2 + y^2 = 4$

46. $x^2y^2 - 2x = 3$

47. $x^2 - y^2 = 36$

48. $1 - xy = x - y$

49. $y^2 = x^3$

50. $y^2 = 10x$

In Exercises 51 and 52, use a graphing utility to graph the equation. Find an equation of the tangent line to the graph at the given point and graph the tangent line in the same viewing window.

51. $\sqrt{x} + \sqrt{y} = 5, \quad (9, 4)$

52. $y^2 = \frac{x-1}{x^2+1}, \quad \left(2, \frac{\sqrt{5}}{5}\right)$

In Exercises 53 and 54, find equations for the tangent line and normal line to the circle at each given point. (The *normal line* at a point is perpendicular to the tangent line at the point.) Use a graphing utility to graph the equation, tangent line, and normal line.

53. $x^2 + y^2 = 25$
$(4, 3), (-3, 4)$

54. $x^2 + y^2 = 36$
$(6, 0), (5, \sqrt{11})$

55. Show that the normal line at any point on the circle $x^2 + y^2 = r^2$ passes through the origin.

56. Two circles of radius 4 are tangent to the graph of $y^2 = 4x$ at the point $(1, 2)$. Find equations of these two circles.

In Exercises 57 and 58, find the points at which the graph of the equation has a vertical or horizontal tangent line.

57. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

58. $4x^2 + y^2 - 8x + 4y + 4 = 0$

Orthogonal Trajectories **In Exercises 59–62, use a graphing utility to sketch the intersecting graphs of the equations and show that they are orthogonal. [Two graphs are *orthogonal* if at their point(s) of intersection their tangent lines are perpendicular to each other.]**

59. $2x^2 + y^2 = 6$
$y^2 = 4x$

60. $y^2 = x^3$
$2x^2 + 3y^2 = 5$

61. $x + y = 0$
$x = \sin y$

62. $x^3 = 3(y - 1)$
$x(3y - 29) = 3$

Orthogonal Trajectories **In Exercises 63 and 64, verify that the two families of curves are orthogonal where *C* and *K* are real numbers. Use a graphing utility to graph the two families for two values of *C* and two values of *K*.**

63. $xy = C, \quad x^2 - y^2 = K$

64. $x^2 + y^2 = C^2, \quad y = Kx$

In Exercises 65–68, differentiate (a) with respect to *x* (*y* is a function of *x*) and (b) with respect to *t* (*x* and *y* are functions of *t*).

65. $2y^2 - 3x^4 = 0$

66. $x^2 - 3xy^2 + y^3 = 10$

67. $\cos \pi y - 3 \sin \pi x = 1$

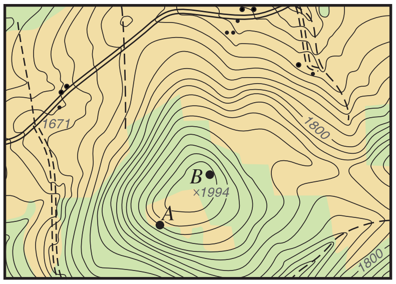
68. $4 \sin x \cos y = 1$

WRITING ABOUT CONCEPTS

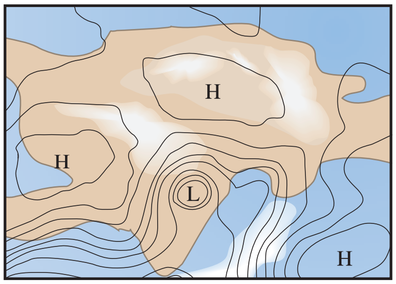
69. Describe the difference between the explicit form of a function and an implicit equation. Give an example of each.

70. In your own words, state the guidelines for implicit differentiation.

71. ***Orthogonal Trajectories*** The figure below shows the topographic map carried by a group of hikers. The hikers are in a wooded area on top of the hill shown on the map and they decide to follow a path of steepest descent (orthogonal trajectories to the contours on the map). Draw their routes if they start from point *A* and if they start from point *B*. If their goal is to reach the road along the top of the map, which starting point should they use? To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



72. ***Weather Map*** The weather map shows several *isobars*—curves that represent areas of constant air pressure. Three high pressures H and one low pressure L are shown on the map. Given that wind speed is greatest along the orthogonal trajectories of the isobars, use the map to determine the areas having high wind speed.



73. Consider the equation $x^4 = 4(4x^2 - y^2)$.

(a) Use a graphing utility to graph the equation.

(b) Find and graph the four tangent lines to the curve for $y = 3$.

(c) Find the exact coordinates of the point of intersection of the two tangent lines in the first quadrant.

CAPSTONE

74. Determine if the statement is true. If it is false, explain why and correct it. For each statement, assume y is a function of x.

(a) $\dfrac{d}{dx}\cos(x^2) = -2x\sin(x^2)$ (b) $\dfrac{d}{dy}\cos(y^2) = 2y\sin(y^2)$

(c) $\dfrac{d}{dx}\cos(y^2) = -2y\sin(y^2)$

75. Let L be any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$. Show that the sum of the x- and y-intercepts of L is c.

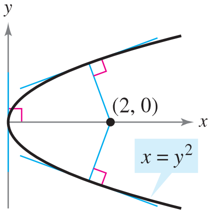
76. Prove (Theorem 2.3) that $d/dx[x^n] = nx^{n-1}$ for the case in which n is a rational number. (*Hint:* Write $y = x^{p/q}$ in the form $y^q = x^p$ and differentiate implicitly. Assume that p and q are integers, where $q > 0$.)

77. ***Slope*** Find all points on the circle $x^2 + y^2 = 100$ where the slope is $\frac{3}{4}$.

78. ***Horizontal Tangent*** Determine the point(s) at which the graph of $y^4 = y^2 - x^2$ has a horizontal tangent.

79. ***Tangent Lines*** Find equations of both tangent lines to the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ that passes through the point $(4, 0)$.

80. ***Normals to a Parabola*** The graph shows the normal lines from the point $(2, 0)$ to the graph of the parabola $x = y^2$. How many normal lines are there from the point $(x_0, 0)$ to the graph of the parabola if (a) $x_0 = \frac{1}{4}$, (b) $x_0 = \frac{1}{2}$, and (c) $x_0 = 1$? For what value of x_0 are two of the normal lines perpendicular to each other?



81. ***Normal Lines*** (a) Find an equation of the normal line to the ellipse $\dfrac{x^2}{32} + \dfrac{y^2}{8} = 1$ at the point $(4, 2)$. (b) Use a graphing utility to graph the ellipse and the normal line. (c) At what other point does the normal line intersect the ellipse?

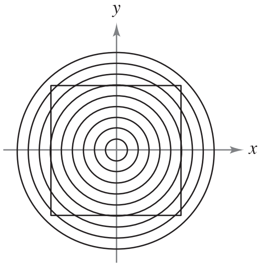
SECTION PROJECT

Optical Illusions

In each graph below, an optical illusion is created by having lines intersect a family of curves. In each case, the lines appear to be curved. Find the value of dy/dx for the given values of x and y.

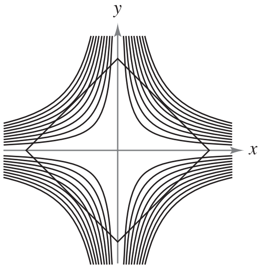
(a) Circles: $x^2 + y^2 = C^2$

$x = 3, y = 4, C = 5$



(b) Hyperbolas: $xy = C$

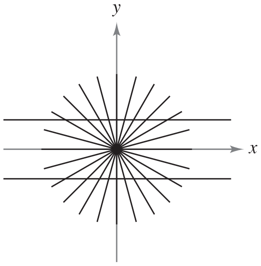
$x = 1, y = 4, C = 4$



(c) Lines: $ax = by$

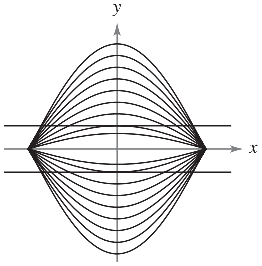
$x = \sqrt{3}, y = 3,$

$a = \sqrt{3}, b = 1$



(d) Cosine curves: $y = C\cos x$

$x = \dfrac{\pi}{3}, y = \dfrac{1}{3}, C = \dfrac{2}{3}$



■ **FOR FURTHER INFORMATION** For more information on the mathematics of optical illusions, see the article "Descriptive Models for Perception of Optical Illusions" by David A. Smith in *The UMAP Journal*.

2.6 Related Rates

- **Find a related rate.**
- **Use related rates to solve real-life problems.**

Finding Related Rates

You have seen how the Chain Rule can be used to find dy/dx implicitly. Another important use of the Chain Rule is to find the rates of change of two or more related variables that are changing with respect to *time*.

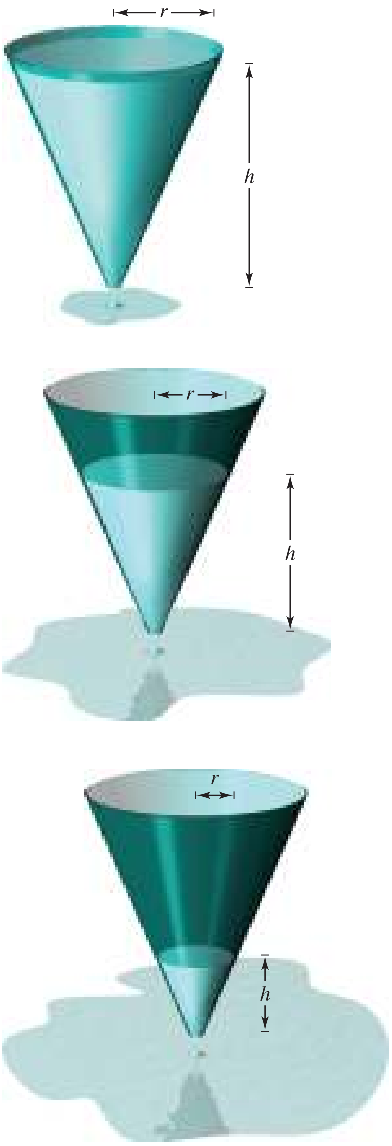
For example, when water is drained out of a conical tank (see Figure 2.33), the volume V, the radius r, and the height h of the water level are all functions of time t. Knowing that these variables are related by the equation

$$V = \frac{\pi}{3} r^2 h \qquad \text{Original equation}$$

you can differentiate implicitly with respect to t to obtain the **related-rate** equation

$$\begin{aligned} \frac{d}{dt}(V) &= \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right) \\ \frac{dV}{dt} &= \frac{\pi}{3}\left[r^2 \frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] && \text{Differentiate with respect to } t. \\ &= \frac{\pi}{3}\left(r^2 \frac{dh}{dt} + 2rh\frac{dr}{dt}\right). \end{aligned}$$

From this equation you can see that the rate of change of V is related to the rates of change of both h and r.



Volume is related to radius and height.
Figure 2.33

FOR FURTHER INFORMATION To learn more about the history of related-rate problems, see the article "The Lengthening Shadow: The Story of Related Rates" by Bill Austin, Don Barry, and David Berman in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

EXPLORATION

Finding a Related Rate In the conical tank shown in Figure 2.33, suppose that the height of the water level is changing at a rate of -0.2 foot per minute and the radius is changing at a rate of -0.1 foot per minute. What is the rate of change in the volume when the radius is $r = 1$ foot and the height is $h = 2$ feet? Does the rate of change in the volume depend on the values of r and h? Explain.

EXAMPLE 1 Two Rates That Are Related

Suppose x and y are both differentiable functions of t and are related by the equation $y = x^2 + 3$. Find dy/dt when $x = 1$, given that $dx/dt = 2$ when $x = 1$.

Solution Using the Chain Rule, you can differentiate both sides of the equation *with respect to t*.

$$\begin{aligned} y &= x^2 + 3 && \text{Write original equation.} \\ \frac{d}{dt}[y] &= \frac{d}{dt}[x^2 + 3] && \text{Differentiate with respect to } t. \\ \frac{dy}{dt} &= 2x\frac{dx}{dt} && \text{Chain Rule} \end{aligned}$$

When $x = 1$ and $dx/dt = 2$, you have

$$\frac{dy}{dt} = 2(1)(2) = 4.$$

■

Problem Solving with Related Rates

In Example 1, you were *given* an equation that related the variables x and y and were asked to find the rate of change of y when $x = 1$.

Equation: $y = x^2 + 3$

Given rate: $\frac{dx}{dt} = 2$ when $x = 1$

Find: $\frac{dy}{dt}$ when $x = 1$

In each of the remaining examples in this section, you must *create* a mathematical model from a verbal description.

EXAMPLE 2 Ripples in a Pond

© Russ Bishop/Alamy

Total area increases as the outer radius increases.
Figure 2.34

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles, as shown in Figure 2.34. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

Solution The variables r and A are related by $A = \pi r^2$. The rate of change of the radius r is $dr/dt = 1$.

Equation: $A = \pi r^2$

Given rate: $\frac{dr}{dt} = 1$

Find: $\frac{dA}{dt}$ when $r = 4$

With this information, you can proceed as in Example 1.

$$\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2] \qquad \text{Differentiate with respect to } t.$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \qquad \text{Chain Rule}$$

$$\frac{dA}{dt} = 2\pi(4)(1) = 8\pi \qquad \text{Substitute 4 for } r \text{ and 1 for } dr/dt.$$

When the radius is 4 feet, the area is changing at a rate of 8π square feet per second. ■

GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t*.
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

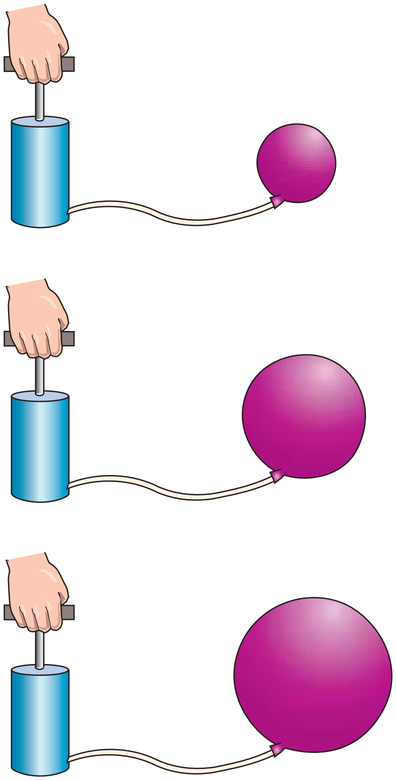
NOTE When using these guidelines, be sure you perform Step 3 before Step 4. Substituting the known values of the variables before differentiating will produce an inappropriate derivative.

The table below lists examples of mathematical models involving rates of change. For instance, the rate of change in the first example is the velocity of a car.

Verbal Statement	Mathematical Model
The velocity of a car after traveling for 1 hour is 50 miles per hour.	$x =$ distance traveled $\dfrac{dx}{dt} = 50$ when $t = 1$
Water is being pumped into a swimming pool at a rate of 10 cubic meters per hour.	$V =$ volume of water in pool $\dfrac{dV}{dt} = 10 \text{ m}^3/\text{hr}$
A gear is revolving at a rate of 25 revolutions per minute (1 revolution $= 2\pi$ rad).	$\theta =$ angle of revolution $\dfrac{d\theta}{dt} = 25(2\pi)$ rad/min

EXAMPLE 3 An Inflating Balloon

Air is being pumped into a spherical balloon (see Figure 2.35) at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.


Inflating a balloon
Figure 2.35

Solution Let V be the volume of the balloon and let r be its radius. Because the volume is increasing at a rate of 4.5 cubic feet per minute, you know that at time t the rate of change of the volume is $dV/dt = \frac{9}{2}$. So, the problem can be stated as shown.

Given rate: $\dfrac{dV}{dt} = \dfrac{9}{2}$ (constant rate)

Find: $\dfrac{dr}{dt}$ when $r = 2$

To find the rate of change of the radius, you must find an equation that relates the radius r to the volume V.

Equation: $V = \dfrac{4}{3}\pi r^3$ Volume of a sphere

Differentiating both sides of the equation with respect to t produces

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad \text{Differentiate with respect to } t.$$

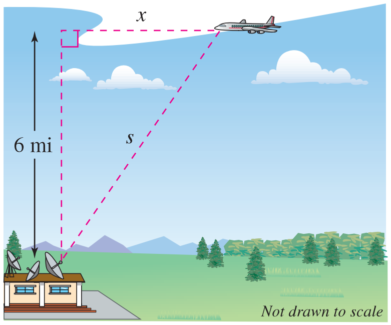
$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right). \qquad \text{Solve for } dr/dt.$$

Finally, when $r = 2$, the rate of change of the radius is

$$\frac{dr}{dt} = \frac{1}{16\pi}\left(\frac{9}{2}\right) \approx 0.09 \text{ foot per minute.}$$ ■

In Example 3, note that the volume is increasing at a *constant* rate but the radius is increasing at a *variable* rate. Just because two rates are related does not mean that they are proportional. In this particular case, the radius is growing more and more slowly as t increases. Do you see why?

EXAMPLE 4 The Speed of an Airplane Tracked by Radar



An airplane is flying at an altitude of 6 miles, *s* miles from the station.
Figure 2.36

An airplane is flying on a flight path that will take it directly over a radar tracking station, as shown in Figure 2.36. If s is decreasing at a rate of 400 miles per hour when $s = 10$ miles, what is the speed of the plane?

Solution Let x be the horizontal distance from the station, as shown in Figure 2.36. Notice that when $s = 10$, $x = \sqrt{10^2 - 36} = 8$.

Given rate: $ds/dt = -400$ when $s = 10$

Find: dx/dt when $s = 10$ and $x = 8$

You can find the velocity of the plane as shown.

Equation: $x^2 + 6^2 = s^2$ Pythagorean Theorem

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$ Differentiate with respect to t.

$$\frac{dx}{dt} = \frac{s}{x}\left(\frac{ds}{dt}\right)$$ Solve for dx/dt.

$$\frac{dx}{dt} = \frac{10}{8}(-400)$$ Substitute for s, x, and ds/dt.

$$= -500 \text{ miles per hour}$$ Simplify.

Because the velocity is -500 miles per hour, the *speed* is 500 miles per hour. ■

NOTE Note that the velocity in Example 4 is negative because x represents a distance that is decreasing. ■

EXAMPLE 5 A Changing Angle of Elevation

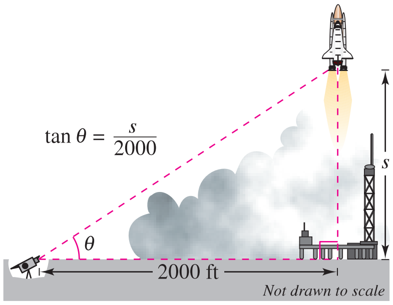
Find the rate of change in the angle of elevation of the camera shown in Figure 2.37 at 10 seconds after lift-off.

Solution Let θ be the angle of elevation, as shown in Figure 2.37. When $t = 10$, the height s of the rocket is $s = 50t^2 = 50(10)^2 = 5000$ feet.

Given rate: $ds/dt = 100t =$ velocity of rocket

Find: $d\theta/dt$ when $t = 10$ and $s = 5000$

Using Figure 2.37, you can relate s and θ by the equation $\tan\theta = s/2000$.



A television camera at ground level is filming the lift-off of a space shuttle that is rising vertically according to the position equation $s = 50t^2$, where s is measured in feet and t is measured in seconds. The camera is 2000 feet from the launch pad.
Figure 2.37

Equation: $$\tan\theta = \frac{s}{2000}$$ See Figure 2.37.

$$(\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{2000}\left(\frac{ds}{dt}\right)$$ Differentiate with respect to t.

$$\frac{d\theta}{dt} = \cos^2\theta\,\frac{100t}{2000}$$ Substitute $100t$ for ds/dt.

$$= \left(\frac{2000}{\sqrt{s^2 + 2000^2}}\right)^2\frac{100t}{2000}$$ $\cos\theta = 2000/\sqrt{s^2 + 2000^2}$

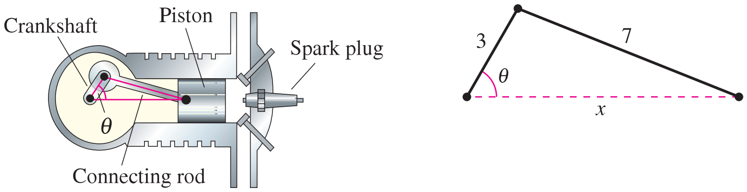
When $t = 10$ and $s = 5000$, you have

$$\frac{d\theta}{dt} = \frac{2000(100)(10)}{5000^2 + 2000^2} = \frac{2}{29} \text{ radian per second.}$$

So, when $t = 10$, θ is changing at a rate of $\frac{2}{29}$ radian per second. ■

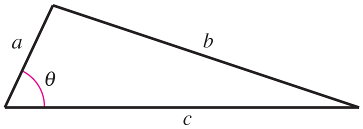
EXAMPLE 6 The Velocity of a Piston

In the engine shown in Figure 2.38, a 7-inch connecting rod is fastened to a crank of radius 3 inches. The crankshaft rotates counterclockwise at a constant rate of 200 revolutions per minute. Find the velocity of the piston when $\theta = \pi/3$.



The velocity of a piston is related to the angle of the crankshaft.
Figure 2.38

Solution Label the distances as shown in Figure 2.38. Because a complete revolution corresponds to 2π radians, it follows that $d\theta/dt = 200(2\pi) = 400\pi$ radians per minute.



Law of Cosines:
$b^2 = a^2 + c^2 - 2ac\cos\theta$
Figure 2.39

Given rate: $\dfrac{d\theta}{dt} = 400\pi$ (constant rate)

Find: $\dfrac{dx}{dt}$ when $\theta = \dfrac{\pi}{3}$

You can use the Law of Cosines (Figure 2.39) to find an equation that relates x and θ.

Equation:

$$\begin{aligned}
7^2 &= 3^2 + x^2 - 2(3)(x)\cos\theta \\
0 &= 2x\frac{dx}{dt} - 6\left(-x\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dx}{dt}\right) \\
(6\cos\theta - 2x)\frac{dx}{dt} &= 6x\sin\theta\frac{d\theta}{dt} \\
\frac{dx}{dt} &= \frac{6x\sin\theta}{6\cos\theta - 2x}\left(\frac{d\theta}{dt}\right)
\end{aligned}$$

When $\theta = \pi/3$, you can solve for x as shown.

$$\begin{aligned}
7^2 &= 3^2 + x^2 - 2(3)(x)\cos\frac{\pi}{3} \\
49 &= 9 + x^2 - 6x\left(\frac{1}{2}\right) \\
0 &= x^2 - 3x - 40 \\
0 &= (x - 8)(x + 5) \\
x &= 8 \qquad \text{Choose positive solution.}
\end{aligned}$$

So, when $x = 8$ and $\theta = \pi/3$, the velocity of the piston is

$$\begin{aligned}
\frac{dx}{dt} &= \frac{6(8)\left(\sqrt{3}/2\right)}{6(1/2) - 16}(400\pi) \\
&= \frac{9600\pi\sqrt{3}}{-13} \\
&\approx -4018 \text{ inches per minute.}
\end{aligned}$$

■

NOTE Note that the velocity in Example 6 is negative because x represents a distance that is decreasing. ■

2.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, assume that x and y are both differentiable functions of t and find the required values of dy/dt and dx/dt.

	Equation	*Find*	*Given*
1.	$y = \sqrt{x}$	(a) $\dfrac{dy}{dt}$ when $x = 4$	$\dfrac{dx}{dt} = 3$
		(b) $\dfrac{dx}{dt}$ when $x = 25$	$\dfrac{dy}{dt} = 2$
2.	$y = 4(x^2 - 5x)$	(a) $\dfrac{dy}{dt}$ when $x = 3$	$\dfrac{dx}{dt} = 2$
		(b) $\dfrac{dx}{dt}$ when $x = 1$	$\dfrac{dy}{dt} = 5$
3.	$xy = 4$	(a) $\dfrac{dy}{dt}$ when $x = 8$	$\dfrac{dx}{dt} = 10$
		(b) $\dfrac{dx}{dt}$ when $x = 1$	$\dfrac{dy}{dt} = -6$
4.	$x^2 + y^2 = 25$	(a) $\dfrac{dy}{dt}$ when $x = 3, y = 4$	$\dfrac{dx}{dt} = 8$
		(b) $\dfrac{dx}{dt}$ when $x = 4, y = 3$	$\dfrac{dy}{dt} = -2$

In Exercises 5–8, a point is moving along the graph of the given function such that dx/dt is 2 centimeters per second. Find dy/dt for the given values of x.

5. $y = 2x^2 + 1$ (a) $x = -1$ (b) $x = 0$ (c) $x = 1$

6. $y = \dfrac{1}{1 + x^2}$ (a) $x = -2$ (b) $x = 0$ (c) $x = 2$

7. $y = \tan x$ (a) $x = -\dfrac{\pi}{3}$ (b) $x = -\dfrac{\pi}{4}$ (c) $x = 0$

8. $y = \cos x$ (a) $x = \dfrac{\pi}{6}$ (b) $x = \dfrac{\pi}{4}$ (c) $x = \dfrac{\pi}{3}$

WRITING ABOUT CONCEPTS

9. Consider the linear function $y = ax + b$. If x changes at a constant rate, does y change at a constant rate? If so, does it change at the same rate as x? Explain.

10. In your own words, state the guidelines for solving related-rate problems.

11. Find the rate of change of the distance between the origin and a moving point on the graph of $y = x^2 + 1$ if $dx/dt = 2$ centimeters per second.

12. Find the rate of change of the distance between the origin and a moving point on the graph of $y = \sin x$ if $dx/dt = 2$ centimeters per second.

13. ***Area*** The radius r of a circle is increasing at a rate of 4 centimeters per minute. Find the rates of change of the area when (a) $r = 8$ centimeters and (b) $r = 32$ centimeters.

14. ***Area*** Let A be the area of a circle of radius r that is changing with respect to time. If dr/dt is constant, is dA/dt constant? Explain.

15. ***Area*** The included angle of the two sides of constant equal length s of an isosceles triangle is θ.

(a) Show that the area of the triangle is given by $A = \frac{1}{2}s^2 \sin \theta$.

(b) If θ is increasing at the rate of $\frac{1}{2}$ radian per minute, find the rates of change of the area when $\theta = \pi/6$ and $\theta = \pi/3$.

(c) Explain why the rate of change of the area of the triangle is not constant even though $d\theta/dt$ is constant.

16. ***Volume*** The radius r of a sphere is increasing at a rate of 3 inches per minute.

(a) Find the rates of change of the volume when $r = 9$ inches and $r = 36$ inches.

(b) Explain why the rate of change of the volume of the sphere is not constant even though dr/dt is constant.

17. ***Volume*** A spherical balloon is inflated with gas at the rate of 800 cubic centimeters per minute. How fast is the radius of the balloon increasing at the instant the radius is (a) 30 centimeters and (b) 60 centimeters?

18. ***Volume*** All edges of a cube are expanding at a rate of 6 centimeters per second. How fast is the volume changing when each edge is (a) 2 centimeters and (b) 10 centimeters?

19. ***Surface Area*** The conditions are the same as in Exercise 18. Determine how fast the *surface area* is changing when each edge is (a) 2 centimeters and (b) 10 centimeters.

20. ***Volume*** The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Find the rates of change of the volume if dr/dt is 2 inches per minute and $h = 3r$ when (a) $r = 6$ inches and (b) $r = 24$ inches.

21. ***Volume*** At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 15 feet high?

22. ***Depth*** A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

23. ***Depth*** A swimming pool is 12 meters long, 6 meters wide, 1 meter deep at the shallow end, and 3 meters deep at the deep end (see figure on next page). Water is being pumped into the pool at $\frac{1}{4}$ cubic meter per minute, and there is 1 meter of water at the deep end.

(a) What percent of the pool is filled?

(b) At what rate is the water level rising?

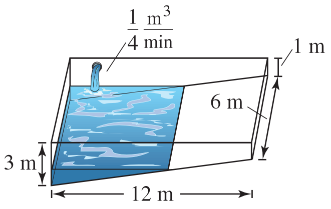


Figure for 23

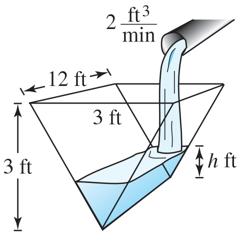


Figure for 24

24. ***Depth*** A trough is 12 feet long and 3 feet across the top (see figure). Its ends are isosceles triangles with altitudes of 3 feet.

(a) If water is being pumped into the trough at 2 cubic feet per minute, how fast is the water level rising when the depth h is 1 foot?

(b) If the water is rising at a rate of $\frac{3}{8}$ inch per minute when $h = 2$, determine the rate at which water is being pumped into the trough.

25. ***Moving Ladder*** A ladder 25 feet long is leaning against the wall of a house (see figure). The base of the ladder is pulled away from the wall at a rate of 2 feet per second.

(a) How fast is the top of the ladder moving down the wall when its base is 7 feet, 15 feet, and 24 feet from the wall?

(b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.

(c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.

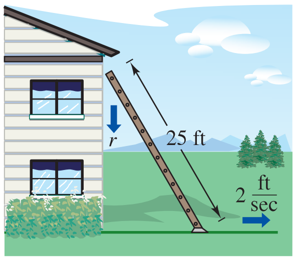


Figure for 25

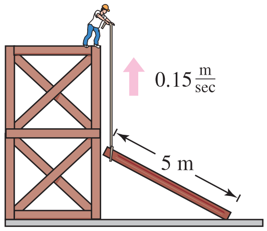
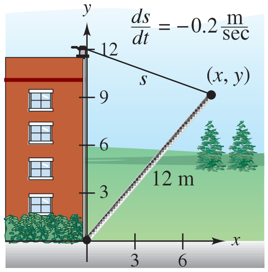


Figure for 26

■ **FOR FURTHER INFORMATION** For more information on the mathematics of moving ladders, see the article "The Falling Ladder Paradox" by Paul Scholten and Andrew Simoson in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

26. ***Construction*** A construction worker pulls a five-meter plank up the side of a building under construction by means of a rope tied to one end of the plank (see figure). Assume the opposite end of the plank follows a path perpendicular to the wall of the building and the worker pulls the rope at a rate of 0.15 meter per second. How fast is the end of the plank sliding along the ground when it is 2.5 meters from the wall of the building?

27. ***Construction*** A winch at the top of a 12-meter building pulls a pipe of the same length to a vertical position, as shown in the figure. The winch pulls in rope at a rate of -0.2 meter per second. Find the rate of vertical change and the rate of horizontal change at the end of the pipe when $y = 6$.



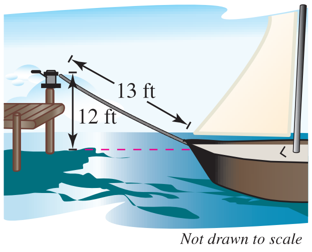


Figure for 27

Figure for 28

28. ***Boating*** A boat is pulled into a dock by means of a winch 12 feet above the deck of the boat (see figure).

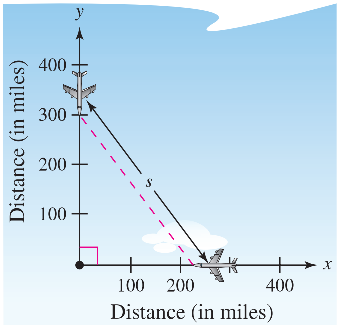
(a) The winch pulls in rope at a rate of 4 feet per second. Determine the speed of the boat when there is 13 feet of rope out. What happens to the speed of the boat as it gets closer to the dock?

(b) Suppose the boat is moving at a constant rate of 4 feet per second. Determine the speed at which the winch pulls in rope when there is a total of 13 feet of rope out. What happens to the speed at which the winch pulls in rope as the boat gets closer to the dock?

29. ***Air Traffic Control*** An air traffic controller spots two planes at the same altitude converging on a point as they fly at right angles to each other (see figure). One plane is 225 miles from the point moving at 450 miles per hour. The other plane is 300 miles from the point moving at 600 miles per hour.

(a) At what rate is the distance between the planes decreasing?

(b) How much time does the air traffic controller have to get one of the planes on a different flight path?



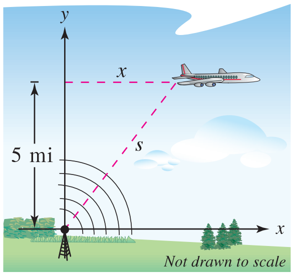


Figure for 29

Figure for 30

30. ***Air Traffic Control*** An airplane is flying at an altitude of 5 miles and passes directly over a radar antenna (see figure). When the plane is 10 miles away ($s = 10$), the radar detects that the distance s is changing at a rate of 240 miles per hour. What is the speed of the plane?

31. ***Sports*** A baseball diamond has the shape of a square with sides 90 feet long (see figure). A player running from second base to third base at a speed of 25 feet per second is 20 feet from third base. At what rate is the player's distance s from home plate changing?

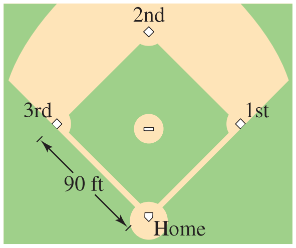


Figure for 31 and 32

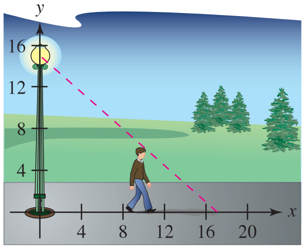


Figure for 33

32. ***Sports*** For the baseball diamond in Exercise 31, suppose the player is running from first to second at a speed of 25 feet per second. Find the rate at which the distance from home plate is changing when the player is 20 feet from second base.

33. ***Shadow Length*** A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground (see figure). When he is 10 feet from the base of the light,

(a) at what rate is the tip of his shadow moving?

(b) at what rate is the length of his shadow changing?

34. ***Shadow Length*** Repeat Exercise 33 for a man 6 feet tall walking at a rate of 5 feet per second *toward* a light that is 20 feet above the ground (see figure).

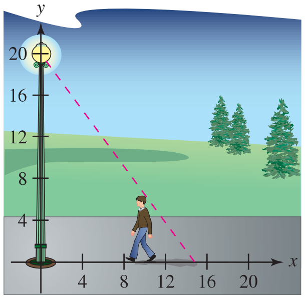


Figure for 34

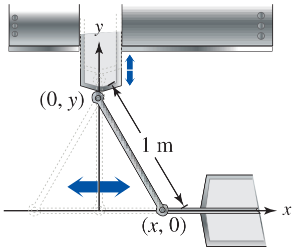


Figure for 35

35. ***Machine Design*** The endpoints of a movable rod of length 1 meter have coordinates $(x, 0)$ and $(0, y)$ (see figure). The position of the end on the x-axis is

$$x(t) = \frac{1}{2} \sin \frac{\pi t}{6}$$

where t is the time in seconds.

(a) Find the time of one complete cycle of the rod.

(b) What is the lowest point reached by the end of the rod on the y-axis?

(c) Find the speed of the y-axis endpoint when the x-axis endpoint is $\left(\frac{1}{4}, 0\right)$.

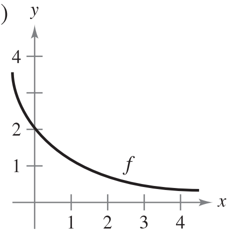
36. ***Machine Design*** Repeat Exercise 35 for a position function of $x(t) = \frac{3}{5} \sin \pi t$. Use the point $\left(\frac{3}{10}, 0\right)$ for part (c).

37. ***Evaporation*** As a spherical raindrop falls, it reaches a layer of dry air and begins to evaporate at a rate that is proportional to its surface area $(S = 4\pi r^2)$. Show that the radius of the raindrop decreases at a constant rate.

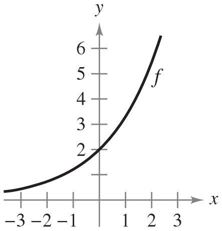
CAPSTONE

38. Using the graph of f, (a) determine whether dy/dt is positive or negative given that dx/dt is negative, and (b) determine whether dx/dt is positive or negative given that dy/dt is positive.

(i)



(ii)



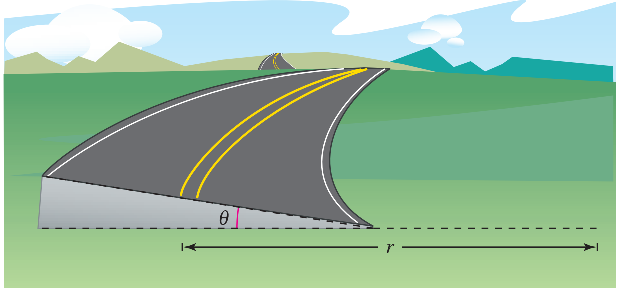
39. ***Electricity*** The combined electrical resistance R of R_1 and R_2, connected in parallel, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

where R, R_1, and R_2 are measured in ohms. R_1 and R_2 are increasing at rates of 1 and 1.5 ohms per second, respectively. At what rate is R changing when $R_1 = 50$ ohms and $R_2 = 75$ ohms?

40. ***Adiabatic Expansion*** When a certain polyatomic gas undergoes adiabatic expansion, its pressure p and volume V satisfy the equation $pV^{1.3} = k$, where k is a constant. Find the relationship between the related rates dp/dt and dV/dt.

41. ***Roadway Design*** Cars on a certain roadway travel on a circular arc of radius r. In order not to rely on friction alone to overcome the centrifugal force, the road is banked at an angle of magnitude θ from the horizontal (see figure). The banking angle must satisfy the equation $rg \tan \theta = v^2$, where v is the velocity of the cars and $g = 32$ feet per second per second is the acceleration due to gravity. Find the relationship between the related rates dv/dt and $d\theta/dt$.



42. ***Angle of Elevation*** A balloon rises at a rate of 4 meters per second from a point on the ground 50 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 50 meters above the ground.

43. ***Angle of Elevation*** A fish is reeled in at a rate of 1 foot per second from a point 10 feet above the water (see figure). At what rate is the angle θ between the line and the water changing when there is a total of 25 feet of line from the end of the rod to the water?

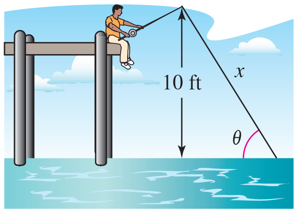


Figure for 43

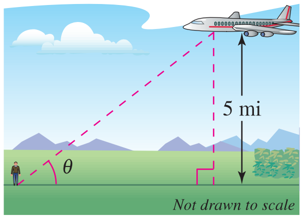


Figure for 44

44. ***Angle of Elevation*** An airplane flies at an altitude of 5 miles toward a point directly over an observer (see figure). The speed of the plane is 600 miles per hour. Find the rates at which the angle of elevation θ is changing when the angle is (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 75°$.

45. ***Linear vs. Angular Speed*** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. How fast is the light beam moving along the wall when the beam makes angles of (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 70°$ with the perpendicular line from the light to the wall?

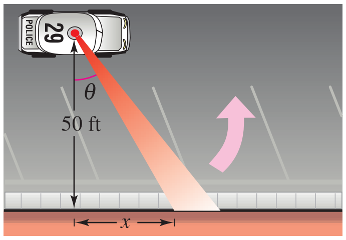


Figure for 45

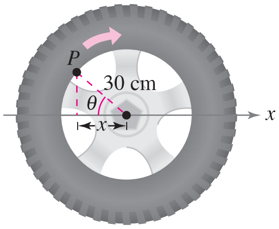


Figure for 46

46. ***Linear vs. Angular Speed*** A wheel of radius 30 centimeters revolves at a rate of 10 revolutions per second. A dot is painted at a point P on the rim of the wheel (see figure).

(a) Find dx/dt as a function of θ.

(b) Use a graphing utility to graph the function in part (a).

(c) When is the absolute value of the rate of change of x greatest? When is it least?

(d) Find dx/dt when $\theta = 30°$ and $\theta = 60°$.

47. ***Flight Control*** An airplane is flying in still air with an airspeed of 275 miles per hour. If it is climbing at an angle of 18°, find the rate at which it is gaining altitude.

48. ***Security Camera*** A security camera is centered 50 feet above a 100-foot hallway (see figure). It is easiest to design the camera with a constant angular rate of rotation, but this results in a variable rate at which the images of the surveillance area are recorded. So, it is desirable to design a system with a variable rate of rotation and a constant rate of movement of the scanning beam along the hallway. Find a model for the variable rate of rotation if $|dx/dt| = 2$ feet per second.

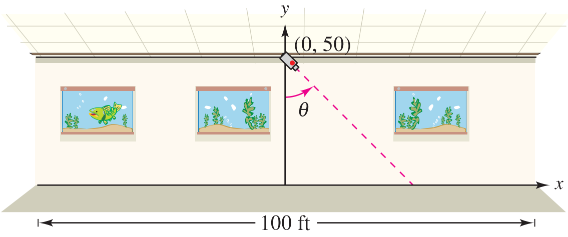


Figure for 48

49. ***Think About It*** Describe the relationship between the rate of change of y and the rate of change of x in each expression. Assume all variables and derivatives are positive.

(a) $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$ (b) $\dfrac{dy}{dt} = x(L - x)\dfrac{dx}{dt}, \quad 0 \le x \le L$

Acceleration **In Exercises 50 and 51, find the acceleration of the specified object. (*Hint:* Recall that if a variable is changing at a constant rate, its acceleration is zero.)**

50. Find the acceleration of the top of the ladder described in Exercise 25 when the base of the ladder is 7 feet from the wall.

51. Find the acceleration of the boat in Exercise 28(a) when there is a total of 13 feet of rope out.

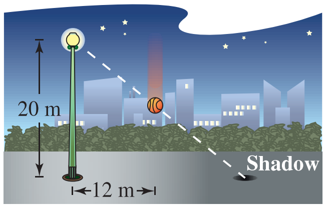
52. ***Modeling Data*** The table shows the numbers (in millions) of single women (never married) s and married women m in the civilian work force in the United States for the years 1997 through 2005. *(Source: U.S. Bureau of Labor Statistics)*

Year	1997	1998	1999	2000	2001	2002	2003	2004	2005
s	16.5	17.1	17.6	17.8	18.0	18.2	18.4	18.6	19.2
m	33.8	33.9	34.4	35.1	35.2	35.5	36.0	35.8	35.9

(a) Use the regression capabilities of a graphing utility to find a model of the form $m(s) = as^3 + bs^2 + cs + d$ for the data, where t is the time in years, with $t = 7$ corresponding to 1997.

(b) Find dm/dt. Then use the model to estimate dm/dt for $t = 10$ if it is predicted that the number of single women in the work force will increase at the rate of 0.75 million per year.

53. ***Moving Shadow*** A ball is dropped from a height of 20 meters, 12 meters away from the top of a 20-meter lamppost (see figure). The ball's shadow, caused by the light at the top of the lamppost, is moving along the level ground. How fast is the shadow moving 1 second after the ball is released? *(Submitted by Dennis Gittinger, St. Philips College, San Antonio, TX)*



2 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find the derivative of the function by using the definition of the derivative.

1. $f(x) = x^2 - 4x + 5$

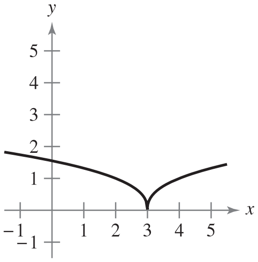
2. $f(x) = \sqrt{x} + 1$

3. $f(x) = \dfrac{x+1}{x-1}$

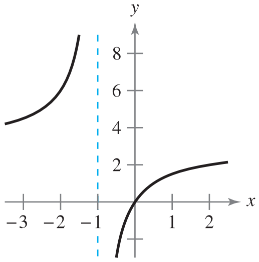
4. $f(x) = \dfrac{6}{x}$

In Exercises 5 and 6, describe the *x*-values at which *f* is differentiable.

5. $f(x) = (x-3)^{2/5}$



6. $f(x) = \dfrac{3x}{x+1}$



7. Sketch the graph of $f(x) = 4 - |x - 2|$.

(a) Is f continuous at $x = 2$?

(b) Is f differentiable at $x = 2$? Explain.

8. Sketch the graph of $f(x) = \begin{cases} x^2 + 4x + 2, & x < -2 \\ 1 - 4x - x^2, & x \geq -2. \end{cases}$

(a) Is f continuous at $x = -2$?

(b) Is f differentiable at $x = -2$? Explain.

In Exercises 9 and 10, find the slope of the tangent line to the graph of the function at the given point.

9. $g(x) = \dfrac{2}{3}x^2 - \dfrac{x}{6}, \quad \left(-1, \dfrac{5}{6}\right)$

10. $h(x) = \dfrac{3x}{8} - 2x^2, \quad \left(-2, -\dfrac{35}{4}\right)$

In Exercises 11 and 12, (a) find an equation of the tangent line to the graph of *f* at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of the graphing utility to confirm your results.

11. $f(x) = x^3 - 1, \quad (-1, -2)$

12. $f(x) = \dfrac{2}{x+1}, \quad (0, 2)$

In Exercises 13 and 14, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).

13. $g(x) = x^2(x-1), \quad c = 2$

14. $f(x) = \dfrac{1}{x+4}, \quad c = 3$

In Exercises 15–30, use the rules of differentiation to find the derivative of the function.

15. $y = 25$

16. $y = -30$

17. $f(x) = x^8$

18. $g(x) = x^{20}$

19. $h(t) = 13t^4$

20. $f(t) = -8t^5$

21. $f(x) = x^3 - 11x^2$

22. $g(s) = 4s^4 - 5s^2$

23. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x}$

24. $f(x) = x^{1/2} - x^{-1/2}$

25. $g(t) = \dfrac{2}{3t^2}$

26. $h(x) = \dfrac{10}{(7x)^2}$

27. $f(\theta) = 4\theta - 5 \sin \theta$

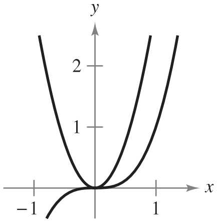
28. $g(\alpha) = 4 \cos \alpha + 6$

29. $f(\theta) = 3 \cos \theta - \dfrac{\sin \theta}{4}$

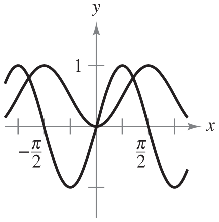
30. $g(\alpha) = \dfrac{5 \sin \alpha}{3} - 2\alpha$

Writing **In Exercises 31 and 32, the figure shows the graphs of a function and its derivative. Label the graphs as *f* or *f′* and write a short paragraph stating the criteria you used in making your selection. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

31.



32.



33. ***Vibrating String*** When a guitar string is plucked, it vibrates with a frequency of $F = 200\sqrt{T}$, where F is measured in vibrations per second and the tension T is measured in pounds. Find the rates of change of F when (a) $T = 4$ and (b) $T = 9$.

34. ***Vertical Motion*** A ball is dropped from a height of 100 feet. One second later, another ball is dropped from a height of 75 feet. Which ball hits the ground first?

35. ***Vertical Motion*** To estimate the height of a building, a weight is dropped from the top of the building into a pool at ground level. How high is the building if the splash is seen 9.2 seconds after the weight is dropped?

36. ***Vertical Motion*** A bomb is dropped from an airplane at an altitude of 14,400 feet. How long will it take for the bomb to reach the ground? (Because of the motion of the plane, the fall will not be vertical, but the time will be the same as that for a vertical fall.) The plane is moving at 600 miles per hour. How far will the bomb move horizontally after it is released from the plane?

37. ***Projectile Motion*** A thrown ball follows a path described by $y = x - 0.02x^2$.

(a) Sketch a graph of the path.

(b) Find the total horizontal distance the ball is thrown.

(c) At what x-value does the ball reach its maximum height? (Use the symmetry of the path.)

(d) Find an equation that gives the instantaneous rate of change of the height of the ball with respect to the horizontal change. Evaluate the equation at $x = 0, 10, 25, 30,$ and 50.

(e) What is the instantaneous rate of change of the height when the ball reaches its maximum height?

38. ***Projectile Motion*** The path of a projectile thrown at an angle of 45° with level ground is

$$y = x - \frac{32}{v_0^2}(x^2)$$

where the initial velocity is v_0 feet per second.

(a) Find the x-coordinate of the point where the projectile strikes the ground. Use the symmetry of the path of the projectile to locate the x-coordinate of the point where the projectile reaches its maximum height.

(b) What is the instantaneous rate of change of the height when the projectile is at its maximum height?

(c) Show that doubling the initial velocity of the projectile multiplies both the maximum height and the range by a factor of 4.

(d) Find the maximum height and range of a projectile thrown with an initial velocity of 70 feet per second. Use a graphing utility to graph the path of the projectile.

39. ***Horizontal Motion*** The position function of a particle moving along the x-axis is

$$x(t) = t^2 - 3t + 2 \quad \text{for} \quad -\infty < t < \infty.$$

(a) Find the velocity of the particle.

(b) Find the open t-interval(s) in which the particle is moving to the left.

(c) Find the position of the particle when the velocity is 0.

(d) Find the speed of the particle when the position is 0.

40. ***Modeling Data*** The speed of a car in miles per hour and the stopping distance in feet are recorded in the table.

Speed, x	20	30	40	50	60
Stopping Distance, y	25	55	105	188	300

(a) Use the regression capabilities of a graphing utility to find a quadratic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use a graphing utility to graph dy/dx.

(d) Use the model to approximate the stopping distance at a speed of 65 miles per hour.

(e) Use the graphs in parts (b) and (c) to explain the change in stopping distance as the speed increases.

In Exercises 41–54, find the derivative of the function.

41. $f(x) = (5x^2 + 8)(x^2 - 4x - 6)$

42. $g(x) = (x^3 + 7x)(x + 3)$

43. $h(x) = \sqrt{x} \sin x$

44. $f(t) = 2t^5 \cos t$

45. $f(x) = \dfrac{x^2 + x - 1}{x^2 - 1}$

46. $f(x) = \dfrac{6x - 5}{x^2 + 1}$

47. $f(x) = \dfrac{1}{9 - 4x^2}$

48. $f(x) = \dfrac{9}{3x^2 - 2x}$

49. $y = \dfrac{x^4}{\cos x}$

50. $y = \dfrac{\sin x}{x^4}$

51. $y = 3x^2 \sec x$

52. $y = 2x - x^2 \tan x$

53. $y = x \cos x - \sin x$

54. $g(x) = 3x \sin x + x^2 \cos x$

In Exercises 55–58, find an equation of the tangent line to the graph of *f* at the given point.

55. $f(x) = \dfrac{2x^3 - 1}{x^2}, \quad (1, 1)$

56. $f(x) = \dfrac{x + 1}{x - 1}, \quad \left(\dfrac{1}{2}, -3\right)$

57. $f(x) = -x \tan x, \quad (0, 0)$

58. $f(x) = \dfrac{1 + \cos x}{1 - \cos x}, \quad \left(\dfrac{\pi}{2}, 1\right)$

59. ***Acceleration*** The velocity of an object in meters per second is $v(t) = 36 - t^2$, $0 \le t \le 6$. Find the velocity and acceleration of the object when $t = 4$.

60. ***Acceleration*** The velocity of an automobile starting from rest is

$$v(t) = \frac{90t}{4t + 10}$$

where v is measured in feet per second. Find the vehicle's velocity and acceleration at each of the following times.

(a) 1 second (b) 5 seconds (c) 10 seconds

In Exercises 61–66, find the second derivative of the function.

61. $g(t) = -8t^3 - 5t + 12$

62. $h(x) = 21x^{-3} + 3x$

63. $f(x) = 15x^{5/2}$

64. $f(x) = 20\sqrt[5]{x}$

65. $f(\theta) = 3 \tan \theta$

66. $h(t) = 10 \cos t - 15 \sin t$

In Exercises 67 and 68, show that the function satisfies the equation.

	Function	*Equation*
67.	$y = 2 \sin x + 3 \cos x$	$y'' + y = 0$
68.	$y = \dfrac{10 - \cos x}{x}$	$xy' + y = \sin x$

In Exercises 69–80, find the derivative of the function.

69. $h(x) = \left(\dfrac{x + 5}{x^2 + 3}\right)^2$

70. $f(x) = \left(x^2 + \dfrac{1}{x}\right)^5$

71. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

72. $h(\theta) = \dfrac{\theta}{(1 - \theta)^3}$

73. $y = 5 \cos(9x + 1)$

74. $y = 1 - \cos 2x + 2 \cos^2 x$

75. $y = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$

76. $y = \dfrac{\sec^7 x}{7} - \dfrac{\sec^5 x}{5}$

77. $y = \dfrac{2}{3} \sin^{3/2} x - \dfrac{2}{7} \sin^{7/2} x$

78. $f(x) = \dfrac{3x}{\sqrt{x^2 + 1}}$

79. $y = \dfrac{\sin \pi x}{x + 2}$

80. $y = \dfrac{\cos(x - 1)}{x - 1}$

In Exercises 81–84, find the derivative of the function at the given point.

81. $f(x) = \sqrt{1 - x^3}, \quad (-2, 3)$

82. $f(x) = \sqrt[3]{x^2 - 1}, \quad (3, 2)$

83. $y = \frac{1}{2}\csc 2x, \quad \left(\frac{\pi}{4}, \frac{1}{2}\right)$

84. $y = \csc 3x + \cot 3x, \quad \left(\frac{\pi}{6}, 1\right)$

CAS **In Exercises 85–88, use a computer algebra system to find the derivative of the function. Use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.**

85. $g(x) = \dfrac{2x}{\sqrt{x+1}}$

86. $f(x) = [(x-2)(x+4)]^2$

87. $f(t) = \sqrt{t+1}\,\sqrt[3]{t+1}$

88. $y = \sqrt{3x}\,(x+2)^3$

CAS **In Exercises 89–92, (a) use a computer algebra system to find the derivative of the function at the given point, (b) find an equation of the tangent line to the graph of the function at the point, and (c) graph the function and its tangent line on the same set of coordinate axes.**

89. $f(t) = t^2(t-1)^5, \quad (2, 4)$

90. $g(x) = x\sqrt{x^2+1}, \quad \left(3, 3\sqrt{10}\right)$

91. $f(x) = \tan\sqrt{1-x}, \quad \left(-2, \tan\sqrt{3}\right)$

92. $f(x) = 2\csc^3\left(\sqrt{x}\right), \quad (1, 2\csc^3 1)$

In Exercises 93–96, find the second derivative of the function.

93. $y = 7x^2 + \cos 2x$

94. $y = \dfrac{1}{x} + \tan x$

95. $f(x) = \cot x$

96. $y = \sin^2 x$

CAS **In Exercises 97–100, use a computer algebra system to find the second derivative of the function.**

97. $f(t) = \dfrac{4t^2}{(1-t)^2}$

98. $g(x) = \dfrac{6x-5}{x^2+1}$

99. $g(\theta) = \tan 3\theta - \sin(\theta - 1)$

100. $h(x) = 5x\sqrt{x^2 - 16}$

101. ***Refrigeration*** The temperature T (in degrees Fahrenheit) of food in a freezer is

$$T = \frac{700}{t^2 + 4t + 10}$$

where t is the time in hours. Find the rate of change of T with respect to t at each of the following times.

(a) $t = 1$ (b) $t = 3$ (c) $t = 5$ (d) $t = 10$

102. ***Fluid Flow*** The emergent velocity v of a liquid flowing from a hole in the bottom of a tank is given by $v = \sqrt{2gh}$, where g is the acceleration due to gravity (32 feet per second per second) and h is the depth of the liquid in the tank. Find the rates of change of v with respect to h when (a) $h = 9$ and (b) $h = 4$. (Note that $g = +32$ feet per second per second. The sign of g depends on how a problem is modeled. In this case, letting g be negative would produce an imaginary value for v.)

In Exercises 103–108, find *dy/dx* by implicit differentiation.

103. $x^2 + 3xy + y^3 = 10$

104. $y^2 = (x - y)(x^2 + y)$

105. $\sqrt{xy} = x - 4y$

106. $y\sqrt{x} - x\sqrt{y} = 25$

107. $x \sin y = y \cos x$

108. $\cos(x + y) = x$

In Exercises 109 and 110, find the equations of the tangent line and the normal line to the graph of the equation at the given point. Use a graphing utility to graph the equation, the tangent line, and the normal line.

109. $x^2 + y^2 = 10, \quad (3, 1)$

110. $x^2 - y^2 = 20, \quad (6, 4)$

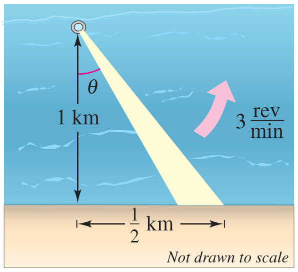
111. A point moves along the curve $y = \sqrt{x}$ in such a way that the y-value is increasing at a rate of 2 units per second. At what rate is x changing for each of the following values?

(a) $x = \frac{1}{2}$ (b) $x = 1$ (c) $x = 4$

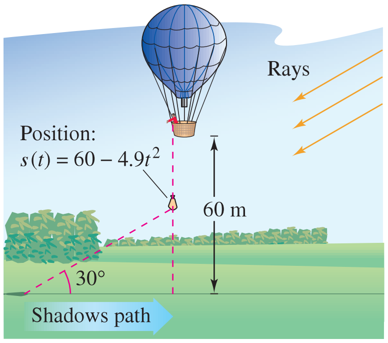
112. ***Surface Area*** The edges of a cube are expanding at a rate of 8 centimeters per second. How fast is the surface area changing when each edge is 6.5 centimeters?

113. ***Depth*** The cross section of a five-meter trough is an isosceles trapezoid with a two-meter lower base, a three-meter upper base, and an altitude of 2 meters. Water is running into the trough at a rate of 1 cubic meter per minute. How fast is the water level rising when the water is 1 meter deep?

114. ***Linear and Angular Velocity*** A rotating beacon is located 1 kilometer off a straight shoreline (see figure). If the beacon rotates at a rate of 3 revolutions per minute, how fast (in kilometers per hour) does the beam of light appear to be moving to a viewer who is $\frac{1}{2}$ kilometer down the shoreline?



115. ***Moving Shadow*** A sandbag is dropped from a balloon at a height of 60 meters when the angle of elevation to the sun is 30° (see figure). Find the rate at which the shadow of the sandbag is traveling along the ground when the sandbag is at a height of 35 meters. [*Hint:* The position of the sandbag is given by $s(t) = 60 - 4.9t^2$.]



P.S. PROBLEM SOLVING

1. Consider the graph of the parabola $y = x^2$.

(a) Find the radius r of the largest possible circle centered on the y-axis that is tangent to the parabola at the origin, as shown in the figure. This circle is called the **circle of curvature** (see Section 12.5). Find the equation of this circle. Use a graphing utility to graph the circle and parabola in the same viewing window to verify your answer.

(b) Find the center $(0, b)$ of the circle of radius 1 centered on the y-axis that is tangent to the parabola at two points, as shown in the figure. Find the equation of this circle. Use a graphing utility to graph the circle and parabola in the same viewing window to verify your answer.

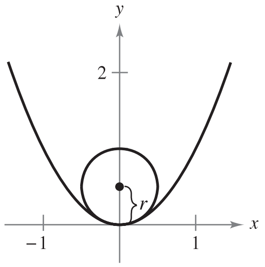


Figure for 1(a)

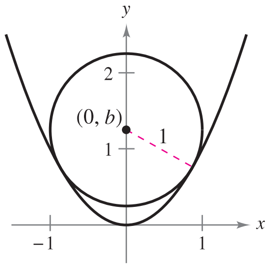


Figure for 1(b)

2. Graph the two parabolas $y = x^2$ and $y = -x^2 + 2x - 5$ in the same coordinate plane. Find equations of the two lines simultaneously tangent to both parabolas.

3. (a) Find the polynomial $P_1(x) = a_0 + a_1x$ whose value and slope agree with the value and slope of $f(x) = \cos x$ at the point $x = 0$.

(b) Find the polynomial $P_2(x) = a_0 + a_1x + a_2x^2$ whose value and first two derivatives agree with the value and first two derivatives of $f(x) = \cos x$ at the point $x = 0$. This polynomial is called the second-degree **Taylor polynomial** of $f(x) = \cos x$ at $x = 0$.

(c) Complete the table comparing the values of $f(x) = \cos x$ and $P_2(x)$. What do you observe?

x	-1.0	-0.1	-0.001	0	0.001	0.1	1.0
$\cos x$							
$P_2(x)$							

(d) Find the third-degree Taylor polynomial of $f(x) = \sin x$ at $x = 0$.

4. (a) Find an equation of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$.

(b) Find an equation of the normal line to $y = x^2$ at the point $(2, 4)$. (The normal line is perpendicular to the tangent line.) Where does this line intersect the parabola a second time?

(c) Find equations of the tangent line and normal line to $y = x^2$ at the point $(0, 0)$.

(d) Prove that for any point $(a, b) \neq (0, 0)$ on the parabola $y = x^2$, the normal line intersects the graph a second time.

5. Find a third-degree polynomial $p(x)$ that is tangent to the line $y = 14x - 13$ at the point $(1, 1)$, and tangent to the line $y = -2x - 5$ at the point $(-1, -3)$.

6. Find a function of the form $f(x) = a + b \cos cx$ that is tangent to the line $y = 1$ at the point $(0, 1)$, and tangent to the line

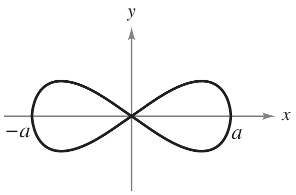
$$y = x + \frac{3}{2} - \frac{\pi}{4}$$

at the point $\left(\frac{\pi}{4}, \frac{3}{2}\right)$.

7. The graph of the **eight curve**

$$x^4 = a^2(x^2 - y^2), \quad a \neq 0$$

is shown below.



(a) Explain how you could use a graphing utility to graph this curve.

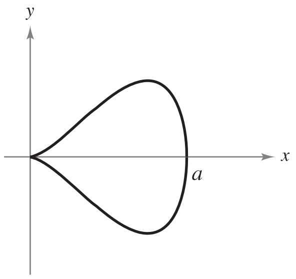
(b) Use a graphing utility to graph the curve for various values of the constant a. Describe how a affects the shape of the curve.

(c) Determine the points on the curve at which the tangent line is horizontal.

8. The graph of the **pear-shaped quartic**

$$b^2y^2 = x^3(a - x), \quad a, b > 0$$

is shown below.



(a) Explain how you could use a graphing utility to graph this curve.

(b) Use a graphing utility to graph the curve for various values of the constants a and b. Describe how a and b affect the shape of the curve.

(c) Determine the points on the curve at which the tangent line is horizontal.

9. A man 6 feet tall walks at a rate of 5 feet per second toward a streetlight that is 30 feet high (see figure). The man's 3-foot-tall child follows at the same speed, but 10 feet behind the man. At times, the shadow behind the child is caused by the man, and at other times, by the child.

(a) Suppose the man is 90 feet from the streetlight. Show that the man's shadow extends beyond the child's shadow.

(b) Suppose the man is 60 feet from the streetlight. Show that the child's shadow extends beyond the man's shadow.

(c) Determine the distance d from the man to the streetlight at which the tips of the two shadows are exactly the same distance from the streetlight.

(d) Determine how fast the tip of the man's shadow is moving as a function of x, the distance between the man and the street- light. Discuss the continuity of this shadow speed function.

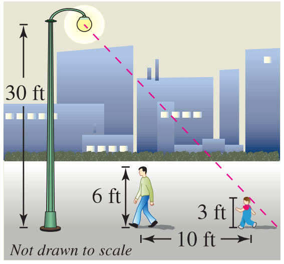


Figure for 9

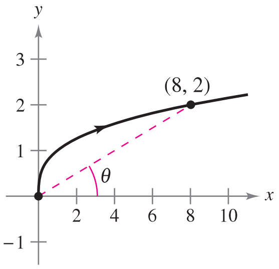


Figure for 10

10. A particle is moving along the graph of $y = \sqrt[3]{x}$ (see figure). When $x = 8$, the y-component of the position of the particle is increasing at the rate of 1 centimeter per second.

(a) How fast is the x-component changing at this moment?

(b) How fast is the distance from the origin changing at this moment?

(c) How fast is the angle of inclination θ changing at this moment?

11. Let L be a differentiable function for all x. Prove that if $L(a + b) = L(a) + L(b)$ for all a and b, then $L'(x) = L'(0)$ for all x. What does the graph of L look like?

12. Let E be a function satisfying $E(0) = E'(0) = 1$. Prove that if $E(a + b) = E(a)E(b)$ for all a and b, then E is differentiable and $E'(x) = E(x)$ for all x. Find an example of a function satisfying $E(a + b) = E(a)E(b)$.

13. The fundamental limit $\lim_{x \to 0} \frac{\sin x}{x} = 1$ assumes that x is measured in radians. What happens if you assume that x is measured in degrees instead of radians?

(a) Set your calculator to *degree* mode and complete the table.

z (in degrees)	0.1	0.01	0.0001
$\frac{\sin z}{z}$			

(b) Use the table to estimate

$$\lim_{z \to 0} \frac{\sin z}{z}$$

for z in degrees. What is the exact value of this limit? (*Hint:* $180° = \pi$ radians)

(c) Use the limit definition of the derivative to find

$$\frac{d}{dz} \sin z$$

for z in degrees.

(d) Define the new functions $S(z) = \sin(cz)$ and $C(z) = \cos(cz)$, where $c = \pi/180$. Find $S(90)$ and $C(180)$. Use the Chain Rule to calculate

$$\frac{d}{dz} S(z).$$

(e) Explain why calculus is made easier by using radians instead of degrees.

14. An astronaut standing on the moon throws a rock upward. The height of the rock is

$$s = -\frac{27}{10}t^2 + 27t + 6$$

where s is measured in feet and t is measured in seconds.

(a) Find expressions for the velocity and acceleration of the rock.

(b) Find the time when the rock is at its highest point by finding the time when the velocity is zero. What is the height of the rock at this time?

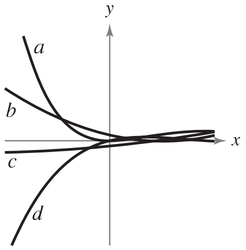
(c) How does the acceleration of the rock compare with the acceleration due to gravity on Earth?

15. If a is the acceleration of an object, the *jerk* j is defined by $j = a'(t)$.

(a) Use this definition to give a physical interpretation of j.

(b) Find j for the slowing vehicle in Exercise 119 in Section 2.3 and interpret the result.

(c) The figure shows the graphs of the position, velocity, acceleration, and jerk functions of a vehicle. Identify each graph and explain your reasoning.



3

Applications of Differentiation

This chapter discusses several applications of the derivative of a function. These applications fall into three basic categories—curve sketching, optimization, and approximation techniques.

In this chapter, you should learn the following.

- How to use a derivative to locate the minimum and maximum values of a function on a closed interval. (**3.1**)
- How numerous results in this chapter depend on two important theorems called *Rolle's Theorem* and the *Mean Value Theorem*. (**3.2**)
- How to use the first derivative to determine whether a function is increasing or decreasing. (**3.3**)
- How to use the second derivative to determine whether the graph of a function is concave upward or concave downward. (**3.4**)
- How to find horizontal asymptotes of the graph of a function. (**3.5**)
- How to graph a function using the techniques from Chapters P–3. (**3.6**)
- How to solve optimization problems. (**3.7**)
- How to use approximation techniques to solve problems. (**3.8** and **3.9**)

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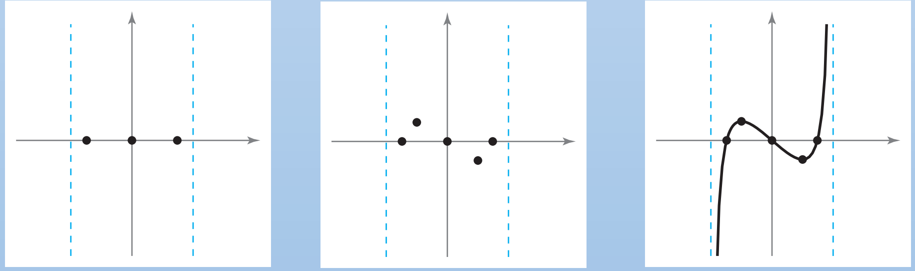
A small aircraft starts its descent from an altitude of 1 mile, 4 miles west of the runway. Given a function that models the glide path of the plane, when would the plane be descending at the greatest rate? (See Section 3.4, Exercise 75.)



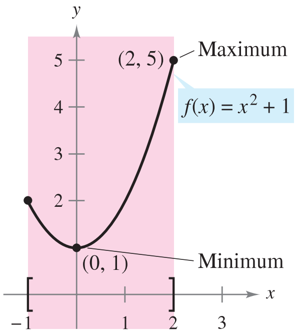
In Chapter 3, you will use calculus to analyze graphs of functions. For example, you can use the derivative of a function to determine the function's maximum and minimum values. You can use limits to identify any asymptotes of the function's graph. In Section 3.6, you will combine these techniques to sketch the graph of a function.

3.1 Extrema on an Interval

- **Understand the definition of extrema of a function on an interval.**
- **Understand the definition of relative extrema of a function on an open interval.**
- **Find extrema on a closed interval.**

Extrema of a Function

In calculus, much effort is devoted to determining the behavior of a function f on an interval I. Does f have a maximum value on I? Does it have a minimum value? Where is the function increasing? Where is it decreasing? In this chapter you will learn how derivatives can be used to answer these questions. You will also see why these questions are important in real-life applications.



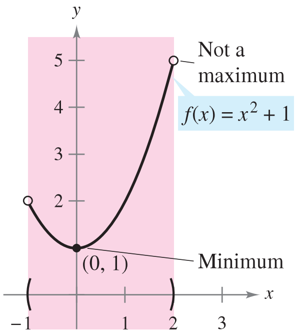
(a) f is continuous, $[-1, 2]$ is closed.

DEFINITION OF EXTREMA

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum of f on I** if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the **maximum of f on I** if $f(c) \geq f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values,** or **extrema** (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum,** or the **global minimum** and **global maximum,** on the interval.

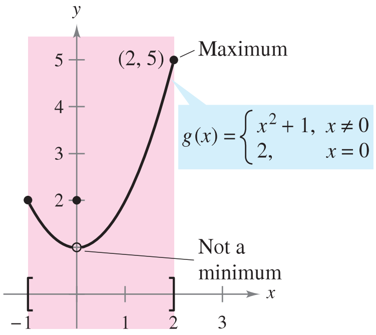


(b) f is continuous, $(-1, 2)$ is open.

A function need not have a minimum or a maximum on an interval. For instance, in Figure 3.1(a) and (b), you can see that the function $f(x) = x^2 + 1$ has both a minimum and a maximum on the closed interval $[-1, 2]$, but does not have a maximum on the open interval $(-1, 2)$. Moreover, in Figure 3.1(c), you can see that continuity (or the lack of it) can affect the existence of an extremum on the interval. This suggests the theorem below. (Although the Extreme Value Theorem is intuitively plausible, a proof of this theorem is not within the scope of this text.)

THEOREM 3.1 THE EXTREME VALUE THEOREM

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.



(c) g is not continuous, $[-1, 2]$ is closed.

Extrema can occur at interior points or endpoints of an interval. Extrema that occur at the endpoints are called **endpoint extrema.**
Figure 3.1

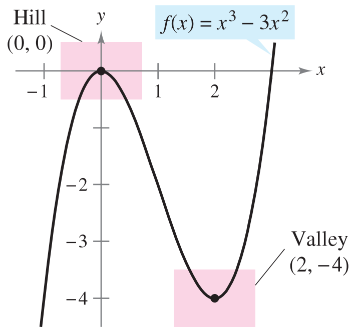
EXPLORATION

Finding Minimum and Maximum Values The Extreme Value Theorem (like the Intermediate Value Theorem) is an *existence theorem* because it tells of the existence of minimum and maximum values but does not show how to find these values. Use the extreme-value capability of a graphing utility to find the minimum and maximum values of each of the following functions. In each case, do you think the x-values are exact or approximate? Explain your reasoning.

a. $f(x) = x^2 - 4x + 5$ on the closed interval $[-1, 3]$

b. $f(x) = x^3 - 2x^2 - 3x - 2$ on the closed interval $[-1, 3]$

Relative Extrema and Critical Numbers



f has a relative maximum at (0, 0) and a relative minimum at (2, −4).
Figure 3.2

In Figure 3.2, the graph of $f(x) = x^3 - 3x^2$ has a **relative maximum** at the point (0, 0) and a **relative minimum** at the point (2, −4). Informally, for a continuous function, you can think of a relative maximum as occurring on a "hill" on the graph, and a relative minimum as occurring in a "valley" on the graph. Such a hill and valley can occur in two ways. If the hill (or valley) is smooth and rounded, the graph has a horizontal tangent line at the high point (or low point). If the hill (or valley) is sharp and peaked, the graph represents a function that is not differentiable at the high point (or low point).

DEFINITION OF RELATIVE EXTREMA

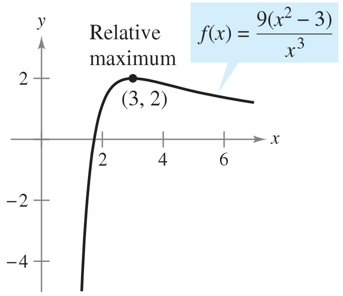
1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f, or you can say that f has a **relative maximum at $(c, f(c))$.**
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f, or you can say that f has a **relative minimum at $(c, f(c))$.**

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called **local maximum** and **local minimum,** respectively.

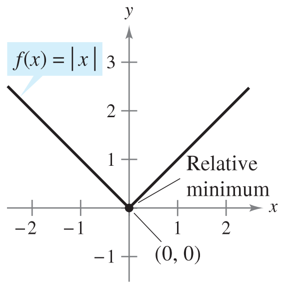
Example 1 examines the derivatives of functions at *given* relative extrema. (Much more is said about *finding* the relative extrema of a function in Section 3.3.)

EXAMPLE 1 The Value of the Derivative at Relative Extrema

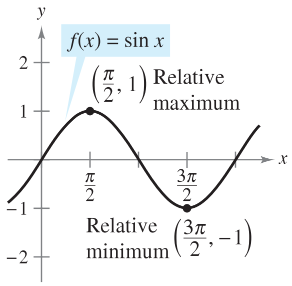
Find the value of the derivative at each relative extremum shown in Figure 3.3.



(a) $f'(3) = 0$



(b) $f'(0)$ does not exist.



(c) $f'\left(\frac{\pi}{2}\right) = 0; f'\left(\frac{3\pi}{2}\right) = 0$

Figure 3.3

Solution

a. The derivative of $f(x) = \dfrac{9(x^2-3)}{x^3}$ is

$$f'(x) = \frac{x^3(18x) - (9)(x^2-3)(3x^2)}{(x^3)^2} \qquad \text{Differentiate using Quotient Rule.}$$

$$= \frac{9(9-x^2)}{x^4}. \qquad \text{Simplify.}$$

At the point (3, 2), the value of the derivative is $f'(3) = 0$ [see Figure 3.3(a)].

b. At $x = 0$, the derivative of $f(x) = |x|$ *does not exist* because the following one-sided limits differ [see Figure 3.3(b)].

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{|x|}{x} = -1 \qquad \text{Limit from the left}$$

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{|x|}{x} = 1 \qquad \text{Limit from the right}$$

c. The derivative of $f(x) = \sin x$ is

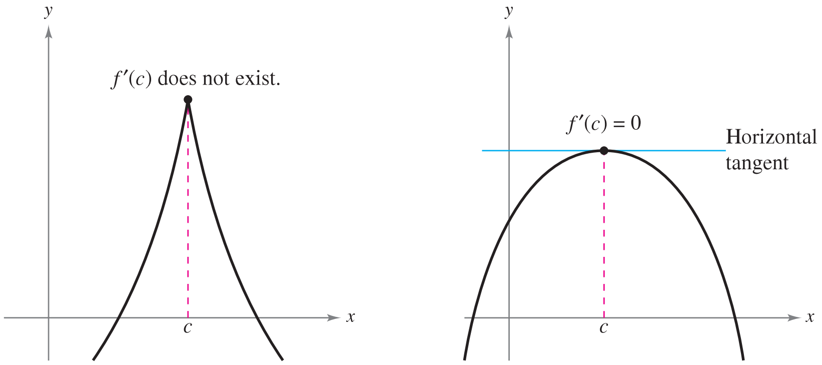
$$f'(x) = \cos x.$$

At the point $(\pi/2, 1)$, the value of the derivative is $f'(\pi/2) = \cos(\pi/2) = 0$. At the point $(3\pi/2, -1)$, the value of the derivative is $f'(3\pi/2) = \cos(3\pi/2) = 0$ [see Figure 3.3(c)]. ■

Note in Example 1 that at each relative extremum, the derivative either is zero or does not exist. The x-values at these special points are called **critical numbers.** Figure 3.4 illustrates the two types of critical numbers. Notice in the definition that the critical number c has to be in the domain of f, but c does not have to be in the domain of f'.

DEFINITION OF A CRITICAL NUMBER

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.



c is a critical number of f.
Figure 3.4

THEOREM 3.2 RELATIVE EXTREMA OCCUR ONLY AT CRITICAL NUMBERS

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

Mary Evans Picture Library

PIERRE DE FERMAT (1601–1665)

For Fermat, who was trained as a lawyer, mathematics was more of a hobby than a profession. Nevertheless, Fermat made many contributions to analytic geometry, number theory, calculus, and probability. In letters to friends, he wrote of many of the fundamental ideas of calculus, long before Newton or Leibniz. For instance, Theorem 3.2 is sometimes attributed to Fermat.

PROOF

Case 1: If f is *not* differentiable at $x = c$, then, by definition, c is a critical number of f and the theorem is valid.

Case 2: If f is differentiable at $x = c$, then $f'(c)$ must be positive, negative, or 0. Suppose $f'(c)$ is positive. Then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0$$

which implies that there exists an interval (a, b) containing c such that

$$\frac{f(x) - f(c)}{x - c} > 0, \text{ for all } x \neq c \text{ in } (a, b).$$ [See Exercise 82(b), Section 1.2.]

Because this quotient is positive, the signs of the denominator and numerator must agree. This produces the following inequalities for x-values in the interval (a, b).

Left of c: $x < c$ and $f(x) < f(c)$ $\Rightarrow$ $f(c)$ is not a relative minimum

Right of c: $x > c$ and $f(x) > f(c)$ $\Rightarrow$ $f(c)$ is not a relative maximum

So, the assumption that $f'(c) > 0$ contradicts the hypothesis that $f(c)$ is a relative extremum. Assuming that $f'(c) < 0$ produces a similar contradiction, you are left with only one possibility—namely, $f'(c) = 0$. So, by definition, c is a critical number of f and the theorem is valid. ■

Finding Extrema on a Closed Interval

Theorem 3.2 states that the relative extrema of a function can occur *only* at the critical numbers of the function. Knowing this, you can use the following guidelines to find extrema on a closed interval.

GUIDELINES FOR FINDING EXTREMA ON A CLOSED INTERVAL

To find the extrema of a continuous function f on a closed interval $[a, b]$, use the following steps.

1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

The next three examples show how to apply these guidelines. Be sure you see that finding the critical numbers of the function is only part of the procedure. Evaluating the function at the critical numbers *and* the endpoints is the other part.

EXAMPLE 2 Finding Extrema on a Closed Interval

Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.

Solution Begin by differentiating the function.

$$f(x) = 3x^4 - 4x^3 \qquad \text{Write original function.}$$
$$f'(x) = 12x^3 - 12x^2 \qquad \text{Differentiate.}$$

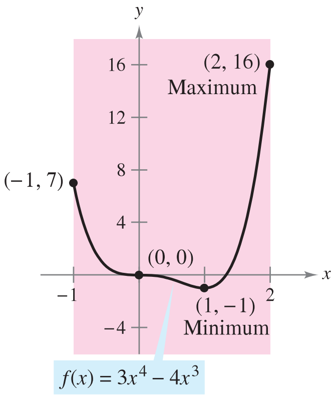
To find the critical numbers of f, you must find all x-values for which $f'(x) = 0$ and all x-values for which $f'(x)$ does not exist.

$$f'(x) = 12x^3 - 12x^2 = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$
$$12x^2(x - 1) = 0 \qquad \text{Factor.}$$
$$x = 0, 1 \qquad \text{Critical numbers}$$

Because f' is defined for all x, you can conclude that these are the only critical numbers of f. By evaluating f at these two critical numbers and at the endpoints of $[-1, 2]$, you can determine that the maximum is $f(2) = 16$ and the minimum is $f(1) = -1$, as shown in the table. The graph of f is shown in Figure 3.5.

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = 7$	$f(0) = 0$	$f(1) = -1$ Minimum	$f(2) = 16$ Maximum

■



On the closed interval $[-1, 2]$, f has a minimum at $(1, -1)$ and a maximum at $(2, 16)$.
Figure 3.5

In Figure 3.5, note that the critical number $x = 0$ does not yield a relative minimum or a relative maximum. This tells you that the converse of Theorem 3.2 is not true. In other words, *the critical numbers of a function need not produce relative extrema.*

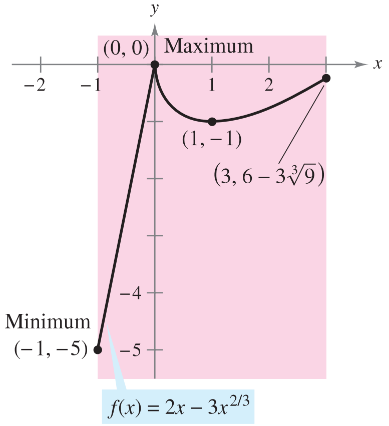
EXAMPLE 3 Finding Extrema on a Closed Interval

Find the extrema of $f(x) = 2x - 3x^{2/3}$ on the interval $[-1, 3]$.

Solution Begin by differentiating the function.

$$\begin{aligned} f(x) &= 2x - 3x^{2/3} && \text{Write original function.} \\ f'(x) &= 2 - \frac{2}{x^{1/3}} = 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right) && \text{Differentiate.} \end{aligned}$$

From this derivative, you can see that the function has two critical numbers in the interval $[-1, 3]$. The number 1 is a critical number because $f'(1) = 0$, and the number 0 is a critical number because $f'(0)$ does not exist. By evaluating f at these two numbers and at the endpoints of the interval, you can conclude that the minimum is $f(-1) = -5$ and the maximum is $f(0) = 0$, as shown in the table. The graph of f is shown in Figure 3.6.



On the closed interval $[-1, 3]$, f has a minimum at $(-1, 5)$ and a maximum at $(0, 0)$.

Figure 3.6

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = -5$ Minimum	$f(0) = 0$ Maximum	$f(1) = -1$	$f(3) = 6 - 3\sqrt[3]{9} \approx -0.24$

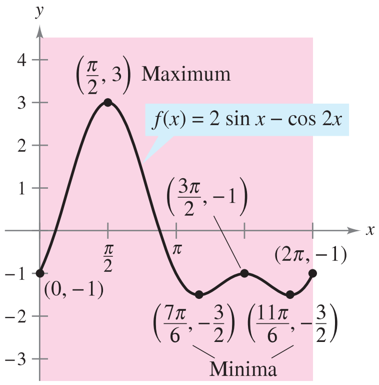
EXAMPLE 4 Finding Extrema on a Closed Interval

Find the extrema of $f(x) = 2 \sin x - \cos 2x$ on the interval $[0, 2\pi]$.

Solution This function is differentiable for all real x, so you can find all critical numbers by differentiating the function and setting $f'(x)$ equal to zero, as shown.

$$\begin{aligned} f(x) &= 2 \sin x - \cos 2x && \text{Write original function.} \\ f'(x) &= 2 \cos x + 2 \sin 2x = 0 && \text{Set } f'(x) \text{ equal to 0.} \\ 2 \cos x + 4 \cos x \sin x &= 0 && \sin 2x = 2 \cos x \sin x \\ 2(\cos x)(1 + 2 \sin x) &= 0 && \text{Factor.} \end{aligned}$$

In the interval $[0, 2\pi]$, the factor $\cos x$ is zero when $x = \pi/2$ and when $x = 3\pi/2$. The factor $(1 + 2 \sin x)$ is zero when $x = 7\pi/6$ and when $x = 11\pi/6$. By evaluating f at these four critical numbers and at the endpoints of the interval, you can conclude that the maximum is $f(\pi/2) = 3$ and the minimum occurs at *two* points, $f(7\pi/6) = -3/2$ and $f(11\pi/6) = -3/2$, as shown in the table. The graph is shown in Figure 3.7.



On the closed interval $[0, 2\pi]$, f has two minima at $(7\pi/6, -3/2)$ and $(11\pi/6, -3/2)$ and a maximum at $(\pi/2, 3)$.

Figure 3.7

Left Endpoint	Critical Number	Critical Number	Critical Number	Critical Number	Right Endpoint
$f(0) = -1$	$f\left(\frac{\pi}{2}\right) = 3$ Maximum	$f\left(\frac{7\pi}{6}\right) = -\frac{3}{2}$ Minimum	$f\left(\frac{3\pi}{2}\right) = -1$	$f\left(\frac{11\pi}{6}\right) = -\frac{3}{2}$ Minimum	$f(2\pi) = -1$

■

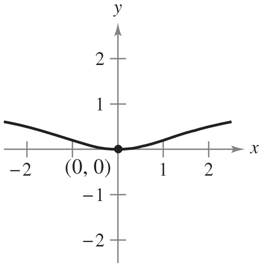
The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

3.1 Exercises

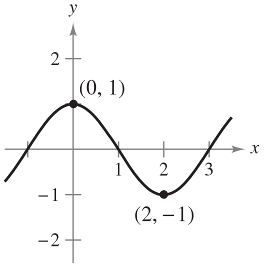
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the value of the derivative (if it exists) at each indicated extremum.

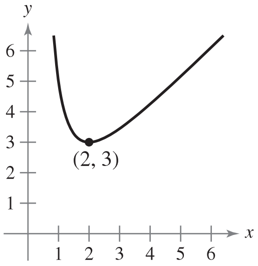
1. $f(x) = \dfrac{x^2}{x^2 + 4}$



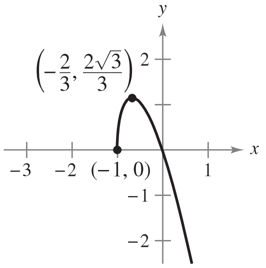
2. $f(x) = \cos \dfrac{\pi x}{2}$



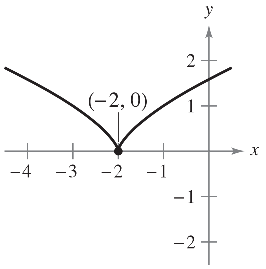
3. $g(x) = x + \dfrac{4}{x^2}$



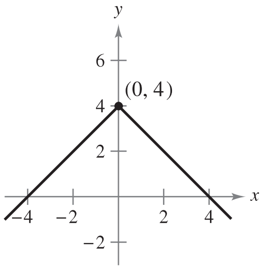
4. $f(x) = -3x\sqrt{x+1}$



5. $f(x) = (x + 2)^{2/3}$

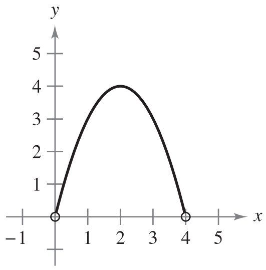


6. $f(x) = 4 - |x|$



In Exercises 7–10, approximate the critical numbers of the function shown in the graph. Determine whether the function has a relative maximum, a relative minimum, an absolute maximum, an absolute minimum, or none of these at each critical number on the interval shown.

7.



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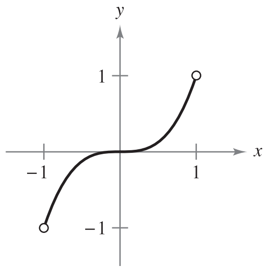

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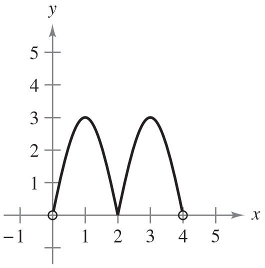

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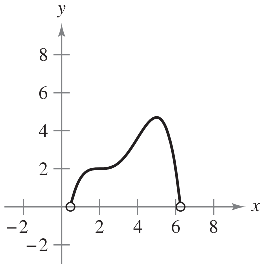


In Exercises 11–16, find any critical numbers of the function.

11. $f(x) = x^3 - 3x^2$

12. $g(x) = x^4 - 4x^2$

13. $g(t) = t\sqrt{4 - t}, \; t < 3$

14. $f(x) = \dfrac{4x}{x^2 + 1}$

15. $h(x) = \sin^2 x + \cos x$
$0 < x < 2\pi$

16. $f(\theta) = 2 \sec \theta + \tan \theta$
$0 < \theta < 2\pi$

In Exercises 17–36, locate the absolute extrema of the function on the closed interval.

17. $f(x) = 3 - x, \; [-1, 2]$

18. $f(x) = \dfrac{2x + 5}{3}, \; [0, 5]$

19. $g(x) = x^2 - 2x, \; [0, 4]$

20. $h(x) = -x^2 + 3x - 5, \; [-2, 1]$

21. $f(x) = x^3 - \dfrac{3}{2}x^2, \; [-1, 2]$

22. $f(x) = x^3 - 12x, \; [0, 4]$

23. $y = 3x^{2/3} - 2x, \; [-1, 1]$

24. $g(x) = \sqrt[3]{x}, \; [-1, 1]$

25. $g(t) = \dfrac{t^2}{t^2 + 3}, \; [-1, 1]$

26. $f(x) = \dfrac{2x}{x^2 + 1}, \; [-2, 2]$

27. $h(s) = \dfrac{1}{s - 2}, \; [0, 1]$

28. $h(t) = \dfrac{t}{t - 2}, \; [3, 5]$

29. $y = 3 - |t - 3|, [-1, 5]$

30. $g(x) = \dfrac{1}{1 + |x + 1|}, \; [-3, 3]$

31. $f(x) = \llbracket x \rrbracket, \; [-2, 2]$

32. $h(x) = \llbracket 2 - x \rrbracket, \; [-2, 2]$

33. $f(x) = \cos \pi x, \; \left[0, \dfrac{1}{6}\right]$

34. $g(x) = \sec x, \; \left[-\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

35. $y = 3 \cos x, \; [0, 2\pi]$

36. $y = \tan\left(\dfrac{\pi x}{8}\right), \; [0, 2]$

In Exercises 37–40, locate the absolute extrema of the function (if any exist) over each interval.

37. $f(x) = 2x - 3$
(a) $[0, 2]$ (b) $[0, 2)$
(c) $(0, 2]$ (d) $(0, 2)$

38. $f(x) = 5 - x$
(a) $[1, 4]$ (b) $[1, 4)$
(c) $(1, 4]$ (d) $(1, 4)$

39. $f(x) = x^2 - 2x$
(a) $[-1, 2]$ (b) $(1, 3]$
(c) $(0, 2)$ (d) $[1, 4)$

40. $f(x) = \sqrt{4 - x^2}$
(a) $[-2, 2]$ (b) $[-2, 0)$
(c) $(-2, 2)$ (d) $[1, 2)$

In Exercises 41–46, use a graphing utility to graph the function. Locate the absolute extrema of the function on the given interval.

41. $f(x) = \begin{cases} 2x + 2, & 0 \le x \le 1 \\ 4x^2, & 1 < x \le 3 \end{cases}, \quad [0, 3]$

42. $f(x) = \begin{cases} 2 - x^2, & 1 \le x < 3 \\ 2 - 3x, & 3 \le x \le 5 \end{cases}, \quad [1, 5]$

43. $f(x) = \dfrac{3}{x - 1}, \quad (1, 4]$ **44.** $f(x) = \dfrac{2}{2 - x}, \quad [0, 2)$

45. $f(x) = x^4 - 2x^3 + x + 1, \quad [-1, 3]$

46. $f(x) = \sqrt{x} + \cos \dfrac{x}{2}, \quad [0, 2\pi]$

CAS **In Exercises 47 and 48, (a) use a computer algebra system to graph the function and approximate any absolute extrema on the given interval. (b) Use the utility to find any critical numbers, and use them to find any absolute extrema not located at the endpoints. Compare the results with those in part (a).**

47. $f(x) = 3.2x^5 + 5x^3 - 3.5x, \quad [0, 1]$

48. $f(x) = \dfrac{4}{3}x\sqrt{3 - x}, \quad [0, 3]$

CAS **In Exercises 49 and 50, use a computer algebra system to find the maximum value of $|f''(x)|$ on the closed interval. (This value is used in the error estimate for the Trapezoidal Rule, as discussed in Section 4.6.)**

49. $f(x) = \sqrt{1 + x^3}, \quad [0, 2]$

50. $f(x) = \dfrac{1}{x^2 + 1}, \quad \left[\dfrac{1}{2}, 3\right]$

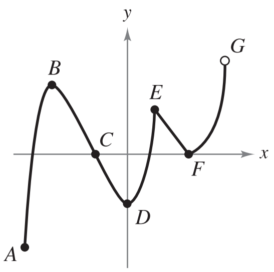
CAS **In Exercises 51 and 52, use a computer algebra system to find the maximum value of $|f^{(4)}(x)|$ on the closed interval. (This value is used in the error estimate for Simpson's Rule, as discussed in Section 4.6.)**

51. $f(x) = (x + 1)^{2/3}, \quad [0, 2]$ **52.** $f(x) = \dfrac{1}{x^2 + 1}, \quad [-1, 1]$

53. ***Writing*** Write a short paragraph explaining why a continuous function on an open interval may not have a maximum or minimum. Illustrate your explanation with a sketch of the graph of such a function.

CAPSTONE

54. Decide whether each labeled point is an absolute maximum or minimum, a relative maximum or minimum, or neither.



WRITING ABOUT CONCEPTS

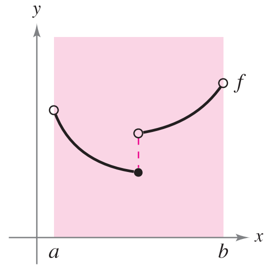
In Exercises 55 and 56, graph a function on the interval $[-2, 5]$ having the given characteristics.

55. Absolute maximum at $x = -2$, absolute minimum at $x = 1$, relative maximum at $x = 3$

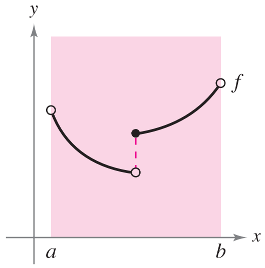
56. Relative minimum at $x = -1$, critical number (but no extremum) at $x = 0$, absolute maximum at $x = 2$, absolute minimum at $x = 5$

In Exercises 57–60, determine from the graph whether f has a minimum in the open interval (a, b).

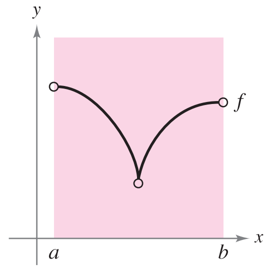
57. (a)



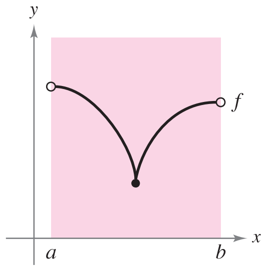
(b)



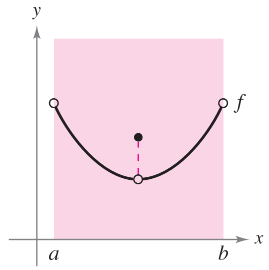
58. (a)



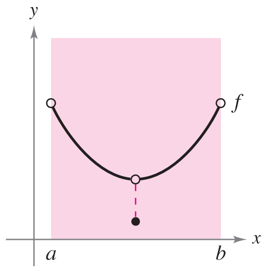
(b)



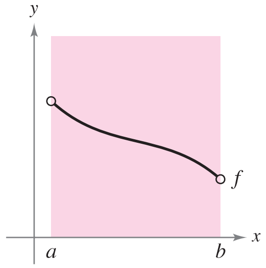
59. (a)



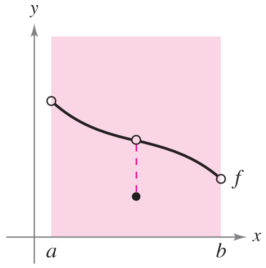
(b)



60. (a)



(b)

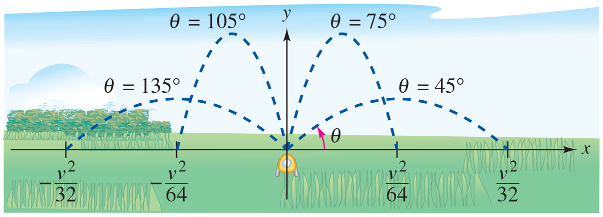


61. ***Power*** The formula for the power output P of a battery is $P = VI - RI^2$, where V is the electromotive force in volts, R is the resistance in ohms, and I is the current in amperes. Find the current that corresponds to a maximum value of P in a battery for which $V = 12$ volts and $R = 0.5$ ohm. Assume that a 15-ampere fuse bounds the output in the interval $0 \le I \le 15$. Could the power output be increased by replacing the 15-ampere fuse with a 20-ampere fuse? Explain.

62. ***Lawn Sprinkler*** A lawn sprinkler is constructed in such a way that $d\theta/dt$ is constant, where θ ranges between 45° and 135° (see figure). The distance the water travels horizontally is

$$x = \frac{v^2 \sin 2\theta}{32}, \quad 45° \le \theta \le 135°$$

where v is the speed of the water. Find dx/dt and explain why this lawn sprinkler does not water evenly. What part of the lawn receives the most water?



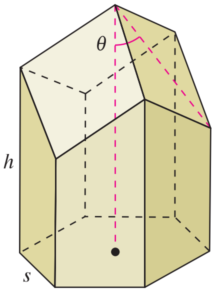
Water sprinkler: $45° \le \theta \le 135°$

■ **FOR FURTHER INFORMATION** For more information on the "calculus of lawn sprinklers," see the article "Design of an Oscillating Sprinkler" by Bart Braden in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

63. ***Honeycomb*** The surface area of a cell in a honeycomb is

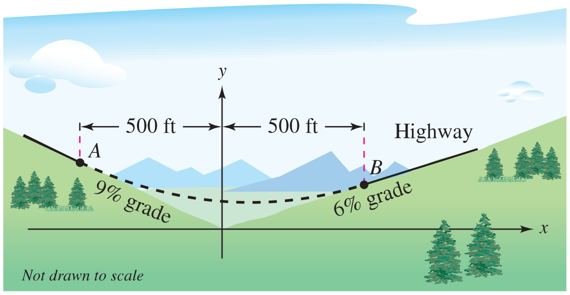
$$S = 6hs + \frac{3s^2}{2}\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right)$$

where h and s are positive constants and θ is the angle at which the upper faces meet the altitude of the cell (see figure). Find the angle θ $(\pi/6 \le \theta \le \pi/2)$ that minimizes the surface area S.



■ **FOR FURTHER INFORMATION** For more information on the geometric structure of a honeycomb cell, see the article "The Design of Honeycombs" by Anthony L. Peressini in UMAP Module 502, published by COMAP, Inc., Suite 210, 57 Bedford Street, Lexington, MA.

64. ***Highway Design*** In order to build a highway, it is necessary to fill a section of a valley where the grades (slopes) of the sides are 9% and 6% (see figure). The top of the filled region will have the shape of a parabolic arc that is tangent to the two slopes at the points A and B. The horizontal distances from A to the y-axis and from B to the y-axis are both 500 feet.



(a) Find the coordinates of A and B.

(b) Find a quadratic function $y = ax^2 + bx + c$, $-500 \le x \le 500$, that describes the top of the filled region.

(c) Construct a table giving the depths d of the fill for $x = -500, -400, -300, -200, -100, 0, 100, 200, 300, 400$, and 500.

(d) What will be the lowest point on the completed highway? Will it be directly over the point where the two hillsides come together?

True or False? **In Exercises 65–68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

65. The maximum of a function that is continuous on a closed interval can occur at two different values in the interval.

66. If a function is continuous on a closed interval, then it must have a minimum on the interval.

67. If $x = c$ is a critical number of the function f, then it is also a critical number of the function $g(x) = f(x) + k$, where k is a constant.

68. If $x = c$ is a critical number of the function f, then it is also a critical number of the function $g(x) = f(x - k)$, where k is a constant.

69. Let the function f be differentiable on an interval I containing c. If f has a maximum value at $x = c$, show that $-f$ has a minimum value at $x = c$.

70. Consider the cubic function $f(x) = ax^3 + bx^2 + cx + d$, where $a \ne 0$. Show that f can have zero, one, or two critical numbers and give an example of each case.

PUTNAM EXAM CHALLENGE

71. Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region $R = \{(x, y);\ 0 \le x \le a,\ 0 \le y \le f(x)\}$ has perimeter k units and area k square units for some real number k.

This problem was composed by the Committee on the Putnam Prize Competition.
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3.2 Rolle's Theorem and the Mean Value Theorem

- Understand and use Rolle's Theorem.
- Understand and use the Mean Value Theorem.

Rolle's Theorem

The Extreme Value Theorem (Section 3.1) states that a continuous function on a closed interval $[a, b]$ must have both a minimum and a maximum on the interval. Both of these values, however, can occur at the endpoints. **Rolle's Theorem,** named after the French mathematician Michel Rolle (1652–1719), gives conditions that guarantee the existence of an extreme value in the *interior* of a closed interval.

ROLLE'S THEOREM

French mathematician Michel Rolle first published the theorem that bears his name in 1691. Before this time, however, Rolle was one of the most vocal critics of calculus, stating that it gave erroneous results and was based on unsound reasoning. Later in life, Rolle came to see the usefulness of calculus.

EXPLORATION

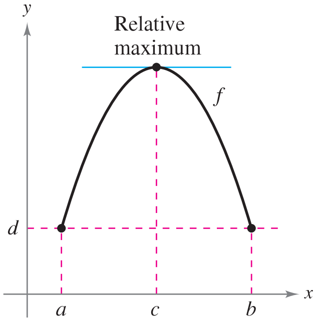
Extreme Values in a Closed Interval Sketch a rectangular coordinate plane on a piece of paper. Label the points $(1, 3)$ and $(5, 3)$. Using a pencil or pen, draw the graph of a differentiable function f that starts at $(1, 3)$ and ends at $(5, 3)$. Is there at least one point on the graph for which the derivative is zero? Would it be possible to draw the graph so that there *isn't* a point for which the derivative is zero? Explain your reasoning.

THEOREM 3.3 ROLLE'S THEOREM

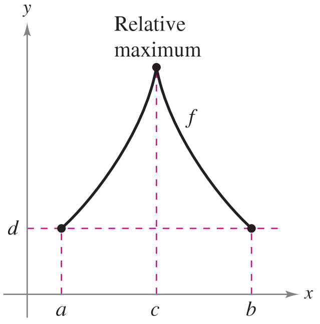
Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If

$$f(a) = f(b)$$

then there is at least one number c in (a, b) such that $f'(c) = 0$.



(a) f is continuous on $[a, b]$ and differentiable on (a, b).



(b) f is continuous on $[a, b]$.

Figure 3.8

PROOF Let $f(a) = d = f(b)$.

Case 1: If $f(x) = d$ for all x in $[a, b]$, f is constant on the interval and, by Theorem 2.2, $f'(x) = 0$ for all x in (a, b).

Case 2: Suppose $f(x) > d$ for some x in (a, b). By the Extreme Value Theorem, you know that f has a maximum at some c in the interval. Moreover, because $f(c) > d$, this maximum does not occur at either endpoint. So, f has a maximum in the *open* interval (a, b). This implies that $f(c)$ is a *relative* maximum and, by Theorem 3.2, c is a critical number of f. Finally, because f is differentiable at c, you can conclude that $f'(c) = 0$.

Case 3: If $f(x) < d$ for some x in (a, b), you can use an argument similar to that in Case 2, but involving the minimum instead of the maximum. ■

From Rolle's Theorem, you can see that if a function f is continuous on $[a, b]$ and differentiable on (a, b), and if $f(a) = f(b)$, there must be at least one x-value between a and b at which the graph of f has a horizontal tangent, as shown in Figure 3.8(a). If the differentiability requirement is dropped from Rolle's Theorem, f will still have a critical number in (a, b), but it may not yield a horizontal tangent. Such a case is shown in Figure 3.8(b).

EXAMPLE 1 Illustrating Rolle's Theorem

Find the two x-intercepts of

$$f(x) = x^2 - 3x + 2$$

and show that $f'(x) = 0$ at some point between the two x-intercepts.

Solution Note that f is differentiable on the entire real line. Setting $f(x)$ equal to 0 produces

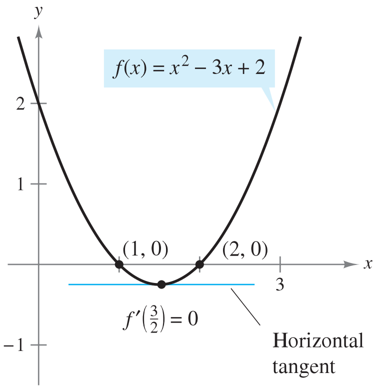
$$x^2 - 3x + 2 = 0 \qquad \text{Set } f(x) \text{ equal to 0.}$$

$$(x - 1)(x - 2) = 0. \qquad \text{Factor.}$$

So, $f(1) = f(2) = 0$, and from Rolle's Theorem you know that there *exists* at least one c in the interval $(1, 2)$ such that $f'(c) = 0$. To *find* such a c, you can solve the equation

$$f'(x) = 2x - 3 = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

and determine that $f'(x) = 0$ when $x = \frac{3}{2}$. Note that this x-value lies in the open interval $(1, 2)$, as shown in Figure 3.9. ■



The x-value for which $f'(x) = 0$ is between the two x-intercepts.
Figure 3.9

Rolle's Theorem states that if f satisfies the conditions of the theorem, there must be *at least* one point between a and b at which the derivative is 0. There may of course be more than one such point, as shown in the next example.

EXAMPLE 2 Illustrating Rolle's Theorem

Let $f(x) = x^4 - 2x^2$. Find all values of c in the interval $(-2, 2)$ such that $f'(c) = 0$.

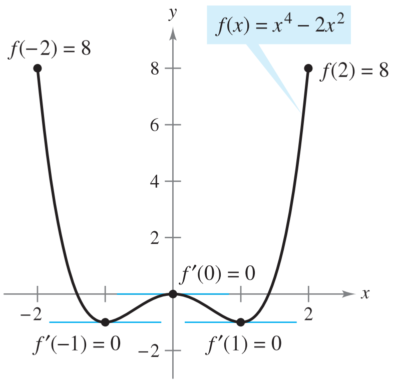
Solution To begin, note that the function satisfies the conditions of Rolle's Theorem. That is, f is continuous on the interval $[-2, 2]$ and differentiable on the interval $(-2, 2)$. Moreover, because $f(-2) = f(2) = 8$, you can conclude that there exists at least one c in $(-2, 2)$ such that $f'(c) = 0$. Setting the derivative equal to 0 produces

$$f'(x) = 4x^3 - 4x = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

$$4x(x - 1)(x + 1) = 0 \qquad \text{Factor.}$$

$$x = 0, 1, -1. \qquad x\text{-values for which } f'(x) = 0$$

So, in the interval $(-2, 2)$, the derivative is zero at three different values of x, as shown in Figure 3.10. ■



$f'(x) = 0$ for more than one x-value in the interval $(-2, 2)$.
Figure 3.10

TECHNOLOGY PITFALL A graphing utility can be used to indicate whether the points on the graphs in Examples 1 and 2 are relative minima or relative maxima of the functions. When using a graphing utility, however, you should keep in mind that it can give misleading pictures of graphs. For example, use a graphing utility to graph

$$f(x) = 1 - (x - 1)^2 - \frac{1}{1000(x - 1)^{1/7} + 1}.$$

With most viewing windows, it appears that the function has a maximum of 1 when $x = 1$ (see Figure 3.11). By evaluating the function at $x = 1$, however, you can see that $f(1) = 0$. To determine the behavior of this function near $x = 1$, you need to examine the graph analytically to get the complete picture.

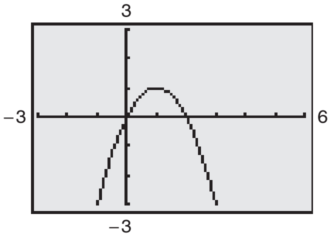


Figure 3.11

The Mean Value Theorem

Rolle's Theorem can be used to prove another theorem—the **Mean Value Theorem.**

THEOREM 3.4 THE MEAN VALUE THEOREM

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

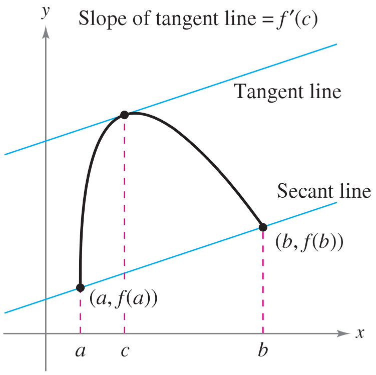


Figure 3.12

PROOF Refer to Figure 3.12. The equation of the secant line that passes through the points $(a, f(a))$ and $(b, f(b))$ is

$$y = \left[\frac{f(b) - f(a)}{b - a}\right](x - a) + f(a).$$

Let $g(x)$ be the difference between $f(x)$ and y. Then

$$\begin{aligned} g(x) &= f(x) - y \\ &= f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a) - f(a). \end{aligned}$$

By evaluating g at a and b, you can see that $g(a) = 0 = g(b)$. Because f is continuous on $[a, b]$, it follows that g is also continuous on $[a, b]$. Furthermore, because f is differentiable, g is also differentiable, and you can apply Rolle's Theorem to the function g. So, there exists a number c in (a, b) such that $g'(c) = 0$, which implies that

$$\begin{aligned} 0 &= g'(c) \\ &= f'(c) - \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

So, there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$ ■

NOTE The "mean" in the Mean Value Theorem refers to the mean (or average) rate of change of f in the interval $[a, b]$. ■

Although the Mean Value Theorem can be used directly in problem solving, it is used more often to prove other theorems. In fact, some people consider this to be the most important theorem in calculus—it is closely related to the Fundamental Theorem of Calculus discussed in Section 4.4. For now, you can get an idea of the versatility of the Mean Value Theorem by looking at the results stated in Exercises 81–89 in this section.

The Mean Value Theorem has implications for both basic interpretations of the derivative. Geometrically, the theorem guarantees the existence of a tangent line that is parallel to the secant line through the points $(a, f(a))$ and $(b, f(b))$, as shown in Figure 3.12. Example 3 illustrates this geometric interpretation of the Mean Value Theorem. In terms of rates of change, the Mean Value Theorem implies that there must be a point in the open interval (a, b) at which the instantaneous rate of change is equal to the average rate of change over the interval $[a, b]$. This is illustrated in Example 4.

Mary Evans Picture Library

JOSEPH-LOUIS LAGRANGE (1736–1813)

The Mean Value Theorem was first proved by the famous mathematician Joseph-Louis Lagrange. Born in Italy, Lagrange held a position in the court of Frederick the Great in Berlin for 20 years. Afterward, he moved to France, where he met emperor Napoleon Bonaparte, who is quoted as saying, "Lagrange is the lofty pyramid of the mathematical sciences."

EXAMPLE 3 Finding a Tangent Line

Given $f(x) = 5 - (4/x)$, find all values of c in the open interval $(1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

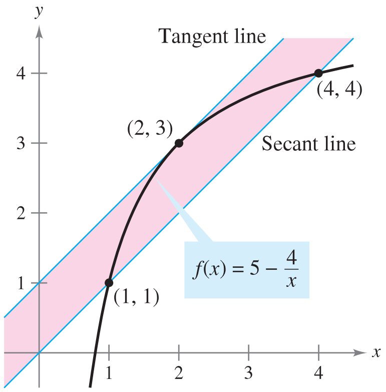
Solution The slope of the secant line through $(1, f(1))$ and $(4, f(4))$ is

$$\frac{f(4) - f(1)}{4 - 1} = \frac{4 - 1}{4 - 1} = 1.$$

Note that the function satisfies the conditions of the Mean Value Theorem. That is, f is continuous on the interval $[1, 4]$ and differentiable on the interval $(1, 4)$. So, there exists at least one number c in $(1, 4)$ such that $f'(c) = 1$. Solving the equation $f'(x) = 1$ yields

$$f'(x) = \frac{4}{x^2} = 1$$

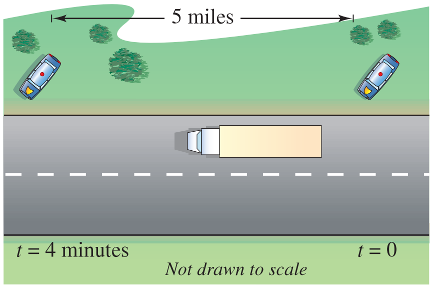
which implies that $x = \pm 2$. So, in the interval $(1, 4)$, you can conclude that $c = 2$, as shown in Figure 3.13.



The tangent line at $(2, 3)$ is parallel to the secant line through $(1, 1)$ and $(4, 4)$.
Figure 3.13

EXAMPLE 4 Finding an Instantaneous Rate of Change

Two stationary patrol cars equipped with radar are 5 miles apart on a highway, as shown in Figure 3.14. As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 miles per hour. Prove that the truck must have exceeded the speed limit (of 55 miles per hour) at some time during the 4 minutes.



At some time t, the instantaneous velocity is equal to the average velocity over 4 minutes.
Figure 3.14

Solution Let $t = 0$ be the time (in hours) when the truck passes the first patrol car. The time when the truck passes the second patrol car is

$$t = \frac{4}{60} = \frac{1}{15} \text{ hour.}$$

By letting $s(t)$ represent the distance (in miles) traveled by the truck, you have $s(0) = 0$ and $s\left(\frac{1}{15}\right) = 5$. So, the average velocity of the truck over the five-mile stretch of highway is

$$\text{Average velocity} = \frac{s(1/15) - s(0)}{(1/15) - 0} = \frac{5}{1/15} = 75 \text{ miles per hour.}$$

Assuming that the position function is differentiable, you can apply the Mean Value Theorem to conclude that the truck must have been traveling at a rate of 75 miles per hour sometime during the 4 minutes. ■

A useful alternative form of the Mean Value Theorem is as follows: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that

$$f(b) = f(a) + (b - a)f'(c).$$

Alternative form of Mean Value Theorem

NOTE When doing the exercises for this section, keep in mind that polynomial functions, rational functions, and trigonometric functions are differentiable at all points in their domains. ■

3.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, explain why Rolle's Theorem does not apply to the function even though there exist a and b such that $f(a) = f(b)$.

1. $f(x) = \left|\frac{1}{x}\right|$, $[-1, 1]$

2. $f(x) = \cot \frac{x}{2}$, $[\pi, 3\pi]$

3. $f(x) = 1 - |x - 1|$, $[0, 2]$

4. $f(x) = \sqrt{(2 - x^{2/3})^3}$, $[-1, 1]$

In Exercises 5–8, find the two x-intercepts of the function f and show that $f'(x) = 0$ at some point between the two x-intercepts.

5. $f(x) = x^2 - x - 2$

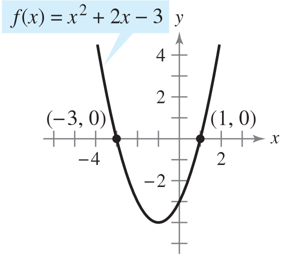
6. $f(x) = x(x - 3)$

7. $f(x) = x\sqrt{x + 4}$

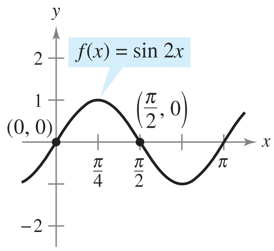
8. $f(x) = -3x\sqrt{x + 1}$

Rolle's Theorem **In Exercises 9 and 10, the graph of f is shown. Apply Rolle's Theorem and find all values of c such that $f'(c) = 0$ at some point between the labeled intercepts.**

9.



10.



In Exercises 11–24, determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$. If Rolle's Theorem cannot be applied, explain why not.

11. $f(x) = -x^2 + 3x$, $[0, 3]$

12. $f(x) = x^2 - 5x + 4$, $[1, 4]$

13. $f(x) = (x - 1)(x - 2)(x - 3)$, $[1, 3]$

14. $f(x) = (x - 3)(x + 1)^2$, $[-1, 3]$

15. $f(x) = x^{2/3} - 1$, $[-8, 8]$

16. $f(x) = 3 - |x - 3|$, $[0, 6]$

17. $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}$, $[-1, 3]$

18. $f(x) = \dfrac{x^2 - 1}{x}$, $[-1, 1]$

19. $f(x) = \sin x$, $[0, 2\pi]$

20. $f(x) = \cos x$, $[0, 2\pi]$

21. $f(x) = \dfrac{6x}{\pi} - 4\sin^2 x$, $\left[0, \dfrac{\pi}{6}\right]$

22. $f(x) = \cos 2x$, $[-\pi, \pi]$

23. $f(x) = \tan x$, $[0, \pi]$

24. $f(x) = \sec x$, $[\pi, 2\pi]$

In Exercises 25–28, use a graphing utility to graph the function on the closed interval $[a, b]$. Determine whether Rolle's Theorem can be applied to f on the interval and, if so, find all values of c in the open interval (a, b) such that $f'(c) = 0$.

25. $f(x) = |x| - 1$, $[-1, 1]$

26. $f(x) = x - x^{1/3}$, $[0, 1]$

27. $f(x) = x - \tan \pi x$, $\left[-\frac{1}{4}, \frac{1}{4}\right]$

28. $f(x) = \dfrac{x}{2} - \sin \dfrac{\pi x}{6}$, $[-1, 0]$

29. ***Vertical Motion*** The height of a ball t seconds after it is thrown upward from a height of 6 feet and with an initial velocity of 48 feet per second is $f(t) = -16t^2 + 48t + 6$.

(a) Verify that $f(1) = f(2)$.

(b) According to Rolle's Theorem, what must the velocity be at some time in the interval $(1, 2)$? Find that time.

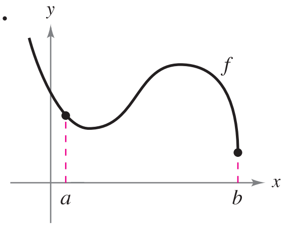
30. ***Reorder Costs*** The ordering and transportation cost C for components used in a manufacturing process is approximated by $C(x) = 10\left(\dfrac{1}{x} + \dfrac{x}{x + 3}\right)$, where C is measured in thousands of dollars and x is the order size in hundreds.

(a) Verify that $C(3) = C(6)$.

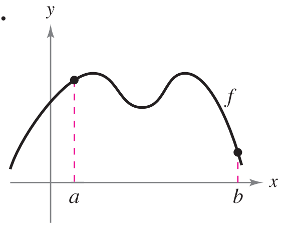
(b) According to Rolle's Theorem, the rate of change of the cost must be 0 for some order size in the interval $(3, 6)$. Find that order size.

In Exercises 31 and 32, copy the graph and sketch the secant line to the graph through the points $(a, f(a))$ and $(b, f(b))$. Then sketch any tangent lines to the graph for each value of c guaranteed by the Mean Value Theorem. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

31.

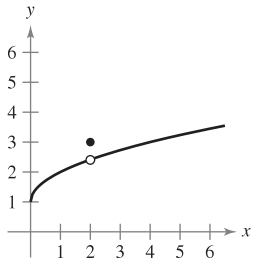


32.

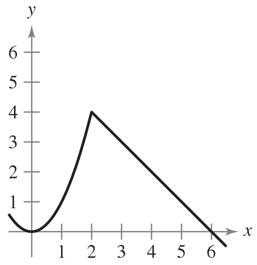


Writing **In Exercises 33–36, explain why the Mean Value Theorem does not apply to the function f on the interval $[0, 6]$.**

33.



34.



35. $f(x) = \dfrac{1}{x - 3}$

36. $f(x) = |x - 3|$

37. ***Mean Value Theorem*** Consider the graph of the function $f(x) = -x^2 + 5$. (a) Find the equation of the secant line joining the points $(-1, 4)$ and $(2, 1)$. (b) Use the Mean Value Theorem to determine a point c in the interval $(-1, 2)$ such that the tangent line at c is parallel to the secant line. (c) Find the equation of the tangent line through c. (d) Then use a graphing utility to graph f, the secant line, and the tangent line.

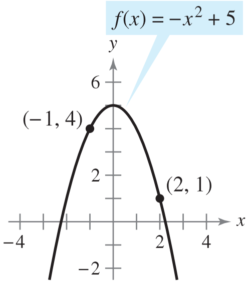


Figure for 37

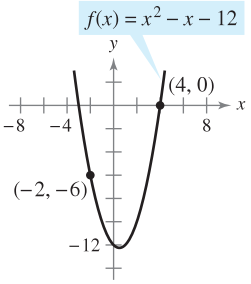


Figure for 38

38. ***Mean Value Theorem*** Consider the graph of the function $f(x) = x^2 - x - 12$. (a) Find the equation of the secant line joining the points $(-2, -6)$ and $(4, 0)$. (b) Use the Mean Value Theorem to determine a point c in the interval $(-2, 4)$ such that the tangent line at c is parallel to the secant line. (c) Find the equation of the tangent line through c. (d) Then use a graphing utility to graph f, the secant line, and the tangent line.

In Exercises 39–48, determine whether the Mean Value Theorem can be applied to f on the closed interval $[a, b]$. If the Mean Value Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. If the Mean Value Theorem cannot be applied, explain why not.

39. $f(x) = x^2, \quad [-2, 1]$

40. $f(x) = x^3, \quad [0, 1]$

41. $f(x) = x^3 + 2x, \quad [-1, 1]$

42. $f(x) = x^4 - 8x, \quad [0, 2]$

43. $f(x) = x^{2/3}, \quad [0, 1]$

44. $f(x) = \dfrac{x + 1}{x}, \quad [-1, 2]$

45. $f(x) = |2x + 1|, \quad [-1, 3]$

46. $f(x) = \sqrt{2 - x}, \quad [-7, 2]$

47. $f(x) = \sin x, \quad [0, \pi]$

48. $f(x) = \cos x + \tan x, \quad [0, \pi]$

In Exercises 49–52, use a graphing utility to (a) graph the function f on the given interval, (b) find and graph the secant line through points on the graph of f at the endpoints of the given interval, and (c) find and graph any tangent lines to the graph of f that are parallel to the secant line.

49. $f(x) = \dfrac{x}{x + 1}, \quad \left[-\frac{1}{2}, 2\right]$

50. $f(x) = x - 2 \sin x, \quad [-\pi, \pi]$

51. $f(x) = \sqrt{x}, \quad [1, 9]$

52. $f(x) = x^4 - 2x^3 + x^2, \quad [0, 6]$

53. ***Vertical Motion*** The height of an object t seconds after it is dropped from a height of 300 meters is $s(t) = -4.9t^2 + 300$.

(a) Find the average velocity of the object during the first 3 seconds.

(b) Use the Mean Value Theorem to verify that at some time during the first 3 seconds of fall the instantaneous velocity equals the average velocity. Find that time.

54. ***Sales*** A company introduces a new product for which the number of units sold S is

$$S(t) = 200\left(5 - \frac{9}{2 + t}\right)$$

where t is the time in months.

(a) Find the average value of $S(t)$ during the first year.

(b) During what month does $S'(t)$ equal the average value during the first year?

WRITING ABOUT CONCEPTS

55. Let f be continuous on $[a, b]$ and differentiable on (a, b). If there exists c in (a, b) such that $f'(c) = 0$, does it follow that $f(a) = f(b)$? Explain.

56. Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Also, suppose that $f(a) = f(b)$ and that c is a real number in the interval such that $f'(c) = 0$. Find an interval for the function g over which Rolle's Theorem can be applied, and find the corresponding critical number of g (k is a constant).

(a) $g(x) = f(x) + k$ (b) $g(x) = f(x - k)$

(c) $g(x) = f(kx)$

57. The function

$$f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \le 1 \end{cases}$$

is differentiable on $(0, 1)$ and satisfies $f(0) = f(1)$. However, its derivative is never zero on $(0, 1)$. Does this contradict Rolle's Theorem? Explain.

58. Can you find a function f such that $f(-2) = -2$, $f(2) = 6$, and $f'(x) < 1$ for all x? Why or why not?

59. ***Speed*** A plane begins its takeoff at 2:00 P.M. on a 2500-mile flight. After 5.5 hours, the plane arrives at its destination. Explain why there are at least two times during the flight when the speed of the plane is 400 miles per hour.

60. ***Temperature*** When an object is removed from a furnace and placed in an environment with a constant temperature of 90°F, its core temperature is 1500°F. Five hours later the core temperature is 390°F. Explain why there must exist a time in the interval when the temperature is decreasing at a rate of 222°F per hour.

61. ***Velocity*** Two bicyclists begin a race at 8:00 A.M. They both finish the race 2 hours and 15 minutes later. Prove that at some time during the race, the bicyclists are traveling at the same velocity.

62. ***Acceleration*** At 9:13 A.M., a sports car is traveling 35 miles per hour. Two minutes later, the car is traveling 85 miles per hour. Prove that at some time during this two-minute interval, the car's acceleration is exactly 1500 miles per hour squared.

63. Consider the function $f(x) = 3\cos^2\left(\dfrac{\pi x}{2}\right)$.

(a) Use a graphing utility to graph f and f'.

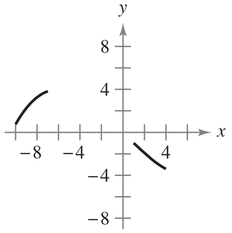
(b) Is f a continuous function? Is f' a continuous function?

(c) Does Rolle's Theorem apply on the interval $[-1, 1]$? Does it apply on the interval $[1, 2]$? Explain.

(d) Evaluate, if possible, $\lim\limits_{x\to3^-} f'(x)$ and $\lim\limits_{x\to3^+} f'(x)$.

CAPSTONE

64. ***Graphical Reasoning*** The figure shows two parts of the graph of a continuous differentiable function f on $[-10, 4]$. The derivative f' is also continuous. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) Explain why f must have at least one zero in $[-10, 4]$.

(b) Explain why f' must also have at least one zero in the interval $[-10, 4]$. What are these zeros called?

(c) Make a possible sketch of the function with one zero of f' on the interval $[-10, 4]$.

(d) Make a possible sketch of the function with two zeros of f' on the interval $[-10, 4]$.

(e) Were the conditions of continuity of f and f' necessary to do parts (a) through (d)? Explain.

Think About It **In Exercises 65 and 66, sketch the graph of an arbitrary function f that satisfies the given condition but does not satisfy the conditions of the Mean Value Theorem on the interval $[-5, 5]$.**

65. f is continuous on $[-5, 5]$.

66. f is not continuous on $[-5, 5]$.

In Exercises 67–70, use the Intermediate Value Theorem and Rolle's Theorem to prove that the equation has exactly one real solution.

67. $x^5 + x^3 + x + 1 = 0$

68. $2x^5 + 7x - 1 = 0$

69. $3x + 1 - \sin x = 0$

70. $2x - 2 - \cos x = 0$

71. Determine the values a, b, and c such that the function f satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 3]$.

$$f(x) = \begin{cases} 1, & x = 0 \\ ax + b, & 0 < x \le 1 \\ x^2 + 4x + c, & 1 < x \le 3 \end{cases}$$

72. Determine the values a, b, c, and d such that the function f satisfies the hypotheses of the Mean Value Theorem on the interval $[-1, 2]$.

$$f(x) = \begin{cases} a, & x = -1 \\ 2, & -1 < x \le 0 \\ bx^2 + c, & 0 < x \le 1 \\ dx + 4, & 1 < x \le 2 \end{cases}$$

Differential Equations **In Exercises 73–76, find a function f that has the derivative $f'(x)$ and whose graph passes through the given point. Explain your reasoning.**

73. $f'(x) = 0$, $(2, 5)$

74. $f'(x) = 4$, $(0, 1)$

75. $f'(x) = 2x$, $(1, 0)$

76. $f'(x) = 2x + 3$, $(1, 0)$

True or False? **In Exercises 77–80, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

77. The Mean Value Theorem can be applied to $f(x) = 1/x$ on the interval $[-1, 1]$.

78. If the graph of a function has three x-intercepts, then it must have at least two points at which its tangent line is horizontal.

79. If the graph of a polynomial function has three x-intercepts, then it must have at least two points at which its tangent line is horizontal.

80. If $f'(x) = 0$ for all x in the domain of f, then f is a constant function.

81. Prove that if $a > 0$ and n is any positive integer, then the polynomial function $p(x) = x^{2n+1} + ax + b$ cannot have two real roots.

82. Prove that if $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

83. Let $p(x) = Ax^2 + Bx + C$. Prove that for any interval $[a, b]$, the value c guaranteed by the Mean Value Theorem is the midpoint of the interval.

84. (a) Let $f(x) = x^2$ and $g(x) = -x^3 + x^2 + 3x + 2$. Then $f(-1) = g(-1)$ and $f(2) = g(2)$. Show that there is at least one value c in the interval $(-1, 2)$ where the tangent line to f at $(c, f(c))$ is parallel to the tangent line to g at $(c, g(c))$. Identify c.

(b) Let f and g be differentiable functions on $[a, b]$ where $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one value c in the interval (a, b) where the tangent line to f at $(c, f(c))$ is parallel to the tangent line to g at $(c, g(c))$.

85. Prove that if f is differentiable on $(-\infty, \infty)$ and $f'(x) < 1$ for all real numbers, then f has at most one fixed point. A fixed point of a function f is a real number c such that $f(c) = c$.

86. Use the result of Exercise 85 to show that $f(x) = \frac{1}{2}\cos x$ has at most one fixed point.

87. Prove that $|\cos a - \cos b| \le |a - b|$ for all a and b.

88. Prove that $|\sin a - \sin b| \le |a - b|$ for all a and b.

89. Let $0 < a < b$. Use the Mean Value Theorem to show that

$$\sqrt{b} - \sqrt{a} < \frac{b - a}{2\sqrt{a}}.$$

3.3 Increasing and Decreasing Functions and the First Derivative Test

- **Determine intervals on which a function is increasing or decreasing.**
- **Apply the First Derivative Test to find relative extrema of a function.**

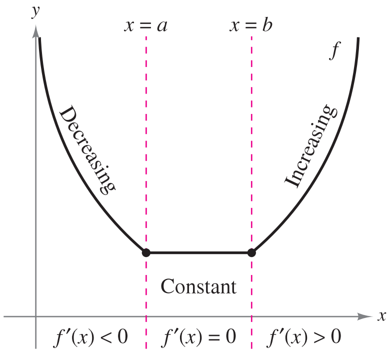
Increasing and Decreasing Functions

In this section you will learn how derivatives can be used to *classify* relative extrema as either relative minima or relative maxima. First, it is important to define increasing and decreasing functions.

DEFINITIONS OF INCREASING AND DECREASING FUNCTIONS

A function f is **increasing** on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.



The derivative is related to the slope of a function.
Figure 3.15

A function is increasing if, *as x moves to the right*, its graph moves up, and is decreasing if its graph moves down. For example, the function in Figure 3.15 is decreasing on the interval $(-\infty, a)$, is constant on the interval (a, b), and is increasing on the interval (b, ∞). As shown in Theorem 3.5 below, a positive derivative implies that the function is increasing; a negative derivative implies that the function is decreasing; and a zero derivative on an entire interval implies that the function is constant on that interval.

THEOREM 3.5 TEST FOR INCREASING AND DECREASING FUNCTIONS

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

PROOF To prove the first case, assume that $f'(x) > 0$ for all x in the interval (a, b) and let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, you know that there exists a number c such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Because $f'(c) > 0$ and $x_2 - x_1 > 0$, you know that

$$f(x_2) - f(x_1) > 0$$

which implies that $f(x_1) < f(x_2)$. So, f is increasing on the interval. The second case has a similar proof (see Exercise 104), and the third case is a consequence of Exercise 82 in Section 3.2. ■

NOTE The conclusions in the first two cases of Theorem 3.5 are valid even if $f'(x) = 0$ at a finite number of x-values in (a, b). ■

EXAMPLE 1 Intervals on Which *f* Is Increasing or Decreasing

Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

Solution Note that *f* is differentiable on the entire real number line. To determine the critical numbers of *f*, set $f'(x)$ equal to zero.

$$f(x) = x^3 - \frac{3}{2}x^2 \qquad \text{Write original function.}$$

$$f'(x) = 3x^2 - 3x = 0 \qquad \text{Differentiate and set } f'(x) \text{ equal to 0.}$$

$$3(x)(x-1) = 0 \qquad \text{Factor.}$$

$$x = 0, 1 \qquad \text{Critical numbers}$$

Because there are no points for which f' does not exist, you can conclude that $x = 0$ and $x = 1$ are the only critical numbers. The table summarizes the testing of the three intervals determined by these two critical numbers.

Interval	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	$x = -1$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-1) = 6 > 0$	$f'\left(\frac{1}{2}\right) = -\frac{3}{4} < 0$	$f'(2) = 6 > 0$
Conclusion	Increasing	Decreasing	Increasing

So, *f* is increasing on the intervals $(-\infty, 0)$ and $(1, \infty)$ and decreasing on the interval $(0, 1)$, as shown in Figure 3.16. ■

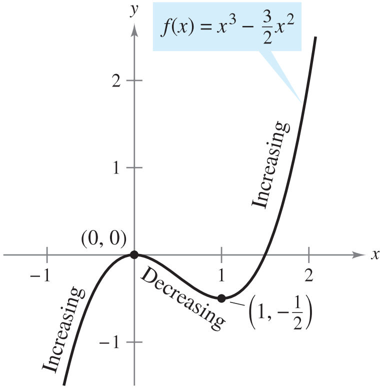


Figure 3.16

Example 1 gives you one example of how to find intervals on which a function is increasing or decreasing. The guidelines below summarize the steps followed in that example.

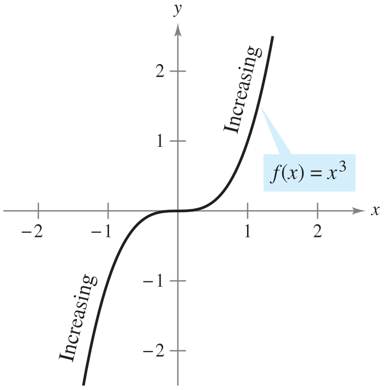
GUIDELINES FOR FINDING INTERVALS ON WHICH A FUNCTION IS INCREASING OR DECREASING

Let *f* be continuous on the interval (a, b). To find the open intervals on which *f* is increasing or decreasing, use the following steps.

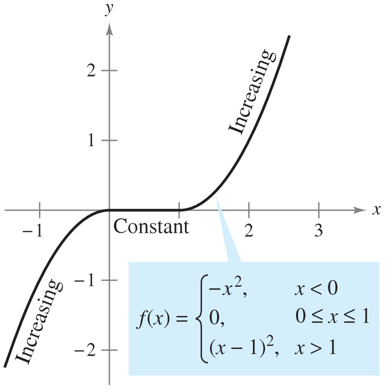
1. Locate the critical numbers of *f* in (a, b), and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use Theorem 3.5 to determine whether *f* is increasing or decreasing on each interval.

These guidelines are also valid if the interval (a, b) is replaced by an interval of the form $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$.

A function is **strictly monotonic** on an interval if it is either increasing on the entire interval or decreasing on the entire interval. For instance, the function $f(x) = x^3$ is strictly monotonic on the entire real number line because it is increasing on the entire real number line, as shown in Figure 3.17(a). The function shown in Figure 3.17(b) is not strictly monotonic on the entire real number line because it is constant on the interval $[0, 1]$.



(a) Strictly monotonic function



(b) Not strictly monotonic

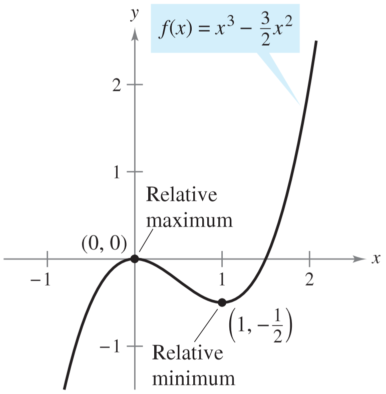
Figure 3.17

The First Derivative Test

After you have determined the intervals on which a function is increasing or decreasing, it is not difficult to locate the relative extrema of the function. For instance, in Figure 3.18 (from Example 1), the function

$$f(x) = x^3 - \frac{3}{2}x^2$$

has a relative maximum at the point $(0, 0)$ because f is increasing immediately to the left of $x = 0$ and decreasing immediately to the right of $x = 0$. Similarly, f has a relative minimum at the point $\left(1, -\frac{1}{2}\right)$ because f is decreasing immediately to the left of $x = 1$ and increasing immediately to the right of $x = 1$. The following theorem, called the First Derivative Test, makes this more explicit.

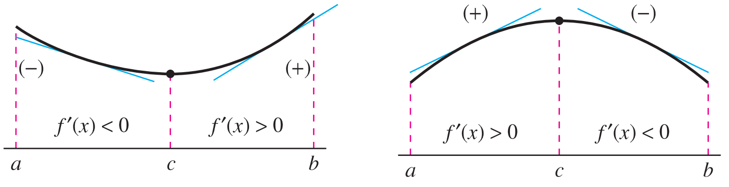


Relative extrema of f
Figure 3.18

THEOREM 3.6 THE FIRST DERIVATIVE TEST

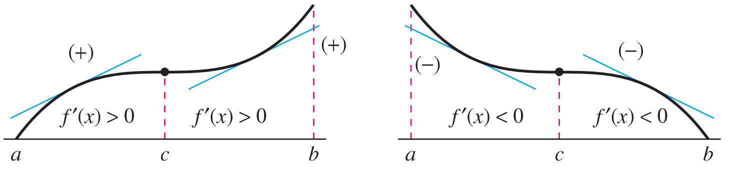
Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

1. If $f'(x)$ changes from negative to positive at c, then f has a *relative minimum* at $(c, f(c))$.
2. If $f'(x)$ changes from positive to negative at c, then f has a *relative maximum* at $(c, f(c))$.
3. If $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a relative minimum nor a relative maximum.



Relative minimum

Relative maximum



Neither relative minimum nor relative maximum

PROOF Assume that $f'(x)$ changes from negative to positive at c. Then there exist a and b in I such that

$$f'(x) < 0 \text{ for all } x \text{ in } (a, c)$$

and

$$f'(x) > 0 \text{ for all } x \text{ in } (c, b).$$

By Theorem 3.5, f is decreasing on $[a, c]$ and increasing on $[c, b]$. So, $f(c)$ is a minimum of f on the open interval (a, b) and, consequently, a relative minimum of f. This proves the first case of the theorem. The second case can be proved in a similar way (see Exercise 105). ■

EXAMPLE 2 Applying the First Derivative Test

Find the relative extrema of the function $f(x) = \frac{1}{2}x - \sin x$ in the interval $(0, 2\pi)$.

Solution Note that f is continuous on the interval $(0, 2\pi)$. To determine the critical numbers of f in this interval, set $f'(x)$ equal to 0.

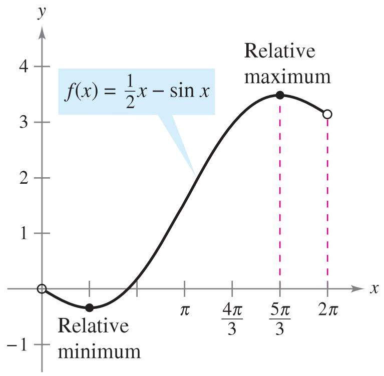
$$f'(x) = \frac{1}{2} - \cos x = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3} \qquad \text{Critical numbers}$$

Because there are no points for which f' does not exist, you can conclude that $x = \pi/3$ and $x = 5\pi/3$ are the only critical numbers. The table summarizes the testing of the three intervals determined by these two critical numbers.

Interval	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \frac{5\pi}{3}$	$\frac{5\pi}{3} < x < 2\pi$
Test Value	$x = \frac{\pi}{4}$	$x = \pi$	$x = \frac{7\pi}{4}$
Sign of $f'(x)$	$f'\left(\frac{\pi}{4}\right) < 0$	$f'(\pi) > 0$	$f'\left(\frac{7\pi}{4}\right) < 0$
Conclusion	Decreasing	Increasing	Decreasing



A relative minimum occurs where f changes from decreasing to increasing, and a relative maximum occurs where f changes from increasing to decreasing.
Figure 3.19

By applying the First Derivative Test, you can conclude that f has a relative minimum at the point where

$$x = \frac{\pi}{3} \qquad x\text{-value where relative minimum occurs}$$

and a relative maximum at the point where

$$x = \frac{5\pi}{3} \qquad x\text{-value where relative maximum occurs}$$

as shown in Figure 3.19. ■

EXPLORATION

Comparing Graphical and Analytic Approaches From Section 3.2, you know that, *by itself*, a graphing utility can give misleading information about the relative extrema of a graph. *Used in conjunction with an analytic approach*, however, a graphing utility can provide a good way to reinforce your conclusions. Use a graphing utility to graph the function in Example 2. Then use the *zoom* and *trace* features to estimate the relative extrema. How close are your graphical approximations?

Note that in Examples 1 and 2 the given functions are differentiable on the entire real number line. For such functions, the only critical numbers are those for which $f'(x) = 0$. Example 3 concerns a function that has two types of critical numbers—those for which $f'(x) = 0$ and those for which f is not differentiable.

EXAMPLE 3 Applying the First Derivative Test

Find the relative extrema of

$$f(x) = (x^2 - 4)^{2/3}.$$

Solution Begin by noting that f is continuous on the entire real number line. The derivative of f

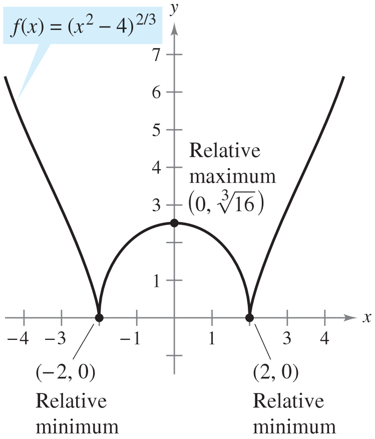
$$f'(x) = \frac{2}{3}(x^2 - 4)^{-1/3}(2x) \qquad \text{General Power Rule}$$

$$= \frac{4x}{3(x^2 - 4)^{1/3}} \qquad \text{Simplify.}$$

is 0 when $x = 0$ and does not exist when $x = \pm 2$. So, the critical numbers are $x = -2$, $x = 0$, and $x = 2$. The table summarizes the testing of the four intervals determined by these three critical numbers.

Interval	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = -1$	$x = 1$	$x = 3$
Sign of $f'(x)$	$f'(-3) < 0$	$f'(-1) > 0$	$f'(1) < 0$	$f'(3) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

By applying the First Derivative Test, you can conclude that f has a relative minimum at the point $(-2, 0)$, a relative maximum at the point $\left(0, \sqrt[3]{16}\right)$, and another relative minimum at the point $(2, 0)$, as shown in Figure 3.20. ■



You can apply the First Derivative Test to find relative extrema.
Figure 3.20

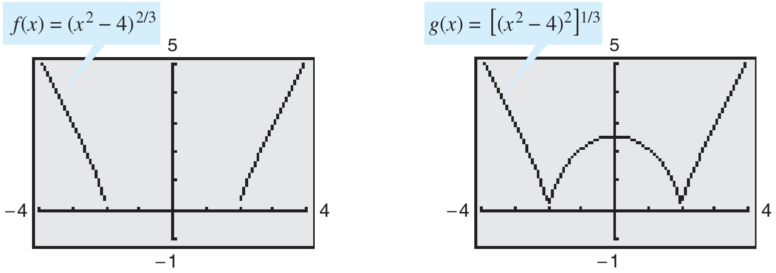
TECHNOLOGY PITFALL When using a graphing utility to graph a function involving radicals or rational exponents, be sure you understand the way the utility evaluates radical expressions. For instance, even though

$$f(x) = (x^2 - 4)^{2/3}$$

and

$$g(x) = [(x^2 - 4)^2]^{1/3}$$

are the same algebraically, some graphing utilities distinguish between these two functions. Which of the graphs shown in Figure 3.21 is incorrect? Why did the graphing utility produce an incorrect graph?



Which graph is incorrect?
Figure 3.21

When using the First Derivative Test, be sure to consider the domain of the function. For instance, in the next example, the function

$$f(x) = \frac{x^4 + 1}{x^2}$$

is not defined when $x = 0$. This x-value must be used with the critical numbers to determine the test intervals.

EXAMPLE 4 Applying the First Derivative Test

Find the relative extrema of $f(x) = \dfrac{x^4 + 1}{x^2}$.

Solution

$$\begin{aligned} f(x) &= x^2 + x^{-2} && \text{Rewrite original function.}\\ f'(x) &= 2x - 2x^{-3} && \text{Differentiate.}\\ &= 2x - \frac{2}{x^3} && \text{Rewrite with positive exponent.}\\ &= \frac{2(x^4 - 1)}{x^3} && \text{Simplify.}\\ &= \frac{2(x^2 + 1)(x - 1)(x + 1)}{x^3} && \text{Factor.} \end{aligned}$$

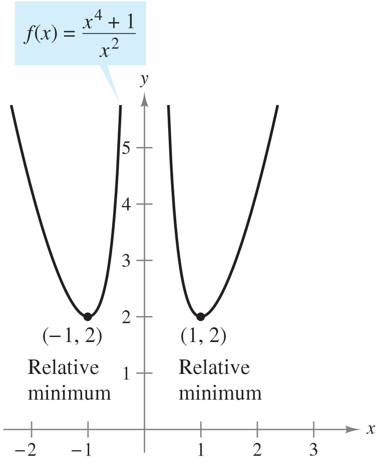
So, $f'(x)$ is zero at $x = \pm 1$. Moreover, because $x = 0$ is not in the domain of f, you should use this x-value along with the critical numbers to determine the test intervals.

$$\begin{aligned} x &= \pm 1 && \text{Critical numbers, } f'(\pm 1) = 0\\ x &= 0 && 0 \text{ is not in the domain of } f. \end{aligned}$$

The table summarizes the testing of the four intervals determined by these three x-values.

Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	$x = -2$	$x = -\frac{1}{2}$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-2) < 0$	$f'\left(-\frac{1}{2}\right) > 0$	$f'\left(\frac{1}{2}\right) < 0$	$f'(2) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

By applying the First Derivative Test, you can conclude that f has one relative minimum at the point $(-1, 2)$ and another at the point $(1, 2)$, as shown in Figure 3.22. ■



x-values that are not in the domain of f, as well as critical numbers, determine test intervals for f'.
Figure 3.22

TECHNOLOGY The most difficult step in applying the First Derivative Test is finding the values for which the derivative is equal to 0. For instance, the values of x for which the derivative of

$$f(x) = \frac{x^4 + 1}{x^2 + 1}$$

is equal to zero are $x = 0$ and $x = \pm\sqrt{\sqrt{2} - 1}$. If you have access to technology that can perform symbolic differentiation and solve equations, use it to apply the First Derivative Test to this function.

© Thomas Kienzle/Wide World Photos

If a projectile is propelled from ground level and air resistance is neglected, the object will travel farthest with an initial angle of 45°. If, however, the projectile is propelled from a point above ground level, the angle that yields a maximum horizontal distance is not 45° (see Example 5).

EXAMPLE 5 The Path of a Projectile

Neglecting air resistance, the path of a projectile that is propelled at an angle θ is

$$y = \frac{g \sec^2 \theta}{2v_0^2}x^2 + (\tan \theta)x + h, \quad 0 \le \theta \le \frac{\pi}{2}$$

where y is the height, x is the horizontal distance, g is the acceleration due to gravity, v_0 is the initial velocity, and h is the initial height. (This equation is derived in Section 12.3.) Let $g = -32$ feet per second per second, $v_0 = 24$ feet per second, and $h = 9$ feet. What value of θ will produce a maximum horizontal distance?

Solution To find the distance the projectile travels, let $y = 0$, and use the Quadratic Formula to solve for x.

$$\frac{g \sec^2 \theta}{2v_0^2}x^2 + (\tan \theta)x + h = 0$$

$$\frac{-32 \sec^2 \theta}{2(24^2)}x^2 + (\tan \theta)x + 9 = 0$$

$$-\frac{\sec^2 \theta}{36}x^2 + (\tan \theta)x + 9 = 0$$

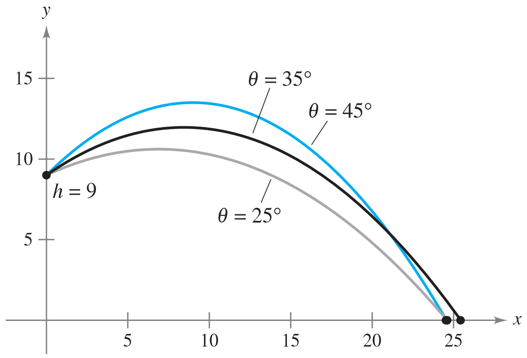
$$x = \frac{-\tan \theta \pm \sqrt{\tan^2 \theta + \sec^2 \theta}}{-\sec^2 \theta/18}$$

$$x = 18 \cos \theta\left(\sin \theta + \sqrt{\sin^2 \theta + 1}\right), \quad x \ge 0$$

At this point, you need to find the value of θ that produces a maximum value of x. Applying the First Derivative Test by hand would be very tedious. Using technology to solve the equation $dx/d\theta = 0$, however, eliminates most of the messy computations. The result is that the maximum value of x occurs when

$$\theta \approx 0.61548 \text{ radian, or } 35.3°.$$

This conclusion is reinforced by sketching the path of the projectile for different values of θ, as shown in Figure 3.23. Of the three paths shown, note that the distance traveled is greatest for $\theta = 35°$.



The path of a projectile with initial angle θ
Figure 3.23

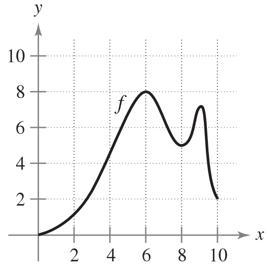
NOTE A computer simulation of this example is given in the premium eBook for this text. Using that simulation, you can experimentally discover that the maximum value of x occurs when $\theta \approx 35.3°$.

3.3 Exercises

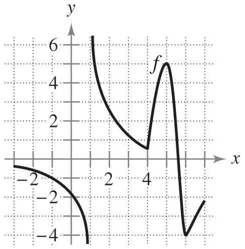
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use the graph of *f* to find (a) the largest open interval on which *f* is increasing, and (b) the largest open interval on which *f* is decreasing.

1.

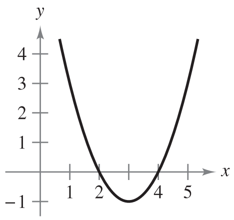


2.

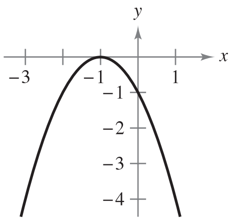


In Exercises 3–8, use the graph to estimate the open intervals on which the function is increasing or decreasing. Then find the open intervals analytically.

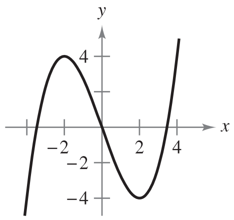
3. $f(x) = x^2 - 6x + 8$



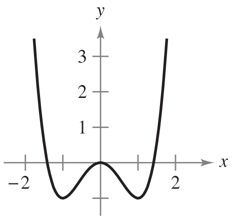
4. $y = -(x + 1)^2$



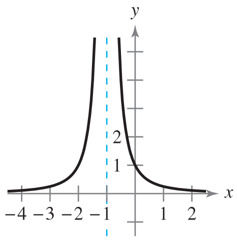
5. $y = \dfrac{x^3}{4} - 3x$



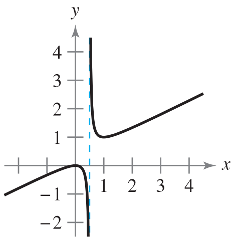
6. $f(x) = x^4 - 2x^2$



7. $f(x) = \dfrac{1}{(x + 1)^2}$



8. $y = \dfrac{x^2}{2x - 1}$



In Exercises 9–16, identify the open intervals on which the function is increasing or decreasing.

9. $g(x) = x^2 - 2x - 8$

10. $h(x) = 27x - x^3$

11. $y = x\sqrt{16 - x^2}$

12. $y = x + \dfrac{4}{x}$

13. $f(x) = \sin x - 1, \quad 0 < x < 2\pi$

14. $h(x) = \cos \dfrac{x}{2}, \quad 0 < x < 2\pi$

15. $y = x - 2 \cos x, \quad 0 < x < 2\pi$

16. $f(x) = \cos^2 x - \cos x, \quad 0 < x < 2\pi$

In Exercises 17–42, (a) find the critical numbers of *f* (if any), (b) find the open interval(s) on which the function is increasing or decreasing, (c) apply the First Derivative Test to identify all relative extrema, and (d) use a graphing utility to confirm your results.

17. $f(x) = x^2 - 4x$

18. $f(x) = x^2 + 6x + 10$

19. $f(x) = -2x^2 + 4x + 3$

20. $f(x) = -(x^2 + 8x + 12)$

21. $f(x) = 2x^3 + 3x^2 - 12x$

22. $f(x) = x^3 - 6x^2 + 15$

23. $f(x) = (x - 1)^2(x + 3)$

24. $f(x) = (x + 2)^2(x - 1)$

25. $f(x) = \dfrac{x^5 - 5x}{5}$

26. $f(x) = x^4 - 32x + 4$

27. $f(x) = x^{1/3} + 1$

28. $f(x) = x^{2/3} - 4$

29. $f(x) = (x + 2)^{2/3}$

30. $f(x) = (x - 3)^{1/3}$

31. $f(x) = 5 - |x - 5|$

32. $f(x) = |x + 3| - 1$

33. $f(x) = 2x + \dfrac{1}{x}$

34. $f(x) = \dfrac{x}{x + 3}$

35. $f(x) = \dfrac{x^2}{x^2 - 9}$

36. $f(x) = \dfrac{x + 4}{x^2}$

37. $f(x) = \dfrac{x^2 - 2x + 1}{x + 1}$

38. $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$

39. $f(x) = \begin{cases} 4 - x^2, & x \le 0 \\ -2x, & x > 0 \end{cases}$

40. $f(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 - 2, & x > -1 \end{cases}$

41. $f(x) = \begin{cases} 3x + 1, & x \le 1 \\ 5 - x^2, & x > 1 \end{cases}$

42. $f(x) = \begin{cases} -x^3 + 1, & x \le 0 \\ -x^2 + 2x, & x > 0 \end{cases}$

In Exercises 43–50, consider the function on the interval $(0, 2\pi)$. For each function, (a) find the open interval(s) on which the function is increasing or decreasing, (b) apply the First Derivative Test to identify all relative extrema, and (c) use a graphing utility to confirm your results.

43. $f(x) = \dfrac{x}{2} + \cos x$

44. $f(x) = \sin x \cos x + 5$

45. $f(x) = \sin x + \cos x$

46. $f(x) = x + 2 \sin x$

47. $f(x) = \cos^2(2x)$

48. $f(x) = \sqrt{3} \sin x + \cos x$

49. $f(x) = \sin^2 x + \sin x$

50. $f(x) = \dfrac{\sin x}{1 + \cos^2 x}$

CAS **In Exercises 51–56, (a) use a computer algebra system to differentiate the function, (b) sketch the graphs of *f* and *f′* on the same set of coordinate axes over the given interval, (c) find the critical numbers of *f* in the open interval, and (d) find the interval(s) on which *f′* is positive and the interval(s) on which it is negative. Compare the behavior of *f* and the sign of *f′*.**

51. $f(x) = 2x\sqrt{9 - x^2}$, $[-3, 3]$

52. $f(x) = 10\left(5 - \sqrt{x^2 - 3x + 16}\right)$, $[0, 5]$

53. $f(t) = t^2 \sin t$, $[0, 2\pi]$ **54.** $f(x) = \dfrac{x}{2} + \cos\dfrac{x}{2}$, $[0, 4\pi]$

55. $f(x) = -3 \sin\dfrac{x}{3}$, $[0, 6\pi]$

56. $f(x) = 2 \sin 3x + 4 \cos 3x$, $[0, \pi]$

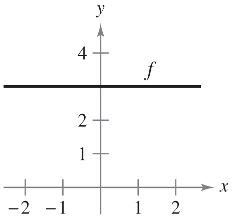
In Exercises 57 and 58, use symmetry, extrema, and zeros to sketch the graph of *f*. How do the functions *f* and *g* differ?

57. $f(x) = \dfrac{x^5 - 4x^3 + 3x}{x^2 - 1}$, $g(x) = x(x^2 - 3)$

58. $f(t) = \cos^2 t - \sin^2 t$, $g(t) = 1 - 2 \sin^2 t$

Think About It **In Exercises 59–64, the graph of *f* is shown in the figure. Sketch a graph of the derivative of *f*. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

59.



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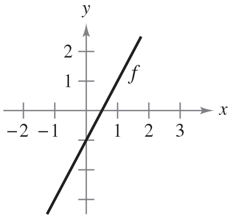

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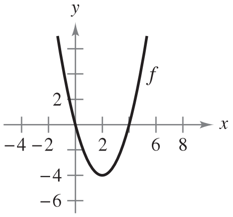

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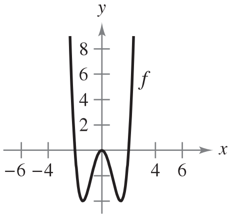

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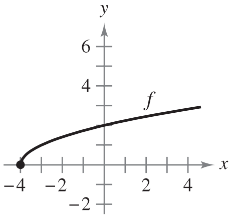

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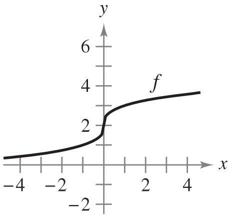
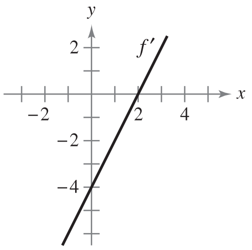
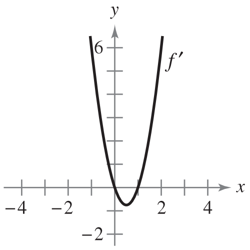


In Exercises 65–68, use the graph of *f′* to (a) identify the interval(s) on which *f* is increasing or decreasing, and (b) estimate the value(s) of *x* at which *f* has a relative maximum or minimum.

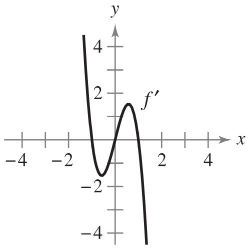
65.



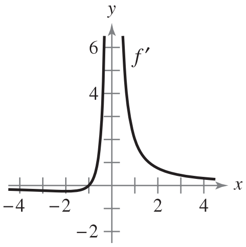
66.



67.

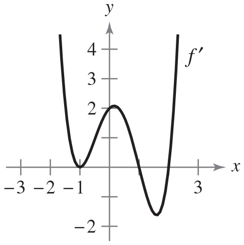


68.

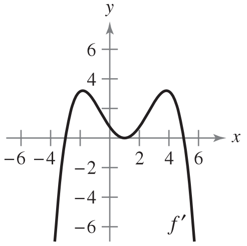


In Exercises 69 and 70, use the graph of *f′* to (a) identify the critical numbers of *f*, and (b) determine whether *f* has a relative maximum, a relative minimum, or neither at each critical number.

69.



70.



WRITING ABOUT CONCEPTS

In Exercises 71–76, assume that *f* is differentiable for all *x*. The signs of *f′* are as follows.

$f'(x) > 0$ on $(-\infty, -4)$

$f'(x) < 0$ on $(-4, 6)$

$f'(x) > 0$ on $(6, \infty)$

Supply the appropriate inequality sign for the indicated value of *c*.

	Function	*Sign of g′(c)*
71.	$g(x) = f(x) + 5$	$g'(0)$ ☐ 0
72.	$g(x) = 3f(x) - 3$	$g'(-5)$ ☐ 0
73.	$g(x) = -f(x)$	$g'(-6)$ ☐ 0
74.	$g(x) = -f(x)$	$g'(0)$ ☐ 0
75.	$g(x) = f(x - 10)$	$g'(0)$ ☐ 0
76.	$g(x) = f(x - 10)$	$g'(8)$ ☐ 0

77. Sketch the graph of the arbitrary function *f* such that

$$f'(x) \begin{cases} > 0, & x < 4 \\ \text{undefined}, & x = 4. \\ < 0, & x > 4 \end{cases}$$

CAPSTONE

78. A differentiable function *f* has one critical number at $x = 5$. Identify the relative extrema of *f* at the critical number if $f'(4) = -2.5$ and $f'(6) = 3$.

Think About It **In Exercises 79 and 80, the function f is differentiable on the indicated interval. The table shows $f'(x)$ for selected values of x. (a) Sketch the graph of f, (b) approximate the critical numbers, and (c) identify the relative extrema.**

79. f is differentiable on $[-1, 1]$

x	-1	-0.75	-0.50	-0.25
$f'(x)$	-10	-3.2	-0.5	0.8

x	0	0.25	0.50	0.75	1
$f'(x)$	5.6	3.6	-0.2	-6.7	-20.1

80. f is differentiable on $[0, \pi]$

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$f'(x)$	3.14	-0.23	-2.45	-3.11	0.69

x	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$f'(x)$	3.00	1.37	-1.14	-2.84

81. ***Rolling a Ball Bearing*** A ball bearing is placed on an inclined plane and begins to roll. The angle of elevation of the plane is θ. The distance (in meters) the ball bearing rolls in t seconds is $s(t) = 4.9(\sin\theta)t^2$.

(a) Determine the speed of the ball bearing after t seconds.

(b) Complete the table and use it to determine the value of θ that produces the maximum speed at a particular time.

θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$s'(t)$							

82. ***Numerical, Graphical, and Analytic Analysis*** The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is

$$C(t) = \frac{3t}{27 + t^3}, \quad t \geq 0.$$

(a) Complete the table and use it to approximate the time when the concentration is greatest.

t	0	0.5	1	1.5	2	2.5	3
$C(t)$							

(b) Use a graphing utility to graph the concentration function and use the graph to approximate the time when the concentration is greatest.

(c) Use calculus to determine analytically the time when the concentration is greatest.

83. ***Numerical, Graphical, and Analytic Analysis*** Consider the functions $f(x) = x$ and $g(x) = \sin x$ on the interval $(0, \pi)$.

(a) Complete the table and make a conjecture about which is the greater function on the interval $(0, \pi)$.

x	0.5	1	1.5	2	2.5	3
$f(x)$						
$g(x)$						

(b) Use a graphing utility to graph the functions and use the graphs to make a conjecture about which is the greater function on the interval $(0, \pi)$.

(c) Prove that $f(x) > g(x)$ on the interval $(0, \pi)$. [*Hint:* Show that $h'(x) > 0$ where $h = f - g$.]

84. ***Numerical, Graphical, and Analytic Analysis*** Consider the functions $f(x) = x$ and $g(x) = \tan x$ on the interval $(0, \pi/2)$.

(a) Complete the table and make a conjecture about which is the greater function on the interval $(0, \pi/2)$.

x	0.25	0.5	0.75	1	1.25	1.5
$f(x)$						
$g(x)$						

(b) Use a graphing utility to graph the functions and use the graphs to make a conjecture about which is the greater function on the interval $(0, \pi/2)$.

(c) Prove that $f(x) < g(x)$ on the interval $(0, \pi/2)$. [*Hint:* Show that $h'(x) > 0$, where $h = g - f$.]

85. ***Trachea Contraction*** Coughing forces the trachea (windpipe) to contract, which affects the velocity v of the air passing through the trachea. The velocity of the air during coughing is $v = k(R - r)r^2$, $0 \leq r < R$, where k is a constant, R is the normal radius of the trachea, and r is the radius during coughing. What radius will produce the maximum air velocity?

86. ***Power*** The electric power P in watts in a direct-current circuit with two resistors R_1 and R_2 connected in parallel is

$$P = \frac{vR_1R_2}{(R_1 + R_2)^2}$$

where v is the voltage. If v and R_1 are held constant, what resistance R_2 produces maximum power?

87. ***Electrical Resistance*** The resistance R of a certain type of resistor is $R = \sqrt{0.001T^4 - 4T + 100}$, where R is measured in ohms and the temperature T is measured in degrees Celsius.

CAS (a) Use a computer algebra system to find dR/dT and the critical number of the function. Determine the minimum resistance for this type of resistor.

(b) Use a graphing utility to graph the function R and use the graph to approximate the minimum resistance for this type of resistor.

88. ***Modeling Data*** The end-of-year assets of the Medicare Hospital Insurance Trust Fund (in billions of dollars) for the years 1995 through 2006 are shown.

1995: 130.3; 1996: 124.9; 1997: 115.6; 1998: 120.4;
1999: 141.4; 2000: 177.5; 2001: 208.7; 2002: 234.8;
2003: 256.0; 2004: 269.3; 2005: 285.8; 2006: 305.4

(Source: U.S. Centers for Medicare and Medicaid Services)

(a) Use the regression capabilities of a graphing utility to find a model of the form $M = at^4 + bt^3 + ct^2 + dt + e$ for the data. (Let $t = 5$ represent 1995.)

(b) Use a graphing utility to plot the data and graph the model.

(c) Find the minimum value of the model and compare the result with the actual data.

Motion Along a Line **In Exercises 89–92, the function $s(t)$ describes the motion of a particle along a line. For each function, (a) find the velocity function of the particle at any time $t \geq 0$, (b) identify the time interval(s) in which the particle is moving in a positive direction, (c) identify the time interval(s) in which the particle is moving in a negative direction, and (d) identify the time(s) at which the particle changes direction.**

89. $s(t) = 6t - t^2$

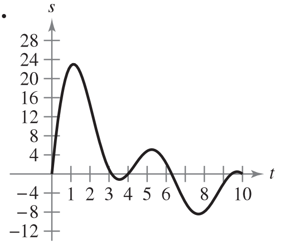
90. $s(t) = t^2 - 7t + 10$

91. $s(t) = t^3 - 5t^2 + 4t$

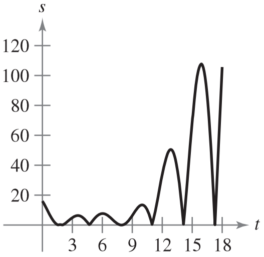
92. $s(t) = t^3 - 20t^2 + 128t - 280$

Motion Along a Line **In Exercises 93 and 94, the graph shows the position of a particle moving along a line. Describe how the particle's position changes with respect to time.**

93.



94.



Creating Polynomial Functions **In Exercises 95–98, find a polynomial function**

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

that has only the specified extrema. (a) Determine the minimum degree of the function and give the criteria you used in determining the degree. (b) Using the fact that the coordinates of the extrema are solution points of the function, and that the x-coordinates are critical numbers, determine a system of linear equations whose solution yields the coefficients of the required function. (c) Use a graphing utility to solve the system of equations and determine the function. (d) Use a graphing utility to confirm your result graphically.

95. Relative minimum: $(0, 0)$; Relative maximum: $(2, 2)$

96. Relative minimum: $(0, 0)$; Relative maximum: $(4, 1000)$

97. Relative minima: $(0, 0)$, $(4, 0)$; Relative maximum: $(2, 4)$

98. Relative minimum: $(1, 2)$; Relative maxima: $(-1, 4)$, $(3, 4)$

True or False? **In Exercises 99–103, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

99. The sum of two increasing functions is increasing.

100. The product of two increasing functions is increasing.

101. Every nth-degree polynomial has $(n - 1)$ critical numbers.

102. An nth-degree polynomial has at most $(n - 1)$ critical numbers.

103. There is a relative maximum or minimum at each critical number.

104. Prove the second case of Theorem 3.5.

105. Prove the second case of Theorem 3.6.

106. Use the definitions of increasing and decreasing functions to prove that $f(x) = x^3$ is increasing on $(-\infty, \infty)$.

107. Use the definitions of increasing and decreasing functions to prove that $f(x) = 1/x$ is decreasing on $(0, \infty)$.

PUTNAM EXAM CHALLENGE

108. Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

This problem was composed by the Committee on the Putnam Prize Competition.
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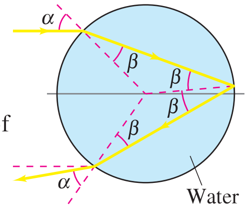
SECTION PROJECT

Rainbows

Rainbows are formed when light strikes raindrops and is reflected and refracted, as shown in the figure. (This figure shows a cross section of a spherical raindrop.) The Law of Refraction states that $(\sin \alpha)/(\sin \beta) = k$, where $k \approx 1.33$ (for water). The angle of deflection is given by $D = \pi + 2\alpha - 4\beta$.



(a) Use a graphing utility to graph

$$D = \pi + 2\alpha - 4\sin^{-1}(1/k \sin \alpha),$$

$$0 \leq \alpha \leq \pi/2.$$

(b) Prove that the minimum angle of deflection occurs when

$$\cos \alpha = \sqrt{\frac{k^2 - 1}{3}}.$$

For water, what is the minimum angle of deflection, $D_{\min}$? (The angle $\pi - D_{\min}$ is called the *rainbow angle*.) What value of α produces this minimum angle? (A ray of sunlight that strikes a raindrop at this angle, α, is called a *rainbow ray*.)

■ **FOR FURTHER INFORMATION** For more information about the mathematics of rainbows, see the article "Somewhere Within the Rainbow" by Steven Janke in *The UMAP Journal*.

3.4 Concavity and the Second Derivative Test

- **Determine intervals on which a function is concave upward or concave downward.**
- **Find any points of inflection of the graph of a function.**
- **Apply the Second Derivative Test to find relative extrema of a function.**

Concavity

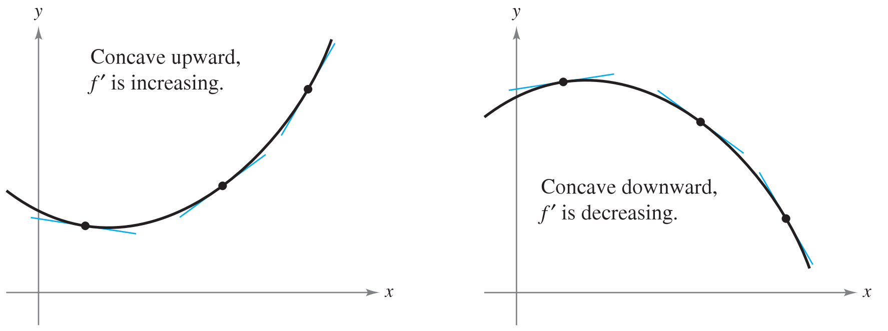
You have already seen that locating the intervals in which a function f increases or decreases helps to describe its graph. In this section, you will see how locating the intervals in which f' increases or decreases can be used to determine where the graph of f is *curving upward* or *curving downward.*

DEFINITION OF CONCAVITY

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.

The following graphical interpretation of concavity is useful. (See Appendix A for a proof of these results.)

1. Let f be differentiable on an open interval I. If the graph of f is concave *upward* on I, then the graph of f lies *above* all of its tangent lines on I. [See Figure 3.24(a).]
2. Let f be differentiable on an open interval I. If the graph of f is concave *downward* on I, then the graph of f lies *below* all of its tangent lines on I. [See Figure 3.24(b).]



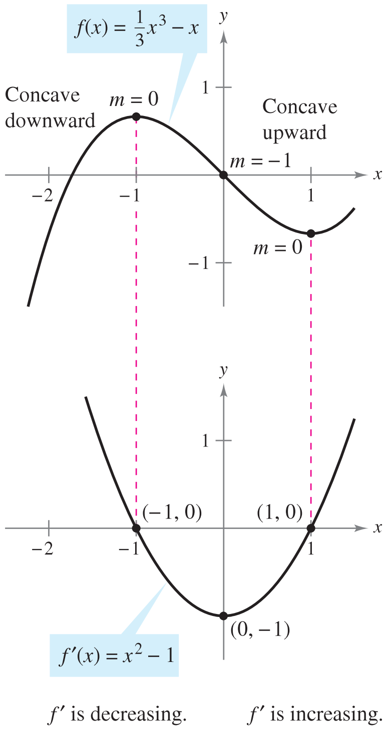
(a) The graph of f lies above its tangent lines. **(b)** The graph of f lies below its tangent lines.

Figure 3.24

To find the open intervals on which the graph of a function f is concave upward or concave downward, you need to find the intervals on which f' is increasing or decreasing. For instance, the graph of

$$f(x) = \tfrac{1}{3}x^3 - x$$

is concave downward on the open interval $(-\infty, 0)$ because $f'(x) = x^2 - 1$ is decreasing there. (See Figure 3.25.) Similarly, the graph of f is concave upward on the interval $(0, \infty)$ because f' is increasing on $(0, \infty)$.



The concavity of f is related to the slope of the derivative.

Figure 3.25

The following theorem shows how to use the *second* derivative of a function f to determine intervals on which the graph of f is concave upward or concave downward. A proof of this theorem (see Appendix A) follows directly from Theorem 3.5 and the definition of concavity.

THEOREM 3.7 TEST FOR CONCAVITY

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

NOTE A third case of Theorem 3.7 could be that if $f''(x) = 0$ for all x in I, then f is linear. Note, however, that concavity is not defined for a line. In other words, a straight line is neither concave upward nor concave downward.

To apply Theorem 3.7, locate the x-values at which $f''(x) = 0$ or f'' does not exist. Second, use these x-values to determine test intervals. Finally, test the sign of $f''(x)$ in each of the test intervals.

EXAMPLE 1 Determining Concavity

Determine the open intervals on which the graph of

$$f(x) = \frac{6}{x^2 + 3}$$

is concave upward or downward.

Solution Begin by observing that f is continuous on the entire real line. Next, find the second derivative of f.

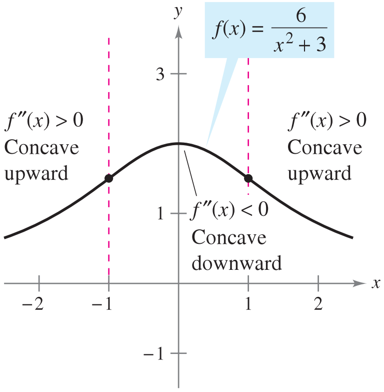
$$f(x) = 6(x^2 + 3)^{-1} \qquad \text{Rewrite original function.}$$

$$f'(x) = (-6)(x^2 + 3)^{-2}(2x) \qquad \text{Differentiate.}$$

$$= \frac{-12x}{(x^2 + 3)^2} \qquad \text{First derivative}$$

$$f''(x) = \frac{(x^2 + 3)^2(-12) - (-12x)(2)(x^2 + 3)(2x)}{(x^2 + 3)^4} \qquad \text{Differentiate.}$$

$$= \frac{36(x^2 - 1)}{(x^2 + 3)^3} \qquad \text{Second derivative}$$



From the sign of f'' you can determine the concavity of the graph of f.
Figure 3.26

Because $f''(x) = 0$ when $x = \pm 1$ and f'' is defined on the entire real line, you should test f'' in the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. The results are shown in the table and in Figure 3.26.

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Test Value	$x = -2$	$x = 0$	$x = 2$
Sign of $f''(x)$	$f''(-2) > 0$	$f''(0) < 0$	$f''(2) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

■

The function given in Example 1 is continuous on the entire real line. If there are x-values at which the function is not continuous, these values should be used, along with the points at which $f''(x) = 0$ or $f''(x)$ does not exist, to form the test intervals.

EXAMPLE 2 Determining Concavity

Determine the open intervals on which the graph of $f(x) = \dfrac{x^2 + 1}{x^2 - 4}$ is concave upward or concave downward.

Solution Differentiating twice produces the following.

$$f(x) = \frac{x^2 + 1}{x^2 - 4} \qquad \text{Write original function.}$$

$$f'(x) = \frac{(x^2 - 4)(2x) - (x^2 + 1)(2x)}{(x^2 - 4)^2} \qquad \text{Differentiate.}$$

$$= \frac{-10x}{(x^2 - 4)^2} \qquad \text{First derivative}$$

$$f''(x) = \frac{(x^2 - 4)^2(-10) - (-10x)(2)(x^2 - 4)(2x)}{(x^2 - 4)^4} \qquad \text{Differentiate.}$$

$$= \frac{10(3x^2 + 4)}{(x^2 - 4)^3} \qquad \text{Second derivative}$$

There are no points at which $f''(x) = 0$, but at $x = \pm 2$ the function f is not continuous, so test for concavity in the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$, as shown in the table. The graph of f is shown in Figure 3.27.

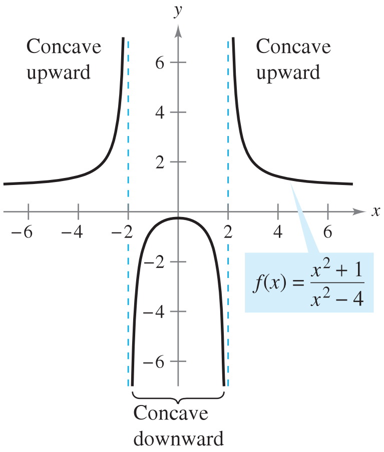


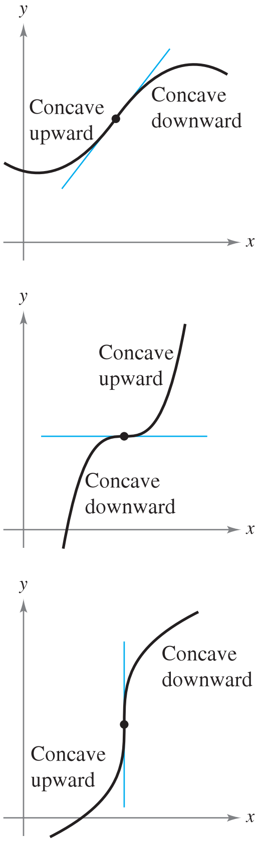
Figure 3.27

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = 0$	$x = 3$
Sign of $f''(x)$	$f''(-3) > 0$	$f''(0) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

■

Points of Inflection

The graph in Figure 3.26 has two points at which the concavity changes. If the tangent line to the graph exists at such a point, that point is a **point of inflection.** Three types of points of inflection are shown in Figure 3.28.



The concavity of f changes at a point of inflection. Note that the graph crosses its tangent line at a point of inflection.
Figure 3.28

DEFINITION OF POINT OF INFLECTION

Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

NOTE The definition of *point of inflection* given above requires that the tangent line exists at the point of inflection. Some books do not require this. For instance, we do not consider the function

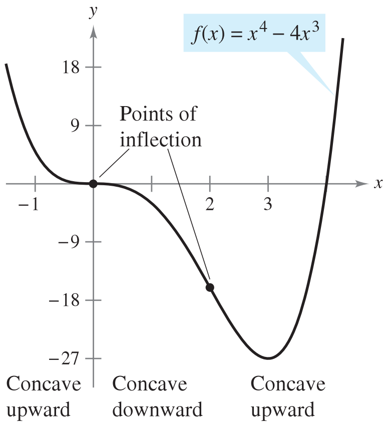
$$f(x) = \begin{cases} x^3, & x < 0 \\ x^2 + 2x, & x \geq 0 \end{cases}$$

to have a point of inflection at the origin, even though the concavity of the graph changes from concave downward to concave upward. ■

To locate *possible* points of inflection, you can determine the values of x for which $f''(x) = 0$ or $f''(x)$ does not exist. This is similar to the procedure for locating relative extrema of f.

THEOREM 3.8 POINTS OF INFLECTION

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f'' does not exist at $x = c$.



Points of inflection can occur where $f''(x) = 0$ or f'' does not exist.
Figure 3.29

EXAMPLE 3 Finding Points of Inflection

Determine the points of inflection and discuss the concavity of the graph of

$$f(x) = x^4 - 4x^3.$$

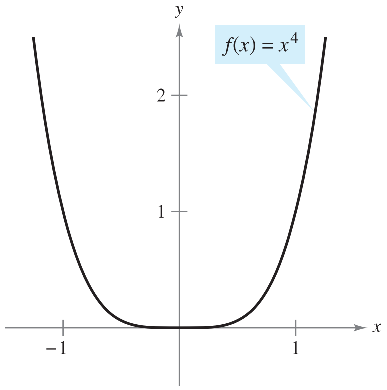
Solution Differentiating twice produces the following.

$$f(x) = x^4 - 4x^3 \quad \text{Write original function.}$$
$$f'(x) = 4x^3 - 12x^2 \quad \text{Find first derivative.}$$
$$f''(x) = 12x^2 - 24x = 12x(x - 2) \quad \text{Find second derivative.}$$

Setting $f''(x) = 0$, you can determine that the possible points of inflection occur at $x = 0$ and $x = 2$. By testing the intervals determined by these x-values, you can conclude that they both yield points of inflection. A summary of this testing is shown in the table, and the graph of f is shown in Figure 3.29.

Interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -1$	$x = 1$	$x = 3$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

■



$f''(x) = 0$, but $(0, 0)$ is not a point of inflection.
Figure 3.30

The converse of Theorem 3.8 is not generally true. That is, it is possible for the second derivative to be 0 at a point that is *not* a point of inflection. For instance, the graph of $f(x) = x^4$ is shown in Figure 3.30. The second derivative is 0 when $x = 0$, but the point $(0, 0)$ is not a point of inflection because the graph of f is concave upward in both intervals $-\infty < x < 0$ and $0 < x < \infty$.

EXPLORATION

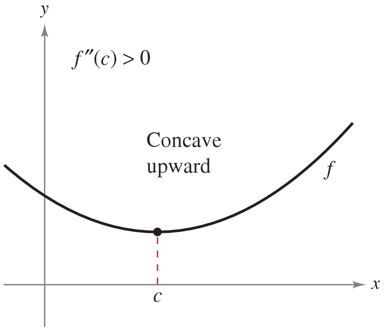
Consider a general cubic function of the form

$$f(x) = ax^3 + bx^2 + cx + d.$$

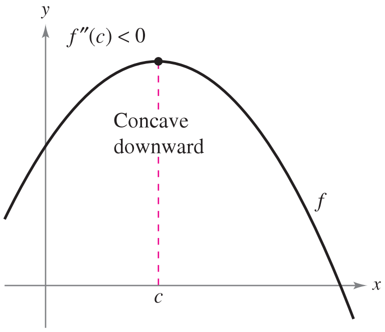
You know that the value of d has a bearing on the location of the graph but has no bearing on the value of the first derivative at given values of x. Graphically, this is true because changes in the value of d shift the graph up or down but do not change its basic shape. Use a graphing utility to graph several cubics with different values of c. Then give a graphical explanation of why changes in c do not affect the values of the second derivative.

The Second Derivative Test

In addition to testing for concavity, the second derivative can be used to perform a simple test for relative maxima and minima. The test is based on the fact that if the graph of a function f is concave upward on an open interval containing c, and $f'(c) = 0$, $f(c)$ must be a relative minimum of f. Similarly, if the graph of a function f is concave downward on an open interval containing c, and $f'(c) = 0$, $f(c)$ must be a relative maximum of f (see Figure 3.31).



If $f'(c) = 0$ and $f''(c) > 0$, $f(c)$ is a relative minimum.



If $f'(c) = 0$ and $f''(c) < 0$, $f(c)$ is a relative maximum.

Figure 3.31

THEOREM 3.9 SECOND DERIVATIVE TEST

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

PROOF If $f'(c) = 0$ and $f''(c) > 0$, there exists an open interval I containing c for which

$$\frac{f'(x) - f'(c)}{x - c} = \frac{f'(x)}{x - c} > 0$$

for all $x \neq c$ in I. If $x < c$, then $x - c < 0$ and $f'(x) < 0$. Also, if $x > c$, then $x - c > 0$ and $f'(x) > 0$. So, $f'(x)$ changes from negative to positive at c, and the First Derivative Test implies that $f(c)$ is a relative minimum. A proof of the second case is left to you. ■

EXAMPLE 4 Using the Second Derivative Test

Find the relative extrema for $f(x) = -3x^5 + 5x^3$.

Solution Begin by finding the critical numbers of f.

$$f'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2) = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

$$x = -1, 0, 1 \qquad \text{Critical numbers}$$

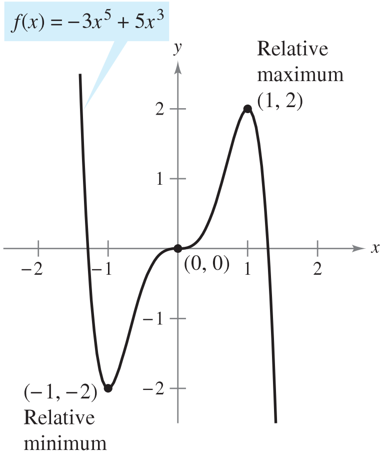
Using

$$f''(x) = -60x^3 + 30x = 30(-2x^3 + x)$$

you can apply the Second Derivative Test as shown below.

Point	$(-1, -2)$	$(1, 2)$	$(0, 0)$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(0) = 0$
Conclusion	Relative minimum	Relative maximum	Test fails

Because the Second Derivative Test fails at $(0, 0)$, you can use the First Derivative Test and observe that f increases to the left and right of $x = 0$. So, $(0, 0)$ is neither a relative minimum nor a relative maximum (even though the graph has a horizontal tangent line at this point). The graph of f is shown in Figure 3.32. ■



(0, 0) is neither a relative minimum nor a relative maximum.

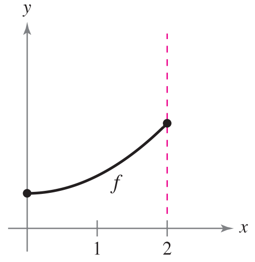
Figure 3.32

3.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, the graph of f is shown. State the signs of f' and f'' on the interval $(0, 2)$.

1.



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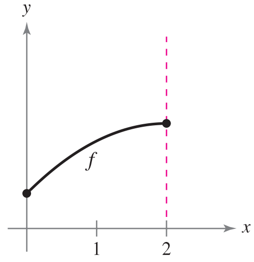

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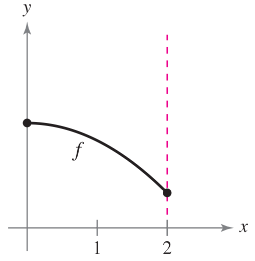

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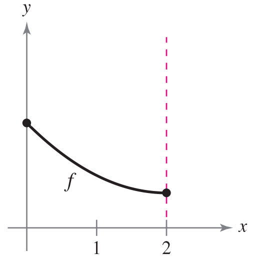


In Exercises 5–18, determine the open intervals on which the graph is concave upward or concave downward.

5. $y = x^2 - x - 2$

6. $y = -x^3 + 3x^2 - 2$

7. $g(x) = 3x^2 - x^3$

8. $h(x) = x^5 - 5x + 2$

9. $f(x) = -x^3 + 6x^2 - 9x - 1$

10. $f(x) = x^5 + 5x^4 - 40x^2$

11. $f(x) = \dfrac{24}{x^2 + 12}$

12. $f(x) = \dfrac{x^2}{x^2 + 1}$

13. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$

14. $y = \dfrac{-3x^5 + 40x^3 + 135x}{270}$

15. $g(x) = \dfrac{x^2 + 4}{4 - x^2}$

16. $h(x) = \dfrac{x^2 - 1}{2x - 1}$

17. $y = 2x - \tan x, \quad \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

18. $y = x + \dfrac{2}{\sin x}, \quad (-\pi, \pi)$

In Exercises 19–36, find the points of inflection and discuss the concavity of the graph of the function.

19. $f(x) = \frac{1}{2}x^4 + 2x^3$

20. $f(x) = -x^4 + 24x^2$

21. $f(x) = x^3 - 6x^2 + 12x$

22. $f(x) = 2x^3 - 3x^2 - 12x + 5$

23. $f(x) = \frac{1}{4}x^4 - 2x^2$

24. $f(x) = 2x^4 - 8x + 3$

25. $f(x) = x(x - 4)^3$

26. $f(x) = (x - 2)^3(x - 1)$

27. $f(x) = x\sqrt{x + 3}$

28. $f(x) = x\sqrt{9 - x}$

29. $f(x) = \dfrac{4}{x^2 + 1}$

30. $f(x) = \dfrac{x + 1}{\sqrt{x}}$

31. $f(x) = \sin \dfrac{x}{2}, \quad [0, 4\pi]$

32. $f(x) = 2 \csc \dfrac{3x}{2}, \quad (0, 2\pi)$

33. $f(x) = \sec\left(x - \dfrac{\pi}{2}\right), \quad (0, 4\pi)$

34. $f(x) = \sin x + \cos x, \quad [0, 2\pi]$

35. $f(x) = 2 \sin x + \sin 2x, \quad [0, 2\pi]$

36. $f(x) = x + 2 \cos x, \quad [0, 2\pi]$

In Exercises 37–52, find all relative extrema. Use the Second Derivative Test where applicable.

37. $f(x) = (x - 5)^2$

38. $f(x) = -(x - 5)^2$

39. $f(x) = 6x - x^2$

40. $f(x) = x^2 + 3x - 8$

41. $f(x) = x^3 - 3x^2 + 3$

42. $f(x) = x^3 - 5x^2 + 7x$

43. $f(x) = x^4 - 4x^3 + 2$

44. $f(x) = -x^4 + 4x^3 + 8x^2$

45. $g(x) = x^2(6 - x)^3$

46. $g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2$

47. $f(x) = x^{2/3} - 3$

48. $f(x) = \sqrt{x^2 + 1}$

49. $f(x) = x + \dfrac{4}{x}$

50. $f(x) = \dfrac{x}{x - 1}$

51. $f(x) = \cos x - x, \quad [0, 4\pi]$

52. $f(x) = 2 \sin x + \cos 2x, \quad [0, 2\pi]$

CAS **In Exercises 53–56, use a computer algebra system to analyze the function over the given interval. (a) Find the first and second derivatives of the function. (b) Find any relative extrema and points of inflection. (c) Graph f, f', and f'' on the same set of coordinate axes and state the relationship between the behavior of f and the signs of f' and f''.**

53. $f(x) = 0.2x^2(x - 3)^3, \quad [-1, 4]$

54. $f(x) = x^2\sqrt{6 - x^2}, \quad \left[-\sqrt{6}, \sqrt{6}\right]$

55. $f(x) = \sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x, \quad [0, \pi]$

56. $f(x) = \sqrt{2x} \sin x, \quad [0, 2\pi]$

WRITING ABOUT CONCEPTS

57. Consider a function f such that f' is increasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.

58. Consider a function f such that f' is decreasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.

59. Sketch the graph of a function f that does *not* have a point of inflection at $(c, f(c))$ even though $f''(c) = 0$.

60. S represents weekly sales of a product. What can be said of S' and S'' for each of the following statements?

(a) The rate of change of sales is increasing.

(b) Sales are increasing at a slower rate.

(c) The rate of change of sales is constant.

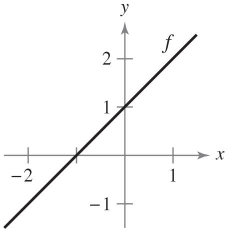
(d) Sales are steady.

(e) Sales are declining, but at a slower rate.

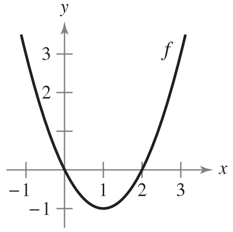
(f) Sales have bottomed out and have started to rise.

In Exercises 61–64, the graph of f is shown. Graph f, f', and f'' on the same set of coordinate axes. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

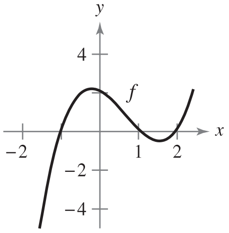
61.



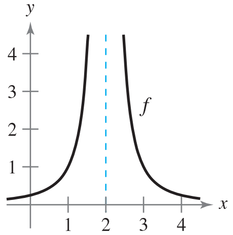
62.



63.



64.



Think About It **In Exercises 65–68, sketch the graph of a function f having the given characteristics.**

65. $f(2) = f(4) = 0$
$f'(x) < 0$ if $x < 3$
$f'(3)$ does not exist.
$f'(x) > 0$ if $x > 3$
$f''(x) < 0, x \neq 3$

66. $f(0) = f(2) = 0$
$f'(x) > 0$ if $x < 1$
$f'(1) = 0$
$f'(x) < 0$ if $x > 1$
$f''(x) < 0$

67. $f(2) = f(4) = 0$
$f'(x) > 0$ if $x < 3$
$f'(3)$ does not exist.
$f'(x) < 0$ if $x > 3$
$f''(x) > 0, x \neq 3$

68. $f(0) = f(2) = 0$
$f'(x) < 0$ if $x < 1$
$f'(1) = 0$
$f'(x) > 0$ if $x > 1$
$f''(x) > 0$

69. ***Think About It*** The figure shows the graph of f''. Sketch a graph of f. (The answer is not unique.) To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

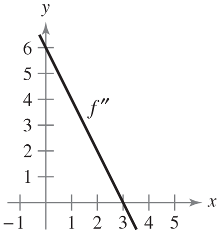


Figure for 69

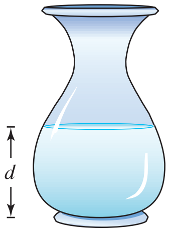


Figure for 70

CAPSTONE

70. ***Think About It*** Water is running into the vase shown in the figure at a constant rate.

(a) Graph the depth d of water in the vase as a function of time.

(b) Does the function have any extrema? Explain.

(c) Interpret the inflection points of the graph of d.

71. ***Conjecture*** Consider the function $f(x) = (x - 2)^n$.

(a) Use a graphing utility to graph f for $n = 1, 2, 3$, and 4. Use the graphs to make a conjecture about the relationship between n and any inflection points of the graph of f.

(b) Verify your conjecture in part (a).

72. (a) Graph $f(x) = \sqrt[3]{x}$ and identify the inflection point.

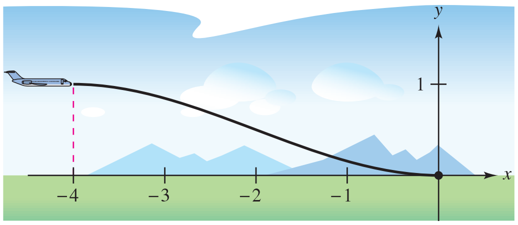
(b) Does $f''(x)$ exist at the inflection point? Explain.

In Exercises 73 and 74, find a, b, c, and d such that the cubic $f(x) = ax^3 + bx^2 + cx + d$ satisfies the given conditions.

73. Relative maximum: $(3, 3)$
Relative minimum: $(5, 1)$
Inflection point: $(4, 2)$

74. Relative maximum: $(2, 4)$
Relative minimum: $(4, 2)$
Inflection point: $(3, 3)$

75. ***Aircraft Glide Path*** A small aircraft starts its descent from an altitude of 1 mile, 4 miles west of the runway (see figure).

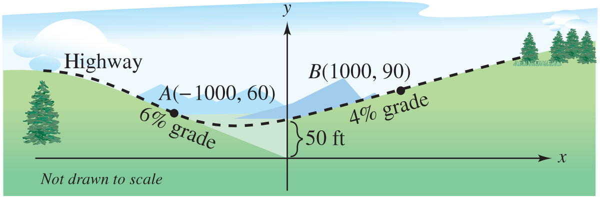


(a) Find the cubic $f(x) = ax^3 + bx^2 + cx + d$ on the interval $[-4, 0]$ that describes a smooth glide path for the landing.

(b) The function in part (a) models the glide path of the plane. When would the plane be descending at the greatest rate?

FOR FURTHER INFORMATION For more information on this type of modeling, see the article "How Not to Land at Lake Tahoe!" by Richard Barshinger in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

76. ***Highway Design*** A section of highway connecting two hillsides with grades of 6% and 4% is to be built between two points that are separated by a horizontal distance of 2000 feet (see figure). At the point where the two hillsides come together, there is a 50-foot difference in elevation.



(a) Design a section of highway connecting the hillsides modeled by the function $f(x) = ax^3 + bx^2 + cx + d$ $(-1000 \le x \le 1000)$. At the points A and B, the slope of the model must match the grade of the hillside.

(b) Use a graphing utility to graph the model.

(c) Use a graphing utility to graph the derivative of the model.

(d) Determine the grade at the steepest part of the transitional section of the highway.

77. ***Beam Deflection*** The deflection D of a beam of length L is $D = 2x^4 - 5Lx^3 + 3L^2x^2$, where x is the distance from one end of the beam. Find the value of x that yields the maximum deflection.

78. ***Specific Gravity*** A model for the specific gravity of water S is

$$S = \frac{5.755}{10^8}T^3 - \frac{8.521}{10^6}T^2 + \frac{6.540}{10^5}T + 0.99987, \quad 0 < T < 25$$

where T is the water temperature in degrees Celsius.

CAS (a) Use a computer algebra system to find the coordinates of the maximum value of the function.

(b) Sketch a graph of the function over the specified domain. (Use a setting in which $0.996 \le S \le 1.001$.)

(c) Estimate the specific gravity of water when $T = 20°$.

79. ***Average Cost*** A manufacturer has determined that the total cost C of operating a factory is $C = 0.5x^2 + 15x + 5000$, where x is the number of units produced. At what level of production will the average cost per unit be minimized? (The average cost per unit is C/x.)

80. ***Inventory Cost*** The total cost C of ordering and storing x units is $C = 2x + (300{,}000/x)$. What order size will produce a minimum cost?

81. ***Sales Growth*** The annual sales S of a new product are given by $S = \dfrac{5000t^2}{8 + t^2}$, $0 \le t \le 3$, where t is time in years.

(a) Complete the table. Then use it to estimate when the annual sales are increasing at the greatest rate.

t	0.5	1	1.5	2	2.5	3
S						

(b) Use a graphing utility to graph the function S. Then use the graph to estimate when the annual sales are increasing at the greatest rate.

(c) Find the exact time when the annual sales are increasing at the greatest rate.

82. ***Modeling Data*** The average typing speed S (in words per minute) of a typing student after t weeks of lessons is shown in the table.

t	5	10	15	20	25	30
S	38	56	79	90	93	94

A model for the data is $S = \dfrac{100t^2}{65 + t^2}$, $t > 0$.

(a) Use a graphing utility to plot the data and graph the model.

(b) Use the second derivative to determine the concavity of S. Compare the result with the graph in part (a).

(c) What is the sign of the first derivative for $t > 0$? By combining this information with the concavity of the model, what inferences can be made about the typing speed as t increases?

Linear and Quadratic Approximations **In Exercises 83–86, use a graphing utility to graph the function. Then graph the linear and quadratic approximations**

$$P_1(x) = f(a) + f'(a)(x - a)$$

and

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2$$

in the same viewing window. Compare the values of f, P_1, and P_2 and their first derivatives at $x = a$. How do the approximations change as you move farther away from $x = a$?

	Function	*Value of a*
83.	$f(x) = 2(\sin x + \cos x)$	$a = \dfrac{\pi}{4}$
84.	$f(x) = 2(\sin x + \cos x)$	$a = 0$
85.	$f(x) = \sqrt{1 - x}$	$a = 0$
86.	$f(x) = \dfrac{\sqrt{x}}{x - 1}$	$a = 2$

87. Use a graphing utility to graph $y = x\sin(1/x)$. Show that the graph is concave downward to the right of $x = 1/\pi$.

88. Show that the point of inflection of $f(x) = x(x - 6)^2$ lies midway between the relative extrema of f.

89. Prove that every cubic function with three distinct real zeros has a point of inflection whose x-coordinate is the average of the three zeros.

90. Show that the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ has exactly one point of inflection (x_0, y_0), where

$$x_0 = \frac{-b}{3a} \quad \text{and} \quad y_0 = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d.$$

Use this formula to find the point of inflection of $p(x) = x^3 - 3x^2 + 2$.

True or False? **In Exercises 91–94, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. The graph of every cubic polynomial has precisely one point of inflection.

92. The graph of $f(x) = 1/x$ is concave downward for $x < 0$ and concave upward for $x > 0$, and thus it has a point of inflection at $x = 0$.

93. If $f'(c) > 0$, then f is concave upward at $x = c$.

94. If $f''(2) = 0$, then the graph of f must have a point of inflection at $x = 2$.

In Exercises 95 and 96, let f and g represent differentiable functions such that $f'' \neq 0$ and $g'' \neq 0$.

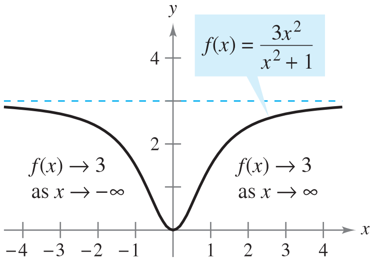
95. Show that if f and g are concave upward on the interval (a, b), then $f + g$ is also concave upward on (a, b).

96. Prove that if f and g are positive, increasing, and concave upward on the interval (a, b), then fg is also concave upward on (a, b).

3.5 Limits at Infinity

- **Determine (finite) limits at infinity.**
- **Determine the horizontal asymptotes, if any, of the graph of a function.**
- **Determine infinite limits at infinity.**

Limits at Infinity



The limit of $f(x)$ as x approaches $-\infty$ or ∞ is 3.
Figure 3.33

This section discusses the "end behavior" of a function on an *infinite* interval. Consider the graph of

$$f(x) = \frac{3x^2}{x^2 + 1}$$

as shown in Figure 3.33. Graphically, you can see that the values of $f(x)$ appear to approach 3 as x increases without bound or decreases without bound. You can come to the same conclusions numerically, as shown in the table.

x decreases without bound. x increases without bound.

x	$-\infty \leftarrow$	-100	-10	-1	0	1	10	100	$\to \infty$
$f(x)$	$3 \leftarrow$	2.9997	2.97	1.5	0	1.5	2.97	2.9997	$\to 3$

$f(x)$ approaches 3. $f(x)$ approaches 3.

The table suggests that the value of $f(x)$ approaches 3 as x increases without bound ($x \to \infty$). Similarly, $f(x)$ approaches 3 as x decreases without bound ($x \to -\infty$). These **limits at infinity** are denoted by

NOTE The statement $\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = L$ means that the limit exists *and* the limit is equal to L.

$$\lim_{x\to-\infty} f(x) = 3 \qquad \text{Limit at negative infinity}$$

and

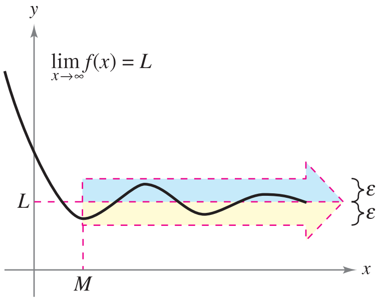
$$\lim_{x\to\infty} f(x) = 3. \qquad \text{Limit at positive infinity}$$

To say that a statement is true as x increases *without bound* means that for some (large) real number M, the statement is true for *all* x in the interval $\{x: x > M\}$. The following definition uses this concept.

DEFINITION OF LIMITS AT INFINITY

Let L be a real number.

1. The statement $\lim_{x\to\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.
2. The statement $\lim_{x\to-\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.



$f(x)$ is within ε units of L as $x \to \infty$.
Figure 3.34

The definition of a limit at infinity is shown in Figure 3.34. In this figure, note that for a given positive number ε there exists a positive number M such that, for $x > M$, the graph of f will lie between the horizontal lines given by $y = L + \varepsilon$ and $y = L - \varepsilon$.

EXPLORATION

Use a graphing utility to graph

$$f(x) = \frac{2x^2 + 4x - 6}{3x^2 + 2x - 16}.$$

Describe all the important features of the graph. Can you find a single viewing window that shows all of these features clearly? Explain your reasoning.

What are the horizontal asymptotes of the graph? How far to the right do you have to move on the graph so that the graph is within 0.001 unit of its horizontal asymptote? Explain your reasoning.

Horizontal Asymptotes

In Figure 3.34, the graph of f approaches the line $y = L$ as x increases without bound. The line $y = L$ is called a **horizontal asymptote** of the graph of f.

DEFINITION OF A HORIZONTAL ASYMPTOTE

The line $y = L$ is a **horizontal asymptote** of the graph of f if

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L.$$

Note that from this definition, it follows that the graph of a *function* of x can have at most two horizontal asymptotes—one to the right and one to the left.

Limits at infinity have many of the same properties of limits discussed in Section 1.3. For example, if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ both exist, then

$$\lim_{x\to\infty} [f(x) + g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$$

and

$$\lim_{x\to\infty} [f(x)g(x)] = \left[\lim_{x\to\infty} f(x)\right]\left[\lim_{x\to\infty} g(x)\right].$$

Similar properties hold for limits at $-\infty$.

When evaluating limits at infinity, the following theorem is helpful. (A proof of this theorem is given in Appendix A.)

THEOREM 3.10 LIMITS AT INFINITY

If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0.$$

EXAMPLE 1 Finding a Limit at Infinity

Find the limit: $\lim_{x\to\infty} \left(5 - \dfrac{2}{x^2}\right)$.

Solution Using Theorem 3.10, you can write

$$\begin{aligned} \lim_{x\to\infty} \left(5 - \frac{2}{x^2}\right) &= \lim_{x\to\infty} 5 - \lim_{x\to\infty} \frac{2}{x^2} && \text{Property of limits} \\ &= 5 - 0 \\ &= 5. \end{aligned}$$

■

EXAMPLE 2 Finding a Limit at Infinity

Find the limit: $\lim_{x\to\infty} \frac{2x-1}{x+1}$.

Solution Note that both the numerator and the denominator approach infinity as x approaches infinity.

$$\begin{tikzcd} & \boxed{\lim_{x\to\infty} (2x-1) \to \infty} \\ \lim_{x\to\infty} \frac{2x-1}{x+1} \arrow[ur] \arrow[dr] & \\ & \boxed{\lim_{x\to\infty} (x+1) \to \infty} \end{tikzcd}$$

NOTE When you encounter an indeterminate form such as the one in Example 2, you should divide the numerator and denominator by the highest power of x in the *denominator*.

This results in $\frac{\infty}{\infty}$, an **indeterminate form.** To resolve this problem, you can divide both the numerator and the denominator by x. After dividing, the limit may be evaluated as shown.

$$\lim_{x\to\infty} \frac{2x-1}{x+1} = \lim_{x\to\infty} \frac{\frac{2x-1}{x}}{\frac{x+1}{x}} \qquad \text{Divide numerator and denominator by } x.$$

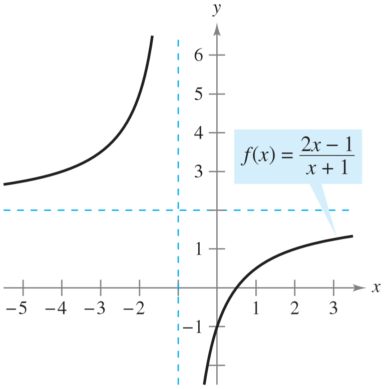
$$= \lim_{x\to\infty} \frac{2-\frac{1}{x}}{1+\frac{1}{x}} \qquad \text{Simplify.}$$

$$= \frac{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} \frac{1}{x}}{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty} \frac{1}{x}} \qquad \text{Take limits of numerator and denominator.}$$

$$= \frac{2-0}{1+0} \qquad \text{Apply Theorem 3.10.}$$

$$= 2$$

So, the line $y = 2$ is a horizontal asymptote to the right. By taking the limit as $x \to -\infty$, you can see that $y = 2$ is also a horizontal asymptote to the left. The graph of the function is shown in Figure 3.35. ■



$y = 2$ is a horizontal asymptote.
Figure 3.35

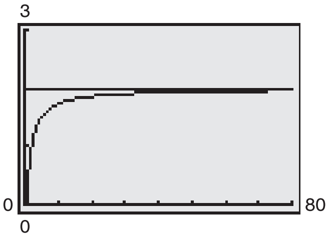
TECHNOLOGY You can test the reasonableness of the limit found in Example 2 by evaluating $f(x)$ for a few large positive values of x. For instance,

$$f(100) \approx 1.9703, \quad f(1000) \approx 1.9970, \quad \text{and} \quad f(10{,}000) \approx 1.9997.$$

Another way to test the reasonableness of the limit is to use a graphing utility. For instance, in Figure 3.36, the graph of

$$f(x) = \frac{2x-1}{x+1}$$

is shown with the horizontal line $y = 2$. Note that as x increases, the graph of f moves closer and closer to its horizontal asymptote.



As x increases, the graph of f moves closer and closer to the line $y = 2$.
Figure 3.36

EXAMPLE 3 A Comparison of Three Rational Functions

Find each limit.

a. $\lim_{x\to\infty} \dfrac{2x+5}{3x^2+1}$ **b.** $\lim_{x\to\infty} \dfrac{2x^2+5}{3x^2+1}$ **c.** $\lim_{x\to\infty} \dfrac{2x^3+5}{3x^2+1}$

Solution In each case, attempting to evaluate the limit produces the indeterminate form ∞/∞.

a. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty} \frac{2x+5}{3x^2+1} = \lim_{x\to\infty} \frac{(2/x)+(5/x^2)}{3+(1/x^2)} = \frac{0+0}{3+0} = \frac{0}{3} = 0$$

b. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1} = \lim_{x\to\infty} \frac{2+(5/x^2)}{3+(1/x^2)} = \frac{2+0}{3+0} = \frac{2}{3}$$

c. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1} = \lim_{x\to\infty} \frac{2x+(5/x^2)}{3+(1/x^2)} = \frac{\infty}{3}$$

You can conclude that the limit *does not exist* because the numerator increases without bound while the denominator approaches 3. ■

The Granger Collection

MARIA GAETANA AGNESI (1718–1799)

Agnesi was one of a handful of women to receive credit for significant contributions to mathematics before the twentieth century. In her early twenties, she wrote the first text that included both differential and integral calculus. By age 30, she was an honorary member of the faculty at the University of Bologna.

For more information on the contributions of women to mathematics, see the article "Why Women Succeed in Mathematics" by Mona Fabricant, Sylvia Svitak, and Patricia Clark Kenschaft in *Mathematics Teacher.* To view this article, go to the website *www.matharticles.com.*

GUIDELINES FOR FINDING LIMITS AT $\pm\infty$ OF RATIONAL FUNCTIONS

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exist.

Use these guidelines to check the results in Example 3. These limits seem reasonable when you consider that for large values of x, the highest-power term of the rational function is the most "influential" in determining the limit. For instance, the limit as x approaches infinity of the function

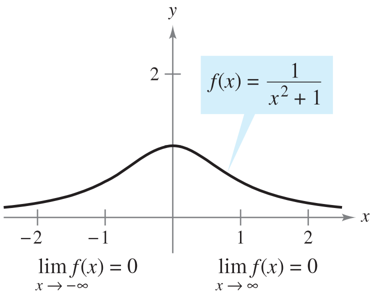
$$f(x) = \frac{1}{x^2+1}$$

is 0 because the denominator overpowers the numerator as x increases or decreases without bound, as shown in Figure 3.37.

The function shown in Figure 3.37 is a special case of a type of curve studied by the Italian mathematician Maria Gaetana Agnesi. The general form of this function is

$$f(x) = \frac{8a^3}{x^2+4a^2} \qquad \text{Witch of Agnesi}$$

and, through a mistranslation of the Italian word *vertéré*, the curve has come to be known as the Witch of Agnesi. Agnesi's work with this curve first appeared in a comprehensive text on calculus that was published in 1748.



f has a horizontal asymptote at $y = 0$.
Figure 3.37

In Figure 3.37, you can see that the function $f(x) = 1/(x^2 + 1)$ approaches the same horizontal asymptote to the right and to the left. This is always true of rational functions. Functions that are not rational, however, may approach different horizontal asymptotes to the right and to the left. This is demonstrated in Example 4.

EXAMPLE 4 A Function with Two Horizontal Asymptotes

Find each limit.

a. $\displaystyle\lim_{x\to\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}}$ **b.** $\displaystyle\lim_{x\to-\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}}$

Solution

a. For $x > 0$, you can write $x = \sqrt{x^2}$. So, dividing both the numerator and the denominator by x produces

$$\frac{3x - 2}{\sqrt{2x^2 + 1}} = \frac{\dfrac{3x - 2}{x}}{\dfrac{\sqrt{2x^2 + 1}}{\sqrt{x^2}}} = \frac{3 - \dfrac{2}{x}}{\sqrt{\dfrac{2x^2 + 1}{x^2}}} = \frac{3 - \dfrac{2}{x}}{\sqrt{2 + \dfrac{1}{x^2}}}$$

and you can take the limit as follows.

$$\lim_{x\to\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}} = \lim_{x\to\infty} \frac{3 - \dfrac{2}{x}}{\sqrt{2 + \dfrac{1}{x^2}}} = \frac{3 - 0}{\sqrt{2 + 0}} = \frac{3}{\sqrt{2}}$$

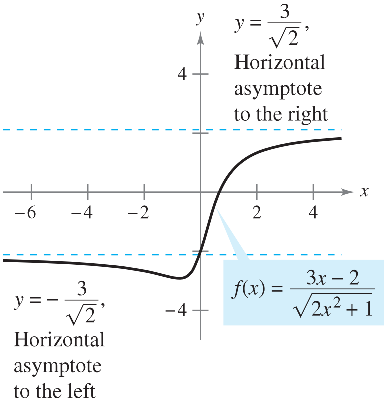
b. For $x < 0$, you can write $x = -\sqrt{x^2}$. So, dividing both the numerator and the denominator by x produces

$$\frac{3x - 2}{\sqrt{2x^2 + 1}} = \frac{\dfrac{3x - 2}{x}}{\dfrac{\sqrt{2x^2 + 1}}{-\sqrt{x^2}}} = \frac{3 - \dfrac{2}{x}}{-\sqrt{\dfrac{2x^2 + 1}{x^2}}} = \frac{3 - \dfrac{2}{x}}{-\sqrt{2 + \dfrac{1}{x^2}}}$$

and you can take the limit as follows.

$$\lim_{x\to-\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}} = \lim_{x\to-\infty} \frac{3 - \dfrac{2}{x}}{-\sqrt{2 + \dfrac{1}{x^2}}} = \frac{3 - 0}{-\sqrt{2 + 0}} = -\frac{3}{\sqrt{2}}$$

The graph of $f(x) = (3x - 2)/\sqrt{2x^2 + 1}$ is shown in Figure 3.38. ■

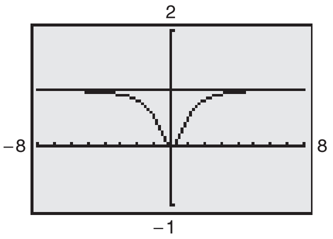


Functions that are not rational may have different right and left horizontal asymptotes.
Figure 3.38

TECHNOLOGY PITFALL If you use a graphing utility to help estimate a limit, be sure that you also confirm the estimate analytically—the pictures shown by a graphing utility can be misleading. For instance, Figure 3.39 shows one view of the graph of

$$y = \frac{2x^3 + 1000x^2 + x}{x^3 + 1000x^2 + x + 1000}.$$

From this view, one could be convinced that the graph has $y = 1$ as a horizontal asymptote. An analytical approach shows that the horizontal asymptote is actually $y = 2$. Confirm this by enlarging the viewing window on the graphing utility.



The horizontal asymptote appears to be the line $y = 1$ but it is actually the line $y = 2$.
Figure 3.39

In Section 1.3 (Example 9), you saw how the Squeeze Theorem can be used to evaluate limits involving trigonometric functions. This theorem is also valid for limits at infinity.

EXAMPLE 5 Limits Involving Trigonometric Functions

Find each limit.

a. $\lim_{x\to\infty} \sin x$ **b.** $\lim_{x\to\infty} \dfrac{\sin x}{x}$

Solution

a. As x approaches infinity, the sine function oscillates between 1 and -1. So, this limit does not exist.

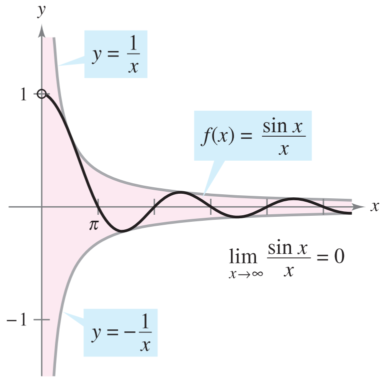
b. Because $-1 \le \sin x \le 1$, it follows that for $x > 0$,

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

where $\lim_{x\to\infty} (-1/x) = 0$ and $\lim_{x\to\infty} (1/x) = 0$. So, by the Squeeze Theorem, you can obtain

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

as shown in Figure 3.40.



As x increases without bound, $f(x)$ approaches 0.
Figure 3.40

EXAMPLE 6 Oxygen Level in a Pond

Suppose that $f(t)$ measures the level of oxygen in a pond, where $f(t) = 1$ is the normal (unpolluted) level and the time t is measured in weeks. When $t = 0$, organic waste is dumped into the pond, and as the waste material oxidizes, the level of oxygen in the pond is

$$f(t) = \frac{t^2 - t + 1}{t^2 + 1}.$$

What percent of the normal level of oxygen exists in the pond after 1 week? After 2 weeks? After 10 weeks? What is the limit as t approaches infinity?

Solution When $t = 1$, 2, and 10, the levels of oxygen are as shown.

$$f(1) = \frac{1^2 - 1 + 1}{1^2 + 1} = \frac{1}{2} = 50\% \qquad \text{1 week}$$

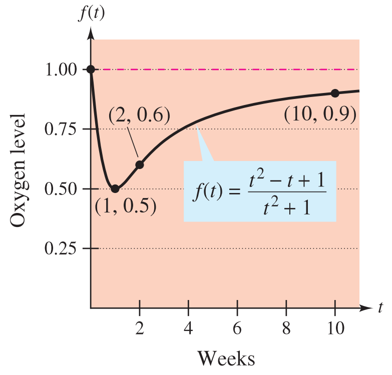
$$f(2) = \frac{2^2 - 2 + 1}{2^2 + 1} = \frac{3}{5} = 60\% \qquad \text{2 weeks}$$

$$f(10) = \frac{10^2 - 10 + 1}{10^2 + 1} = \frac{91}{101} \approx 90.1\% \qquad \text{10 weeks}$$

To find the limit as t approaches infinity, divide the numerator and the denominator by t^2 to obtain

$$\lim_{t\to\infty} \frac{t^2 - t + 1}{t^2 + 1} = \lim_{t\to\infty} \frac{1 - (1/t) + (1/t^2)}{1 + (1/t^2)} = \frac{1 - 0 + 0}{1 + 0} = 1 = 100\%.$$

See Figure 3.41. ■



The level of oxygen in a pond approaches the normal level of 1 as t approaches ∞.
Figure 3.41

Infinite Limits at Infinity

Many functions do not approach a finite limit as x increases (or decreases) without bound. For instance, no polynomial function has a finite limit at infinity. The following definition is used to describe the behavior of polynomial and other functions at infinity.

NOTE Determining whether a function has an infinite limit at infinity is useful in analyzing the "end behavior" of its graph. You will see examples of this in Section 3.6 on curve sketching.

DEFINITION OF INFINITE LIMITS AT INFINITY

Let f be a function defined on the interval (a, ∞).

1. The statement $\lim_{x\to\infty} f(x) = \infty$ means that for each positive number M, there is a corresponding number $N > 0$ such that $f(x) > M$ whenever $x > N$.
2. The statement $\lim_{x\to\infty} f(x) = -\infty$ means that for each negative number M, there is a corresponding number $N > 0$ such that $f(x) < M$ whenever $x > N$.

Similar definitions can be given for the statements $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

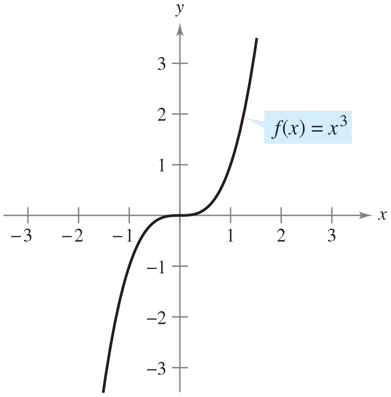


Figure 3.42

EXAMPLE 7 Finding Infinite Limits at Infinity

Find each limit.

a. $\lim_{x\to\infty} x^3$ **b.** $\lim_{x\to-\infty} x^3$

Solution

a. As x increases without bound, x^3 also increases without bound. So, you can write $\lim_{x\to\infty} x^3 = \infty$.

b. As x decreases without bound, x^3 also decreases without bound. So, you can write $\lim_{x\to-\infty} x^3 = -\infty$.

The graph of $f(x) = x^3$ in Figure 3.42 illustrates these two results. These results agree with the Leading Coefficient Test for polynomial functions as described in Section P.3.

EXAMPLE 8 Finding Infinite Limits at Infinity

Find each limit.

a. $\lim_{x\to\infty} \frac{2x^2 - 4x}{x + 1}$ **b.** $\lim_{x\to-\infty} \frac{2x^2 - 4x}{x + 1}$

Solution One way to evaluate each of these limits is to use long division to rewrite the improper rational function as the sum of a polynomial and a rational function.

$$\textbf{a.}\quad \lim_{x\to\infty} \frac{2x^2 - 4x}{x + 1} = \lim_{x\to\infty}\left(2x - 6 + \frac{6}{x + 1}\right) = \infty$$

$$\textbf{b.}\quad \lim_{x\to-\infty} \frac{2x^2 - 4x}{x + 1} = \lim_{x\to-\infty}\left(2x - 6 + \frac{6}{x + 1}\right) = -\infty$$

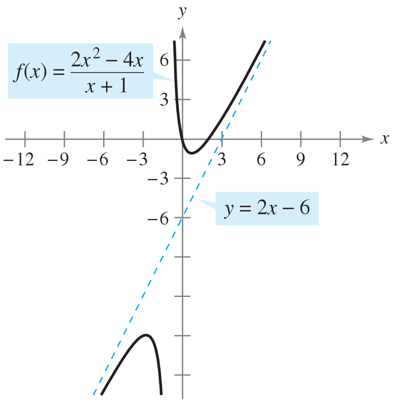


Figure 3.43

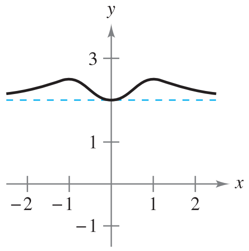
The statements above can be interpreted as saying that as x approaches $\pm\infty$, the function $f(x) = (2x^2 - 4x)/(x + 1)$ behaves like the function $g(x) = 2x - 6$. In Section 3.6, you will see that this is graphically described by saying that the line $y = 2x - 6$ is a slant asymptote of the graph of f, as shown in Figure 3.43. ■

3.5 Exercises

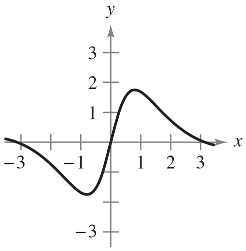
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, match the function with one of the graphs [(a), (b), (c), (d), (e), or (f)] using horizontal asymptotes as an aid.

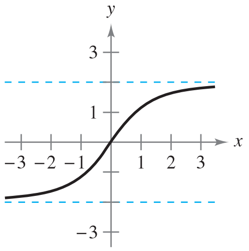
(a)



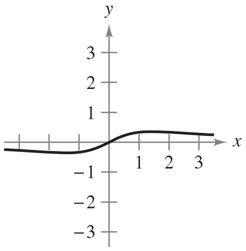
(b)



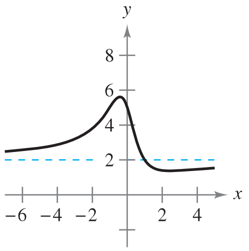
(c)



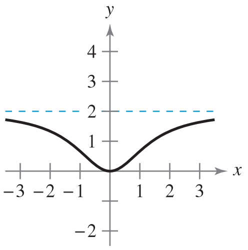
(d)



(e)



(f)



1. $f(x) = \dfrac{2x^2}{x^2+2}$

2. $f(x) = \dfrac{2x}{\sqrt{x^2+2}}$

3. $f(x) = \dfrac{x}{x^2+2}$

4. $f(x) = 2 + \dfrac{x^2}{x^4+1}$

5. $f(x) = \dfrac{4\sin x}{x^2+1}$

6. $f(x) = \dfrac{2x^2-3x+5}{x^2+1}$

Numerical and Graphical Analysis **In Exercises 7–12, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit graphically.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

7. $f(x) = \dfrac{4x+3}{2x-1}$

8. $f(x) = \dfrac{2x^2}{x+1}$

9. $f(x) = \dfrac{-6x}{\sqrt{4x^2+5}}$

10. $f(x) = \dfrac{20x}{\sqrt{9x^2-1}}$

11. $f(x) = 5 - \dfrac{1}{x^2+1}$

12. $f(x) = 4 + \dfrac{3}{x^2+2}$

In Exercises 13 and 14, find $\lim\limits_{x\to\infty} h(x)$, if possible.

13. $f(x) = 5x^3 - 3x^2 + 10x$

(a) $h(x) = \dfrac{f(x)}{x^2}$

(b) $h(x) = \dfrac{f(x)}{x^3}$

(c) $h(x) = \dfrac{f(x)}{x^4}$

14. $f(x) = -4x^2 + 2x - 5$

(a) $h(x) = \dfrac{f(x)}{x}$

(b) $h(x) = \dfrac{f(x)}{x^2}$

(c) $h(x) = \dfrac{f(x)}{x^3}$

In Exercises 15–18, find each limit, if possible.

15. (a) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x^3-1}$

(b) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x^2-1}$

(c) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x-1}$

16. (a) $\lim\limits_{x\to\infty} \dfrac{3-2x}{3x^3-1}$

(b) $\lim\limits_{x\to\infty} \dfrac{3-2x}{3x-1}$

(c) $\lim\limits_{x\to\infty} \dfrac{3-2x^2}{3x-1}$

17. (a) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x^2-4}$

(b) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x^{3/2}-4}$

(c) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x-4}$

18. (a) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4x^2+1}$

(b) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4x^{3/2}+1}$

(c) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4\sqrt{x}+1}$

In Exercises 19–38, find the limit.

19. $\lim\limits_{x\to\infty} \left(4 + \dfrac{3}{x}\right)$

20. $\lim\limits_{x\to-\infty} \left(\dfrac{5}{x} - \dfrac{x}{3}\right)$

21. $\lim\limits_{x\to\infty} \dfrac{2x-1}{3x+2}$

22. $\lim\limits_{x\to\infty} \dfrac{x^2+3}{2x^2-1}$

23. $\lim\limits_{x\to\infty} \dfrac{x}{x^2-1}$

24. $\lim\limits_{x\to\infty} \dfrac{5x^3+1}{10x^3-3x^2+7}$

25. $\lim\limits_{x\to-\infty} \dfrac{5x^2}{x+3}$

26. $\lim\limits_{x\to-\infty} \left(\dfrac{1}{2}x - \dfrac{4}{x^2}\right)$

27. $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2-x}}$

28. $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2+1}}$

29. $\lim\limits_{x\to-\infty} \dfrac{2x+1}{\sqrt{x^2-x}}$

30. $\lim\limits_{x\to-\infty} \dfrac{-3x+1}{\sqrt{x^2+x}}$

31. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2-1}}{2x-1}$

32. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^4-1}}{x^3-1}$

33. $\lim\limits_{x\to\infty} \dfrac{x+1}{(x^2+1)^{1/3}}$

34. $\lim\limits_{x\to-\infty} \dfrac{2x}{(x^6-1)^{1/3}}$

35. $\lim\limits_{x\to\infty} \dfrac{1}{2x+\sin x}$

36. $\lim\limits_{x\to\infty} \cos\dfrac{1}{x}$

37. $\lim\limits_{x\to\infty} \dfrac{\sin 2x}{x}$

38. $\lim\limits_{x\to\infty} \dfrac{x-\cos x}{x}$

In Exercises 39–42, use a graphing utility to graph the function and identify any horizontal asymptotes.

39. $f(x) = \dfrac{|x|}{x+1}$

40. $f(x) = \dfrac{|3x+2|}{x-2}$

41. $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$

42. $f(x) = \dfrac{\sqrt{9x^2-2}}{2x+1}$

In Exercises 43 and 44, find the limit. (*Hint:* Let $x = 1/t$ and find the limit as $t \to 0^+$.)

43. $\displaystyle\lim_{x\to\infty} x \sin \frac{1}{x}$

44. $\displaystyle\lim_{x\to\infty} x \tan \frac{1}{x}$

In Exercises 45–48, find the limit. (*Hint:* Treat the expression as a fraction whose denominator is 1, and rationalize the numerator.) Use a graphing utility to verify your result.

45. $\displaystyle\lim_{x\to-\infty} \left(x + \sqrt{x^2+3}\right)$

46. $\displaystyle\lim_{x\to\infty} \left(x - \sqrt{x^2+x}\right)$

47. $\displaystyle\lim_{x\to-\infty} \left(3x + \sqrt{9x^2-x}\right)$

48. $\displaystyle\lim_{x\to\infty} \left(4x - \sqrt{16x^2-x}\right)$

Numerical, Graphical, and Analytic Analysis **In Exercises 49–52, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit. Finally, find the limit analytically and compare your results with the estimates.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

49. $f(x) = x - \sqrt{x(x-1)}$

50. $f(x) = x^2 - x\sqrt{x(x-1)}$

51. $f(x) = x \sin \dfrac{1}{2x}$

52. $f(x) = \dfrac{x+1}{x\sqrt{x}}$

WRITING ABOUT CONCEPTS

In Exercises 53 and 54, describe in your own words what the statement means.

53. $\displaystyle\lim_{x\to\infty} f(x) = 4$

54. $\displaystyle\lim_{x\to-\infty} f(x) = 2$

55. Sketch a graph of a differentiable function f that satisfies the following conditions and has $x = 2$ as its only critical number.

$$f'(x) < 0 \text{ for } x < 2 \qquad f'(x) > 0 \text{ for } x > 2$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 6$$

56. Is it possible to sketch a graph of a function that satisfies the conditions of Exercise 55 and has *no* points of inflection? Explain.

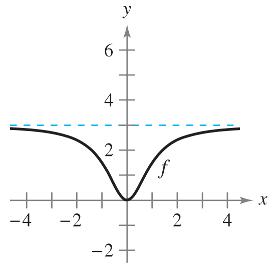
57. If f is a continuous function such that $\displaystyle\lim_{x\to\infty} f(x) = 5$, find, if possible, $\displaystyle\lim_{x\to-\infty} f(x)$ for each specified condition.

(a) The graph of f is symmetric with respect to the y-axis.

(b) The graph of f is symmetric with respect to the origin.

CAPSTONE

58. The graph of a function f is shown below. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) Sketch f'.

(b) Use the graphs to estimate $\displaystyle\lim_{x\to\infty} f(x)$ and $\displaystyle\lim_{x\to\infty} f'(x)$.

(c) Explain the answers you gave in part (b).

In Exercises 59–76, sketch the graph of the equation using extrema, intercepts, symmetry, and asymptotes. Then use a graphing utility to verify your result.

59. $y = \dfrac{x}{1-x}$

60. $y = \dfrac{x-4}{x-3}$

61. $y = \dfrac{x+1}{x^2-4}$

62. $y = \dfrac{2x}{9-x^2}$

63. $y = \dfrac{x^2}{x^2+16}$

64. $y = \dfrac{x^2}{x^2-16}$

65. $y = \dfrac{2x^2}{x^2-4}$

66. $y = \dfrac{2x^2}{x^2+4}$

67. $xy^2 = 9$

68. $x^2y = 9$

69. $y = \dfrac{3x}{1-x}$

70. $y = \dfrac{3x}{1-x^2}$

71. $y = 2 - \dfrac{3}{x^2}$

72. $y = 1 + \dfrac{1}{x}$

73. $y = 3 + \dfrac{2}{x}$

74. $y = 4\left(1 - \dfrac{1}{x^2}\right)$

75. $y = \dfrac{x^3}{\sqrt{x^2-4}}$

76. $y = \dfrac{x}{\sqrt{x^2-4}}$

CAS **In Exercises 77–84, use a computer algebra system to analyze the graph of the function. Label any extrema and/or asymptotes that exist.**

77. $f(x) = 9 - \dfrac{5}{x^2}$

78. $f(x) = \dfrac{1}{x^2-x-2}$

79. $f(x) = \dfrac{x-2}{x^2-4x+3}$

80. $f(x) = \dfrac{x+1}{x^2+x+1}$

81. $f(x) = \dfrac{3x}{\sqrt{4x^2+1}}$

82. $g(x) = \dfrac{2x}{\sqrt{3x^2+1}}$

83. $g(x) = \sin\left(\dfrac{x}{x-2}\right), \quad x > 3$

84. $f(x) = \dfrac{2 \sin 2x}{x}$

In Exercises 85 and 86, (a) use a graphing utility to graph f and g in the same viewing window, (b) verify algebraically that f and g represent the same function, and (c) zoom out sufficiently far so that the graph appears as a line. What equation does this line appear to have? (Note that the points at which the function is not continuous are not readily seen when you zoom out.)

85. $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x-3)}$

$g(x) = x + \dfrac{2}{x(x-3)}$

86. $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$

$g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

87. ***Engine Efficiency*** The efficiency of an internal combustion engine is

$$\text{Efficiency } (\%) = 100\left[1 - \frac{1}{(v_1/v_2)^c}\right]$$

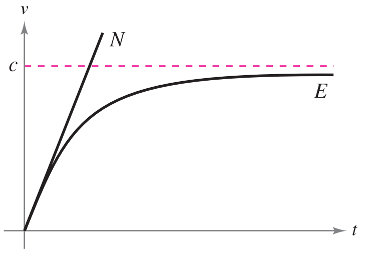
where v_1/v_2 is the ratio of the uncompressed gas to the compressed gas and c is a positive constant dependent on the engine design. Find the limit of the efficiency as the compression ratio approaches infinity.

88. ***Average Cost*** A business has a cost of $C = 0.5x + 500$ for producing x units. The average cost per unit is

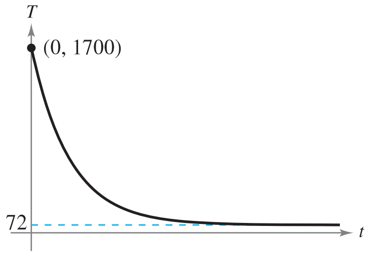
$$\overline{C} = \frac{C}{x}.$$

Find the limit of $\overline{C}$ as x approaches infinity.

89. ***Physics*** Newton's First Law of Motion and Einstein's Special Theory of Relativity differ concerning a particle's behavior as its velocity approaches the speed of light c. In the graph, functions N and E represent the velocity v, with respect to time t, of a particle accelerated by a constant force as predicted by Newton and Einstein. Write limit statements that describe these two theories.



90. ***Temperature*** The graph shows the temperature T, in degrees Fahrenheit, of molten glass t seconds after it is removed from a kiln.



(a) Find $\lim\limits_{t \to 0^+} T$. What does this limit represent?

(b) Find $\lim\limits_{t \to \infty} T$. What does this limit represent?

(c) Will the temperature of the glass ever actually reach room temperature? Why?

91. ***Modeling Data*** The table shows the world record times for the mile run, where t represents the year, with $t = 0$ corresponding to 1900, and y is the time in minutes and seconds.

t	23	33	45	54	58	66	79	85	99
y	4:10.4	4:07.6	4:01.3	3:59.4	3:54.5	3:51.3	3:48.9	3:46.3	3:43.1

A model for the data is

$$y = \frac{3.351t^2 + 42.461t - 543.730}{t^2}$$

where the seconds have been changed to decimal parts of a minute.

(a) Use a graphing utility to plot the data and graph the model.

(b) Does there appear to be a limiting time for running 1 mile? Explain.

92. ***Modeling Data*** The average typing speeds S (in words per minute) of a typing student after t weeks of lessons are shown in the table.

t	5	10	15	20	25	30
S	28	56	79	90	93	94

A model for the data is $S = \dfrac{100t^2}{65 + t^2}, \quad t > 0.$

(a) Use a graphing utility to plot the data and graph the model.

(b) Does there appear to be a limiting typing speed? Explain.

93. ***Modeling Data*** A heat probe is attached to the heat exchanger of a heating system. The temperature T (in degrees Celsius) is recorded t seconds after the furnace is started. The results for the first 2 minutes are recorded in the table.

t	0	15	30	45	60	75	90	105	120
T	25.2°	36.9°	45.5°	51.4°	56.0°	59.6°	62.0°	64.0°	65.2°

(a) Use the regression capabilities of a graphing utility to find a model of the form $T_1 = at^2 + bt + c$ for the data.

(b) Use a graphing utility to graph T_1.

(c) A rational model for the data is $T_2 = \dfrac{1451 + 86t}{58 + t}$. Use a graphing utility to graph T_2.

(d) Find $T_1(0)$ and $T_2(0)$.

(e) Find $\lim\limits_{t \to \infty} T_2$.

(f) Interpret the result in part (e) in the context of the problem. Is it possible to do this type of analysis using T_1? Explain.

94. ***Modeling Data*** A container holds 5 liters of a 25% brine solution. The table shows the concentrations C of the mixture after adding x liters of a 75% brine solution to the container.

x	0	0.5	1	1.5	2
C	0.25	0.295	0.333	0.365	0.393

x	2.5	3	3.5	4
C	0.417	0.438	0.456	0.472

(a) Use the regression features of a graphing utility to find a model of the form $C_1 = ax^2 + bx + c$ for the data.

(b) Use a graphing utility to graph C_1.

(c) A rational model for these data is $C_2 = \dfrac{5 + 3x}{20 + 4x}$. Use a graphing utility to graph C_2.

(d) Find $\lim\limits_{x\to\infty} C_1$ and $\lim\limits_{x\to\infty} C_2$. Which model do you think best represents the concentration of the mixture? Explain.

(e) What is the limiting concentration?

95. A line with slope m passes through the point $(0, 4)$.

(a) Write the distance d between the line and the point $(3, 1)$ as a function of m.

(b) Use a graphing utility to graph the equation in part (a).

(c) Find $\lim\limits_{m\to\infty} d(m)$ and $\lim\limits_{m\to-\infty} d(m)$. Interpret the results geometrically.

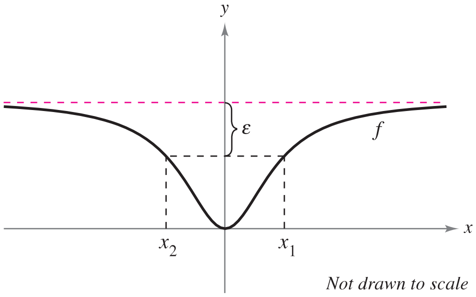
96. A line with slope m passes through the point $(0, -2)$.

(a) Write the distance d between the line and the point $(4, 2)$ as a function of m.

(b) Use a graphing utility to graph the equation in part (a).

(c) Find $\lim\limits_{m\to\infty} d(m)$ and $\lim\limits_{m\to-\infty} d(m)$. Interpret the results geometrically.

97. The graph of $f(x) = \dfrac{2x^2}{x^2 + 2}$ is shown.



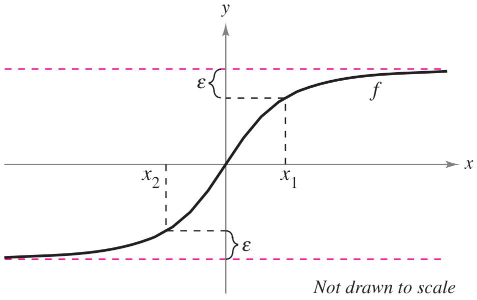
(a) Find $L = \lim\limits_{x\to\infty} f(x)$.

(b) Determine x_1 and x_2 in terms of ε.

(c) Determine M, where $M > 0$, such that $|f(x) - L| < \varepsilon$ for $x > M$.

(d) Determine N, where $N < 0$, such that $|f(x) - L| < \varepsilon$ for $x < N$.

98. The graph of $f(x) = \dfrac{6x}{\sqrt{x^2 + 2}}$ is shown.



(a) Find $L = \lim\limits_{x\to\infty} f(x)$ and $K = \lim\limits_{x\to-\infty} f(x)$.

(b) Determine x_1 and x_2 in terms of ε.

(c) Determine M, where $M > 0$, such that $|f(x) - L| < \varepsilon$ for $x > M$.

(d) Determine N, where $N < 0$, such that $|f(x) - K| < \varepsilon$ for $x < N$.

99. Consider $\lim\limits_{x\to\infty} \dfrac{3x}{\sqrt{x^2 + 3}}$. Use the definition of limits at infinity to find values of M that correspond to (a) $\varepsilon = 0.5$ and (b) $\varepsilon = 0.1$.

100. Consider $\lim\limits_{x\to-\infty} \dfrac{3x}{\sqrt{x^2 + 3}}$. Use the definition of limits at infinity to find values of N that correspond to (a) $\varepsilon = 0.5$ and (b) $\varepsilon = 0.1$.

In Exercises 101–104, use the definition of limits at infinity to prove the limit.

101. $\lim\limits_{x\to\infty} \dfrac{1}{x^2} = 0$

102. $\lim\limits_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$

103. $\lim\limits_{x\to-\infty} \dfrac{1}{x^3} = 0$

104. $\lim\limits_{x\to-\infty} \dfrac{1}{x - 2} = 0$

105. Prove that if $p(x) = a_nx^n + \cdots + a_1x + a_0$ and $q(x) = b_mx^m + \cdots + b_1x + b_0$ $(a_n \neq 0, b_m \neq 0)$, then

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m \\ \pm\infty, & n > m \end{cases}.$$

106. Use the definition of infinite limits at infinity to prove that $\lim\limits_{x\to\infty} x^3 = \infty$.

True or False? **In Exercises 107 and 108, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

107. If $f'(x) > 0$ for all real numbers x, then f increases without bound.

108. If $f''(x) < 0$ for all real numbers x, then f decreases without bound.

3.6 A Summary of Curve Sketching

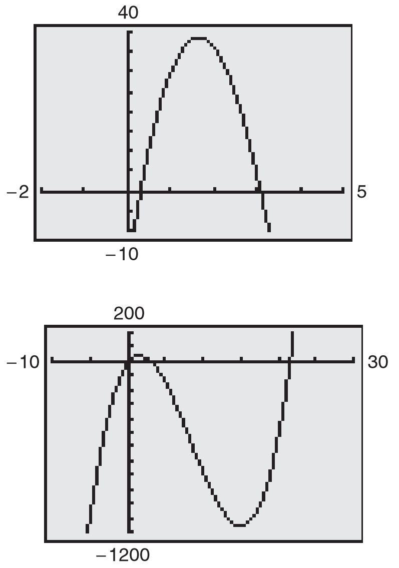
- **Analyze and sketch the graph of a function.**

Analyzing the Graph of a Function

It would be difficult to overstate the importance of using graphs in mathematics. Descartes's introduction of analytic geometry contributed significantly to the rapid advances in calculus that began during the mid-seventeenth century. In the words of Lagrange, "As long as algebra and geometry traveled separate paths their advance was slow and their applications limited. But when these two sciences joined company, they drew from each other fresh vitality and thenceforth marched on at a rapid pace toward perfection."

So far, you have studied several concepts that are useful in analyzing the graph of a function.

• x-intercepts and y-intercepts	(Section P.1)
• Symmetry	(Section P.1)
• Domain and range	(Section P.3)
• Continuity	(Section 1.4)
• Vertical asymptotes	(Section 1.5)
• Differentiability	(Section 2.1)
• Relative extrema	(Section 3.1)
• Concavity	(Section 3.4)
• Points of inflection	(Section 3.4)
• Horizontal asymptotes	(Section 3.5)
• Infinite limits at infinity	(Section 3.5)



Different viewing windows for the graph of $f(x) = x^3 - 25x^2 + 74x - 20$
Figure 3.44

When you are sketching the graph of a function, either by hand or with a graphing utility, remember that normally you cannot show the *entire* graph. The decision as to which part of the graph you choose to show is often crucial. For instance, which of the viewing windows in Figure 3.44 better represents the graph of

$$f(x) = x^3 - 25x^2 + 74x - 20?$$

By seeing both views, it is clear that the second viewing window gives a more complete representation of the graph. But would a third viewing window reveal other interesting portions of the graph? To answer this, you need to use calculus to interpret the first and second derivatives. Here are some guidelines for determining a good viewing window for the graph of a function.

GUIDELINES FOR ANALYZING THE GRAPH OF A FUNCTION

1. Determine the domain and range of the function.
2. Determine the intercepts, asymptotes, and symmetry of the graph.
3. Locate the x-values for which $f'(x)$ and $f''(x)$ either are zero or do not exist. Use the results to determine relative extrema and points of inflection.

NOTE In these guidelines, note the importance of *algebra* (as well as calculus) for solving the equations $f(x) = 0$, $f'(x) = 0$, and $f''(x) = 0$. ■

EXAMPLE 1 Sketching the Graph of a Rational Function

Analyze and sketch the graph of $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$.

Solution

First derivative: $f'(x) = \dfrac{20x}{(x^2 - 4)^2}$

Second derivative: $f''(x) = \dfrac{-20(3x^2 + 4)}{(x^2 - 4)^3}$

x-intercepts: $(-3, 0), (3, 0)$

y-intercept: $\left(0, \frac{9}{2}\right)$

Vertical asymptotes: $x = -2, x = 2$

Horizontal asymptote: $y = 2$

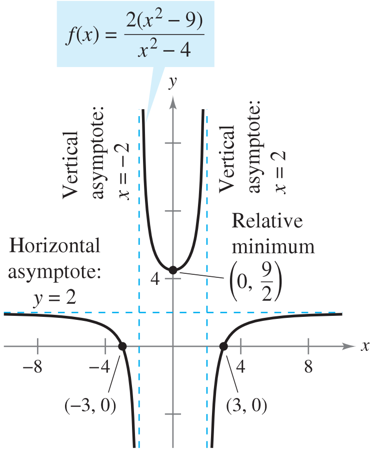
Critical number: $x = 0$

Possible points of inflection: None

Domain: All real numbers except $x = \pm 2$

Symmetry: With respect to y-axis

Test intervals: $(-\infty, -2), (-2, 0), (0, 2), (2, \infty)$



Using calculus, you can be certain that you have determined all characteristics of the graph of f.
Figure 3.45

The table shows how the test intervals are used to determine several characteristics of the graph. The graph of f is shown in Figure 3.45.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < -2$		$-$	$-$	Decreasing, concave downward
$x = -2$	Undef.	Undef.	Undef.	Vertical asymptote
$-2 < x < 0$		$-$	$+$	Decreasing, concave upward
$x = 0$	$\frac{9}{2}$	0	$+$	Relative minimum
$0 < x < 2$		$+$	$+$	Increasing, concave upward
$x = 2$	Undef.	Undef.	Undef.	Vertical asymptote
$2 < x < \infty$		$+$	$-$	Increasing, concave downward

■

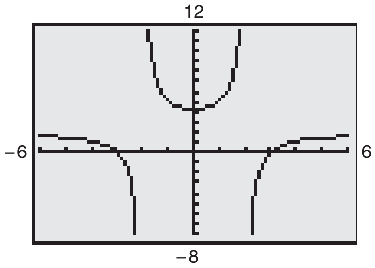
FOR FURTHER INFORMATION For more information on the use of technology to graph rational functions, see the article "Graphs of Rational Functions for Computer Assisted Calculus" by Stan Byrd and Terry Walters in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

Be sure you understand all of the implications of creating a table such as that shown in Example 1. By using calculus, you can *be sure* that the graph has no relative extrema or points of inflection other than those shown in Figure 3.45.

TECHNOLOGY PITFALL Without using the type of analysis outlined in Example 1, it is easy to obtain an incomplete view of a graph's basic characteristics. For instance, Figure 3.46 shows a view of the graph of

$$g(x) = \frac{2(x^2 - 9)(x - 20)}{(x^2 - 4)(x - 21)}.$$

From this view, it appears that the graph of g is about the same as the graph of f shown in Figure 3.45. The graphs of these two functions, however, differ significantly. Try enlarging the viewing window to see the differences.



By not using calculus you may overlook important characteristics of the graph of g.
Figure 3.46

EXAMPLE 2 Sketching the Graph of a Rational Function

Analyze and sketch the graph of $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$.

Solution

First derivative: $f'(x) = \dfrac{x(x-4)}{(x-2)^2}$

Second derivative: $f''(x) = \dfrac{8}{(x-2)^3}$

x-intercepts: None

y-intercept: $(0, -2)$

Vertical asymptote: $x = 2$

Horizontal asymptotes: None

End behavior: $\lim\limits_{x\to-\infty} f(x) = -\infty, \ \lim\limits_{x\to\infty} f(x) = \infty$

Critical numbers: $x = 0, x = 4$

Possible points of inflection: None

Domain: All real numbers except $x = 2$

Test intervals: $(-\infty, 0), (0, 2), (2, 4), (4, \infty)$

The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.47.

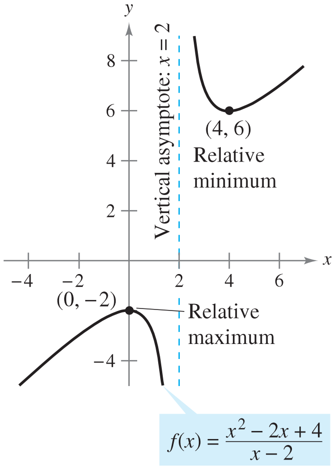


Figure 3.47

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		$+$	$-$	Increasing, concave downward
$x = 0$	-2	0	$-$	Relative maximum
$0 < x < 2$		$-$	$-$	Decreasing, concave downward
$x = 2$	Undef.	Undef.	Undef.	Vertical asymptote
$2 < x < 4$		$-$	$+$	Decreasing, concave upward
$x = 4$	6	0	$+$	Relative minimum
$4 < x < \infty$		$+$	$+$	Increasing, concave upward

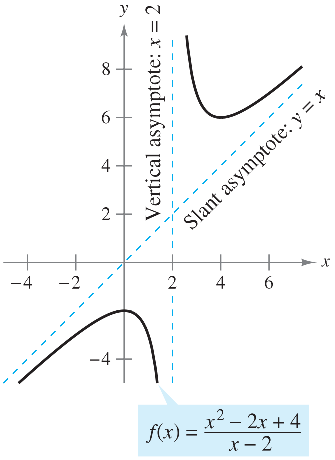
■

Although the graph of the function in Example 2 has no horizontal asymptote, it does have a slant asymptote. The graph of a rational function (having no common factors and whose denominator is of degree 1 or greater) has a **slant asymptote** if the degree of the numerator exceeds the degree of the denominator by exactly 1. To find the slant asymptote, use long division to rewrite the rational function as the sum of a first-degree polynomial and another rational function.

$$f(x) = \frac{x^2 - 2x + 4}{x - 2} \qquad \text{Write original equation.}$$

$$= x + \frac{4}{x - 2} \qquad \text{Rewrite using long division.}$$

In Figure 3.48, note that the graph of f approaches the slant asymptote $y = x$ as x approaches $-\infty$ or ∞.



A slant asymptote

Figure 3.48

EXAMPLE 3 Sketching the Graph of a Radical Function

Analyze and sketch the graph of $f(x) = \dfrac{x}{\sqrt{x^2 + 2}}$.

Solution

$$f'(x) = \frac{2}{(x^2 + 2)^{3/2}} \qquad f''(x) = -\frac{6x}{(x^2 + 2)^{5/2}}$$

The graph has only one intercept, $(0, 0)$. It has no vertical asymptotes, but it has two horizontal asymptotes: $y = 1$ (to the right) and $y = -1$ (to the left). The function has no critical numbers and one possible point of inflection (at $x = 0$). The domain of the function is all real numbers, and the graph is symmetric with respect to the origin. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.49.

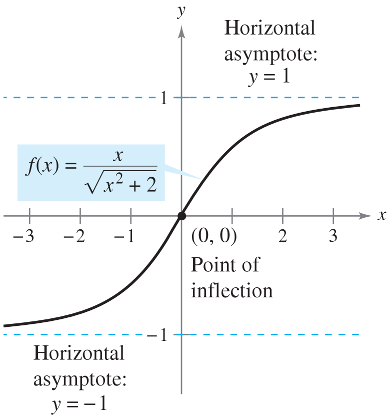


Figure 3.49

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		$+$	$+$	Increasing, concave upward
$x = 0$	0	$\dfrac{1}{\sqrt{2}}$	0	Point of inflection
$0 < x < \infty$		$+$	$-$	Increasing, concave downward

EXAMPLE 4 Sketching the Graph of a Radical Function

Analyze and sketch the graph of $f(x) = 2x^{5/3} - 5x^{4/3}$.

Solution

$$f'(x) = \frac{10}{3}x^{1/3}(x^{1/3} - 2) \qquad f''(x) = \frac{20(x^{1/3} - 1)}{9x^{2/3}}$$

The function has two intercepts: $(0, 0)$ and $\left(\frac{125}{8}, 0\right)$. There are no horizontal or vertical asymptotes. The function has two critical numbers ($x = 0$ and $x = 8$) and two possible points of inflection ($x = 0$ and $x = 1$). The domain is all real numbers. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.50.

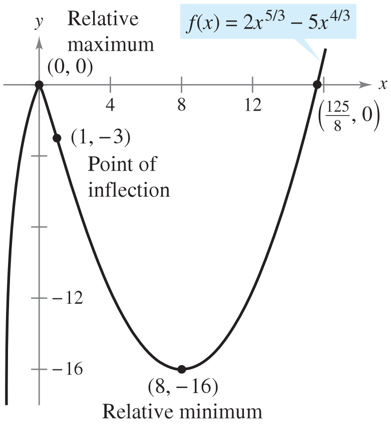


Figure 3.50

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		$+$	$-$	Increasing, concave downward
$x = 0$	0	0	Undef.	Relative maximum
$0 < x < 1$		$-$	$-$	Decreasing, concave downward
$x = 1$	-3	$-$	0	Point of inflection
$1 < x < 8$		$-$	$+$	Decreasing, concave upward
$x = 8$	-16	0	$+$	Relative minimum
$8 < x < \infty$		$+$	$+$	Increasing, concave upward

■

EXAMPLE 5 Sketching the Graph of a Polynomial Function

Analyze and sketch the graph of $f(x) = x^4 - 12x^3 + 48x^2 - 64x$.

Solution Begin by factoring to obtain

$$\begin{aligned} f(x) &= x^4 - 12x^3 + 48x^2 - 64x \\ &= x(x-4)^3. \end{aligned}$$

Then, using the factored form of $f(x)$, you can perform the following analysis.

First derivative: $f'(x) = 4(x-1)(x-4)^2$

Second derivative: $f''(x) = 12(x-4)(x-2)$

x-intercepts: $(0, 0), (4, 0)$

y-intercept: $(0, 0)$

Vertical asymptotes: None

Horizontal asymptotes: None

End behavior: $\lim_{x\to-\infty} f(x) = \infty, \ \lim_{x\to\infty} f(x) = \infty$

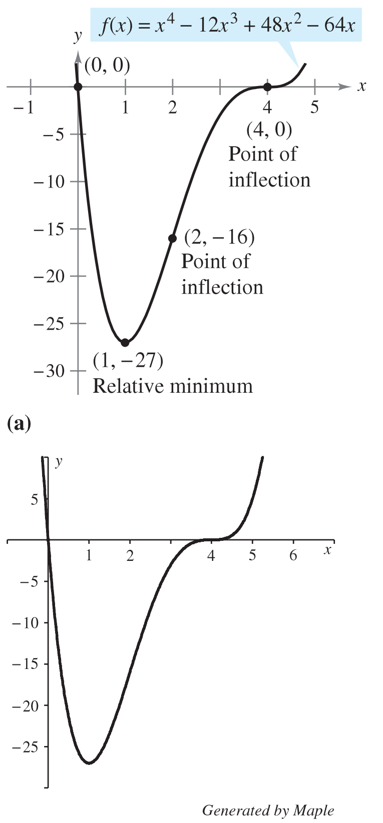
Critical numbers: $x = 1, x = 4$

Possible points of inflection: $x = 2, x = 4$

Domain: All real numbers

Test intervals: $(-\infty, 1), (1, 2), (2, 4), (4, \infty)$

The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.51(a). Using a computer algebra system such as *Maple* [see Figure 3.51(b)] can help you verify your analysis.



(a)

Generated by Maple

(b)

A polynomial function of even degree must have at least one relative extremum.

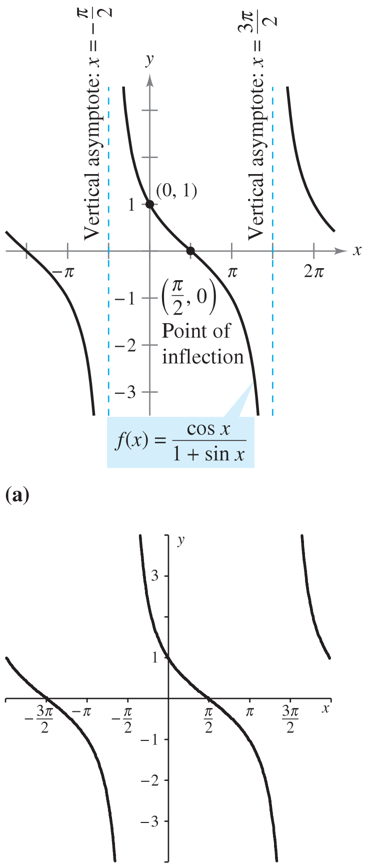
Figure 3.51

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 1$		−	+	Decreasing, concave upward
$x = 1$	−27	0	+	Relative minimum
$1 < x < 2$		+	+	Increasing, concave upward
$x = 2$	−16	+	0	Point of inflection
$2 < x < 4$		+	−	Increasing, concave downward
$x = 4$	0	0	0	Point of inflection
$4 < x < \infty$		+	+	Increasing, concave upward

■

The fourth-degree polynomial function in Example 5 has one relative minimum and no relative maxima. In general, a polynomial function of degree n can have *at most* $n - 1$ relative extrema, and *at most* $n - 2$ points of inflection. Moreover, polynomial functions of even degree must have *at least* one relative extremum.

Remember from the Leading Coefficient Test described in Section P.3 that the “end behavior” of the graph of a polynomial function is determined by its leading coefficient and its degree. For instance, because the polynomial in Example 5 has a positive leading coefficient, the graph rises to the right. Moreover, because the degree is even, the graph also rises to the left.



(a)

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(b)

Figure 3.52

EXAMPLE 6 Sketching the Graph of a Trigonometric Function

Analyze and sketch the graph of $f(x) = \dfrac{\cos x}{1 + \sin x}$.

Solution Because the function has a period of 2π, you can restrict the analysis of the graph to any interval of length 2π. For convenience, choose $(-\pi/2, 3\pi/2)$.

First derivative: $f'(x) = -\dfrac{1}{1 + \sin x}$

Second derivative: $f''(x) = \dfrac{\cos x}{(1 + \sin x)^2}$

Period: 2π

x-intercept: $\left(\dfrac{\pi}{2}, 0\right)$

y-intercept: $(0, 1)$

Vertical asymptotes: $x = -\dfrac{\pi}{2},\ x = \dfrac{3\pi}{2}$ See Note below.

Horizontal asymptotes: None

Critical numbers: None

Possible points of inflection: $x = \dfrac{\pi}{2}$

Domain: All real numbers except $x = \dfrac{3 + 4n}{2}\pi$

Test intervals: $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

The analysis of the graph of f on the interval $(-\pi/2, 3\pi/2)$ is shown in the table, and the graph is shown in Figure 3.52(a). Compare this with the graph generated by the computer algebra system *Maple* in Figure 3.52(b).

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$x = -\frac{\pi}{2}$	Undef.	Undef.	Undef.	Vertical asymptote
$-\frac{\pi}{2} < x < \frac{\pi}{2}$		−	+	Decreasing, concave upward
$x = \frac{\pi}{2}$	0	$-\frac{1}{2}$	0	Point of inflection
$\frac{\pi}{2} < x < \frac{3\pi}{2}$		−	−	Decreasing, concave downward
$x = \frac{3\pi}{2}$	Undef.	Undef.	Undef.	Vertical asymptote

NOTE By substituting $-\pi/2$ or $3\pi/2$ into the function, you obtain the form 0/0. This is called an indeterminate form, which you will study in Section 8.7. To determine that the function has vertical asymptotes at these two values, you can rewrite the function as follows.

$$f(x) = \frac{\cos x}{1 + \sin x} = \frac{(\cos x)(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} = \frac{(\cos x)(1 - \sin x)}{\cos^2 x} = \frac{1 - \sin x}{\cos x}$$

In this form, it is clear that the graph of f has vertical asymptotes at $x = -\pi/2$ and $3\pi/2$.

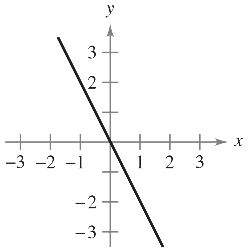
3.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

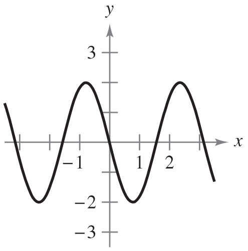
In Exercises 1–4, match the graph of f in the left column with that of its derivative in the right column.

Graph of f *Graph of f'*

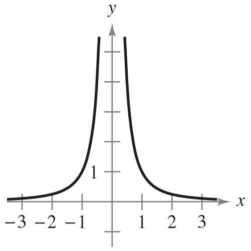
1.



(a)



2.



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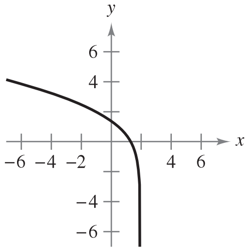
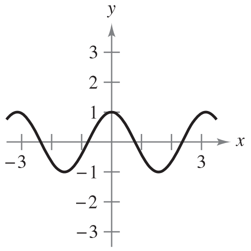
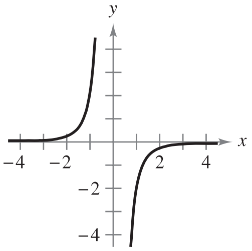


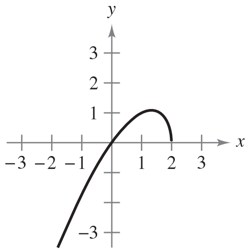
3.



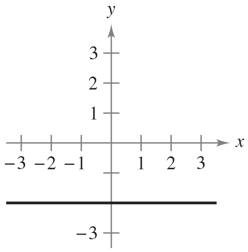
(c)



4.



(d)



In Exercises 5–32, analyze and sketch a graph of the function. Label any intercepts, relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.

5. $y = \dfrac{1}{x-2} - 3$

6. $y = \dfrac{x}{x^2+1}$

7. $y = \dfrac{x^2}{x^2+3}$

8. $y = \dfrac{x^2+1}{x^2-4}$

9. $y = \dfrac{3x}{x^2-1}$

10. $f(x) = \dfrac{x-3}{x}$

11. $g(x) = x - \dfrac{8}{x^2}$

12. $f(x) = x + \dfrac{32}{x^2}$

13. $f(x) = \dfrac{x^2+1}{x}$

14. $f(x) = \dfrac{x^3}{x^2-9}$

15. $y = \dfrac{x^2-6x+12}{x-4}$

16. $y = \dfrac{2x^2-5x+5}{x-2}$

17. $y = x\sqrt{4-x}$

18. $g(x) = x\sqrt{9-x}$

19. $h(x) = x\sqrt{4-x^2}$

20. $g(x) = x\sqrt{9-x^2}$

21. $y = 3x^{2/3} - 2x$

22. $y = 3(x-1)^{2/3} - (x-1)^2$

23. $y = x^3 - 3x^2 + 3$

24. $y = -\frac{1}{3}(x^3 - 3x + 2)$

25. $y = 2 - x - x^3$

26. $f(x) = \frac{1}{3}(x-1)^3 + 2$

27. $y = 3x^4 + 4x^3$

28. $y = 3x^4 - 6x^2 + \frac{5}{3}$

29. $y = x^5 - 5x$

30. $y = (x-1)^5$

31. $y = |2x-3|$

32. $y = |x^2 - 6x + 5|$

CAS **In Exercises 33–36, use a computer algebra system to analyze and graph the function. Identify any relative extrema, points of inflection, and asymptotes.**

33. $f(x) = \dfrac{20x}{x^2+1} - \dfrac{1}{x}$

34. $f(x) = x + \dfrac{4}{x^2+1}$

35. $f(x) = \dfrac{-2x}{\sqrt{x^2+7}}$

36. $f(x) = \dfrac{4x}{\sqrt{x^2+15}}$

In Exercises 37–46, sketch a graph of the function over the given interval. Use a graphing utility to verify your graph.

37. $f(x) = 2x - 4\sin x, \quad 0 \le x \le 2\pi$

38. $f(x) = -x + 2\cos x, \quad 0 \le x \le 2\pi$

39. $y = \sin x - \frac{1}{18}\sin 3x, \quad 0 \le x \le 2\pi$

40. $y = \cos x - \frac{1}{4}\cos 2x, \quad 0 \le x \le 2\pi$

41. $y = 2x - \tan x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

42. $y = 2(x-2) + \cot x, \quad 0 < x < \pi$

43. $y = 2(\csc x + \sec x), \quad 0 < x < \dfrac{\pi}{2}$

44. $y = \sec^2\left(\dfrac{\pi x}{8}\right) - 2\tan\left(\dfrac{\pi x}{8}\right) - 1, \quad -3 < x < 3$

45. $g(x) = x\tan x, \quad -\dfrac{3\pi}{2} < x < \dfrac{3\pi}{2}$

46. $g(x) = x\cot x, \quad -2\pi < x < 2\pi$

WRITING ABOUT CONCEPTS

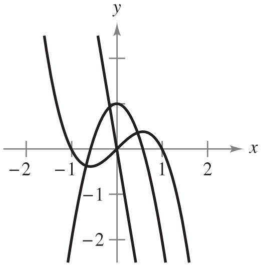
47. Suppose $f'(t) < 0$ for all t in the interval $(2, 8)$. Explain why $f(3) > f(5)$.

48. Suppose $f(0) = 3$ and $2 \le f'(x) \le 4$ for all x in the interval $[-5, 5]$. Determine the greatest and least possible values of $f(2)$.

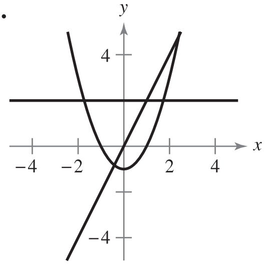
WRITING ABOUT CONCEPTS (continued)

In Exercises 49 and 50, the graphs of f, f', and f'' are shown on the same set of coordinate axes. Which is which? Explain your reasoning. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

49.



50.



In Exercises 51–54, use a graphing utility to graph the function. Use the graph to determine whether it is possible for the graph of a function to cross its horizontal asymptote. Do you think it is possible for the graph of a function to cross its vertical asymptote? Why or why not?

51. $f(x) = \dfrac{4(x-1)^2}{x^2-4x+5}$

52. $g(x) = \dfrac{3x^4-5x+3}{x^4+1}$

53. $h(x) = \dfrac{\sin 2x}{x}$

54. $f(x) = \dfrac{\cos 3x}{4x}$

In Exercises 55 and 56, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function may indicate that there should be one.

55. $h(x) = \dfrac{6-2x}{3-x}$

56. $g(x) = \dfrac{x^2+x-2}{x-1}$

In Exercises 57–60, use a graphing utility to graph the function and determine the slant asymptote of the graph. Zoom out repeatedly and describe how the graph on the display appears to change. Why does this occur?

57. $f(x) = -\dfrac{x^2-3x-1}{x-2}$

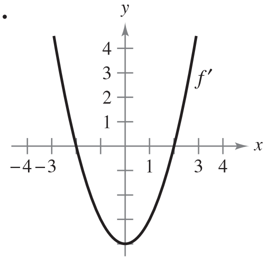
58. $g(x) = \dfrac{2x^2-8x-15}{x-5}$

59. $f(x) = \dfrac{2x^3}{x^2+1}$

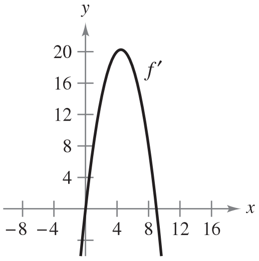
60. $h(x) = \dfrac{-x^3+x^2+4}{x^2}$

Graphical Reasoning **In Exercises 61–64, use the graph of f' to sketch a graph of f and the graph of f''. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

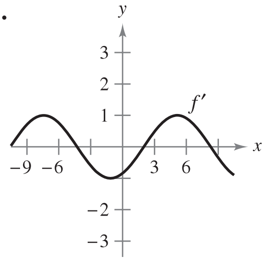
61.



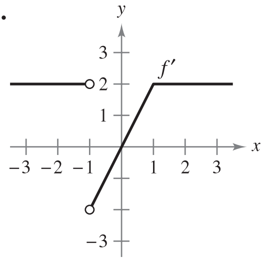
62.



63.



64.



(Submitted by Bill Fox, Moberly Area Community College, Moberly, MO)

CAS **65.** ***Graphical Reasoning*** Consider the function

$$f(x) = \frac{\cos^2 \pi x}{\sqrt{x^2+1}}, \quad 0 < x < 4.$$

(a) Use a computer algebra system to graph the function and use the graph to approximate the critical numbers visually.

(b) Use a computer algebra system to find f' and approximate the critical numbers. Are the results the same as the visual approximation in part (a)? Explain.

66. ***Graphical Reasoning*** Consider the function

$$f(x) = \tan(\sin \pi x).$$

(a) Use a graphing utility to graph the function.

(b) Identify any symmetry of the graph.

(c) Is the function periodic? If so, what is the period?

(d) Identify any extrema on $(-1, 1)$.

(e) Use a graphing utility to determine the concavity of the graph on $(0, 1)$.

Think About It **In Exercises 67–70, create a function whose graph has the given characteristics. (There is more than one correct answer.)**

67. Vertical asymptote: $x = 3$
Horizontal asymptote: $y = 0$

68. Vertical asymptote: $x = -5$
Horizontal asymptote: None

69. Vertical asymptote: $x = 3$
Slant asymptote: $y = 3x + 2$

70. Vertical asymptote: $x = 2$
Slant asymptote: $y = -x$

71. ***Graphical Reasoning*** The graph of f is shown in the figure on the next page.

(a) For which values of x is $f'(x)$ zero? Positive? Negative?

(b) For which values of x is $f''(x)$ zero? Positive? Negative?

(c) On what interval is f' an increasing function?

(d) For which value of x is $f'(x)$ minimum? For this value of x, how does the rate of change of f compare with the rates of change of f for other values of x? Explain.

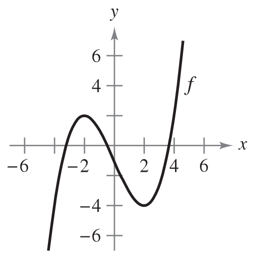


Figure for 71

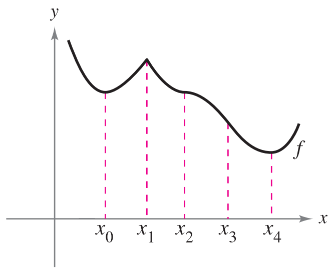


Figure for 72

CAPSTONE

72. ***Graphical Reasoning*** Identify the real numbers $x_0, x_1, x_2, x_3,$ and x_4 in the figure such that each of the following is true.

(a) $f'(x) = 0$ (b) $f''(x) = 0$

(c) $f'(x)$ does not exist. (d) f has a relative maximum.

(e) f has a point of inflection.

73. ***Graphical Reasoning*** Consider the function

$$f(x) = \frac{ax}{(x-b)^2}.$$

Determine the effect on the graph of f as a and b are changed. Consider cases where a and b are both positive or both negative, and cases where a and b have opposite signs.

74. Consider the function $f(x) = \frac{1}{2}(ax)^2 - ax,\ a \neq 0.$

(a) Determine the changes (if any) in the intercepts, extrema, and concavity of the graph of f when a is varied.

(b) In the same viewing window, use a graphing utility to graph the function for four different values of a.

75. ***Investigation*** Consider the function

$$f(x) = \frac{2x^n}{x^4 + 1}$$

for nonnegative integer values of n.

(a) Discuss the relationship between the value of n and the symmetry of the graph.

(b) For which values of n will the x-axis be the horizontal asymptote?

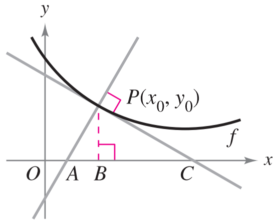
(c) For which value of n will $y = 2$ be the horizontal asymptote?

(d) What is the asymptote of the graph when $n = 5$?

(e) Use a graphing utility to graph f for the indicated values of n in the table. Use the graph to determine the number of extrema M and the number of inflection points N of the graph.

n	0	1	2	3	4	5
M						
N						

76. ***Investigation*** Let $P(x_0, y_0)$ be an arbitrary point on the graph of f such that $f'(x_0) \neq 0$, as shown in the figure. Verify each statement.



(a) The x-intercept of the tangent line is $\left(x_0 - \dfrac{f(x_0)}{f'(x_0)}, 0\right)$.

(b) The y-intercept of the tangent line is $(0, f(x_0) - x_0 f'(x_0))$.

(c) The x-intercept of the normal line is $(x_0 + f(x_0)f'(x_0), 0)$.

(d) The y-intercept of the normal line is $\left(0, y_0 + \dfrac{x_0}{f'(x_0)}\right)$.

(e) $|BC| = \left|\dfrac{f(x_0)}{f'(x_0)}\right|$ (f) $|PC| = \left|\dfrac{f(x_0)\sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|$

(g) $|AB| = |f(x_0)f'(x_0)|$

(h) $|AP| = |f(x_0)|\sqrt{1 + [f'(x_0)]^2}$

77. ***Modeling Data*** The data in the table show the number N of bacteria in a culture at time t, where t is measured in days.

t	1	2	3	4	5	6	7	8
N	25	200	804	1756	2296	2434	2467	2473

A model for these data is given by

$$N = \frac{24{,}670 - 35{,}153t + 13{,}250t^2}{100 - 39t + 7t^2}, \quad 1 \le t \le 8.$$

(a) Use a graphing utility to plot the data and graph the model.

(b) Use the model to estimate the number of bacteria when $t = 10$.

(c) Approximate the day when the number of bacteria is greatest.

CAS (d) Use a computer algebra system to determine the time when the rate of increase in the number of bacteria is greatest.

(e) Find $\lim_{t\to\infty} N(t)$.

Slant Asymptotes **In Exercises 78 and 79, the graph of the function has two slant asymptotes. Identify each slant asymptote. Then graph the function and its asymptotes.**

78. $y = \sqrt{4 + 16x^2}$ **79.** $y = \sqrt{x^2 + 6x}$

PUTNAM EXAM CHALLENGE

80. Let $f(x)$ be defined for $a \le x \le b$. Assuming appropriate properties of continuity and derivability, prove for $a < x < b$ that

$$\frac{\dfrac{f(x) - f(a)}{x - a} - \dfrac{f(b) - f(a)}{b - a}}{x - b} = \frac{1}{2}f''(\beta)$$

where β is some number between a and b.

This problem was composed by the Committee on the Putnam Prize Competition.
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3.7 Optimization Problems

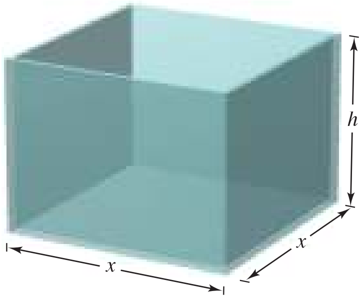
- **Solve applied minimum and maximum problems.**

Applied Minimum and Maximum Problems

One of the most common applications of calculus involves the determination of minimum and maximum values. Consider how frequently you hear or read terms such as greatest profit, least cost, least time, greatest voltage, optimum size, least size, greatest strength, and greatest distance. Before outlining a general problem-solving strategy for such problems, let's look at an example.

EXAMPLE 1 Finding Maximum Volume

A manufacturer wants to design an open box having a square base and a surface area of 108 square inches, as shown in Figure 3.53. What dimensions will produce a box with maximum volume?



Open box with square base:
$S = x^2 + 4xh = 108$
Figure 3.53

Solution Because the box has a square base, its volume is

$$V = x^2h. \qquad \text{Primary equation}$$

This equation is called the **primary equation** because it gives a formula for the quantity to be optimized. The surface area of the box is

$$S = (\text{area of base}) + (\text{area of four sides})$$
$$S = x^2 + 4xh = 108. \qquad \text{Secondary equation}$$

Because V is to be maximized, you want to write V as a function of just one variable. To do this, you can solve the equation $x^2 + 4xh = 108$ for h in terms of x to obtain $h = (108 - x^2)/(4x)$. Substituting into the primary equation produces

$$\begin{aligned} V &= x^2h && \text{Function of two variables} \\ &= x^2\left(\frac{108 - x^2}{4x}\right) && \text{Substitute for } h. \\ &= 27x - \frac{x^3}{4}. && \text{Function of one variable} \end{aligned}$$

Before finding which x-value will yield a maximum value of V, you should determine the *feasible domain*. That is, what values of x make sense in this problem? You know that $V \geq 0$. You also know that x must be nonnegative and that the area of the base $(A = x^2)$ is at most 108. So, the feasible domain is

$$0 \leq x \leq \sqrt{108}. \qquad \text{Feasible domain}$$

To maximize V, find the critical numbers of the volume function on the interval $\left(0, \sqrt{108}\right)$.

$$\begin{aligned} \frac{dV}{dx} = 27 - \frac{3x^2}{4} &= 0 && \text{Set derivative equal to 0.} \\ 3x^2 &= 108 && \text{Simplify.} \\ x &= \pm 6 && \text{Critical numbers} \end{aligned}$$

So, the critical numbers are $x = \pm 6$. You do not need to consider $x = -6$ because it is outside the domain. Evaluating V at the critical number $x = 6$ and at the endpoints of the domain produces $V(0) = 0$, $V(6) = 108$, and $V\left(\sqrt{108}\right) = 0$. So, V is maximum when $x = 6$ and the dimensions of the box are $6 \times 6 \times 3$ inches. ■

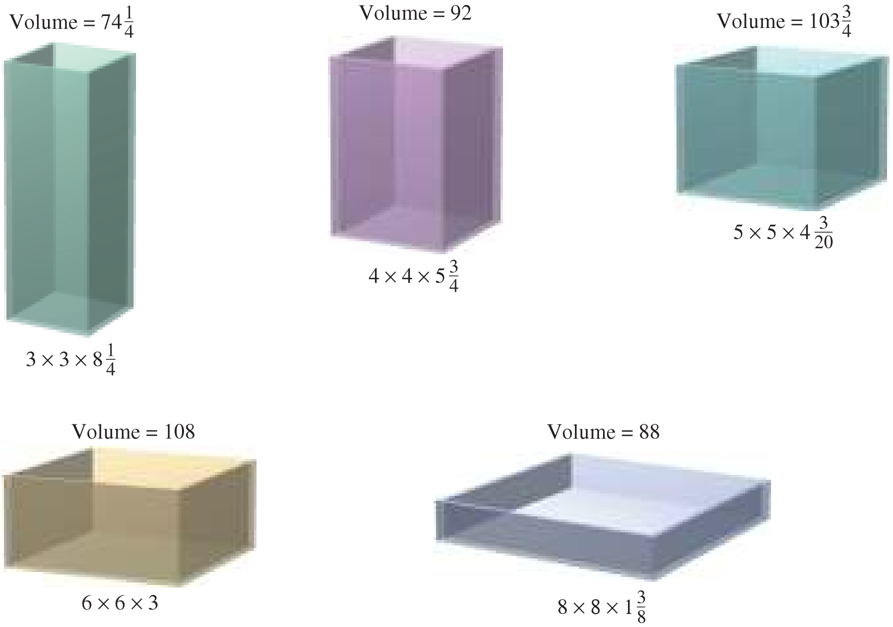
TECHNOLOGY You can verify your answer in Example 1 by using a graphing utility to graph the volume function

$$V = 27x - \frac{x^3}{4}.$$

Use a viewing window in which $0 \leq x \leq \sqrt{108} \approx 10.4$ and $0 \leq y \leq 120$, and use the *trace* feature to determine the maximum value of V.

In Example 1, you should realize that there are infinitely many open boxes having 108 square inches of surface area. To begin solving the problem, you might ask yourself which basic shape would seem to yield a maximum volume. Should the box be tall, squat, or nearly cubical?

You might even try calculating a few volumes, as shown in Figure 3.54, to see if you can get a better feeling for what the optimum dimensions should be. Remember that you are not ready to begin solving a problem until you have clearly identified what the problem is.


Volume = $74\frac{1}{4}$
$3 \times 3 \times 8\frac{1}{4}$
Volume = 92
$4 \times 4 \times 5\frac{3}{4}$
Volume = $103\frac{3}{4}$
$5 \times 5 \times 4\frac{3}{20}$
Volume = 108
$6 \times 6 \times 3$
Volume = 88
$8 \times 8 \times 1\frac{3}{8}$

Which box has the greatest volume?
Figure 3.54

Example 1 illustrates the following guidelines for solving applied minimum and maximum problems.

GUIDELINES FOR SOLVING APPLIED MINIMUM AND MAXIMUM PROBLEMS

1. Identify all *given* quantities and all quantities *to be determined*. If possible, make a sketch.
2. Write a **primary equation** for the quantity that is to be maximized or minimized. (A review of several useful formulas from geometry is presented inside the back cover.)
3. Reduce the primary equation to one having a *single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques discussed in Sections 3.1 through 3.4.

NOTE When performing Step 5, recall that to determine the maximum or minimum value of a continuous function f on a closed interval, you should compare the values of f at its critical numbers with the values of f at the endpoints of the interval.

EXAMPLE 2 Finding Minimum Distance

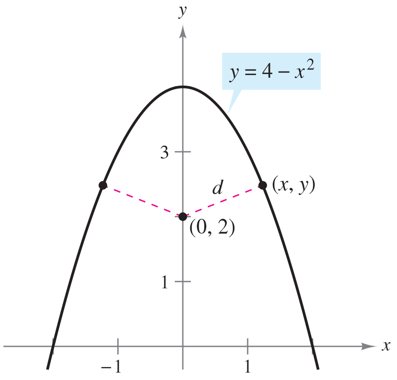
Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?

Solution Figure 3.55 shows that there are two points at a minimum distance from the point $(0, 2)$. The distance between the point $(0, 2)$ and a point (x, y) on the graph of $y = 4 - x^2$ is given by

$$d = \sqrt{(x-0)^2 + (y-2)^2}. \qquad \text{Primary equation}$$

Using the secondary equation $y = 4 - x^2$, you can rewrite the primary equation as

$$d = \sqrt{x^2 + (4 - x^2 - 2)^2} = \sqrt{x^4 - 3x^2 + 4}.$$



The quantity to be minimized is distance: $d = \sqrt{(x-0)^2 + (y-2)^2}$.
Figure 3.55

Because d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of $f(x) = x^4 - 3x^2 + 4$. Note that the domain of f is the entire real line. So, there are no endpoints of the domain to consider. Moreover, setting $f'(x)$ equal to 0 yields

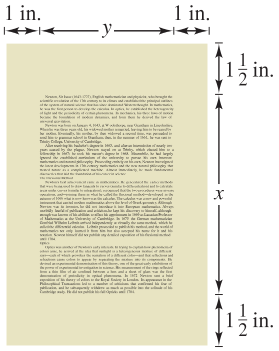
$$f'(x) = 4x^3 - 6x = 2x(2x^2 - 3) = 0$$

$$x = 0, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}.$$

The First Derivative Test verifies that $x = 0$ yields a relative maximum, whereas both $x = \sqrt{3/2}$ and $x = -\sqrt{3/2}$ yield a minimum distance. So, the closest points are $\left(\sqrt{3/2}, 5/2\right)$ and $\left(-\sqrt{3/2}, 5/2\right)$.

EXAMPLE 3 Finding Minimum Area

A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be $1\frac{1}{2}$ inches, and the margins on the left and right are to be 1 inch (see Figure 3.56). What should the dimensions of the page be so that the least amount of paper is used?



The quantity to be minimized is area: $A = (x + 3)(y + 2)$.
Figure 3.56

Solution Let A be the area to be minimized.

$$A = (x + 3)(y + 2) \qquad \text{Primary equation}$$

The printed area inside the margins is given by

$$24 = xy. \qquad \text{Secondary equation}$$

Solving this equation for y produces $y = 24/x$. Substitution into the primary equation produces

$$A = (x + 3)\left(\frac{24}{x} + 2\right) = 30 + 2x + \frac{72}{x}. \qquad \text{Function of one variable}$$

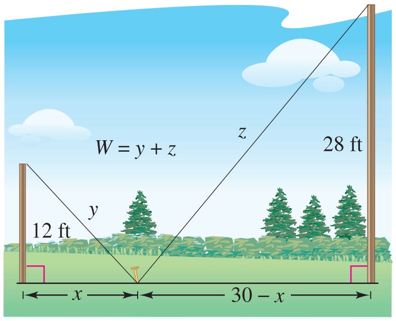
Because x must be positive, you are interested only in values of A for $x > 0$. To find the critical numbers, differentiate with respect to x.

$$\frac{dA}{dx} = 2 - \frac{72}{x^2} = 0 \quad \Rightarrow \quad x^2 = 36$$

So, the critical numbers are $x = \pm 6$. You do not have to consider $x = -6$ because it is outside the domain. The First Derivative Test confirms that A is a minimum when $x = 6$. So, $y = \frac{24}{6} = 4$ and the dimensions of the page should be $x + 3 = 9$ inches by $y + 2 = 6$ inches. ■

EXAMPLE 4 Finding Minimum Length

Two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least amount of wire?



The quantity to be minimized is length. From the diagram, you can see that x varies between 0 and 30.
Figure 3.57

Solution Let W be the wire length to be minimized. Using Figure 3.57, you can write

$$W = y + z. \qquad \text{Primary equation}$$

In this problem, rather than solving for y in terms of z (or vice versa), you can solve for both y and z in terms of a third variable x, as shown in Figure 3.57. From the Pythagorean Theorem, you obtain

$$\begin{aligned} x^2 + 12^2 &= y^2 \\ (30 - x)^2 + 28^2 &= z^2 \end{aligned}$$

which implies that

$$\begin{aligned} y &= \sqrt{x^2 + 144} \\ z &= \sqrt{x^2 - 60x + 1684}. \end{aligned}$$

So, W is given by

$$\begin{aligned} W &= y + z \\ &= \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}, \quad 0 \le x \le 30. \end{aligned}$$

Differentiating W with respect to x yields

$$\frac{dW}{dx} = \frac{x}{\sqrt{x^2 + 144}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}}.$$

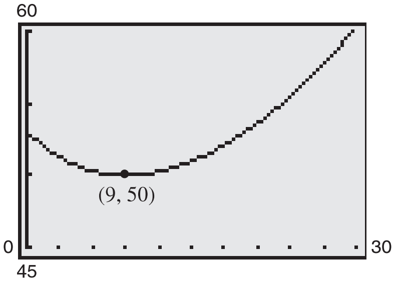
By letting $dW/dx = 0$, you obtain

$$\begin{aligned} \frac{x}{\sqrt{x^2 + 144}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}} &= 0 \\ x\sqrt{x^2 - 60x + 1684} &= (30 - x)\sqrt{x^2 + 144} \\ x^2(x^2 - 60x + 1684) &= (30 - x)^2(x^2 + 144) \\ x^4 - 60x^3 + 1684x^2 &= x^4 - 60x^3 + 1044x^2 - 8640x + 129{,}600 \\ 640x^2 + 8640x - 129{,}600 &= 0 \\ 320(x - 9)(2x + 45) &= 0 \\ x &= 9, -22.5. \end{aligned}$$

Because $x = -22.5$ is not in the domain and

$$W(0) \approx 53.04, \quad W(9) = 50, \quad \text{and} \quad W(30) \approx 60.31$$

you can conclude that the wire should be staked at 9 feet from the 12-foot pole. ■



You can confirm the minimum value of W with a graphing utility.
Figure 3.58

TECHNOLOGY From Example 4, you can see that applied optimization problems can involve a lot of algebra. If you have access to a graphing utility, you can confirm that $x = 9$ yields a minimum value of W by graphing

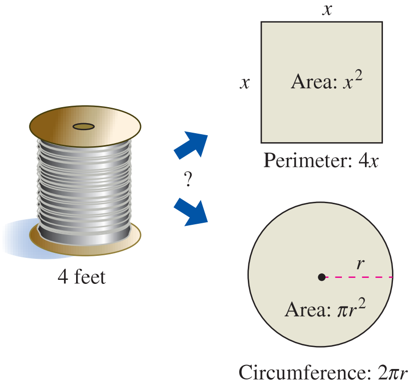
$$W = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}$$

as shown in Figure 3.58.

In each of the first four examples, the extreme value occurred at a critical number. Although this happens often, remember that an extreme value can also occur at an endpoint of an interval, as shown in Example 5.

EXAMPLE 5 An Endpoint Maximum

Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?



The quantity to be maximized is area: $A = x^2 + \pi r^2$.
Figure 3.59

Solution The total area (see Figure 3.59) is given by

$$A = (\text{area of square}) + (\text{area of circle})$$
$$A = x^2 + \pi r^2. \qquad \text{Primary equation}$$

Because the total length of wire is 4 feet, you obtain

$$4 = (\text{perimeter of square}) + (\text{circumference of circle})$$
$$4 = 4x + 2\pi r.$$

So, $r = 2(1 - x)/\pi$, and by substituting into the primary equation you have

$$\begin{aligned} A &= x^2 + \pi\left[\frac{2(1 - x)}{\pi}\right]^2 \\ &= x^2 + \frac{4(1 - x)^2}{\pi} \\ &= \frac{1}{\pi}[(\pi + 4)x^2 - 8x + 4]. \end{aligned}$$

The feasible domain is $0 \le x \le 1$ restricted by the square's perimeter. Because

$$\frac{dA}{dx} = \frac{2(\pi + 4)x - 8}{\pi}$$

the only critical number in $(0, 1)$ is $x = 4/(\pi + 4) \approx 0.56$. So, using

$$A(0) \approx 1.273, \qquad A(0.56) \approx 0.56, \qquad \text{and} \qquad A(1) = 1$$

you can conclude that the maximum area occurs when $x = 0$. That is, *all* the wire is used for the circle. ■

EXPLORATION

What would the answer be if Example 5 asked for the dimensions needed to enclose the *minimum* total area?

Let's review the primary equations developed in the first five examples. As applications go, these five examples are fairly simple, and yet the resulting primary equations are quite complicated.

$$V = 27x - \frac{x^3}{4} \qquad W = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}$$

$$d = \sqrt{x^4 - 3x^2 + 4} \qquad A = \frac{1}{\pi}[(\pi + 4)x^2 - 8x + 4]$$

$$A = 30 + 2x + \frac{72}{x}$$

You must expect that real-life applications often involve equations that are *at least as complicated* as these five. Remember that one of the main goals of this course is to learn to use calculus to analyze equations that initially seem formidable.

3.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. ***Numerical, Graphical, and Analytic Analysis*** Find two positive numbers whose sum is 110 and whose product is a maximum.

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

First Number x	Second Number	Product P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$

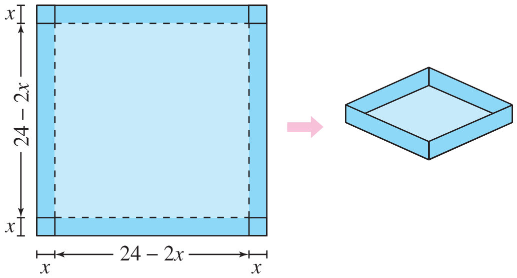
(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the solution. (*Hint:* Use the *table* feature of the graphing utility.)

(c) Write the product P as a function of x.

(d) Use a graphing utility to graph the function in part (c) and estimate the solution from the graph.

(e) Use calculus to find the critical number of the function in part (c). Then find the two numbers.

2. ***Numerical, Graphical, and Analytic Analysis*** An open box of maximum volume is to be made from a square piece of material, 24 inches on a side, by cutting equal squares from the corners and turning up the sides (see figure).



(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.) Use the table to guess the maximum volume.

Height x	Length and Width	Volume V
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$

(b) Write the volume V as a function of x.

(c) Use calculus to find the critical number of the function in part (b) and find the maximum value.

(d) Use a graphing utility to graph the function in part (b) and verify the maximum volume from the graph.

In Exercises 3–8, find two positive numbers that satisfy the given requirements.

3. The sum is S and the product is a maximum.

4. The product is 185 and the sum is a minimum.

5. The product is 147 and the sum of the first number plus three times the second number is a minimum.

6. The second number is the reciprocal of the first number and the sum is a minimum.

7. The sum of the first number and twice the second number is 108 and the product is a maximum.

8. The sum of the first number squared and the second number is 54 and the product is a maximum.

In Exercises 9 and 10, find the length and width of a rectangle that has the given perimeter and a maximum area.

9. Perimeter: 80 meters

10. Perimeter: P units

In Exercises 11 and 12, find the length and width of a rectangle that has the given area and a minimum perimeter.

11. Area: 32 square feet

12. Area: A square centimeters

In Exercises 13–16, find the point on the graph of the function that is closest to the given point.

	Function	*Point*		*Function*	*Point*
13.	$f(x) = x^2$	$\left(2, \frac{1}{2}\right)$	**14.**	$f(x) = (x - 1)^2$	$(-5, 3)$
15.	$f(x) = \sqrt{x}$	$(4, 0)$	**16.**	$f(x) = \sqrt{x - 8}$	$(12, 0)$

17. ***Area*** A rectangular page is to contain 30 square inches of print. The margins on each side are 1 inch. Find the dimensions of the page such that the least amount of paper is used.

18. ***Area*** A rectangular page is to contain 36 square inches of print. The margins on each side are $1\frac{1}{2}$ inches. Find the dimensions of the page such that the least amount of paper is used.

19. ***Chemical Reaction*** In an autocatalytic chemical reaction, the product formed is a catalyst for the reaction. If Q_0 is the amount of the original substance and x is the amount of catalyst formed, the rate of chemical reaction is

$$\frac{dQ}{dx} = kx(Q_0 - x).$$

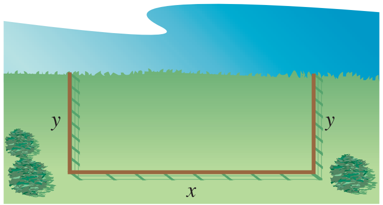
For what value of x will the rate of chemical reaction be greatest?

20. ***Traffic Control*** On a given day, the flow rate F (cars per hour) on a congested roadway is

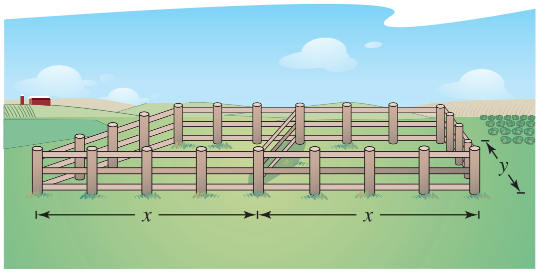
$$F = \frac{v}{22 + 0.02v^2}$$

where v is the speed of the traffic in miles per hour. What speed will maximize the flow rate on the road?

21. ***Area*** A farmer plans to fence a rectangular pasture adjacent to a river (see figure). The pasture must contain 245,000 square meters in order to provide enough grass for the herd. What dimensions will require the least amount of fencing if no fencing is needed along the river?



22. ***Maximum Area*** A rancher has 400 feet of fencing with which to enclose two adjacent rectangular corrals (see figure). What dimensions should be used so that the enclosed area will be a maximum?

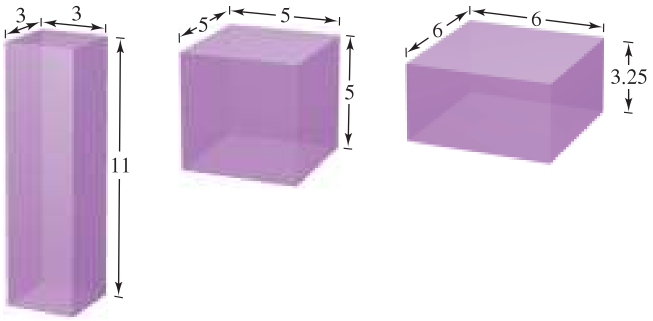


23. ***Maximum Volume***

(a) Verify that each of the rectangular solids shown in the figure has a surface area of 150 square inches.

(b) Find the volume of each solid.

(c) Determine the dimensions of a rectangular solid (with a square base) of maximum volume if its surface area is 150 square inches.



24. ***Maximum Volume*** Determine the dimensions of a rectangular solid (with a square base) with maximum volume if its surface area is 337.5 square centimeters.

25. ***Maximum Area*** A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window (see figure). Find the dimensions of a Norman window of maximum area if the total perimeter is 16 feet.

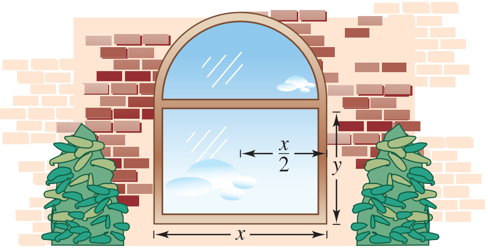
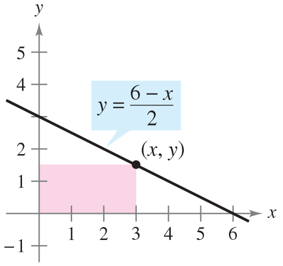


Figure for 25

26. ***Maximum Area*** A rectangle is bounded by the x- and y-axes and the graph of $y = (6 - x)/2$ (see figure). What length and width should the rectangle have so that its area is a maximum?



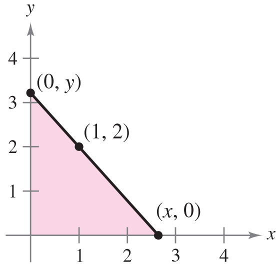


Figure for 26 **Figure for 27**

27. ***Minimum Length*** A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(1, 2)$ (see figure).

(a) Write the length L of the hypotenuse as a function of x.

(b) Use a graphing utility to approximate x graphically such that the length of the hypotenuse is a minimum.

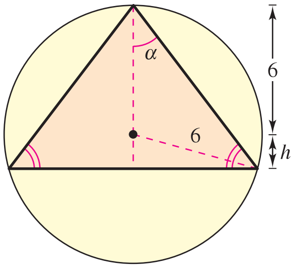
(c) Find the vertices of the triangle such that its area is a minimum.

28. ***Maximum Area*** Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 6 (see figure).

(a) Solve by writing the area as a function of h.

(b) Solve by writing the area as a function of α.

(c) Identify the type of triangle of maximum area.



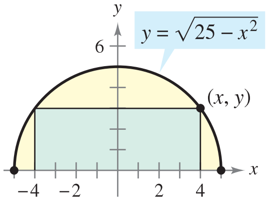


Figure for 28 **Figure for 29**

29. ***Maximum Area*** A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$ (see figure). What length and width should the rectangle have so that its area is a maximum?

30. ***Area*** Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius r (see Exercise 29).

31. ***Numerical, Graphical, and Analytic Analysis*** An exercise room consists of a rectangle with a semicircle on each end. A 200-meter running track runs around the outside of the room.

(a) Draw a figure to represent the problem. Let x and y represent the length and width of the rectangle.

(b) Analytically complete six rows of a table such as the one below. (The first two rows are shown.) Use the table to guess the maximum area of the rectangular region.

Length x	Width y	Area xy
10	$\frac{2}{\pi}(100 - 10)$	$(10)\frac{2}{\pi}(100 - 10) \approx 573$
20	$\frac{2}{\pi}(100 - 20)$	$(20)\frac{2}{\pi}(100 - 20) \approx 1019$

(c) Write the area A as a function of x.

(d) Use calculus to find the critical number of the function in part (c) and find the maximum value.

(e) Use a graphing utility to graph the function in part (c) and verify the maximum area from the graph.

32. ***Numerical, Graphical, and Analytic Analysis*** A right circular cylinder is to be designed to hold 22 cubic inches of a soft drink (approximately 12 fluid ounces).

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

Radius r	Height	Surface Area S
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$

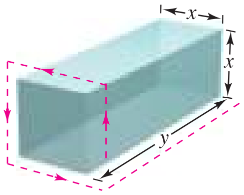
(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the minimum surface area. (*Hint:* Use the *table* feature of the graphing utility.)

(c) Write the surface area S as a function of r.

(d) Use a graphing utility to graph the function in part (c) and estimate the minimum surface area from the graph.

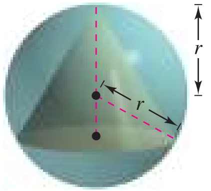
(e) Use calculus to find the critical number of the function in part (c) and find dimensions that will yield the minimum surface area.

33. ***Maximum Volume*** A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches (see figure). Find the dimensions of the package of maximum volume that can be sent. (Assume the cross section is square.)



34. ***Maximum Volume*** Rework Exercise 33 for a cylindrical package. (The cross section is circular.)

35. ***Maximum Volume*** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius r.



36. ***Maximum Volume*** Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius r.

WRITING ABOUT CONCEPTS

37. A shampoo bottle is a right circular cylinder. Because the surface area of the bottle does not change when it is squeezed, is it true that the volume remains the same? Explain.

CAPSTONE

38. The perimeter of a rectangle is 20 feet. Of all possible dimensions, the maximum area is 25 square feet when its length and width are both 5 feet. Are there dimensions that yield a minimum area? Explain.

39. ***Minimum Surface Area*** A solid is formed by adjoining two hemispheres to the ends of a right circular cylinder. The total volume of the solid is 14 cubic centimeters. Find the radius of the cylinder that produces the minimum surface area.

40. ***Minimum Cost*** An industrial tank of the shape described in Exercise 39 must have a volume of 4000 cubic feet. The hemispherical ends cost twice as much per square foot of surface area as the sides. Find the dimensions that will minimize cost.

41. ***Minimum Area*** The sum of the perimeters of an equilateral triangle and a square is 10. Find the dimensions of the triangle and the square that produce a minimum total area.

42. ***Maximum Area*** Twenty feet of wire is to be used to form two figures. In each of the following cases, how much wire should be used for each figure so that the total enclosed area is maximum?

(a) Equilateral triangle and square

(b) Square and regular pentagon

(c) Regular pentagon and regular hexagon

(d) Regular hexagon and circle

What can you conclude from this pattern? {*Hint:* The area of a regular polygon with n sides of length x is $A = (n/4)[\cot(\pi/n)]x^2$.}

43. ***Beam Strength*** A wooden beam has a rectangular cross section of height h and width w (see figure on the next page). The strength S of the beam is directly proportional to the width and the square of the height. What are the dimensions of the strongest beam that can be cut from a round log of diameter 20 inches? (*Hint*: $S = kh^2w$, where k is the proportionality constant.)

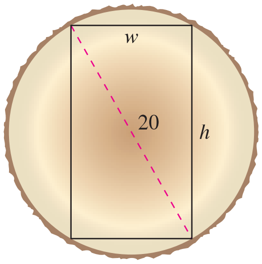


Figure for 43

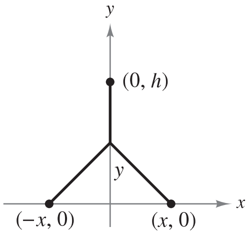


Figure for 44

44. ***Minimum Length*** Two factories are located at the coordinates $(-x, 0)$ and $(x, 0)$, and their power supply is at $(0, h)$ (see figure). Find y such that the total length of power line from the power supply to the factories is a minimum.

45. ***Projectile Range*** The range R of a projectile fired with an initial velocity v_0 at an angle θ with the horizontal is $R = \dfrac{v_0^2 \sin 2\theta}{g}$, where g is the acceleration due to gravity. Find the angle θ such that the range is a maximum.

46. ***Conjecture*** Consider the functions $f(x) = \frac{1}{2}x^2$ and $g(x) = \frac{1}{16}x^4 - \frac{1}{2}x^2$ on the domain $[0, 4]$.

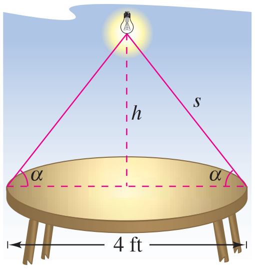
(a) Use a graphing utility to graph the functions on the specified domain.

(b) Write the vertical distance d between the functions as a function of x and use calculus to find the value of x for which d is maximum.

(c) Find the equations of the tangent lines to the graphs of f and g at the critical number found in part (b). Graph the tangent lines. What is the relationship between the lines?

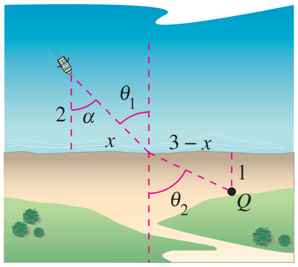
(d) Make a conjecture about the relationship between tangent lines to the graphs of two functions at the value of x at which the vertical distance between the functions is greatest, and prove your conjecture.

47. ***Illumination*** A light source is located over the center of a circular table of diameter 4 feet (see figure). Find the height h of the light source such that the illumination I at the perimeter of the table is maximum if $I = k(\sin \alpha)/s^2$, where s is the slant height, α is the angle at which the light strikes the table, and k is a constant.



48. ***Illumination*** The illumination from a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two light sources of intensities I_1 and I_2 are d units apart. What point on the line segment joining the two sources has the least illumination?

49. ***Minimum Time*** A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q, located 3 miles down the coast and 1 mile inland (see figure). He can row at 2 miles per hour and walk at 4 miles per hour. Toward what point on the coast should he row in order to reach point Q in the least time?



50. ***Minimum Time*** Consider Exercise 49 if the point Q is on the shoreline rather than 1 mile inland.

(a) Write the travel time T as a function of α.

(b) Use the result of part (a) to find the minimum time to reach Q.

(c) The man can row at v_1 miles per hour and walk at v_2 miles per hour. Write the time T as a function of α. Show that the critical number of T depends only on v_1 and v_2 and not on the distances. Explain how this result would be more beneficial to the man than the result of Exercise 49.

(d) Describe how to apply the result of part (c) to minimizing the cost of constructing a power transmission cable that costs c_1 dollars per mile under water and c_2 dollars per mile over land.

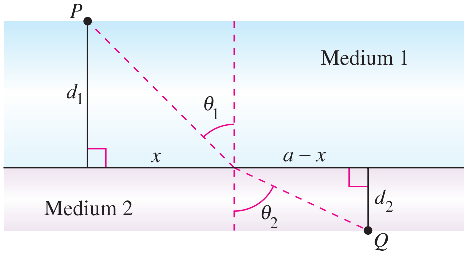
51. ***Minimum Time*** The conditions are the same as in Exercise 49 except that the man can row at v_1 miles per hour and walk at v_2 miles per hour. If θ_1 and θ_2 are the magnitudes of the angles, show that the man will reach point Q in the least time when

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$

52. ***Minimum Time*** When light waves traveling in a transparent medium strike the surface of a second transparent medium, they change direction. This change of direction is called *refraction* and is defined by **Snell's Law of Refraction,**

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

where θ_1 and θ_2 are the magnitudes of the angles shown in the figure and v_1 and v_2 are the velocities of light in the two media. Show that this problem is equivalent to that in Exercise 51, and that light waves traveling from P to Q follow the path of minimum time.



53. Sketch the graph of $f(x) = 2 - 2\sin x$ on the interval $[0, \pi/2]$.

(a) Find the distance from the origin to the y-intercept and the distance from the origin to the x-intercept.

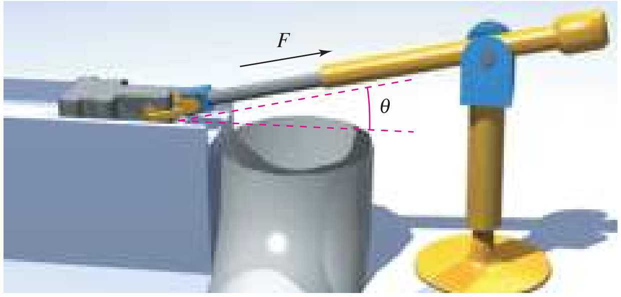
(b) Write the distance d from the origin to a point on the graph of f as a function of x. Use your graphing utility to graph d and find the minimum distance.

(c) Use calculus and the *zero* or *root* feature of a graphing utility to find the value of x that minimizes the function d on the interval $[0, \pi/2]$. What is the minimum distance?

(Submitted by Tim Chapell, Penn Valley Community College, Kansas City, MO)

54. ***Minimum Cost*** An offshore oil well is 2 kilometers off the coast. The refinery is 4 kilometers down the coast. Laying pipe in the ocean is twice as expensive as on land. What path should the pipe follow in order to minimize the cost?

55. ***Minimum Force*** A component is designed to slide a block of steel with weight W across a table and into a chute (see figure). The motion of the block is resisted by a frictional force proportional to its apparent weight. (Let k be the constant of proportionality.) Find the minimum force F needed to slide the block, and find the corresponding value of θ. (*Hint:* $F\cos\theta$ is the force in the direction of motion, and $F\sin\theta$ is the amount of force tending to lift the block. So, the apparent weight of the block is $W - F\sin\theta$.)



56. ***Maximum Volume*** A sector with central angle θ is cut from a circle of radius 12 inches (see figure), and the edges of the sector are brought together to form a cone. Find the magnitude of θ such that the volume of the cone is a maximum.

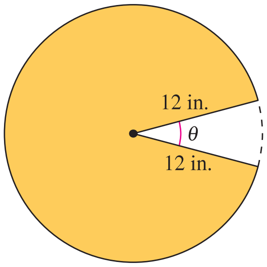


Figure for 56

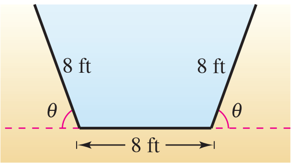


Figure for 57

57. ***Numerical, Graphical, and Analytic Analysis*** The cross sections of an irrigation canal are isosceles trapezoids of which three sides are 8 feet long (see figure). Determine the angle of elevation θ of the sides such that the area of the cross sections is a maximum by completing the following.

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 10°$	$8\sin 10°$	≈ 22.1
8	$8 + 16\cos 20°$	$8\sin 20°$	≈ 42.5

(b) Use a graphing utility to generate additional rows of the table and estimate the maximum cross-sectional area. (*Hint:* Use the *table* feature of the graphing utility.)

(c) Write the cross-sectional area A as a function of θ.

(d) Use calculus to find the critical number of the function in part (c) and find the angle that will yield the maximum cross-sectional area.

(e) Use a graphing utility to graph the function in part (c) and verify the maximum cross-sectional area.

58. ***Maximum Profit*** Assume that the amount of money deposited in a bank is proportional to the square of the interest rate the bank pays on this money. Furthermore, the bank can reinvest this money at 12%. Find the interest rate the bank should pay to maximize profit. (Use the simple interest formula.)

59. ***Minimum Cost*** The ordering and transportation cost C of the components used in manufacturing a product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$$

where C is measured in thousands of dollars and x is the order size in hundreds. Find the order size that minimizes the cost. (*Hint:* Use the *root* feature of a graphing utility.)

60. ***Diminishing Returns*** The profit P (in thousands of dollars) for a company spending an amount s (in thousands of dollars) on advertising is

$$P = -\tfrac{1}{10}s^3 + 6s^2 + 400.$$

(a) Find the amount of money the company should spend on advertising in order to yield a maximum profit.

(b) The *point of diminishing returns* is the point at which the rate of growth of the profit function begins to decline. Find the point of diminishing returns.

Minimum Distance **In Exercises 61–63, consider a fuel distribution center located at the origin of the rectangular coordinate system (units in miles; see figures on next page). The center supplies three factories with coordinates (4, 1), (5, 6), and (10, 3). A trunk line will run from the distribution center along the line $y = mx$, and feeder lines will run to the three factories. The objective is to find m such that the lengths of the feeder lines are minimized.**

61. Minimize the sum of the squares of the lengths of the vertical feeder lines (see figure) given by

$$S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2.$$

Find the equation of the trunk line by this method and then determine the sum of the lengths of the feeder lines.

62. Minimize the sum of the absolute values of the lengths of the vertical feeder lines (see figure) given by

$$S_2 = |4m - 1| + |5m - 6| + |10m - 3|.$$

Find the equation of the trunk line by this method and then determine the sum of the lengths of the feeder lines. (*Hint:* Use a graphing utility to graph the function S_2 and approximate the required critical number.)

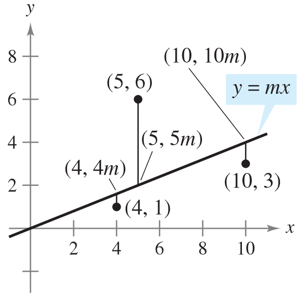


Figure for 61 and 62

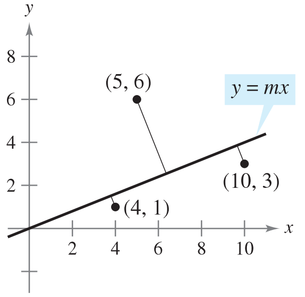


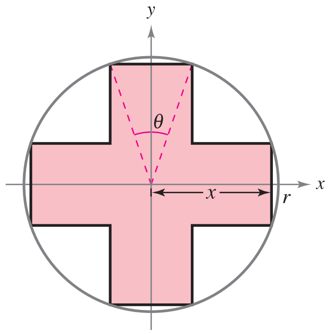
Figure for 63

63. Minimize the sum of the perpendicular distances (see figure and Exercises 87–92 in Section P.2) from the trunk line to the factories given by

$$S_3 = \frac{|4m - 1|}{\sqrt{m^2 + 1}} + \frac{|5m - 6|}{\sqrt{m^2 + 1}} + \frac{|10m - 3|}{\sqrt{m^2 + 1}}.$$

Find the equation of the trunk line by this method and then determine the sum of the lengths of the feeder lines. (*Hint:* Use a graphing utility to graph the function S_3 and approximate the required critical number.)

64. ***Maximum Area*** Consider a symmetric cross inscribed in a circle of radius r (see figure).



(a) Write the area A of the cross as a function of x and find the value of x that maximizes the area.

(b) Write the area A of the cross as a function of θ and find the value of θ that maximizes the area.

(c) Show that the critical numbers of parts (a) and (b) yield the same maximum area. What is that area?

PUTNAM EXAM CHALLENGE

65. Find the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \le 13x^2$. Explain your reasoning.

66. Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)} \text{ for } x > 0.$$

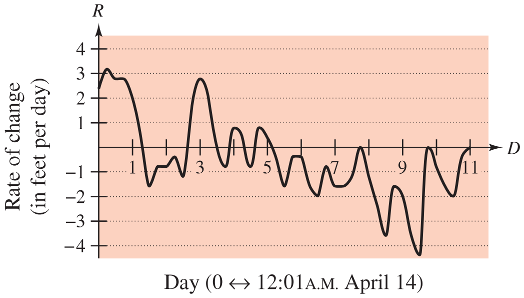
These problems were composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.

SECTION PROJECT

Connecticut River

Whenever the Connecticut River reaches a level of 105 feet above sea level, two Northampton, Massachusetts flood control station operators begin a round-the-clock river watch. Every 2 hours, they check the height of the river, using a scale marked off in tenths of a foot, and record the data in a log book. In the spring of 1996, the flood watch lasted from April 4, when the river reached 105 feet and was rising at 0.2 foot per hour, until April 25, when the level subsided again to 105 feet. Between those dates, their log shows that the river rose and fell several times, at one point coming close to the 115-foot mark. If the river had reached 115 feet, the city would have closed down Mount Tom Road (Route 5, south of Northampton).

The graph below shows the rate of change of the level of the river during one portion of the flood watch. Use the graph to answer each question.



(a) On what date was the river rising most rapidly? How do you know?

(b) On what date was the river falling most rapidly? How do you know?

(c) There were two dates in a row on which the river rose, then fell, then rose again during the course of the day. On which days did this occur, and how do you know?

(d) At 1 minute past midnight, April 14, the river level was 111.0 feet. Estimate its height 24 hours later and 48 hours later. Explain how you made your estimates.

(e) The river crested at 114.4 feet. On what date do you think this occurred?

(Submitted by Mary Murphy, Smith College, Northampton, MA)

3.8 Newton's Method

- **Approximate a zero of a function using Newton's Method.**

Newton's Method

In this section you will study a technique for approximating the real zeros of a function. The technique is called **Newton's Method,** and it uses tangent lines to approximate the graph of the function near its x-intercepts.

To see how Newton's Method works, consider a function f that is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). If $f(a)$ and $f(b)$ differ in sign, then, by the Intermediate Value Theorem, f must have at least one zero in the interval (a, b). Suppose you estimate this zero to occur at

$$x = x_1 \qquad \text{First estimate}$$

as shown in Figure 3.60(a). Newton's Method is based on the assumption that the graph of f and the tangent line at $(x_1, f(x_1))$ both cross the x-axis at *about* the same point. Because you can easily calculate the x-intercept for this tangent line, you can use it as a second (and, usually, better) estimate of the zero of f. The tangent line passes through the point $(x_1, f(x_1))$ with a slope of $f'(x_1)$. In point-slope form, the equation of the tangent line is therefore

$$y - f(x_1) = f'(x_1)(x - x_1)$$
$$y = f'(x_1)(x - x_1) + f(x_1).$$

Letting $y = 0$ and solving for x produces

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

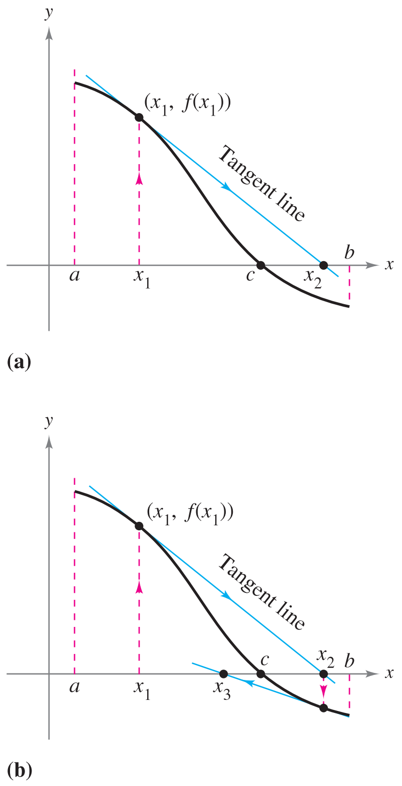
So, from the initial estimate x_1 you obtain a new estimate

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}. \qquad \text{Second estimate [see Figure 3.60(b)]}$$

You can improve on x_2 and calculate yet a third estimate

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}. \qquad \text{Third estimate}$$

Repeated application of this process is called Newton's Method.



The x-intercept of the tangent line approximates the zero of f.
Figure 3.60

NEWTON'S METHOD FOR APPROXIMATING THE ZEROS OF A FUNCTION

Let $f(c) = 0$, where f is differentiable on an open interval containing c. Then, to approximate c, use the following steps.

1. Make an initial estimate x_1 that is close to c. (A graph is helpful.)
2. Determine a new approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

3. If $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called an **iteration.**

NEWTON'S METHOD

Isaac Newton first described the method for approximating the real zeros of a function in his text *Method of Fluxions.* Although the book was written in 1671, it was not published until 1736. Meanwhile, in 1690, Joseph Raphson (1648–1715) published a paper describing a method for approximating the real zeros of a function that was very similar to Newton's. For this reason, the method is often referred to as the Newton-Raphson method.

NOTE For many functions, just a few iterations of Newton's Method will produce approximations having very small errors, as shown in Example 1.

EXAMPLE 1 Using Newton's Method

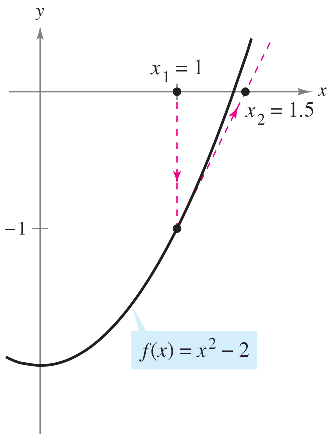
Calculate three iterations of Newton's Method to approximate a zero of $f(x) = x^2 - 2$. Use $x_1 = 1$ as the initial guess.

Solution Because $f(x) = x^2 - 2$, you have $f'(x) = 2x$, and the iterative process is given by the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n}.$$

The calculations for three iterations are shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.000000	−1.000000	2.000000	−0.500000	1.500000
2	1.500000	0.250000	3.000000	0.083333	1.416667
3	1.416667	0.006945	2.833334	0.002451	1.414216
4	1.414216				



The first iteration of Newton's Method
Figure 3.61

Of course, in this case you know that the two zeros of the function are $\pm\sqrt{2}$. To six decimal places, $\sqrt{2} = 1.414214$. So, after only three iterations of Newton's Method, you have obtained an approximation that is within 0.000002 of an actual root. The first iteration of this process is shown in Figure 3.61.

EXAMPLE 2 Using Newton's Method

Use Newton's Method to approximate the zeros of

$$f(x) = 2x^3 + x^2 - x + 1.$$

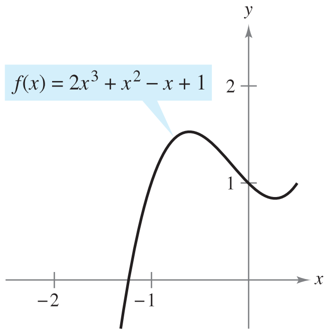
Continue the iterations until two successive approximations differ by less than 0.0001.

Solution Begin by sketching a graph of f, as shown in Figure 3.62. From the graph, you can observe that the function has only one zero, which occurs near $x = -1.2$. Next, differentiate f and form the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n^3 + x_n^2 - x_n + 1}{6x_n^2 + 2x_n - 1}.$$

The calculations are shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−1.20000	0.18400	5.24000	0.03511	−1.23511
2	−1.23511	−0.00771	5.68276	−0.00136	−1.23375
3	−1.23375	0.00001	5.66533	0.00000	−1.23375
4	−1.23375				

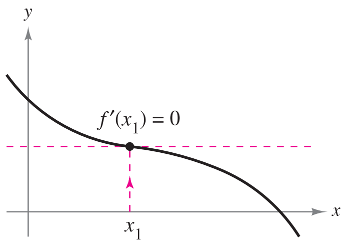


After three iterations of Newton's Method, the zero of f is approximated to the desired accuracy.
Figure 3.62

Because two successive approximations differ by less than the required 0.0001, you can estimate the zero of f to be -1.23375. ■

When, as in Examples 1 and 2, the approximations approach a limit, the sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$ is said to **converge.** Moreover, if the limit is c, it can be shown that c must be a zero of f.

Newton's Method does not always yield a convergent sequence. One way it can fail to do so is shown in Figure 3.63. Because Newton's Method involves division by $f'(x_n)$, it is clear that the method will fail if the derivative is zero for any x_n in the sequence. When you encounter this problem, you can usually overcome it by choosing a different value for x_1. Another way Newton's Method can fail is shown in the next example.



Newton's Method fails to converge if $f'(x_n) = 0$.
Figure 3.63

EXAMPLE 3 An Example in Which Newton's Method Fails

The function $f(x) = x^{1/3}$ is not differentiable at $x = 0$. Show that Newton's Method fails to converge using $x_1 = 0.1$.

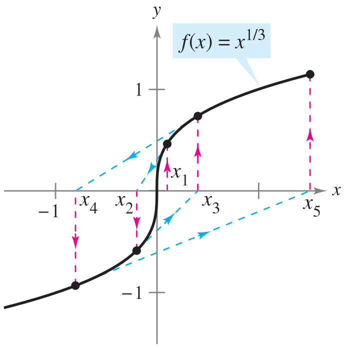
Solution Because $f'(x) = \frac{1}{3}x^{-2/3}$, the iterative formula is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} \\ &= x_n - 3x_n \\ &= -2x_n. \end{aligned}$$

The calculations are shown in the table. This table and Figure 3.64 indicate that x_n continues to increase in magnitude as $n \to \infty$, and so the limit of the sequence does not exist.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.10000	0.46416	1.54720	0.30000	−0.20000
2	−0.20000	−0.58480	0.97467	−0.60000	0.40000
3	0.40000	0.73681	0.61401	1.20000	−0.80000
4	−0.80000	−0.92832	0.38680	−2.40000	1.60000

■



Newton's Method fails to converge for every x-value other than the actual zero of f.
Figure 3.64

NOTE In Example 3, the initial estimate $x_1 = 0.1$ fails to produce a convergent sequence. Try showing that Newton's Method also fails for every other choice of x_1 (other than the actual zero). ■

It can be shown that a condition sufficient to produce convergence of Newton's Method to a zero of f is that

$$\left|\frac{f(x)\,f''(x)}{[f'(x)]^2}\right| < 1 \qquad \text{Condition for convergence}$$

on an open interval containing the zero. For instance, in Example 1 this test would yield $f(x) = x^2 - 2, f'(x) = 2x, f''(x) = 2$, and

$$\left|\frac{f(x)\,f''(x)}{[f'(x)]^2}\right| = \left|\frac{(x^2 - 2)(2)}{4x^2}\right| = \left|\frac{1}{2} - \frac{1}{x^2}\right|. \qquad \text{Example 1}$$

On the interval $(1, 3)$, this quantity is less than 1 and therefore the convergence of Newton's Method is guaranteed. On the other hand, in Example 3, you have $f(x) = x^{1/3}, f'(x) = \frac{1}{3}x^{-2/3}, f''(x) = -\frac{2}{9}x^{-5/3}$, and

$$\left|\frac{f(x)\,f''(x)}{[f'(x)]^2}\right| = \left|\frac{x^{1/3}(-2/9)(x^{-5/3})}{(1/9)(x^{-4/3})}\right| = 2 \qquad \text{Example 3}$$

which is not less than 1 for any value of x, so you cannot conclude that Newton's Method will converge.

The Granger Collection

NIELS HENRIK ABEL (1802–1829)

The Granger Collection

EVARISTE GALOIS (1811–1832)

Although the lives of both Abel and Galois were brief, their work in the fields of analysis and abstract algebra was far-reaching.

Algebraic Solutions of Polynomial Equations

The zeros of some functions, such as

$$f(x) = x^3 - 2x^2 - x + 2$$

can be found by simple algebraic techniques, such as factoring. The zeros of other functions, such as

$$f(x) = x^3 - x + 1$$

cannot be found by *elementary* algebraic methods. This particular function has only one real zero, and by using more advanced algebraic techniques you can determine the zero to be

$$x = -\sqrt[3]{\frac{3 - \sqrt{23/3}}{6}} - \sqrt[3]{\frac{3 + \sqrt{23/3}}{6}}.$$

Because the *exact* solution is written in terms of square roots and cube roots, it is called a **solution by radicals.**

NOTE Try approximating the real zero of $f(x) = x^3 - x + 1$ and compare your result with the exact solution shown above. ■

The determination of radical solutions of a polynomial equation is one of the fundamental problems of algebra. The earliest such result is the Quadratic Formula, which dates back at least to Babylonian times. The general formula for the zeros of a cubic function was developed much later. In the sixteenth century an Italian mathematician, Jerome Cardan, published a method for finding radical solutions to cubic and quartic equations. Then, for 300 years, the problem of finding a general quintic formula remained open. Finally, in the nineteenth century, the problem was answered independently by two young mathematicians. Niels Henrik Abel, a Norwegian mathematician, and Evariste Galois, a French mathematician, proved that it is not possible to solve a *general* fifth- (or higher-) degree polynomial equation by radicals. Of course, you can solve particular fifth-degree equations such as $x^5 - 1 = 0$, but Abel and Galois were able to show that no general *radical* solution exists.

3.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, complete two iterations of Newton's Method for the function using the given initial guess.

1. $f(x) = x^2 - 5, \quad x_1 = 2.2$

2. $f(x) = x^3 - 3, \quad x_1 = 1.4$

3. $f(x) = \cos x, \quad x_1 = 1.6$

4. $f(x) = \tan x, \quad x_1 = 0.1$

In Exercises 5–14, approximate the zero(s) of the function. Use Newton's Method and continue the process until two successive approximations differ by less than 0.001. Then find the zero(s) using a graphing utility and compare the results.

5. $f(x) = x^3 + 4$

6. $f(x) = 2 - x^3$

7. $f(x) = x^3 + x - 1$

8. $f(x) = x^5 + x - 1$

9. $f(x) = 5\sqrt{x-1} - 2x$

10. $f(x) = x - 2\sqrt{x+1}$

11. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

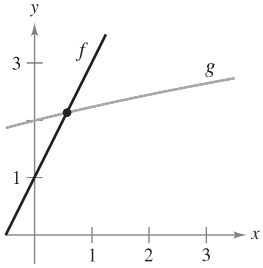
12. $f(x) = x^4 + x^3 - 1$

13. $f(x) = -x + \sin x$

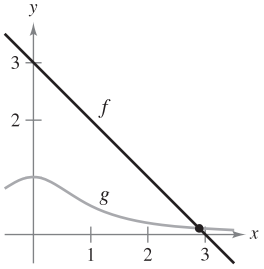
14. $f(x) = x^3 - \cos x$

In Exercises 15–18, apply Newton's Method to approximate the *x*-value(s) of the indicated point(s) of intersection of the two graphs. Continue the process until two successive approximations differ by less than 0.001. [*Hint:* Let $h(x) = f(x) - g(x)$.]

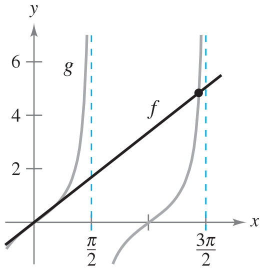
15. $f(x) = 2x + 1$
$g(x) = \sqrt{x+4}$



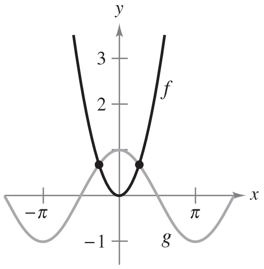
16. $f(x) = 3 - x$
$g(x) = 1/(x^2 + 1)$



17. $f(x) = x$
$g(x) = \tan x$



18. $f(x) = x^2$
$g(x) = \cos x$



19. ***Mechanic's Rule*** The Mechanic's Rule for approximating $\sqrt{a}$, $a > 0$, is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \quad n = 1, 2, 3 \ldots$$

where x_1 is an approximation of $\sqrt{a}$.

(a) Use Newton's Method and the function $f(x) = x^2 - a$ to derive the Mechanic's Rule.

(b) Use the Mechanic's Rule to approximate $\sqrt{5}$ and $\sqrt{7}$ to three decimal places.

20. (a) Use Newton's Method and the function $f(x) = x^n - a$ to obtain a general rule for approximating $x = \sqrt[n]{a}$.

(b) Use the general rule found in part (a) to approximate $\sqrt[4]{6}$ and $\sqrt[3]{15}$ to three decimal places.

In Exercises 21–24, apply Newton's Method using the given initial guess, and explain why the method fails.

21. $y = 2x^3 - 6x^2 + 6x - 1, \quad x_1 = 1$

22. $y = x^3 - 2x - 2, \quad x_1 = 0$

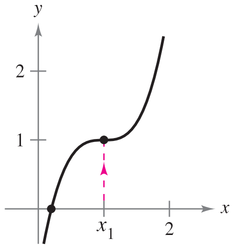


Figure for 21

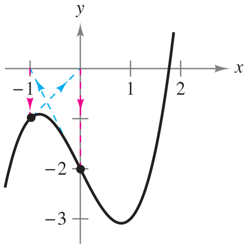


Figure for 22

23. $f(x) = -x^3 + 6x^2 - 10x + 6, \quad x_1 = 2$

24. $f(x) = 2\sin x + \cos 2x, \quad x_1 = \dfrac{3\pi}{2}$

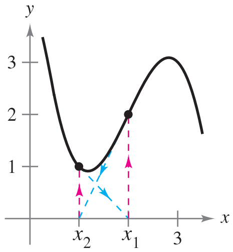


Figure for 23

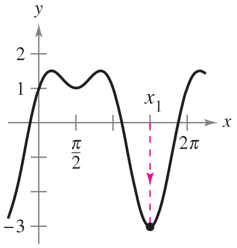


Figure for 24

Fixed Point **In Exercises 25 and 26, approximate the fixed point of the function to two decimal places. [A *fixed point* x_0 of a function *f* is a value of *x* such that $f(x_0) = x_0$.]**

25. $f(x) = \cos x$

26. $f(x) = \cot x, \quad 0 < x < \pi$

27. Use Newton's Method to show that the equation $x_{n+1} = x_n(2 - ax_n)$ can be used to approximate $1/a$ if x_1 is an initial guess of the reciprocal of a. Note that this method of approximating reciprocals uses only the operations of multiplication and subtraction. [*Hint:* Consider $f(x) = (1/x) - a$.]

28. Use the result of Exercise 27 to approximate (a) $\frac{1}{3}$ and (b) $\frac{1}{11}$ to three decimal places.

WRITING ABOUT CONCEPTS

29. Consider the function $f(x) = x^3 - 3x^2 + 3$.

(a) Use a graphing utility to graph f.

(b) Use Newton's Method with $x_1 = 1$ as an initial guess.

(c) Repeat part (b) using $x_1 = \frac{1}{4}$ as an initial guess and observe that the result is different.

(d) To understand why the results in parts (b) and (c) are different, sketch the tangent lines to the graph of f at the points $(1, f(1))$ and $\left(\frac{1}{4}, f\left(\frac{1}{4}\right)\right)$. Find the x-intercept of each tangent line and compare the intercepts with the first iteration of Newton's Method using the respective initial guesses.

(e) Write a short paragraph summarizing how Newton's Method works. Use the results of this exercise to describe why it is important to select the initial guess carefully.

30. Repeat the steps in Exercise 29 for the function $f(x) = \sin x$ with initial guesses of $x_1 = 1.8$ and $x_1 = 3$.

31. In your own words and using a sketch, describe Newton's Method for approximating the zeros of a function.

CAPSTONE

32. Under what conditions will Newton's Method fail?

In Exercises 33 and 34, approximate the critical number of f on the interval $(0, \pi)$. Sketch the graph of f, labeling any extrema.

33. $f(x) = x \cos x$

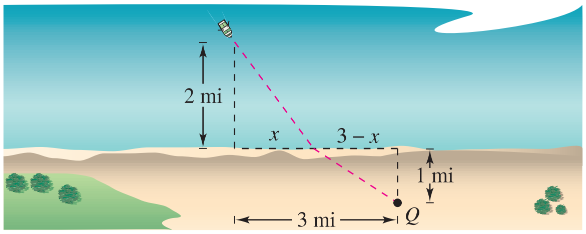
34. $f(x) = x \sin x$

Exercises 35–38 present problems similar to exercises from the previous sections of this chapter. In each case, use Newton's Method to approximate the solution.

35. ***Minimum Distance*** Find the point on the graph of $f(x) = 4 - x^2$ that is closest to the point $(1, 0)$.

36. ***Minimum Distance*** Find the point on the graph of $f(x) = x^2$ that is closest to the point $(4, -3)$.

37. ***Minimum Time*** You are in a boat 2 miles from the nearest point on the coast (see figure). You are to go to a point Q, which is 3 miles down the coast and 1 mile inland. You can row at 3 miles per hour and walk at 4 miles per hour. Toward what point on the coast should you row in order to reach Q in the least time?



38. ***Medicine*** The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is given by $C = (3t^2 + t)/(50 + t^3)$. When is the concentration greatest?

39. ***Crime*** The total number of arrests T (in thousands) for all males ages 14 to 27 in 2006 is approximated by the model

$$T = 0.602x^3 - 41.44x^2 + 922.8x - 6330, \quad 14 \le x \le 27$$

where x is the age in years (see figure). Approximate the two ages that had total arrests of 225 thousand. (*Source: U.S. Department of Justice*)

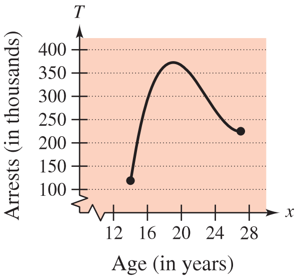


Figure for 39

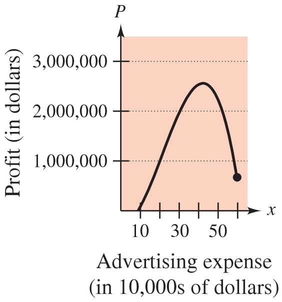


Figure for 40

40. ***Advertising Costs*** A manufacturer of digital audio players estimates that the profit for selling a particular model is

$$P = -76x^3 + 4830x^2 - 320{,}000, \quad 0 \le x \le 60$$

where P is the profit in dollars and x is the advertising expense in tens of thousands of dollars (see figure). Find the smaller of two advertising amounts that yield a profit P of \$2,500,000.

True or False? **In Exercises 41–44, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

41. The zeros of $f(x) = p(x)/q(x)$ coincide with the zeros of $p(x)$.

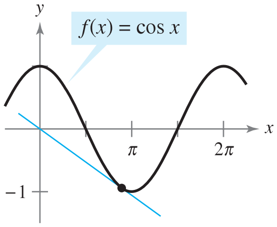
42. If the coefficients of a polynomial function are all positive, then the polynomial has no positive zeros.

43. If $f(x)$ is a cubic polynomial such that $f'(x)$ is never zero, then any initial guess will force Newton's Method to converge to the zero of f.

44. The roots of $\sqrt{f(x)} = 0$ coincide with the roots of $f(x) = 0$.

45. ***Tangent Lines*** The graph of $f(x) = -\sin x$ has infinitely many tangent lines that pass through the origin. Use Newton's Method to approximate to three decimal places the slope of the tangent line having the greatest slope.

46. ***Point of Tangency*** The graph of $f(x) = \cos x$ and a tangent line to f through the origin are shown. Find the coordinates of the point of tangency to three decimal places.



3.9 Differentials

- **Understand the concept of a tangent line approximation.**
- **Compare the value of the differential, *dy*, with the actual change in *y*, Δ*y*.**
- **Estimate a propagated error using a differential.**
- **Find the differential of a function using differentiation formulas.**

EXPLORATION

Tangent Line Approximation
Use a graphing utility to graph

$$f(x) = x^2.$$

In the same viewing window, graph the tangent line to the graph of f at the point $(1, 1)$. Zoom in twice on the point of tangency. Does your graphing utility distinguish between the two graphs? Use the *trace* feature to compare the two graphs. As the x-values get closer to 1, what can you say about the y-values?

Tangent Line Approximations

Newton's Method (Section 3.8) is an example of the use of a tangent line to a graph to approximate the graph. In this section, you will study other situations in which the graph of a function can be approximated by a straight line.

To begin, consider a function f that is differentiable at c. The equation for the tangent line at the point $(c, f(c))$ is given by

$$y - f(c) = f'(c)(x - c)$$

$$\boxed{y = f(c) + f'(c)(x - c)}$$

and is called the **tangent line approximation** (or **linear approximation**) **of *f* at *c*.** Because c is a constant, y is a linear function of x. Moreover, by restricting the values of x to those sufficiently close to c, the values of y can be used as approximations (to any desired degree of accuracy) of the values of the function f. In other words, as $x \to c$, the limit of y is $f(c)$.

EXAMPLE 1 Using a Tangent Line Approximation

Find the tangent line approximation of

$$f(x) = 1 + \sin x$$

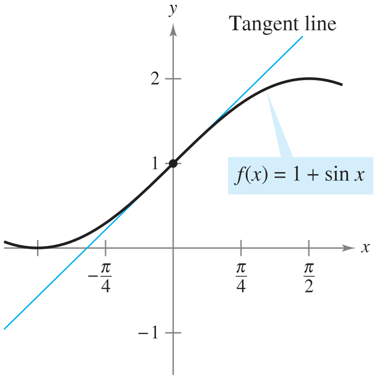
at the point $(0, 1)$. Then use a table to compare the y-values of the linear function with those of $f(x)$ on an open interval containing $x = 0$.

Solution The derivative of f is

$$f'(x) = \cos x. \qquad \text{First derivative}$$

So, the equation of the tangent line to the graph of f at the point $(0, 1)$ is

$$y - f(0) = f'(0)(x - 0)$$
$$y - 1 = (1)(x - 0)$$
$$y = 1 + x. \qquad \text{Tangent line approximation}$$



The tangent line approximation of f at the point $(0, 1)$
Figure 3.65

The table compares the values of y given by this linear approximation with the values of $f(x)$ near $x = 0$. Notice that the closer x is to 0, the better the approximation is. This conclusion is reinforced by the graph shown in Figure 3.65.

x	-0.5	-0.1	-0.01	0	0.01	0.1	0.5
$f(x) = 1 + \sin x$	0.521	0.9002	0.9900002	1	1.0099998	1.0998	1.479
$y = 1 + x$	0.5	0.9	0.99	1	1.01	1.1	1.5

■

NOTE Be sure you see that this linear approximation of $f(x) = 1 + \sin x$ depends on the point of tangency. At a different point on the graph of f, you would obtain a different tangent line approximation. ■

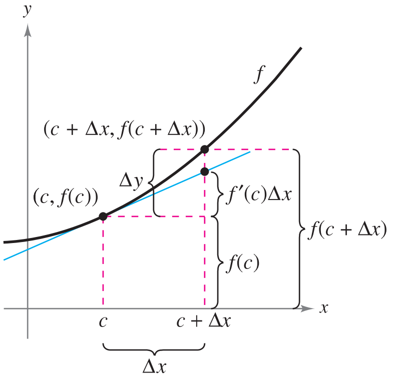
Differentials

When the tangent line to the graph of f at the point $(c, f(c))$

$$y = f(c) + f'(c)(x - c) \qquad \text{Tangent line at } (c, f(c))$$

is used as an approximation of the graph of f, the quantity $x - c$ is called the change in x, and is denoted by Δx, as shown in Figure 3.66. When Δx is small, the change in y (denoted by Δy) can be approximated as shown.

$$\begin{aligned} \Delta y &= f(c + \Delta x) - f(c) && \text{Actual change in } y \\ &\approx f'(c)\Delta x && \text{Approximate change in } y \end{aligned}$$



When Δx is small, $\Delta y = f(c + \Delta x) - f(c)$ is approximated by $f'(c)\Delta x$.
Figure 3.66

For such an approximation, the quantity Δx is traditionally denoted by dx, and is called the **differential of *x*.** The expression $f'(x)\,dx$ is denoted by dy, and is called the **differential of *y*.**

DEFINITION OF DIFFERENTIALS

Let $y = f(x)$ represent a function that is differentiable on an open interval containing x. The **differential of *x*** (denoted by dx) is any nonzero real number. The **differential of *y*** (denoted by dy) is

$$dy = f'(x)\,dx.$$

In many types of applications, the differential of y can be used as an approximation of the change in y. That is,

$$\Delta y \approx dy \qquad \text{or} \qquad \Delta y \approx f'(x)dx.$$

EXAMPLE 2 Comparing Δy and dy

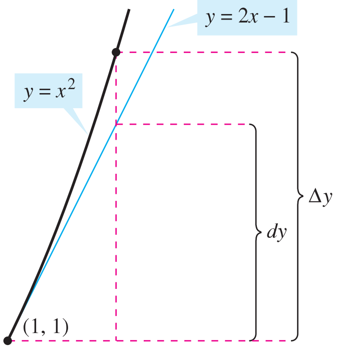
Let $y = x^2$. Find dy when $x = 1$ and $dx = 0.01$. Compare this value with Δy for $x = 1$ and $\Delta x = 0.01$.

Solution Because $y = f(x) = x^2$, you have $f'(x) = 2x$, and the differential dy is given by

$$dy = f'(x)\,dx = f'(1)(0.01) = 2(0.01) = 0.02. \qquad \text{Differential of } y$$

Now, using $\Delta x = 0.01$, the change in y is

$$\Delta y = f(x + \Delta x) - f(x) = f(1.01) - f(1) = (1.01)^2 - 1^2 = 0.0201.$$



The change in y, Δy, is approximated by the differential of y, dy.
Figure 3.67

Figure 3.67 shows the geometric comparison of dy and Δy. Try comparing other values of dy and Δy. You will see that the values become closer to each other as dx (or Δx) approaches 0. ■

In Example 2, the tangent line to the graph of $f(x) = x^2$ at $x = 1$ is

$$y = 2x - 1 \qquad \text{or} \qquad g(x) = 2x - 1. \qquad \text{Tangent line to the graph of } f \text{ at } x = 1.$$

For x-values near 1, this line is close to the graph of f, as shown in Figure 3.67. For instance,

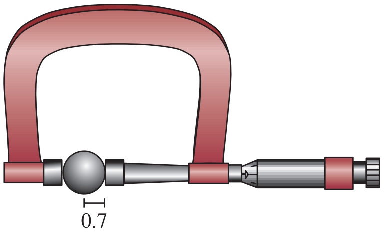
$$f(1.01) = 1.01^2 = 1.0201 \qquad \text{and} \qquad g(1.01) = 2(1.01) - 1 = 1.02.$$

Error Propagation

Physicists and engineers tend to make liberal use of the approximation of Δy by dy. One way this occurs in practice is in the estimation of errors propagated by physical measuring devices. For example, if you let x represent the measured value of a variable and let $x + \Delta x$ represent the exact value, then Δx is the *error in measurement*. Finally, if the measured value x is used to compute another value $f(x)$, the difference between $f(x + \Delta x)$ and $f(x)$ is the **propagated error.**

$$\underbrace{f(x + \overset{\text{Measurement error}}{\overbrace{\Delta x}})}_{\text{Exact value}} - \underbrace{f(x)}_{\text{Measured value}} = \overset{\text{Propagated error}}{\overbrace{\Delta y}}$$

EXAMPLE 3 Estimation of Error



Ball bearing with measured radius that is correct to within 0.01 inch.
Figure 3.68

The measured radius of a ball bearing is 0.7 inch, as shown in Figure 3.68. If the measurement is correct to within 0.01 inch, estimate the propagated error in the volume V of the ball bearing.

Solution The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere. So, you can write

$$r = 0.7 \qquad \text{Measured radius}$$

and

$$-0.01 \le \Delta r \le 0.01. \qquad \text{Possible error}$$

To approximate the propagated error in the volume, differentiate V to obtain $dV/dr = 4\pi r^2$ and write

$$\begin{aligned} \Delta V &\approx dV && \text{Approximate } \Delta V \text{ by } dV. \\ &= 4\pi r^2\,dr \\ &= 4\pi(0.7)^2(\pm 0.01) && \text{Substitute for } r \text{ and } dr. \\ &\approx \pm 0.06158 \text{ cubic inch.} \end{aligned}$$

So, the volume has a propagated error of about 0.06 cubic inch. ■

Would you say that the propagated error in Example 3 is large or small? The answer is best given in *relative* terms by comparing dV with V. The ratio

$$\begin{aligned} \frac{dV}{V} &= \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3} && \text{Ratio of } dV \text{ to } V \\ &= \frac{3\,dr}{r} && \text{Simplify.} \\ &\approx \frac{3}{0.7}(\pm 0.01) && \text{Substitute for } dr \text{ and } r. \\ &\approx \pm 0.0429 \end{aligned}$$

is called the **relative error.** The corresponding **percent error** is approximately 4.29%.

Calculating Differentials

Each of the differentiation rules that you studied in Chapter 2 can be written in **differential form.** For example, suppose u and v are differentiable functions of x. By the definition of differentials, you have

$$du = u'\,dx \quad \text{and} \quad dv = v'\,dx.$$

So, you can write the differential form of the Product Rule as shown below.

$$\begin{aligned} d[uv] &= \frac{d}{dx}[uv]\,dx && \text{Differential of } uv \\ &= [uv' + vu']\,dx && \text{Product Rule} \\ &= uv'\,dx + vu'\,dx \\ &= u\,dv + v\,du \end{aligned}$$

DIFFERENTIAL FORMULAS

Let u and v be differentiable functions of x.

Constant multiple: $d[cu] = c\,du$

Sum or difference: $d[u \pm v] = du \pm dv$

Product: $d[uv] = u\,dv + v\,du$

Quotient: $d\left[\dfrac{u}{v}\right] = \dfrac{v\,du - u\,dv}{v^2}$

EXAMPLE 4 Finding Differentials

	Function	*Derivative*	*Differential*
a.	$y = x^2$	$\dfrac{dy}{dx} = 2x$	$dy = 2x\,dx$
b.	$y = 2 \sin x$	$\dfrac{dy}{dx} = 2 \cos x$	$dy = 2 \cos x\,dx$
c.	$y = x \cos x$	$\dfrac{dy}{dx} = -x \sin x + \cos x$	$dy = (-x \sin x + \cos x)\,dx$
d.	$y = \dfrac{1}{x}$	$\dfrac{dy}{dx} = -\dfrac{1}{x^2}$	$dy = -\dfrac{dx}{x^2}$ ■

Mary Evans Picture Library

GOTTFRIED WILHELM LEIBNIZ (1646–1716)

Both Leibniz and Newton are credited with creating calculus. It was Leibniz, however, who tried to broaden calculus by developing rules and formal notation. He often spent days choosing an appropriate notation for a new concept.

The notation in Example 4 is called the **Leibniz notation** for derivatives and differentials, named after the German mathematician Gottfried Wilhelm Leibniz. The beauty of this notation is that it provides an easy way to remember several important calculus formulas by making it seem as though the formulas were derived from algebraic manipulations of differentials. For instance, in Leibniz notation, the *Chain Rule*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

would appear to be true because the du's divide out. Even though this reasoning is *incorrect*, the notation does help one remember the Chain Rule.

EXAMPLE 5 Finding the Differential of a Composite Function

$$\begin{aligned} y = f(x) &= \sin 3x && \text{Original function} \\ f'(x) &= 3\cos 3x && \text{Apply Chain Rule.} \\ dy = f'(x)\,dx &= 3\cos 3x\,dx && \text{Differential form} \end{aligned}$$

EXAMPLE 6 Finding the Differential of a Composite Function

$$\begin{aligned} y = f(x) &= (x^2+1)^{1/2} && \text{Original function} \\ f'(x) &= \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}} && \text{Apply Chain Rule.} \\ dy = f'(x)\,dx &= \frac{x}{\sqrt{x^2+1}}\,dx && \text{Differential form} \end{aligned}$$

■

Differentials can be used to approximate function values. To do this for the function given by $y = f(x)$, use the formula

$$f(x+\Delta x) \approx f(x) + dy = f(x) + f'(x)\,dx$$

which is derived from the approximation $\Delta y = f(x+\Delta x) - f(x) \approx dy$. The key to using this formula is to choose a value for x that makes the calculations easier, as shown in Example 7. (This formula is equivalent to the tangent line approximation given earlier in this section.)

EXAMPLE 7 Approximating Function Values

Use differentials to approximate $\sqrt{16.5}$.

Solution Using $f(x) = \sqrt{x}$, you can write

$$f(x+\Delta x) \approx f(x) + f'(x)\,dx = \sqrt{x} + \frac{1}{2\sqrt{x}}\,dx.$$

Now, choosing $x = 16$ and $dx = 0.5$, you obtain the following approximation.

$$f(x+\Delta x) = \sqrt{16.5} \approx \sqrt{16} + \frac{1}{2\sqrt{16}}(0.5) = 4 + \left(\frac{1}{8}\right)\left(\frac{1}{2}\right) = 4.0625$$

■

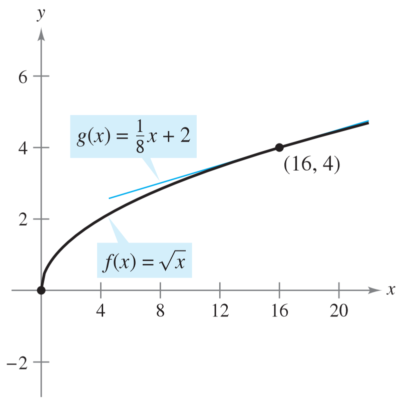


Figure 3.69

The tangent line approximation to $f(x) = \sqrt{x}$ at $x = 16$ is the line $g(x) = \frac{1}{8}x + 2$. For x-values near 16, the graphs of f and g are close together, as shown in Figure 3.69. For instance,

$$f(16.5) = \sqrt{16.5} \approx 4.0620 \quad \text{and} \quad g(16.5) = \frac{1}{8}(16.5) + 2 = 4.0625.$$

In fact, if you use a graphing utility to zoom in near the point of tangency (16, 4), you will see that the two graphs appear to coincide. Notice also that as you move farther away from the point of tangency, the linear approximation becomes less accurate.

3.9 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the equation of the tangent line T to the graph of f at the given point. Use this linear approximation to complete the table.

x	1.9	1.99	2	2.01	2.1
$f(x)$					
$T(x)$					

1. $f(x) = x^2, \quad (2, 4)$

2. $f(x) = \dfrac{6}{x^2}, \quad \left(2, \dfrac{3}{2}\right)$

3. $f(x) = x^5, \quad (2, 32)$

4. $f(x) = \sqrt{x}, \quad \left(2, \sqrt{2}\right)$

5. $f(x) = \sin x, \quad (2, \sin 2)$

6. $f(x) = \csc x, \quad (2, \csc 2)$

In Exercises 7–10, use the information to evaluate and compare Δy and dy.

7.	$y = x^3$	$x = 1$	$\Delta x = dx = 0.1$
8.	$y = 1 - 2x^2$	$x = 0$	$\Delta x = dx = -0.1$
9.	$y = x^4 + 1$	$x = -1$	$\Delta x = dx = 0.01$
10.	$y = 2 - x^4$	$x = 2$	$\Delta x = dx = 0.01$

In Exercises 11–20, find the differential dy of the given function.

11. $y = 3x^2 - 4$

12. $y = 3x^{2/3}$

13. $y = \dfrac{x + 1}{2x - 1}$

14. $y = \sqrt{9 - x^2}$

15. $y = x\sqrt{1 - x^2}$

16. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

17. $y = 3x - \sin^2 x$

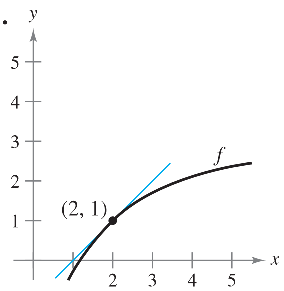
18. $y = x \cos x$

19. $y = \dfrac{1}{3} \cos\left(\dfrac{6\pi x - 1}{2}\right)$

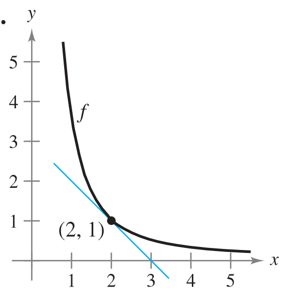
20. $y = \dfrac{\sec^2 x}{x^2 + 1}$

In Exercises 21–24, use differentials and the graph of f to approximate (a) $f(1.9)$ and (b) $f(2.04)$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

21.



22.



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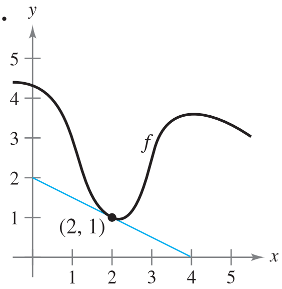
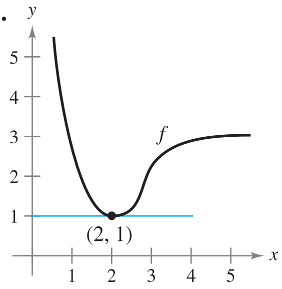
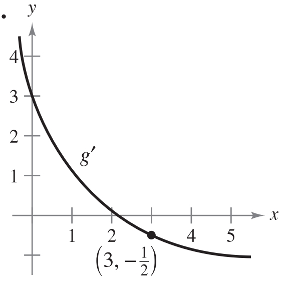


24.

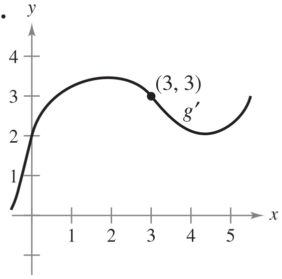


In Exercises 25 and 26, use differentials and the graph of g' to approximate (a) $g(2.93)$ and (b) $g(3.1)$ given that $g(3) = 8$.

25.



26.



27. ***Area*** The measurement of the side of a square floor tile is 10 inches, with a possible error of $\frac{1}{32}$ inch. Use differentials to approximate the possible propagated error in computing the area of the square.

28. ***Area*** The measurements of the base and altitude of a triangle are found to be 36 and 50 centimeters, respectively. The possible error in each measurement is 0.25 centimeter. Use differentials to approximate the possible propagated error in computing the area of the triangle.

29. ***Area*** The measurement of the radius of the end of a log is found to be 16 inches, with a possible error of $\frac{1}{4}$ inch. Use differentials to approximate the possible propagated error in computing the area of the end of the log.

30. ***Volume and Surface Area*** The measurement of the edge of a cube is found to be 15 inches, with a possible error of 0.03 inch. Use differentials to approximate the maximum possible propagated error in computing (a) the volume of the cube and (b) the surface area of the cube.

31. ***Area*** The measurement of a side of a square is found to be 12 centimeters, with a possible error of 0.05 centimeter.

(a) Approximate the percent error in computing the area of the square.

(b) Estimate the maximum allowable percent error in measuring the side if the error in computing the area cannot exceed 2.5%.

32. ***Circumference*** The measurement of the circumference of a circle is found to be 64 centimeters, with a possible error of 0.9 centimeter.

(a) Approximate the percent error in computing the area of the circle.

(b) Estimate the maximum allowable percent error in measuring the circumference if the error in computing the area cannot exceed 3%.

33. ***Volume and Surface Area*** The radius of a spherical balloon is measured as 8 inches, with a possible error of 0.02 inch. Use differentials to approximate the maximum possible error in calculating (a) the volume of the sphere, (b) the surface area of the sphere, and (c) the relative errors in parts (a) and (b).

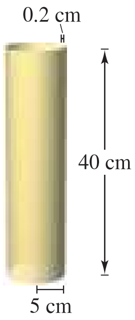
34. ***Stopping Distance*** The total stopping distance T of a vehicle is

$$T = 2.5x + 0.5x^2$$

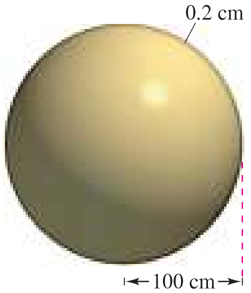
where T is in feet and x is the speed in miles per hour. Approximate the change and percent change in total stopping distance as speed changes from $x = 25$ to $x = 26$ miles per hour.

Volume **In Exercises 35 and 36, the thickness of each shell is 0.2 centimeter. Use differentials to approximate the volume of each shell.**

35.



36.



37. ***Pendulum*** The period of a pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum in feet, g is the acceleration due to gravity, and T is the time in seconds. The pendulum has been subjected to an increase in temperature such that the length has increased by $\frac{1}{2}\%$.

(a) Find the approximate percent change in the period.

(b) Using the result in part (a), find the approximate error in this pendulum clock in 1 day.

38. ***Ohm's Law*** A current of I amperes passes through a resistor of R ohms. **Ohm's Law** states that the voltage E applied to the resistor is $E = IR$. If the voltage is constant, show that the magnitude of the relative error in R caused by a change in I is equal in magnitude to the relative error in I.

39. ***Triangle Measurements*** The measurement of one side of a right triangle is found to be 9.5 inches, and the angle opposite that side is $26°45'$ with a possible error of $15'$.

(a) Approximate the percent error in computing the length of the hypotenuse.

(b) Estimate the maximum allowable percent error in measuring the angle if the error in computing the length of the hypotenuse cannot exceed 2%.

40. ***Area*** Approximate the percent error in computing the area of the triangle in Exercise 39.

41. ***Projectile Motion*** The range R of a projectile is

$$R = \frac{v_0^2}{32}(\sin 2\theta)$$

where v_0 is the initial velocity in feet per second and θ is the angle of elevation. If $v_0 = 2500$ feet per second and θ is changed from 10° to 11°, use differentials to approximate the change in the range.

42. ***Surveying*** A surveyor standing 50 feet from the base of a large tree measures the angle of elevation to the top of the tree as 71.5°. How accurately must the angle be measured if the percent error in estimating the height of the tree is to be less than 6%?

In Exercises 43–46, use differentials to approximate the value of the expression. Compare your answer with that of a calculator.

43. $\sqrt{99.4}$

44. $\sqrt[3]{26}$

45. $\sqrt[4]{624}$

46. $(2.99)^3$

In Exercises 47 and 48, verify the tangent line approximation of the function at the given point. Then use a graphing utility to graph the function and its approximation in the same viewing window.

	Function	*Approximation*	*Point*
47.	$f(x) = \sqrt{x + 4}$	$y = 2 + \frac{x}{4}$	$(0, 2)$
48.	$f(x) = \tan x$	$y = x$	$(0, 0)$

WRITING ABOUT CONCEPTS

49. Describe the change in accuracy of dy as an approximation for Δy when Δx is decreased.

50. When using differentials, what is meant by the terms *propagated error*, *relative error*, and *percent error*?

51. Give a short explanation of why the approximation is valid.

(a) $\sqrt{4.02} \approx 2 + \frac{1}{4}(0.02)$

(b) $\tan 0.05 \approx 0 + 1(0.05)$

CAPSTONE

52. Would you use $y = x$ to approximate $f(x) = \sin x$ near $x = 0$? Why or why not?

True or False? **In Exercises 53–56, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

53. If $y = x + c$, then $dy = dx$.

54. If $y = ax + b$, then $\Delta y/\Delta x = dy/dx$.

55. If y is differentiable, then $\lim_{\Delta x \to 0} (\Delta y - dy) = 0$.

56. If $y = f(x)$, f is increasing and differentiable, and $\Delta x > 0$, then $\Delta y \ge dy$.

3 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. Give the definition of a critical number, and graph a function f showing the different types of critical numbers.

2. Consider the odd function f that is continuous and differentiable and has the functional values shown in the table.

x	-5	-4	-1	0	2	3	6
$f(x)$	1	3	2	0	-1	-4	0

(a) Determine $f(4)$.

(b) Determine $f(-3)$.

(c) Plot the points and make a possible sketch of the graph of f on the interval $[-6, 6]$. What is the smallest number of critical points in the interval? Explain.

(d) Does there exist at least one real number c in the interval $(-6, 6)$ where $f'(c) = -1$? Explain.

(e) Is it possible that $\lim_{x \to 0} f(x)$ does not exist? Explain.

(f) Is it necessary that $f'(x)$ exists at $x = 2$? Explain.

In Exercises 3–6, find the absolute extrema of the function on the closed interval. Use a graphing utility to graph the function over the given interval to confirm your results.

3. $f(x) = x^2 + 5x, \quad [-4, 0]$

4. $h(x) = 3\sqrt{x} - x, \quad [0, 9]$

5. $g(x) = 2x + 5\cos x, \quad [0, 2\pi]$

6. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}, \quad [0, 2]$

In Exercises 7–10, determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$. If Rolle's Theorem cannot be applied, explain why not.

7. $f(x) = 2x^2 - 7, \quad [0, 4]$

8. $f(x) = (x - 2)(x + 3)^2, \quad [-3, 2]$

9. $f(x) = \dfrac{x^2}{1 - x^2}, \quad [-2, 2]$

10. $f(x) = |x - 2| - 2, \quad [0, 4]$

11. Consider the function $f(x) = 3 - |x - 4|$.

(a) Graph the function and verify that $f(1) = f(7)$.

(b) Note that $f'(x)$ is not equal to zero for any x in $[1, 7]$. Explain why this does not contradict Rolle's Theorem.

12. Can the Mean Value Theorem be applied to the function $f(x) = 1/x^2$ on the interval $[-2, 1]$? Explain.

In Exercises 13–18, determine whether the Mean Value Theorem can be applied to f on the closed interval $[a, b]$. If the Mean Value Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. If the Mean Value Theorem cannot be applied, explain why not.

13. $f(x) = x^{2/3}, \quad [1, 8]$

14. $f(x) = \dfrac{1}{x}, \quad [1, 4]$

15. $f(x) = |5 - x|, \quad [2, 6]$

16. $f(x) = 2x - 3\sqrt{x}, \quad [-1, 1]$

17. $f(x) = x - \cos x, \quad \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

18. $f(x) = \sqrt{x} - 2x, \quad [0, 4]$

19. For the function $f(x) = Ax^2 + Bx + C$, determine the value of c guaranteed by the Mean Value Theorem on the interval $[x_1, x_2]$.

20. Demonstrate the result of Exercise 19 for $f(x) = 2x^2 - 3x + 1$ on the interval $[0, 4]$.

In Exercises 21–26, find the critical numbers (if any) and the open intervals on which the function is increasing or decreasing.

21. $f(x) = x^2 + 3x - 12$

22. $h(x) = (x + 2)^{1/3} + 8$

23. $f(x) = (x - 1)^2(x - 3)$

24. $g(x) = (x + 1)^3$

25. $h(x) = \sqrt{x}(x - 3), \quad x > 0$

26. $f(x) = \sin x + \cos x, \quad [0, 2\pi]$

In Exercises 27–30, use the First Derivative Test to find any relative extrema of the function. Use a graphing utility to confirm your results.

27. $f(x) = 4x^3 - 5x$

28. $g(x) = \dfrac{x^3 - 8x}{4}$

29. $h(t) = \dfrac{1}{4}t^4 - 8t$

30. $g(x) = \dfrac{3}{2}\sin\left(\dfrac{\pi x}{2} - 1\right), \quad [0, 4]$

31. ***Harmonic Motion*** The height of an object attached to a spring is given by the harmonic equation

$$y = \tfrac{1}{3}\cos 12t - \tfrac{1}{4}\sin 12t$$

where y is measured in inches and t is measured in seconds.

(a) Calculate the height and velocity of the object when $t = \pi/8$ second.

(b) Show that the maximum displacement of the object is $\frac{5}{12}$ inch.

(c) Find the period P of y. Also, find the frequency f (number of oscillations per second) if $f = 1/P$.

32. ***Writing*** The general equation giving the height of an oscillating object attached to a spring is

$$y = A\sin\sqrt{\frac{k}{m}}\,t + B\cos\sqrt{\frac{k}{m}}\,t$$

where k is the spring constant and m is the mass of the object.

(a) Show that the maximum displacement of the object is $\sqrt{A^2 + B^2}$.

(b) Show that the object oscillates with a frequency of

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

In Exercises 33–36, determine the points of inflection and discuss the concavity of the graph of the function.

33. $f(x) = x^3 - 9x^2$

34. $g(x) = x\sqrt{x+5}$

35. $f(x) = x + \cos x, \quad [0, 2\pi]$

36. $f(x) = (x+2)^2(x-4)$

In Exercises 37–40, use the Second Derivative Test to find all relative extrema.

37. $f(x) = (x+9)^2$

38. $h(x) = x - 2\cos x, \quad [0, 4\pi]$

39. $g(x) = 2x^2(1 - x^2)$

40. $h(t) = t - 4\sqrt{t+1}$

Think About It **In Exercises 41 and 42, sketch the graph of a function *f* having the given characteristics.**

41. $f(0) = f(6) = 0$
$f'(3) = f'(5) = 0$
$f'(x) > 0$ if $x < 3$
$f'(x) > 0$ if $3 < x < 5$
$f'(x) < 0$ if $x > 5$
$f''(x) < 0$ if $x < 3$ or $x > 4$
$f''(x) > 0$ if $3 < x < 4$

42. $f(0) = 4,\ f(6) = 0$
$f'(x) < 0$ if $x < 2$ or $x > 4$
$f'(2)$ does not exist.
$f'(4) = 0$
$f'(x) > 0$ if $2 < x < 4$
$f''(x) < 0$ if $x \neq 2$

43. ***Writing*** A newspaper headline states that "The rate of growth of the national deficit is decreasing." What does this mean? What does it imply about the graph of the deficit as a function of time?

44. ***Inventory Cost*** The cost of inventory depends on the ordering and storage costs according to the inventory model

$$C = \left(\frac{Q}{x}\right)s + \left(\frac{x}{2}\right)r.$$

Determine the order size that will minimize the cost, assuming that sales occur at a constant rate, Q is the number of units sold per year, r is the cost of storing one unit for 1 year, s is the cost of placing an order, and x is the number of units per order.

45. ***Modeling Data*** Outlays for national defense D (in billions of dollars) for selected years from 1970 through 2005 are shown in the table, where t is time in years, with $t = 0$ corresponding to 1970. *(Source: U.S. Office of Management and Budget)*

t	0	5	10	15	20	25	30	35
D	81.7	86.5	134.0	252.7	299.3	272.1	294.5	495.3

(a) Use the regression capabilities of a graphing utility to fit a model of the form

$$D = at^4 + bt^3 + ct^2 + dt + e$$

to the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) For the years shown in the table, when does the model indicate that the outlay for national defense was at a maximum? When was it at a minimum?

(d) For the years shown in the table, when does the model indicate that the outlay for national defense was increasing at the greatest rate?

46. ***Modeling Data*** The manager of a store recorded the annual sales S (in thousands of dollars) of a product over a period of 7 years, as shown in the table, where t is the time in years, with $t = 1$ corresponding to 2001.

t	1	2	3	4	5	6	7
S	5.4	6.9	11.5	15.5	19.0	22.0	23.6

(a) Use the regression capabilities of a graphing utility to find a model of the form $S = at^3 + bt^2 + ct + d$ for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use calculus and the model to find the time t when sales were increasing at the greatest rate.

(d) Do you think the model would be accurate for predicting future sales? Explain.

In Exercises 47–56, find the limit.

47. $\displaystyle\lim_{x\to\infty}\left(8 + \frac{1}{x}\right)$

48. $\displaystyle\lim_{x\to\infty}\frac{3 - x}{2x + 5}$

49. $\displaystyle\lim_{x\to\infty}\frac{2x^2}{3x^2 + 5}$

50. $\displaystyle\lim_{x\to\infty}\frac{2x}{3x^2 + 5}$

51. $\displaystyle\lim_{x\to-\infty}\frac{3x^2}{x + 5}$

52. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2 + x}}{-2x}$

53. $\displaystyle\lim_{x\to\infty}\frac{5\cos x}{x}$

54. $\displaystyle\lim_{x\to\infty}\frac{3x}{\sqrt{x^2 + 4}}$

55. $\displaystyle\lim_{x\to-\infty}\frac{6x}{x + \cos x}$

56. $\displaystyle\lim_{x\to-\infty}\frac{x}{2\sin x}$

In Exercises 57–60, find any vertical and horizontal asymptotes of the graph of the function. Use a graphing utility to verify your results.

57. $f(x) = \dfrac{3}{x} - 2$

58. $g(x) = \dfrac{5x^2}{x^2 + 2}$

59. $h(x) = \dfrac{2x + 3}{x - 4}$

60. $f(x) = \dfrac{3x}{\sqrt{x^2 + 2}}$

In Exercises 61–64, use a graphing utility to graph the function. Use the graph to approximate any relative extrema or asymptotes.

61. $f(x) = x^3 + \dfrac{243}{x}$

62. $f(x) = |x^3 - 3x^2 + 2x|$

63. $f(x) = \dfrac{x - 1}{1 + 3x^2}$

64. $g(x) = \dfrac{\pi^2}{3} - 4\cos x + \cos 2x$

In Exercises 65–82, analyze and sketch the graph of the function.

65. $f(x) = 4x - x^2$

66. $f(x) = 4x^3 - x^4$

67. $f(x) = x\sqrt{16 - x^2}$

68. $f(x) = (x^2 - 4)^2$

69. $f(x) = (x - 1)^3(x - 3)^2$

70. $f(x) = (x - 3)(x + 2)^3$

71. $f(x) = x^{1/3}(x + 3)^{2/3}$

72. $f(x) = (x - 2)^{1/3}(x + 1)^{2/3}$

73. $f(x) = \dfrac{5 - 3x}{x - 2}$

74. $f(x) = \dfrac{2x}{1 + x^2}$

75. $f(x) = \dfrac{4}{1 + x^2}$

76. $f(x) = \dfrac{x^2}{1 + x^4}$

77. $f(x) = x^3 + x + \dfrac{4}{x}$

78. $f(x) = x^2 + \dfrac{1}{x}$

79. $f(x) = |x^2 - 9|$

80. $f(x) = |x - 1| + |x - 3|$

81. $f(x) = x + \cos x, \quad 0 \le x \le 2\pi$

82. $f(x) = \dfrac{1}{\pi}(2 \sin \pi x - \sin 2\pi x), \quad -1 \le x \le 1$

83. Find the maximum and minimum points on the graph of

$$x^2 + 4y^2 - 2x - 16y + 13 = 0$$

(a) without using calculus.

(b) using calculus.

84. Consider the function $f(x) = x^n$ for positive integer values of n.

(a) For what values of n does the function have a relative minimum at the origin?

(b) For what values of n does the function have a point of inflection at the origin?

85. ***Distance*** At noon, ship A is 100 kilometers due east of ship B. Ship A is sailing west at 12 kilometers per hour, and ship B is sailing south at 10 kilometers per hour. At what time will the ships be nearest to each other, and what will this distance be?

86. ***Maximum Area*** Find the dimensions of the rectangle of maximum area, with sides parallel to the coordinate axes, that can be inscribed in the ellipse given by

$$\frac{x^2}{144} + \frac{y^2}{16} = 1.$$

87. ***Minimum Length*** A right triangle in the first quadrant has the coordinate axes as sides, and the hypotenuse passes through the point $(1, 8)$. Find the vertices of the triangle such that the length of the hypotenuse is minimum.

88. ***Minimum Length*** The wall of a building is to be braced by a beam that must pass over a parallel fence 5 feet high and 4 feet from the building. Find the length of the shortest beam that can be used.

89. ***Maximum Area*** Three sides of a trapezoid have the same length s. Of all such possible trapezoids, show that the one of maximum area has a fourth side of length $2s$.

90. ***Maximum Area*** Show that the greatest area of any rectangle inscribed in a triangle is one-half the area of the triangle.

91. ***Distance*** Find the length of the longest pipe that can be carried level around a right-angle corner at the intersection of two corridors of widths 4 feet and 6 feet. (Do not use trigonometry.)

92. ***Distance*** Rework Exercise 91, given corridors of widths a meters and b meters.

93. ***Distance*** A hallway of width 6 feet meets a hallway of width 9 feet at right angles. Find the length of the longest pipe that can be carried level around this corner. [*Hint:* If L is the length of the pipe, show that

$$L = 6 \csc \theta + 9 \csc\left(\frac{\pi}{2} - \theta\right)$$

where θ is the angle between the pipe and the wall of the narrower hallway.]

94. ***Length*** Rework Exercise 93, given that one hallway is of width a meters and the other is of width b meters. Show that the result is the same as in Exercise 92.

Minimum Cost **In Exercises 95 and 96, find the speed v, in miles per hour, that will minimize costs on a 110-mile delivery trip. The cost per hour for fuel is C dollars, and the driver is paid W dollars per hour. (Assume there are no costs other than wages and fuel.)**

95. Fuel cost: $C = \dfrac{v^2}{600}$

Driver: $W = \$5$

96. Fuel cost: $C = \dfrac{v^2}{500}$

Driver: $W = \$7.50$

In Exercises 97 and 98, use Newton's Method to approximate any real zeros of the function accurate to three decimal places. Use the *zero* or *root* feature of a graphing utility to verify your results.

97. $f(x) = x^3 - 3x - 1$

98. $f(x) = x^3 + 2x + 1$

In Exercises 99 and 100, use Newton's Method to approximate, to three decimal places, the x-value(s) of the point(s) of intersection of the equations. Use a graphing utility to verify your results.

99. $y = x^4$

$y = x + 3$

100. $y = \sin \pi x$

$y = 1 - x$

In Exercises 101 and 102, find the differential dy.

101. $y = x(1 - \cos x)$

102. $y = \sqrt{36 - x^2}$

103. ***Surface Area and Volume*** The diameter of a sphere is measured as 18 centimeters, with a maximum possible error of 0.05 centimeter. Use differentials to approximate the possible propagated error and percent error in calculating the surface area and the volume of the sphere.

104. ***Demand Function*** A company finds that the demand for its commodity is

$$p = 75 - \frac{1}{4}x.$$

If x changes from 7 to 8, find and compare the values of Δp and dp.

P.S. PROBLEM SOLVING

1. Graph the fourth-degree polynomial $p(x) = x^4 + ax^2 + 1$ for various values of the constant a.

(a) Determine the values of a for which p has exactly one relative minimum.

(b) Determine the values of a for which p has exactly one relative maximum.

(c) Determine the values of a for which p has exactly two relative minima.

(d) Show that the graph of p cannot have exactly two relative extrema.

2. (a) Graph the fourth-degree polynomial $p(x) = ax^4 - 6x^2$ for $a = -3, -2, -1, 0, 1, 2,$ and 3. For what values of the constant a does p have a relative minimum or relative maximum?

(b) Show that p has a relative maximum for all values of the constant a.

(c) Determine analytically the values of a for which p has a relative minimum.

(d) Let $(x, y) = (x, p(x))$ be a relative extremum of p. Show that (x, y) lies on the graph of $y = -3x^2$. Verify this result graphically by graphing $y = -3x^2$ together with the seven curves from part (a).

3. Let $f(x) = \dfrac{c}{x} + x^2$. Determine all values of the constant c such that f has a relative minimum, but no relative maximum.

4. (a) Let $f(x) = ax^2 + bx + c$, $a \neq 0$, be a quadratic polynomial. How many points of inflection does the graph of f have?

(b) Let $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, be a cubic polynomial. How many points of inflection does the graph of f have?

(c) Suppose the function $y = f(x)$ satisfies the equation

$$\frac{dy}{dx} = ky\left(1 - \frac{y}{L}\right)$$ where k and L are positive constants.

Show that the graph of f has a point of inflection at the point where $y = \dfrac{L}{2}$. (This equation is called the **logistic differential equation.**)

5. Prove Darboux's Theorem: Let f be differentiable on the closed interval $[a, b]$ such that $f'(a) = y_1$ and $f'(b) = y_2$. If d lies between y_1 and y_2, then there exists c in (a, b) such that $f'(c) = d$.

6. Let f and g be functions that are continuous on $[a, b]$ and differentiable on (a, b). Prove that if $f(a) = g(a)$ and $g'(x) > f'(x)$ for all x in (a, b), then $g(b) > f(b)$.

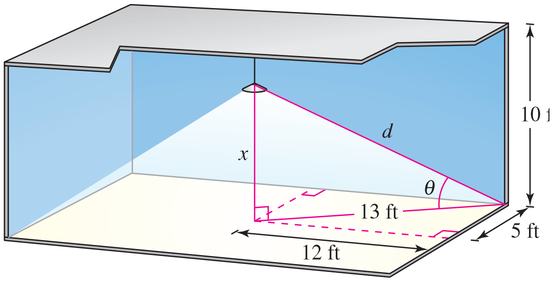
7. Prove the following **Extended Mean Value Theorem.** If f and f' are continuous on the closed interval $[a, b]$, and if f'' exists in the open interval (a, b), then there exists a number c in (a, b) such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{1}{2}f''(c)(b - a)^2.$$

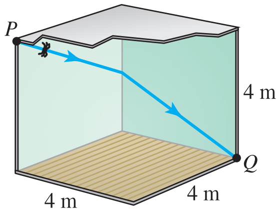
8. (a) Let $V = x^3$. Find dV and ΔV. Show that for small values of x, the difference $\Delta V - dV$ is very small in the sense that there exists ε such that $\Delta V - dV = \varepsilon\Delta x$, where $\varepsilon \to 0$ as $\Delta x \to 0$.

(b) Generalize this result by showing that if $y = f(x)$ is a differentiable function, then $\Delta y - dy = \varepsilon\Delta x$, where $\varepsilon \to 0$ as $\Delta x \to 0$.

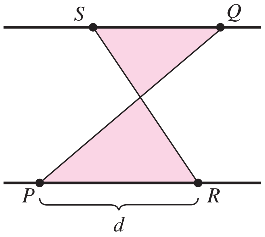
9. The amount of illumination of a surface is proportional to the intensity of the light source, inversely proportional to the square of the distance from the light source, and proportional to $\sin\theta$, where θ is the angle at which the light strikes the surface. A rectangular room measures 10 feet by 24 feet, with a 10-foot ceiling (see figure). Determine the height at which the light should be placed to allow the corners of the floor to receive as much light as possible.



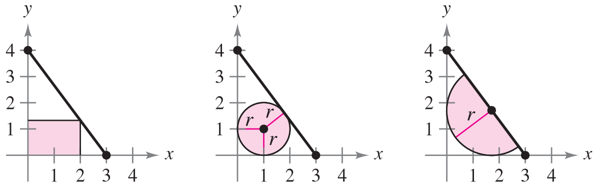
10. Consider a room in the shape of a cube, 4 meters on each side. A bug at point P wants to walk to point Q at the opposite corner, as shown in the figure. Use calculus to determine the shortest path. Can you solve the problem without calculus?



11. The line joining P and Q crosses the two parallel lines, as shown in the figure. The point R is d units from P. How far from Q should the point S be positioned so that the sum of the areas of the two shaded triangles is a minimum? So that the sum is a maximum?



12. The figures show a rectangle, a circle, and a semicircle inscribed in a triangle bounded by the coordinate axes and the first-quadrant portion of the line with intercepts $(3, 0)$ and $(0, 4)$. Find the dimensions of each inscribed figure such that its area is maximum. State whether calculus was helpful in finding the required dimensions. Explain your reasoning.

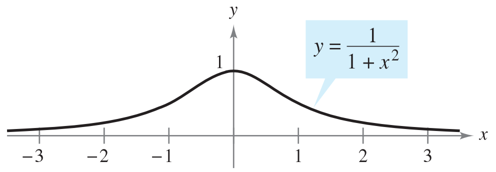


13. (a) Prove that $\lim_{x\to\infty} x^2 = \infty$.

(b) Prove that $\lim_{x\to\infty} \left(\dfrac{1}{x^2}\right) = 0$.

(c) Let L be a real number. Prove that if $\lim_{x\to\infty} f(x) = L$, then $\lim_{y\to 0^+} f\left(\dfrac{1}{y}\right) = L$.

14. Find the point on the graph of $y = \dfrac{1}{1 + x^2}$ (see figure) where the tangent line has the greatest slope, and the point where the tangent line has the least slope.



15. (a) Let x be a positive number. Use the *table* feature of a graphing utility to verify that $\sqrt{1 + x} < \frac{1}{2}x + 1$.

(b) Use the Mean Value Theorem to prove that $\sqrt{1 + x} < \frac{1}{2}x + 1$ for all positive real numbers x.

16. (a) Let x be a positive number. Use the *table* feature of a graphing utility to verify that $\sin x < x$.

(b) Use the Mean Value Theorem to prove that $\sin x < x$ for all positive real numbers x.

17. The police department must determine the speed limit on a bridge such that the flow rate of cars is maximum per unit time. The greater the speed limit, the farther apart the cars must be in order to keep a safe stopping distance. Experimental data on the stopping distances d (in meters) for various speeds v (in kilometers per hour) are shown in the table.

v	20	40	60	80	100
d	5.1	13.7	27.2	44.2	66.4

(a) Convert the speeds v in the table to speeds s in meters per second. Use the regression capabilities of a graphing utility to find a model of the form $d(s) = as^2 + bs + c$ for the data.

(b) Consider two consecutive vehicles of average length 5.5 meters, traveling at a safe speed on the bridge. Let T be the difference between the times (in seconds) when the front bumpers of the vehicles pass a given point on the bridge. Verify that this difference in times is given by

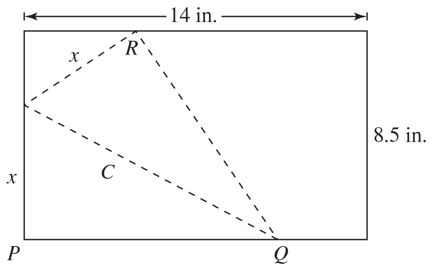
$$T = \frac{d(s)}{s} + \frac{5.5}{s}.$$

(c) Use a graphing utility to graph the function T and estimate the speed s that minimizes the time between vehicles.

(d) Use calculus to determine the speed that minimizes T. What is the minimum value of T? Convert the required speed to kilometers per hour.

(e) Find the optimal distance between vehicles for the posted speed limit determined in part (d).

18. A legal-sized sheet of paper (8.5 inches by 14 inches) is folded so that corner P touches the opposite 14-inch edge at R (see figure). $\left(\textit{Note: } PQ = \sqrt{C^2 - x^2}.\right)$



(a) Show that $C^2 = \dfrac{2x^3}{2x - 8.5}$.

(b) What is the domain of C?

(c) Determine the x-value that minimizes C.

(d) Determine the minimum length C.

19. The polynomial $P(x) = c_0 + c_1(x - a) + c_2(x - a)^2$ is the quadratic approximation of the function f at $(a, f(a))$ if $P(a) = f(a)$, $P'(a) = f'(a)$, and $P''(a) = f''(a)$.

(a) Find the quadratic approximation of

$$f(x) = \frac{x}{x + 1}$$

at $(0, 0)$.

(b) Use a graphing utility to graph $P(x)$ and $f(x)$ in the same viewing window.

20. Let $x > 0$ and $n > 1$ be real numbers. Prove that $(1 + x)^n > 1 + nx$.

4 Integration

In this chapter, you will study an important process of calculus that is closely related to differentiation–*integration*. You will learn new methods and rules for solving definite and indefinite integrals, including the Fundamental Theorem of Calculus. Then you will apply these rules to find such things as the position function for an object and the average value of a function.

In this chapter, you should learn the following.

- How to evaluate indefinite integrals using basic integration rules. (4.1)
- How to evaluate a sum and approximate the area of a plane region. (4.2)
- How to evaluate a definite integral using a limit. (4.3)
- How to evaluate a definite integral using the Fundamental Theorem of Calculus. (4.4)
- How to evaluate different types of definite and indefinite integrals using a variety of methods. (4.5)
- How to approximate a definite integral using the Trapezoidal Rule and Simpson's Rule. (4.6)

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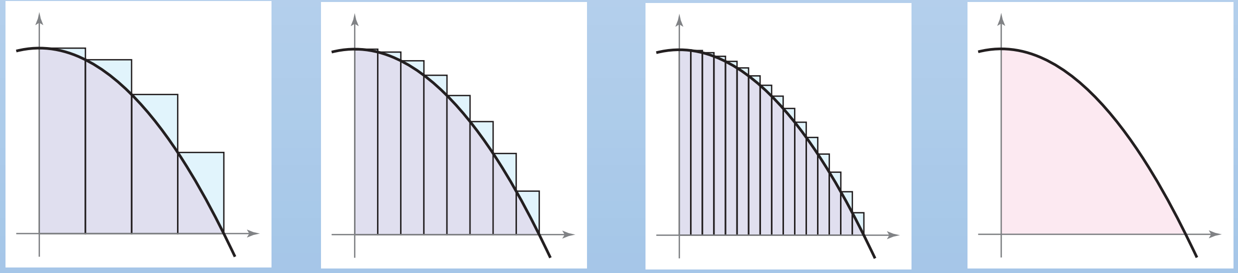
Although its official nickname is the Emerald City, Seattle is sometimes called the Rainy City due to its weather. But there are several cities, including New York and Boston, that typically get more annual precipitation. How could you use integration to calculate the normal annual precipitation for the Seattle area? (See Section 4.5, Exercise 117.)



The area of a parabolic region can be approximated as the sum of the areas of rectangles. As you increase the number of rectangles, the approximation tends to become more and more accurate. In Section 4.2, you will learn how the limit process can be used to find areas of a wide variety of regions.

4.1 Antiderivatives and Indefinite Integration

- **Write the general solution of a differential equation.**
- **Use indefinite integral notation for antiderivatives.**
- **Use basic integration rules to find antiderivatives.**
- **Find a particular solution of a differential equation.**

EXPLORATION

Finding Antiderivatives For each derivative, describe the original function F.

a. $F'(x) = 2x$ **b.** $F'(x) = x$

c. $F'(x) = x^2$ **d.** $F'(x) = \dfrac{1}{x^2}$

e. $F'(x) = \dfrac{1}{x^3}$ **f.** $F'(x) = \cos x$

What strategy did you use to find F?

Antiderivatives

Suppose you were asked to find a function F whose derivative is $f(x) = 3x^2$. From your knowledge of derivatives, you would probably say that

$$F(x) = x^3 \text{ because } \frac{d}{dx}[x^3] = 3x^2.$$

The function F is an *antiderivative* of f.

DEFINITION OF ANTIDERIVATIVE

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Note that F is called *an* antiderivative of f, rather than *the* antiderivative of f. To see why, observe that

$$F_1(x) = x^3, \quad F_2(x) = x^3 - 5, \quad \text{and} \quad F_3(x) = x^3 + 97$$

are all antiderivatives of $f(x) = 3x^2$. In fact, for any constant C, the function given by $F(x) = x^3 + C$ is an antiderivative of f.

THEOREM 4.1 REPRESENTATION OF ANTIDERIVATIVES

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$, for all x in I where C is a constant.

PROOF The proof of Theorem 4.1 in one direction is straightforward. That is, if $G(x) = F(x) + C$, $F'(x) = f(x)$, and C is a constant, then

$$G'(x) = \frac{d}{dx}[F(x) + C] = F'(x) + 0 = f(x).$$

To prove this theorem in the other direction, assume that G is an antiderivative of f. Define a function H such that

$$H(x) = G(x) - F(x).$$

For any two points a and b $(a < b)$ in the interval, H is continuous on $[a, b]$ and differentiable on (a, b). By the Mean Value Theorem,

$$H'(c) = \frac{H(b) - H(a)}{b - a}$$

for some c in (a, b). However, $H'(c) = 0$, so $H(a) = H(b)$. Because a and b are arbitrary points in the interval, you know that H is a constant function C. So, $G(x) - F(x) = C$ and it follows that $G(x) = F(x) + C$. ■

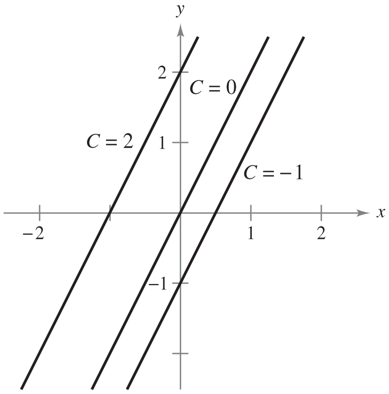
Using Theorem 4.1, you can represent the entire family of antiderivatives of a function by adding a constant to a *known* antiderivative. For example, knowing that $D_x[x^2] = 2x$, you can represent the family of *all* antiderivatives of $f(x) = 2x$ by

$$G(x) = x^2 + C \qquad \text{Family of all antiderivatives of } f(x) = 2x$$

where C is a constant. The constant C is called the **constant of integration.** The family of functions represented by G is the **general antiderivative** of f, and $G(x) = x^2 + C$ is the **general solution** of the *differential equation*

$$G'(x) = 2x. \qquad \text{Differential equation}$$

A **differential equation** in x and y is an equation that involves x, y, and derivatives of y. For instance, $y' = 3x$ and $y' = x^2 + 1$ are examples of differential equations.



Functions of the form $y = 2x + C$
Figure 4.1

EXAMPLE 1 Solving a Differential Equation

Find the general solution of the differential equation $y' = 2$.

Solution To begin, you need to find a function whose derivative is 2. One such function is

$$y = 2x. \qquad 2x \text{ is } \textit{an} \text{ antiderivative of } 2.$$

Now, you can use Theorem 4.1 to conclude that the general solution of the differential equation is

$$y = 2x + C. \qquad \text{General solution}$$

The graphs of several functions of the form $y = 2x + C$ are shown in Figure 4.1. ■

Notation for Antiderivatives

When solving a differential equation of the form

$$\frac{dy}{dx} = f(x)$$

it is convenient to write it in the equivalent differential form

$$dy = f(x)\,dx.$$

The operation of finding all solutions of this equation is called **antidifferentiation** (or **indefinite integration**) and is denoted by an integral sign $\int$. The general solution is denoted by

$$y = \int \underset{\substack{\uparrow\\ \boxed{\text{Integrand}}}}{f(x)}\, \overset{\substack{\boxed{\text{Variable of integration}}\\ \downarrow}}{dx} = \underset{\substack{\uparrow\\ \boxed{\text{An antiderivative of } f(x)}}}{F(x)} + \overset{\substack{\boxed{\text{Constant of integration}}\\ \downarrow}}{C}.$$

The expression $\int f(x)\,dx$ is read as the *antiderivative of f with respect to x.* So, the differential dx serves to identify x as the variable of integration. The term **indefinite integral** is a synonym for antiderivative.

NOTE In this text, the notation $\int f(x)\,dx = F(x) + C$ means that F is an antiderivative of f *on an interval.*

Basic Integration Rules

The inverse nature of integration and differentiation can be verified by substituting $F'(x)$ for $f(x)$ in the indefinite integration definition to obtain

$$\int F'(x)\,dx = F(x) + C.$$

Integration is the "inverse" of differentiation.

Moreover, if $\int f(x)\,dx = F(x) + C$, then

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

Differentiation is the "inverse" of integration.

These two equations allow you to obtain integration formulas directly from differentiation formulas, as shown in the following summary.

BASIC INTEGRATION RULES

Differentiation Formula	*Integration Formula*	
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$	
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$	
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$	
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	Power Rule
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$	
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$	
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$	
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$	
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$	
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$	

NOTE Note that the Power Rule for Integration has the restriction that $n \neq -1$. The evaluation of $\int 1/x\,dx$ must wait until the introduction of the natural logarithmic function in Chapter 5. ■

EXAMPLE 2 Applying the Basic Integration Rules

Describe the antiderivatives of $3x$.

Solution

$$
\begin{aligned}
\int 3x\,dx &= 3\int x\,dx && \text{Constant Multiple Rule}\\
&= 3\int x^1\,dx && \text{Rewrite } x \text{ as } x^1.\\
&= 3\left(\frac{x^2}{2}\right) + C && \text{Power Rule } (n = 1)\\
&= \frac{3}{2}x^2 + C && \text{Simplify.}
\end{aligned}
$$

So, the antiderivatives of $3x$ are of the form $\frac{3}{2}x^2 + C$, where C is any constant. ■

When indefinite integrals are evaluated, a strict application of the basic integration rules tends to produce complicated constants of integration. For instance, in Example 2, you could have written

$$\int 3x\,dx = 3\int x\,dx = 3\left(\frac{x^2}{2} + C\right) = \frac{3}{2}x^2 + 3C.$$

However, because C represents *any* constant, it is both cumbersome and unnecessary to write $3C$ as the constant of integration. So, $\frac{3}{2}x^2 + 3C$ is written in the simpler form, $\frac{3}{2}x^2 + C$.

In Example 2, note that the general pattern of integration is similar to that of differentiation.

Original integral ⇨ Rewrite ⇨ Integrate ⇨ Simplify

EXAMPLE 3 Rewriting Before Integrating

	Original Integral	*Rewrite*	*Integrate*	*Simplify*
a.	$\int \frac{1}{x^3}\,dx$	$\int x^{-3}\,dx$	$\frac{x^{-2}}{-2} + C$	$-\frac{1}{2x^2} + C$
b.	$\int \sqrt{x}\,dx$	$\int x^{1/2}\,dx$	$\frac{x^{3/2}}{3/2} + C$	$\frac{2}{3}x^{3/2} + C$
c.	$\int 2\sin x\,dx$	$2\int \sin x\,dx$	$2(-\cos x) + C$	$-2\cos x + C$

■

TECHNOLOGY Some software programs, such as *Maple*, *Mathematica*, and the *TI-89*, are capable of performing integration symbolically. If you have access to such a symbolic integration utility, try using it to evaluate the indefinite integrals in Example 3.

Remember that you can check your answer to an antidifferentiation problem by differentiating. For instance, in Example 3(b), you can check that $\frac{2}{3}x^{3/2} + C$ is the correct antiderivative by differentiating the answer to obtain

$$D_x\left[\frac{2}{3}x^{3/2} + C\right] = \left(\frac{2}{3}\right)\left(\frac{3}{2}\right)x^{1/2} = \sqrt{x}. \qquad \text{Use differentiation to check antiderivative.}$$

The icon ↻ indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

The basic integration rules listed on page 250 allow you to integrate any polynomial function, as shown in Example 4.

EXAMPLE 4 Integrating Polynomial Functions

a.

$$\begin{aligned}\int dx &= \int 1\,dx && \text{Integrand is understood to be 1.}\\ &= x + C && \text{Integrate.}\end{aligned}$$

b.

$$\begin{aligned}\int (x+2)\,dx &= \int x\,dx + \int 2\,dx && \\ &= \frac{x^2}{2} + C_1 + 2x + C_2 && \text{Integrate.}\\ &= \frac{x^2}{2} + 2x + C && C = C_1 + C_2\end{aligned}$$

The second line in the solution is usually omitted.

c.

$$\begin{aligned}\int (3x^4 - 5x^2 + x)\,dx &= 3\left(\frac{x^5}{5}\right) - 5\left(\frac{x^3}{3}\right) + \frac{x^2}{2} + C && \text{Integrate.}\\ &= \frac{3}{5}x^5 - \frac{5}{3}x^3 + \frac{1}{2}x^2 + C && \text{Simplify.}\end{aligned}$$

EXAMPLE 5 Rewriting Before Integrating

$$\begin{aligned}\int \frac{x+1}{\sqrt{x}}\,dx &= \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx && \text{Rewrite as two fractions.}\\ &= \int (x^{1/2} + x^{-1/2})\,dx && \text{Rewrite with fractional exponents.}\\ &= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C && \text{Integrate.}\\ &= \frac{2}{3}x^{3/2} + 2x^{1/2} + C && \text{Simplify.}\\ &= \frac{2}{3}\sqrt{x}(x+3) + C && \blacksquare\end{aligned}$$

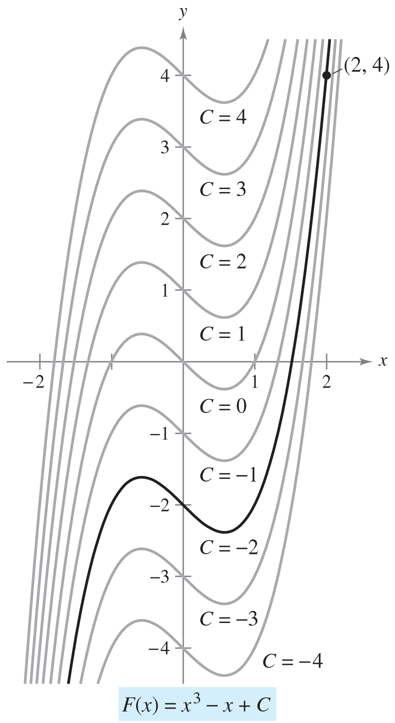
STUDY TIP Remember that you can check your answer by differentiating.

NOTE When integrating quotients, do not integrate the numerator and denominator separately. This is no more valid in integration than it is in differentiation. For instance, in Example 5, be sure you understand that

$$\int \frac{x+1}{\sqrt{x}}\,dx = \frac{2}{3}\sqrt{x}(x+3) + C \text{ is not the same as } \frac{\int (x+1)\,dx}{\int \sqrt{x}\,dx} = \frac{\frac{1}{2}x^2 + x + C_1}{\frac{2}{3}x\sqrt{x} + C_2}. \qquad \blacksquare$$

EXAMPLE 6 Rewriting Before Integrating

$$\begin{aligned}\int \frac{\sin x}{\cos^2 x}\,dx &= \int \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) dx && \text{Rewrite as a product.}\\ &= \int \sec x \tan x\,dx && \text{Rewrite using trigonometric identities.}\\ &= \sec x + C && \text{Integrate.} \quad \blacksquare\end{aligned}$$



The particular solution that satisfies the initial condition $F(2) = 4$ is $F(x) = x^3 - x - 2$.
Figure 4.2

Initial Conditions and Particular Solutions

You have already seen that the equation $y = \int f(x)\,dx$ has many solutions (each differing from the others by a constant). This means that the graphs of any two antiderivatives of f are vertical translations of each other. For example, Figure 4.2 shows the graphs of several antiderivatives of the form

$$y = \int (3x^2 - 1)\,dx = x^3 - x + C \qquad \text{General solution}$$

for various integer values of C. Each of these antiderivatives is a solution of the differential equation

$$\frac{dy}{dx} = 3x^2 - 1.$$

In many applications of integration, you are given enough information to determine a **particular solution.** To do this, you need only know the value of $y = F(x)$ for one value of x. This information is called an **initial condition.** For example, in Figure 4.2, only one curve passes through the point $(2, 4)$. To find this curve, you can use the following information.

$$F(x) = x^3 - x + C \qquad \text{General solution}$$
$$F(2) = 4 \qquad \text{Initial condition}$$

By using the initial condition in the general solution, you can determine that $F(2) = 8 - 2 + C = 4$, which implies that $C = -2$. So, you obtain

$$F(x) = x^3 - x - 2. \qquad \text{Particular solution}$$

EXAMPLE 7 Finding a Particular Solution

Find the general solution of

$$F'(x) = \frac{1}{x^2}, \quad x > 0$$

and find the particular solution that satisfies the initial condition $F(1) = 0$.

Solution To find the general solution, integrate to obtain

$$\begin{aligned} F(x) &= \int \frac{1}{x^2}\,dx && F(x) = \int F'(x)\,dx \\ &= \int x^{-2}\,dx && \text{Rewrite as a power.} \\ &= \frac{x^{-1}}{-1} + C && \text{Integrate.} \\ &= -\frac{1}{x} + C, \quad x > 0. && \text{General solution} \end{aligned}$$

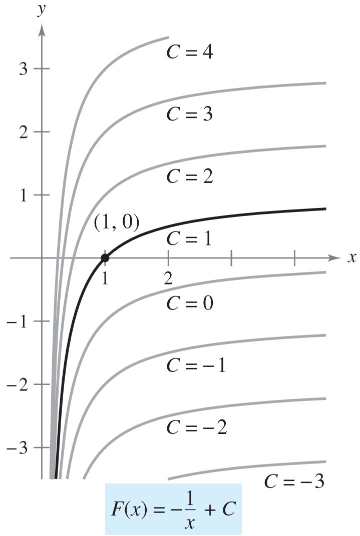
Using the initial condition $F(1) = 0$, you can solve for C as follows.

$$F(1) = -\frac{1}{1} + C = 0 \quad \Rightarrow \quad C = 1$$

So, the particular solution, as shown in Figure 4.3, is

$$F(x) = -\frac{1}{x} + 1, \quad x > 0. \qquad \text{Particular solution}$$

■



The particular solution that satisfies the initial condition $F(1) = 0$ is $F(x) = -(1/x) + 1$, $x > 0$.
Figure 4.3

So far in this section you have been using x as the variable of integration. In applications, it is often convenient to use a different variable. For instance, in the following example involving *time*, the variable of integration is t.

EXAMPLE 8 Solving a Vertical Motion Problem

A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet.

a. Find the position function giving the height s as a function of the time t.

b. When does the ball hit the ground?

Solution

a. Let $t = 0$ represent the initial time. The two given initial conditions can be written as follows.

$$s(0) = 80 \qquad \text{Initial height is 80 feet.}$$
$$s'(0) = 64 \qquad \text{Initial velocity is 64 feet per second.}$$

Using -32 feet per second per second as the acceleration due to gravity, you can write

$$s''(t) = -32$$
$$s'(t) = \int s''(t)\,dt = \int -32\,dt = -32t + C_1.$$

Using the initial velocity, you obtain $s'(0) = 64 = -32(0) + C_1$, which implies that $C_1 = 64$. Next, by integrating $s'(t)$, you obtain

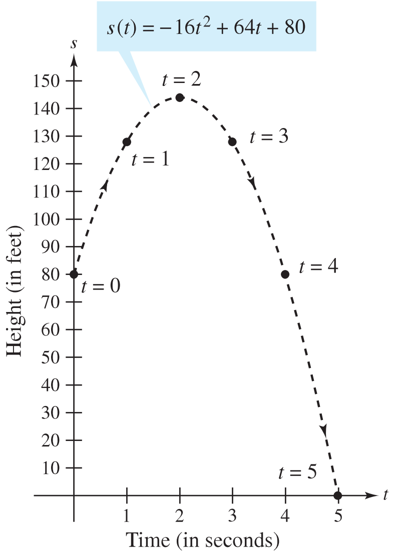
$$s(t) = \int s'(t)\,dt = \int (-32t + 64)\,dt = -16t^2 + 64t + C_2.$$

Using the initial height, you obtain

$$s(0) = 80 = -16(0^2) + 64(0) + C_2$$

which implies that $C_2 = 80$. So, the position function is

$$s(t) = -16t^2 + 64t + 80. \qquad \text{See Figure 4.4.}$$



Height of a ball at time t
Figure 4.4

b. Using the position function found in part (a), you can find the time at which the ball hits the ground by solving the equation $s(t) = 0$.

$$s(t) = -16t^2 + 64t + 80 = 0$$
$$-16(t + 1)(t - 5) = 0$$
$$t = -1, 5$$

Because t must be positive, you can conclude that the ball hits the ground 5 seconds after it was thrown. ■

NOTE In Example 8, note that the position function has the form

$$s(t) = \tfrac{1}{2}gt^2 + v_0t + s_0$$

where $g = -32$, v_0 is the initial velocity, and s_0 is the initial height, as presented in Section 2.2.

Example 8 shows how to use calculus to analyze vertical motion problems in which the acceleration is determined by a gravitational force. You can use a similar strategy to analyze other linear motion problems (vertical or horizontal) in which the acceleration (or deceleration) is the result of some other force, as you will see in Exercises 81–89.

Before you begin the exercise set, be sure you realize that one of the most important steps in integration is *rewriting the integrand* in a form that fits the basic integration rules. To illustrate this point further, here are some additional examples.

Original Integral	*Rewrite*	*Integrate*	*Simplify*
$\int \frac{2}{\sqrt{x}}\,dx$	$2\int x^{-1/2}\,dx$	$2\left(\frac{x^{1/2}}{1/2}\right) + C$	$4x^{1/2} + C$
$\int (t^2 + 1)^2\,dt$	$\int (t^4 + 2t^2 + 1)\,dt$	$\frac{t^5}{5} + 2\left(\frac{t^3}{3}\right) + t + C$	$\frac{1}{5}t^5 + \frac{2}{3}t^3 + t + C$
$\int \frac{x^3 + 3}{x^2}\,dx$	$\int (x + 3x^{-2})\,dx$	$\frac{x^2}{2} + 3\left(\frac{x^{-1}}{-1}\right) + C$	$\frac{1}{2}x^2 - \frac{3}{x} + C$
$\int \sqrt[3]{x}(x - 4)\,dx$	$\int (x^{4/3} - 4x^{1/3})\,dx$	$\frac{x^{7/3}}{7/3} - 4\left(\frac{x^{4/3}}{4/3}\right) + C$	$\frac{3}{7}x^{7/3} - 3x^{4/3}$

4.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, verify the statement by showing that the derivative of the right side equals the integrand of the left side.

1. $\int \left(-\frac{6}{x^4}\right) dx = \frac{2}{x^3} + C$

2. $\int \left(8x^3 + \frac{1}{2x^2}\right) dx = 2x^4 - \frac{1}{2x} + C$

3. $\int (x - 4)(x + 4)\,dx = \frac{1}{3}x^3 - 16x + C$

4. $\int \frac{x^2 - 1}{x^{3/2}}\,dx = \frac{2(x^2 + 3)}{3\sqrt{x}} + C$

In Exercises 5–8, find the general solution of the differential equation and check the result by differentiation.

5. $\frac{dy}{dt} = 9t^2$

6. $\frac{dr}{d\theta} = \pi$

7. $\frac{dy}{dx} = x^{3/2}$

8. $\frac{dy}{dx} = 2x^{-3}$

In Exercises 9–14, complete the table.

	Original Integral	*Rewrite*	*Integrate*	*Simplify*
9.	$\int \sqrt[3]{x}\,dx$			
10.	$\int \frac{1}{4x^2}\,dx$			
11.	$\int \frac{1}{x\sqrt{x}}\,dx$			
12.	$\int x(x^3 + 1)\,dx$			
13.	$\int \frac{1}{2x^3}\,dx$			
14.	$\int \frac{1}{(3x)^2}\,dx$			

In Exercises 15–34, find the indefinite integral and check the result by differentiation.

15. $\int (x + 7)\,dx$

16. $\int (13 - x)\,dx$

17. $\int (2x - 3x^2)\,dx$

18. $\int (8x^3 - 9x^2 + 4)\,dx$

19. $\int (x^5 + 1)\,dx$

20. $\int (x^3 - 10x - 3)\,dx$

21. $\int (x^{3/2} + 2x + 1)\,dx$

22. $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx$

23. $\int \sqrt[3]{x^2}\,dx$

24. $\int \left(\sqrt[4]{x^3} + 1\right) dx$

25. $\int \frac{1}{x^5}\,dx$

26. $\int \frac{1}{x^6}\,dx$

27. $\int \frac{x + 6}{\sqrt{x}}\,dx$

28. $\int \frac{x^2 + 2x - 3}{x^4}\,dx$

29. $\int (x + 1)(3x - 2)\,dx$

30. $\int (2t^2 - 1)^2\,dt$

31. $\int y^2\sqrt{y}\,dy$

32. $\int (1 + 3t)t^2\,dt$

33. $\int dx$

34. $\int 14\,dt$

In Exercises 35–44, find the indefinite integral and check the result by differentiation.

35. $\int (5\cos x + 4\sin x)\,dx$

36. $\int (t^2 - \cos t)\,dt$

37. $\int (1 - \csc t\cot t)\,dt$

38. $\int (\theta^2 + \sec^2\theta)\,d\theta$

39. $\int (\sec^2\theta - \sin\theta)\,d\theta$

40. $\int \sec y(\tan y - \sec y)\,dy$

41. $\int (\tan^2 y + 1)\, dy$

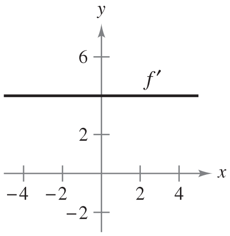
42. $\int (4x - \csc^2 x)\, dx$

43. $\int \frac{\cos x}{1 - \cos^2 x}\, dx$

44. $\int \frac{\sin x}{1 - \sin^2 x}\, dx$

In Exercises 45–48, the graph of the derivative of a function is given. Sketch the graphs of *two* functions that have the given derivative. (There is more than one correct answer.) To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

45.



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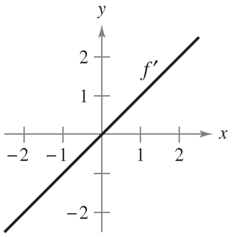

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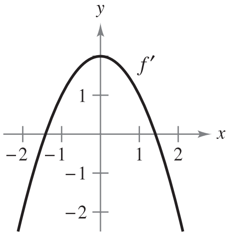

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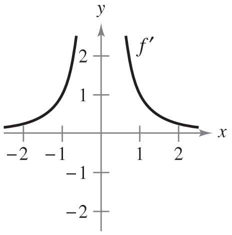
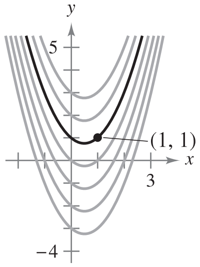
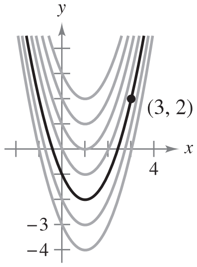


In Exercises 49 and 50, find the equation of y, given the derivative and the indicated point on the curve.

49. $\frac{dy}{dx} = 2x - 1$

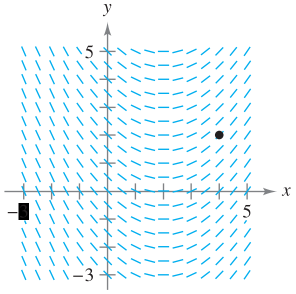


50. $\frac{dy}{dx} = 2(x - 1)$

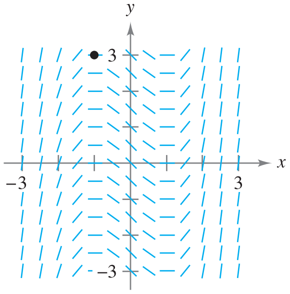


Slope Fields **In Exercises 51–54, a differential equation, a point, and a slope field are given. A *slope field* (or *direction field*) consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the slopes of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the indicated point. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).**

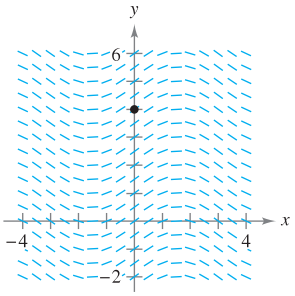
51. $\frac{dy}{dx} = \frac{1}{2}x - 1,\ (4, 2)$



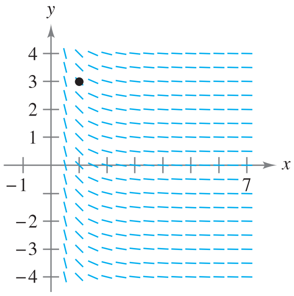
52. $\frac{dy}{dx} = x^2 - 1,\ (-1, 3)$



53. $\frac{dy}{dx} = \cos x,\ (0, 4)$



54. $\frac{dy}{dx} = -\frac{1}{x^2},\ x > 0,\ (1, 3)$



Slope Fields **In Exercises 55 and 56, (a) use a graphing utility to graph a slope field for the differential equation, (b) use integration and the given point to find the particular solution of the differential equation, and (c) graph the solution and the slope field in the same viewing window.**

55. $\frac{dy}{dx} = 2x,\ (-2, -2)$

56. $\frac{dy}{dx} = 2\sqrt{x},\ (4, 12)$

In Exercises 57–64, solve the differential equation.

57. $f'(x) = 6x,\ f(0) = 8$

58. $g'(x) = 6x^2,\ g(0) = -1$

59. $h'(t) = 8t^3 + 5,\ h(1) = -4$

60. $f'(s) = 10s - 12s^3,\ f(3) = 2$

61. $f''(x) = 2,\ f'(2) = 5,\ f(2) = 10$

62. $f''(x) = x^2,\ f'(0) = 8,\ f(0) = 4$

63. $f''(x) = x^{-3/2},\ f'(4) = 2,\ f(0) = 0$

64. $f''(x) = \sin x,\ f'(0) = 1,\ f(0) = 6$

65. ***Tree Growth*** An evergreen nursery usually sells a certain type of shrub after 6 years of growth and shaping. The growth rate during those 6 years is approximated by $dh/dt = 1.5t + 5$, where t is the time in years and h is the height in centimeters. The seedlings are 12 centimeters tall when planted $(t = 0)$.

(a) Find the height after t years.

(b) How tall are the shrubs when they are sold?

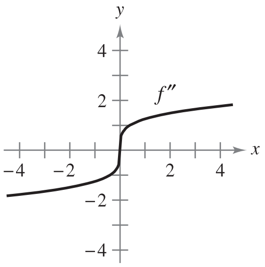
66. ***Population Growth*** The rate of growth dP/dt of a population of bacteria is proportional to the square root of t, where P is the population size and t is the time in days $(0 \le t \le 10)$. That is, $dP/dt = k\sqrt{t}$. The initial size of the population is 500. After 1 day the population has grown to 600. Estimate the population after 7 days.

WRITING ABOUT CONCEPTS

67. What is the difference, if any, between finding the antiderivative of $f(x)$ and evaluating the integral $\int f(x)\,dx$?

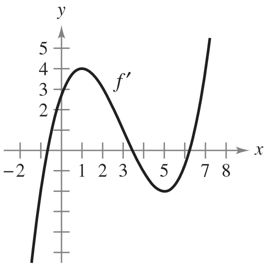
68. Consider $f(x) = \tan^2 x$ and $g(x) = \sec^2 x$. What do you notice about the derivatives of $f(x)$ and $g(x)$? What can you conclude about the relationship between $f(x)$ and $g(x)$?

69. The graphs of f and f' each pass through the origin. Use the graph of f'' shown in the figure to sketch the graphs of f and f'. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



CAPSTONE

70. Use the graph of f' shown in the figure to answer the following, given that $f(0) = -4$.



(a) Approximate the slope of f at $x = 4$. Explain.

(b) Is it possible that $f(2) = -1$? Explain.

(c) Is $f(5) - f(4) > 0$? Explain.

(d) Approximate the value of x where f is maximum. Explain.

(e) Approximate any intervals in which the graph of f is concave upward and any intervals in which it is concave downward. Approximate the x-coordinates of any points of inflection.

(f) Approximate the x-coordinate of the minimum of $f''(x)$.

(g) Sketch an approximate graph of f. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

Vertical Motion **In Exercises 71–74, use $a(t) = -32$ feet per second per second as the acceleration due to gravity. (Neglect air resistance.)**

71. A ball is thrown vertically upward from a height of 6 feet with an initial velocity of 60 feet per second. How high will the ball go?

72. Show that the height above the ground of an object thrown upward from a point s_0 feet above the ground with an initial velocity of v_0 feet per second is given by the function

$$f(t) = -16t^2 + v_0 t + s_0.$$

73. With what initial velocity must an object be thrown upward (from ground level) to reach the top of the Washington Monument (approximately 550 feet)?

74. A balloon, rising vertically with a velocity of 16 feet per second, releases a sandbag at the instant it is 64 feet above the ground.

(a) How many seconds after its release will the bag strike the ground?

(b) At what velocity will it hit the ground?

Vertical Motion **In Exercises 75–78, use $a(t) = -9.8$ meters per second per second as the acceleration due to gravity. (Neglect air resistance.)**

75. Show that the height above the ground of an object thrown upward from a point s_0 meters above the ground with an initial velocity of v_0 meters per second is given by the function

$$f(t) = -4.9t^2 + v_0 t + s_0.$$

76. The Grand Canyon is 1800 meters deep at its deepest point. A rock is dropped from the rim above this point. Write the height of the rock as a function of the time t in seconds. How long will it take the rock to hit the canyon floor?

77. A baseball is thrown upward from a height of 2 meters with an initial velocity of 10 meters per second. Determine its maximum height.

78. With what initial velocity must an object be thrown upward (from a height of 2 meters) to reach a maximum height of 200 meters?

79. ***Lunar Gravity*** On the moon, the acceleration due to gravity is -1.6 meters per second per second. A stone is dropped from a cliff on the moon and hits the surface of the moon 20 seconds later. How far did it fall? What was its velocity at impact?

80. ***Escape Velocity*** The minimum velocity required for an object to escape Earth's gravitational pull is obtained from the solution of the equation

$$\int v\,dv = -GM\int \frac{1}{y^2}\,dy$$

where v is the velocity of the object projected from Earth, y is the distance from the center of Earth, G is the gravitational constant, and M is the mass of Earth. Show that v and y are related by the equation

$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

where v_0 is the initial velocity of the object and R is the radius of Earth.

Rectilinear Motion **In Exercises 81–84, consider a particle moving along the x-axis where $x(t)$ is the position of the particle at time t, $x'(t)$ is its velocity, and $x''(t)$ is its acceleration.**

81. $x(t) = t^3 - 6t^2 + 9t - 2, \quad 0 \le t \le 5$

(a) Find the velocity and acceleration of the particle.

(b) Find the open t-intervals on which the particle is moving to the right.

(c) Find the velocity of the particle when the acceleration is 0.

82. Repeat Exercise 81 for the position function

$$x(t) = (t-1)(t-3)^2, \quad 0 \le t \le 5$$

83. A particle moves along the x-axis at a velocity of $v(t) = 1/\sqrt{t}$, $t > 0$. At time $t = 1$, its position is $x = 4$. Find the acceleration and position functions for the particle.

84. A particle, initially at rest, moves along the x-axis such that its acceleration at time $t > 0$ is given by $a(t) = \cos t$. At the time $t = 0$, its position is $x = 3$.

(a) Find the velocity and position functions for the particle.

(b) Find the values of t for which the particle is at rest.

85. ***Acceleration*** The maker of an automobile advertises that it takes 13 seconds to accelerate from 25 kilometers per hour to 80 kilometers per hour. Assuming constant acceleration, compute the following.

(a) The acceleration in meters per second per second

(b) The distance the car travels during the 13 seconds

86. ***Deceleration*** A car traveling at 45 miles per hour is brought to a stop, at constant deceleration, 132 feet from where the brakes are applied.

(a) How far has the car moved when its speed has been reduced to 30 miles per hour?

(b) How far has the car moved when its speed has been reduced to 15 miles per hour?

(c) Draw the real number line from 0 to 132, and plot the points found in parts (a) and (b). What can you conclude?

87. ***Acceleration*** At the instant the traffic light turns green, a car that has been waiting at an intersection starts with a constant acceleration of 6 feet per second per second. At the same instant, a truck traveling with a constant velocity of 30 feet per second passes the car.

(a) How far beyond its starting point will the car pass the truck?

(b) How fast will the car be traveling when it passes the truck?

88. ***Acceleration*** Assume that a fully loaded plane starting from rest has a constant acceleration while moving down a runway. The plane requires 0.7 mile of runway and a speed of 160 miles per hour in order to lift off. What is the plane's acceleration?

89. ***Airplane Separation*** Two airplanes are in a straight-line landing pattern and, according to FAA regulations, must keep at least a three-mile separation. Airplane A is 10 miles from touchdown and is gradually decreasing its speed from 150 miles per hour to a landing speed of 100 miles per hour. Airplane B is 17 miles from touchdown and is gradually decreasing its speed from 250 miles per hour to a landing speed of 115 miles per hour.

(a) Assuming the deceleration of each airplane is constant, find the position functions s_A and s_B for airplane A and airplane B. Let $t = 0$ represent the times when the airplanes are 10 and 17 miles from the airport.

(b) Use a graphing utility to graph the position functions.

(c) Find a formula for the magnitude of the distance d between the two airplanes as a function of t. Use a graphing utility to graph d. Is $d < 3$ for some time prior to the landing of airplane A? If so, find that time.

True or False? **In Exercises 90–95, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

90. Each antiderivative of an nth-degree polynomial function is an $(n+1)$th-degree polynomial function.

91. If $p(x)$ is a polynomial function, then p has exactly one antiderivative whose graph contains the origin.

92. If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, then $F(x) = G(x) + C$.

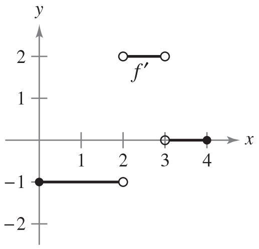
93. If $f'(x) = g(x)$, then $\int g(x)\,dx = f(x) + C$.

94. $\int f(x)g(x)\,dx = \int f(x)\,dx \int g(x)\,dx$

95. The antiderivative of $f(x)$ is unique.

96. Find a function f such that the graph of f has a horizontal tangent at $(2, 0)$ and $f''(x) = 2x$.

97. The graph of f' is shown. Sketch the graph of f given that f is continuous and $f(0) = 1$.



98. If $f'(x) = \begin{cases} 1, & 0 \le x < 2 \\ 3x, & 2 \le x \le 5 \end{cases}$, f is continuous, and $f(1) = 3$, find f. Is f differentiable at $x = 2$?

99. Let $s(x)$ and $c(x)$ be two functions satisfying $s'(x) = c(x)$ and $c'(x) = -s(x)$ for all x. If $s(0) = 0$ and $c(0) = 1$, prove that $[s(x)]^2 + [c(x)]^2 = 1$.

PUTNAM EXAM CHALLENGE

100. Suppose f and g are nonconstant, differentiable, real-valued functions on R. Furthermore, suppose that for each pair of real numbers x and y, $f(x+y) = f(x)f(y) - g(x)g(y)$ and $g(x+y) = f(x)g(y) + g(x)f(y)$. If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

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4.2 Area

- Use sigma notation to write and evaluate a sum.
- Understand the concept of area.
- Approximate the area of a plane region.
- Find the area of a plane region using limits.

Sigma Notation

In the preceding section, you studied antidifferentiation. In this section, you will look further into a problem introduced in Section 1.1—that of finding the area of a region in the plane. At first glance, these two ideas may seem unrelated, but you will discover in Section 4.4 that they are closely related by an extremely important theorem called the Fundamental Theorem of Calculus.

This section begins by introducing a concise notation for sums. This notation is called **sigma notation** because it uses the uppercase Greek letter sigma, written as Σ.

SIGMA NOTATION

The sum of n terms $a_1, a_2, a_3, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation,** a_i is the ***i*th term** of the sum, and the **upper and lower bounds of summation** are n and 1.

NOTE The upper and lower bounds must be constant with respect to the index of summation. However, the lower bound doesn't have to be 1. Any integer less than or equal to the upper bound is legitimate. ■

EXAMPLE 1 Examples of Sigma Notation

a. $\displaystyle\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6$

b. $\displaystyle\sum_{i=0}^{5} (i + 1) = 1 + 2 + 3 + 4 + 5 + 6$

c. $\displaystyle\sum_{j=3}^{7} j^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

d. $\displaystyle\sum_{k=1}^{n} \frac{1}{n}(k^2 + 1) = \frac{1}{n}(1^2 + 1) + \frac{1}{n}(2^2 + 1) + \cdots + \frac{1}{n}(n^2 + 1)$

e. $\displaystyle\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$

From parts (a) and (b), notice that the same sum can be represented in different ways using sigma notation. ■

Although any variable can be used as the index of summation i, j, and k are often used. Notice in Example 1 that the index of summation does not appear in the terms of the expanded sum.

■ **FOR FURTHER INFORMATION** For a geometric interpretation of summation formulas, see the article, "Looking at $\displaystyle\sum_{k=1}^{n} k$ and $\displaystyle\sum_{k=1}^{n} k^2$ Geometrically" by Eric Hegblom in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

THE SUM OF THE FIRST 100 INTEGERS

A teacher of Carl Friedrich Gauss (1777–1855) asked him to add all the integers from 1 to 100. When Gauss returned with the correct answer after only a few moments, the teacher could only look at him in astounded silence. This is what Gauss did:

$$\begin{array}{rcrcrcccr} 1 & + & 2 & + & 3 & + & \cdots & + & 100 \\ 100 & + & 99 & + & 98 & + & \cdots & + & 1 \\ \hline 101 & + & 101 & + & 101 & + & \cdots & + & 101 \end{array}$$

$$\frac{100 \times 101}{2} = 5050$$

This is generalized by Theorem 4.2, where

$$\sum_{i=1}^{100} i = \frac{100(101)}{2} = 5050.$$

The following properties of summation can be derived using the associative and commutative properties of addition and the distributive property of addition over multiplication. (In the first property, k is a constant.)

1. $\displaystyle\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

2. $\displaystyle\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

The next theorem lists some useful formulas for sums of powers. A proof of this theorem is given in Appendix A.

THEOREM 4.2 SUMMATION FORMULAS

1. $\displaystyle\sum_{i=1}^{n} c = cn$

2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

EXAMPLE 2 Evaluating a Sum

Evaluate $\displaystyle\sum_{i=1}^{n} \frac{i+1}{n^2}$ for $n = 10, 100, 1000$, and $10{,}000$.

Solution Applying Theorem 4.2, you can write

$$\sum_{i=1}^{n} \frac{i+1}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n}(i+1)$$ Factor the constant $1/n^2$ out of sum.

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right)$$ Write as two sums.

$$= \frac{1}{n^2}\left[\frac{n(n+1)}{2} + n\right]$$ Apply Theorem 4.2.

$$= \frac{1}{n^2}\left[\frac{n^2+3n}{2}\right]$$ Simplify.

$$= \frac{n+3}{2n}.$$ Simplify.

Now you can evaluate the sum by substituting the appropriate values of n, as shown in the table at the left. ■

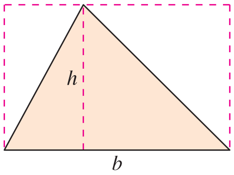
n	$\displaystyle\sum_{i=1}^{n} \frac{i+1}{n^2} = \frac{n+3}{2n}$
10	0.65000
100	0.51500
1,000	0.50150
10,000	0.50015

In the table, note that the sum appears to approach a limit as n increases. Although the discussion of limits at infinity in Section 3.5 applies to a variable x, where x can be any real number, many of the same results hold true for limits involving the variable n, where n is restricted to positive integer values. So, to find the limit of $(n+3)/2n$ as n approaches infinity, you can write

$$\lim_{n\to\infty} \frac{n+3}{2n} = \lim_{n\to\infty}\left(\frac{n}{2n} + \frac{3}{2n}\right) = \lim_{n\to\infty}\left(\frac{1}{2} + \frac{3}{2n}\right) = \frac{1}{2} + 0 = \frac{1}{2}.$$

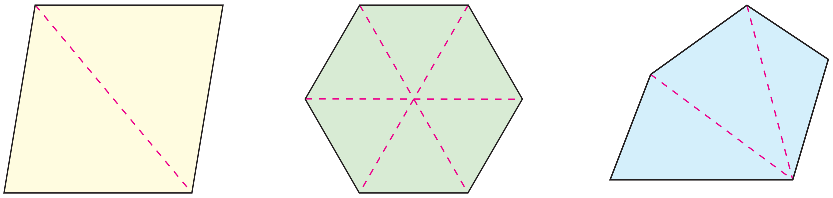
Area

In Euclidean geometry, the simplest type of plane region is a rectangle. Although people often say that the *formula* for the area of a rectangle is $A = bh$, it is actually more proper to say that this is the *definition* of the **area of a rectangle.**

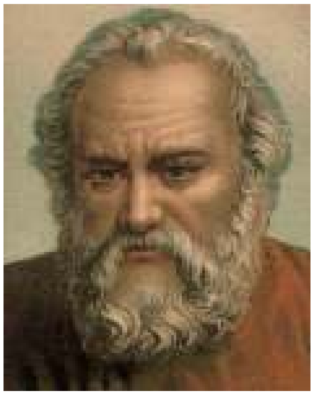


Triangle: $A = \frac{1}{2}bh$
Figure 4.5

From this definition, you can develop formulas for the areas of many other plane regions. For example, to determine the area of a triangle, you can form a rectangle whose area is twice that of the triangle, as shown in Figure 4.5. Once you know how to find the area of a triangle, you can determine the area of any polygon by subdividing the polygon into triangular regions, as shown in Figure 4.6.



Parallelogram Hexagon Polygon
Figure 4.6



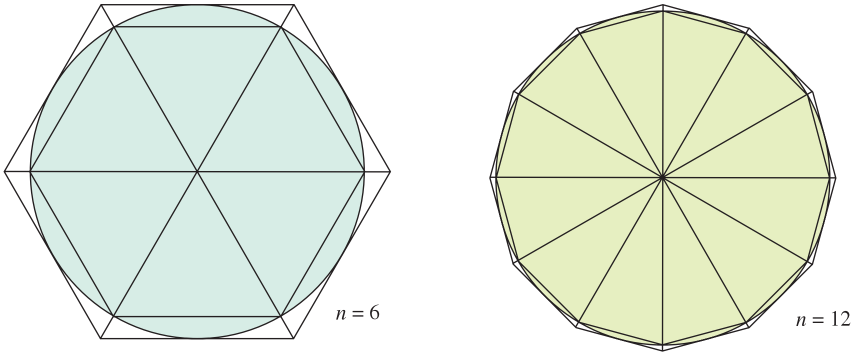
Mary Evans Picture Library

ARCHIMEDES (287–212 B.C.)

Archimedes used the method of exhaustion to derive formulas for the areas of ellipses, parabolic segments, and sectors of a spiral. He is considered to have been the greatest applied mathematician of antiquity.

Finding the areas of regions other than polygons is more difficult. The ancient Greeks were able to determine formulas for the areas of some general regions (principally those bounded by conics) by the *exhaustion* method. The clearest description of this method was given by Archimedes. Essentially, the method is a limiting process in which the area is squeezed between two polygons—one inscribed in the region and one circumscribed about the region.

For instance, in Figure 4.7 the area of a circular region is approximated by an n-sided inscribed polygon and an n-sided circumscribed polygon. For each value of n, the area of the inscribed polygon is less than the area of the circle, and the area of the circumscribed polygon is greater than the area of the circle. Moreover, as n increases, the areas of both polygons become better and better approximations of the area of the circle.



The exhaustion method for finding the area of a circular region
Figure 4.7

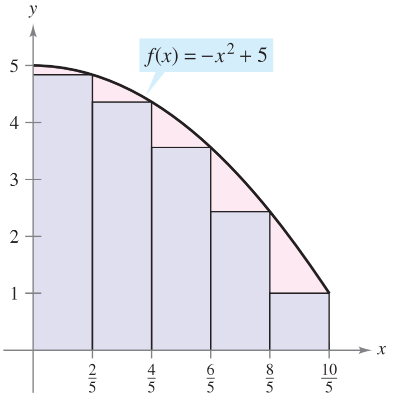
A process that is similar to that used by Archimedes to determine the area of a plane region is used in the remaining examples in this section.

■ **FOR FURTHER INFORMATION** For an alternative development of the formula for the area of a circle, see the article "Proof Without Words: Area of a Disk is πR^2" by Russell Jay Hendel in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

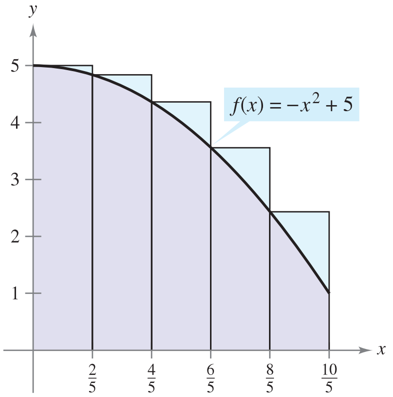
The Area of a Plane Region

Recall from Section 1.1 that the origins of calculus are connected to two classic problems: the tangent line problem and the area problem. Example 3 begins the investigation of the area problem.

EXAMPLE 3 Approximating the Area of a Plane Region



(a) The area of the parabolic region is greater than the area of the rectangles.



(b) The area of the parabolic region is less than the area of the rectangles.

Figure 4.8

Use the five rectangles in Figure 4.8(a) and (b) to find *two* approximations of the area of the region lying between the graph of

$$f(x) = -x^2 + 5$$

and the x-axis between $x = 0$ and $x = 2$.

Solution

a. The right endpoints of the five intervals are $\frac{2}{5}i$, where $i = 1, 2, 3, 4, 5$. The width of each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained by evaluating f at the right endpoint of each interval.

$$\left[0, \frac{2}{5}\right], \left[\frac{2}{5}, \frac{4}{5}\right], \left[\frac{4}{5}, \frac{6}{5}\right], \left[\frac{6}{5}, \frac{8}{5}\right], \left[\frac{8}{5}, \frac{10}{5}\right]$$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

Evaluate f at the right endpoints of these intervals.

The sum of the areas of the five rectangles is

$$\sum_{i=1}^{5} \overbrace{f\left(\frac{2i}{5}\right)}^{\text{Height}} \overbrace{\left(\frac{2}{5}\right)}^{\text{Width}} = \sum_{i=1}^{5} \left[-\left(\frac{2i}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{162}{25} = 6.48.$$

Because each of the five rectangles lies inside the parabolic region, you can conclude that the area of the parabolic region is greater than 6.48.

b. The left endpoints of the five intervals are $\frac{2}{5}(i - 1)$, where $i = 1, 2, 3, 4, 5$. The width of each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained by evaluating f at the left endpoint of each interval. So, the sum is

$$\sum_{i=1}^{5} \overbrace{f\left(\frac{2i-2}{5}\right)}^{\text{Height}} \overbrace{\left(\frac{2}{5}\right)}^{\text{Width}} = \sum_{i=1}^{5} \left[-\left(\frac{2i-2}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{202}{25} = 8.08.$$

Because the parabolic region lies within the union of the five rectangular regions, you can conclude that the area of the parabolic region is less than 8.08.

By combining the results in parts (a) and (b), you can conclude that

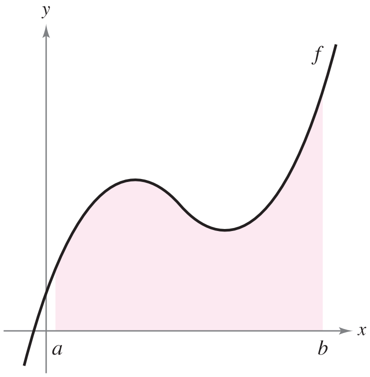
$$6.48 < (\text{Area of region}) < 8.08.$$ ■

NOTE By increasing the number of rectangles used in Example 3, you can obtain closer and closer approximations of the area of the region. For instance, using 25 rectangles of width $\frac{2}{25}$ each, you can conclude that

$$7.17 < (\text{Area of region}) < 7.49.$$ ■

Upper and Lower Sums

The procedure used in Example 3 can be generalized as follows. Consider a plane region bounded above by the graph of a nonnegative, continuous function $y = f(x)$, as shown in Figure 4.9. The region is bounded below by the x-axis, and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$.



The region under a curve
Figure 4.9

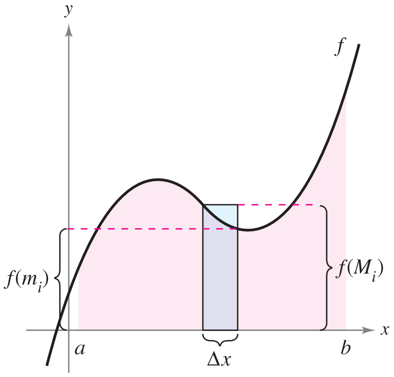
To approximate the area of the region, begin by subdividing the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$, as shown in Figure 4.10. The endpoints of the intervals are as follows.

$$\overbrace{a + 0(\Delta x)}^{a = x_0} < \overbrace{a + 1(\Delta x)}^{x_1} < \overbrace{a + 2(\Delta x)}^{x_2} < \cdot \cdot \cdot < \overbrace{a + n(\Delta x)}^{x_n = b}$$

Because f is continuous, the Extreme Value Theorem guarantees the existence of a minimum and a maximum value of $f(x)$ in *each* subinterval.

$f(m_i)$ = Minimum value of $f(x)$ in ith subinterval

$f(M_i)$ = Maximum value of $f(x)$ in ith subinterval



The interval $[a, b]$ is divided into n subintervals of width $\Delta x = \dfrac{b - a}{n}$.

Figure 4.10

Next, define an **inscribed rectangle** lying *inside* the ith subregion and a **circumscribed rectangle** extending *outside* the ith subregion. The height of the ith inscribed rectangle is $f(m_i)$ and the height of the ith circumscribed rectangle is $f(M_i)$. For *each* i, the area of the inscribed rectangle is less than or equal to the area of the circumscribed rectangle.

$$\begin{pmatrix}\text{Area of inscribed}\\ \text{rectangle}\end{pmatrix} = f(m_i)\,\Delta x \le f(M_i)\,\Delta x = \begin{pmatrix}\text{Area of circumscribed}\\ \text{rectangle}\end{pmatrix}$$

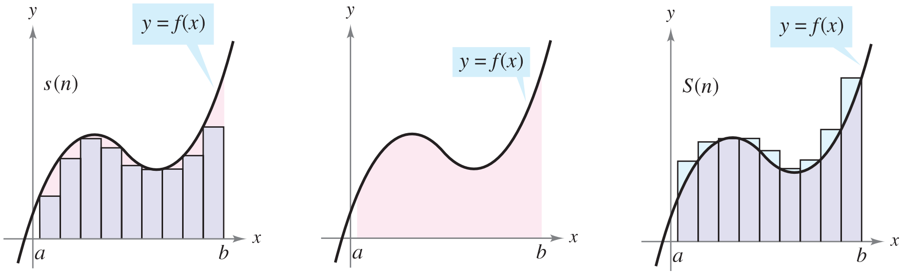
The sum of the areas of the inscribed rectangles is called a **lower sum,** and the sum of the areas of the circumscribed rectangles is called an **upper sum.**

$$\text{Lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x \qquad \text{Area of inscribed rectangles}$$

$$\text{Upper sum} = S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x \qquad \text{Area of circumscribed rectangles}$$

From Figure 4.11, you can see that the lower sum $s(n)$ is less than or equal to the upper sum $S(n)$. Moreover, the actual area of the region lies between these two sums.

$$s(n) \le (\text{Area of region}) \le S(n)$$



Area of inscribed rectangles is less than area of region.

Area of region

Area of circumscribed rectangles is greater than area of region.

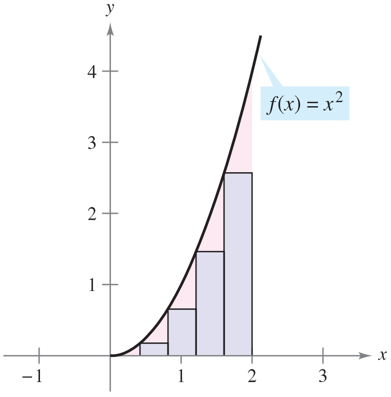
Figure 4.11

EXAMPLE 4 Finding Upper and Lower Sums for a Region

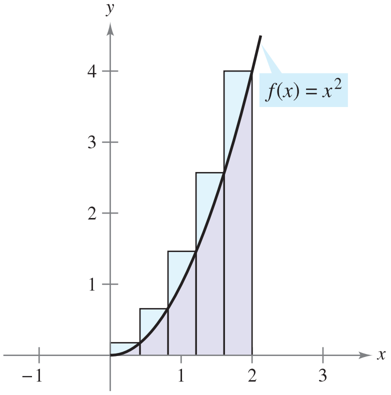
Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ and the x-axis between $x = 0$ and $x = 2$.

Solution To begin, partition the interval $[0, 2]$ into n subintervals, each of width

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}.$$



Inscribed rectangles



Circumscribed rectangles

Figure 4.12

Figure 4.12 shows the endpoints of the subintervals and several inscribed and circumscribed rectangles. Because f is increasing on the interval $[0, 2]$, the minimum value on each subinterval occurs at the left endpoint, and the maximum value occurs at the right endpoint.

Left Endpoints	*Right Endpoints*
$m_i = 0 + (i-1)\left(\frac{2}{n}\right) = \frac{2(i-1)}{n}$	$M_i = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$

Using the left endpoints, the lower sum is

$$\begin{aligned} s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x &= \sum_{i=1}^{n} f\left[\frac{2(i-1)}{n}\right]\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n} \left[\frac{2(i-1)}{n}\right]^2\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n} \left(\frac{8}{n^3}\right)(i^2 - 2i + 1) \\ &= \frac{8}{n^3}\left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right) \\ &= \frac{8}{n^3}\left\{\frac{n(n+1)(2n+1)}{6} - 2\left[\frac{n(n+1)}{2}\right] + n\right\} \\ &= \frac{4}{3n^3}(2n^3 - 3n^2 + n) \\ &= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}. \qquad \text{Lower sum} \end{aligned}$$

Using the right endpoints, the upper sum is

$$\begin{aligned} S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n} \left(\frac{8}{n^3}\right)i^2 \\ &= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\ &= \frac{4}{3n^3}(2n^3 + 3n^2 + n) \\ &= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}. \qquad \text{Upper sum} \end{aligned}$$

■

EXPLORATION

For the region given in Example 4, evaluate the lower sum

$$s(n) = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}$$

and the upper sum

$$S(n) = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

for $n = 10$, 100, and 1000. Use your results to determine the area of the region.

Example 4 illustrates some important things about lower and upper sums. First, notice that for any value of n, the lower sum is less than (or equal to) the upper sum.

$$s(n) = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2} < \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} = S(n)$$

Second, the difference between these two sums lessens as n increases. In fact, if you take the limits as $n \to \infty$, both the upper sum and the lower sum approach $\frac{8}{3}$.

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty}\left(\frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3} \qquad \text{Lower sum limit}$$

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3} \qquad \text{Upper sum limit}$$

The next theorem shows that the equivalence of the limits (as $n \to \infty$) of the upper and lower sums is not mere coincidence. It is true for all functions that are continuous and nonnegative on the closed interval $[a, b]$. The proof of this theorem is best left to a course in advanced calculus.

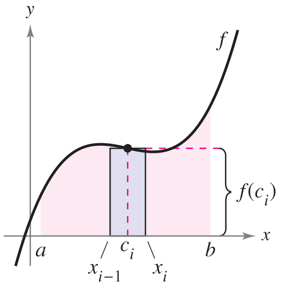
THEOREM 4.3 LIMITS OF THE LOWER AND UPPER SUMS

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\begin{aligned}\lim_{n\to\infty} s(n) &= \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\,\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\,\Delta x \\ &= \lim_{n\to\infty} S(n)\end{aligned}$$

where $\Delta x = (b - a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

Because the same limit is attained for both the minimum value $f(m_i)$ and the maximum value $f(M_i)$, it follows from the Squeeze Theorem (Theorem 1.8) that the choice of x in the ith subinterval does not affect the limit. This means that you are free to choose an *arbitrary* x-value in the ith subinterval, as in the following *definition of the area of a region in the plane*.



The width of the ith subinterval is $\Delta x = x_i - x_{i-1}$.
Figure 4.13

DEFINITION OF THE AREA OF A REGION IN THE PLANE

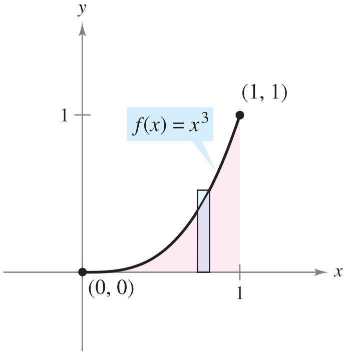
Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x, \quad x_{i-1} \le c_i \le x_i$$

where $\Delta x = (b - a)/n$ (see Figure 4.13).

EXAMPLE 5 Finding Area by the Limit Definition

Find the area of the region bounded by the graph $f(x) = x^3$, the x-axis, and the vertical lines $x = 0$ and $x = 1$, as shown in Figure 4.14.



The area of the region bounded by the graph of f, the x-axis, $x = 0$, and $x = 1$ is $\frac{1}{4}$.
Figure 4.14

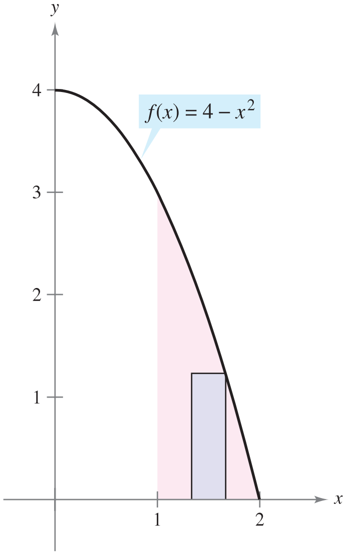
Solution Begin by noting that f is continuous and nonnegative on the interval $[0, 1]$. Next, partition the interval $[0, 1]$ into n subintervals, each of width $\Delta x = 1/n$. According to the definition of area, you can choose any x-value in the ith subinterval. For this example, the right endpoints $c_i = i/n$ are convenient.

$$\begin{aligned}
\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\left(\frac{1}{n}\right) && \text{Right endpoints: } c_i = \frac{i}{n}\\
&= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n} i^3\\
&= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]\\
&= \lim_{n\to\infty}\left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right)\\
&= \frac{1}{4}
\end{aligned}$$

The area of the region is $\frac{1}{4}$.

EXAMPLE 6 Finding Area by the Limit Definition

Find the area of the region bounded by the graph of $f(x) = 4 - x^2$, the x-axis, and the vertical lines $x = 1$ and $x = 2$, as shown in Figure 4.15.



The area of the region bounded by the graph of f, the x-axis, $x = 1$, and $x = 2$ is $\frac{5}{3}$.
Figure 4.15

Solution The function f is continuous and nonnegative on the interval $[1, 2]$, and so begin by partitioning the interval into n subintervals, each of width $\Delta x = 1/n$. Choosing the right endpoint

$$c_i = a + i\Delta x = 1 + \frac{i}{n} \qquad \text{Right endpoints}$$

of each subinterval, you obtain

$$\begin{aligned}
\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[4 - \left(1 + \frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(3 - \frac{2i}{n} - \frac{i^2}{n^2}\right)\left(\frac{1}{n}\right)\\
&= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n} 3 - \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2\right)\\
&= \lim_{n\to\infty}\left[3 - \left(1 + \frac{1}{n}\right) - \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)\right]\\
&= 3 - 1 - \frac{1}{3}\\
&= \frac{5}{3}.
\end{aligned}$$

The area of the region is $\frac{5}{3}$. ■

The last example in this section looks at a region that is bounded by the y-axis (rather than by the x-axis).

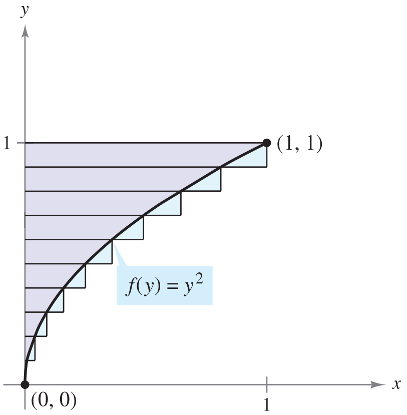
EXAMPLE 7 A Region Bounded by the y-axis

Find the area of the region bounded by the graph of $f(y) = y^2$ and the y-axis for $0 \le y \le 1$, as shown in Figure 4.16.

Solution When f is a continuous, nonnegative function of y, you still can use the same basic procedure shown in Examples 5 and 6. Begin by partitioning the interval $[0, 1]$ into n subintervals, each of width $\Delta y = 1/n$. Then, using the upper endpoints $c_i = i/n$, you obtain

$$\begin{aligned}\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta y &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) \qquad \text{Upper endpoints: } c_i = \frac{i}{n}\\ &= \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n} i^2\\ &= \lim_{n\to\infty}\frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\\ &= \lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)\\ &= \frac{1}{3}.\end{aligned}$$

The area of the region is $\frac{1}{3}$. ■



The area of the region bounded by the graph of f and the y-axis for $0 \le y \le 1$ is $\frac{1}{3}$.
Figure 4.16

4.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the sum. Use the summation capabilities of a graphing utility to verify your result.

1. $\displaystyle\sum_{i=1}^{6}(3i+2)$

2. $\displaystyle\sum_{k=5}^{8}k(k-4)$

3. $\displaystyle\sum_{k=0}^{4}\frac{1}{k^2+1}$

4. $\displaystyle\sum_{j=4}^{7}\frac{2}{j}$

5. $\displaystyle\sum_{k=1}^{4}c$

6. $\displaystyle\sum_{i=1}^{4}\left[(i-1)^2+(i+1)^3\right]$

In Exercises 7–14, use sigma notation to write the sum.

7. $\dfrac{1}{5(1)}+\dfrac{1}{5(2)}+\dfrac{1}{5(3)}+\cdots+\dfrac{1}{5(11)}$

8. $\dfrac{9}{1+1}+\dfrac{9}{1+2}+\dfrac{9}{1+3}+\cdots+\dfrac{9}{1+14}$

9. $\left[7\left(\frac{1}{6}\right)+5\right]+\left[7\left(\frac{2}{6}\right)+5\right]+\cdots+\left[7\left(\frac{6}{6}\right)+5\right]$

10. $\left[1-\left(\frac{1}{4}\right)^2\right]+\left[1-\left(\frac{2}{4}\right)^2\right]+\cdots+\left[1-\left(\frac{4}{4}\right)^2\right]$

11. $\left[\left(\frac{2}{n}\right)^3-\frac{2}{n}\right]\left(\frac{2}{n}\right)+\cdots+\left[\left(\frac{2n}{n}\right)^3-\frac{2n}{n}\right]\left(\frac{2}{n}\right)$

12. $\left[1-\left(\frac{2}{n}-1\right)^2\right]\left(\frac{2}{n}\right)+\cdots+\left[1-\left(\frac{2n}{n}-1\right)^2\right]\left(\frac{2}{n}\right)$

13. $\left[2\left(1+\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right)+\cdots+\left[2\left(1+\frac{3n}{n}\right)^2\right]\left(\frac{3}{n}\right)$

14. $\left(\frac{1}{n}\right)\sqrt{1-\left(\frac{0}{n}\right)^2}+\cdots+\left(\frac{1}{n}\right)\sqrt{1-\left(\frac{n-1}{n}\right)^2}$

In Exercises 15–22, use the properties of summation and Theorem 4.2 to evaluate the sum. Use the summation capabilities of a graphing utility to verify your result.

15. $\displaystyle\sum_{i=1}^{12}7$

16. $\displaystyle\sum_{i=1}^{30}-18$

17. $\displaystyle\sum_{i=1}^{24}4i$

18. $\displaystyle\sum_{i=1}^{16}(5i-4)$

19. $\displaystyle\sum_{i=1}^{20}(i-1)^2$

20. $\displaystyle\sum_{i=1}^{10}(i^2-1)$

21. $\displaystyle\sum_{i=1}^{15}i(i-1)^2$

22. $\displaystyle\sum_{i=1}^{10}i(i^2+1)$

In Exercises 23 and 24, use the summation capabilities of a graphing utility to evaluate the sum. Then use the properties of summation and Theorem 4.2 to verify the sum.

23. $\displaystyle\sum_{i=1}^{20}(i^2+3)$

24. $\displaystyle\sum_{i=1}^{15}(i^3-2i)$

25. Consider the function $f(x) = 3x + 2$.

(a) Estimate the area between the graph of f and the x-axis between $x = 0$ and $x = 3$ using six rectangles and right endpoints. Sketch the graph and the rectangles.

(b) Repeat part (a) using left endpoints.

26. Consider the function $g(x) = x^2 + x - 4$.

(a) Estimate the area between the graph of g and the x-axis between $x = 2$ and $x = 4$ using four rectangles and right endpoints. Sketch the graph and the rectangles.

(b) Repeat part (a) using left endpoints.

In Exercises 27–32, use left and right endpoints and the given number of rectangles to find two approximations of the area of the region between the graph of the function and the x-axis over the given interval.

27. $f(x) = 2x + 5$, $[0, 2]$, 4 rectangles

28. $f(x) = 9 - x$, $[2, 4]$, 6 rectangles

29. $g(x) = 2x^2 - x - 1$, $[2, 5]$, 6 rectangles

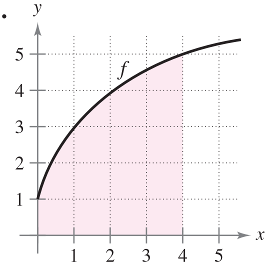
30. $g(x) = x^2 + 1$, $[1, 3]$, 8 rectangles

31. $f(x) = \cos x$, $\left[0, \dfrac{\pi}{2}\right]$, 4 rectangles

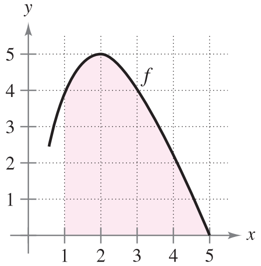
32. $g(x) = \sin x$, $[0, \pi]$, 6 rectangles

In Exercises 33–36, bound the area of the shaded region by approximating the upper and lower sums. Use rectangles of width 1.

33.



34.



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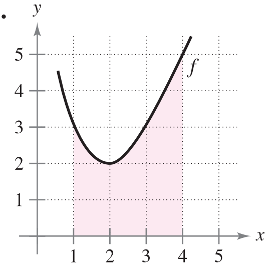
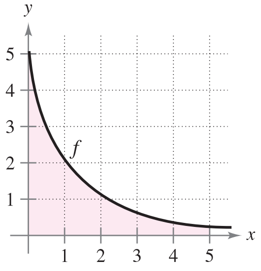


36.



In Exercises 37–40, find the limit of $s(n)$ as $n \to \infty$.

37. $s(n) = \dfrac{81}{n^4}\left[\dfrac{n^2(n+1)^2}{4}\right]$

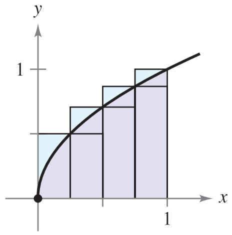
38. $s(n) = \dfrac{64}{n^3}\left[\dfrac{n(n+1)(2n+1)}{6}\right]$

39. $s(n) = \dfrac{18}{n^2}\left[\dfrac{n(n+1)}{2}\right]$

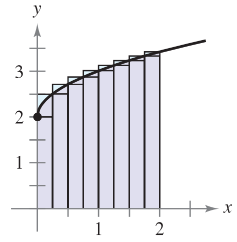
40. $s(n) = \dfrac{1}{n^2}\left[\dfrac{n(n+1)}{2}\right]$

In Exercises 41–44, use upper and lower sums to approximate the area of the region using the given number of subintervals (of equal width).

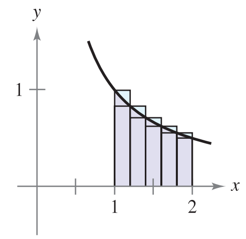
41. $y = \sqrt{x}$



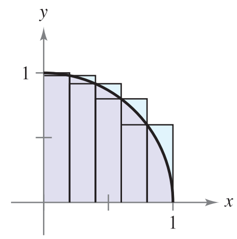
42. $y = \sqrt{x} + 2$



43. $y = \dfrac{1}{x}$



44. $y = \sqrt{1 - x^2}$



In Exercises 45–48, use the summation formulas to rewrite the expression without the summation notation. Use the result to find the sums for $n = 10$, 100, 1000, and 10,000.

45. $\displaystyle\sum_{i=1}^{n} \frac{2i+1}{n^2}$

46. $\displaystyle\sum_{j=1}^{n} \frac{4j+3}{n^2}$

47. $\displaystyle\sum_{k=1}^{n} \frac{6k(k-1)}{n^3}$

48. $\displaystyle\sum_{i=1}^{n} \frac{4i^2(i-1)}{n^4}$

In Exercises 49–54, find a formula for the sum of n terms. Use the formula to find the limit as $n \to \infty$.

49. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{24i}{n^2}$

50. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$

51. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(i-1)^2$

52. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$

53. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)\left(\frac{2}{n}\right)$

54. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right)$

55. ***Numerical Reasoning*** Consider a triangle of area 2 bounded by the graphs of $y = x$, $y = 0$, and $x = 2$.

(a) Sketch the region.

(b) Divide the interval $[0, 2]$ into n subintervals of equal width and show that the endpoints are

$$0 < 1\left(\frac{2}{n}\right) < \cdots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right).$$

(c) Show that $s(n) = \displaystyle\sum_{i=1}^{n} \left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(d) Show that $S(n) = \displaystyle\sum_{i=1}^{n} \left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(e) Complete the table.

n	5	10	50	100
$s(n)$				
$S(n)$				

(f) Show that $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n) = 2$.

56. ***Numerical Reasoning*** Consider a trapezoid of area 4 bounded by the graphs of $y = x$, $y = 0$, $x = 1$, and $x = 3$.

(a) Sketch the region.

(b) Divide the interval $[1, 3]$ into n subintervals of equal width and show that the endpoints are

$$1 < 1 + 1\left(\frac{2}{n}\right) < \cdots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right).$$

(c) Show that $s(n) = \displaystyle\sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(d) Show that $S(n) = \displaystyle\sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(e) Complete the table.

n	5	10	50	100
$s(n)$				
$S(n)$				

(f) Show that $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n) = 4$.

In Exercises 57–66, use the limit process to find the area of the region between the graph of the function and the x-axis over the given interval. Sketch the region.

57. $y = -4x + 5, \quad [0, 1]$

58. $y = 3x - 2, \quad [2, 5]$

59. $y = x^2 + 2, \quad [0, 1]$

60. $y = x^2 + 1, \quad [0, 3]$

61. $y = 25 - x^2, \quad [1, 4]$

62. $y = 4 - x^2, \quad [-2, 2]$

63. $y = 27 - x^3, \quad [1, 3]$

64. $y = 2x - x^3, \quad [0, 1]$

65. $y = x^2 - x^3, \quad [-1, 1]$

66. $y = x^2 - x^3, \quad [-1, 0]$

In Exercises 67–72, use the limit process to find the area of the region between the graph of the function and the y-axis over the given y-interval. Sketch the region.

67. $f(y) = 4y, 0 \le y \le 2$

68. $g(y) = \frac{1}{2}y, 2 \le y \le 4$

69. $f(y) = y^2, 0 \le y \le 5$

70. $f(y) = 4y - y^2, 1 \le y \le 2$

71. $g(y) = 4y^2 - y^3, 1 \le y \le 3$

72. $h(y) = y^3 + 1, 1 \le y \le 2$

In Exercises 73–76, use the *Midpoint Rule*

$$\textbf{Area} \approx \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x$$

with $n = 4$ to approximate the area of the region bounded by the graph of the function and the x-axis over the given interval.

73. $f(x) = x^2 + 3, \quad [0, 2]$

74. $f(x) = x^2 + 4x, \quad [0, 4]$

75. $f(x) = \tan x, \quad \left[0, \frac{\pi}{4}\right]$

76. $f(x) = \sin x, \quad \left[0, \frac{\pi}{2}\right]$

Programming **Write a program for a graphing utility to approximate areas by using the Midpoint Rule. Assume that the function is positive over the given interval and that the subintervals are of equal width. In Exercises 77–80, use the program to approximate the area of the region between the graph of the function and the x-axis over the given interval, and complete the table.**

n	4	8	12	16	20
Approximate Area					

77. $f(x) = \sqrt{x}, \quad [0, 4]$

78. $f(x) = \dfrac{8}{x^2 + 1}, \quad [2, 6]$

79. $f(x) = \tan\left(\dfrac{\pi x}{8}\right), \quad [1, 3]$

80. $f(x) = \cos\sqrt{x}, \quad [0, 2]$

WRITING ABOUT CONCEPTS

Approximation **In Exercises 81 and 82, determine which value best approximates the area of the region between the x-axis and the graph of the function over the given interval. (Make your selection on the basis of a sketch of the region and not by performing calculations.)**

81. $f(x) = 4 - x^2, \quad [0, 2]$

(a) -2 (b) 6 (c) 10 (d) 3 (e) 8

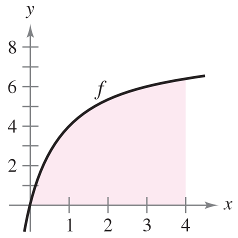
82. $f(x) = \sin\dfrac{\pi x}{4}, \quad [0, 4]$

(a) 3 (b) 1 (c) -2 (d) 8 (e) 6

83. In your own words and using appropriate figures, describe the methods of upper sums and lower sums in approximating the area of a region.

84. Give the definition of the area of a region in the plane.

85. ***Graphical Reasoning*** Consider the region bounded by the graphs of $f(x) = 8x/(x + 1)$, $x = 0$, $x = 4$, and $y = 0$, as shown in the figure. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) Redraw the figure, and complete and shade the rectangles representing the lower sum when $n = 4$. Find this lower sum.

(b) Redraw the figure, and complete and shade the rectangles representing the upper sum when $n = 4$. Find this upper sum.

(c) Redraw the figure, and complete and shade the rectangles whose heights are determined by the functional values at the midpoint of each subinterval when $n = 4$. Find this sum using the Midpoint Rule.

(d) Verify the following formulas for approximating the area of the region using n subintervals of equal width.

Lower sum: $s(n) = \sum_{i=1}^{n} f\left[(i-1)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

Upper sum: $S(n) = \sum_{i=1}^{n} f\left[(i)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

Midpoint Rule: $M(n) = \sum_{i=1}^{n} f\left[\left(i-\frac{1}{2}\right)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

(e) Use a graphing utility and the formulas in part (d) to complete the table.

n	4	8	20	100	200
$s(n)$					
$S(n)$					
$M(n)$					

(f) Explain why $s(n)$ increases and $S(n)$ decreases for increasing values of n, as shown in the table in part (e).

CAPSTONE

86. Consider a function $f(x)$ that is increasing on the interval $[1, 4]$. The interval $[1, 4]$ is divided into 12 subintervals.

(a) What are the left endpoints of the first and last subintervals?

(b) What are the right endpoints of the first two subintervals?

(c) When using the right endpoints, will the rectangles lie above or below the graph of $f(x)$? Use a graph to explain your answer.

(d) What can you conclude about the heights of the rectangles if a function is constant on the given interval?

True or False? **In Exercises 87 and 88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

87. The sum of the first n positive integers is $n(n+1)/2$.

88. If f is continuous and nonnegative on $[a, b]$, then the limits as $n \to \infty$ of its lower sum $s(n)$ and upper sum $S(n)$ both exist and are equal.

89. ***Writing*** Use the figure to write a short paragraph explaining why the formula $1 + 2 + \cdot \cdot \cdot + n = \frac{1}{2}n(n+1)$ is valid for all positive integers n.

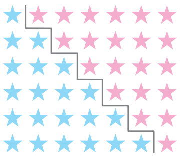


Figure for 89

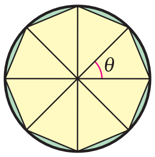


Figure for 90

90. ***Graphical Reasoning*** Consider an n-sided regular polygon inscribed in a circle of radius r. Join the vertices of the polygon to the center of the circle, forming n congruent triangles (see figure).

(a) Determine the central angle θ in terms of n.

(b) Show that the area of each triangle is $\frac{1}{2}r^2 \sin \theta$.

(c) Let A_n be the sum of the areas of the n triangles. Find $\lim_{n \to \infty} A_n$.

91. ***Modeling Data*** The table lists the measurements of a lot bounded by a stream and two straight roads that meet at right angles, where x and y are measured in feet (see figure).

x	0	50	100	150	200	250	300
y	450	362	305	268	245	156	0

(a) Use the regression capabilities of a graphing utility to find a model of the form $y = ax^3 + bx^2 + cx + d$.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model in part (a) to estimate the area of the lot.

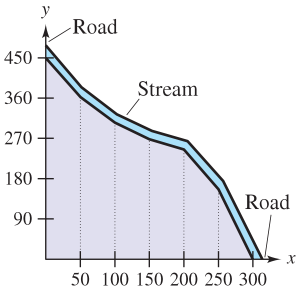
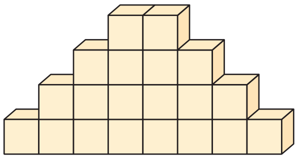


Figure for 91



n is even.

Figure for 92

92. ***Building Blocks*** A child places n cubic building blocks in a row to form the base of a triangular design (see figure). Each successive row contains two fewer blocks than the preceding row. Find a formula for the number of blocks used in the design. (*Hint:* The number of building blocks in the design depends on whether n is odd or even.)

93. Prove each formula by mathematical induction. (You may need to review the method of proof by induction from a precalculus text.)

(a) $\sum_{i=1}^{n} 2i = n(n+1)$ (b) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

PUTNAM EXAM CHALLENGE

94. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Write your answer in the form $(a\sqrt{b} + c)/d$, where a, b, c, and d are positive integers.

This problem was composed by the Committee on the Putnam Prize Competition.
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4.3 Riemann Sums and Definite Integrals

- **Understand the definition of a Riemann sum.**
- **Evaluate a definite integral using limits.**
- **Evaluate a definite integral using properties of definite integrals.**

Riemann Sums

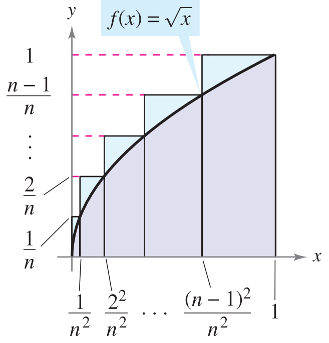
In the definition of area given in Section 4.2, the partitions have subintervals of *equal width*. This was done only for computational convenience. The following example shows that it is not necessary to have subintervals of equal width.

EXAMPLE 1 A Partition with Subintervals of Unequal Widths

Consider the region bounded by the graph of $f(x) = \sqrt{x}$ and the x-axis for $0 \le x \le 1$, as shown in Figure 4.17. Evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

where c_i is the right endpoint of the partition given by $c_i = i^2/n^2$ and Δx_i is the width of the ith interval.



The subintervals do not have equal widths.
Figure 4.17

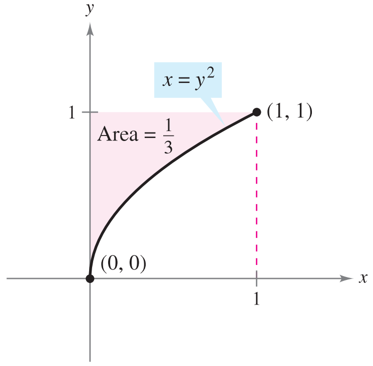
Solution The width of the ith interval is given by

$$\begin{aligned}\Delta x_i &= \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2}\\ &= \frac{i^2 - i^2 + 2i - 1}{n^2}\\ &= \frac{2i-1}{n^2}.\end{aligned}$$

So, the limit is

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{i^2}{n^2}}\left(\frac{2i-1}{n^2}\right)\\ &= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} (2i^2 - i)\\ &= \lim_{n\to\infty} \frac{1}{n^3}\left[2\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{n(n+1)}{2}\right]\\ &= \lim_{n\to\infty} \frac{4n^3 + 3n^2 - n}{6n^3}\\ &= \frac{2}{3}.\end{aligned}$$

■



The area of the region bounded by the graph of $x = y^2$ and the y-axis for $0 \le y \le 1$ is $\frac{1}{3}$.
Figure 4.18

From Example 7 in Section 4.2, you know that the region shown in Figure 4.18 has an area of $\frac{1}{3}$. Because the square bounded by $0 \le x \le 1$ and $0 \le y \le 1$ has an area of 1, you can conclude that the area of the region shown in Figure 4.17 has an area of $\frac{2}{3}$. This agrees with the limit found in Example 1, even though that example used a partition having subintervals of unequal widths. The reason this particular partition gave the proper area is that as n increases, the *width of the largest subinterval approaches zero*. This is a key feature of the development of definite integrals.

The Granger Collection

GEORG FRIEDRICH BERNHARD RIEMANN (1826–1866)

German mathematician Riemann did his most famous work in the areas of non-Euclidean geometry, differential equations, and number theory. It was Riemann's results in physics and mathematics that formed the structure on which Einstein's General Theory of Relativity is based.

In the preceding section, the limit of a sum was used to define the area of a region in the plane. Finding area by this means is only one of *many* applications involving the limit of a sum. A similar approach can be used to determine quantities as diverse as arc lengths, average values, centroids, volumes, work, and surface areas. The following definition is named after Georg Friedrich Bernhard Riemann. Although the definite integral had been defined and used long before the time of Riemann, he generalized the concept to cover a broader category of functions.

In the following definition of a Riemann sum, note that the function f has no restrictions other than being defined on the interval $[a, b]$. (In the preceding section, the function f was assumed to be continuous and nonnegative because we were dealing with the area under a curve.)

DEFINITION OF RIEMANN SUM

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is *any* point in the ith subinterval $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^{n} f(c_i)\, \Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ.

NOTE The sums in Section 4.2 are examples of Riemann sums, but there are more general Riemann sums than those covered there. ■

The width of the largest subinterval of a partition Δ is the **norm** of the partition and is denoted by $\|\Delta\|$. If every subinterval is of equal width, the partition is **regular** and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b-a}{n}. \qquad \text{Regular partition}$$

For a general partition, the norm is related to the number of subintervals of $[a, b]$ in the following way.

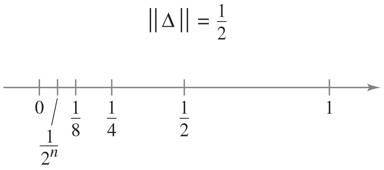
$$\frac{b-a}{\|\Delta\|} \le n \qquad \text{General partition}$$

So, the number of subintervals in a partition approaches infinity as the norm of the partition approaches 0. That is, $\|\Delta\| \to 0$ implies that $n \to \infty$.

The converse of this statement is not true. For example, let Δ_n be the partition of the interval $[0, 1]$ given by

$$0 < \frac{1}{2^n} < \frac{1}{2^{n-1}} < \cdot \cdot \cdot < \frac{1}{8} < \frac{1}{4} < \frac{1}{2} < 1.$$

As shown in Figure 4.19, for any positive value of n, the norm of the partition Δ_n is $\frac{1}{2}$. So, letting n approach infinity does not force $\|\Delta\|$ to approach 0. In a regular partition, however, the statements $\|\Delta\| \to 0$ and $n \to \infty$ are equivalent.



$n \to \infty$ does not imply that $\|\Delta\| \to 0$.
Figure 4.19

Definite Integrals

To define the definite integral, consider the following limit.

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = L$$

To say that this limit exists means there exists a real number L such that for each $\varepsilon > 0$ there exists a $\delta > 0$ so that for every partition with $\|\Delta\| < \delta$ it follows that

$$\left| L - \sum_{i=1}^{n} f(c_i)\,\Delta x_i \right| < \varepsilon$$

regardless of the choice of c_i in the ith subinterval of each partition Δ.

FOR FURTHER INFORMATION For insight into the history of the definite integral, see the article "The Evolution of Integration" by A. Shenitzer and J. Steprāns in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

DEFINITION OF DEFINITE INTEGRAL

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

It is not a coincidence that the notation for definite integrals is similar to that used for indefinite integrals. You will see why in the next section when the Fundamental Theorem of Calculus is introduced. For now it is important to see that definite integrals and indefinite integrals are different concepts. A definite integral is a *number*, whereas an indefinite integral is a *family of functions*.

Though Riemann sums were defined for functions with very few restrictions, a sufficient condition for a function f to be integrable on $[a, b]$ is that it is continuous on $[a, b]$. A proof of this theorem is beyond the scope of this text.

STUDY TIP Later in this chapter, you will learn convenient methods for calculating $\int_a^b f(x)\,dx$ for continuous functions. For now, you must use the limit definition.

THEOREM 4.4 CONTINUITY IMPLIES INTEGRABILITY

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,dx$ exists.

EXPLORATION

The Converse of Theorem 4.4 Is the converse of Theorem 4.4 true? That is, if a function is integrable, does it have to be continuous? Explain your reasoning and give examples.

Describe the relationships among continuity, differentiability, and integrability. Which is the strongest condition? Which is the weakest? Which conditions imply other conditions?

EXAMPLE 2 Evaluating a Definite Integral as a Limit

Evaluate the definite integral $\int_{-2}^{1} 2x\,dx$.

Solution The function $f(x) = 2x$ is integrable on the interval $[-2, 1]$ because it is continuous on $[-2, 1]$. Moreover, the definition of integrability implies that any partition whose norm approaches 0 can be used to determine the limit. For computational convenience, define Δ by subdividing $[-2, 1]$ into n subintervals of equal width

$$\Delta x_i = \Delta x = \frac{b-a}{n} = \frac{3}{n}.$$

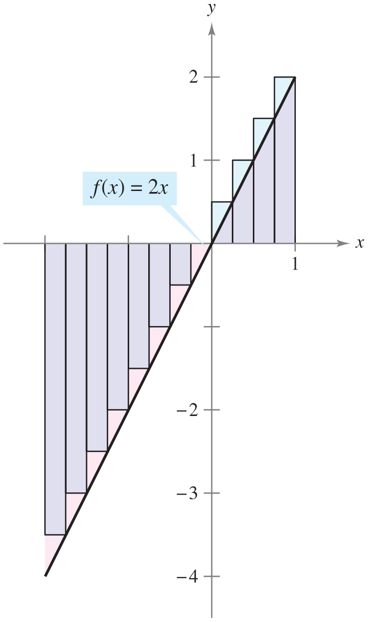
Choosing c_i as the right endpoint of each subinterval produces

$$c_i = a + i(\Delta x) = -2 + \frac{3i}{n}.$$

So, the definite integral is given by

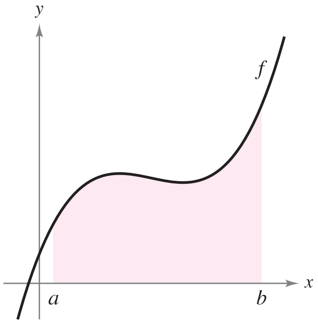
$$\begin{aligned}\int_{-2}^{1} 2x\,dx &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(-2+\frac{3i}{n}\right)\left(\frac{3}{n}\right) \\ &= \lim_{n\to\infty} \frac{6}{n}\sum_{i=1}^{n}\left(-2+\frac{3i}{n}\right) \\ &= \lim_{n\to\infty} \frac{6}{n}\left\{-2n + \frac{3}{n}\left[\frac{n(n+1)}{2}\right]\right\} \\ &= \lim_{n\to\infty}\left(-12 + 9 + \frac{9}{n}\right) \\ &= -3.\end{aligned}$$

■



Because the definite integral is negative, it does not represent the area of the region.
Figure 4.20

Because the definite integral in Example 2 is negative, it *does not* represent the area of the region shown in Figure 4.20. Definite integrals can be positive, negative, or zero. For a definite integral to be interpreted as an area (as defined in Section 4.2), the function f must be continuous and nonnegative on $[a, b]$, as stated in the following theorem. The proof of this theorem is straightforward—you simply use the definition of area given in Section 4.2, because it is a Riemann sum.



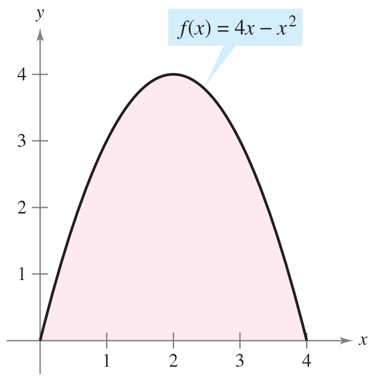
You can use a definite integral to find the area of the region bounded by the graph of f, the x-axis, $x = a$, and $x = b$.
Figure 4.21

THEOREM 4.5 THE DEFINITE INTEGRAL AS THE AREA OF A REGION

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)\,dx.$$

(See Figure 4.21.)



Area $= \int_0^4 (4x - x^2)\, dx$
Figure 4.22

As an example of Theorem 4.5, consider the region bounded by the graph of

$$f(x) = 4x - x^2$$

and the x-axis, as shown in Figure 4.22. Because f is continuous and nonnegative on the closed interval $[0, 4]$, the area of the region is

$$\text{Area} = \int_0^4 (4x - x^2)\, dx.$$

A straightforward technique for evaluating a definite integral such as this will be discussed in Section 4.4. For now, however, you can evaluate a definite integral in two ways—you can use the limit definition *or* you can check to see whether the definite integral represents the area of a common geometric region such as a rectangle, triangle, or semicircle.

EXAMPLE 3 Areas of Common Geometric Figures

Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

a. $\int_1^3 4\, dx$ **b.** $\int_0^3 (x + 2)\, dx$ **c.** $\int_{-2}^2 \sqrt{4 - x^2}\, dx$

Solution A sketch of each region is shown in Figure 4.23.

a. This region is a rectangle of height 4 and width 2.

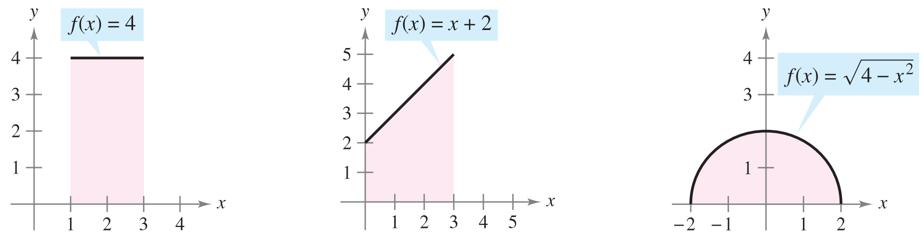
$$\int_1^3 4\, dx = (\text{Area of rectangle}) = 4(2) = 8$$

b. This region is a trapezoid with an altitude of 3 and parallel bases of lengths 2 and 5. The formula for the area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$.

$$\int_0^3 (x + 2)\, dx = (\text{Area of trapezoid}) = \frac{1}{2}(3)(2 + 5) = \frac{21}{2}$$

c. This region is a semicircle of radius 2. The formula for the area of a semicircle is $\frac{1}{2}\pi r^2$.

$$\int_{-2}^2 \sqrt{4 - x^2}\, dx = (\text{Area of semicircle}) = \frac{1}{2}\pi(2^2) = 2\pi$$



(a) **(b)** **(c)**
Figure 4.23

NOTE The variable of integration in a definite integral is sometimes called a *dummy variable* because it can be replaced by any other variable without changing the value of the integral. For instance, the definite integrals

$$\int_0^3 (x + 2)\, dx$$

and

$$\int_0^3 (t + 2)\, dt$$

have the same value.

Properties of Definite Integrals

The definition of the definite integral of f on the interval $[a, b]$ specifies that $a < b$. Now, however, it is convenient to extend the definition to cover cases in which $a = b$ or $a > b$. Geometrically, the following two definitions seem reasonable. For instance, it makes sense to define the area of a region of zero width and finite height to be 0.

DEFINITIONS OF TWO SPECIAL DEFINITE INTEGRALS

1. If f is defined at $x = a$, then we define $\displaystyle\int_a^a f(x)\,dx = 0$.
2. If f is integrable on $[a, b]$, then we define $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

EXAMPLE 4 Evaluating Definite Integrals

a. Because the sine function is defined at $x = \pi$, and the upper and lower limits of integration are equal, you can write

$$\int_{\pi}^{\pi} \sin x\,dx = 0.$$

b. The integral $\int_3^0 (x + 2)\,dx$ is the same as that given in Example 3(b) except that the upper and lower limits are interchanged. Because the integral in Example 3(b) has a value of $\frac{21}{2}$, you can write

$$\int_3^0 (x + 2)\,dx = -\int_0^3 (x + 2)\,dx = -\frac{21}{2}.$$

■

In Figure 4.24, the larger region can be divided at $x = c$ into two subregions whose intersection is a line segment. Because the line segment has zero area, it follows that the area of the larger region is equal to the sum of the areas of the two smaller regions.

THEOREM 4.6 ADDITIVE INTERVAL PROPERTY

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

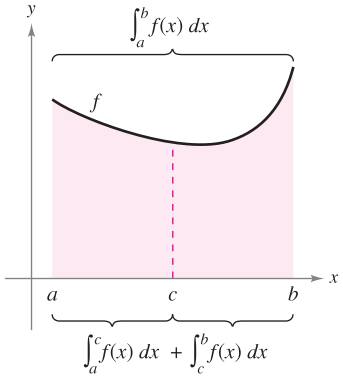


Figure 4.24

EXAMPLE 5 Using the Additive Interval Property

$$\begin{aligned}\int_{-1}^{1} |x|\,dx &= \int_{-1}^{0} -x\,dx + \int_0^1 x\,dx && \text{Theorem 4.6}\\ &= \frac{1}{2} + \frac{1}{2} && \text{Area of a triangle}\\ &= 1\end{aligned}$$

■

Because the definite integral is defined as the limit of a sum, it inherits the properties of summation given at the top of page 260.

THEOREM 4.7 PROPERTIES OF DEFINITE INTEGRALS

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

Note that Property 2 of Theorem 4.7 can be extended to cover any finite number of functions. For example,

$$\int_a^b [f(x) + g(x) + h(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \int_a^b h(x)\,dx.$$

EXAMPLE 6 Evaluation of a Definite Integral

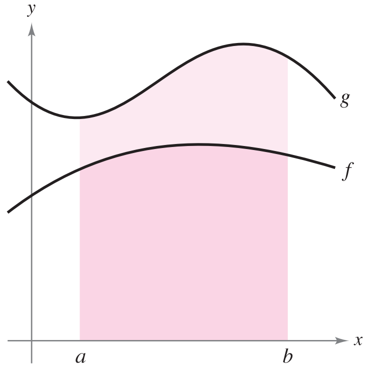
Evaluate $\displaystyle\int_1^3 (-x^2 + 4x - 3)\,dx$ using each of the following values.

$$\int_1^3 x^2\,dx = \frac{26}{3}, \qquad \int_1^3 x\,dx = 4, \qquad \int_1^3 dx = 2$$

Solution

$$\begin{aligned}\int_1^3 (-x^2 + 4x - 3)\,dx &= \int_1^3 (-x^2)\,dx + \int_1^3 4x\,dx + \int_1^3 (-3)\,dx\\ &= -\int_1^3 x^2\,dx + 4\int_1^3 x\,dx - 3\int_1^3 dx\\ &= -\left(\frac{26}{3}\right) + 4(4) - 3(2)\\ &= \frac{4}{3}\end{aligned}$$

■



$\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

Figure 4.25

If f and g are continuous on the closed interval $[a, b]$ and

$0 \le f(x) \le g(x)$

for $a \le x \le b$, the following properties are true. First, the area of the region bounded by the graph of f and the x-axis (between a and b) must be nonnegative. Second, this area must be less than or equal to the area of the region bounded by the graph of g and the x-axis (between a and b), as shown in Figure 4.25. These two properties are generalized in Theorem 4.8. (A proof of this theorem is given in Appendix A.)

THEOREM 4.8 PRESERVATION OF INEQUALITY

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \le \int_a^b f(x)\,dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \le g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

4.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use Example 1 as a model to evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

over the region bounded by the graphs of the equations.

1. $f(x) = \sqrt{x}, \quad y = 0, \quad x = 0, \quad x = 3$
(*Hint:* Let $c_i = 3i^2/n^2$.)

2. $f(x) = \sqrt[3]{x}, \quad y = 0, \quad x = 0, \quad x = 1$
(*Hint:* Let $c_i = i^3/n^3$.)

In Exercises 3–8, evaluate the definite integral by the limit definition.

3. $\displaystyle\int_2^6 8\,dx$

4. $\displaystyle\int_{-2}^3 x\,dx$

5. $\displaystyle\int_{-1}^1 x^3\,dx$

6. $\displaystyle\int_1^4 4x^2\,dx$

7. $\displaystyle\int_1^2 (x^2 + 1)\,dx$

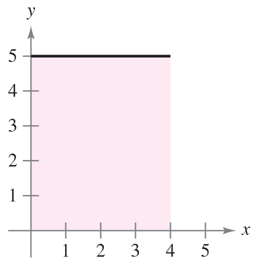
8. $\displaystyle\int_{-2}^1 (2x^2 + 3)\,dx$

In Exercises 9–12, write the limit as a definite integral on the interval $[a, b]$, where c_i is any point in the ith subinterval.

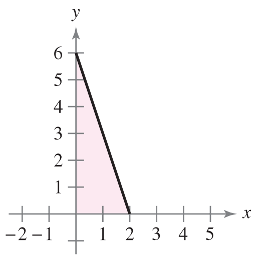
	Limit	*Interval*
9.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (3c_i + 10)\,\Delta x_i$	$[-1, 5]$
10.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 6c_i(4 - c_i)^2\,\Delta x_i$	$[0, 4]$
11.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{c_i^2 + 4}\,\Delta x_i$	$[0, 3]$
12.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \left(\frac{3}{c_i^2}\right)\Delta x_i$	$[1, 3]$

In Exercises 13–22, set up a definite integral that yields the area of the region. (Do not evaluate the integral.)

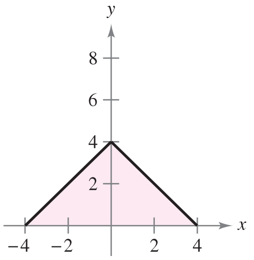
13. $f(x) = 5$



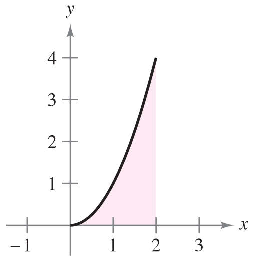
14. $f(x) = 6 - 3x$



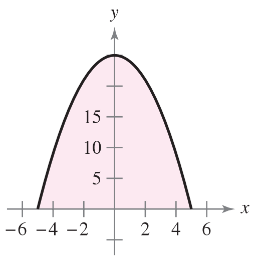
15. $f(x) = 4 - |x|$



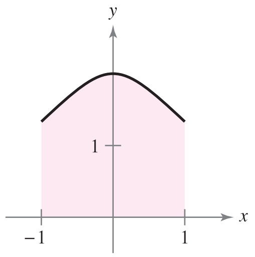
16. $f(x) = x^2$



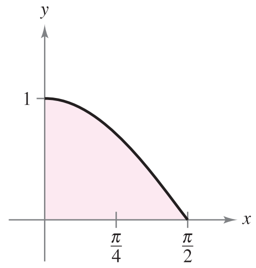
17. $f(x) = 25 - x^2$



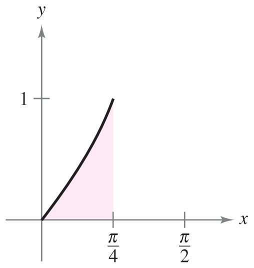
18. $f(x) = \dfrac{4}{x^2 + 2}$



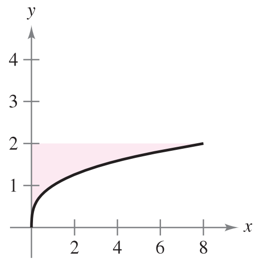
19. $f(x) = \cos x$



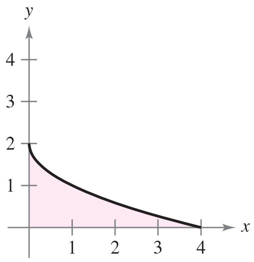
20. $f(x) = \tan x$



21. $g(y) = y^3$



22. $f(y) = (y-2)^2$



In Exercises 23–32, sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral ($a > 0, r > 0$)

23. $\displaystyle\int_0^3 4\,dx$

24. $\displaystyle\int_{-a}^a 4\,dx$

25. $\displaystyle\int_0^4 x\,dx$

26. $\displaystyle\int_0^4 \frac{x}{2}\,dx$

27. $\displaystyle\int_0^2 (3x+4)\,dx$

28. $\displaystyle\int_0^6 (6-x)\,dx$

29. $\displaystyle\int_{-1}^1 (1-|x|)\,dx$

30. $\displaystyle\int_{-a}^a (a-|x|)\,dx$

31. $\displaystyle\int_{-7}^7 \sqrt{49-x^2}\,dx$

32. $\displaystyle\int_{-r}^r \sqrt{r^2-x^2}\,dx$

In Exercises 33–40, evaluate the integral using the following values.

$$\int_2^4 x^3\,dx = 60, \qquad \int_2^4 x\,dx = 6, \qquad \int_2^4 dx = 2$$

33. $\displaystyle\int_4^2 x\,dx$

34. $\displaystyle\int_2^2 x^3\,dx$

35. $\displaystyle\int_2^4 8x\,dx$

36. $\displaystyle\int_2^4 25\,dx$

37. $\displaystyle\int_2^4 (x-9)\,dx$

38. $\displaystyle\int_2^4 (x^3+4)\,dx$

39. $\displaystyle\int_2^4 \left(\tfrac{1}{2}x^3-3x+2\right)dx$

40. $\displaystyle\int_2^4 (10+4x-3x^3)\,dx$

41. Given $\int_0^5 f(x)\,dx = 10$ and $\int_5^7 f(x)\,dx = 3$, evaluate

(a) $\displaystyle\int_0^7 f(x)\,dx.$ (b) $\displaystyle\int_5^0 f(x)\,dx.$

(c) $\displaystyle\int_5^5 f(x)\,dx.$ (d) $\displaystyle\int_0^5 3f(x)\,dx.$

42. Given $\int_0^3 f(x)\,dx = 4$ and $\int_3^6 f(x)\,dx = -1$, evaluate

(a) $\displaystyle\int_0^6 f(x)\,dx.$ (b) $\displaystyle\int_6^3 f(x)\,dx.$

(c) $\displaystyle\int_3^3 f(x)\,dx.$ (d) $\displaystyle\int_3^6 -5f(x)\,dx.$

43. Given $\int_2^6 f(x)\,dx = 10$ and $\int_2^6 g(x)\,dx = -2$, evaluate

(a) $\displaystyle\int_2^6 [f(x)+g(x)]\,dx.$ (b) $\displaystyle\int_2^6 [g(x)-f(x)]\,dx.$

(c) $\displaystyle\int_2^6 2g(x)\,dx.$ (d) $\displaystyle\int_2^6 3f(x)\,dx.$

44. Given $\int_{-1}^1 f(x)\,dx = 0$ and $\int_0^1 f(x)\,dx = 5$, evaluate

(a) $\displaystyle\int_{-1}^0 f(x)\,dx.$ (b) $\displaystyle\int_0^1 f(x)\,dx - \int_{-1}^0 f(x)\,dx.$

(c) $\displaystyle\int_{-1}^1 3f(x)\,dx.$ (d) $\displaystyle\int_0^1 3f(x)\,dx.$

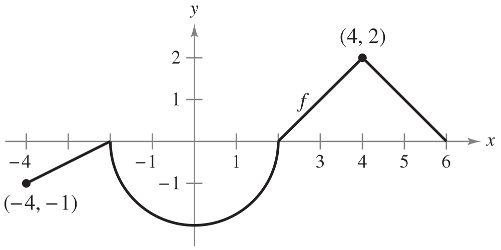
45. Use the table of values to find lower and upper estimates of $\int_0^{10} f(x)\,dx$. Assume that f is a decreasing function.

x	0	2	4	6	8	10
$f(x)$	32	24	12	-4	-20	-36

46. Use the table of values to estimate $\int_0^6 f(x)\,dx$. Use three equal subintervals and the (a) left endpoints, (b) right endpoints, and (c) midpoints. If f is an increasing function, how does each estimate compare with the actual value? Explain your reasoning.

x	0	1	2	3	4	5	6
$f(x)$	-6	0	8	18	30	50	80

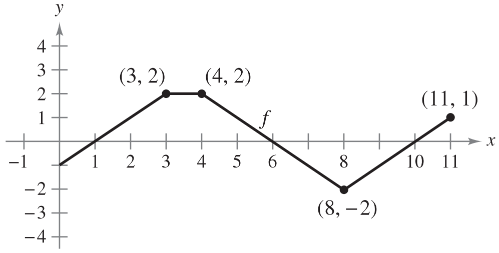
47. ***Think About It*** The graph of f consists of line segments and a semicircle, as shown in the figure. Evaluate each definite integral by using geometric formulas.



(a) $\displaystyle\int_0^2 f(x)\,dx$ (b) $\displaystyle\int_2^6 f(x)\,dx$ (c) $\displaystyle\int_{-4}^2 f(x)\,dx$

(d) $\displaystyle\int_{-4}^6 f(x)\,dx$ (e) $\displaystyle\int_{-4}^6 |f(x)|\,dx$ (f) $\displaystyle\int_{-4}^6 [f(x)+2]\,dx$

48. ***Think About It*** The graph of f consists of line segments, as shown in the figure. Evaluate each definite integral by using geometric formulas.



(a) $\displaystyle\int_0^1 -f(x)\,dx$ (b) $\displaystyle\int_3^4 3f(x)\,dx$

(c) $\displaystyle\int_0^7 f(x)\,dx$ (d) $\displaystyle\int_5^{11} f(x)\,dx$

(e) $\displaystyle\int_0^{11} f(x)\,dx$ (f) $\displaystyle\int_4^{10} f(x)\,dx$

49. ***Think About It*** Consider the function f that is continuous on the interval $[-5, 5]$ and for which

$$\int_0^5 f(x)\,dx = 4.$$

Evaluate each integral.

(a) $\displaystyle\int_0^5 [f(x) + 2]\,dx$ (b) $\displaystyle\int_{-2}^3 f(x + 2)\,dx$

(c) $\displaystyle\int_{-5}^5 f(x)\,dx$ (f is even.) (d) $\displaystyle\int_{-5}^5 f(x)\,dx$ (f is odd.)

50. ***Think About It*** A function f is defined below. Use geometric formulas to find $\int_0^8 f(x)\,dx$.

$$f(x) = \begin{cases} 4, & x < 4 \\ x, & x \ge 4 \end{cases}$$

51. ***Think About It*** A function f is defined below. Use geometric formulas to find $\int_0^{12} f(x)\,dx$.

$$f(x) = \begin{cases} 6, & x > 6 \\ -\frac{1}{2}x + 9, & x \le 6 \end{cases}$$

CAPSTONE

52. Find possible values of a and b that make the statement true. If possible, use a graph to support your answer. (There may be more than one correct answer.)

(a) $\displaystyle\int_{-2}^1 f(x)\,dx + \int_1^5 f(x)\,dx = \int_a^b f(x)\,dx$

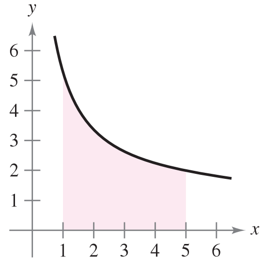
(b) $\displaystyle\int_{-3}^3 f(x)\,dx + \int_3^6 f(x)\,dx - \int_a^b f(x)\,dx = \int_{-1}^6 f(x)\,dx$

(c) $\displaystyle\int_a^b \sin x\,dx < 0$

(d) $\displaystyle\int_a^b \cos x\,dx = 0$

WRITING ABOUT CONCEPTS

In Exercises 53 and 54, use the figure to fill in the blank with the symbol <, >, or =.



53. The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the left endpoint of the ith subinterval.

$$\sum_{i=1}^n f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

54. The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the right endpoint of the ith subinterval.

$$\sum_{i=1}^n f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

55. Determine whether the function $f(x) = \dfrac{1}{x - 4}$ is integrable on the interval $[3, 5]$. Explain.

56. Give an example of a function that is integrable on the interval $[-1, 1]$, but not continuous on $[-1, 1]$.

In Exercises 57–60, determine which value best approximates the definite integral. Make your selection on the basis of a sketch.

57. $\displaystyle\int_0^4 \sqrt{x}\,dx$

(a) 5 (b) -3 (c) 10 (d) 2 (e) 8

58. $\displaystyle\int_0^{1/2} 4\cos \pi x\,dx$

(a) 4 (b) $\frac{4}{3}$ (c) 16 (d) 2π (e) -6

59. $\displaystyle\int_0^1 2\sin \pi x\,dx$

(a) 6 (b) $\frac{1}{2}$ (c) 4 (d) $\frac{5}{4}$

60. $\displaystyle\int_0^9 \left(1 + \sqrt{x}\right)dx$

(a) -3 (b) 9 (c) 27 (d) 3

Programming **Write a program for your graphing utility to approximate a definite integral using the Riemann sum**

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

where the subintervals are of equal width. The output should give three approximations of the integral, where c_i is the left-hand endpoint $L(n)$, the midpoint $M(n)$, and the right-hand endpoint $R(n)$ of each subinterval. In Exercises 61–64, use the program to approximate the definite integral and complete the table.

n	4	8	12	16	20
$L(n)$					
$M(n)$					
$R(n)$					

61. $\displaystyle\int_0^3 x\sqrt{3-x}\,dx$

62. $\displaystyle\int_0^3 \frac{5}{x^2+1}\,dx$

63. $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx$

64. $\displaystyle\int_0^3 x\sin x\,dx$

True or False? **In Exercises 65–70, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

65. $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

66. $\displaystyle\int_a^b f(x)g(x)\,dx = \left[\int_a^b f(x)\,dx\right]\left[\int_a^b g(x)\,dx\right]$

67. If the norm of a partition approaches zero, then the number of subintervals approaches infinity.

68. If f is increasing on $[a, b]$, then the minimum value of $f(x)$ on $[a, b]$ is $f(a)$.

69. The value of $\int_a^b f(x)\,dx$ must be positive.

70. The value of $\int_2^2 \sin(x^2)\,dx$ is 0.

71. Find the Riemann sum for $f(x) = x^2 + 3x$ over the interval $[0, 8]$, where $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 7$, and $x_4 = 8$, and where $c_1 = 1$, $c_2 = 2$, $c_3 = 5$, and $c_4 = 8$.

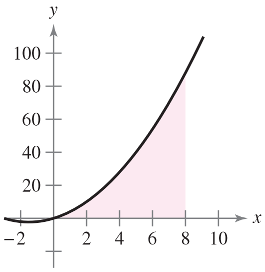


Figure for 71

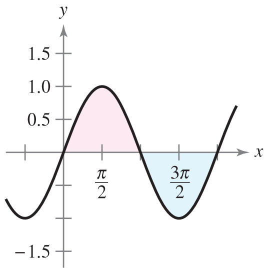


Figure for 72

72. Find the Riemann sum for $f(x) = \sin x$ over the interval $[0, 2\pi]$, where $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/3$, $x_3 = \pi$, and $x_4 = 2\pi$, and where $c_1 = \pi/6$, $c_2 = \pi/3$, $c_3 = 2\pi/3$, and $c_4 = 3\pi/2$.

73. Prove that $\displaystyle\int_a^b x\,dx = \frac{b^2 - a^2}{2}$.

74. Prove that $\displaystyle\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}$.

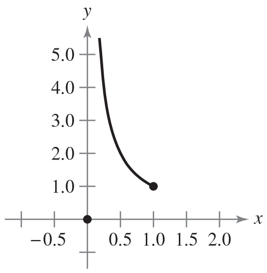
75. ***Think About It*** Determine whether the Dirichlet function

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

is integrable on the interval $[0, 1]$. Explain.

76. Suppose the function f is defined on $[0, 1]$, as shown in the figure.

$$f(x) = \begin{cases} 0, & x = 0 \\ \dfrac{1}{x}, & 0 < x \le 1 \end{cases}$$



Show that $\int_0^1 f(x)\,dx$ does not exist. Why doesn't this contradict Theorem 4.4?

77. Find the constants a and b that maximize the value of

$$\int_a^b (1 - x^2)\,dx.$$

Explain your reasoning.

78. Evaluate, if possible, the integral $\displaystyle\int_0^2 [\![x]\!]\,dx$.

79. Determine

$$\lim_{n\to\infty} \frac{1}{n^3}\left[1^2 + 2^2 + 3^2 + \cdot\cdot\cdot + n^2\right]$$

by using an appropriate Riemann sum.

PUTNAM EXAM CHALLENGE

80. For each continuous function $f\colon [0, 1] \to R$, let $I(f) = \int_0^1 x^2 f(x)\,dx$ and $J(x) = \int_0^1 x(f(x))^2\,dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

This problem was composed by the Committee on the Putnam Prize Competition.
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4.4 The Fundamental Theorem of Calculus

- **Evaluate a definite integral using the Fundamental Theorem of Calculus.**
- **Understand and use the Mean Value Theorem for Integrals.**
- **Find the average value of a function over a closed interval.**
- **Understand and use the Second Fundamental Theorem of Calculus.**
- **Understand and use the Net Change Theorem.**

The Fundamental Theorem of Calculus

You have now been introduced to the two major branches of calculus: differential calculus (introduced with the tangent line problem) and integral calculus (introduced with the area problem). At this point, these two problems might seem unrelated—but there is a very close connection. The connection was discovered independently by Isaac Newton and Gottfried Leibniz and is stated in a theorem that is appropriately called the **Fundamental Theorem of Calculus.**

Informally, the theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations. To see how Newton and Leibniz might have anticipated this relationship, consider the approximations shown in Figure 4.26. The slope of the tangent line was defined using the *quotient* $\Delta y/\Delta x$ (the slope of the secant line). Similarly, the area of a region under a curve was defined using the *product* $\Delta y \Delta x$ (the area of a rectangle). So, at least in the primitive approximation stage, the operations of differentiation and definite integration appear to have an inverse relationship in the same sense that division and multiplication are inverse operations. The Fundamental Theorem of Calculus states that the limit processes (used to define the derivative and definite integral) preserve this inverse relationship.

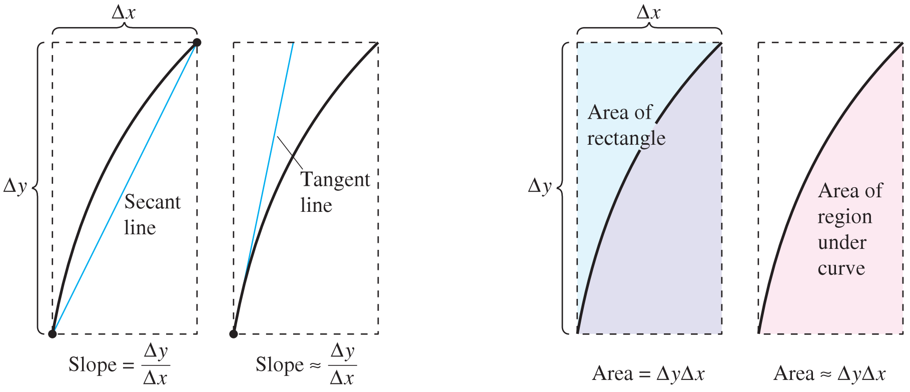
EXPLORATION

Integration and Antidifferentiation Throughout this chapter, you have been using the integral sign to denote an antiderivative (a family of functions) and a definite integral (a number).

Antidifferentiation: $\int f(x)\,dx$

Definite integration: $\int_a^b f(x)\,dx$

The use of this same symbol for both operations makes it appear that they are related. In the early work with calculus, however, it was not known that the two operations were related. Do you think the symbol $\int$ was first applied to antidifferentiation or to definite integration? Explain your reasoning. (*Hint:* The symbol was first used by Leibniz and was derived from the letter *S*.)



Δx
Δy
Secant line
Tangent line
Slope $= \frac{\Delta y}{\Delta x}$
Slope $\approx \frac{\Delta y}{\Delta x}$
Δx
Δy
Area of rectangle
Area of region under curve
Area $= \Delta y \Delta x$
Area $\approx \Delta y \Delta x$

(a) Differentiation **(b)** Definite integration

Differentiation and definite integration have an "inverse" relationship.

Figure 4.26

THEOREM 4.9 THE FUNDAMENTAL THEOREM OF CALCULUS

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

PROOF The key to the proof is in writing the difference $F(b) - F(a)$ in a convenient form. Let Δ be any partition of $[a, b]$.

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

By pairwise subtraction and addition of like terms, you can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \cdot \cdot \cdot - F(x_1) + F(x_1) - F(x_0) \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]. \end{aligned}$$

By the Mean Value Theorem, you know that there exists a number c_i in the ith subinterval such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}.$$

Because $F'(c_i) = f(c_i)$, you can let $\Delta x_i = x_i - x_{i-1}$ and obtain

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i.$$

This important equation tells you that by repeatedly applying the Mean Value Theorem, you can always find a collection of c_i's such that the *constant* $F(b) - F(a)$ is a Riemann sum of f on $[a, b]$ for any partition. Theorem 4.4 guarantees that the limit of Riemann sums over the partition with $\|\Delta\| \to 0$ exists. So, taking the limit (as $\|\Delta\| \to 0$) produces

$$F(b) - F(a) = \int_a^b f(x)\,dx.$$ ■

The following guidelines can help you understand the use of the Fundamental Theorem of Calculus.

GUIDELINES FOR USING THE FUNDAMENTAL THEOREM OF CALCULUS

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.

2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\begin{aligned} \int_a^b f(x)\,dx &= F(x)\Big]_a^b \\ &= F(b) - F(a) \end{aligned}$$

For instance, to evaluate $\int_1^3 x^3\,dx$, you can write

$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\begin{aligned} \int_a^b f(x)\,dx &= \Big[F(x) + C\Big]_a^b \\ &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a). \end{aligned}$$

EXAMPLE 1 Evaluating a Definite Integral

Evaluate each definite integral.

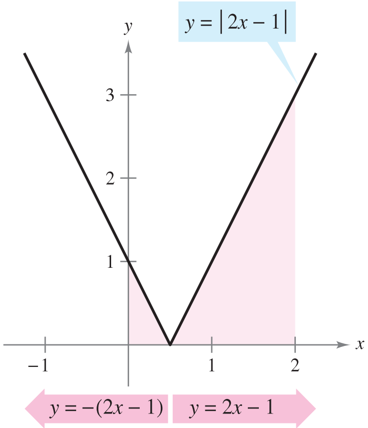
a. $\displaystyle\int_1^2 (x^2 - 3)\,dx$ **b.** $\displaystyle\int_1^4 3\sqrt{x}\,dx$ **c.** $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx$

Solution

a. $$\int_1^2 (x^2 - 3)\,dx = \left[\frac{x^3}{3} - 3x\right]_1^2 = \left(\frac{8}{3} - 6\right) - \left(\frac{1}{3} - 3\right) = -\frac{2}{3}$$

b. $$\int_1^4 3\sqrt{x}\,dx = 3\int_1^4 x^{1/2}\,dx = 3\left[\frac{x^{3/2}}{3/2}\right]_1^4 = 2(4)^{3/2} - 2(1)^{3/2} = 14$$

c. $$\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big]_0^{\pi/4} = 1 - 0 = 1$$



The definite integral of y on $[0, 2]$ is $\frac{5}{2}$.
Figure 4.27

EXAMPLE 2 A Definite Integral Involving Absolute Value

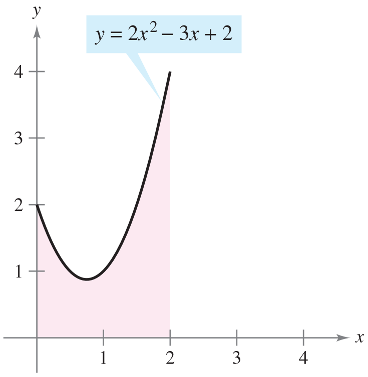
Evaluate $\displaystyle\int_0^2 |2x - 1|\,dx.$

Solution Using Figure 4.27 and the definition of absolute value, you can rewrite the integrand as shown.

$$|2x - 1| = \begin{cases} -(2x - 1), & x < \frac{1}{2} \\ 2x - 1, & x \geq \frac{1}{2} \end{cases}$$

From this, you can rewrite the integral in two parts.

$$\begin{aligned}\int_0^2 |2x - 1|\,dx &= \int_0^{1/2} -(2x - 1)\,dx + \int_{1/2}^2 (2x - 1)\,dx \\ &= \Big[-x^2 + x\Big]_0^{1/2} + \Big[x^2 - x\Big]_{1/2}^2 \\ &= \left(-\frac{1}{4} + \frac{1}{2}\right) - (0 + 0) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right) \\ &= \frac{5}{2}\end{aligned}$$



The area of the region bounded by the graph of y, the x-axis, $x = 0$, and $x = 2$ is $\frac{10}{3}$.
Figure 4.28

EXAMPLE 3 Using the Fundamental Theorem to Find Area

Find the area of the region bounded by the graph of $y = 2x^2 - 3x + 2$, the x-axis, and the vertical lines $x = 0$ and $x = 2$, as shown in Figure 4.28.

Solution Note that $y > 0$ on the interval $[0, 2]$.

$$\text{Area} = \int_0^2 (2x^2 - 3x + 2)\,dx \qquad \text{Integrate between } x = 0 \text{ and } x = 2.$$

$$= \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^2 \qquad \text{Find antiderivative.}$$

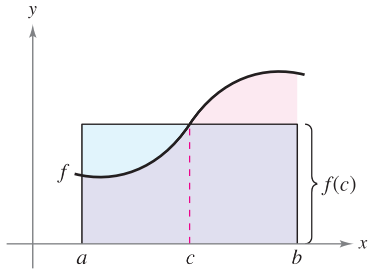
$$= \left(\frac{16}{3} - 6 + 4\right) - (0 - 0 + 0) \qquad \text{Apply Fundamental Theorem.}$$

$$= \frac{10}{3} \qquad \text{Simplify.}$$

■

The Mean Value Theorem for Integrals

In Section 4.2, you saw that the area of a region under a curve is greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle. The Mean Value Theorem for Integrals states that somewhere "between" the inscribed and circumscribed rectangles there is a rectangle whose area is precisely equal to the area of the region under the curve, as shown in Figure 4.29.



Mean value rectangle:

$$f(c)(b-a)=\int_a^b f(x)\,dx$$

Figure 4.29

THEOREM 4.10 MEAN VALUE THEOREM FOR INTEGRALS

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

PROOF

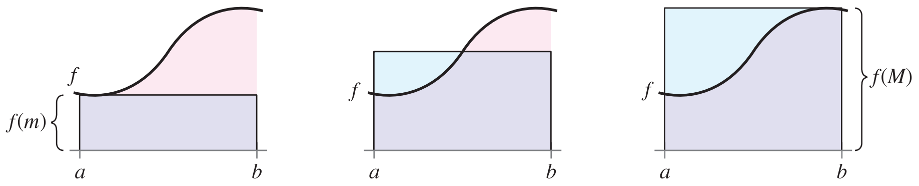
Case 1: If f is constant on the interval $[a, b]$, the theorem is clearly valid because c can be any point in $[a, b]$.

Case 2: If f is not constant on $[a, b]$, then, by the Extreme Value Theorem, you can choose $f(m)$ and $f(M)$ to be the minimum and maximum values of f on $[a, b]$. Because $f(m) \le f(x) \le f(M)$ for all x in $[a, b]$, you can apply Theorem 4.8 to write the following.

$$\begin{aligned}
\int_a^b f(m)\,dx \le &\int_a^b f(x)\,dx \le \int_a^b f(M)\,dx && \text{See Figure 4.30.}\\
f(m)(b-a) \le &\int_a^b f(x)\,dx \le f(M)(b-a)\\
f(m) \le &\frac{1}{b-a}\int_a^b f(x)\,dx \le f(M)
\end{aligned}$$

From the third inequality, you can apply the Intermediate Value Theorem to conclude that there exists some c in $[a, b]$ such that

$$f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx \quad \text{or} \quad f(c)(b-a)=\int_a^b f(x)\,dx.$$



Inscribed rectangle (less than actual area)

$$\int_a^b f(m)\,dx = f(m)(b-a)$$

Mean value rectangle (equal to actual area)

$$\int_a^b f(x)\,dx$$

Circumscribed rectangle (greater than actual area)

$$\int_a^b f(M)\,dx = f(M)(b-a)$$

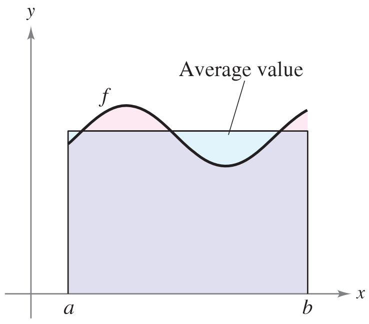
Figure 4.30

■

NOTE Notice that Theorem 4.10 does not specify how to determine c. It merely guarantees the existence of at least one number c in the interval. ■

Average Value of a Function

The value of $f(c)$ given in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.



$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Figure 4.31

DEFINITION OF THE AVERAGE VALUE OF A FUNCTION ON AN INTERVAL

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

NOTE Notice in Figure 4.31 that the area of the region under the graph of f is equal to the area of the rectangle whose height is the average value. ■

To see why the average value of f is defined in this way, suppose that you partition $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. If c_i is any point in the ith subinterval, the arithmetic average (or mean) of the function values at the c_i's is given by

$$a_n = \frac{1}{n}[f(c_1) + f(c_2) + \cdot \cdot \cdot + f(c_n)].$$

Average of $f(c_1), . . . , f(c_n)$

By multiplying and dividing by $(b - a)$, you can write the average as

$$a_n = \frac{1}{n}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{b-a}\right) = \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{n}\right)$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\,\Delta x.$$

Finally, taking the limit as $n \to \infty$ produces the average value of f on the interval $[a, b]$, as given in the definition above.

This development of the average value of a function on an interval is only one of many practical uses of definite integrals to represent summation processes. In Chapter 7, you will study other applications, such as volume, arc length, centers of mass, and work.

EXAMPLE 4 Finding the Average Value of a Function

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

Solution The average value is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-1}\int_1^4 (3x^2 - 2x)\,dx$$

$$= \frac{1}{3}\Big[x^3 - x^2\Big]_1^4$$

$$= \frac{1}{3}[64 - 16 - (1 - 1)] = \frac{48}{3} = 16.$$

(See Figure 4.32.) ■

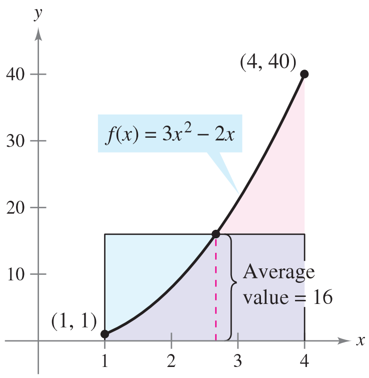


Figure 4.32

George Hall/Corbis

The first person to fly at a speed greater than the speed of sound was Charles Yeager. On October 14, 1947, Yeager was clocked at 295.9 meters per second at an altitude of 12.2 kilometers. If Yeager had been flying at an altitude below 11.275 kilometers, this speed would not have "broken the sound barrier." The photo above shows an F-14 *Tomcat*, a supersonic, twin-engine strike fighter. Currently, the *Tomcat* can reach heights of 15.24 kilometers and speeds up to 2 mach (707.78 meters per second).

EXAMPLE 5 The Speed of Sound

At different altitudes in Earth's atmosphere, sound travels at different speeds. The speed of sound $s(x)$ (in meters per second) can be modeled by

$$s(x) = \begin{cases} -4x + 341, & 0 \le x < 11.5 \\ 295, & 11.5 \le x < 22 \\ \frac{3}{4}x + 278.5, & 22 \le x < 32 \\ \frac{3}{2}x + 254.5, & 32 \le x < 50 \\ -\frac{3}{2}x + 404.5, & 50 \le x \le 80 \end{cases}$$

where x is the altitude in kilometers (see Figure 4.33). What is the average speed of sound over the interval $[0, 80]$?

Solution Begin by integrating $s(x)$ over the interval $[0, 80]$. To do this, you can break the integral into five parts.

$$\int_0^{11.5} s(x)\,dx = \int_0^{11.5} (-4x + 341)\,dx = \Big[-2x^2 + 341x\Big]_0^{11.5} = 3657$$

$$\int_{11.5}^{22} s(x)\,dx = \int_{11.5}^{22} (295)\,dx = \Big[295x\Big]_{11.5}^{22} = 3097.5$$

$$\int_{22}^{32} s(x)\,dx = \int_{22}^{32} \left(\tfrac{3}{4}x + 278.5\right)dx = \Big[\tfrac{3}{8}x^2 + 278.5x\Big]_{22}^{32} = 2987.5$$

$$\int_{32}^{50} s(x)\,dx = \int_{32}^{50} \left(\tfrac{3}{2}x + 254.5\right)dx = \Big[\tfrac{3}{4}x^2 + 254.5x\Big]_{32}^{50} = 5688$$

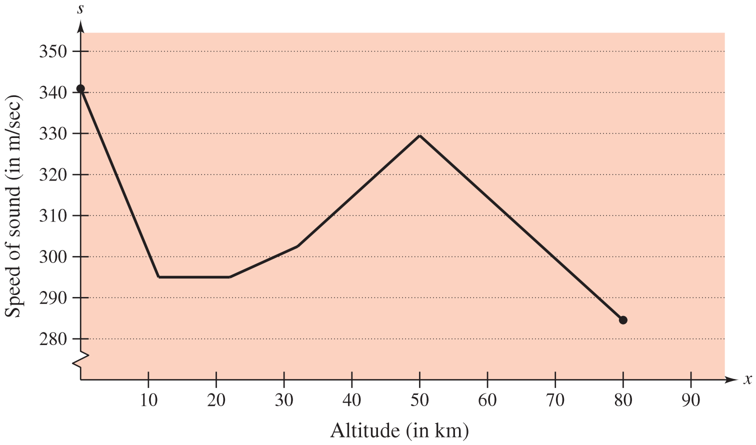
$$\int_{50}^{80} s(x)\,dx = \int_{50}^{80} \left(-\tfrac{3}{2}x + 404.5\right)dx = \Big[-\tfrac{3}{4}x^2 + 404.5x\Big]_{50}^{80} = 9210$$

By adding the values of the five integrals, you have

$$\int_0^{80} s(x)\,dx = 24{,}640.$$

So, the average speed of sound from an altitude of 0 kilometers to an altitude of 80 kilometers is

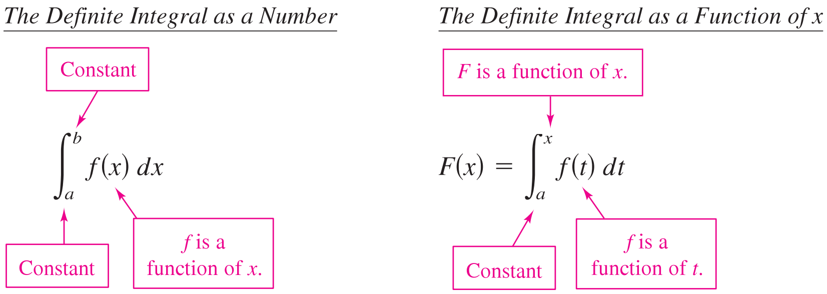
$$\text{Average speed} = \frac{1}{80}\int_0^{80} s(x)\,dx = \frac{24{,}640}{80} = 308 \text{ meters per second.}$$



Speed of sound depends on altitude.
Figure 4.33 ■

The Second Fundamental Theorem of Calculus

Earlier you saw that the definite integral of f on the interval $[a, b]$ was defined using the constant b as the upper limit of integration and x as the variable of integration. However, a slightly different situation may arise in which the variable x is used in the upper limit of integration. To avoid the confusion of using x in two different ways, t is temporarily used as the variable of integration. (Remember that the definite integral is *not* a function of its variable of integration.)



The Definite Integral as a Number
Constant
$\int_a^b f(x)\,dx$
Constant
f is a function of x.
The Definite Integral as a Function of x
F is a function of x.
$F(x) = \int_a^x f(t)\,dt$
Constant
f is a function of t.

EXPLORATION

Use a graphing utility to graph the function

$$F(x) = \int_0^x \cos t\,dt$$

for $0 \le x \le \pi$. Do you recognize this graph? Explain.

EXAMPLE 6 The Definite Integral as a Function

Evaluate the function

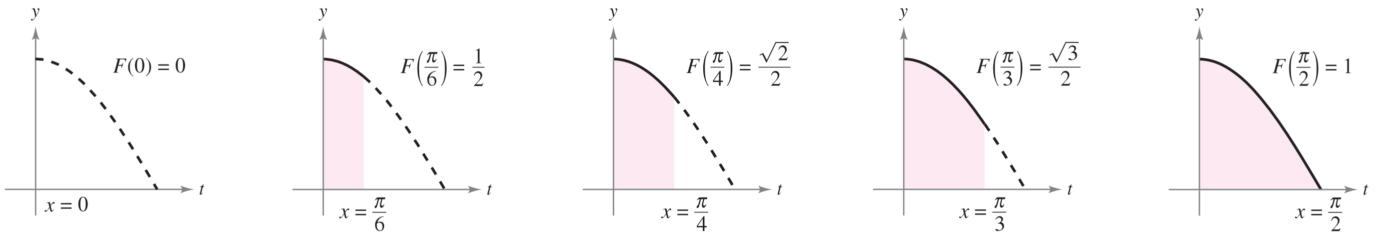
$$F(x) = \int_0^x \cos t\,dt$$

at $x = 0, \pi/6, \pi/4, \pi/3$, and $\pi/2$.

Solution You could evaluate five different definite integrals, one for each of the given upper limits. However, it is much simpler to fix x (as a constant) temporarily to obtain

$$\int_0^x \cos t\,dt = \sin t\Big]_0^x = \sin x - \sin 0 = \sin x.$$

Now, using $F(x) = \sin x$, you can obtain the results shown in Figure 4.34.



$F(x) = \int_0^x \cos t\,dt$ is the area under the curve $f(t) = \cos t$ from 0 to x.

Figure 4.34

■

You can think of the function $F(x)$ as *accumulating* the area under the curve $f(t) = \cos t$ from $t = 0$ to $t = x$. For $x = 0$, the area is 0 and $F(0) = 0$. For $x = \pi/2$, $F(\pi/2) = 1$ gives the accumulated area under the cosine curve on the entire interval $[0, \pi/2]$. This interpretation of an integral as an **accumulation function** is used often in applications of integration.

In Example 6, note that the derivative of F is the original integrand (with only the variable changed). That is,

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}[\sin x] = \frac{d}{dx}\left[\int_0^x \cos t\,dt\right] = \cos x.$$

This result is generalized in the following theorem, called the **Second Fundamental Theorem of Calculus.**

THEOREM 4.11 THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

PROOF Begin by defining F as

$$F(x) = \int_a^x f(t)\,dt.$$

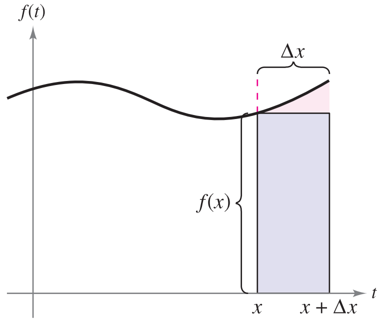
Then, by the definition of the derivative, you can write

$$\begin{aligned} F'(x) &= \lim_{\Delta x\to 0}\frac{F(x+\Delta x)-F(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt\right] \\ &= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\,dt + \int_x^a f(t)\,dt\right] \\ &= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\int_x^{x+\Delta x} f(t)\,dt\right]. \end{aligned}$$

From the Mean Value Theorem for Integrals (assuming $\Delta x > 0$), you know there exists a number c in the interval $[x, x + \Delta x]$ such that the integral in the expression above is equal to $f(c)\,\Delta x$. Moreover, because $x \le c \le x + \Delta x$, it follows that $c \to x$ as $\Delta x \to 0$. So, you obtain

$$\begin{aligned} F'(x) &= \lim_{\Delta x\to 0}\left[\frac{1}{\Delta x}f(c)\,\Delta x\right] \\ &= \lim_{\Delta x\to 0} f(c) \\ &= f(x). \end{aligned}$$

A similar argument can be made for $\Delta x < 0$. ■



$$f(x)\,\Delta x \approx \int_x^{x+\Delta x} f(t)\,dt$$

Figure 4.35

NOTE Using the area model for definite integrals, you can view the approximation

$$f(x)\,\Delta x \approx \int_x^{x+\Delta x} f(t)\,dt$$

as saying that the area of the rectangle of height $f(x)$ and width Δx is approximately equal to the area of the region lying between the graph of f and the x-axis on the interval $[x, x + \Delta x]$, as shown in Figure 4.35. ■

Note that the Second Fundamental Theorem of Calculus tells you that if a function is continuous, you can be sure that it has an antiderivative. This antiderivative need not, however, be an elementary function. (Recall the discussion of elementary functions in Section P.3.)

EXAMPLE 7 Using the Second Fundamental Theorem of Calculus

Evaluate $\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right]$.

Solution Note that $f(t) = \sqrt{t^2+1}$ is continuous on the entire real line. So, using the Second Fundamental Theorem of Calculus, you can write

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right] = \sqrt{x^2+1}.$$ ■

The differentiation shown in Example 7 is a straightforward application of the Second Fundamental Theorem of Calculus. The next example shows how this theorem can be combined with the Chain Rule to find the derivative of a function.

EXAMPLE 8 Using the Second Fundamental Theorem of Calculus

Find the derivative of $F(x) = \displaystyle\int_{\pi/2}^{x^3} \cos t\,dt$.

Solution Using $u = x^3$, you can apply the Second Fundamental Theorem of Calculus with the Chain Rule as shown.

$$\begin{aligned}
F'(x) &= \frac{dF}{du}\frac{du}{dx} && \text{Chain Rule}\\
&= \frac{d}{du}[F(x)]\frac{du}{dx} && \text{Definition of } \frac{dF}{du}\\
&= \frac{d}{du}\left[\int_{\pi/2}^{x^3} \cos t\,dt\right]\frac{du}{dx} && \text{Substitute } \int_{\pi/2}^{x^3} \cos t\,dt \text{ for } F(x).\\
&= \frac{d}{du}\left[\int_{\pi/2}^{u} \cos t\,dt\right]\frac{du}{dx} && \text{Substitute } u \text{ for } x^3.\\
&= (\cos u)(3x^2) && \text{Apply Second Fundamental Theorem of Calculus.}\\
&= (\cos x^3)(3x^2) && \text{Rewrite as function of } x.
\end{aligned}$$ ■

Because the integrand in Example 8 is easily integrated, you can verify the derivative as follows.

$$F(x) = \int_{\pi/2}^{x^3} \cos t\,dt = \sin t\Big]_{\pi/2}^{x^3} = \sin x^3 - \sin\frac{\pi}{2} = (\sin x^3) - 1$$

In this form, you can apply the Power Rule to verify that the derivative is the same as that obtained in Example 8.

$$F'(x) = (\cos x^3)(3x^2)$$

Net Change Theorem

The Fundamental Theorem of Calculus (Theorem 4.9) states that if f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

But because $F'(x) = f(x)$, this statement can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

where the quantity $F(b) - F(a)$ represents the *net change of F* on the interval $[a, b]$.

THEOREM 4.12 THE NET CHANGE THEOREM

The definite integral of the rate of change of a quantity $F'(x)$ gives the total change, or **net change,** in that quantity on the interval $[a, b]$.

$$\int_a^b F'(x)\,dx = F(b) - F(a) \qquad \text{Net change of } F$$

EXAMPLE 9 Using the Net Change Theorem

A chemical flows into a storage tank at a rate of $180 + 3t$ liters per minute, where $0 \le t \le 60$. Find the amount of the chemical that flows into the tank during the first 20 minutes.

Solution Let $c(t)$ be the amount of the chemical in the tank at time t. Then $c'(t)$ represents the rate at which the chemical flows into the tank at time t. During the first 20 minutes, the amount that flows into the tank is

$$\begin{aligned}\int_0^{20} c'(t)\,dt &= \int_0^{20} (180 + 3t)\,dt\\ &= \left[180t + \frac{3}{2}t^2\right]_0^{20}\\ &= 3600 + 600 = 4200.\end{aligned}$$

So, the amount that flows into the tank during the first 20 minutes is 4200 liters. ■

Another way to illustrate the Net Change Theorem is to examine the velocity of a particle moving along a straight line where $s(t)$ is the position at time t. Then its velocity is $v(t) = s'(t)$ and

$$\int_a^b v(t)\,dt = s(b) - s(a).$$

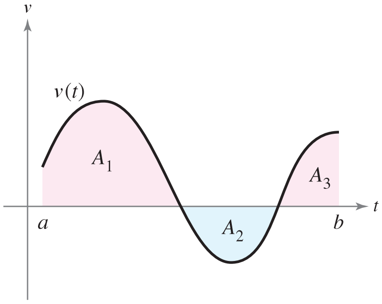
This definite integral represents the net change in position, or **displacement,** of the particle.

When calculating the *total* distance traveled by the particle, you must consider the intervals where $v(t) \le 0$ and the intervals where $v(t) \ge 0$. When $v(t) \le 0$, the particle moves to the left, and when $v(t) \ge 0$, the particle moves to the right. To calculate the total distance traveled, integrate the absolute value of velocity $|v(t)|$. So, the displacement of a particle and the total distance traveled by a particle over $[a, b]$ can be written as

$$\textbf{Displacement on } [a, b] = \int_a^b v(t)\,dt = A_1 - A_2 + A_3$$

$$\textbf{Total distance traveled on } [a, b] = \int_a^b |v(t)|\,dt = A_1 + A_2 + A_3$$

(see Figure 4.36).



A_1, A_2, and A_3 are the areas of the shaded regions.
Figure 4.36

EXAMPLE 10 Solving a Particle Motion Problem

A particle is moving along a line so that its velocity is $v(t) = t^3 - 10t^2 + 29t - 20$ feet per second at time t.

a. What is the displacement of the particle on the time interval $1 \le t \le 5$?

b. What is the total distance traveled by the particle on the time interval $1 \le t \le 5$?

Solution

a. By definition, you know that the displacement is

$$\begin{aligned}\int_1^5 v(t)\,dt &= \int_1^5 (t^3 - 10t^2 + 29t - 20)\,dt \\ &= \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_1^5 \\ &= \frac{25}{12} - \left(-\frac{103}{12}\right) \\ &= \frac{128}{12} \\ &= \frac{32}{3}.\end{aligned}$$

So, the particle moves $\frac{32}{3}$ feet to the right.

b. To find the total distance traveled, calculate $\int_1^5 |v(t)|\,dt$. Using Figure 4.37 and the fact that $v(t)$ can be factored as $(t - 1)(t - 4)(t - 5)$, you can determine that $v(t) \ge 0$ on $[1, 4]$ and $v(t) \le 0$ on $[4, 5]$. So, the total distance traveled is

$$\begin{aligned}\int_1^5 |v(t)|\,dt &= \int_1^4 v(t)\,dt - \int_4^5 v(t)\,dt \\ &= \int_1^4 (t^3 - 10t^2 + 29t - 20)\,dt - \int_4^5 (t^3 - 10t^2 + 29t - 20)\,dt \\ &= \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_1^4 - \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_4^5 \\ &= \frac{45}{4} - \left(-\frac{7}{12}\right) \\ &= \frac{71}{6} \text{ feet.}\end{aligned}$$

■

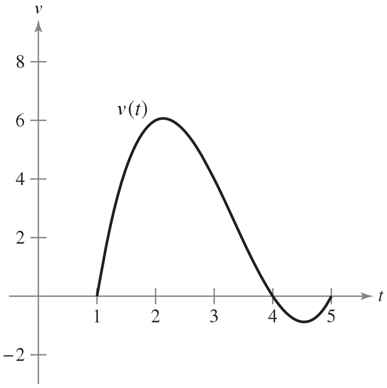


Figure 4.37

4.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Graphical Reasoning **In Exercises 1–4, use a graphing utility to graph the integrand. Use the graph to determine whether the definite integral is positive, negative, or zero.**

1. $\displaystyle\int_0^{\pi} \frac{4}{x^2+1}\,dx$ **2.** $\displaystyle\int_0^{\pi} \cos x\,dx$

3. $\displaystyle\int_{-2}^{2} x\sqrt{x^2+1}\,dx$ **4.** $\displaystyle\int_{-2}^{2} x\sqrt{2-x}\,dx$

In Exercises 5–26, evaluate the definite integral of the algebraic function. Use a graphing utility to verify your result.

5. $\displaystyle\int_0^{2} 6x\,dx$ **6.** $\displaystyle\int_4^{9} 5\,dv$

7. $\displaystyle\int_{-1}^{0} (2x-1)\,dx$ **8.** $\displaystyle\int_2^{5} (-3v+4)\,dv$

9. $\displaystyle\int_{-1}^{1} (t^2-2)\,dt$ **10.** $\displaystyle\int_1^{7} (6x^2+2x-3)\,dx$

11. $\displaystyle\int_0^{1} (2t-1)^2\,dt$ **12.** $\displaystyle\int_{-1}^{1} (t^3-9t)\,dt$

13. $\displaystyle\int_1^{2} \left(\frac{3}{x^2}-1\right)dx$ **14.** $\displaystyle\int_{-2}^{-1} \left(u-\frac{1}{u^2}\right)du$

15. $\displaystyle\int_1^{4} \frac{u-2}{\sqrt{u}}\,du$ **16.** $\displaystyle\int_{-3}^{3} v^{1/3}\,dv$

17. $\displaystyle\int_{-1}^{1} (\sqrt[3]{t}-2)\,dt$ **18.** $\displaystyle\int_1^{8} \sqrt{\frac{2}{x}}\,dx$

19. $\displaystyle\int_0^{1} \frac{x-\sqrt{x}}{3}\,dx$ **20.** $\displaystyle\int_0^{2} (2-t)\sqrt{t}\,dt$

21. $\displaystyle\int_{-1}^{0} \left(t^{1/3}-t^{2/3}\right)dt$ **22.** $\displaystyle\int_{-8}^{-1} \frac{x-x^2}{2\sqrt[3]{x}}\,dx$

23. $\displaystyle\int_0^{5} |2x-5|\,dx$ **24.** $\displaystyle\int_1^{4} (3-|x-3|)\,dx$

25. $\displaystyle\int_0^{4} |x^2-9|\,dx$ **26.** $\displaystyle\int_0^{4} |x^2-4x+3|\,dx$

In Exercises 27–34, evaluate the definite integral of the trigonometric function. Use a graphing utility to verify your result.

27. $\displaystyle\int_0^{\pi} (1+\sin x)\,dx$ **28.** $\displaystyle\int_0^{\pi} (2+\cos x)\,dx$

29. $\displaystyle\int_0^{\pi/4} \frac{1-\sin^2\theta}{\cos^2\theta}\,d\theta$ **30.** $\displaystyle\int_0^{\pi/4} \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta$

31. $\displaystyle\int_{-\pi/6}^{\pi/6} \sec^2 x\,dx$

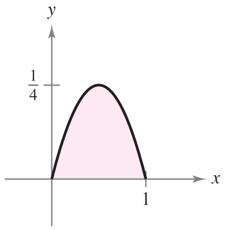
32. $\displaystyle\int_{\pi/4}^{\pi/2} (2-\csc^2 x)\,dx$

33. $\displaystyle\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,d\theta$

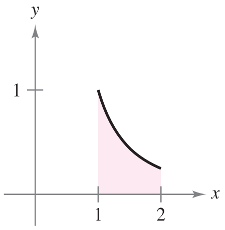
34. $\displaystyle\int_{-\pi/2}^{\pi/2} (2t+\cos t)\,dt$

In Exercises 35–38, determine the area of the given region.

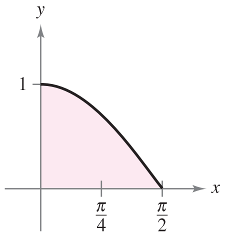
35. $y = x - x^2$



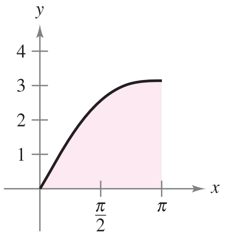
36. $y = \dfrac{1}{x^2}$



37. $y = \cos x$



38. $y = x + \sin x$



In Exercises 39–44, find the area of the region bounded by the graphs of the equations.

39. $y = 5x^2 + 2, \quad x = 0, \quad x = 2, \quad y = 0$

40. $y = x^3 + x, \quad x = 2, \quad y = 0$

41. $y = 1 + \sqrt[3]{x}, \quad x = 0, \quad x = 8, \quad y = 0$

42. $y = (3 - x)\sqrt{x}, \quad y = 0$

43. $y = -x^2 + 4x, \quad y = 0$ **44.** $y = 1 - x^4, \quad y = 0$

In Exercises 45–50, find the value(s) of *c* guaranteed by the Mean Value Theorem for Integrals for the function over the given interval.

45. $f(x) = x^3, \quad [0, 3]$ **46.** $f(x) = \dfrac{9}{x^3}, \quad [1, 3]$

47. $f(x) = \sqrt{x}, \quad [4, 9]$ **48.** $f(x) = x - 2\sqrt{x}, \quad [0, 2]$

49. $f(x) = 2\sec^2 x, \quad [-\pi/4, \pi/4]$

50. $f(x) = \cos x, \quad [-\pi/3, \pi/3]$

In Exercises 51–56, find the average value of the function over the given interval and all values of *x* in the interval for which the function equals its average value.

51. $f(x) = 9 - x^2, \quad [-3, 3]$

52. $f(x) = \dfrac{4(x^2+1)}{x^2}, \quad [1, 3]$

53. $f(x) = x^3, \quad [0, 1]$

54. $f(x) = 4x^3 - 3x^2, \quad [-1, 2]$

55. $f(x) = \sin x, \quad [0, \pi]$

56. $f(x) = \cos x, \quad [0, \pi/2]$

57. ***Velocity*** The graph shows the velocity, in feet per second, of a car accelerating from rest. Use the graph to estimate the distance the car travels in 8 seconds.

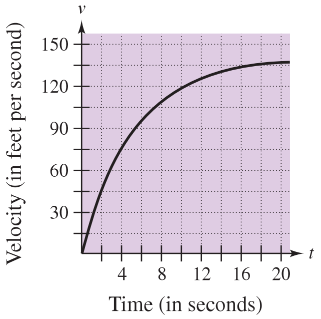


Figure for 57

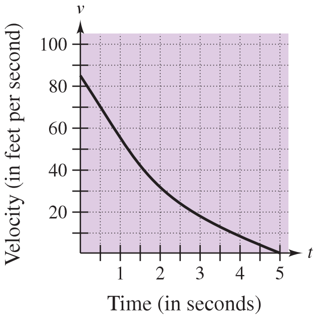
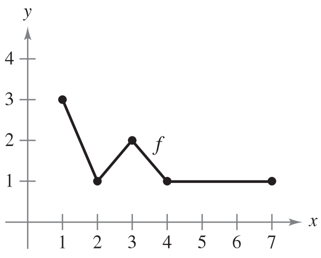


Figure for 58

58. ***Velocity*** The graph shows the velocity, in feet per second, of a decelerating car after the driver applies the brakes. Use the graph to estimate how far the car travels before it comes to a stop.

WRITING ABOUT CONCEPTS

59. The graph of f is shown in the figure.



(a) Evaluate $\int_1^7 f(x)\,dx$.

(b) Determine the average value of f on the interval $[1, 7]$.

(c) Determine the answers to parts (a) and (b) if the graph is translated two units upward.

60. If $r'(t)$ represents the rate of growth of a dog in pounds per year, what does $r(t)$ represent? What does $\int_2^6 r'(t)\,dt$ represent about the dog?

61. ***Force*** The force F (in newtons) of a hydraulic cylinder in a press is proportional to the square of $\sec x$, where x is the distance (in meters) that the cylinder is extended in its cycle. The domain of F is $[0, \pi/3]$, and $F(0) = 500$.

(a) Find F as a function of x.

(b) Find the average force exerted by the press over the interval $[0, \pi/3]$.

62. ***Blood Flow*** The velocity v of the flow of blood at a distance r from the central axis of an artery of radius R is

$$v = k(R^2 - r^2)$$

where k is the constant of proportionality. Find the average rate of flow of blood along a radius of the artery. (Use 0 and R as the limits of integration.)

63. ***Respiratory Cycle*** The volume V, in liters, of air in the lungs during a five-second respiratory cycle is approximated by the model $V = 0.1729t + 0.1522t^2 - 0.0374t^3$, where t is the time in seconds. Approximate the average volume of air in the lungs during one cycle.

64. ***Average Sales*** A company fits a model to the monthly sales data for a seasonal product. The model is

$$S(t) = \frac{t}{4} + 1.8 + 0.5\sin\left(\frac{\pi t}{6}\right), \quad 0 \le t \le 24$$

where S is sales (in thousands) and t is time in months.

(a) Use a graphing utility to graph $f(t) = 0.5\sin(\pi t/6)$ for $0 \le t \le 24$. Use the graph to explain why the average value of $f(t)$ is 0 over the interval.

(b) Use a graphing utility to graph $S(t)$ and the line $g(t) = t/4 + 1.8$ in the same viewing window. Use the graph and the result of part (a) to explain why g is called the *trend line*.

65. ***Modeling Data*** An experimental vehicle is tested on a straight track. It starts from rest, and its velocity v (in meters per second) is recorded every 10 seconds for 1 minute (see table).

t	0	10	20	30	40	50	60
v	0	5	21	40	62	78	83

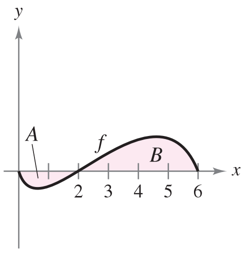
(a) Use a graphing utility to find a model of the form $v = at^3 + bt^2 + ct + d$ for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the Fundamental Theorem of Calculus to approximate the distance traveled by the vehicle during the test.

CAPSTONE

66. The graph of f is shown in the figure. The shaded region A has an area of 1.5, and $\int_0^6 f(x)\,dx = 3.5$. Use this information to fill in the blanks.



(a) $\displaystyle\int_0^2 f(x)\,dx = \boxed{}$

(b) $\displaystyle\int_2^6 f(x)\,dx = \boxed{}$

(c) $\displaystyle\int_0^6 |f(x)|\,dx = \boxed{}$

(d) $\displaystyle\int_0^2 -2f(x)\,dx = \boxed{}$

(e) $\displaystyle\int_0^6 [2 + f(x)]\,dx = \boxed{}$

(f) The average value of f over the interval $[0, 6]$ is $\boxed{}$.

In Exercises 67–72, find F as a function of x and evaluate it at $x = 2$, $x = 5$, and $x = 8$.

67. $F(x) = \displaystyle\int_0^x (4t - 7)\,dt$

68. $F(x) = \displaystyle\int_2^x (t^3 + 2t - 2)\,dt$

69. $F(x) = \int_1^x \frac{20}{v^2}\,dv$

70. $F(x) = \int_2^x -\frac{2}{t^3}\,dt$

71. $F(x) = \int_1^x \cos\theta\,d\theta$

72. $F(x) = \int_0^x \sin\theta\,d\theta$

73. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown in the figure.

(a) Estimate $g(0)$, $g(2)$, $g(4)$, $g(6)$, and $g(8)$.

(b) Find the largest open interval on which g is increasing. Find the largest open interval on which g is decreasing.

(c) Identify any extrema of g.

(d) Sketch a rough graph of g.

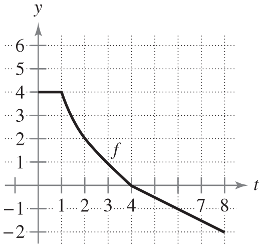


Figure for 73

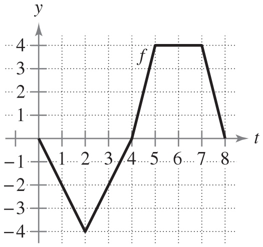


Figure for 74

74. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown in the figure.

(a) Estimate $g(0)$, $g(2)$, $g(4)$, $g(6)$, and $g(8)$.

(b) Find the largest open interval on which g is increasing. Find the largest open interval on which g is decreasing.

(c) Identify any extrema of g.

(d) Sketch a rough graph of g.

In Exercises 75–80, (a) integrate to find F as a function of x and (b) demonstrate the Second Fundamental Theorem of Calculus by differentiating the result in part (a).

75. $F(x) = \int_0^x (t+2)\,dt$

76. $F(x) = \int_0^x t(t^2+1)\,dt$

77. $F(x) = \int_8^x \sqrt[3]{t}\,dt$

78. $F(x) = \int_4^x \sqrt{t}\,dt$

79. $F(x) = \int_{\pi/4}^x \sec^2 t\,dt$

80. $F(x) = \int_{\pi/3}^x \sec t\tan t\,dt$

In Exercises 81–86, use the Second Fundamental Theorem of Calculus to find $F'(x)$.

81. $F(x) = \int_{-2}^x (t^2 - 2t)\,dt$

82. $F(x) = \int_1^x \frac{t^2}{t^2+1}\,dt$

83. $F(x) = \int_{-1}^x \sqrt{t^4+1}\,dt$

84. $F(x) = \int_1^x \sqrt[4]{t}\,dt$

85. $F(x) = \int_0^x t\cos t\,dt$

86. $F(x) = \int_0^x \sec^3 t\,dt$

In Exercises 87–92, find $F'(x)$.

87. $F(x) = \int_x^{x+2} (4t+1)\,dt$

88. $F(x) = \int_{-x}^x t^3\,dt$

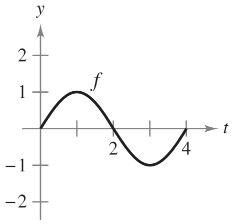
89. $F(x) = \int_0^{\sin x} \sqrt{t}\,dt$

90. $F(x) = \int_2^{x^2} \frac{1}{t^3}\,dt$

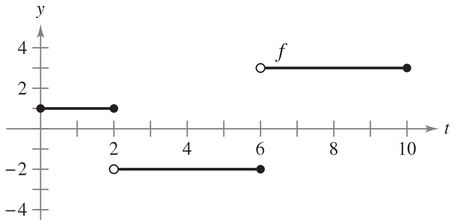
91. $F(x) = \int_0^{x^3} \sin t^2\,dt$

92. $F(x) = \int_0^{x^2} \sin\theta^2\,d\theta$

93. ***Graphical Analysis*** Sketch an approximate graph of g on the interval $0 \le x \le 4$, where $g(x) = \int_0^x f(t)\,dt$. Identify the x-coordinate of an extremum of g. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



94. Use the graph of the function f shown in the figure and the function g defined by $g(x) = \int_0^x f(t)\,dt$.



(a) Complete the table.

x	1	2	3	4	5	6	7	8	9	10
$g(x)$										

(b) Plot the points from the table in part (a) and graph g.

(c) Where does g have its minimum? Explain.

(d) Where does g have a maximum? Explain.

(e) On what interval does g increase at the greatest rate? Explain.

(f) Identify the zeros of g.

95. ***Cost*** The total cost C (in dollars) of purchasing and maintaining a piece of equipment for x years is

$$C(x) = 5000\left(25 + 3\int_0^x t^{1/4}\,dt\right).$$

(a) Perform the integration to write C as a function of x.

(b) Find $C(1)$, $C(5)$, and $C(10)$.

96. ***Area*** The area A between the graph of the function $g(t) = 4 - 4/t^2$ and the t-axis over the interval $[1, x]$ is

$$A(x) = \int_1^x \left(4 - \frac{4}{t^2}\right) dt.$$

(a) Find the horizontal asymptote of the graph of g.

(b) Integrate to find A as a function of x. Does the graph of A have a horizontal asymptote? Explain.

In Exercises 97–102, the velocity function, in feet per second, is given for a particle moving along a straight line. Find (a) the displacement and (b) the total distance that the particle travels over the given interval.

97. $v(t) = 5t - 7, \quad 0 \le t \le 3$

98. $v(t) = t^2 - t - 12, \quad 1 \le t \le 5$

99. $v(t) = t^3 - 10t^2 + 27t - 18, \quad 1 \le t \le 7$

100. $v(t) = t^3 - 8t^2 + 15t, \quad 0 \le t \le 5$

101. $v(t) = \dfrac{1}{\sqrt{t}}, \quad 1 \le t \le 4$ **102.** $v(t) = \cos t, \quad 0 \le t \le 3\pi$

103. A particle is moving along the x-axis. The position of the particle at time t is given by $x(t) = t^3 - 6t^2 + 9t - 2$, $0 \le t \le 5$. Find the total distance the particle travels in 5 units of time.

104. Repeat Exercise 103 for the position function given by $x(t) = (t - 1)(t - 3)^2$, $0 \le t \le 5$.

105. ***Water Flow*** Water flows from a storage tank at a rate of $500 - 5t$ liters per minute. Find the amount of water that flows out of the tank during the first 18 minutes.

106. ***Oil Leak*** At 1:00 P.M., oil begins leaking from a tank at a rate of $4 + 0.75t$ gallons per hour.

(a) How much oil is lost from 1:00 P.M. to 4:00 P.M.?

(b) How much oil is lost from 4:00 P.M. to 7:00 P.M.?

(c) Compare your answers from parts (a) and (b). What do you notice?

In Exercises 107–110, describe why the statement is incorrect.

107. $\xcancel{\displaystyle\int_{-1}^{1} x^{-2}\,dx = \left[-x^{-1}\right]_{-1}^{1} = (-1) - 1 = -2}$

108. $\xcancel{\displaystyle\int_{-2}^{1} \frac{2}{x^3}\,dx = \left[-\frac{1}{x^2}\right]_{-2}^{1} = -\frac{3}{4}}$

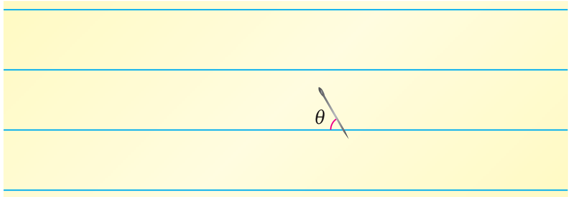
109. $\xcancel{\displaystyle\int_{\pi/4}^{3\pi/4} \sec^2 x\,dx = \left[\tan x\right]_{\pi/4}^{3\pi/4} = -2}$

110. $\xcancel{\displaystyle\int_{\pi/2}^{3\pi/2} \csc x \cot x\,dx = \left[-\csc x\right]_{\pi/2}^{3\pi/2} = 2}$

111. ***Buffon's Needle Experiment*** A horizontal plane is ruled with parallel lines 2 inches apart. A two-inch needle is tossed randomly onto the plane. The probability that the needle will touch a line is

$$P = \frac{2}{\pi}\int_0^{\pi/2} \sin\theta\,d\theta$$

where θ is the acute angle between the needle and any one of the parallel lines. Find this probability.



112. Prove that $\dfrac{d}{dx}\left[\displaystyle\int_{u(x)}^{v(x)} f(t)\,dt\right] = f(v(x))v'(x) - f(u(x))u'(x)$.

True or False? **In Exercises 113 and 114, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

113. If $F'(x) = G'(x)$ on the interval $[a, b]$, then $F(b) - F(a) = G(b) - G(a)$.

114. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

115. Show that the function

$$f(x) = \int_0^{1/x} \frac{1}{t^2 + 1}\,dt + \int_0^x \frac{1}{t^2 + 1}\,dt$$

is constant for $x > 0$.

116. Find the function $f(x)$ and all values of c such that

$$\int_c^x f(t)\,dt = x^2 + x - 2.$$

117. Let $G(x) = \displaystyle\int_0^x \left[s\int_0^s f(t)dt\right]ds$, where f is continuous for all real t. Find (a) $G(0)$, (b) $G'(0)$, (c) $G''(x)$, and (d) $G''(0)$.

SECTION PROJECT

Demonstrating the Fundamental Theorem

Use a graphing utility to graph the function $y_1 = \sin^2 t$ on the interval $0 \le t \le \pi$. Let $F(x)$ be the following function of x.

$$F(x) = \int_0^x \sin^2 t\,dt$$

(a) Complete the table. Explain why the values of F are increasing.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$F(x)$							

(b) Use the integration capabilities of a graphing utility to graph F.

(c) Use the differentiation capabilities of a graphing utility to graph $F'(x)$. How is this graph related to the graph in part (b)?

(d) Verify that the derivative of $y = (1/2)t - (\sin 2t)/4$ is $\sin^2 t$. Graph y and write a short paragraph about how this graph is related to those in parts (b) and (c).

4.5 Integration by Substitution

- **Use pattern recognition to find an indefinite integral.**
- **Use a change of variables to find an indefinite integral.**
- **Use the General Power Rule for Integration to find an indefinite integral.**
- **Use a change of variables to evaluate a definite integral.**
- **Evaluate a definite integral involving an even or odd function.**

Pattern Recognition

In this section you will study techniques for integrating composite functions. The discussion is split into two parts—*pattern recognition* and *change of variables*. Both techniques involve a ***u*-substitution.** With pattern recognition you perform the substitution mentally, and with change of variables you write the substitution steps.

The role of substitution in integration is comparable to the role of the Chain Rule in differentiation. Recall that for differentiable functions given by $y = F(u)$ and $u = g(x)$, the Chain Rule states that

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x).$$

From the definition of an antiderivative, it follows that

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

These results are summarized in the following theorem.

THEOREM 4.13 ANTIDIFFERENTIATION OF A COMPOSITE FUNCTION

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

NOTE The statement of Theorem 4.13 doesn't tell how to distinguish between $f(g(x))$ and $g'(x)$ in the integrand. As you become more experienced at integration, your skill in doing this will increase. Of course, part of the key is familiarity with derivatives.

Examples 1 and 2 show how to apply Theorem 4.13 *directly*, by recognizing the presence of $f(g(x))$ and $g'(x)$. Note that the composite function in the integrand has an *outside function* f and an *inside function* g. Moreover, the derivative $g'(x)$ is present as a factor of the integrand.

$$\int \overset{\text{Outside function}}{f}(\underset{\text{Inside function}}{g(x)})\underbrace{g'(x)}_{\text{Derivative of inside function}}\,dx = F(g(x)) + C$$

EXAMPLE 1 Recognizing the $f(g(x))g'(x)$ Pattern

Find $\int (x^2 + 1)^2(2x)\,dx$.

Solution Letting $g(x) = x^2 + 1$, you obtain

$$g'(x) = 2x$$

and

$$f(g(x)) = f(x^2 + 1) = (x^2 + 1)^2.$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Power Rule for Integration and Theorem 4.13, you can write

$$\int \overbrace{(x^2 + 1)^2}^{f(g(x))}\overbrace{(2x)}^{g'(x)}\,dx = \frac{1}{3}(x^2 + 1)^3 + C.$$

Try using the Chain Rule to check that the derivative of $\frac{1}{3}(x^2 + 1)^3 + C$ is the integrand of the original integral.

EXAMPLE 2 Recognizing the $f(g(x))g'(x)$ Pattern

Find $\int 5 \cos 5x\,dx$.

Solution Letting $g(x) = 5x$, you obtain

$$g'(x) = 5$$

and

$$f(g(x)) = f(5x) = \cos 5x.$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Cosine Rule for Integration and Theorem 4.13, you can write

$$\int \overbrace{(\cos (5x))}^{f(g(x))}\overbrace{(5)}^{g'(x)}\,dx = \sin 5x + C.$$

You can check this by differentiating $\sin 5x + C$ to obtain the original integrand. ■

TECHNOLOGY Try using a computer algebra system, such as *Maple*, *Mathematica*, or the *TI-89*, to solve the integrals given in Examples 1 and 2. Do you obtain the same antiderivatives that are listed in the examples?

EXPLORATION

Recognizing Patterns The integrand in each of the following integrals fits the pattern $f(g(x))g'(x)$. Identify the pattern and use the result to evaluate the integral.

a. $\int 2x(x^2 + 1)^4\,dx$ **b.** $\int 3x^2\sqrt{x^3 + 1}\,dx$ **c.** $\int \sec^2 x(\tan x + 3)\,dx$

The next three integrals are similar to the first three. Show how you can multiply and divide by a constant to evaluate these integrals.

d. $\int x(x^2 + 1)^4\,dx$ **e.** $\int x^2\sqrt{x^3 + 1}\,dx$ **f.** $\int 2\sec^2 x(\tan x + 3)\,dx$

The integrands in Examples 1 and 2 fit the $f(g(x))g'(x)$ pattern exactly—you only had to recognize the pattern. You can extend this technique considerably with the Constant Multiple Rule

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

Many integrands contain the essential part (the variable part) of $g'(x)$ but are missing a constant multiple. In such cases, you can multiply and divide by the necessary constant multiple, as shown in Example 3.

EXAMPLE 3 Multiplying and Dividing by a Constant

Find $\int x(x^2+1)^2\,dx$.

Solution This is similar to the integral given in Example 1, except that the integrand is missing a factor of 2. Recognizing that $2x$ is the derivative of x^2+1, you can let $g(x) = x^2+1$ and supply the $2x$ as follows.

$$\begin{aligned}
\int x(x^2+1)^2\,dx &= \int (x^2+1)^2\left(\frac{1}{2}\right)(2x)\,dx && \text{Multiply and divide by 2.}\\
&= \frac{1}{2}\int \overbrace{(x^2+1)^2}^{f(g(x))}\,\overbrace{(2x)}^{g'(x)}\,dx && \text{Constant Multiple Rule}\\
&= \frac{1}{2}\left[\frac{(x^2+1)^3}{3}\right] + C && \text{Integrate.}\\
&= \frac{1}{6}(x^2+1)^3 + C && \text{Simplify.}
\end{aligned}$$

■

In practice, most people would not write as many steps as are shown in Example 3. For instance, you could evaluate the integral by simply writing

$$\begin{aligned}
\int x(x^2+1)^2\,dx &= \frac{1}{2}\int (x^2+1)^2\,2x\,dx\\
&= \frac{1}{2}\left[\frac{(x^2+1)^3}{3}\right] + C\\
&= \frac{1}{6}(x^2+1)^3 + C.
\end{aligned}$$

NOTE Be sure you see that the *Constant* Multiple Rule applies only to *constants*. You cannot multiply and divide by a variable and then move the variable outside the integral sign. For instance,

$$\int (x^2+1)^2\,dx \neq \frac{1}{2x}\int (x^2+1)^2(2x)\,dx.$$

After all, if it were legitimate to move variable quantities outside the integral sign, you could move the entire integrand out and simplify the whole process. But the result would be incorrect.

■

Change of Variables

With a formal **change of variables,** you completely rewrite the integral in terms of u and du (or any other convenient variable). Although this procedure can involve more written steps than the pattern recognition illustrated in Examples 1 to 3, it is useful for complicated integrands. The change of variables technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $du = g'(x)\,dx$, and the integral in Theorem 4.13 takes the form

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C.$$

EXAMPLE 4 Change of Variables

Find $\int \sqrt{2x-1}\,dx$.

Solution First, let u be the inner function, $u = 2x - 1$. Then calculate the differential du to be $du = 2\,dx$. Now, using $\sqrt{2x-1} = \sqrt{u}$ and $dx = du/2$, substitute to obtain

$$\begin{aligned}
\int \sqrt{2x-1}\,dx &= \int \sqrt{u}\left(\frac{du}{2}\right) && \text{Integral in terms of } u\\
&= \frac{1}{2}\int u^{1/2}\,du && \text{Constant Multiple Rule}\\
&= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C && \text{Antiderivative in terms of } u\\
&= \frac{1}{3}u^{3/2} + C && \text{Simplify.}\\
&= \frac{1}{3}(2x-1)^{3/2} + C. && \text{Antiderivative in terms of } x
\end{aligned}$$

STUDY TIP Because integration is usually more difficult than differentiation, you should always check your answer to an integration problem by differentiating. For instance, in Example 4 you should differentiate $\frac{1}{3}(2x-1)^{3/2} + C$ to verify that you obtain the original integrand.

EXAMPLE 5 Change of Variables

Find $\int x\sqrt{2x-1}\,dx$.

Solution As in the previous example, let $u = 2x - 1$ and obtain $dx = du/2$. Because the integrand contains a factor of x, you must also solve for x in terms of u, as shown.

$$u = 2x - 1 \quad \Rightarrow \quad x = (u+1)/2 \qquad \text{Solve for } x \text{ in terms of } u.$$

Now, using substitution, you obtain

$$\begin{aligned}
\int x\sqrt{2x-1}\,dx &= \int \left(\frac{u+1}{2}\right) u^{1/2}\left(\frac{du}{2}\right)\\
&= \frac{1}{4}\int \left(u^{3/2} + u^{1/2}\right) du\\
&= \frac{1}{4}\left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C\\
&= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C.
\end{aligned}$$

■

To complete the change of variables in Example 5, you solved for x in terms of u. Sometimes this is very difficult. Fortunately it is not always necessary, as shown in the next example.

EXAMPLE 6 Change of Variables

Find $\int \sin^2 3x \cos 3x \, dx$.

Solution Because $\sin^2 3x = (\sin 3x)^2$, you can let $u = \sin 3x$. Then

$$du = (\cos 3x)(3)\, dx.$$

Now, because $\cos 3x \, dx$ is part of the original integral, you can write

$$\frac{du}{3} = \cos 3x \, dx.$$

Substituting u and $du/3$ in the original integral yields

$$\begin{aligned}\int \sin^2 3x \cos 3x \, dx &= \int u^2 \frac{du}{3} \\ &= \frac{1}{3}\int u^2 \, du \\ &= \frac{1}{3}\left(\frac{u^3}{3}\right) + C \\ &= \frac{1}{9}\sin^3 3x + C.\end{aligned}$$

You can check this by differentiating.

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{9}\sin^3 3x\right] &= \left(\frac{1}{9}\right)(3)(\sin 3x)^2(\cos 3x)(3) \\ &= \sin^2 3x \cos 3x\end{aligned}$$

Because differentiation produces the original integrand, you know that you have obtained the correct antiderivative. ■

STUDY TIP When making a change of variables, be sure that your answer is written using the same variables as in the original integrand. For instance, in Example 6, you should not leave your answer as

$$\tfrac{1}{9}u^3 + C$$

but rather, replace u by $\sin 3x$.

The steps used for integration by substitution are summarized in the following guidelines.

GUIDELINES FOR MAKING A CHANGE OF VARIABLES

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\, dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

The General Power Rule for Integration

One of the most common u-substitutions involves quantities in the integrand that are raised to a power. Because of the importance of this type of substitution, it is given a special name—the **General Power Rule for Integration.** A proof of this rule follows directly from the (simple) Power Rule for Integration, together with Theorem 4.13.

THEOREM 4.14 THE GENERAL POWER RULE FOR INTEGRATION

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

EXAMPLE 7 Substitution and the General Power Rule

a. $$\int 3(3x-1)^4\,dx = \int \overbrace{(3x-1)^4}^{u^4}\overbrace{(3)\,dx}^{du} = \overbrace{\frac{(3x-1)^5}{5}}^{u^5/5} + C$$

b. $$\int (2x+1)(x^2+x)\,dx = \int \overbrace{(x^2+x)^1}^{u^1}\overbrace{(2x+1)\,dx}^{du} = \overbrace{\frac{(x^2+x)^2}{2}}^{u^2/2} + C$$

c. $$\int 3x^2\sqrt{x^3-2}\,dx = \int \overbrace{(x^3-2)^{1/2}}^{u^{1/2}}\overbrace{(3x^2)\,dx}^{du} = \overbrace{\frac{(x^3-2)^{3/2}}{3/2}}^{u^{3/2}/(3/2)} + C = \frac{2}{3}(x^3-2)^{3/2} + C$$

d. $$\int \frac{-4x}{(1-2x^2)^2}\,dx = \int \overbrace{(1-2x^2)^{-2}}^{u^{-2}}\overbrace{(-4x)\,dx}^{du} = \overbrace{\frac{(1-2x^2)^{-1}}{-1}}^{u^{-1}/(-1)} + C = -\frac{1}{1-2x^2} + C$$

e. $$\int \cos^2 x \sin x\,dx = -\int \overbrace{(\cos x)^2}^{u^2}\overbrace{(-\sin x)\,dx}^{du} = -\overbrace{\frac{(\cos x)^3}{3}}^{u^3/3} + C \qquad \blacksquare$$

EXPLORATION

Suppose you were asked to find one of the following integrals. Which one would you choose? Explain your reasoning.

a. $\int \sqrt{x^3+1}\,dx$ or

$\int x^2\sqrt{x^3+1}\,dx$

b. $\int \tan(3x)\sec^2(3x)\,dx$ or

$\int \tan(3x)\,dx$

Some integrals whose integrands involve quantities raised to powers cannot be found by the General Power Rule. Consider the two integrals

$$\int x(x^2+1)^2\,dx \quad \text{and} \quad \int (x^2+1)^2\,dx.$$

The substitution $u = x^2 + 1$ works in the first integral but not in the second. In the second, the substitution fails because the integrand lacks the factor x needed for du. Fortunately, *for this particular integral*, you can expand the integrand as $(x^2+1)^2 = x^4 + 2x^2 + 1$ and use the (simple) Power Rule to integrate each term.

Change of Variables for Definite Integrals

When using u-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable u rather than to convert the antiderivative back to the variable x and evaluate at the original limits. This change of variables is stated explicitly in the next theorem. The proof follows from Theorem 4.13 combined with the Fundamental Theorem of Calculus.

THEOREM 4.15 CHANGE OF VARIABLES FOR DEFINITE INTEGRALS

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

EXAMPLE 8 Change of Variables

Evaluate $\displaystyle\int_0^1 x(x^2+1)^3\,dx$.

Solution To evaluate this integral, let $u = x^2 + 1$. Then, you obtain

$$u = x^2 + 1 \implies du = 2x\,dx.$$

Before substituting, determine the new upper and lower limits of integration.

Lower Limit	*Upper Limit*
When $x = 0$, $u = 0^2 + 1 = 1$.	When $x = 1$, $u = 1^2 + 1 = 2$.

Now, you can substitute to obtain

$$\begin{aligned}\int_0^1 x(x^2+1)^3\,dx &= \frac{1}{2}\int_0^1 (x^2+1)^3(2x)\,dx && \text{Integration limits for } x\\ &= \frac{1}{2}\int_1^2 u^3\,du && \text{Integration limits for } u\\ &= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2\\ &= \frac{1}{2}\left(4-\frac{1}{4}\right)\\ &= \frac{15}{8}.\end{aligned}$$

Try rewriting the antiderivative $\frac{1}{2}(u^4/4)$ in terms of the variable x and evaluate the definite integral at the original limits of integration, as shown.

$$\begin{aligned}\frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 &= \frac{1}{2}\left[\frac{(x^2+1)^4}{4}\right]_0^1\\ &= \frac{1}{2}\left(4-\frac{1}{4}\right) = \frac{15}{8}\end{aligned}$$

Notice that you obtain the same result. ■

EXAMPLE 9 Change of Variables

Evaluate $A = \displaystyle\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$.

Solution To evaluate this integral, let $u = \sqrt{2x-1}$. Then, you obtain

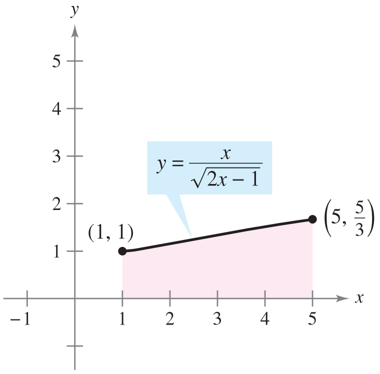
$$\begin{aligned} u^2 &= 2x - 1 \\ u^2 + 1 &= 2x \\ \frac{u^2+1}{2} &= x \\ u\,du &= dx. && \text{Differentiate each side.} \end{aligned}$$

Before substituting, determine the new upper and lower limits of integration.

Lower Limit	*Upper Limit*
When $x = 1$, $u = \sqrt{2-1} = 1$.	When $x = 5$, $u = \sqrt{10-1} = 3$.

Now, substitute to obtain

$$\begin{aligned} \int_1^5 \frac{x}{\sqrt{2x-1}}\,dx &= \int_1^3 \frac{1}{u}\left(\frac{u^2+1}{2}\right)u\,du \\ &= \frac{1}{2}\int_1^3 (u^2+1)\,du \\ &= \frac{1}{2}\left[\frac{u^3}{3} + u\right]_1^3 \\ &= \frac{1}{2}\left(9 + 3 - \frac{1}{3} - 1\right) \\ &= \frac{16}{3}. \end{aligned}$$

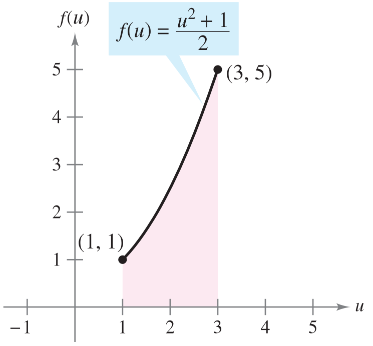


The region before substitution has an area of $\frac{16}{3}$.
Figure 4.38

Geometrically, you can interpret the equation

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^3 \frac{u^2+1}{2}\,du$$

to mean that the two *different* regions shown in Figures 4.38 and 4.39 have the *same* area.



The region after substitution has an area of $\frac{16}{3}$.
Figure 4.39

When evaluating definite integrals by substitution, it is possible for the upper limit of integration of the u-variable form to be smaller than the lower limit. If this happens, don't rearrange the limits. Simply evaluate as usual. For example, after substituting $u = \sqrt{1-x}$ in the integral

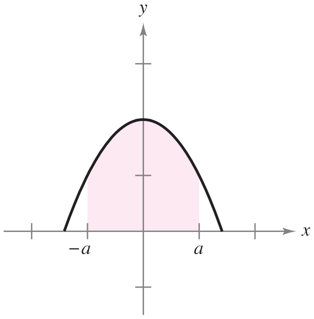
$$\int_0^1 x^2(1-x)^{1/2}\,dx$$

you obtain $u = \sqrt{1-1} = 0$ when $x = 1$, and $u = \sqrt{1-0} = 1$ when $x = 0$. So, the correct u-variable form of this integral is

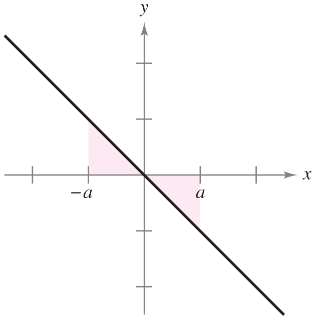
$$-2\int_1^0 (1-u^2)^2u^2\,du.$$

Integration of Even and Odd Functions

Even with a change of variables, integration can be difficult. Occasionally, you can simplify the evaluation of a definite integral over an interval that is symmetric about the y-axis or about the origin by recognizing the integrand to be an even or odd function (see Figure 4.40).



Even function



Odd function

Figure 4.40

THEOREM 4.16 INTEGRATION OF EVEN AND ODD FUNCTIONS

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$
2. If f is an *odd* function, then $\int_{-a}^{a} f(x)\,dx = 0.$

PROOF Because f is even, you know that $f(x) = f(-x)$. Using Theorem 4.13 with the substitution $u = -x$ produces

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)(-du) = -\int_{a}^{0} f(u)\,du = \int_{0}^{a} f(u)\,du = \int_{0}^{a} f(x)\,dx.$$

Finally, using Theorem 4.6, you obtain

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx\\ &= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.\end{aligned}$$

This proves the first property. The proof of the second property is left to you (see Exercise 137). ■

EXAMPLE 10 Integration of an Odd Function

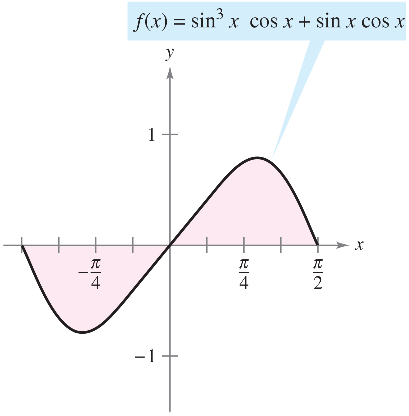
Evaluate $\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,dx.$

Solution Letting $f(x) = \sin^3 x \cos x + \sin x \cos x$ produces

$$\begin{aligned}f(-x) &= \sin^3(-x)\cos(-x) + \sin(-x)\cos(-x)\\ &= -\sin^3 x \cos x - \sin x \cos x = -f(x).\end{aligned}$$

So, f is an odd function, and because f is symmetric about the origin over $[-\pi/2, \pi/2]$, you can apply Theorem 4.16 to conclude that

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,dx = 0.$$ ■



Because f is an odd function, $\int_{-\pi/2}^{\pi/2} f(x)\,dx = 0.$

Figure 4.41

NOTE From Figure 4.41 you can see that the two regions on either side of the y-axis have the same area. However, because one lies below the x-axis and one lies above it, integration produces a cancellation effect. (More will be said about this in Section 7.1.) ■

4.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, complete the table by identifying *u* and *du* for the integral.

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
1. $\int (8x^2+1)^2(16x)\,dx$		
2. $\int x^2\sqrt{x^3+1}\,dx$		
3. $\int \frac{x}{\sqrt{x^2+1}}\,dx$		
4. $\int \sec 2x \tan 2x\,dx$		
5. $\int \tan^2 x \sec^2 x\,dx$		
6. $\int \frac{\cos x}{\sin^2 x}\,dx$		

In Exercises 7–10, determine whether it is necessary to use substitution to evaluate the integral. (Do not evaluate the integral.)

7. $\int \sqrt{x}(6-x)\,dx$

8. $\int x\sqrt{x+4}\,dx$

9. $\int x\sqrt[3]{1+x^2}\,dx$

10. $\int x\cos x^2\,dx$

In Exercises 11–38, find the indefinite integral and check the result by differentiation.

11. $\int (1+6x)^4(6)\,dx$

12. $\int (x^2-9)^3(2x)\,dx$

13. $\int \sqrt{25-x^2}\,(-2x)\,dx$

14. $\int \sqrt[3]{3-4x^2}(-8x)\,dx$

15. $\int x^3(x^4+3)^2\,dx$

16. $\int x^2(x^3+5)^4\,dx$

17. $\int x^2(x^3-1)^4\,dx$

18. $\int x(5x^2+4)^3\,dx$

19. $\int t\sqrt{t^2+2}\,dt$

20. $\int t^3\sqrt{t^4+5}\,dt$

21. $\int 5x\sqrt[3]{1-x^2}\,dx$

22. $\int u^2\sqrt{u^3+2}\,du$

23. $\int \frac{x}{(1-x^2)^3}\,dx$

24. $\int \frac{x^3}{(1+x^4)^2}\,dx$

25. $\int \frac{x^2}{(1+x^3)^2}\,dx$

26. $\int \frac{x^2}{(16-x^3)^2}\,dx$

27. $\int \frac{x}{\sqrt{1-x^2}}\,dx$

28. $\int \frac{x^3}{\sqrt{1+x^4}}\,dx$

29. $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt$

30 $\int \left[x^2+\frac{1}{(3x)^2}\right]dx$

31. $\int \frac{1}{\sqrt{2x}}\,dx$

32. $\int \frac{1}{2\sqrt{x}}\,dx$

33. $\int \frac{x^2+5x-8}{\sqrt{x}}\,dx$

34. $\int \frac{t-9t^2}{\sqrt{t}}\,dt$

35. $\int t^2\left(t-\frac{8}{t}\right)dt$

36. $\int \left(\frac{t^3}{3}+\frac{1}{4t^2}\right)dt$

37. $\int (9-y)\sqrt{y}\,dy$

38. $\int 4\pi y(6+y^{3/2})\,dy$

In Exercises 39–42, solve the differential equation.

39. $\frac{dy}{dx} = 4x + \frac{4x}{\sqrt{16-x^2}}$

40. $\frac{dy}{dx} = \frac{10x^2}{\sqrt{1+x^3}}$

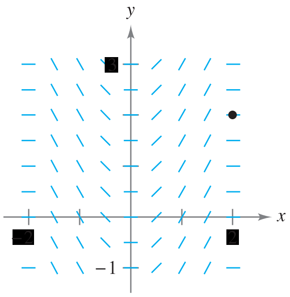
41. $\frac{dy}{dx} = \frac{x+1}{(x^2+2x-3)^2}$

42. $\frac{dy}{dx} = \frac{x-4}{\sqrt{x^2-8x+1}}$

Slope Fields **In Exercises 43–46, a differential equation, a point, and a slope field are given. A *slope field* consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the directions of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).**

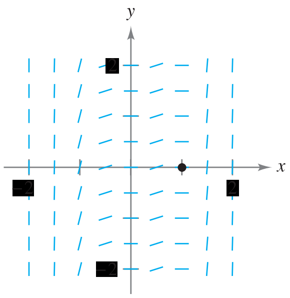
43. $\frac{dy}{dx} = x\sqrt{4-x^2}$

$(2, 2)$



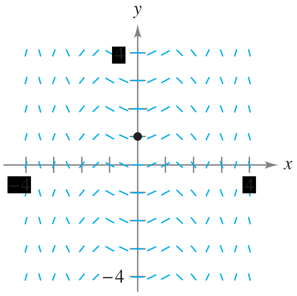
44. $\frac{dy}{dx} = x^2(x^3-1)^2$

$(1, 0)$



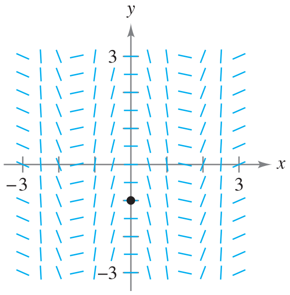
45. $\frac{dy}{dx} = x\cos x^2$

$(0, 1)$



46. $\frac{dy}{dx} = -2\sec(2x)\tan(2x)$

$(0, -1)$



In Exercises 47–60, find the indefinite integral.

47. $\int \pi \sin \pi x\, dx$ **48.** $\int 4x^3 \sin x^4\, dx$

49. $\int \sin 4x\, dx$ **50.** $\int \cos 8x\, dx$

51. $\int \frac{1}{\theta^2} \cos \frac{1}{\theta}\, d\theta$ **52.** $\int x \sin x^2\, dx$

53. $\int \sin 2x \cos 2x\, dx$

54. $\int \sec(1 - x) \tan(1 - x)\, dx$

55. $\int \tan^4 x \sec^2 x\, dx$ **56.** $\int \sqrt{\tan x} \sec^2 x\, dx$

57. $\int \frac{\csc^2 x}{\cot^3 x}\, dx$ **58.** $\int \frac{\sin x}{\cos^3 x}\, dx$

59. $\int \cot^2 x\, dx$ **60.** $\int \csc^2\left(\frac{x}{2}\right) dx$

In Exercises 61–66, find an equation for the function *f* that has the given derivative and whose graph passes through the given point.

	Derivative	*Point*
61.	$f'(x) = -\sin \frac{x}{2}$	$(0, 6)$
62.	$f'(x) = \pi \sec \pi x \tan \pi x$	$\left(\frac{1}{3}, 1\right)$
63.	$f'(x) = 2 \sin 4x$	$\left(\frac{\pi}{4}, -\frac{1}{2}\right)$
64.	$f'(x) = \sec^2(2x)$	$\left(\frac{\pi}{2}, 2\right)$
65.	$f'(x) = 2x(4x^2 - 10)^2$	$(2, 10)$
66.	$f'(x) = -2x\sqrt{8 - x^2}$	$(2, 7)$

In Exercises 67–74, find the indefinite integral by the method shown in Example 5.

67. $\int x\sqrt{x + 6}\, dx$ **68.** $\int x\sqrt{4x + 1}\, dx$

69. $\int x^2\sqrt{1 - x}\, dx$ **70.** $\int (x + 1)\sqrt{2 - x}\, dx$

71. $\int \frac{x^2 - 1}{\sqrt{2x - 1}}\, dx$ **72.** $\int \frac{2x + 1}{\sqrt{x + 4}}\, dx$

73. $\int \frac{-x}{(x + 1) - \sqrt{x + 1}}\, dx$ **74.** $\int t\sqrt[3]{t + 10}\, dt$

In Exercises 75–86, evaluate the definite integral. Use a graphing utility to verify your result.

75. $\int_{-1}^{1} x(x^2 + 1)^3\, dx$ **76.** $\int_{-2}^{4} x^2(x^3 + 8)^2\, dx$

77. $\int_{1}^{2} 2x^2\sqrt{x^3 + 1}\, dx$ **78.** $\int_{0}^{1} x\sqrt{1 - x^2}\, dx$

79. $\int_{0}^{4} \frac{1}{\sqrt{2x + 1}}\, dx$ **80.** $\int_{0}^{2} \frac{x}{\sqrt{1 + 2x^2}}\, dx$

81. $\int_{1}^{9} \frac{1}{\sqrt{x}\left(1 + \sqrt{x}\right)^2}\, dx$ **82.** $\int_{0}^{2} x\sqrt[3]{4 + x^2}\, dx$

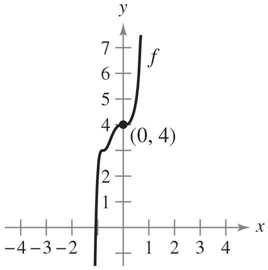
83. $\int_{1}^{2} (x - 1)\sqrt{2 - x}\, dx$ **84.** $\int_{1}^{5} \frac{x}{\sqrt{2x - 1}}\, dx$

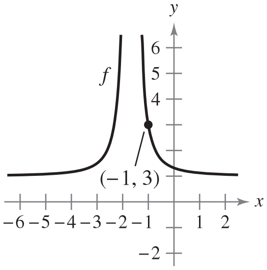
85. $\int_{0}^{\pi/2} \cos\left(\frac{2x}{3}\right) dx$

86. $\int_{\pi/3}^{\pi/2} (x + \cos x)\, dx$

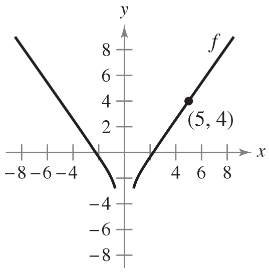
Differential Equations **In Exercises 87–90, the graph of a function *f* is shown. Use the differential equation and the given point to find an equation of the function.**

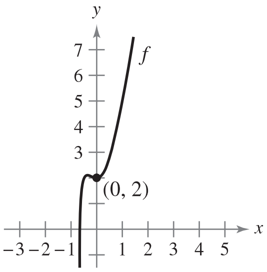
87. $\frac{dy}{dx} = 18x^2(2x^3 + 1)^2$ **88.** $\frac{dy}{dx} = \frac{-48}{(3x + 5)^3}$





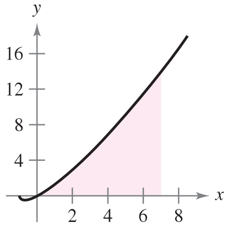
89. $\frac{dy}{dx} = \frac{2x}{\sqrt{2x^2 - 1}}$ **90.** $\frac{dy}{dx} = 4x + \frac{9x^2}{(3x^3 + 1)^{(3/2)}}$

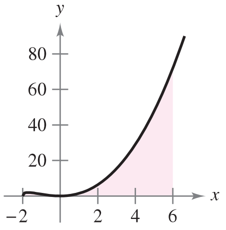




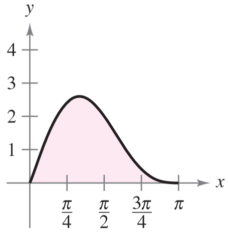
In Exercises 91–96, find the area of the region. Use a graphing utility to verify your result.

91. $\int_{0}^{7} x\sqrt[3]{x + 1}\, dx$ **92.** $\int_{-2}^{6} x^2\sqrt[3]{x + 2}\, dx$

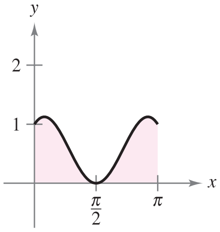




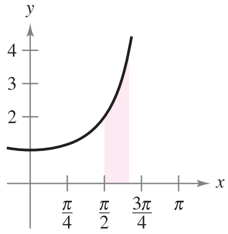
93. $y = 2 \sin x + \sin 2x$



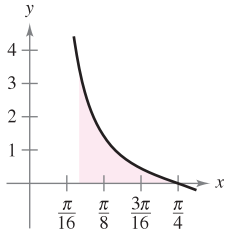
94. $y = \sin x + \cos 2x$



95. $\displaystyle\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) dx$



96. $\displaystyle\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x \, dx$



In Exercises 97–102, use a graphing utility to evaluate the integral. Graph the region whose area is given by the definite integral.

97. $\displaystyle\int_0^6 \frac{x}{\sqrt{4x+1}} \, dx$

98. $\displaystyle\int_0^2 x^3\sqrt{2x+3} \, dx$

99. $\displaystyle\int_3^7 x\sqrt{x-3} \, dx$

100. $\displaystyle\int_1^5 x^2\sqrt{x-1} \, dx$

101. $\displaystyle\int_1^4 \left(\theta + \sin\frac{\theta}{4}\right) d\theta$

102. $\displaystyle\int_0^{\pi/6} \cos 3x \, dx$

In Exercises 103–106, evaluate the integral using the properties of even and odd functions as an aid.

103. $\displaystyle\int_{-2}^{2} x^2(x^2+1) \, dx$

104. $\displaystyle\int_{-2}^{2} x(x^2+1)^3 \, dx$

105. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x \, dx$

106. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx$

107. Use $\int_0^4 x^2 \, dx = \frac{64}{3}$ to evaluate each definite integral without using the Fundamental Theorem of Calculus.

(a) $\displaystyle\int_{-4}^{0} x^2 \, dx$

(b) $\displaystyle\int_{-4}^{4} x^2 \, dx$

(c) $\displaystyle\int_{0}^{4} -x^2 \, dx$

(d) $\displaystyle\int_{-4}^{0} 3x^2 \, dx$

108. Use the symmetry of the graphs of the sine and cosine functions as an aid in evaluating each definite integral.

(a) $\displaystyle\int_{-\pi/4}^{\pi/4} \sin x \, dx$

(b) $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x \, dx$

(c) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \, dx$

(d) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx$

In Exercises 109 and 110, write the integral as the sum of the integral of an odd function and the integral of an even function. Use this simplification to evaluate the integral.

109. $\displaystyle\int_{-3}^{3} (x^3 + 4x^2 - 3x - 6) \, dx$

110. $\displaystyle\int_{-\pi/2}^{\pi/2} (\sin 4x + \cos 4x) \, dx$

WRITING ABOUT CONCEPTS

111. Describe why

$$\int x(5-x^2)^3 \, dx \neq \int u^3 \, du$$

where $u = 5 - x^2$.

112. Without integrating, explain why $\displaystyle\int_{-2}^{2} x(x^2+1)^2 \, dx = 0$.

113. If f is continuous and $\displaystyle\int_0^8 f(x) \, dx = 32$, find $\displaystyle\int_0^4 f(2x) \, dx$.

CAPSTONE

114. ***Writing*** Find the indefinite integral in two ways. Explain any difference in the forms of the answers.

(a) $\displaystyle\int (2x-1)^2 \, dx$

(b) $\displaystyle\int \sin x \cos x \, dx$

(c) $\displaystyle\int \tan x \sec^2 x \, dx$

115. ***Cash Flow*** The rate of disbursement dQ/dt of a 2 million dollar federal grant is proportional to the square of $100 - t$. Time t is measured in days $(0 \le t \le 100)$, and Q is the amount that remains to be disbursed. Find the amount that remains to be disbursed after 50 days. Assume that all the money will be disbursed in 100 days.

116. ***Depreciation*** The rate of depreciation dV/dt of a machine is inversely proportional to the square of $t + 1$, where V is the value of the machine t years after it was purchased. The initial value of the machine was $500,000, and its value decreased $100,000 in the first year. Estimate its value after 4 years.

117. ***Precipitation*** The normal monthly precipitation at the Seattle-Tacoma airport can be approximated by the model

$$R = 2.876 + 2.202 \sin(0.576t + 0.847)$$

where R is measured in inches and t is the time in months, with $t = 0$ corresponding to January 1. (*Source: U.S. National Oceanic and Atmospheric Administration*)

(a) Determine the extrema of the function over a one-year period.

(b) Use integration to approximate the normal annual precipitation. (*Hint:* Integrate over the interval $[0, 12]$.)

(c) Approximate the average monthly precipitation during the months of October, November, and December.

118. ***Sales*** The sales S (in thousands of units) of a seasonal product are given by the model

$$S = 74.50 + 43.75 \sin \frac{\pi t}{6}$$

where t is the time in months, with $t = 1$ corresponding to January. Find the average sales for each time period.

(a) The first quarter $(0 \le t \le 3)$

(b) The second quarter $(3 \le t \le 6)$

(c) The entire year $(0 \le t \le 12)$

119. ***Water Supply*** A model for the flow rate of water at a pumping station on a given day is

$$R(t) = 53 + 7 \sin\left(\frac{\pi t}{6} + 3.6\right) + 9 \cos\left(\frac{\pi t}{12} + 8.9\right)$$

where $0 \le t \le 24$. R is the flow rate in thousands of gallons per hour, and t is the time in hours.

(a) Use a graphing utility to graph the rate function and approximate the maximum flow rate at the pumping station.

(b) Approximate the total volume of water pumped in 1 day.

120. ***Electricity*** The oscillating current in an electrical circuit is

$$I = 2 \sin(60\pi t) + \cos(120\pi t)$$

where I is measured in amperes and t is measured in seconds. Find the average current for each time interval.

(a) $0 \le t \le \frac{1}{60}$

(b) $0 \le t \le \frac{1}{240}$

(c) $0 \le t \le \frac{1}{30}$

Probability **In Exercises 121 and 122, the function**

$$f(x) = kx^n(1 - x)^m, \quad 0 \le x \le 1$$

where $n > 0$, $m > 0$, and k is a constant, can be used to represent various probability distributions. If k is chosen such that

$$\int_0^1 f(x)\,dx = 1$$

the probability that x will fall between a and b $(0 \le a \le b \le 1)$ is

$$P_{a,b} = \int_a^b f(x)\,dx.$$

121. The probability that a person will remember between $100a\%$ and $100b\%$ of material learned in an experiment is

$$P_{a,b} = \int_a^b \frac{15}{4} x\sqrt{1 - x}\,dx$$

where x represents the proportion remembered. (See figure.)

(a) For a randomly chosen individual, what is the probability that he or she will recall between 50% and 75% of the material?

(b) What is the median percent recall? That is, for what value of b is it true that the probability of recalling 0 to b is 0.5?

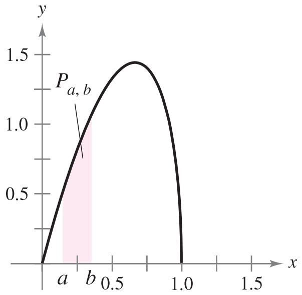


Figure for 121

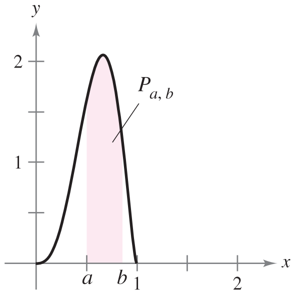
122. The probability that ore samples taken from a region contain between $100a\%$ and $100b\%$ iron is

$$P_{a,b} = \int_a^b \frac{1155}{32} x^3(1 - x)^{3/2}\,dx$$

where x represents the proportion of iron. (See figure.) What is the probability that a sample will contain between

(a) 0% and 25% iron?

(b) 50% and 100% iron?



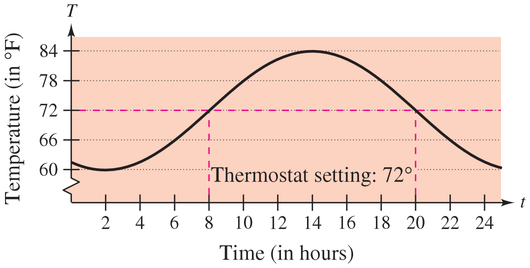
123. ***Temperature*** The temperature in degrees Fahrenheit in a house is

$$T = 72 + 12 \sin\left[\frac{\pi(t - 8)}{12}\right]$$

where t is time in hours, with $t = 0$ representing midnight. The hourly cost of cooling a house is \$0.10 per degree.

(a) Find the cost C of cooling the house if its thermostat is set at 72°F by evaluating the integral

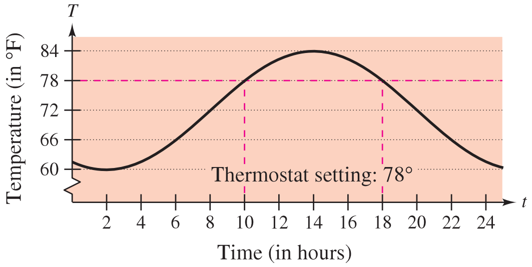
$$C = 0.1\int_8^{20} \left[72 + 12 \sin \frac{\pi(t - 8)}{12} - 72\right] dt. \text{ (See figure.)}$$



(b) Find the savings from resetting the thermostat to 78°F by evaluating the integral

$$C = 0.1\int_{10}^{18}\left[72 + 12\sin\frac{\pi(t-8)}{12} - 78\right]dt.$$

(See figure.)



124. ***Manufacturing*** A manufacturer of fertilizer finds that national sales of fertilizer follow the seasonal pattern

$$F = 100{,}000\left[1 + \sin\frac{2\pi(t-60)}{365}\right]$$

where F is measured in pounds and t represents the time in days, with $t = 1$ corresponding to January 1. The manufacturer wants to set up a schedule to produce a uniform amount of fertilizer each day. What should this amount be?

125. ***Graphical Analysis*** Consider the functions f and g, where

$$f(x) = 6\sin x\cos^2 x \quad \text{and} \quad g(t) = \int_0^t f(x)\,dx.$$

(a) Use a graphing utility to graph f and g in the same viewing window.

(b) Explain why g is nonnegative.

(c) Identify the points on the graph of g that correspond to the extrema of f.

(d) Does each of the zeros of f correspond to an extremum of g? Explain.

(e) Consider the function

$$h(t) = \int_{\pi/2}^t f(x)\,dx.$$

Use a graphing utility to graph h. What is the relationship between g and h? Verify your conjecture.

126. Find $\displaystyle\lim_{n\to+\infty}\sum_{i=1}^{n}\frac{\sin(i\pi/n)}{n}$ by evaluating an appropriate definite integral over the interval $[0, 1]$.

127. (a) Show that $\int_0^1 x^2(1-x)^5\,dx = \int_0^1 x^5(1-x)^2\,dx$.

(b) Show that $\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$.

128. (a) Show that $\int_0^{\pi/2}\sin^2 x\,dx = \int_0^{\pi/2}\cos^2 x\,dx$.

(b) Show that $\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx$, where n is a positive integer.

True or False? **In Exercises 129–134, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

129. $\displaystyle\int (2x+1)^2\,dx = \tfrac{1}{3}(2x+1)^3 + C$

130. $\displaystyle\int x(x^2+1)\,dx = \tfrac{1}{2}x^2\left(\tfrac{1}{3}x^3 + x\right) + C$

131. $\displaystyle\int_{-10}^{10}(ax^3 + bx^2 + cx + d)\,dx = 2\int_0^{10}(bx^2 + d)\,dx$

132. $\displaystyle\int_a^b \sin x\,dx = \int_a^{b+2\pi}\sin x\,dx$

133. $\displaystyle 4\int \sin x\cos x\,dx = -\cos 2x + C$

134. $\displaystyle\int \sin^2 2x\cos 2x\,dx = \tfrac{1}{3}\sin^3 2x + C$

135. Assume that f is continuous everywhere and that c is a constant. Show that

$$\int_{ca}^{cb} f(x)\,dx = c\int_a^b f(cx)\,dx.$$

136. (a) Verify that $\sin u - u\cos u + C = \int u\sin u\,du$.

(b) Use part (a) to show that $\int_0^{\pi^2}\sin\sqrt{x}\,dx = 2\pi$.

137. Complete the proof of Theorem 4.16.

138. Show that if f is continuous on the entire real number line, then

$$\int_a^b f(x+h)\,dx = \int_{a+h}^{b+h} f(x)\,dx.$$

PUTNAM EXAM CHALLENGE

139. If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$$

show that the equation

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

has at least one real zero.

140. Find all the continuous positive functions $f(x)$, for $0 \le x \le 1$, such that

$$\int_0^1 f(x)\,dx = 1$$

$$\int_0^1 f(x)x\,dx = \alpha$$

$$\int_0^1 f(x)x^2\,dx = \alpha^2$$

where α is a real number.

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4.6 Numerical Integration

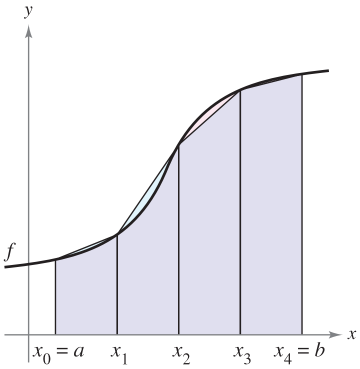
- **Approximate a definite integral using the Trapezoidal Rule.**
- **Approximate a definite integral using Simpson's Rule.**
- **Analyze the approximate errors in the Trapezoidal Rule and Simpson's Rule.**

The Trapezoidal Rule

Some elementary functions simply do not have antiderivatives that are elementary functions. For example, there is no elementary function that has any of the following functions as its derivative.

$$\sqrt[3]{x}\sqrt{1-x}, \qquad \sqrt{x}\cos x, \qquad \frac{\cos x}{x}, \qquad \sqrt{1-x^3}, \qquad \sin x^2$$

If you need to evaluate a definite integral involving a function whose antiderivative cannot be found, then while the Fundamental Theorem of Calculus is still true, it cannot be easily applied. In this case, it is easier to resort to an approximation technique. Two such techniques are described in this section.



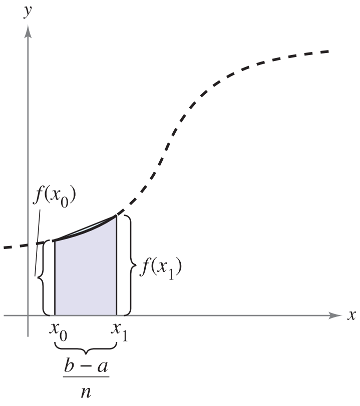
The area of the region can be approximated using four trapezoids.
Figure 4.42

One way to approximate a definite integral is to use n trapezoids, as shown in Figure 4.42. In the development of this method, assume that f is continuous and positive on the interval $[a, b]$. So, the definite integral

$$\int_a^b f(x)\,dx$$

represents the area of the region bounded by the graph of f and the x-axis, from $x = a$ to $x = b$. First, partition the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$, such that

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_n = b.$$



The area of the first trapezoid is
$\left[\frac{f(x_0) + f(x_1)}{2}\right]\left(\frac{b-a}{n}\right).$
Figure 4.43

Then form a trapezoid for each subinterval (see Figure 4.43). The area of the ith trapezoid is

$$\text{Area of } i\text{th trapezoid} = \left[\frac{f(x_{i-1}) + f(x_i)}{2}\right]\left(\frac{b-a}{n}\right).$$

This implies that the sum of the areas of the n trapezoids is

$$\begin{aligned}\text{Area} &= \left(\frac{b-a}{n}\right)\left[\frac{f(x_0) + f(x_1)}{2} + \cdot \cdot \cdot + \frac{f(x_{n-1}) + f(x_n)}{2}\right]\\ &= \left(\frac{b-a}{2n}\right)[f(x_0) + f(x_1) + f(x_1) + f(x_2) + \cdot \cdot \cdot + f(x_{n-1}) + f(x_n)]\\ &= \left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + 2f(x_2) + \cdot \cdot \cdot + 2f(x_{n-1}) + f(x_n)].\end{aligned}$$

Letting $\Delta x = (b - a)/n$, you can take the limit as $n \to \infty$ to obtain

$$\begin{aligned}&\lim_{n\to\infty}\left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + \cdot \cdot \cdot + 2f(x_{n-1}) + f(x_n)]\\ &\quad = \lim_{n\to\infty}\left[\frac{[f(a) - f(b)]\,\Delta x}{2} + \sum_{i=1}^{n} f(x_i)\,\Delta x\right]\\ &\quad = \lim_{n\to\infty}\frac{[f(a) - f(b)](b-a)}{2n} + \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x\\ &\quad = 0 + \int_a^b f(x)\,dx.\end{aligned}$$

The result is summarized in the following theorem.

THEOREM 4.17 THE TRAPEZOIDAL RULE

Let f be continuous on $[a, b]$. The Trapezoidal Rule for approximating $\int_a^b f(x)\,dx$ is given by

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdot\cdot\cdot + 2f(x_{n-1}) + f(x_n)\right].$$

Moreover, as $n \to \infty$, the right-hand side approaches $\int_a^b f(x)\,dx$.

NOTE Observe that the coefficients in the Trapezoidal Rule have the following pattern.

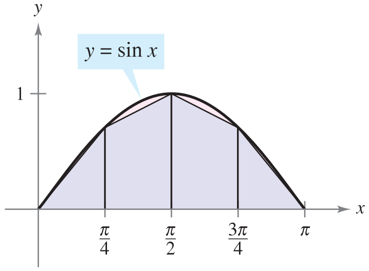
$$1 \quad 2 \quad 2 \quad 2 \quad . \, . \, . \quad 2 \quad 2 \quad 1$$

EXAMPLE 1 Approximation with the Trapezoidal Rule

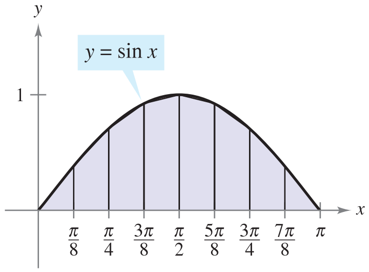
Use the Trapezoidal Rule to approximate

$$\int_0^\pi \sin x\,dx.$$

Compare the results for $n = 4$ and $n = 8$, as shown in Figure 4.44.



Four subintervals



Eight subintervals

Trapezoidal approximations
Figure 4.44

Solution When $n = 4$, $\Delta x = \pi/4$, and you obtain

$$\begin{aligned}\int_0^\pi \sin x\,dx &\approx \frac{\pi}{8}\left(\sin 0 + 2\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 2\sin\frac{3\pi}{4} + \sin\pi\right)\\ &= \frac{\pi}{8}\left(0 + \sqrt{2} + 2 + \sqrt{2} + 0\right) = \frac{\pi\left(1+\sqrt{2}\right)}{4} \approx 1.896.\end{aligned}$$

When $n = 8$, $\Delta x = \pi/8$, and you obtain

$$\begin{aligned}\int_0^\pi \sin x\,dx \approx \frac{\pi}{16}\Big(&\sin 0 + 2\sin\frac{\pi}{8} + 2\sin\frac{\pi}{4} + 2\sin\frac{3\pi}{8} + 2\sin\frac{\pi}{2}\\ &+ 2\sin\frac{5\pi}{8} + 2\sin\frac{3\pi}{4} + 2\sin\frac{7\pi}{8} + \sin\pi\Big)\\ = \frac{\pi}{16}&\left(2 + 2\sqrt{2} + 4\sin\frac{\pi}{8} + 4\sin\frac{3\pi}{8}\right) \approx 1.974.\end{aligned}$$

For this particular integral, you could have found an antiderivative and determined that the exact area of the region is 2.

TECHNOLOGY Most graphing utilities and computer algebra systems have built-in programs that can be used to approximate the value of a definite integral. Try using such a program to approximate the integral in Example 1. How close is your approximation?

When you use such a program, you need to be aware of its limitations. Often, you are given no indication of the degree of accuracy of the approximation. Other times, you may be given an approximation that is completely wrong. For instance, try using a built-in numerical integration program to evaluate

$$\int_{-1}^{2} \frac{1}{x}\,dx.$$

Your calculator should give an error message. Does yours?

It is interesting to compare the Trapezoidal Rule with the Midpoint Rule given in Section 4.2 (Exercises 73–76). For the Trapezoidal Rule, you average the function values at the endpoints of the subintervals, but for the Midpoint Rule you take the function values of the subinterval midpoints.

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x \qquad \text{Midpoint Rule}$$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \left(\frac{f(x_i) + f(x_{i-1})}{2}\right)\Delta x \qquad \text{Trapezoidal Rule}$$

NOTE There are two important points that should be made concerning the Trapezoidal Rule (or the Midpoint Rule). First, the approximation tends to become more accurate as n increases. For instance, in Example 1, if $n = 16$, the Trapezoidal Rule yields an approximation of 1.994. Second, although you could have used the Fundamental Theorem to evaluate the integral in Example 1, this theorem cannot be used to evaluate an integral as simple as $\int_0^{\pi} \sin x^2\,dx$ because $\sin x^2$ has no elementary antiderivative. Yet, the Trapezoidal Rule can be applied easily to estimate this integral. ■

Simpson's Rule

One way to view the trapezoidal approximation of a definite integral is to say that on each subinterval you approximate f by a *first*-degree polynomial. In Simpson's Rule, named after the English mathematician Thomas Simpson (1710–1761), you take this procedure one step further and approximate f by *second*-degree polynomials.

Before presenting Simpson's Rule, we list a theorem for evaluating integrals of polynomials of degree 2 (or less).

THEOREM 4.18 INTEGRAL OF $p(x) = Ax^2 + Bx + C$

If $p(x) = Ax^2 + Bx + C$, then

$$\int_a^b p(x)\,dx = \left(\frac{b-a}{6}\right)\left[p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right].$$

PROOF

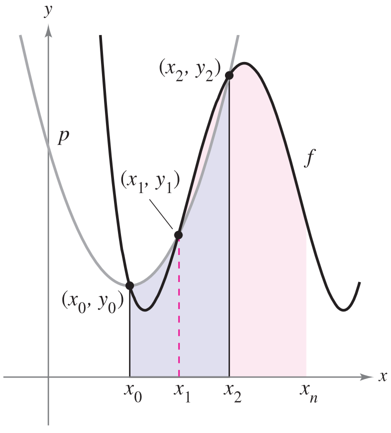
$$\begin{aligned}\int_a^b p(x)\,dx &= \int_a^b (Ax^2 + Bx + C)\,dx\\ &= \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_a^b\\ &= \frac{A(b^3 - a^3)}{3} + \frac{B(b^2 - a^2)}{2} + C(b-a)\\ &= \left(\frac{b-a}{6}\right)[2A(a^2 + ab + b^2) + 3B(b+a) + 6C]\end{aligned}$$

By expansion and collection of terms, the expression inside the brackets becomes

$$\underbrace{(Aa^2 + Ba + C)}_{p(a)} + \underbrace{4\left[A\left(\frac{b+a}{2}\right)^2 + B\left(\frac{b+a}{2}\right) + C\right]}_{4p\left(\frac{a+b}{2}\right)} + \underbrace{(Ab^2 + Bb + C)}_{p(b)}$$

and you can write

$$\int_a^b p(x)\,dx = \left(\frac{b-a}{6}\right)\left[p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right].$$ ■



$$\int_{x_0}^{x_2} p(x)\,dx \approx \int_{x_0}^{x_2} f(x)\,dx$$

Figure 4.45

To develop Simpson's Rule for approximating a definite integral, you again partition the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$. This time, however, n is required to be even, and the subintervals are grouped in pairs such that

$$a = \underbrace{x_0 < x_1 < x_2}_{[x_0,\, x_2]} < \underbrace{x_3 < x_4}_{[x_2,\, x_4]} < \cdot \cdot \cdot < \underbrace{x_{n-2} < x_{n-1} < x_n}_{[x_{n-2},\, x_n]} = b.$$

On each (double) subinterval $[x_{i-2}, x_i]$, you can approximate f by a polynomial p of degree less than or equal to 2. (See Exercise 56.) For example, on the subinterval $[x_0, x_2]$, choose the polynomial of least degree passing through the points (x_0, y_0), (x_1, y_1), and (x_2, y_2), as shown in Figure 4.45. Now, using p as an approximation of f on this subinterval, you have, by Theorem 4.18,

$$\begin{aligned}\int_{x_0}^{x_2} f(x)\,dx \approx \int_{x_0}^{x_2} p(x)\,dx &= \frac{x_2 - x_0}{6}\left[p(x_0) + 4p\left(\frac{x_0 + x_2}{2}\right) + p(x_2)\right]\\ &= \frac{2[(b - a)/n]}{6}[p(x_0) + 4p(x_1) + p(x_2)]\\ &= \frac{b - a}{3n}[f(x_0) + 4f(x_1) + f(x_2)].\end{aligned}$$

Repeating this procedure on the entire interval $[a, b]$ produces the following theorem.

THEOREM 4.19 SIMPSON'S RULE

Let f be continuous on $[a, b]$ and let n be an even integer. Simpson's Rule for approximating $\int_a^b f(x)\,dx$ is

$$\int_a^b f(x)\,dx \approx \frac{b - a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdot \cdot \cdot + 4f(x_{n-1}) + f(x_n)].$$

Moreover, as $n \to \infty$, the right-hand side approaches $\int_a^b f(x)\,dx$.

NOTE Observe that the coefficients in Simpson's Rule have the following pattern.

1 4 2 4 2 4 . . . 4 2 4 1 ■

In Example 1, the Trapezoidal Rule was used to estimate $\int_0^\pi \sin x\,dx$. In the next example, Simpson's Rule is applied to the same integral.

EXAMPLE 2 Approximation with Simpson's Rule

Use Simpson's Rule to approximate

$$\int_0^\pi \sin x\,dx.$$

Compare the results for $n = 4$ and $n = 8$.

Solution When $n = 4$, you have

$$\int_0^\pi \sin x\,dx \approx \frac{\pi}{12}\left(\sin 0 + 4\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 4\sin\frac{3\pi}{4} + \sin\pi\right) \approx 2.005.$$

When $n = 8$, you have $\int_0^\pi \sin x\,dx \approx 2.0003$. ■

NOTE In Example 1, the Trapezoidal Rule with $n = 8$ approximates $\int_0^\pi \sin x\,dx$ as 1.974. In Example 2, Simpson's Rule with $n = 8$ gives an approximation of 2.0003. The antiderivative would produce the true value of 2.

Error Analysis

If you must use an approximation technique, it is important to know how accurate you can expect the approximation to be. The following theorem, which is listed without proof, gives the formulas for estimating the errors involved in the use of Simpson's Rule and the Trapezoidal Rule. In general, when using an approximation, you can think of the error E as the difference between $\int_a^b f(x)\,dx$ and the approximation.

THEOREM 4.20 ERRORS IN THE TRAPEZOIDAL RULE AND SIMPSON'S RULE

If f has a continuous second derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)\,dx$ by the Trapezoidal Rule is

$$|E| \le \frac{(b-a)^3}{12n^2}\left[\max|f''(x)|\right], \quad a \le x \le b. \qquad \text{Trapezoidal Rule}$$

Moreover, if f has a continuous fourth derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)\,dx$ by Simpson's Rule is

$$|E| \le \frac{(b-a)^5}{180n^4}\left[\max|f^{(4)}(x)|\right], \quad a \le x \le b. \qquad \text{Simpson's Rule}$$

Theorem 4.20 states that the errors generated by the Trapezoidal Rule and Simpson's Rule have upper bounds dependent on the extreme values of $f''(x)$ and $f^{(4)}(x)$ in the interval $[a, b]$. Furthermore, these errors can be made arbitrarily small by *increasing* n, provided that f'' and $f^{(4)}$ are continuous and therefore bounded in $[a, b]$.

TECHNOLOGY If you have access to a computer algebra system, use it to evaluate the definite integral in Example 3. You should obtain a value of

$$\int_0^1 \sqrt{1+x^2}\,dx = \tfrac{1}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$$
$$\approx 1.14779.$$

("ln" represents the natural logarithmic function, which you will study in Section 5.1.)

EXAMPLE 3 The Approximate Error in the Trapezoidal Rule

Determine a value of n such that the Trapezoidal Rule will approximate the value of $\int_0^1 \sqrt{1+x^2}\,dx$ with an error that is less than or equal to 0.01.

Solution Begin by letting $f(x) = \sqrt{1+x^2}$ and finding the second derivative of f.

$$f'(x) = x(1+x^2)^{-1/2} \quad \text{and} \quad f''(x) = (1+x^2)^{-3/2}$$

The maximum value of $|f''(x)|$ on the interval $[0, 1]$ is $|f''(0)| = 1$. So, by Theorem 4.20, you can write

$$|E| \le \frac{(b-a)^3}{12n^2}|f''(0)| = \frac{1}{12n^2}(1) = \frac{1}{12n^2}.$$

To obtain an error E that is less than 0.01, you must choose n such that $1/(12n^2) \le 1/100$.

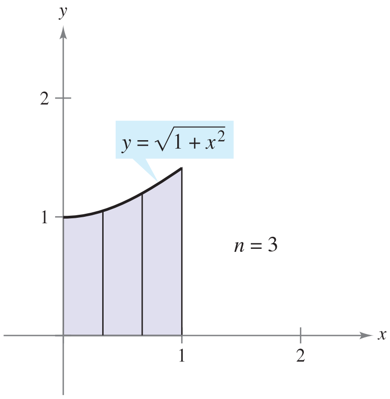
$$100 \le 12n^2 \quad \Rightarrow \quad n \ge \sqrt{\frac{100}{12}} \approx 2.89$$

So, you can choose $n = 3$ (because n must be greater than or equal to 2.89) and apply the Trapezoidal Rule, as shown in Figure 4.46, to obtain

$$\int_0^1 \sqrt{1+x^2}\,dx \approx \frac{1}{6}\left[\sqrt{1+0^2} + 2\sqrt{1+\left(\tfrac{1}{3}\right)^2} + 2\sqrt{1+\left(\tfrac{2}{3}\right)^2} + \sqrt{1+1^2}\right]$$
$$\approx 1.154.$$

So, by adding and subtracting the error from this estimate, you know that

$$1.144 \le \int_0^1 \sqrt{1+x^2}\,dx \le 1.164.$$ ■



$$1.144 \le \int_0^1 \sqrt{1+x^2}\,dx \le 1.164$$

Figure 4.46

4.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, use the Trapezoidal Rule and Simpson's Rule to approximate the value of the definite integral for the given value of *n*. Round your answer to four decimal places and compare the results with the exact value of the definite integral.

1. $\int_0^2 x^2\,dx, \quad n = 4$

2. $\int_1^2 \left(\frac{x^2}{4} + 1\right) dx, \quad n = 4$

3. $\int_0^2 x^3\,dx, \quad n = 4$

4. $\int_2^3 \frac{2}{x^2}\,dx, \quad n = 4$

5. $\int_1^3 x^3\,dx, \quad n = 6$

6. $\int_0^8 \sqrt[3]{x}\,dx, \quad n = 8$

7. $\int_4^9 \sqrt{x}\,dx, \quad n = 8$

8. $\int_1^4 (4 - x^2)\,dx, \quad n = 6$

9. $\int_0^1 \frac{2}{(x+2)^2}\,dx, \quad n = 4$

10. $\int_0^2 x\sqrt{x^2+1}\,dx, \quad n = 4$

In Exercises 11–20, approximate the definite integral using the Trapezoidal Rule and Simpson's Rule with $n = 4$. Compare these results with the approximation of the integral using a graphing utility.

11. $\int_0^2 \sqrt{1 + x^3}\,dx$

12. $\int_0^2 \frac{1}{\sqrt{1 + x^3}}\,dx$

13. $\int_0^1 \sqrt{x}\,\sqrt{1 - x}\,dx$

14. $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx$

15. $\int_0^{\sqrt{\pi/2}} \sin x^2\,dx$

16. $\int_0^{\sqrt{\pi/4}} \tan x^2\,dx$

17. $\int_3^{3.1} \cos x^2\,dx$

18. $\int_0^{\pi/2} \sqrt{1 + \sin^2 x}\,dx$

19. $\int_0^{\pi/4} x\tan x\,dx$

20. $\int_0^{\pi} f(x)\,dx, \quad f(x) = \begin{cases} \dfrac{\sin x}{x}, & x > 0 \\ 1, & x = 0 \end{cases}$

WRITING ABOUT CONCEPTS

21. The Trapezoidal Rule and Simpson's Rule yield approximations of a definite integral $\int_a^b f(x)\,dx$ based on polynomial approximations of f. What is the degree of the polynomials used for each?

22. Describe the size of the error when the Trapezoidal Rule is used to approximate $\int_a^b f(x)\,dx$ when $f(x)$ is a linear function. Use a graph to explain your answer.

In Exercises 23–28, use the error formulas in Theorem 4.20 to estimate the errors in approximating the integral, with $n = 4$, using (a) the Trapezoidal Rule and (b) Simpson's Rule.

23. $\int_1^3 2x^3\,dx$

24. $\int_3^5 (5x + 2)\,dx$

25. $\int_0^1 \frac{1}{x+1}\,dx$

26. $\int_2^4 \frac{1}{(x-1)^2}\,dx$

27. $\int_0^{\pi} \cos x\,dx$

28. $\int_0^1 \sin(\pi x)\,dx$

In Exercises 29–34, use the error formulas in Theorem 4.20 to find *n* such that the error in the approximation of the definite integral is less than or equal to 0.00001 using (a) the Trapezoidal Rule and (b) Simpson's Rule.

29. $\int_1^3 \frac{1}{x}\,dx$

30. $\int_0^1 \frac{1}{1+x}\,dx$

31. $\int_0^2 \sqrt{x+2}\,dx$

32. $\int_1^3 \frac{1}{\sqrt{x}}\,dx$

33. $\int_0^1 \cos(\pi x)\,dx$

34. $\int_0^{\pi/2} \sin x\,dx$

CAS **In Exercises 35–38, use a computer algebra system and the error formulas to find *n* such that the error in the approximation of the definite integral is less than or equal to 0.00001 using (a) the Trapezoidal Rule and (b) Simpson's Rule.**

35. $\int_0^2 \sqrt{1+x}\,dx$

36. $\int_0^2 (x+1)^{2/3}\,dx$

37. $\int_0^1 \tan x^2\,dx$

38. $\int_0^1 \sin x^2\,dx$

39. Approximate the area of the shaded region using (a) the Trapezoidal Rule and (b) Simpson's Rule with $n = 4$.

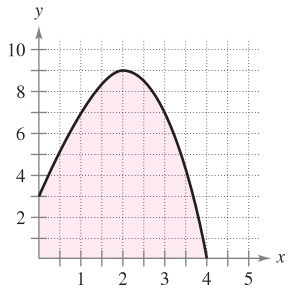


Figure for 39

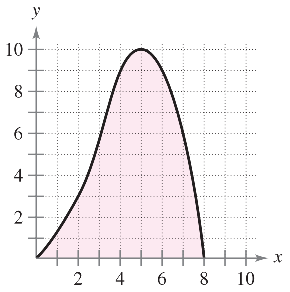


Figure for 40

40. Approximate the area of the shaded region using (a) the Trapezoidal Rule and (b) Simpson's Rule with $n = 8$.

41. ***Programming*** Write a program for a graphing utility to approximate a definite integral using the Trapezoidal Rule and Simpson's Rule. Start with the program written in Section 4.3, Exercises 61–64, and note that the Trapezoidal Rule can be written as $T(n) = \frac{1}{2}[L(n) + R(n)]$ and Simpson's Rule can be written as

$$S(n) = \tfrac{1}{3}[T(n/2) + 2M(n/2)].$$

[Recall that $L(n)$, $M(n)$, and $R(n)$ represent the Riemann sums using the left-hand endpoints, midpoints, and right-hand endpoints of subintervals of equal width.]

Programming **In Exercises 42–44, use the program in Exercise 41 to approximate the definite integral and complete the table.**

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4					
8					
10					
12					
16					
20					

42. $\int_0^4 \sqrt{2+3x^2}\,dx$ **43.** $\int_0^1 \sqrt{1-x^2}\,dx$ **44.** $\int_0^4 \sin\sqrt{x}\,dx$

45. ***Area*** Use Simpson's Rule with $n = 14$ to approximate the area of the region bounded by the graphs of $y = \sqrt{x}\cos x$, $y = 0$, $x = 0$, and $x = \pi/2$.

CAPSTONE

46. Consider a function $f(x)$ that is concave upward on the interval $[0, 2]$ and a function $g(x)$ that is concave downward on $[0, 2]$.

(a) Using the Trapezoidal Rule, which integral would be overestimated? Which integral would be underestimated? Assume $n = 4$. Use graphs to explain your answer.

(b) Which rule would you use for more accurate approximations of $\int_0^2 f(x)\,dx$ and $\int_0^2 g(x)\,dx$, the Trapezoidal Rule or Simpson's Rule? Explain your reasoning.

47. ***Circumference*** The **elliptic integral**

$$8\sqrt{3}\int_0^{\pi/2} \sqrt{1-\tfrac{2}{3}\sin^2\theta}\,d\theta$$

gives the circumference of an ellipse. Use Simpson's Rule with $n = 8$ to approximate the circumference.

48. ***Work*** To determine the size of the motor required to operate a press, a company must know the amount of work done when the press moves an object linearly 5 feet. The variable force to move the object is $F(x) = 100x\sqrt{125 - x^3}$, where F is given in pounds and x gives the position of the unit in feet. Use Simpson's Rule with $n = 12$ to approximate the work W (in foot-pounds) done through one cycle if $W = \int_0^5 F(x)\,dx$.

49. The table lists several measurements gathered in an experiment to approximate an unknown continuous function $y = f(x)$.

x	0.00	0.25	0.50	0.75	1.00
y	4.32	4.36	4.58	5.79	6.14

x	1.25	1.50	1.75	2.00
y	7.25	7.64	8.08	8.14

(a) Approximate the integral $\int_0^2 f(x)\,dx$ using the Trapezoidal Rule and Simpson's Rule.

(b) Use a graphing utility to find a model of the form $y = ax^3 + bx^2 + cx + d$ for the data. Integrate the resulting polynomial over $[0, 2]$ and compare the result with the integral from part (a).

Approximation of Pi **In Exercises 50 and 51, use Simpson's Rule with $n = 6$ to approximate π using the given equation. (In Section 5.7, you will be able to evaluate the integral using inverse trigonometric functions.)**

50. $\pi = \int_0^{1/2} \frac{6}{\sqrt{1-x^2}}\,dx$ **51.** $\pi = \int_0^1 \frac{4}{1+x^2}\,dx$

Area **In Exercises 52 and 53, use the Trapezoidal Rule to estimate the number of square meters of land in a lot where x and y are measured in meters, as shown in the figures. The land is bounded by a stream and two straight roads that meet at right angles.**

52.

x	0	100	200	300	400	500
y	125	125	120	112	90	90

x	600	700	800	900	1000
y	95	88	75	35	0

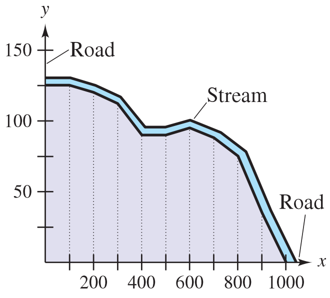


Figure for 52

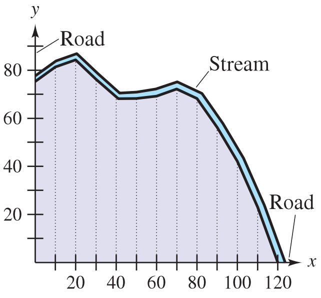


Figure for 53

53.

x	0	10	20	30	40	50	60
y	75	81	84	76	67	68	69

x	70	80	90	100	110	120
y	72	68	56	42	23	0

54. Prove that Simpson's Rule is exact when approximating the integral of a cubic polynomial function, and demonstrate the result for $\int_0^1 x^3\,dx$, $n = 2$.

CAS **55.** Use Simpson's Rule with $n = 10$ and a computer algebra system to approximate t in the integral equation $\int_0^t \sin\sqrt{x}\,dx = 2$.

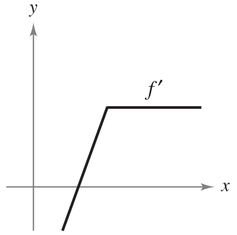
56. Prove that you can find a polynomial $p(x) = Ax^2 + Bx + C$ that passes through any three points (x_1, y_1), (x_2, y_2), and (x_3, y_3), where the x_i's are distinct.

4 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use the graph of f' to sketch a graph of f. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

1.



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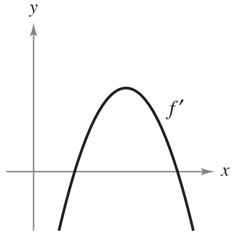


In Exercises 3–8, find the indefinite integral.

3. $\displaystyle\int (4x^2 + x + 3)\, dx$

4. $\displaystyle\int \frac{2}{\sqrt[3]{3x}}\, dx$

5. $\displaystyle\int \frac{x^4 + 8}{x^3}\, dx$

6. $\displaystyle\int \frac{x^4 - 4x^2 + 1}{x^2}\, dx$

7. $\displaystyle\int (2x - 9 \sin x)\, dx$

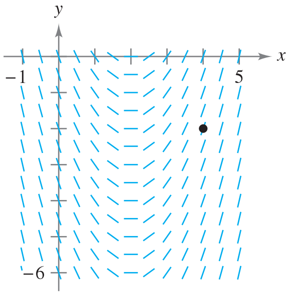
8. $\displaystyle\int (5 \cos x - 2 \sec^2 x)\, dx$

9. Find the particular solution of the differential equation $f'(x) = -6x$ whose graph passes through the point $(1, -2)$.

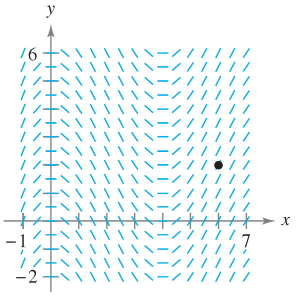
10. Find the particular solution of the differential equation $f''(x) = 6(x - 1)$ whose graph passes through the point $(2, 1)$ and is tangent to the line $3x - y - 5 = 0$ at that point.

Slope Fields **In Exercises 11 and 12, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution.**

11. $\dfrac{dy}{dx} = 2x - 4, \quad (4, -2)$



12. $\dfrac{dy}{dx} = \dfrac{1}{2}x^2 - 2x, \quad (6, 2)$



13. ***Velocity and Acceleration*** An airplane taking off from a runway travels 3600 feet before lifting off. The airplane starts from rest, moves with constant acceleration, and makes the run in 30 seconds. With what speed does it lift off?

14. ***Velocity and Acceleration*** The speed of a car traveling in a straight line is reduced from 45 to 30 miles per hour in a distance of 264 feet. Find the distance in which the car can be brought to rest from 30 miles per hour, assuming the same constant deceleration.

15. ***Velocity and Acceleration*** A ball is thrown vertically upward from ground level with an initial velocity of 96 feet per second.

(a) How long will it take the ball to rise to its maximum height? What is the maximum height?

(b) After how many seconds is the velocity of the ball one-half the initial velocity?

(c) What is the height of the ball when its velocity is one-half the initial velocity?

16. ***Modeling Data*** The table shows the velocities (in miles per hour) of two cars on an entrance ramp to an interstate highway. The time t is in seconds.

t	0	5	10	15	20	25	30
v_1	0	2.5	7	16	29	45	65
v_2	0	21	38	51	60	64	65

(a) Rewrite the velocities in feet per second.

(b) Use the regression capabilities of a graphing utility to find quadratic models for the data in part (a).

(c) Approximate the distance traveled by each car during the 30 seconds. Explain the difference in the distances.

In Exercises 17 and 18, use sigma notation to write the sum.

17. $\dfrac{1}{3(1)} + \dfrac{1}{3(2)} + \dfrac{1}{3(3)} + \cdots + \dfrac{1}{3(10)}$

18. $\left(\dfrac{3}{n}\right)\left(\dfrac{1+1}{n}\right)^2 + \left(\dfrac{3}{n}\right)\left(\dfrac{2+1}{n}\right)^2 + \cdots + \left(\dfrac{3}{n}\right)\left(\dfrac{n+1}{n}\right)^2$

In Exercises 19–22, use the properties of summation and Theorem 4.2 to evaluate the sum.

19. $\displaystyle\sum_{i=1}^{20} 2i$

20. $\displaystyle\sum_{i=1}^{20} (4i - 1)$

21. $\displaystyle\sum_{i=1}^{20} (i + 1)^2$

22. $\displaystyle\sum_{i=1}^{12} i(i^2 - 1)$

23. Write in sigma notation (a) the sum of the first ten positive odd integers, (b) the sum of the cubes of the first n positive integers, and (c) $6 + 10 + 14 + 18 + \cdots + 42$.

24. Evaluate each sum for $x_1 = 2$, $x_2 = -1$, $x_3 = 5$, $x_4 = 3$, and $x_5 = 7$.

(a) $\displaystyle\frac{1}{5}\sum_{i=1}^{5} x_i$

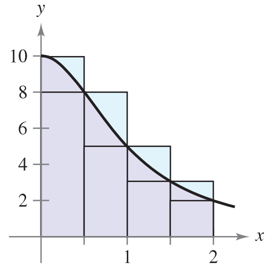
(b) $\displaystyle\sum_{i=1}^{5} \frac{1}{x_i}$

(c) $\displaystyle\sum_{i=1}^{5} (2x_i - x_i^2)$

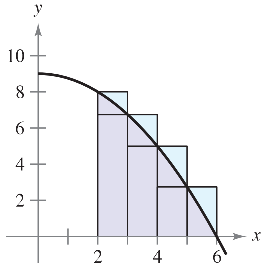
(d) $\displaystyle\sum_{i=2}^{5} (x_i - x_{i-1})$

In Exercises 25 and 26, use upper and lower sums to approximate the area of the region using the indicated number of subintervals of equal width.

25. $y = \dfrac{10}{x^2 + 1}$



26. $y = 9 - \frac{1}{4}x^2$



In Exercises 27–30, use the limit process to find the area of the region between the graph of the function and the x-axis over the given interval. Sketch the region.

27. $y = 8 - 2x, \quad [0, 3]$

28. $y = x^2 + 3, \quad [0, 2]$

29. $y = 5 - x^2, \quad [-2, 1]$

30. $y = \frac{1}{4}x^3, \quad [2, 4]$

31. Use the limit process to find the area of the region bounded by $x = 5y - y^2$, $x = 0$, $y = 2$, and $y = 5$.

32. Consider the region bounded by $y = mx$, $y = 0$, $x = 0$, and $x = b$.

(a) Find the upper and lower sums to approximate the area of the region when $\Delta x = b/4$.

(b) Find the upper and lower sums to approximate the area of the region when $\Delta x = b/n$.

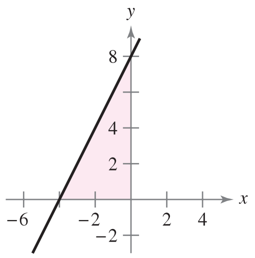
(c) Find the area of the region by letting n approach infinity in both sums in part (b). Show that in each case you obtain the formula for the area of a triangle.

In Exercises 33 and 34, write the limit as a definite integral on the interval $[a, b]$, where c_i is any point in the ith subinterval.

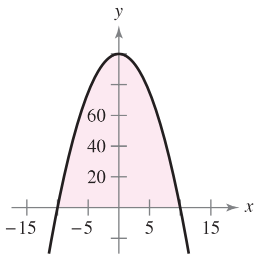
Limit	*Interval*
33. $\displaystyle\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (2c_i - 3)\,\Delta x_i$	$[4, 6]$
34. $\displaystyle\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 3c_i(9 - c_i^2)\,\Delta x_i$	$[1, 3]$

In Exercises 35 and 36, set up a definite integral that yields the area of the region. (Do not evaluate the integral.)

35. $f(x) = 2x + 8$



36. $f(x) = 100 - x^2$



In Exercises 37 and 38, sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral.

37. $\displaystyle\int_0^5 (5 - |x - 5|)\,dx$

38. $\displaystyle\int_{-6}^{6} \sqrt{36 - x^2}\,dx$

39. Given $\displaystyle\int_4^8 f(x)\,dx = 12$ and $\displaystyle\int_4^8 g(x)\,dx = 5$, evaluate

(a) $\displaystyle\int_4^8 [f(x) + g(x)]\,dx.$

(b) $\displaystyle\int_4^8 [f(x) - g(x)]\,dx.$

(c) $\displaystyle\int_4^8 [2f(x) - 3g(x)]\,dx.$

(d) $\displaystyle\int_4^8 7f(x)\,dx.$

40. Given $\displaystyle\int_0^3 f(x)\,dx = 4$ and $\displaystyle\int_3^6 f(x)\,dx = -1$, evaluate

(a) $\displaystyle\int_0^6 f(x)\,dx.$

(b) $\displaystyle\int_6^3 f(x)\,dx.$

(c) $\displaystyle\int_4^4 f(x)\,dx.$

(d) $\displaystyle\int_3^6 -10f(x)\,dx.$

In Exercises 41–48, use the Fundamental Theorem of Calculus to evaluate the definite integral.

41. $\displaystyle\int_0^8 (3 + x)\,dx$

42. $\displaystyle\int_{-3}^{3} (t^2 + 1)\,dt$

43. $\displaystyle\int_{-1}^{1} (4t^3 - 2t)\,dt$

44. $\displaystyle\int_{-2}^{-1} (x^4 + 3x^2 - 4)\,dx$

45. $\displaystyle\int_4^9 x\sqrt{x}\,dx$

46. $\displaystyle\int_1^2 \left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx$

47. $\displaystyle\int_0^{3\pi/4} \sin\theta\,d\theta$

48. $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2 t\,dt$

In Exercises 49–54, sketch the graph of the region whose area is given by the integral, and find the area.

49. $\displaystyle\int_2^4 (3x - 4)\,dx$

50. $\displaystyle\int_0^6 (8 - x)\,dx$

51. $\displaystyle\int_3^4 (x^2 - 9)\,dx$

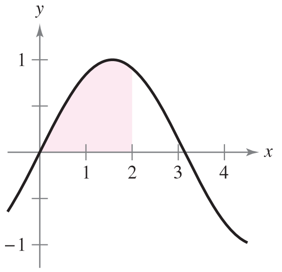
52. $\displaystyle\int_{-2}^{3} (-x^2 + x + 6)\,dx$

53. $\displaystyle\int_0^1 (x - x^3)\,dx$

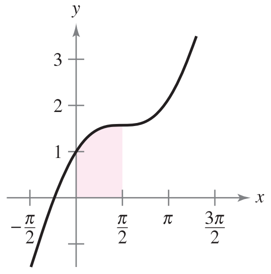
54. $\displaystyle\int_0^1 \sqrt{x}(1 - x)\,dx$

In Exercises 55 and 56, determine the area of the given region.

55. $y = \sin x$



56. $y = x + \cos x$



In Exercises 57 and 58, sketch the region bounded by the graphs of the equations, and determine its area.

57. $y = \dfrac{4}{\sqrt{x}}, \quad y = 0, \quad x = 1, \quad x = 9$

58. $y = \sec^2 x, \quad y = 0, \quad x = 0, \quad x = \dfrac{\pi}{3}$

In Exercises 59 and 60, find the average value of the function over the given interval. Find the values of x at which the function assumes its average value, and graph the function.

59. $f(x) = \dfrac{1}{\sqrt{x}}, \quad [4, 9]$

60. $f(x) = x^3, \quad [0, 2]$

In Exercises 61–64, use the Second Fundamental Theorem of Calculus to find $F'(x)$.

61. $F(x) = \displaystyle\int_0^x t^2\sqrt{1 + t^3}\, dt$

62. $F(x) = \displaystyle\int_1^x \frac{1}{t^2}\, dt$

63. $F(x) = \displaystyle\int_{-3}^x (t^2 + 3t + 2)\, dt$

64. $F(x) = \displaystyle\int_0^x \csc^2 t\, dt$

In Exercises 65–76, find the indefinite integral.

65. $\displaystyle\int (3 - x^2)^3\, dx$

66. $\displaystyle\int \left(x + \frac{1}{x}\right)^2 dx$

67. $\displaystyle\int \frac{x^2}{\sqrt{x^3 + 3}}\, dx$

68. $\displaystyle\int 3x^2\sqrt{2x^3 - 5}\, dx$

69. $\displaystyle\int x(1 - 3x^2)^4\, dx$

70. $\displaystyle\int \frac{x + 4}{(x^2 + 8x - 7)^2}\, dx$

71. $\displaystyle\int \sin^3 x \cos x\, dx$

72. $\displaystyle\int x \sin 3x^2\, dx$

73. $\displaystyle\int \frac{\cos\theta}{\sqrt{1 - \sin\theta}}\, d\theta$

74. $\displaystyle\int \frac{\sin x}{\sqrt{\cos x}}\, dx$

75. $\displaystyle\int (1 + \sec \pi x)^2 \sec \pi x \tan \pi x\, dx$

76. $\displaystyle\int \sec 2x \tan 2x\, dx$

In Exercises 77–84, evaluate the definite integral. Use a graphing utility to verify your result.

77. $\displaystyle\int_{-2}^1 x(x^2 - 6)\, dx$

78. $\displaystyle\int_0^1 x^2(x^3 - 2)^3\, dx$

79. $\displaystyle\int_0^3 \frac{1}{\sqrt{1 + x}}\, dx$

80. $\displaystyle\int_3^6 \frac{x}{3\sqrt{x^2 - 8}}\, dx$

81. $2\pi\displaystyle\int_0^1 (y + 1)\sqrt{1 - y}\, dy$

82. $2\pi\displaystyle\int_{-1}^0 x^2\sqrt{x + 1}\, dx$

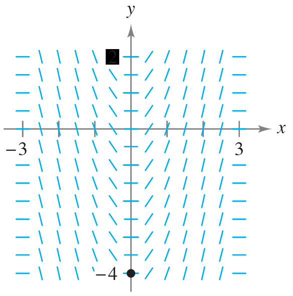
83. $\displaystyle\int_0^\pi \cos\frac{x}{2}\, dx$

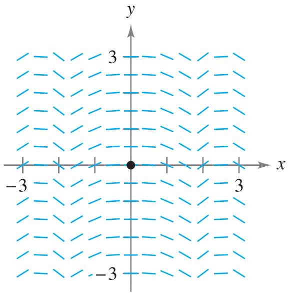
84. $\displaystyle\int_{-\pi/4}^{\pi/4} \sin 2x\, dx$

Slope Fields **In Exercises 85 and 86, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution.**

85. $\dfrac{dy}{dx} = x\sqrt{9 - x^2}, \quad (0, -4)$

86. $\dfrac{dy}{dx} = -\dfrac{1}{2}x\sin(x^2), \quad (0, 0)$

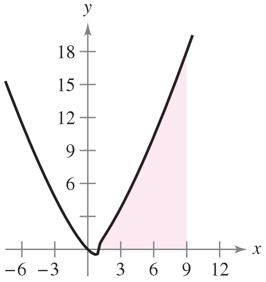


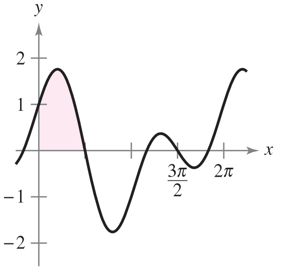


In Exercises 87 and 88, find the area of the region. Use a graphing utility to verify your result.

87. $\displaystyle\int_1^9 x\sqrt[3]{x - 1}\, dx$

88. $\displaystyle\int_0^{\pi/2} [\cos x + \sin(2x)]\, dx$





89. ***Precipitation*** The normal monthly precipitation in Portland, Oregon can be approximated by the model

$$R = 2.880 + 2.125\sin(0.578t + 0.745)$$

where R is measured in inches and t is the time in months, with $t = 0$ corresponding to January 1. *(Source: U.S. National Oceanic and Atmospheric Administration)*

(a) Write an integral and approximate the normal annual precipitation.

(b) Approximate the average monthly precipitation during the months of September and October.

90. ***Respiratory Cycle*** After exercising for a few minutes, a person has a respiratory cycle for which the rate of air intake is

$$v = 1.75\sin\frac{\pi t}{2}.$$

Find the volume, in liters, of air inhaled during one cycle by integrating the function over the interval $[0, 2]$.

In Exercises 91–94, use the Trapezoidal Rule and Simpson's Rule with $n = 4$, and use the integration capabilities of a graphing utility, to approximate the definite integral. Compare the results.

91. $\displaystyle\int_2^3 \frac{2}{1 + x^2}\, dx$

92. $\displaystyle\int_0^1 \frac{x^{3/2}}{3 - x^2}\, dx$

93. $\displaystyle\int_0^{\pi/2} \sqrt{x}\cos x\, dx$

94. $\displaystyle\int_0^\pi \sqrt{1 + \sin^2 x}\, dx$

P.S. PROBLEM SOLVING

1. Let $L(x) = \int_1^x \frac{1}{t}\, dt,\ x > 0.$

(a) Find $L(1)$.

(b) Find $L'(x)$ and $L'(1)$.

(c) Use a graphing utility to approximate the value of x (to three decimal places) for which $L(x) = 1$.

(d) Prove that $L(x_1x_2) = L(x_1) + L(x_2)$ for all positive values of x_1 and x_2.

2. Let $F(x) = \int_2^x \sin t^2\, dt.$

(a) Use a graphing utility to complete the table.

x	0	1.0	1.5	1.9	2.0
$F(x)$					

x	2.1	2.5	3.0	4.0	5.0
$F(x)$					

(b) Let $G(x) = \frac{1}{x-2}F(x) = \frac{1}{x-2}\int_2^x \sin t^2\, dt$. Use a graphing utility to complete the table and estimate $\lim_{x\to 2} G(x)$.

x	1.9	1.95	1.99	2.01	2.1
$G(x)$					

(c) Use the definition of the derivative to find the exact value of the limit $\lim_{x\to 2} G(x)$.

In Exercises 3 and 4, (a) write the area under the graph of the given function defined on the given interval as a limit. Then (b) evaluate the sum in part (a), and (c) evaluate the limit using the result of part (b).

3. $y = x^4 - 4x^3 + 4x^2, \quad [0, 2]$

$$\left(\textit{Hint: } \sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\right)$$

4. $y = \frac{1}{2}x^5 + 2x^3, \quad [0, 2]$

$$\left(\textit{Hint: } \sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}\right)$$

5. The **Fresnel function** S is defined by the integral

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt.$$

(a) Graph the function $y = \sin\left(\frac{\pi x^2}{2}\right)$ on the interval $[0, 3]$.

(b) Use the graph in part (a) to sketch the graph of S on the interval $[0, 3]$.

(c) Locate all relative extrema of S on the interval $(0, 3)$.

(d) Locate all points of inflection of S on the interval $(0, 3)$.

6. The **Two-Point Gaussian Quadrature Approximation** for f is

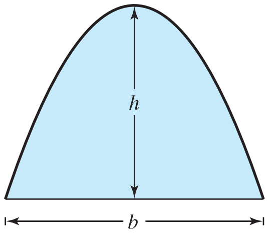
$$\int_{-1}^{1} f(x)\, dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

(a) Use this formula to approximate $\int_{-1}^{1} \cos x\, dx$. Find the error of the approximation.

(b) Use this formula to approximate $\int_{-1}^{1} \frac{1}{1+x^2}\, dx$.

(c) Prove that the Two-Point Gaussian Quadrature Approximation is exact for all polynomials of degree 3 or less.

7. Archimedes showed that the area of a parabolic arch is equal to $\frac{2}{3}$ the product of the base and the height (see figure).



(a) Graph the parabolic arch bounded by $y = 9 - x^2$ and the x-axis. Use an appropriate integral to find the area A.

(b) Find the base and height of the arch and verify Archimedes' formula.

(c) Prove Archimedes' formula for a general parabola.

8. Galileo Galilei (1564–1642) stated the following proposition concerning falling objects:

> *The time in which any space is traversed by a uniformly accelerating body is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed of the accelerating body and the speed just before acceleration began.*

Use the techniques of this chapter to verify this proposition.

9. The graph of the function f consists of the three line segments joining the points $(0, 0)$, $(2, -2)$, $(6, 2)$, and $(8, 3)$. The function F is defined by the integral

$$F(x) = \int_0^x f(t)\, dt.$$

(a) Sketch the graph of f.

(b) Complete the table.

x	0	1	2	3	4	5	6	7	8
$F(x)$									

(c) Find the extrema of F on the interval $[0, 8]$.

(d) Determine all points of inflection of F on the interval $(0, 8)$.

10. A car travels in a straight line for 1 hour. Its velocity v in miles per hour at six-minute intervals is shown in the table.

t (hours)	0	0.1	0.2	0.3	0.4	0.5
v (mi/h)	0	10	20	40	60	50

t (hours)	0.6	0.7	0.8	0.9	1.0
v (mi/h)	40	35	40	50	65

(a) Produce a reasonable graph of the velocity function v by graphing these points and connecting them with a smooth curve.

(b) Find the open intervals over which the acceleration a is positive.

(c) Find the average acceleration of the car (in miles per hour squared) over the interval $[0, 0.4]$.

(d) What does the integral $\int_0^1 v(t)\,dt$ signify? Approximate this integral using the Trapezoidal Rule with five subintervals.

(e) Approximate the acceleration at $t = 0.8$.

11. Prove $\displaystyle\int_0^x f(t)(x - t)\,dt = \int_0^x \left(\int_0^t f(v)\,dv\right) dt.$

12. Prove $\displaystyle\int_a^b f(x)f'(x)\,dx = \tfrac{1}{2}\left([f(b)]^2 - [f(a)]^2\right).$

13. Use an appropriate Riemann sum to evaluate the limit

$$\lim_{n\to\infty} \frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdot \cdot \cdot + \sqrt{n}}{n^{3/2}}.$$

14. Use an appropriate Riemann sum to evaluate the limit

$$\lim_{n\to\infty} \frac{1^5 + 2^5 + 3^5 + \cdot \cdot \cdot + n^5}{n^6}.$$

15. Suppose that f is integrable on $[a, b]$ and $0 < m \le f(x) \le M$ for all x in the interval $[a, b]$. Prove that

$$m(a - b) \le \int_a^b f(x)\,dx \le M(b - a).$$

Use this result to estimate $\int_0^1 \sqrt{1 + x^4}\,dx$.

16. Let f be continuous on the interval $[0, b]$, where $f(x) + f(b - x) \neq 0$ on $[0, b]$.

(a) Show that $\displaystyle\int_0^b \frac{f(x)}{f(x) + f(b - x)}\,dx = \frac{b}{2}.$

(b) Use the result in part (a) to evaluate

$$\int_0^1 \frac{\sin x}{\sin(1 - x) + \sin x}\,dx.$$

(c) Use the result in part (a) to evaluate

$$\int_0^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{3 - x}}\,dx.$$

17. Verify that

$$\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}$$

by showing the following.

(a) $(1 + i)^3 - i^3 = 3i^2 + 3i + 1$

(b) $\displaystyle(n + 1)^3 = \sum_{i=1}^{n} (3i^2 + 3i + 1) + 1$

(c) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}$

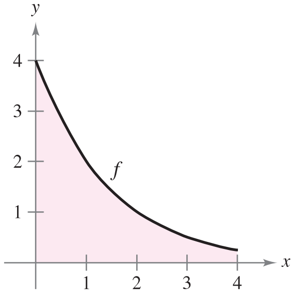
18. Prove that if f is a continuous function on a closed interval $[a, b]$, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

19. Let $I = \int_0^4 f(x)\,dx$, where f is shown in the figure. Let $L(n)$ and $R(n)$ represent the Riemann sums using the left-hand endpoints and right-hand endpoints of n subintervals of equal width. (Assume n is even.) Let $T(n)$ and $S(n)$ be the corresponding values of the Trapezoidal Rule and Simpson's Rule.

(a) For any n, list $L(n)$, $R(n)$, $T(n)$, and I in increasing order.

(b) Approximate $S(4)$.



20. The **sine integral function**

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$$

is often used in engineering. The function $f(t) = \dfrac{\sin t}{t}$ is not defined at $t = 0$, but its limit is 1 as $t \to 0$. So, define $f(0) = 1$. Then f is continuous everywhere.

(a) Use a graphing utility to graph $\text{Si}(x)$.

(b) At what values of x does $\text{Si}(x)$ have relative maxima?

(c) Find the coordinates of the first inflection point where $x > 0$.

(d) Decide whether $\text{Si}(x)$ has any horizontal asymptotes. If so, identify each.

21. Determine the limits of integration where $a \le b$ such that

$$\int_a^b (x^2 - 16)\,dx$$

has minimal value.

5

Logarithmic, Exponential, and Other Transcendental Functions

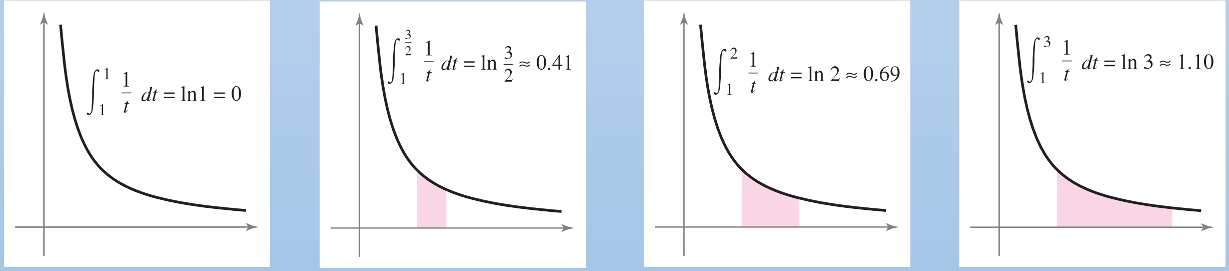
So far in this text, you have studied two types of elementary functions—algebraic functions and trigonometric functions. This chapter concludes the introduction of elementary functions. As each new type is introduced, you will study its properties, its derivative, and its antiderivative.

In this chapter, you should learn the following.

- The properties of the natural logarithmic function. How to find the derivative and antiderivative of the natural logarithmic function. (5.1, 5.2)
- How to determine whether a function has an inverse function. (5.3)
- The properties of the natural exponential function. How to find the derivative and antiderivative of the natural exponential function. (5.4)
- The properties, derivatives, and antiderivatives of logarithmic and exponential functions that have bases other than *e*. (5.5)
- The properties of inverse trigonometric functions. How to find derivatives and antiderivatives of inverse trigonometric functions. (5.6, 5.7)
- The properties of hyperbolic functions. How to find derivatives and antiderivatives of hyperbolic functions. (5.8)

Owaki-Kulla/Photolibrary

The Gateway Arch in St. Louis, Missouri is over 600 feet high and covered with 886 tons of quarter-inch stainless steel. A mathematical equation used to construct the arch involves which function? (See Section 5.8, Section Project.)



In Section 5.1, you will see how the function $f(x) = 1/x$ can be used to define the natural logarithmic function. To do this, consider the definite integral $\int_1^x 1/t\,dt$. When $x < 1$, the value of this definite integral is negative. When $x = 1$, the value is 0. When $x > 1$, the value is positive.

5.1 The Natural Logarithmic Function: Differentiation

- **Develop and use properties of the natural logarithmic function.**
- **Understand the definition of the number *e*.**
- **Find derivatives of functions involving the natural logarithmic function.**

The Natural Logarithmic Function

Recall that the General Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \qquad \text{General Power Rule}$$

has an important disclaimer—it doesn't apply when $n = -1$. Consequently, you have not yet found an antiderivative for the function $f(x) = 1/x$. In this section, you will use the Second Fundamental Theorem of Calculus to *define* such a function. This antiderivative is a function that you have not encountered previously in the text. It is neither algebraic nor trigonometric, but falls into a new class of functions called *logarithmic functions*. This particular function is the **natural logarithmic function.**

The Granger Collection

JOHN NAPIER (1550–1617)

Logarithms were invented by the Scottish mathematician John Napier. Napier coined the term *logarithm*, from the two Greek words *logos* (or ratio) and *arithmos* (or number), to describe the theory that he spent 20 years developing and that first appeared in the book *Mirifici Logarithmorum canonis descriptio* (A Description of the Marvelous Rule of Logarithms). Although he did not introduce the *natural* logarithmic function, it is sometimes called the *Napierian* logarithm.

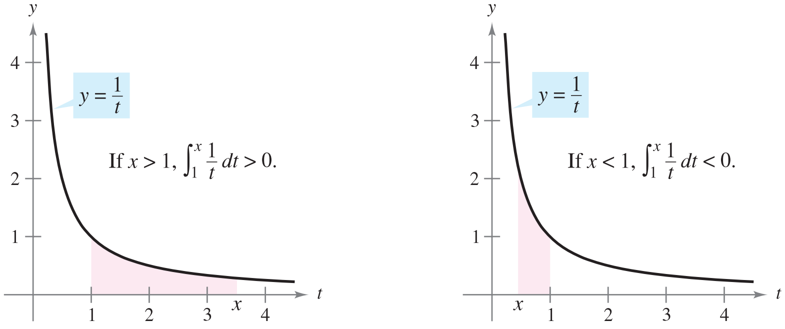
DEFINITION OF THE NATURAL LOGARITHMIC FUNCTION

The **natural logarithmic function** is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

The domain of the natural logarithmic function is the set of all positive real numbers.

From this definition, you can see that $\ln x$ is positive for $x > 1$ and negative for $0 < x < 1$, as shown in Figure 5.1. Moreover, $\ln(1) = 0$, because the upper and lower limits of integration are equal when $x = 1$.

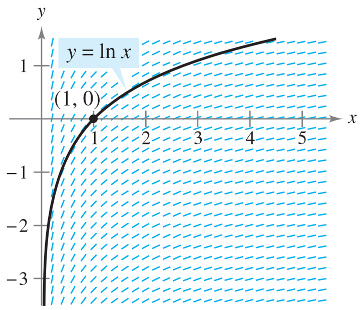


If $x > 1$, then $\ln x > 0$. If $0 < x < 1$, then $\ln x < 0$.

Figure 5.1

EXPLORATION

Graphing the Natural Logarithmic Function Using *only* the definition of the natural logarithmic function, sketch a graph of the function. Explain your reasoning.



Each small line segment has a slope of $\frac{1}{x}$.

Figure 5.2

To sketch the graph of $y = \ln x$, you can think of the natural logarithmic function as an *antiderivative* given by the differential equation

$$\frac{dy}{dx} = \frac{1}{x}.$$

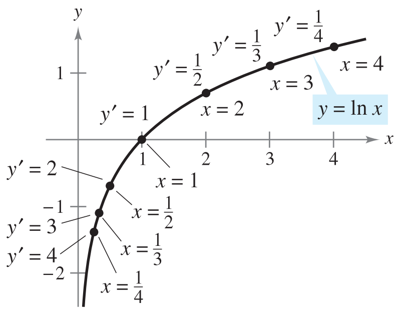
Figure 5.2 is a computer-generated graph, called a *slope (or direction) field*, showing small line segments of slope $1/x$. The graph of $y = \ln x$ is the solution that passes through the point $(1, 0)$. You will study slope fields in Section 6.1.

The following theorem lists some basic properties of the natural logarithmic function.

THEOREM 5.1 PROPERTIES OF THE NATURAL LOGARITHMIC FUNCTION

The natural logarithmic function has the following properties.

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.



The natural logarithmic function is increasing, and its graph is concave downward.

Figure 5.3

PROOF The domain of $f(x) = \ln x$ is $(0, \infty)$ by definition. Moreover, the function is continuous because it is differentiable. It is increasing because its derivative

$$f'(x) = \frac{1}{x} \qquad \text{First derivative}$$

is positive for $x > 0$, as shown in Figure 5.3. It is concave downward because

$$f''(x) = -\frac{1}{x^2} \qquad \text{Second derivative}$$

is negative for $x > 0$. The proof that f is one-to-one is given in Appendix A. The following limits imply that its range is the entire real line.

$$\lim_{x \to 0^+} \ln x = -\infty \qquad \text{and} \qquad \lim_{x \to \infty} \ln x = \infty$$

Verification of these two limits is given in Appendix A. ■

Using the definition of the natural logarithmic function, you can prove several important properties involving operations with natural logarithms. If you are already familiar with logarithms, you will recognize that these properties are characteristic of all logarithms.

THEOREM 5.2 LOGARITHMIC PROPERTIES

If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

PROOF The first property has already been discussed. The proof of the second property follows from the fact that two antiderivatives of the same function differ at most by a constant. From the Second Fundamental Theorem of Calculus and the definition of the natural logarithmic function, you know that

$$\frac{d}{dx}[\ln x] = \frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] = \frac{1}{x}.$$

So, consider the two derivatives

$$\frac{d}{dx}[\ln(ax)] = \frac{a}{ax} = \frac{1}{x}$$

and

$$\frac{d}{dx}[\ln a + \ln x] = 0 + \frac{1}{x} = \frac{1}{x}.$$

Because $\ln(ax)$ and $(\ln a + \ln x)$ are both antiderivatives of $1/x$, they must differ at most by a constant.

$$\ln(ax) = \ln a + \ln x + C$$

By letting $x = 1$, you can see that $C = 0$. The third property can be proved similarly by comparing the derivatives of $\ln(x^n)$ and $n \ln x$. Finally, using the second and third properties, you can prove the fourth property.

$$\ln\left(\frac{a}{b}\right) = \ln[a(b^{-1})] = \ln a + \ln(b^{-1}) = \ln a - \ln b$$ ■

Example 1 shows how logarithmic properties can be used to expand logarithmic expressions.

EXAMPLE 1 Expanding Logarithmic Expressions

a. $\ln\frac{10}{9} = \ln 10 - \ln 9$ Property 4

b. $\ln\sqrt{3x+2} = \ln(3x+2)^{1/2}$ Rewrite with rational exponent.

$= \frac{1}{2}\ln(3x+2)$ Property 3

c. $\ln\frac{6x}{5} = \ln(6x) - \ln 5$ Property 4

$= \ln 6 + \ln x - \ln 5$ Property 2

d. $$\ln\frac{(x^2+3)^2}{x\sqrt[3]{x^2+1}} = \ln(x^2+3)^2 - \ln\left(x\sqrt[3]{x^2+1}\right)$$

$$= 2\ln(x^2+3) - [\ln x + \ln(x^2+1)^{1/3}]$$

$$= 2\ln(x^2+3) - \ln x - \ln(x^2+1)^{1/3}$$

$$= 2\ln(x^2+3) - \ln x - \frac{1}{3}\ln(x^2+1)$$ ■

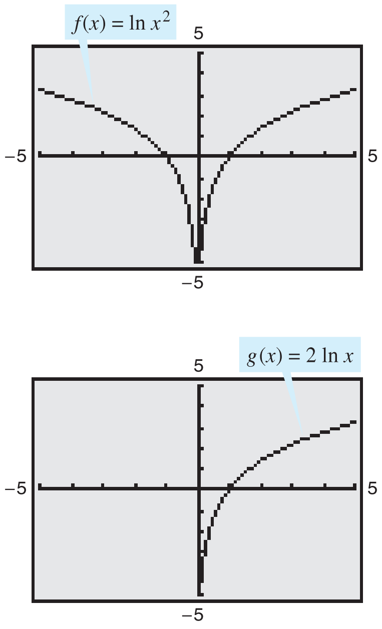
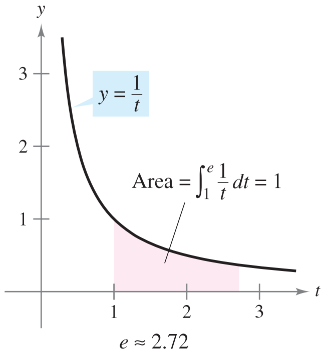


Figure 5.4

When using the properties of logarithms to rewrite logarithmic functions, you must check to see whether the domain of the rewritten function is the same as the domain of the original. For instance, the domain of $f(x) = \ln x^2$ is all real numbers except $x = 0$, and the domain of $g(x) = 2\ln x$ is all positive real numbers. (See Figure 5.4.)

The Number *e*



e is the base for the natural logarithm because $\ln e = 1$.
Figure 5.5

It is likely that you have studied logarithms in an algebra course. There, without the benefit of calculus, logarithms would have been defined in terms of a **base** number. For example, common logarithms have a base of 10 and therefore $\log_{10}10 = 1$. (You will learn more about this in Section 5.5.)

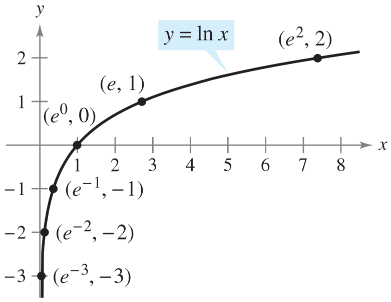
The **base for the natural logarithm** is defined using the fact that the natural logarithmic function is continuous, is one-to-one, and has a range of $(-\infty, \infty)$. So, there must be a unique real number x such that $\ln x = 1$, as shown in Figure 5.5. This number is denoted by the letter e. It can be shown that e is irrational and has the following decimal approximation.

$$e \approx 2.71828182846$$

DEFINITION OF *e*

The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\,dt = 1.$$



If $x = e^n$, then $\ln x = n$.
Figure 5.6

■ **FOR FURTHER INFORMATION** To learn more about the number e, see the article "Unexpected Occurrences of the Number e" by Harris S. Shultz and Bill Leonard in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*. ■

Once you know that $\ln e = 1$, you can use logarithmic properties to evaluate the natural logarithms of several other numbers. For example, by using the property

$$\begin{aligned}\ln(e^n) &= n \ln e\\ &= n(1)\\ &= n\end{aligned}$$

you can evaluate $\ln(e^n)$ for various values of n, as shown in the table and in Figure 5.6.

x	$\frac{1}{e^3} \approx 0.050$	$\frac{1}{e^2} \approx 0.135$	$\frac{1}{e} \approx 0.368$	$e^0 = 1$	$e \approx 2.718$	$e^2 \approx 7.389$
$\ln x$	-3	-2	-1	0	1	2

The logarithms shown in the table above are convenient because the x-values are integer powers of e. Most logarithmic expressions are, however, best evaluated with a calculator.

EXAMPLE 2 Evaluating Natural Logarithmic Expressions

a. $\ln 2 \approx 0.693$

b. $\ln 32 \approx 3.466$

c. $\ln 0.1 \approx -2.303$ ■

The Derivative of the Natural Logarithmic Function

The derivative of the natural logarithmic function is given in Theorem 5.3. The first part of the theorem follows from the definition of the natural logarithmic function as an antiderivative. The second part of the theorem is simply the Chain Rule version of the first part.

THEOREM 5.3 DERIVATIVE OF THE NATURAL LOGARITHMIC FUNCTION

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}, \quad x > 0$ **2.** $\dfrac{d}{dx}[\ln u] = \dfrac{1}{u}\dfrac{du}{dx} = \dfrac{u'}{u}, \quad u > 0$

EXAMPLE 3 Differentiation of Logarithmic Functions

a. $\dfrac{d}{dx}[\ln(2x)] = \dfrac{u'}{u} = \dfrac{2}{2x} = \dfrac{1}{x}$ $u = 2x$

b. $\dfrac{d}{dx}[\ln(x^2 + 1)] = \dfrac{u'}{u} = \dfrac{2x}{x^2 + 1}$ $u = x^2 + 1$

c. $\dfrac{d}{dx}[x \ln x] = x\left(\dfrac{d}{dx}[\ln x]\right) + (\ln x)\left(\dfrac{d}{dx}[x]\right)$ Product Rule

$$= x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x$$

d. $\dfrac{d}{dx}[(\ln x)^3] = 3(\ln x)^2 \dfrac{d}{dx}[\ln x]$ Chain Rule

$$= 3(\ln x)^2 \frac{1}{x}$$

■

EXPLORATION

Use a graphing utility to graph

$$y_1 = \frac{1}{x}$$

and

$$y_2 = \frac{d}{dx}[\ln x]$$

in the same viewing window, in which $0.1 \le x \le 5$ and $-2 \le y \le 8$. Explain why the graphs appear to be identical.

Napier used logarithmic properties to simplify *calculations* involving products, quotients, and powers. Of course, given the availability of calculators, there is now little need for this particular application of logarithms. However, there is great value in using logarithmic properties to simplify *differentiation* involving products, quotients, and powers.

EXAMPLE 4 Logarithmic Properties as Aids to Differentiation

Differentiate $f(x) = \ln\sqrt{x + 1}$.

Solution Because

$$f(x) = \ln\sqrt{x + 1} = \ln(x + 1)^{1/2} = \frac{1}{2}\ln(x + 1)$$ Rewrite before differentiating.

you can write

$$f'(x) = \frac{1}{2}\left(\frac{1}{x + 1}\right) = \frac{1}{2(x + 1)}.$$ Differentiate. ■

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

EXAMPLE 5 Logarithmic Properties as Aids to Differentiation

Differentiate $f(x) = \ln \dfrac{x(x^2+1)^2}{\sqrt{2x^3-1}}$.

Solution

$$f(x) = \ln \frac{x(x^2+1)^2}{\sqrt{2x^3-1}} \qquad \text{Write original function.}$$

$$= \ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1) \qquad \text{Rewrite before differentiating.}$$

$$f'(x) = \frac{1}{x} + 2\left(\frac{2x}{x^2+1}\right) - \frac{1}{2}\left(\frac{6x^2}{2x^3-1}\right) \qquad \text{Differentiate.}$$

$$= \frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1} \qquad \text{Simplify.}$$

■

NOTE In Examples 4 and 5, be sure you see the benefit of applying logarithmic properties *before* differentiating. Consider, for instance, the difficulty of direct differentiation of the function given in Example 5. ■

On occasion, it is convenient to use logarithms as aids in differentiating *nonlogarithmic* functions. This procedure is called **logarithmic differentiation.**

EXAMPLE 6 Logarithmic Differentiation

Find the derivative of

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2.$$

Solution Note that $y > 0$ for all $x \neq 2$. So, $\ln y$ is defined. Begin by taking the natural logarithm of each side of the equation. Then apply logarithmic properties and differentiate implicitly. Finally, solve for y'.

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2 \qquad \text{Write original equation.}$$

$$\ln y = \ln \frac{(x-2)^2}{\sqrt{x^2+1}} \qquad \text{Take natural log of each side.}$$

$$\ln y = 2\ln(x-2) - \frac{1}{2}\ln(x^2+1) \qquad \text{Logarithmic properties}$$

$$\frac{y'}{y} = 2\left(\frac{1}{x-2}\right) - \frac{1}{2}\left(\frac{2x}{x^2+1}\right) \qquad \text{Differentiate.}$$

$$= \frac{x^2+2x+2}{(x-2)(x^2+1)} \qquad \text{Simplify.}$$

$$y' = y\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] \qquad \text{Solve for } y'.$$

$$= \frac{(x-2)^2}{\sqrt{x^2+1}}\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] \qquad \text{Substitute for } y.$$

$$= \frac{(x-2)(x^2+2x+2)}{(x^2+1)^{3/2}} \qquad \text{Simplify.}$$

■

Because the natural logarithm is undefined for negative numbers, you will often encounter expressions of the form $\ln|u|$. The following theorem states that you can differentiate functions of the form $y = \ln|u|$ as if the absolute value notation was not present.

THEOREM 5.4 DERIVATIVE INVOLVING ABSOLUTE VALUE

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}.$$

PROOF If $u > 0$, then $|u| = u$, and the result follows from Theorem 5.3. If $u < 0$, then $|u| = -u$, and you have

$$\begin{aligned}\frac{d}{dx}[\ln|u|] &= \frac{d}{dx}[\ln(-u)]\\ &= \frac{-u'}{-u}\\ &= \frac{u'}{u}.\end{aligned}$$

■

EXAMPLE 7 Derivative Involving Absolute Value

Find the derivative of

$f(x) = \ln|\cos x|$.

Solution Using Theorem 5.4, let $u = \cos x$ and write

$$\begin{aligned}\frac{d}{dx}[\ln|\cos x|] &= \frac{u'}{u} && \frac{d}{dx}[\ln|u|] = \frac{u'}{u}\\ &= \frac{-\sin x}{\cos x} && u = \cos x\\ &= -\tan x. && \text{Simplify.}\end{aligned}$$

EXAMPLE 8 Finding Relative Extrema

Locate the relative extrema of

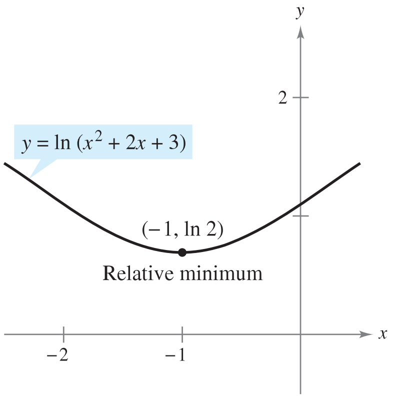
$y = \ln(x^2 + 2x + 3)$.

Solution Differentiating y, you obtain

$$\frac{dy}{dx} = \frac{2x + 2}{x^2 + 2x + 3}.$$

Because $dy/dx = 0$ when $x = -1$, you can apply the First Derivative Test and conclude that the point $(-1, \ln 2)$ is a relative minimum. Because there are no other critical points, it follows that this is the only relative extremum (see Figure 5.7).

■



The derivative of y changes from negative to positive at $x = -1$.
Figure 5.7

5.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. Complete the table below. Use a graphing utility and Simpson's Rule with $n = 10$ to approximate the integral $\int_1^x (1/t)\,dt$.

x	0.5	1.5	2	2.5	3	3.5	4
$\int_1^x (1/t)\,dt$							

2. (a) Plot the points generated in Exercise 1 and connect them with a smooth curve. Compare the result with the graph of $y = \ln x$.

(b) Use a graphing utility to graph $y = \int_1^x (1/t)\,dt$ for $0.2 \le x \le 4$. Compare the result with the graph of $y = \ln x$.

In Exercises 3–6, use a graphing utility to evaluate the logarithm by (a) using the natural logarithm key and (b) using the integration capabilities to evaluate the integral $\int_1^x (1/t)\,dt$.

3. ln 45

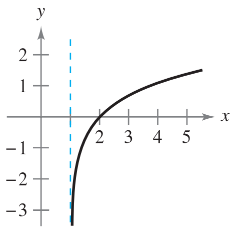
4. ln 8.3

5. ln 0.8

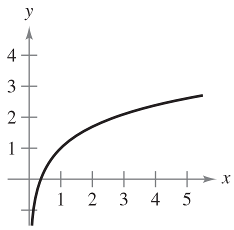
6. ln 0.6

In Exercises 7–10, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

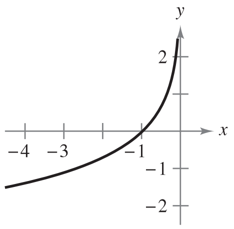
(a)



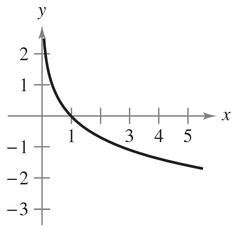
(b)



(c)



(d)



7. $f(x) = \ln x + 1$

8. $f(x) = -\ln x$

9. $f(x) = \ln(x - 1)$

10. $f(x) = -\ln(-x)$

In Exercises 11–18, sketch the graph of the function and state its domain.

11. $f(x) = 3\ln x$

12. $f(x) = -2\ln x$

13. $f(x) = \ln 2x$

14. $f(x) = \ln|x|$

15. $f(x) = \ln(x - 1)$

16. $g(x) = 2 + \ln x$

17. $h(x) = \ln(x + 2)$

18. $f(x) = \ln(x - 2) + 1$

In Exercises 19 and 20, use the properties of logarithms to approximate the indicated logarithms, given that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$.

19. (a) ln 6 (b) $\ln\frac{2}{3}$ (c) ln 81 (d) $\ln\sqrt{3}$

20. (a) ln 0.25 (b) ln 24 (c) $\ln\sqrt[3]{12}$ (d) $\ln\frac{1}{72}$

In Exercises 21–30, use the properties of logarithms to expand the logarithmic expression.

21. $\ln\dfrac{x}{4}$

22. $\ln\sqrt{x^5}$

23. $\ln\dfrac{xy}{z}$

24. $\ln(xyz)$

25. $\ln\left(x\sqrt{x^2 + 5}\right)$

26. $\ln\sqrt{a - 1}$

27. $\ln\sqrt{\dfrac{x - 1}{x}}$

28. $\ln(3e^2)$

29. $\ln z(z - 1)^2$

30. $\ln\dfrac{1}{e}$

In Exercises 31–36, write the expression as a logarithm of a single quantity.

31. $\ln(x - 2) - \ln(x + 2)$

32. $3\ln x + 2\ln y - 4\ln z$

33. $\frac{1}{3}[2\ln(x + 3) + \ln x - \ln(x^2 - 1)]$

34. $2[\ln x - \ln(x + 1) - \ln(x - 1)]$

35. $2\ln 3 - \frac{1}{2}\ln(x^2 + 1)$

36. $\frac{3}{2}[\ln(x^2 + 1) - \ln(x + 1) - \ln(x - 1)]$

In Exercises 37 and 38, (a) verify that $f = g$ by using a graphing utility to graph f and g in the same viewing window and (b) verify that $f = g$ algebraically.

37. $f(x) = \ln\dfrac{x^2}{4}, \quad x > 0, \quad g(x) = 2\ln x - \ln 4$

38. $f(x) = \ln\sqrt{x(x^2 + 1)}, \quad g(x) = \frac{1}{2}[\ln x + \ln(x^2 + 1)]$

In Exercises 39–42, find the limit.

39. $\lim_{x\to3^+}\ln(x - 3)$

40. $\lim_{x\to6^-}\ln(6 - x)$

41. $\lim_{x\to2^-}\ln[x^2(3 - x)]$

42. $\lim_{x\to5^+}\ln\dfrac{x}{\sqrt{x - 4}}$

In Exercises 43–46, find an equation of the tangent line to the graph of the logarithmic function at the point (1, 0).

43. $y = \ln x^3$

44. $y = \ln x^{3/2}$

45. $y = x^4$

46. $y = \ln x^{1/2}$

In Exercises 47–76, find the derivative of the function.

47. $f(x) = \ln(3x)$

48. $f(x) = \ln(x - 1)$

49. $g(x) = \ln x^2$

50. $h(x) = \ln(2x^2 + 1)$

51. $y = (\ln x)^4$

52. $y = x^2\ln x$

53. $y = \ln(t+1)^2$

54. $y = \ln\sqrt{x^2-4}$

55. $y = \ln\left(x\sqrt{x^2-1}\right)$

56. $y = \ln[t(t^2+3)^3]$

57. $f(x) = \ln\left(\dfrac{x}{x^2+1}\right)$

58. $f(x) = \ln\left(\dfrac{2x}{x+3}\right)$

59. $g(t) = \dfrac{\ln t}{t^2}$

60. $h(t) = \dfrac{\ln t}{t}$

61. $y = \ln(\ln x^2)$

62. $y = \ln(\ln x)$

63. $y = \ln\sqrt{\dfrac{x+1}{x-1}}$

64. $y = \ln\sqrt[3]{\dfrac{x-1}{x+1}}$

65. $f(x) = \ln\left(\dfrac{\sqrt{4+x^2}}{x}\right)$

66. $f(x) = \ln\left(x+\sqrt{4+x^2}\right)$

67. $y = \dfrac{-\sqrt{x^2+1}}{x} + \ln\left(x+\sqrt{x^2+1}\right)$

68. $y = \dfrac{-\sqrt{x^2+4}}{2x^2} - \dfrac{1}{4}\ln\left(\dfrac{2+\sqrt{x^2+4}}{x}\right)$

69. $y = \ln|\sin x|$

70. $y = \ln|\csc x|$

71. $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

72. $y = \ln|\sec x + \tan x|$

73. $y = \ln\left|\dfrac{-1+\sin x}{2+\sin x}\right|$

74. $y = \ln\sqrt{2+\cos^2 x}$

75. $f(x) = \displaystyle\int_2^{\ln(2x)} (t+1)\,dt$

76. $g(x) = \displaystyle\int_1^{\ln x} (t^2+3)\,dt$

In Exercises 77–82, (a) find an equation of the tangent line to the graph of *f* at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

77. $f(x) = 3x^2 - \ln x, \quad (1, 3)$

78. $f(x) = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right), \quad (0, 4)$

79. $f(x) = \ln\sqrt{1+\sin^2 x}, \quad \left(\dfrac{\pi}{4}, \ln\sqrt{\dfrac{3}{2}}\right)$

80. $f(x) = \sin 2x \ln x^2, \quad (1, 0)$

81. $f(x) = x^3 \ln x, \quad (1, 0)$

82. $f(x) = \dfrac{1}{2}x \ln x^2, \quad (-1, 0)$

In Exercises 83–86, use implicit differentiation to find *dy/dx*.

83. $x^2 - 3\ln y + y^2 = 10$

84. $\ln xy + 5x = 30$

85. $4x^3 + \ln y^2 + 2y = 2x$

86. $4xy + \ln x^2 y = 7$

In Exercises 87 and 88, use implicit differentiation to find an equation of the tangent line to the graph at the given point.

87. $x + y - 1 = \ln(x^2+y^2), \quad (1, 0)$

88. $y^2 + \ln xy = 2, \quad (e, 1)$

In Exercises 89 and 90, show that the function is a solution of the differential equation.

	Function	*Differential Equation*
89.	$y = 2\ln x + 3$	$xy'' + y' = 0$
90.	$y = x\ln x - 4x$	$x + y - xy' = 0$

In Exercises 91–96, locate any relative extrema and inflection points. Use a graphing utility to confirm your results.

91. $y = \dfrac{x^2}{2} - \ln x$

92. $y = x - \ln x$

93. $y = x\ln x$

94. $y = \dfrac{\ln x}{x}$

95. $y = \dfrac{x}{\ln x}$

96. $y = x^2 \ln \dfrac{x}{4}$

Linear and Quadratic Approximations **In Exercises 97 and 98, use a graphing utility to graph the function. Then graph**

$$P_1(x) = f(1) + f'(1)(x-1)$$

and

$$P_2(x) = f(1) + f'(1)(x-1) + \tfrac{1}{2}f''(1)(x-1)^2$$

in the same viewing window. Compare the values of *f*, P_1, and P_2 and their first derivatives at $x = 1$.

97. $f(x) = \ln x$

98. $f(x) = x\ln x$

In Exercises 99 and 100, use Newton's Method to approximate, to three decimal places, the *x*-coordinate of the point of intersection of the graphs of the two equations. Use a graphing utility to verify your result.

99. $y = \ln x, \quad y = -x$

100. $y = \ln x, \quad y = 3 - x$

In Exercises 101–106, use logarithmic differentiation to find *dy/dx*.

101. $y = x\sqrt{x^2+1}, \quad x > 0$

102. $y = \sqrt{x^2(x+1)(x+2)}, \quad x > 0$

103. $y = \dfrac{x^2\sqrt{3x-2}}{(x+1)^2}, \quad x > \dfrac{2}{3}$

104. $y = \sqrt{\dfrac{x^2-1}{x^2+1}}, \quad x > 1$

105. $y = \dfrac{x(x-1)^{3/2}}{\sqrt{x+1}}, \quad x > 1$

106. $y = \dfrac{(x+1)(x-2)}{(x-1)(x+2)}, \quad x > 2$

WRITING ABOUT CONCEPTS

107. In your own words, state the properties of the natural logarithmic function.

108. Define the base for the natural logarithmic function.

109. Let f be a function that is positive and differentiable on the entire real line. Let $g(x) = \ln f(x)$.

(a) If g is increasing, must f be increasing? Explain.

(b) If the graph of f is concave upward, must the graph of g be concave upward? Explain.

110. Consider the function $f(x) = x - 2\ln x$ on $[1, 3]$.

(a) Explain why Rolle's Theorem (Section 3.2) does not apply.

(b) Do you think the conclusion of Rolle's Theorem is true for f? Explain

True or False? **In Exercises 111–114, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

111. $\ln(x + 25) = \ln x + \ln 25$

112. $\ln xy = \ln x \ln y$

113. If $y = \ln \pi$, then $y' = 1/\pi$.

114. If $y = \ln e$, then $y' = 1$.

115. ***Home Mortgage*** The term t (in years) of a \$200,000 home mortgage at 7.5% interest can be approximated by

$$t = 13.375 \ln\left(\frac{x}{x - 1250}\right), \quad x > 1250$$

where x is the monthly payment in dollars.

(a) Use a graphing utility to graph the model.

(b) Use the model to approximate the term of a home mortgage for which the monthly payment is \$1398.43. What is the total amount paid?

(c) Use the model to approximate the term of a home mortgage for which the monthly payment is \$1611.19. What is the total amount paid?

(d) Find the instantaneous rates of change of t with respect to x when $x = \$1398.43$ and $x = \$1611.19$.

(e) Write a short paragraph describing the benefit of the higher monthly payment.

116. ***Sound Intensity*** The relationship between the number of decibels β and the intensity of a sound I in watts per centimeter squared is

$$\beta = 10 \log_{10}\left(\frac{I}{10^{-16}}\right).$$

Use the properties of logarithms to write the formula in simpler form, and determine the number of decibels of a sound with an intensity of 10^{-10} watt per square centimeter.

117. ***Modeling Data*** The table shows the temperatures T (°F) at which water boils at selected pressures p (pounds per square inch). *(Source: Standard Handbook of Mechanical Engineers)*

p	5	10	14.696 (1 atm)	20	30	40	60	80	100
T	162.24°	193.21°	212.00°	227.96°	250.33°	267.25°	292.71°	312.03°	327.81°

A model that approximates the data is

$$T = 87.97 + 34.96 \ln p + 7.91\sqrt{p}.$$

(a) Use a graphing utility to plot the data and graph the model.

(b) Find the rates of change of T with respect to p when $p = 10$ and $p = 70$.

(c) Use a graphing utility to graph T'. Find $\lim\limits_{p\to\infty} T'(p)$ and interpret the result in the context of the problem.

118. ***Modeling Data*** The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is one atmosphere (1.033227 kilograms per square centimeter). The table shows the pressures p (in atmospheres) at selected altitudes h (in kilometers).

h	0	5	10	15	20	25
p	1	0.55	0.25	0.12	0.06	0.02

(a) Use a graphing utility to find a model of the form $p = a + b \ln h$ for the data. Explain why the result is an error message.

(b) Use a graphing utility to find the logarithmic model $h = a + b \ln p$ for the data.

(c) Use a graphing utility to plot the data and graph the model.

(d) Use the model to estimate the altitude when $p = 0.75$.

(e) Use the model to estimate the pressure when $h = 13$.

(f) Use the model to find the rates of change of pressure when $h = 5$ and $h = 20$. Interpret the results.

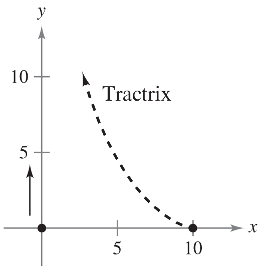
119. ***Tractrix*** A person walking along a dock drags a boat by a 10-meter rope. The boat travels along a path known as a *tractrix* (see figure). The equation of this path is

$$y = 10 \ln\left(\frac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2}.$$

(a) Use a graphing utility to graph the function.

(b) What are the slopes of this path when $x = 5$ and $x = 9$?

(c) What does the slope of the path approach as $x \to 10$?



CAPSTONE

120. Given that $f(x) = \ln x^a$, where a is a real number such that $a > 0$, determine the rates of change of f when (a) $x = 10$ and (b) $x = 100$.

121. ***Conjecture*** Use a graphing utility to graph f and g in the same viewing window and determine which is increasing at the greater rate for large values of x. What can you conclude about the rate of growth of the natural logarithmic function?

(a) $f(x) = \ln x, \quad g(x) = \sqrt{x}$ (b) $f(x) = \ln x, \quad g(x) = \sqrt[4]{x}$

122. To approximate e^x, you can use a function of the form $f(x) = \dfrac{a + bx}{1 + cx}$. (This function is known as a **Padé approximation.**) The values of $f(0)$, $f'(0)$, and $f''(0)$ are equal to the corresponding values of e^x. Show that these values are equal to 1 and find the values of a, b, and c such that $f(0) = f'(0) = f''(0) = 1$. Then use a graphing utility to compare the graphs of f and e^x.

5.2 The Natural Logarithmic Function: Integration

- **Use the Log Rule for Integration to integrate a rational function.**
- **Integrate trigonometric functions.**

EXPLORATION

Integrating Rational Functions Early in Chapter 4, you learned rules that allowed you to integrate *any* polynomial function. The Log Rule presented in this section goes a long way toward enabling you to integrate rational functions. For instance, each of the following functions can be integrated with the Log Rule.

$\dfrac{2}{x}$	Example 1
$\dfrac{1}{4x-1}$	Example 2
$\dfrac{x}{x^2+1}$	Example 3
$\dfrac{3x^2+1}{x^3+x}$	Example 4(a)
$\dfrac{x+1}{x^2+2x}$	Example 4(c)
$\dfrac{1}{3x+2}$	Example 4(d)
$\dfrac{x^2+x+1}{x^2+1}$	Example 5
$\dfrac{2x}{(x+1)^2}$	Example 6

There are still some rational functions that cannot be integrated using the Log Rule. Give examples of these functions, and explain your reasoning.

Log Rule for Integration

The differentiation rules

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \qquad \text{and} \qquad \frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$

that you studied in the preceding section produce the following integration rule.

THEOREM 5.5 LOG RULE FOR INTEGRATION

Let u be a differentiable function of x.

1. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$ **2.** $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

Because $du = u'\,dx$, the second formula can also be written as

$$\int \frac{u'}{u}\,dx = \ln|u| + C. \qquad \text{Alternative form of Log Rule}$$

EXAMPLE 1 Using the Log Rule for Integration

$$\begin{aligned}
\int \frac{2}{x}\,dx &= 2\int \frac{1}{x}\,dx && \text{Constant Multiple Rule}\\
&= 2\ln|x| + C && \text{Log Rule for Integration}\\
&= \ln(x^2) + C && \text{Property of logarithms}
\end{aligned}$$

Because x^2 cannot be negative, the absolute value notation is unnecessary in the final form of the antiderivative.

EXAMPLE 2 Using the Log Rule with a Change of Variables

Find $\displaystyle\int \frac{1}{4x-1}\,dx$.

Solution If you let $u = 4x - 1$, then $du = 4\,dx$.

$$\begin{aligned}
\int \frac{1}{4x-1}\,dx &= \frac{1}{4}\int \left(\frac{1}{4x-1}\right)4\,dx && \text{Multiply and divide by 4.}\\
&= \frac{1}{4}\int \frac{1}{u}\,du && \text{Substitute: } u = 4x - 1.\\
&= \frac{1}{4}\ln|u| + C && \text{Apply Log Rule.}\\
&= \frac{1}{4}\ln|4x-1| + C && \text{Back-substitute.}
\end{aligned}$$

■

Example 3 uses the alternative form of the Log Rule. To apply this rule, look for quotients in which the numerator is the derivative of the denominator.

EXAMPLE 3 Finding Area with the Log Rule

Find the area of the region bounded by the graph of

$$y = \frac{x}{x^2 + 1}$$

the x-axis, and the line $x = 3$.

Solution In Figure 5.8, you can see that the area of the region is given by the definite integral

$$\int_0^3 \frac{x}{x^2 + 1}\,dx.$$

If you let $u = x^2 + 1$, then $u' = 2x$. To apply the Log Rule, multiply and divide by 2 as shown.

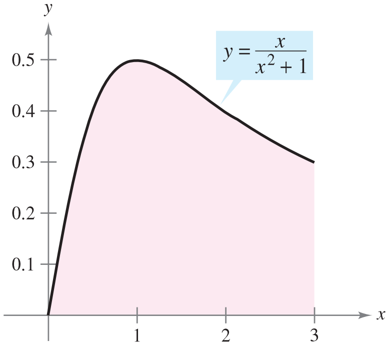
$$\int_0^3 \frac{x}{x^2 + 1}\,dx = \frac{1}{2}\int_0^3 \frac{2x}{x^2 + 1}\,dx \qquad \text{Multiply and divide by 2.}$$

$$= \frac{1}{2}\Big[\ln(x^2 + 1)\Big]_0^3 \qquad \int \frac{u'}{u}\,dx = \ln|u| + C$$

$$= \frac{1}{2}(\ln 10 - \ln 1)$$

$$= \frac{1}{2}\ln 10 \qquad \ln 1 = 0$$

$$\approx 1.151$$



$$\text{Area} = \int_0^3 \frac{x}{x^2 + 1}\,dx$$

The area of the region bounded by the graph of y, the x-axis, and $x = 3$ is $\frac{1}{2}\ln 10$.
Figure 5.8

EXAMPLE 4 Recognizing Quotient Forms of the Log Rule

a. $\displaystyle\int \frac{3x^2 + 1}{x^3 + x}\,dx = \ln|x^3 + x| + C \qquad u = x^3 + x$

b. $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx = \ln|\tan x| + C \qquad u = \tan x$

c. $\displaystyle\int \frac{x + 1}{x^2 + 2x}\,dx = \frac{1}{2}\int \frac{2x + 2}{x^2 + 2x}\,dx \qquad u = x^2 + 2x$

$$= \frac{1}{2}\ln|x^2 + 2x| + C$$

d. $\displaystyle\int \frac{1}{3x + 2}\,dx = \frac{1}{3}\int \frac{3}{3x + 2}\,dx \qquad u = 3x + 2$

$$= \frac{1}{3}\ln|3x + 2| + C$$

■

With antiderivatives involving logarithms, it is easy to obtain forms that look quite different but are still equivalent. For instance, both of the following are equivalent to the antiderivative listed in Example 4(d).

$$\ln|(3x + 2)^{1/3}| + C \qquad \text{and} \qquad \ln|3x + 2|^{1/3} + C$$

Integrals to which the Log Rule can be applied often appear in disguised form. For instance, if a rational function has a *numerator of degree greater than or equal to that of the denominator*, division may reveal a form to which you can apply the Log Rule. This is shown in Example 5.

EXAMPLE 5 Using Long Division Before Integrating

Find $\displaystyle\int \frac{x^2+x+1}{x^2+1}\,dx$.

Solution Begin by using long division to rewrite the integrand.

$$\frac{x^2+x+1}{x^2+1} \quad \Rightarrow \quad \begin{array}{r} 1 \\ x^2+1\,\overline{\smash{)}\,x^2+x+1} \\ \underline{x^2 \qquad\;\; +1} \\ x \qquad \end{array} \quad \Rightarrow \quad 1+\frac{x}{x^2+1}$$

Now, you can integrate to obtain

$$\begin{aligned}
\int \frac{x^2+x+1}{x^2+1}\,dx &= \int\left(1+\frac{x}{x^2+1}\right)dx && \text{Rewrite using long division.}\\
&= \int dx + \frac{1}{2}\int\frac{2x}{x^2+1}\,dx && \text{Rewrite as two integrals.}\\
&= x + \frac{1}{2}\ln(x^2+1) + C. && \text{Integrate.}
\end{aligned}$$

Check this result by differentiating to obtain the original integrand. ■

The next example presents another instance in which the use of the Log Rule is disguised. In this case, a change of variables helps you recognize the Log Rule.

EXAMPLE 6 Change of Variables with the Log Rule

Find $\displaystyle\int \frac{2x}{(x+1)^2}\,dx$.

Solution If you let $u = x+1$, then $du = dx$ and $x = u-1$.

$$\begin{aligned}
\int \frac{2x}{(x+1)^2}\,dx &= \int \frac{2(u-1)}{u^2}\,du && \text{Substitute.}\\
&= 2\int\left(\frac{u}{u^2}-\frac{1}{u^2}\right)du && \text{Rewrite as two fractions.}\\
&= 2\int\frac{du}{u} - 2\int u^{-2}\,du && \text{Rewrite as two integrals.}\\
&= 2\ln|u| - 2\left(\frac{u^{-1}}{-1}\right) + C && \text{Integrate.}\\
&= 2\ln|u| + \frac{2}{u} + C && \text{Simplify.}\\
&= 2\ln|x+1| + \frac{2}{x+1} + C && \text{Back-substitute.}
\end{aligned}$$

Check this result by differentiating to obtain the original integrand. ■

TECHNOLOGY If you have access to a computer algebra system, use it to find the indefinite integrals in Examples 5 and 6. How does the form of the antiderivative that it gives you compare with that given in Examples 5 and 6?

As you study the methods shown in Examples 5 and 6, be aware that both methods involve rewriting a disguised integrand so that it fits one or more of the basic integration formulas. Throughout the remaining sections of Chapter 5 and in Chapter 8, much time will be devoted to integration techniques. To master these techniques, you must recognize the "form-fitting" nature of integration. In this sense, integration is not nearly as straightforward as differentiation. Differentiation takes the form

"Here is the question; what is the answer?"

Integration is more like

"Here is the answer; what is the question?"

The following are guidelines you can use for integration.

STUDY TIP Keep in mind that you can check your answer to an integration problem by differentiating the answer. For instance, in Example 7, the derivative of $y = \ln|\ln x| + C$ is $y' = 1/(x \ln x)$.

GUIDELINES FOR INTEGRATION

1. Learn a basic list of integration formulas. (Including those given in this section, you now have 12 formulas: the Power Rule, the Log Rule, and ten trigonometric rules. By the end of Section 5.7, this list will have expanded to 20 basic rules.)
2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of u that will make the integrand conform to the formula.
3. If you cannot find a u-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division. Be creative.
4. If you have access to computer software that will find antiderivatives symbolically, use it.

EXAMPLE 7 *u*-Substitution and the Log Rule

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x \ln x}$.

Solution The solution can be written as an indefinite integral.

$$y = \int \frac{1}{x \ln x}\, dx$$

Because the integrand is a quotient whose denominator is raised to the first power, you should try the Log Rule. There are three basic choices for u. The choices $u = x$ and $u = x \ln x$ fail to fit the u'/u form of the Log Rule. However, the third choice does fit. Letting $u = \ln x$ produces $u' = 1/x$, and you obtain the following.

$$\int \frac{1}{x \ln x}\, dx = \int \frac{1/x}{\ln x}\, dx \qquad \text{Divide numerator and denominator by } x.$$

$$= \int \frac{u'}{u}\, dx \qquad \text{Substitute: } u = \ln x.$$

$$= \ln|u| + C \qquad \text{Apply Log Rule.}$$

$$= \ln|\ln x| + C \qquad \text{Back-substitute.}$$

So, the solution is $y = \ln|\ln x| + C$. ■

Integrals of Trigonometric Functions

In Section 4.1, you looked at six trigonometric integration rules—the six that correspond directly to differentiation rules. With the Log Rule, you can now complete the set of basic trigonometric integration formulas.

EXAMPLE 8 Using a Trigonometric Identity

Find $\int \tan x \, dx$.

Solution This integral does not seem to fit any formulas on our basic list. However, by using a trigonometric identity, you obtain

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx.$$

Knowing that $D_x[\cos x] = -\sin x$, you can let $u = \cos x$ and write

$$\begin{aligned}
\int \tan x \, dx &= -\int \frac{-\sin x}{\cos x} \, dx && \text{Trigonometric identity} \\
&= -\int \frac{u'}{u} \, dx && \text{Substitute: } u = \cos x. \\
&= -\ln|u| + C && \text{Apply Log Rule.} \\
&= -\ln|\cos x| + C. && \text{Back-substitute.}
\end{aligned}$$

■

Example 8 uses a trigonometric identity to derive an integration rule for the tangent function. The next example takes a rather unusual step (multiplying and dividing by the same quantity) to derive an integration rule for the secant function.

EXAMPLE 9 Derivation of the Secant Formula

Find $\int \sec x \, dx$.

Solution Consider the following procedure.

$$\begin{aligned}
\int \sec x \, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx
\end{aligned}$$

Letting u be the denominator of this quotient produces

$$u = \sec x + \tan x \quad \Rightarrow \quad u' = \sec x \tan x + \sec^2 x.$$

So, you can conclude that

$$\begin{aligned}
\int \sec x \, dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx && \text{Rewrite integrand.} \\
&= \int \frac{u'}{u} \, dx && \text{Substitute: } u = \sec x + \tan x. \\
&= \ln|u| + C && \text{Apply Log Rule.} \\
&= \ln|\sec x + \tan x| + C. && \text{Back-substitute.}
\end{aligned}$$

■

With the results of Examples 8 and 9, you now have integration formulas for $\sin x$, $\cos x$, $\tan x$, and $\sec x$. All six trigonometric rules are summarized below. (For proofs of $\cot u$ and $\csc u$, see Exercises 91 and 92.)

NOTE Using trigonometric identities and properties of logarithms, you could rewrite these six integration rules in other forms. For instance, you could write

$$\int \csc u\,du = \ln|\csc u - \cot u| + C.$$

(See Exercises 93–96.)

INTEGRALS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

EXAMPLE 10 Integrating Trigonometric Functions

Evaluate $\displaystyle\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,dx$.

Solution Using $1 + \tan^2 x = \sec^2 x$, you can write

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,dx &= \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx \\
&= \int_0^{\pi/4} \sec x\,dx && \sec x \ge 0 \text{ for } 0 \le x \le \frac{\pi}{4}. \\
&= \ln|\sec x + \tan x|\Big]_0^{\pi/4} \\
&= \ln\left(\sqrt{2} + 1\right) - \ln 1 \\
&\approx 0.881.
\end{aligned}$$

EXAMPLE 11 Finding an Average Value

Find the average value of $f(x) = \tan x$ on the interval $\left[0, \dfrac{\pi}{4}\right]$.

Solution

$$\begin{aligned}
\text{Average value} &= \frac{1}{(\pi/4) - 0}\int_0^{\pi/4} \tan x\,dx && \text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx \\
&= \frac{4}{\pi}\int_0^{\pi/4} \tan x\,dx && \text{Simplify.} \\
&= \frac{4}{\pi}\Big[-\ln|\cos x|\Big]_0^{\pi/4} && \text{Integrate.} \\
&= -\frac{4}{\pi}\left[\ln\left(\frac{\sqrt{2}}{2}\right) - \ln(1)\right] \\
&= -\frac{4}{\pi}\ln\left(\frac{\sqrt{2}}{2}\right) \\
&\approx 0.441
\end{aligned}$$

The average value is about 0.441, as shown in Figure 5.9. ■

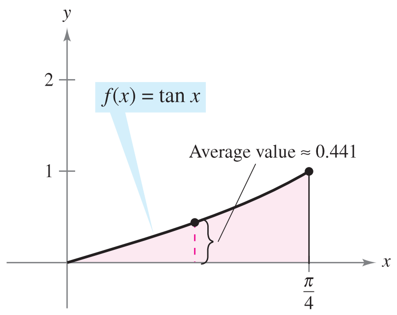


Figure 5.9

5.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–26, find the indefinite integral.

1. $\displaystyle\int \frac{5}{x}\,dx$

2. $\displaystyle\int \frac{10}{x}\,dx$

3. $\displaystyle\int \frac{1}{x+1}\,dx$

4. $\displaystyle\int \frac{1}{x-5}\,dx$

5. $\displaystyle\int \frac{1}{2x+5}\,dx$

6. $\displaystyle\int \frac{1}{4-3x}\,dx$

7. $\displaystyle\int \frac{x}{x^2-3}\,dx$

8. $\displaystyle\int \frac{x^2}{5-x^3}\,dx$

9. $\displaystyle\int \frac{4x^3+3}{x^4+3x}\,dx$

10. $\displaystyle\int \frac{x^2-2x}{x^3-3x^2}\,dx$

11. $\displaystyle\int \frac{x^2-4}{x}\,dx$

12. $\displaystyle\int \frac{x}{\sqrt{9-x^2}}\,dx$

13. $\displaystyle\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx$

14. $\displaystyle\int \frac{x(x+2)}{x^3+3x^2-4}\,dx$

15. $\displaystyle\int \frac{x^2-3x+2}{x+1}\,dx$

16. $\displaystyle\int \frac{2x^2+7x-3}{x-2}\,dx$

17. $\displaystyle\int \frac{x^3-3x^2+5}{x-3}\,dx$

18. $\displaystyle\int \frac{x^3-6x-20}{x+5}\,dx$

19. $\displaystyle\int \frac{x^4+x-4}{x^2+2}\,dx$

20. $\displaystyle\int \frac{x^3-3x^2+4x-9}{x^2+3}\,dx$

21. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

22. $\displaystyle\int \frac{1}{x\ln x^3}\,dx$

23. $\displaystyle\int \frac{1}{\sqrt{x}+1}\,dx$

24. $\displaystyle\int \frac{1}{x^{2/3}(1+x^{1/3})}\,dx$

25. $\displaystyle\int \frac{2x}{(x-1)^2}\,dx$

26. $\displaystyle\int \frac{x(x-2)}{(x-1)^3}\,dx$

In Exercises 27–30, find the indefinite integral by *u*-substitution. (*Hint:* Let *u* be the denominator of the integrand.)

27. $\displaystyle\int \frac{1}{1+\sqrt{2x}}\,dx$

28. $\displaystyle\int \frac{1}{1+\sqrt{3x}}\,dx$

29. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx$

30. $\displaystyle\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx$

In Exercises 31–40, find the indefinite integral.

31. $\displaystyle\int \cot\frac{\theta}{3}\,d\theta$

32. $\displaystyle\int \tan 5\theta\,d\theta$

33. $\displaystyle\int \csc 2x\,dx$

34. $\displaystyle\int \sec\frac{x}{2}\,dx$

35. $\displaystyle\int (\cos 3\theta - 1)\,d\theta$

36. $\displaystyle\int \left(2-\tan\frac{\theta}{4}\right)d\theta$

37. $\displaystyle\int \frac{\cos t}{1+\sin t}\,dt$

38. $\displaystyle\int \frac{\csc^2 t}{\cot t}\,dt$

39. $\displaystyle\int \frac{\sec x\tan x}{\sec x-1}\,dx$

40. $\displaystyle\int (\sec 2x+\tan 2x)\,dx$

In Exercises 41–46, solve the differential equation. Use a graphing utility to graph three solutions, one of which passes through the given point.

41. $\dfrac{dy}{dx}=\dfrac{4}{x},\quad (1, 2)$

42. $\dfrac{dy}{dx}=\dfrac{x-2}{x},\quad (-1, 0)$

43. $\dfrac{dy}{dx}=\dfrac{3}{2-x},\quad (1, 0)$

44. $\dfrac{dy}{dx}=\dfrac{2x}{x^2-9},\quad (0, 4)$

45. $\dfrac{ds}{d\theta}=\tan 2\theta,\quad (0, 2)$

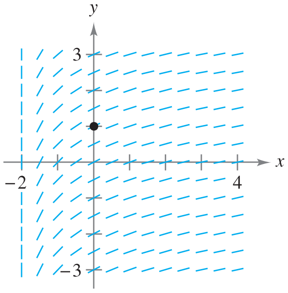
46. $\dfrac{dr}{dt}=\dfrac{\sec^2 t}{\tan t+1},\quad (\pi, 4)$

47. Determine the function f if $f''(x)=\dfrac{2}{x^2}$, $f(1)=1$, $f'(1)=1$, $x>0$.

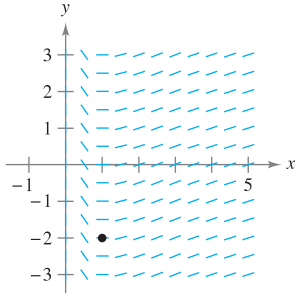
48. Determine the function f if $f''(x)=-\dfrac{4}{(x-1)^2}-2$, $f(2)=3$, $f'(2)=0$, $x>1$.

Slope Fields **In Exercises 49–52, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

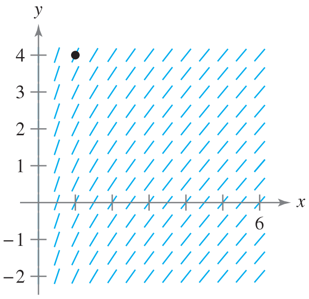
49. $\dfrac{dy}{dx}=\dfrac{1}{x+2},\quad (0, 1)$



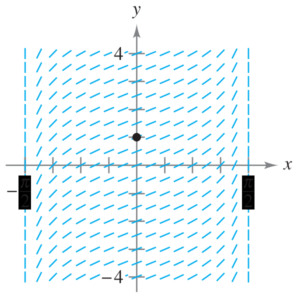
50. $\dfrac{dy}{dx}=\dfrac{\ln x}{x},\quad (1, -2)$



51. $\dfrac{dy}{dx}=1+\dfrac{1}{x},\quad (1, 4)$



52. $\dfrac{dy}{dx}=\sec x,\quad (0, 1)$



In Exercises 53–60, evaluate the definite integral. Use a graphing utility to verify your result.

53. $\displaystyle\int_0^4 \frac{5}{3x+1}\,dx$

54. $\displaystyle\int_{-1}^1 \frac{1}{2x+3}\,dx$

55. $\displaystyle\int_1^e \frac{(1+\ln x)^2}{x}\,dx$

56. $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx$

57. $\displaystyle\int_0^2 \frac{x^2-2}{x+1}\,dx$

58. $\displaystyle\int_0^1 \frac{x-1}{x+1}\,dx$

59. $\displaystyle\int_1^2 \frac{1-\cos\theta}{\theta-\sin\theta}\,d\theta$

60. $\displaystyle\int_{0.1}^{0.2} (\csc 2\theta-\cot 2\theta)^2\,d\theta$

CAS **In Exercises 61–66, use a computer algebra system to find or evaluate the integral.**

61. $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$

62. $\displaystyle\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

63. $\displaystyle\int \frac{\sqrt{x}}{x-1}\,dx$

64. $\displaystyle\int \frac{x^2}{x-1}\,dx$

65. $\displaystyle\int_{\pi/4}^{\pi/2} (\csc x-\sin x)\,dx$

66. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin^2 x-\cos^2 x}{\cos x}\,dx$

In Exercises 67–70, find $F'(x)$.

67. $\displaystyle F(x)=\int_1^x \frac{1}{t}\,dt$

68. $\displaystyle F(x)=\int_0^x \tan t\,dt$

69. $\displaystyle F(x)=\int_1^{3x} \frac{1}{t}\,dt$

70. $\displaystyle F(x)=\int_1^{x^2} \frac{1}{t}\,dt$

Approximation **In Exercises 71 and 72, determine which value best approximates the area of the region between the x-axis and the graph of the function over the given interval. (Make your selection on the basis of a sketch of the region and not by performing any calculations.)**

71. $f(x)=\sec x,\quad [0, 1]$

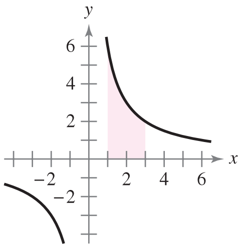
(a) 6 (b) -6 (c) $\frac{1}{2}$ (d) 1.25 (e) 3

72. $f(x)=\dfrac{2x}{x^2+1},\quad [0, 4]$

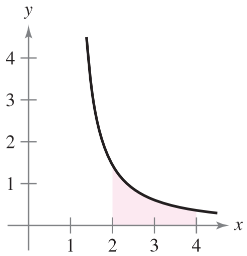
(a) 3 (b) 7 (c) -2 (d) 5 (e) 1

Area **In Exercises 73–76, find the area of the given region. Use a graphing utility to verify your result.**

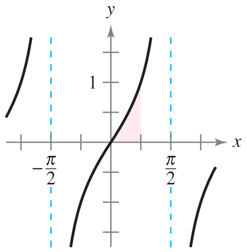
73. $y=\dfrac{6}{x}$



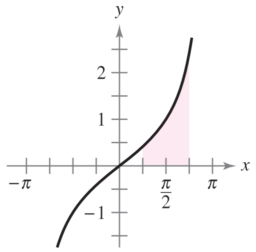
74. $y=\dfrac{2}{x\ln x}$



75. $y=\tan x$



76. $y=\dfrac{\sin x}{1+\cos x}$



Area **In Exercises 77–80, find the area of the region bounded by the graphs of the equations. Use a graphing utility to verify your result.**

77. $y=\dfrac{x^2+4}{x},\ x=1,\ x=4,\ y=0$

78. $y=\dfrac{x+6}{x},\ x=1,\ x=5,\ y=0$

79. $y=2\sec\dfrac{\pi x}{6},\ x=0,\ x=2,\ y=0$

80. $y=2x-\tan 0.3x,\ x=1,\ x=4,\ y=0$

Numerical Integration **In Exercises 81–84, use the Trapezoidal Rule and Simpson's Rule to approximate the value of the definite integral. Let $n=4$ and round your answer to four decimal places. Use a graphing utility to verify your result.**

81. $\displaystyle\int_1^5 \frac{12}{x}\,dx$

82. $\displaystyle\int_0^4 \frac{8x}{x^2+4}\,dx$

83. $\displaystyle\int_2^6 \ln x\,dx$

84. $\displaystyle\int_{-\pi/3}^{\pi/3} \sec x\,dx$

WRITING ABOUT CONCEPTS

In Exercises 85–88, state the integration formula you would use to perform the integration. Do not integrate.

85. $\displaystyle\int \sqrt[3]{x}\,dx$

86. $\displaystyle\int \frac{x}{(x^2+4)^3}\,dx$

87. $\displaystyle\int \frac{x}{x^2+4}\,dx$

88. $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx$

89. Find a value of x such that $\displaystyle\int_1^x \frac{3}{t}\,dt=\int_{1/4}^x \frac{1}{t}\,dt.$

CAPSTONE

90. Find a value of x such that

$$\int_1^x \frac{1}{t}\,dt$$

is equal to (a) $\ln 5$ and (b) 1.

91. Show that $\int \cot u\,du = \ln|\sin u| + C$.

92. Show that $\int \csc u\,du = -\ln|\csc u + \cot u| + C$.

In Exercises 93–96, show that the two formulas are equivalent.

93. $\int \tan x\,dx = -\ln|\cos x| + C$

$\int \tan x\,dx = \ln|\sec x| + C$

94. $\int \cot x\,dx = \ln|\sin x| + C$

$\int \cot x\,dx = -\ln|\csc x| + C$

95. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

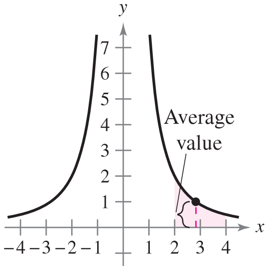
$\int \sec x\,dx = -\ln|\sec x - \tan x| + C$

96. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

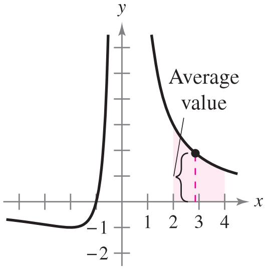
$\int \csc x\,dx = \ln|\csc x - \cot x| + C$

In Exercises 97–100, find the average value of the function over the given interval.

97. $f(x) = \dfrac{8}{x^2}$, $[2, 4]$



98. $f(x) = \dfrac{4(x+1)}{x^2}$, $[2, 4]$



99. $f(x) = \dfrac{2\ln x}{x}$, $[1, e]$

100. $f(x) = \sec\dfrac{\pi x}{6}$, $[0, 2]$

101. ***Population Growth*** A population of bacteria is changing at a rate of

$$\frac{dP}{dt} = \frac{3000}{1 + 0.25t}$$

where t is the time in days. The initial population (when $t = 0$) is 1000. Write an equation that gives the population at any time t, and find the population when $t = 3$ days.

102. ***Heat Transfer*** Find the time required for an object to cool from 300°F to 250°F by evaluating

$$t = \frac{10}{\ln 2}\int_{250}^{300} \frac{1}{T - 100}\,dT$$

where t is time in minutes.

103. ***Average Price*** The demand equation for a product is

$$p = \frac{90{,}000}{400 + 3x}.$$

Find the *average* price p on the interval $40 \le x \le 50$.

104. ***Sales*** The rate of change in sales S is inversely proportional to time t $(t > 1)$ measured in weeks. Find S as a function of t if sales after 2 and 4 weeks are 200 units and 300 units, respectively.

True or False? **In Exercises 105–108, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

105. $(\ln x)^{1/2} = \frac{1}{2}(\ln x)$

106. $\int \ln x\,dx = (1/x) + C$

107. $\int \frac{1}{x}\,dx = \ln|cx|, \quad c \neq 0$

108. $\int_{-1}^{2} \frac{1}{x}\,dx = \Big[\ln|x|\Big]_{-1}^{2} = \ln 2 - \ln 1 = \ln 2$

109. ***Orthogonal Trajectory***

(a) Use a graphing utility to graph the equation $2x^2 - y^2 = 8$.

(b) Evaluate the integral to find y^2 in terms of x.

$$y^2 = e^{-\int(1/x)\,dx}$$

For a particular value of the constant of integration, graph the result in the same viewing window used in part (a).

(c) Verify that the tangents to the graphs in parts (a) and (b) are perpendicular at the points of intersection.

110. Graph the function

$$f(x) = \frac{x}{1 + x^2}$$

on the interval $[0, \infty)$.

(a) Find the area bounded by the graph of f and the line $y = \frac{1}{2}x$.

(b) Determine the values of the slope m such that the line $y = mx$ and the graph of f enclose a finite region.

(c) Calculate the area of this region as a function of m.

111. ***Napier's Inequality*** For $0 < x < y$, show that

$$\frac{1}{y} < \frac{\ln y - \ln x}{y - x} < \frac{1}{x}.$$

112. Prove that the function

$$F(x) = \int_{x}^{2x} \frac{1}{t}\,dt$$

is constant on the interval $(0, \infty)$.

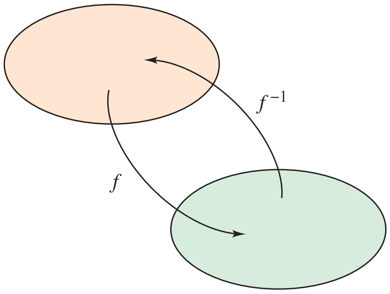
5.3 Inverse Functions

- **Verify that one function is the inverse function of another function.**
- **Determine whether a function has an inverse function.**
- **Find the derivative of an inverse function.**

Inverse Functions

Recall from Section P.3 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 3$ from $A = \{1, 2, 3, 4\}$ to $B = \{4, 5, 6, 7\}$ can be written as

$$f: \{(1, 4), (2, 5), (3, 6), (4, 7)\}.$$



Domain of f = range of f^{-1}
Domain of f^{-1} = range of f
Figure 5.10

By interchanging the first and second coordinates of each ordered pair, you can form the **inverse function** of f. This function is denoted by f^{-1}. It is a function from B to A, and can be written as

$$f^{-1}: \{(4, 1), (5, 2), (6, 3), (7, 4)\}.$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 5.10. The functions f and f^{-1} have the effect of "undoing" each other. That is, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x$$

EXPLORATION

Finding Inverse Functions
Explain how to "undo" each of the following functions. Then use your explanation to write the inverse function of f.

a. $f(x) = x - 5$
b. $f(x) = 6x$
c. $f(x) = \dfrac{x}{2}$
d. $f(x) = 3x + 2$
e. $f(x) = x^3$
f. $f(x) = 4(x - 2)$

Use a graphing utility to graph each function and its inverse function in the same "square" viewing window. What observation can you make about each pair of graphs?

DEFINITION OF INVERSE FUNCTION

A function g is the **inverse function** of the function f if

$$f(g(x)) = x \quad \text{for each } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \quad \text{for each } x \text{ in the domain of } f.$$

The function g is denoted by f^{-1} (read "f inverse").

NOTE Although the notation used to denote an inverse function resembles *exponential notation*, it is a different use of -1 as a superscript. That is, in general, $f^{-1}(x) \neq 1/f(x)$. ■

Here are some important observations about inverse functions.

1. If g is the inverse function of f, then f is the inverse function of g.
2. The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
3. A function need not have an inverse function, but if it does, the inverse function is unique (see Exercise 108).

You can think of f^{-1} as undoing what has been done by f. For example, subtraction can be used to undo addition, and division can be used to undo multiplication. Use the definition of an inverse function to check the following.

$$f(x) = x + c \qquad \text{and} \qquad f^{-1}(x) = x - c \quad \text{are inverse functions of each other.}$$

$$f(x) = cx \qquad \text{and} \qquad f^{-1}(x) = \frac{x}{c},\ c \neq 0, \quad \text{are inverse functions of each other.}$$

EXAMPLE 1 Verifying Inverse Functions

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \qquad \text{and} \qquad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

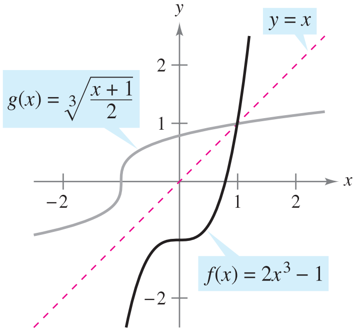
Solution Because the domains and ranges of both f and g consist of all real numbers, you can conclude that both composite functions exist for all x. The composition of f with g is given by

$$\begin{aligned} f(g(x)) &= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \\ &= 2\left(\frac{x+1}{2}\right) - 1 \\ &= x + 1 - 1 \\ &= x. \end{aligned}$$

The composition of g with f is given by

$$\begin{aligned} g(f(x)) &= \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}} \\ &= \sqrt[3]{\frac{2x^3}{2}} \\ &= \sqrt[3]{x^3} \\ &= x. \end{aligned}$$

Because $f(g(x)) = x$ and $g(f(x)) = x$, you can conclude that f and g are inverse functions of each other (see Figure 5.11).



f and *g* are inverse functions of each other.
Figure 5.11

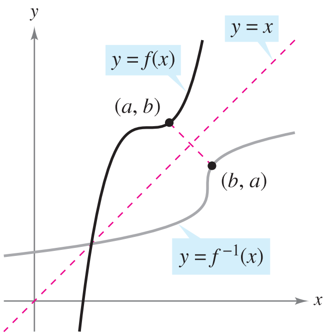
STUDY TIP In Example 1, try comparing the functions f and g verbally.

For f: First cube x, then multiply by 2, then subtract 1.

For g: First add 1, then divide by 2, then take the cube root.

Do you see the "undoing pattern"?

In Figure 5.11, the graphs of f and $g = f^{-1}$ appear to be mirror images of each other with respect to the line $y = x$. The graph of f^{-1} is a **reflection** of the graph of f in the line $y = x$. This idea is generalized in the following theorem.



The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.
Figure 5.12

THEOREM 5.6 REFLECTIVE PROPERTY OF INVERSE FUNCTIONS

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

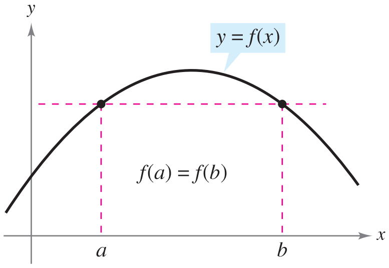
PROOF If (a, b) is on the graph of f, then $f(a) = b$ and you can write

$$f^{-1}(b) = f^{-1}(f(a)) = a.$$

So, (b, a) is on the graph of f^{-1}, as shown in Figure 5.12. A similar argument will prove the theorem in the other direction.

Existence of an Inverse Function

Not every function has an inverse function, and Theorem 5.6 suggests a graphical test for those that do—the **Horizontal Line Test** for an inverse function. This test states that a function f has an inverse function if and only if every horizontal line intersects the graph of f at most once (see Figure 5.13). The following theorem formally states why the Horizontal Line Test is valid. (Recall from Section 3.3 that a function is *strictly monotonic* if it is either increasing on its entire domain or decreasing on its entire domain.)



If a horizontal line intersects the graph of f twice, then f is not one-to-one.
Figure 5.13

THEOREM 5.7 THE EXISTENCE OF AN INVERSE FUNCTION

1. A function has an inverse function if and only if it is one-to-one.
2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

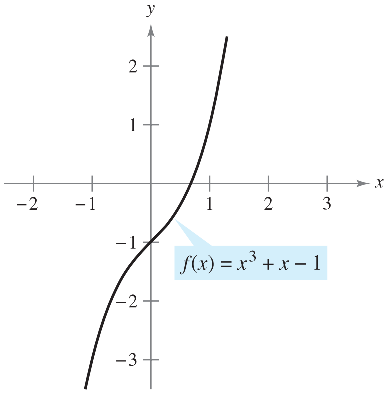
PROOF To prove the second part of the theorem, recall from Section P.3 that f is one-to-one if for x_1 and x_2 in its domain

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

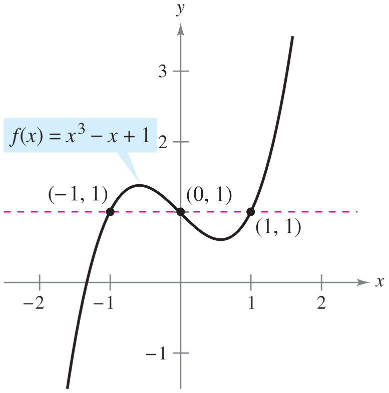
Now, choose x_1 and x_2 in the domain of f. If $x_1 \neq x_2$, then, because f is strictly monotonic, it follows that either

$$f(x_1) < f(x_2) \qquad \text{or} \qquad f(x_1) > f(x_2).$$

In either case, $f(x_1) \neq f(x_2)$. So, f is one-to-one on the interval. The proof of the first part of the theorem is left as an exercise (see Exercise 109). ■



(a) Because f is increasing over its entire domain, it has an inverse function.



(b) Because f is not one-to-one, it does not have an inverse function.
Figure 5.14

EXAMPLE 2 The Existence of an Inverse Function

Which of the functions has an inverse function?

a. $f(x) = x^3 + x - 1$ **b.** $f(x) = x^3 - x + 1$

Solution

a. From the graph of f shown in Figure 5.14(a), it appears that f is increasing over its entire domain. To verify this, note that the derivative, $f'(x) = 3x^2 + 1$, is positive for all real values of x. So, f is strictly monotonic and it must have an inverse function.

b. From the graph of f shown in Figure 5.14(b), you can see that the function does not pass the horizontal line test. In other words, it is not one-to-one. For instance, f has the same value when $x = -1$, 0, and 1.

$$f(-1) = f(1) = f(0) = 1 \qquad \text{Not one-to-one}$$

So, by Theorem 5.7, f does not have an inverse function. ■

NOTE Often it is easier to prove that a function *has* an inverse function than to find the inverse function. For instance, it would be difficult algebraically to find the inverse function of the function in Example 2(a). ■

The following guidelines suggest a procedure for finding an inverse function.

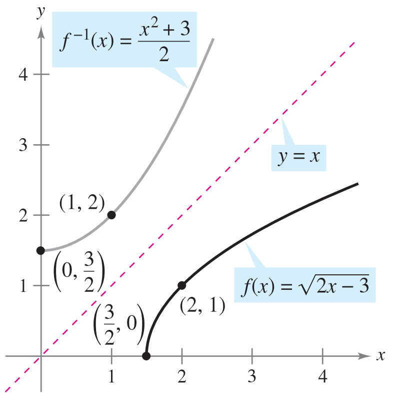
GUIDELINES FOR FINDING AN INVERSE FUNCTION

1. Use Theorem 5.7 to determine whether the function given by $y = f(x)$ has an inverse function.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} as the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

EXAMPLE 3 Finding an Inverse Function

Find the inverse function of

$$f(x) = \sqrt{2x - 3}.$$



The domain of f^{-1}, $[0, \infty)$, is the range of f.
Figure 5.15

Solution From the graph of f in Figure 5.15, it appears that f is increasing over its entire domain, $\left[\frac{3}{2}, \infty\right)$. To verify this, note that $f'(x) = \dfrac{1}{\sqrt{2x-3}}$ is positive on the domain of f. So, f is strictly monotonic and it must have an inverse function. To find an equation for the inverse function, let $y = f(x)$ and solve for x in terms of y.

$$\sqrt{2x-3} = y \qquad \text{Let } y = f(x).$$

$$2x - 3 = y^2 \qquad \text{Square each side.}$$

$$x = \frac{y^2+3}{2} \qquad \text{Solve for } x.$$

$$y = \frac{x^2+3}{2} \qquad \text{Interchange } x \text{ and } y.$$

$$f^{-1}(x) = \frac{x^2+3}{2} \qquad \text{Replace } y \text{ by } f^{-1}(x).$$

The domain of f^{-1} is the range of f, which is $[0, \infty)$. You can verify this result as shown.

$$f(f^{-1}(x)) = \sqrt{2\left(\frac{x^2+3}{2}\right) - 3} = \sqrt{x^2} = x, \quad x \ge 0$$

$$f^{-1}(f(x)) = \frac{\left(\sqrt{2x-3}\right)^2 + 3}{2} = \frac{2x-3+3}{2} = x, \quad x \ge \frac{3}{2}$$

■

NOTE Remember that any letter can be used to represent the independent variable. So,

$$f^{-1}(y) = \frac{y^2+3}{2}$$

$$f^{-1}(x) = \frac{x^2+3}{2}$$

$$f^{-1}(s) = \frac{s^2+3}{2}$$

all represent the same function. ■

Theorem 5.7 is useful in the following type of problem. Suppose you are given a function that is *not* one-to-one on its domain. By restricting the domain to an interval on which the function is strictly monotonic, you can conclude that the new function *is* one-to-one on the restricted domain.

EXAMPLE 4 Testing Whether a Function Is One-to-One

Show that the sine function

$$f(x) = \sin x$$

is not one-to-one on the entire real line. Then show that $[-\pi/2, \pi/2]$ is the largest interval, centered at the origin, on which f is strictly monotonic.

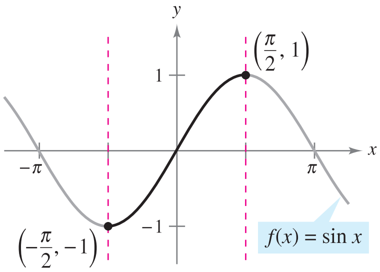
Solution It is clear that f is not one-to-one, because many different x-values yield the same y-value. For instance,

$$\sin(0) = 0 = \sin(\pi).$$

Moreover, f is increasing on the open interval $(-\pi/2, \pi/2)$, because its derivative

$$f'(x) = \cos x$$

is positive there. Finally, because the left and right endpoints correspond to relative extrema of the sine function, you can conclude that f is increasing on the closed interval $[-\pi/2, \pi/2]$ *and* that on any larger interval the function is not strictly monotonic (see Figure 5.16). ■



f is one-to-one on the interval $[-\pi/2, \pi/2]$.
Figure 5.16

Derivative of an Inverse Function

The next two theorems discuss the derivative of an inverse function. The reasonableness of Theorem 5.8 follows from the reflective property of inverse functions, as shown in Figure 5.12. Proofs of the two theorems are given in Appendix A.

THEOREM 5.8 CONTINUITY AND DIFFERENTIABILITY OF INVERSE FUNCTIONS

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

EXPLORATION

Graph the inverse functions

$$f(x) = x^3$$

and

$$g(x) = x^{1/3}.$$

Calculate the slopes of f at $(1, 1)$, $(2, 8)$, and $(3, 27)$, and the slopes of g at $(1, 1)$, $(8, 2)$, and $(27, 3)$. What do you observe? What happens at $(0, 0)$?

THEOREM 5.9 THE DERIVATIVE OF AN INVERSE FUNCTION

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

EXAMPLE 5 Evaluating the Derivative of an Inverse Function

Let $f(x) = \frac{1}{4}x^3 + x - 1$.

a. What is the value of $f^{-1}(x)$ when $x = 3$?

b. What is the value of $(f^{-1})'(x)$ when $x = 3$?

Solution Notice that f is one-to-one and therefore has an inverse function.

a. Because $f(x) = 3$ when $x = 2$, you know that $f^{-1}(3) = 2$.

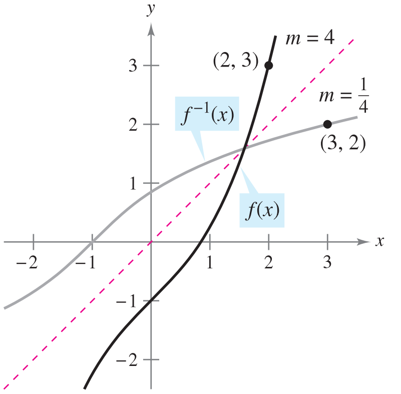
b. Because the function f is differentiable and has an inverse function, you can apply Theorem 5.9 (with $g = f^{-1}$) to write

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)}.$$

Moreover, using $f'(x) = \frac{3}{4}x^2 + 1$, you can conclude that

$$(f^{-1})'(3) = \frac{1}{f'(2)} = \frac{1}{\frac{3}{4}(2^2) + 1} = \frac{1}{4}.$$

■



The graphs of the inverse functions f and f^{-1} have reciprocal slopes at points (a, b) and (b, a).
Figure 5.17

In Example 5, note that at the point $(2, 3)$ the slope of the graph of f is 4 and at the point $(3, 2)$ the slope of the graph of f^{-1} is $\frac{1}{4}$ (see Figure 5.17). This reciprocal relationship (which follows from Theorem 5.9) can be written as shown below. If $y = g(x) = f^{-1}(x)$, then $f(y) = x$ and $f'(y) = \frac{dx}{dy}$. Theorem 5.9 says that

$$g'(x) = \frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{f'(y)} = \frac{1}{(dx/dy)}.$$

So, $\boxed{\dfrac{dy}{dx} = \dfrac{1}{dx/dy}.}$

EXAMPLE 6 Graphs of Inverse Functions Have Reciprocal Slopes

Let $f(x) = x^2$ (for $x \geq 0$) and let $f^{-1}(x) = \sqrt{x}$. Show that the slopes of the graphs of f and f^{-1} are reciprocals at each of the following points.

a. $(2, 4)$ and $(4, 2)$ **b.** $(3, 9)$ and $(9, 3)$

Solution The derivatives of f and f^{-1} are given by

$$f'(x) = 2x \qquad \text{and} \qquad (f^{-1})'(x) = \frac{1}{2\sqrt{x}}.$$

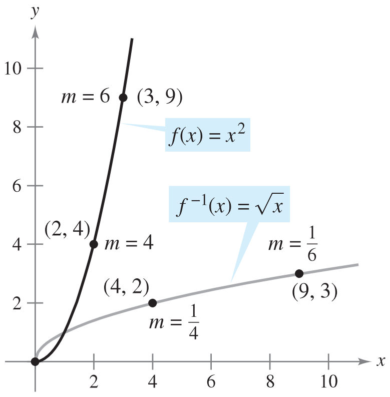
a. At $(2, 4)$, the slope of the graph of f is $f'(2) = 2(2) = 4$. At $(4, 2)$, the slope of the graph of f^{-1} is

$$(f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4}.$$

b. At $(3, 9)$, the slope of the graph of f is $f'(3) = 2(3) = 6$. At $(9, 3)$, the slope of the graph of f^{-1} is

$$(f^{-1})'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}.$$

So, in both cases, the slopes are reciprocals, as shown in Figure 5.18. ■



At $(0, 0)$, the derivative of f is 0, and the derivative of f^{-1} does not exist.
Figure 5.18

5.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, show that *f* and *g* are inverse functions (a) analytically and (b) graphically.

1. $f(x) = 5x + 1, \quad g(x) = \dfrac{x-1}{5}$

2. $f(x) = 3 - 4x, \quad g(x) = \dfrac{3-x}{4}$

3. $f(x) = x^3, \quad g(x) = \sqrt[3]{x}$

4. $f(x) = 1 - x^3, \quad g(x) = \sqrt[3]{1-x}$

5. $f(x) = \sqrt{x-4}, \quad g(x) = x^2 + 4, \quad x \geq 0$

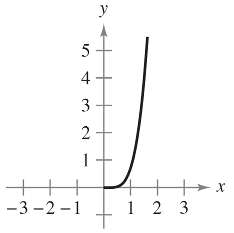
6. $f(x) = 16 - x^2, \quad x \geq 0, \quad g(x) = \sqrt{16-x}$

7. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$

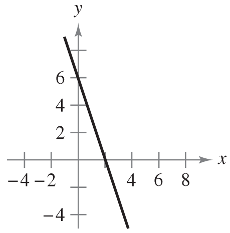
8. $f(x) = \dfrac{1}{1+x}, \quad x \geq 0, \quad g(x) = \dfrac{1-x}{x}, \quad 0 < x \leq 1$

In Exercises 9–12, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]

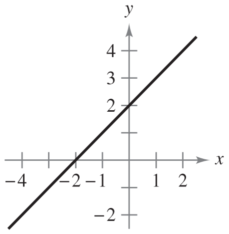
(a)



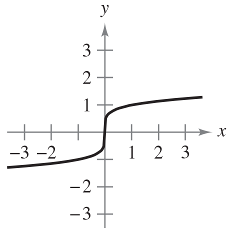
(b)



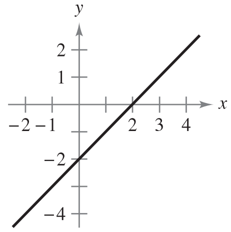
(c)



(d)



9.



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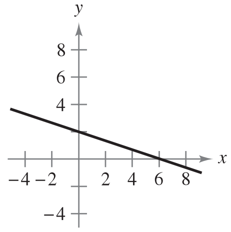

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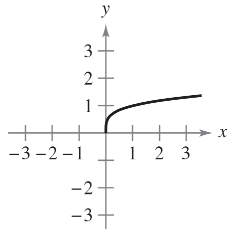

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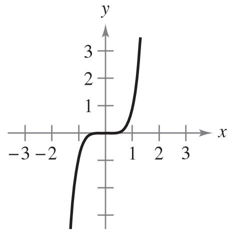


In Exercises 13–22, use a graphing utility to graph the function. Then use the Horizontal Line Test to determine whether the function is one-to-one on its entire domain and therefore has an inverse function.

13. $f(x) = \frac{3}{4}x + 6$

14. $f(x) = 5x - 3$

15. $f(\theta) = \sin \theta$

16. $f(x) = \dfrac{x^2}{x^2+4}$

17. $h(s) = \dfrac{1}{s-2} - 3$

18. $g(t) = \dfrac{1}{\sqrt{t^2+1}}$

19. $f(x) = \ln x$

20. $f(x) = 5x\sqrt{x-1}$

21. $g(x) = (x+5)^3$

22. $h(x) = |x+4| - |x-4|$

In Exercises 23–30, (a) find the inverse function of *f*, (b) graph *f* and f^{-1} on the same set of coordinate axes, (c) describe the relationship between the graphs, and (d) state the domain and range of *f* and f^{-1}.

23. $f(x) = 2x - 3$

24. $f(x) = 3x$

25. $f(x) = x^5$

26. $f(x) = x^3 - 1$

27. $f(x) = \sqrt{x}$

28. $f(x) = x^2, \quad x \geq 0$

29. $f(x) = \sqrt{4-x^2}, \quad 0 \leq x \leq 2$

30. $f(x) = \sqrt{x^2-4}, \quad x \geq 2$

In Exercises 31–36, (a) find the inverse function of *f*, (b) use a graphing utility to graph *f* and f^{-1} in the same viewing window, (c) describe the relationship between the graphs, and (d) state the domain and range of *f* and f^{-1}.

31. $f(x) = \sqrt[3]{x-1}$

32. $f(x) = 3\sqrt[5]{2x-1}$

33. $f(x) = x^{2/3}, \quad x \geq 0$

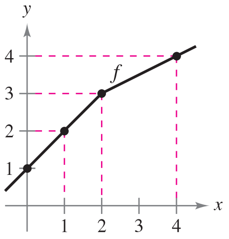
34. $f(x) = x^{3/5}$

35. $f(x) = \dfrac{x}{\sqrt{x^2+7}}$

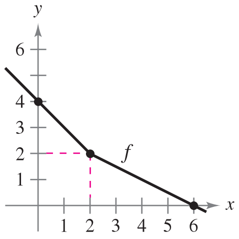
36. $f(x) = \dfrac{x+2}{x}$

In Exercises 37 and 38, use the graph of the function *f* to make a table of values for the given points. Then make a second table that can be used to find f^{-1}, and sketch the graph of f^{-1}. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

37.



38.



39. ***Cost*** You need 50 pounds of two commodities costing \$1.25 and \$1.60 per pound.

(a) Verify that the total cost is $y = 1.25x + 1.60(50 - x)$, where x is the number of pounds of the less expensive commodity.

(b) Find the inverse function of the cost function. What does each variable represent in the inverse function?

(c) What is the domain of the inverse function? Validate or explain your answer using the context of the problem.

(d) Determine the number of pounds of the less expensive commodity purchased if the total cost is \$73.

40. ***Temperature*** The formula $C = \frac{5}{9}(F - 32)$, where $F \geq -459.6$, represents Celsius temperature C as a function of Fahrenheit temperature F.

(a) Find the inverse function of C.

(b) What does the inverse function represent?

(c) What is the domain of the inverse function? Validate or explain your answer using the context of the problem.

(d) The temperature is 22°C. What is the corresponding temperature in degrees Fahrenheit?

In Exercises 41–46, use the derivative to determine whether the function is strictly monotonic on its entire domain and therefore has an inverse function.

41. $f(x) = 2 - x - x^3$
42. $f(x) = x^3 - 6x^2 + 12x$

43. $f(x) = \dfrac{x^4}{4} - 2x^2$
44. $f(x) = (x + a)^3 + b$

45. $f(x) = \ln(x - 3)$
46. $f(x) = \cos \dfrac{3x}{2}$

In Exercises 47–52, show that f is strictly monotonic on the given interval and therefore has an inverse function on that interval.

47. $f(x) = (x - 4)^2, \quad [4, \infty)$
48. $f(x) = |x + 2|, \quad [-2, \infty)$

49. $f(x) = \dfrac{4}{x^2}, \quad (0, \infty)$
50. $f(x) = \cot x, \quad (0, \pi)$

51. $f(x) = \cos x, \quad [0, \pi]$
52. $f(x) = \sec x, \quad \left[0, \dfrac{\pi}{2}\right)$

In Exercises 53 and 54, find the inverse function of f over the given interval. Use a graphing utility to graph f and f^{-1} in the same viewing window. Describe the relationship between the graphs.

53. $f(x) = \dfrac{x}{x^2 - 4}, \quad (-2, 2)$
54. $f(x) = 2 - \dfrac{3}{x^2}, \quad (0, 10)$

Graphical Reasoning **In Exercises 55–58, (a) use a graphing utility to graph the function, (b) use the *drawing* feature of a graphing utility to draw the inverse function of the function, and (c) determine whether the graph of the inverse relation is an inverse function. Explain your reasoning.**

55. $f(x) = x^3 + x + 4$
56. $h(x) = x\sqrt{4 - x^2}$

57. $g(x) = \dfrac{3x^2}{x^2 + 1}$
58. $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$

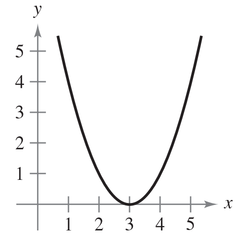
In Exercises 59–62, determine whether the function is one-to-one. If it is, find its inverse function.

59. $f(x) = \sqrt{x - 2}$
60. $f(x) = -3$

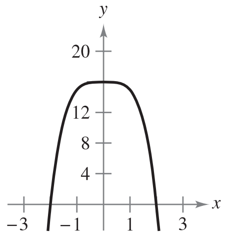
61. $f(x) = |x - 2|, \quad x \leq 2$
62. $f(x) = ax + b, \quad a \neq 0$

In Exercises 63–66, delete part of the domain so that the function that remains is one-to-one. Find the inverse function of the remaining function and give the domain of the inverse function. (*Note:* There is more than one correct answer.)

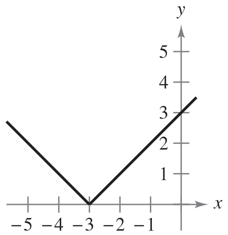
63. $f(x) = (x - 3)^2$



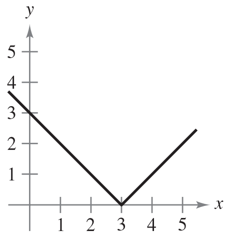
64. $f(x) = 16 - x^4$



65. $f(x) = |x + 3|$



66. $f(x) = |x - 3|$



Think About It **In Exercises 67–70, decide whether the function has an inverse function. If so, what is the inverse function?**

67. $g(t)$ is the volume of water that has passed through a water line t minutes after a control valve is opened.

68. $h(t)$ is the height of the tide t hours after midnight, where $0 \leq t < 24$.

69. $C(t)$ is the cost of a long distance call lasting t minutes.

70. $A(r)$ is the area of a circle of radius r.

In Exercises 71–80, verify that f has an inverse. Then use the function f and the given real number a to find $(f^{-1})'(a)$. (*Hint:* See Example 5.)

71. $f(x) = x^3 - 1, \quad a = 26$
72. $f(x) = 5 - 2x^3, \quad a = 7$

73. $f(x) = x^3 + 2x - 1, \quad a = 2$

74. $f(x) = \frac{1}{27}(x^5 + 2x^3), \quad a = -11$

75. $f(x) = \sin x, \quad -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}, \quad a = \dfrac{1}{2}$

76. $f(x) = \cos 2x, \quad 0 \leq x \leq \dfrac{\pi}{2}, \quad a = 1$

77. $f(x) = \dfrac{x + 6}{x - 2}, \quad x > 2, \quad a = 3$

78. $f(x) = \dfrac{x+3}{x+1}, \quad x > -1, \quad a = 2$

79. $f(x) = x^3 - \dfrac{4}{x}, \quad x > 0, \quad a = 6$

80. $f(x) = \sqrt{x-4}, \quad a = 2$

In Exercises 81–84, (a) find the domains of f and f^{-1}, (b) find the ranges of f and f^{-1}, (c) graph f and f^{-1}, and (d) show that the slopes of the graphs of f and f^{-1} are reciprocals at the given points.

	Functions	*Point*
81.	$f(x) = x^3$	$\left(\frac{1}{2}, \frac{1}{8}\right)$
	$f^{-1}(x) = \sqrt[3]{x}$	$\left(\frac{1}{8}, \frac{1}{2}\right)$
82.	$f(x) = 3 - 4x$	$(1, -1)$
	$f^{-1}(x) = \dfrac{3-x}{4}$	$(-1, 1)$
83.	$f(x) = \sqrt{x-4}$	$(5, 1)$
	$f^{-1}(x) = x^2 + 4, \quad x \geq 0$	$(1, 5)$
84.	$f(x) = \dfrac{4}{1+x^2}, \quad x \geq 0$	$(1, 2)$
	$f^{-1}(x) = \sqrt{\dfrac{4-x}{x}}$	$(2, 1)$

In Exercises 85 and 86, find dy/dx for the equation at the given point.

85. $x = y^3 - 7y^2 + 2, \quad (-4, 1)$

86. $x = 2\ln(y^2 - 3), \quad (0, 2)$

In Exercises 87–90, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the given value.

87. $(f^{-1} \circ g^{-1})(1)$

88. $(g^{-1} \circ f^{-1})(-3)$

89. $(f^{-1} \circ f^{-1})(6)$

90. $(g^{-1} \circ g^{-1})(-4)$

In Exercises 91–94, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the given function.

91. $g^{-1} \circ f^{-1}$

92. $f^{-1} \circ g^{-1}$

93. $(f \circ g)^{-1}$

94. $(g \circ f)^{-1}$

WRITING ABOUT CONCEPTS

95. Describe how to find the inverse function of a one-to-one function given by an equation in x and y. Give an example.

96. Describe the relationship between the graph of a function and the graph of its inverse function.

In Exercises 97 and 98, the derivative of the function has the same sign for all x in its domain, but the function is not one-to-one. Explain.

97. $f(x) = \tan x$

98. $f(x) = \dfrac{x}{x^2 - 4}$

99. ***Think About It*** The function $f(x) = k(2 - x - x^3)$ is one-to-one and $f^{-1}(3) = -2$. Find k.

CAPSTONE

100. ***Think About It*** The point $(1, 3)$ lies on the graph of f, and the slope of the tangent line through this point is $m = 2$. Assume f^{-1} exists. What is the slope of the tangent line to the graph of f^{-1} at the point $(3, 1)$?

True or False? **In Exercises 101–104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

101. If f is an even function, then f^{-1} exists.

102. If the inverse function of f exists, then the y-intercept of f is an x-intercept of f^{-1}.

103. If $f(x) = x^n$, where n is odd, then f^{-1} exists.

104. There exists no function f such that $f = f^{-1}$.

105. (a) Show that $f(x) = 2x^3 + 3x^2 - 36x$ is not one-to-one on $(-\infty, \infty)$.

(b) Determine the greatest value c such that f is one-to-one on $(-c, c)$.

106. Let f and g be one-to-one functions. Prove that (a) $f \circ g$ is one-to-one and (b) $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

107. Prove that if f has an inverse function, then $(f^{-1})^{-1} = f$.

108. Prove that if a function has an inverse function, then the inverse function is unique.

109. Prove that a function has an inverse function if and only if it is one-to-one.

110. Is the converse of the second part of Theorem 5.7 true? That is, if a function is one-to-one (and therefore has an inverse function), then must the function be strictly monotonic? If so, prove it. If not, give a counterexample.

111. Let f be twice-differentiable and one-to-one on an open interval I. Show that its inverse function g satisfies

$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}.$$

If f is increasing and concave downward, what is the concavity of $f^{-1} = g$?

112. If $f(x) = \displaystyle\int_2^x \frac{dt}{\sqrt{1+t^4}}$, find $(f^{-1})'(0)$.

113. Show that $f(x) = \displaystyle\int_2^x \sqrt{1+t^2}\,dt$ is one-to-one and find $(f^{-1})'(0)$.

114. Let $y = \dfrac{x-2}{x-1}$. Show that y is its own inverse function. What can you conclude about the graph of f? Explain.

115. Let $f(x) = \dfrac{ax+b}{cx+d}$.

(a) Show that f is one-to-one if and only if $bc - ad \neq 0$.

(b) Given $bc - ad \neq 0$, find f^{-1}.

(c) Determine the values of a, b, c, and d such that $f = f^{-1}$.

5.4 Exponential Functions: Differentiation and Integration

- **Develop properties of the natural exponential function.**
- **Differentiate natural exponential functions.**
- **Integrate natural exponential functions.**

The Natural Exponential Function

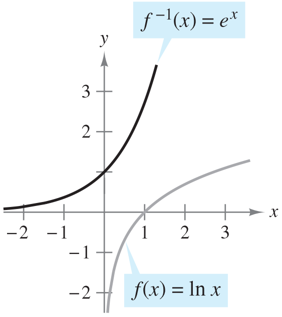
The function $f(x) = \ln x$ is increasing on its entire domain, and therefore it has an inverse function f^{-1}. The domain of f^{-1} is the set of all reals, and the range is the set of positive reals, as shown in Figure 5.19. So, for any real number x,

$$f(f^{-1}(x)) = \ln[f^{-1}(x)] = x. \qquad x \text{ is any real number.}$$

If x happens to be rational, then

$$\ln(e^x) = x \ln e = x(1) = x. \qquad x \text{ is a rational number.}$$

Because the natural logarithmic function is one-to-one, you can conclude that $f^{-1}(x)$ and e^x agree for *rational* values of x. The following definition extends the meaning of e^x to include *all* real values of x.



The inverse function of the natural logarithmic function is the natural exponential function.
Figure 5.19

DEFINITION OF THE NATURAL EXPONENTIAL FUNCTION

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

$$f^{-1}(x) = e^x.$$

That is,

$$y = e^x \quad \text{if and only if} \quad x = \ln y.$$

THE NUMBER *e*

The symbol *e* was first used by mathematician Leonhard Euler to represent the base of natural logarithms in a letter to another mathematician, Christian Goldbach, in 1731.

The inverse relationship between the natural logarithmic function and the natural exponential function can be summarized as follows.

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x \qquad \text{Inverse relationship}$$

EXAMPLE 1 Solving Exponential Equations

Solve $7 = e^{x+1}$.

Solution You can convert from exponential form to logarithmic form by *taking the natural logarithm of each side* of the equation.

$$\begin{aligned} 7 &= e^{x+1} && \text{Write original equation.} \\ \ln 7 &= \ln(e^{x+1}) && \text{Take natural logarithm of each side.} \\ \ln 7 &= x + 1 && \text{Apply inverse property.} \\ -1 + \ln 7 &= x && \text{Solve for } x. \\ 0.946 &\approx x && \text{Use a calculator.} \end{aligned}$$

Check this solution in the original equation. ■

EXAMPLE 2 Solving a Logarithmic Equation

Solve $\ln(2x - 3) = 5$.

Solution To convert from logarithmic form to exponential form, you can *exponentiate each side* of the logarithmic equation.

$$\begin{aligned} \ln(2x-3) &= 5 && \text{Write original equation.} \\ e^{\ln(2x-3)} &= e^5 && \text{Exponentiate each side.} \\ 2x - 3 &= e^5 && \text{Apply inverse property.} \\ x &= \tfrac{1}{2}(e^5 + 3) && \text{Solve for } x. \\ x &\approx 75.707 && \text{Use a calculator.} \end{aligned}$$

■

The familiar rules for operating with rational exponents can be extended to the natural exponential function, as shown in the following theorem.

THEOREM 5.10 OPERATIONS WITH EXPONENTIAL FUNCTIONS

Let a and b be any real numbers.

1. $e^a e^b = e^{a+b}$
2. $\dfrac{e^a}{e^b} = e^{a-b}$

PROOF To prove Property 1, you can write

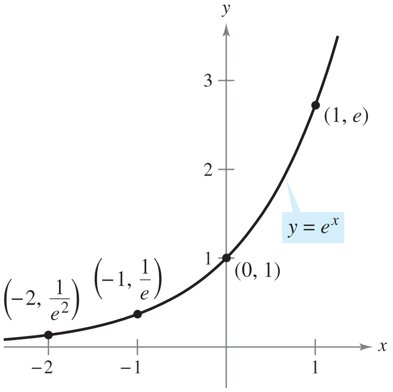
$$\begin{aligned} \ln(e^a e^b) &= \ln(e^a) + \ln(e^b) \\ &= a + b \\ &= \ln(e^{a+b}). \end{aligned}$$

Because the natural logarithmic function is one-to-one, you can conclude that

$$e^a e^b = e^{a+b}.$$

The proof of the other property is given in Appendix A. ■

In Section 5.3, you learned that an inverse function f^{-1} shares many properties with f. So, the natural exponential function inherits the following properties from the natural logarithmic function (see Figure 5.20).



The natural exponential function is increasing, and its graph is concave upward.
Figure 5.20

PROPERTIES OF THE NATURAL EXPONENTIAL FUNCTION

1. The domain of $f(x) = e^x$ is $(-\infty, \infty)$, and the range is $(0, \infty)$.
2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.
3. The graph of $f(x) = e^x$ is concave upward on its entire domain.
4. $\displaystyle\lim_{x \to -\infty} e^x = 0$ and $\displaystyle\lim_{x \to \infty} e^x = \infty$

Derivatives of Exponential Functions

One of the most intriguing (and useful) characteristics of the natural exponential function is that *it is its own derivative*. In other words, it is a solution to the differential equation $y' = y$. This result is stated in the next theorem.

FOR FURTHER INFORMATION To find out about derivatives of exponential functions of order 1/2, see the article "A Child's Garden of Fractional Derivatives" by Marcia Kleinz and Thomas J. Osler in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

THEOREM 5.11 DERIVATIVES OF THE NATURAL EXPONENTIAL FUNCTION

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[e^x] = e^x$
2. $\dfrac{d}{dx}[e^u] = e^u\dfrac{du}{dx}$

PROOF To prove Property 1, use the fact that $\ln e^x = x$, and differentiate each side of the equation.

$$\ln e^x = x \qquad \text{Definition of exponential function}$$

$$\frac{d}{dx}[\ln e^x] = \frac{d}{dx}[x] \qquad \text{Differentiate each side with respect to } x.$$

$$\frac{1}{e^x}\frac{d}{dx}[e^x] = 1$$

$$\frac{d}{dx}[e^x] = e^x$$

The derivative of e^u follows from the Chain Rule. ■

NOTE You can interpret this theorem geometrically by saying that the slope of the graph of $f(x) = e^x$ at any point (x, e^x) is equal to the y-coordinate of the point. ■

EXAMPLE 3 Differentiating Exponential Functions

a. $\dfrac{d}{dx}[e^{2x-1}] = e^u\dfrac{du}{dx} = 2e^{2x-1}$ $\qquad u = 2x - 1$

b. $\dfrac{d}{dx}[e^{-3/x}] = e^u\dfrac{du}{dx} = \left(\dfrac{3}{x^2}\right)e^{-3/x} = \dfrac{3e^{-3/x}}{x^2}$ $\qquad u = -\dfrac{3}{x}$

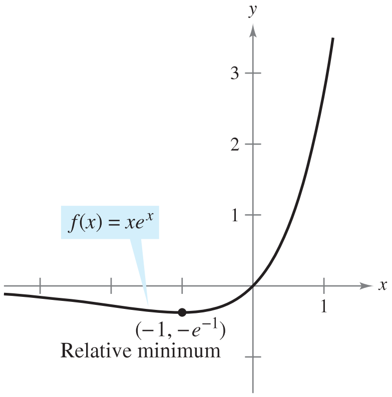
EXAMPLE 4 Locating Relative Extrema

Find the relative extrema of $f(x) = xe^x$.

Solution The derivative of f is given by

$$f'(x) = x(e^x) + e^x(1) \qquad \text{Product Rule}$$
$$= e^x(x + 1).$$

Because e^x is never 0, the derivative is 0 only when $x = -1$. Moreover, by the First Derivative Test, you can determine that this corresponds to a relative minimum, as shown in Figure 5.21. Because the derivative $f'(x) = e^x(x + 1)$ is defined for all x, there are no other critical points. ■



The derivative of f changes from negative to positive at $x = -1$.
Figure 5.21

EXAMPLE 5 The Standard Normal Probability Density Function

Show that the *standard normal probability density function*

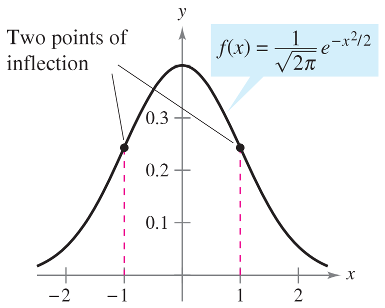
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

has points of inflection when $x = \pm 1$.

Solution To locate possible points of inflection, find the x-values for which the second derivative is 0.

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} && \text{Write original function.}\\
f'(x) &= \frac{1}{\sqrt{2\pi}}(-x)e^{-x^2/2} && \text{First derivative}\\
f''(x) &= \frac{1}{\sqrt{2\pi}}\left[(-x)(-x)e^{-x^2/2} + (-1)e^{-x^2/2}\right] && \text{Product Rule}\\
&= \frac{1}{\sqrt{2\pi}}(e^{-x^2/2})(x^2 - 1) && \text{Second derivative}
\end{aligned}$$

So, $f''(x) = 0$ when $x = \pm 1$, and you can apply the techniques of Chapter 3 to conclude that these values yield the two points of inflection shown in Figure 5.22. ■



The bell-shaped curve given by a standard normal probability density function

Figure 5.22

NOTE The general form of a normal probability density function (whose mean is 0) is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

where σ is the standard deviation (σ is the lowercase Greek letter sigma). This "bell-shaped curve" has points of inflection when $x = \pm\sigma$. ■

EXAMPLE 6 Shares Traded

The numbers y of shares traded (in millions) on the New York Stock Exchange from 1990 through 2005 can be modeled by

$$y = 39{,}811e^{0.1715t}$$

where t represents the year, with $t = 0$ corresponding to 1990. At what rate was the number of shares traded changing in 2000? *(Source: New York Stock Exchange, Inc.)*

Solution The derivative of the given model is

$$\begin{aligned}
y' &= (0.1715)(39{,}811)e^{0.1715t}\\
&\approx 6828e^{0.1715t}.
\end{aligned}$$

By evaluating the derivative when $t = 10$, you can conclude that the rate of change in 2000 was about

37,941 million shares per year.

The graph of this model is shown in Figure 5.23. ■

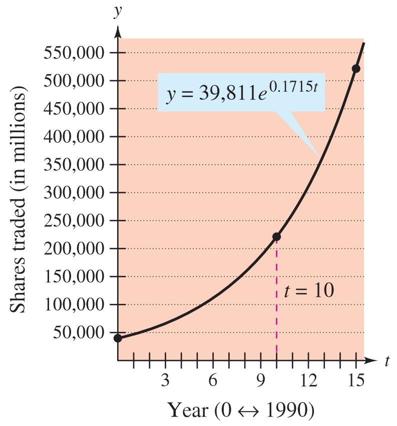


Figure 5.23

Integrals of Exponential Functions

Each differentiation formula in Theorem 5.11 has a corresponding integration formula.

THEOREM 5.12 INTEGRATION RULES FOR EXPONENTIAL FUNCTIONS

Let u be a differentiable function of x.

1. $\displaystyle\int e^x\,dx = e^x + C$ **2.** $\displaystyle\int e^u\,du = e^u + C$

EXAMPLE 7 Integrating Exponential Functions

Find $\displaystyle\int e^{3x+1}\,dx$.

Solution If you let $u = 3x + 1$, then $du = 3\,dx$.

$$\begin{aligned}
\int e^{3x+1}dx &= \frac{1}{3}\int e^{3x+1}(3)\,dx && \text{Multiply and divide by 3.}\\
&= \frac{1}{3}\int e^u\,du && \text{Substitute: } u = 3x+1.\\
&= \frac{1}{3}e^u + C && \text{Apply Exponential Rule.}\\
&= \frac{e^{3x+1}}{3} + C && \text{Back-substitute.}
\end{aligned}$$

■

NOTE In Example 7, the missing *constant* factor 3 was introduced to create $du = 3\,dx$. However, remember that you cannot introduce a missing *variable* factor in the integrand. For instance,

$$\int e^{-x^2}\,dx \neq \frac{1}{x}\int e^{-x^2}\,(x\,dx).$$

■

EXAMPLE 8 Integrating Exponential Functions

Find $\displaystyle\int 5xe^{-x^2}\,dx$.

Solution If you let $u = -x^2$, then $du = -2x\,dx$ or $x\,dx = -du/2$.

$$\begin{aligned}
\int 5xe^{-x^2}dx &= \int 5e^{-x^2}(x\,dx) && \text{Regroup integrand.}\\
&= \int 5e^u\left(-\frac{du}{2}\right) && \text{Substitute: } u = -x^2.\\
&= -\frac{5}{2}\int e^u\,du && \text{Constant Multiple Rule}\\
&= -\frac{5}{2}e^u + C && \text{Apply Exponential Rule.}\\
&= -\frac{5}{2}e^{-x^2} + C && \text{Back-substitute.}
\end{aligned}$$

■

EXAMPLE 9 Integrating Exponential Functions

a. $$\int \frac{e^{1/x}}{x^2}\,dx = -\int \overbrace{e^{1/x}}^{e^u}\overbrace{\left(-\frac{1}{x^2}\right)dx}^{du} \qquad u = \frac{1}{x}$$
$$= -e^{1/x} + C$$

b. $$\int \sin x\, e^{\cos x}\,dx = -\int \overbrace{e^{\cos x}}^{e^u}\overbrace{(-\sin x\,dx)}^{du} \qquad u = \cos x$$
$$= -e^{\cos x} + C$$

EXAMPLE 10 Finding Areas Bounded by Exponential Functions

Evaluate each definite integral.

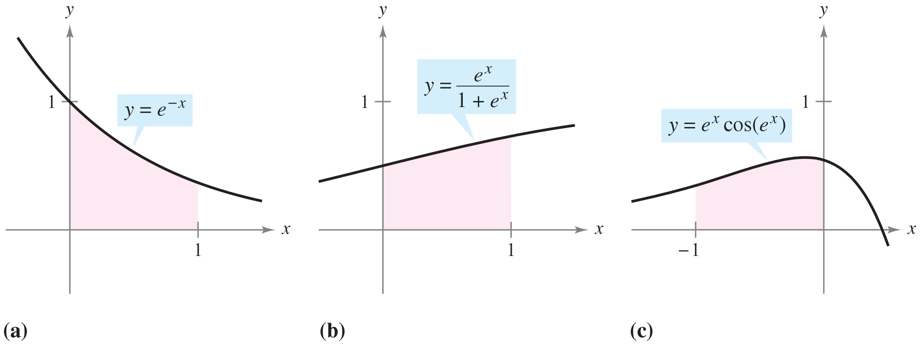
a. $\int_0^1 e^{-x}\,dx$ **b.** $\int_0^1 \frac{e^x}{1+e^x}\,dx$ **c.** $\int_{-1}^0 [e^x \cos(e^x)]\,dx$

Solution

a. $$\int_0^1 e^{-x}\,dx = -e^{-x}\Big]_0^1 \qquad \text{See Figure 5.24(a).}$$
$$= -e^{-1} - (-1)$$
$$= 1 - \frac{1}{e}$$
$$\approx 0.632$$

b. $$\int_0^1 \frac{e^x}{1+e^x}\,dx = \ln(1+e^x)\Big]_0^1 \qquad \text{See Figure 5.24(b).}$$
$$= \ln(1+e) - \ln 2$$
$$\approx 0.620$$

c. $$\int_{-1}^0 [e^x\cos(e^x)]\,dx = \sin(e^x)\Big]_{-1}^0 \qquad \text{See Figure 5.24(c).}$$
$$= \sin 1 - \sin(e^{-1})$$
$$\approx 0.482$$



(a) (b) (c)

Figure 5.24

5.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–16, solve for x accurate to three decimal places.

1. $e^{\ln x} = 4$

2. $e^{\ln 2x} = 12$

3. $e^x = 12$

4. $4e^x = 83$

5. $9 - 2e^x = 7$

6. $-6 + 3e^x = 8$

7. $50e^{-x} = 30$

8. $200e^{-4x} = 15$

9. $\dfrac{800}{100 - e^{x/2}} = 50$

10. $\dfrac{5000}{1 + e^{2x}} = 2$

11. $\ln x = 2$

12. $\ln x^2 = 10$

13. $\ln(x - 3) = 2$

14. $\ln 4x = 1$

15. $\ln\sqrt{x + 2} = 1$

16. $\ln(x - 2)^2 = 12$

In Exercises 17–22, sketch the graph of the function.

17. $y = e^{-x}$

18. $y = \frac{1}{2}e^x$

19. $y = e^x + 2$

20. $y = e^{x-1}$

21. $y = e^{-x^2}$

22. $y = e^{-x/2}$

23. Use a graphing utility to graph $f(x) = e^x$ and the given function in the same viewing window. How are the two graphs related?

(a) $g(x) = e^{x-2}$ (b) $h(x) = -\frac{1}{2}e^x$ (c) $q(x) = e^{-x} + 3$

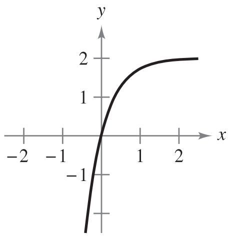
24. Use a graphing utility to graph the function. Use the graph to determine any asymptotes of the function.

(a) $f(x) = \dfrac{8}{1 + e^{-0.5x}}$

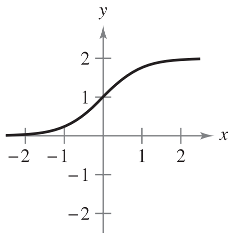
(b) $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

In Exercises 25–28, match the equation with the correct graph. Assume that a and C are positive real numbers. [The graphs are labeled (a), (b), (c), and (d).]

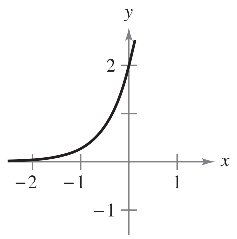
(a)



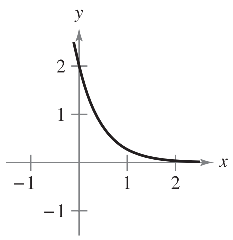
(b)



(c)



(d)



25. $y = Ce^{ax}$

26. $y = Ce^{-ax}$

27. $y = C(1 - e^{-ax})$

28. $y = \dfrac{C}{1 + e^{-ax}}$

In Exercises 29–32, illustrate that the functions are inverses of each other by graphing both functions on the same set of coordinate axes.

29. $f(x) = e^{2x}$
$g(x) = \ln\sqrt{x}$

30. $f(x) = e^{x/3}$
$g(x) = \ln x^3$

31. $f(x) = e^x - 1$
$g(x) = \ln(x + 1)$

32. $f(x) = e^{x-1}$
$g(x) = 1 + \ln x$

33. ***Graphical Analysis*** Use a graphing utility to graph

$$f(x) = \left(1 + \frac{0.5}{x}\right)^x \quad \text{and} \quad g(x) = e^{0.5}$$

in the same viewing window. What is the relationship between f and g as $x \to \infty$?

34. ***Conjecture*** Use the result of Exercise 33 to make a conjecture about the value of

$$\left(1 + \frac{r}{x}\right)^x$$

as $x \to \infty$.

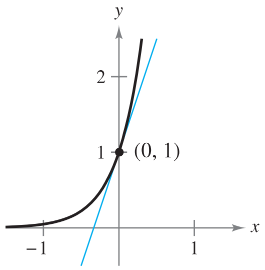
In Exercises 35 and 36, compare the given number with the number e. Is the number less than or greater than e?

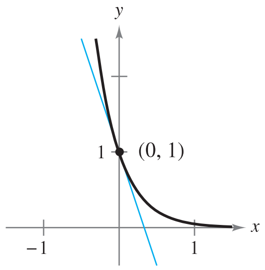
35. $\left(1 + \dfrac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$ (See Exercise 34.)

36. $1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120} + \dfrac{1}{720} + \dfrac{1}{5040}$

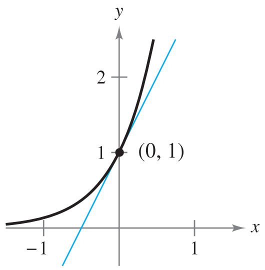
In Exercises 37 and 38, find an equation of the tangent line to the graph of the function at the point (0, 1).

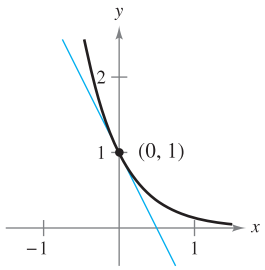
37. (a) $y = e^{3x}$ (b) $y = e^{-3x}$





38. (a) $y = e^{2x}$ (b) $y = e^{-2x}$





In Exercises 39–60, find the derivative.

39. $f(x) = e^{2x}$

40. $y = e^{-5x}$

41. $y = e^{\sqrt{x}}$

42. $y = e^{-x^2}$

43. $y = e^{x-4}$

44. $f(x) = 3e^{1-x^2}$

45. $y = e^x \ln x$

46. $y = xe^x$

47. $y = x^3 e^x$

48. $y = x^2 e^{-x}$

49. $g(t) = (e^{-t} + e^t)^3$

50. $g(t) = e^{-3/t^2}$

51. $y = \ln(1 + e^{2x})$

52. $y = \ln\left(\dfrac{1 + e^x}{1 - e^x}\right)$

53. $y = \dfrac{2}{e^x + e^{-x}}$

54. $y = \dfrac{e^x - e^{-x}}{2}$

55. $y = \dfrac{e^x + 1}{e^x - 1}$

56. $y = \dfrac{e^{2x}}{e^{2x} + 1}$

57. $y = e^x(\sin x + \cos x)$

58. $y = \ln e^x$

59. $F(x) = \displaystyle\int_{\pi}^{\ln x} \cos e^t\, dt$

60. $F(x) = \displaystyle\int_0^{e^{2x}} \ln(t + 1)\, dt$

In Exercises 61–68, find an equation of the tangent line to the graph of the function at the given point.

61. $f(x) = e^{1-x}$, $(1, 1)$

62. $y = e^{-2x+x^2}$, $(2, 1)$

63. $y = \ln(e^{x^2})$, $(-2, 4)$

64. $y = \ln \dfrac{e^x + e^{-x}}{2}$, $(0, 0)$

65. $y = x^2e^x - 2xe^x + 2e^x$, $(1, e)$

66. $y = xe^x - e^x$, $(1, 0)$

67. $f(x) = e^{-x} \ln x$, $(1, 0)$

68. $f(x) = e^3 \ln x$, $(1, 0)$

In Exercises 69 and 70, use implicit differentiation to find dy/dx.

69. $xe^y - 10x + 3y = 0$

70. $e^{xy} + x^2 - y^2 = 10$

In Exercises 71 and 72, find an equation of the tangent line to the graph of the function at the given point.

71. $xe^y + ye^x = 1$, $(0, 1)$

72. $1 + \ln xy = e^{x-y}$, $(1, 1)$

In Exercises 73 and 74, find the second derivative of the function.

73. $f(x) = (3 + 2x)e^{-3x}$

74. $g(x) = \sqrt{x} + e^x \ln x$

In Exercises 75–78, show that the function $y = f(x)$ is a solution of the differential equation.

75. $y = 4e^{-x}$

$y'' - y = 0$

76. $y = e^{3x} + e^{-3x}$

$y'' - 9y = 0$

77. $y = e^x\left(\cos \sqrt{2}x + \sin \sqrt{2}x\right)$

$y'' - 2y' + 3y = 0$

78. $y = e^x(3 \cos 2x - 4 \sin 2x)$

$y'' - 2y' + 5y = 0$

In Exercises 79–86, find the extrema and the points of inflection (if any exist) of the function. Use a graphing utility to graph the function and confirm your results.

79. $f(x) = \dfrac{e^x + e^{-x}}{2}$

80. $f(x) = \dfrac{e^x - e^{-x}}{2}$

81. $g(x) = \dfrac{1}{\sqrt{2\pi}} e^{-(x-2)^2/2}$

82. $g(x) = \dfrac{1}{\sqrt{2\pi}} e^{-(x-3)^2/2}$

83. $f(x) = x^2e^{-x}$

84. $f(x) = xe^{-x}$

85. $g(t) = 1 + (2 + t)e^{-t}$

86. $f(x) = -2 + e^{3x}(4 - 2x)$

87. ***Area*** Find the area of the largest rectangle that can be inscribed under the curve $y = e^{-x^2}$ in the first and second quadrants.

88. ***Area*** Perform the following steps to find the maximum area of the rectangle shown in the figure.

(a) Solve for c in the equation $f(c) = f(c + x)$.

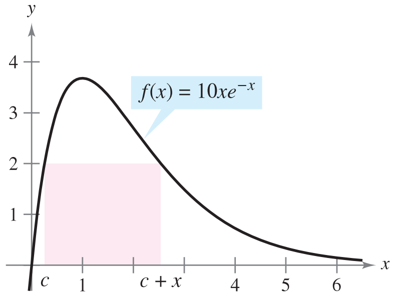
(b) Use the result in part (a) to write the area A as a function of x. [*Hint*: $A = xf(c)$]

(c) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions of the rectangle of maximum area. Determine the maximum area.

(d) Use a graphing utility to graph the expression for c found in part (a). Use the graph to approximate

$$\lim_{x\to 0^+} c \quad \text{and} \quad \lim_{x\to\infty} c.$$

Use this result to describe the changes in dimensions and position of the rectangle for $0 < x < \infty$.



89. Find a point on the graph of the function $f(x) = e^{2x}$ such that the tangent line to the graph at that point passes through the origin. Use a graphing utility to graph f and the tangent line in the same viewing window.

90. Find the point on the graph of $y = e^{-x}$ where the normal line to the curve passes through the origin. (Use Newton's Method or the *zero* or *root* feature of a graphing utility.)

91. ***Depreciation*** The value V of an item t years after it is purchased is $V = 15{,}000e^{-0.6286t}$, $0 \le t \le 10$.

(a) Use a graphing utility to graph the function.

(b) Find the rates of change of V with respect to t when $t = 1$ and $t = 5$.

(c) Use a graphing utility to graph the tangent lines to the function when $t = 1$ and $t = 5$.

92. ***Harmonic Motion*** The displacement from equilibrium of a mass oscillating on the end of a spring suspended from a ceiling is $y = 1.56e^{-0.22t} \cos 4.9t$, where y is the displacement in feet and t is the time in seconds. Use a graphing utility to graph the displacement function on the interval $[0, 10]$. Find a value of t past which the displacement is less than 3 inches from equilibrium.

93. ***Modeling Data*** A meteorologist measures the atmospheric pressure P (in kilograms per square meter) at altitude h (in kilometers). The data are shown below.

h	0	5	10	15	20
P	10,332	5583	2376	1240	517

(a) Use a graphing utility to plot the points $(h, \ln P)$. Use the regression capabilities of the graphing utility to find a linear model for the revised data points.

(b) The line in part (a) has the form $\ln P = ah + b$. Write the equation in exponential form.

(c) Use a graphing utility to plot the original data and graph the exponential model in part (b).

(d) Find the rate of change of the pressure when $h = 5$ and $h = 18$.

94. ***Modeling Data*** The table lists the approximate values V of a mid-sized sedan for the years 2003 through 2009. The variable t represents the time in years, with $t = 3$ corresponding to 2003.

t	3	4	5	6
V	\$23,046	\$20,596	\$18,851	\$17,001

t	7	8	9
V	\$15,226	\$14,101	\$12,841

(a) Use the regression capabilities of a graphing utility to fit linear and quadratic models to the data. Plot the data and graph the models.

(b) What does the slope represent in the linear model in part (a)?

(c) Use the regression capabilities of a graphing utility to fit an exponential model to the data.

(d) Determine the horizontal asymptote of the exponential model found in part (c). Interpret its meaning in the context of the problem.

(e) Find the rate of decrease in the value of the sedan when $t = 4$ and $t = 8$ using the exponential model.

Linear and Quadratic Approximations **In Exercises 95 and 96, use a graphing utility to graph the function. Then graph**

$$P_1(x) = f(0) + f'(0)(x - 0) \quad \text{and}$$
$$P_2(x) = f(0) + f'(0)(x - 0) + \tfrac{1}{2}f''(0)(x - 0)^2$$

in the same viewing window. Compare the values of f, P_1, and P_2 and their first derivatives at $x = 0$.

95. $f(x) = e^x$

96. $f(x) = e^{x/2}$

Stirling's Formula **For large values of n,**

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot (n - 1) \cdot n$$

can be approximated by Stirling's Formula, $n! \approx \left(\dfrac{n}{e}\right)^n \sqrt{2\pi n}$.

In Exercises 97 and 98, find the exact value of $n!$, and then approximate $n!$ using Stirling's Formula.

97. $n = 12$

98. $n = 15$

In Exercises 99–116, find the indefinite integral.

99. $\displaystyle\int e^{5x}(5)\,dx$

100. $\displaystyle\int e^{-x^4}(-4x^3)\,dx$

101. $\displaystyle\int e^{2x-1}\,dx$

102. $\displaystyle\int e^{1-3x}\,dx$

103. $\displaystyle\int x^2 e^{x^3}\,dx$

104. $\displaystyle\int e^x(e^x + 1)^2\,dx$

105. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

106. $\displaystyle\int \frac{e^{1/x^2}}{x^3}\,dx$

107. $\displaystyle\int \frac{e^{-x}}{1 + e^{-x}}\,dx$

108. $\displaystyle\int \frac{e^{2x}}{1 + e^{2x}}\,dx$

109. $\displaystyle\int e^x\sqrt{1 - e^x}\,dx$

110. $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

111. $\displaystyle\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$

112. $\displaystyle\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\,dx$

113. $\displaystyle\int \frac{5 - e^x}{e^{2x}}\,dx$

114. $\displaystyle\int \frac{e^{2x} + 2e^x + 1}{e^x}\,dx$

115. $\displaystyle\int e^{-x}\tan(e^{-x})\,dx$

116. $\displaystyle\int \ln(e^{2x-1})\,dx$

In Exercises 117–126, evaluate the definite integral. Use a graphing utility to verify your result.

117. $\displaystyle\int_0^1 e^{-2x}\,dx$

118. $\displaystyle\int_3^4 e^{3-x}\,dx$

119. $\displaystyle\int_0^1 xe^{-x^2}\,dx$

120. $\displaystyle\int_{-2}^0 x^2 e^{x^3/2}\,dx$

121. $\displaystyle\int_1^3 \frac{e^{3/x}}{x^2}\,dx$

122. $\displaystyle\int_0^{\sqrt{2}} xe^{-(x^2/2)}\,dx$

123. $\displaystyle\int_0^3 \frac{2e^{2x}}{1 + e^{2x}}\,dx$

124. $\displaystyle\int_0^1 \frac{e^x}{5 - e^x}\,dx$

125. $\displaystyle\int_0^{\pi/2} e^{\sin \pi x}\cos \pi x\,dx$

126. $\displaystyle\int_{\pi/3}^{\pi/2} e^{\sec 2x}\sec 2x\tan 2x\,dx$

Differential Equations **In Exercises 127 and 128, solve the differential equation.**

127. $\dfrac{dy}{dx} = xe^{ax^2}$

128. $\dfrac{dy}{dx} = (e^x - e^{-x})^2$

Differential Equations **In Exercises 129 and 130, find the particular solution that satisfies the initial conditions.**

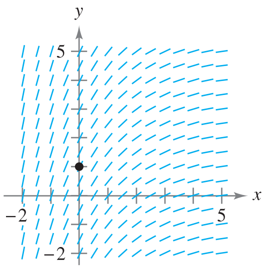
129. $f''(x) = \frac{1}{2}(e^x + e^{-x})$,
$f(0) = 1, f'(0) = 0$

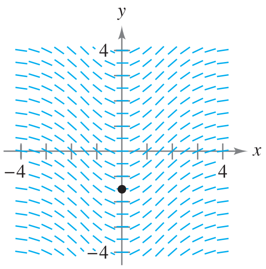
130. $f''(x) = \sin x + e^{2x}$,
$f(0) = \frac{1}{4}, f'(0) = \frac{1}{2}$

Slope Fields **In Exercises 131 and 132, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

131. $\dfrac{dy}{dx} = 2e^{-x/2}, \quad (0, 1)$

132. $\dfrac{dy}{dx} = xe^{-0.2x^2}, \quad \left(0, -\dfrac{3}{2}\right)$





Area **In Exercises 133–136, find the area of the region bounded by the graphs of the equations. Use a graphing utility to graph the region and verify your result.**

133. $y = e^x, y = 0, x = 0, x = 5$

134. $y = e^{-x}, y = 0, x = a, x = b$

135. $y = xe^{-x^2/4}, y = 0, x = 0, x = \sqrt{6}$

136. $y = e^{-2x} + 2, y = 0, x = 0, x = 2$

Numerical Integration **In Exercises 137 and 138, approximate the integral using the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule with $n = 12$. Use a graphing utility to verify your results.**

137. $\int_0^4 \sqrt{x}\, e^x\, dx$

138. $\int_0^2 2xe^{-x}\, dx$

139. ***Probability*** A car battery has an average lifetime of 48 months with a standard deviation of 6 months. The battery lives are normally distributed. The probability that a given battery will last between 48 months and 60 months is $0.0665 \int_{48}^{60} e^{-0.0139(t-48)^2}\, dt$. Use the integration capabilities of a graphing utility to approximate the integral. Interpret the resulting probability.

140. ***Probability*** The median waiting time (in minutes) for people waiting for service in a convenience store is given by the solution of the equation $\int_0^x 0.3e^{-0.3t}\, dt = \frac{1}{2}$. Solve the equation.

141. ***Horizontal Motion*** The position function of a particle moving along the x-axis is $x(t) = Ae^{kt} + Be^{-kt}$ where A, B, and k are positive constants.

(a) During what times t is the particle closest to the origin?

(b) Show that the acceleration of the particle is proportional to the position of the particle. What is the constant of proportionality?

142. ***Modeling Data*** A valve on a storage tank is opened for 4 hours to release a chemical in a manufacturing process. The flow rate R (in liters per hour) at time t (in hours) is given in the table.

t	0	1	2	3	4
R	425	240	118	71	36

Table for 142

(a) Use the regression capabilities of a graphing utility to find a linear model for the points $(t, \ln R)$. Write the resulting equation of the form $\ln R = at + b$ in exponential form.

(b) Use a graphing utility to plot the data and graph the exponential model.

(c) Use the definite integral to approximate the number of liters of chemical released during the 4 hours.

WRITING ABOUT CONCEPTS

143. In your own words, state the properties of the natural exponential function.

144. Is there a function f such that $f(x) = f'(x)$? If so, identify it.

145. Without integrating, state the integration formula you can use to integrate each of the following.

(a) $\displaystyle\int \frac{e^x}{e^x + 1}\, dx$ (b) $\displaystyle\int xe^{x^2}\, dx$

146. Consider the function $f(x) = \dfrac{2}{1 + e^{1/x}}$.

(a) Use a graphing utility to graph f.

(b) Write a short paragraph explaining why the graph has a horizontal asymptote at $y = 1$ and why the function has a nonremovable discontinuity at $x = 0$.

147. Given $e^x \geq 1$ for $x \geq 0$, it follows that $\int_0^x e^t\, dt \geq \int_0^x 1\, dt$. Perform this integration to derive the inequality $e^x \geq 1 + x$ for $x \geq 0$.

CAPSTONE

148. Describe the relationship between the graphs of $f(x) = \ln x$ and $g(x) = e^x$.

149. Find, to three decimal places, the value of x such that $e^{-x} = x$. (Use Newton's Method or the *zero* or *root* feature of a graphing utility.)

150. Find the value of a such that the area bounded by $y = e^{-x}$, the x-axis, $x = -a$, and $x = a$ is $\frac{8}{3}$.

151. Verify that the function

$$y = \frac{L}{1 + ae^{-x/b}}, \quad a > 0, \quad b > 0, \quad L > 0$$

increases at a maximum rate when $y = L/2$.

152. Let $f(x) = \dfrac{\ln x}{x}$.

(a) Graph f on $(0, \infty)$ and show that f is strictly decreasing on (e, ∞).

(b) Show that if $e \leq A < B$, then $A^B > B^A$.

(c) Use part (b) to show that $e^\pi > \pi^e$.

5.5 Bases Other Than *e* and Applications

- **Define exponential functions that have bases other than *e*.**
- **Differentiate and integrate exponential functions that have bases other than *e*.**
- **Use exponential functions to model compound interest and exponential growth.**

Bases Other than *e*

The **base** of the natural exponential function is e. This "natural" base can be used to assign a meaning to a general base a.

DEFINITION OF EXPONENTIAL FUNCTION TO BASE *a*

If a is a positive real number $(a \neq 1)$ and x is any real number, then the **exponential function to the base *a*** is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}.$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

These functions obey the usual laws of exponents. For instance, here are some familiar properties.

1. $a^0 = 1$

2. $a^x a^y = a^{x+y}$

3. $\dfrac{a^x}{a^y} = a^{x-y}$

4. $(a^x)^y = a^{xy}$

When modeling the half-life of a radioactive sample, it is convenient to use $\frac{1}{2}$ as the base of the exponential model. (*Half-life* is the number of years required for half of the atoms in a sample of radioactive material to decay.)

EXAMPLE 1 Radioactive Half-Life Model

The half-life of carbon-14 is about 5715 years. A sample contains 1 gram of carbon-14. How much will be present in 10,000 years?

Solution Let $t = 0$ represent the present time and let y represent the amount (in grams) of carbon-14 in the sample. Using a base of $\frac{1}{2}$, you can model y by the equation

$$y = \left(\frac{1}{2}\right)^{t/5715}.$$

Notice that when $t = 5715$, the amount is reduced to half of the original amount.

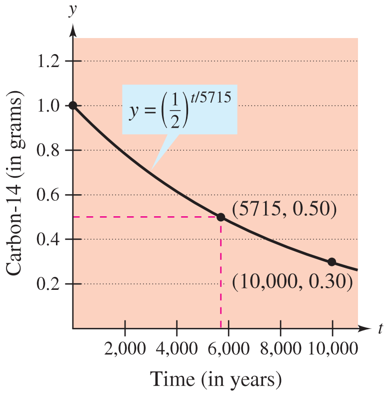
$$y = \left(\frac{1}{2}\right)^{5715/5715} = \frac{1}{2} \text{ gram}$$

When $t = 11{,}430$, the amount is reduced to a quarter of the original amount, and so on. To find the amount of carbon-14 after 10,000 years, substitute 10,000 for t.

$$y = \left(\frac{1}{2}\right)^{10{,}000/5715}$$

$$\approx 0.30 \text{ gram}$$

The graph of y is shown in Figure 5.25. ■



The half-life of carbon-14 is about 5715 years.
Figure 5.25

Logarithmic functions to bases other than e can be defined in much the same way as exponential functions to other bases are defined.

NOTE In precalculus, you learned that $\log_a x$ is the value to which a must be raised to produce x. This agrees with the definition given here because

$$\begin{aligned} a^{\log_a x} &= a^{(1/\ln a)\ln x} \\ &= (e^{\ln a})^{(1/\ln a)\ln x} \\ &= e^{(\ln a/\ln a)\ln x} \\ &= e^{\ln x} \\ &= x. \end{aligned}$$

DEFINITION OF LOGARITHMIC FUNCTION TO BASE *a*

If a is a positive real number $(a \neq 1)$ and x is any positive real number, then the **logarithmic function to the base *a*** is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

Logarithmic functions to the base a have properties similar to those of the natural logarithmic function given in Theorem 5.2. (Assume x and y are positive numbers and n is rational.)

1. $\log_a 1 = 0$ Log of 1

2. $\log_a xy = \log_a x + \log_a y$ Log of a product

3. $\log_a x^n = n \log_a x$ Log of a power

4. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$ Log of a quotient

From the definitions of the exponential and logarithmic functions to the base a, it follows that $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions of each other.

PROPERTIES OF INVERSE FUNCTIONS

1. $y = a^x$ if and only if $x = \log_a y$

2. $a^{\log_a x} = x$, for $x > 0$

3. $\log_a a^x = x$, for all x

The logarithmic function to the base 10 is called the **common logarithmic function.** So, for common logarithms, $y = 10^x$ if and only if $x = \log_{10} y$.

EXAMPLE 2 Bases Other Than *e*

Solve for x in each equation.

a. $3^x = \dfrac{1}{81}$

b. $\log_2 x = -4$

Solution

a. To solve this equation, you can apply the logarithmic function to the base 3 to each side of the equation.

$$\begin{aligned} 3^x &= \frac{1}{81} \\ \log_3 3^x &= \log_3 \frac{1}{81} \\ x &= \log_3 3^{-4} \\ x &= -4 \end{aligned}$$

b. To solve this equation, you can apply the exponential function to the base 2 to each side of the equation.

$$\begin{aligned} \log_2 x &= -4 \\ 2^{\log_2 x} &= 2^{-4} \\ x &= \frac{1}{2^4} \\ x &= \frac{1}{16} \end{aligned}$$

■

Differentiation and Integration

To differentiate exponential and logarithmic functions to other bases, you have three options: (1) use the definitions of a^x and $\log_a x$ and differentiate using the rules for the natural exponential and logarithmic functions, (2) use logarithmic differentiation, or (3) use the following differentiation rules for bases other than e.

THEOREM 5.13 DERIVATIVES FOR BASES OTHER THAN *e*

Let a be a positive real number $(a \neq 1)$ and let u be a differentiable function of x.

1. $\dfrac{d}{dx}[a^x] = (\ln a)a^x$

2. $\dfrac{d}{dx}[a^u] = (\ln a)a^u\dfrac{du}{dx}$

3. $\dfrac{d}{dx}[\log_a x] = \dfrac{1}{(\ln a)x}$

4. $\dfrac{d}{dx}[\log_a u] = \dfrac{1}{(\ln a)u}\dfrac{du}{dx}$

PROOF By definition, $a^x = e^{(\ln a)x}$. So, you can prove the first rule by letting $u = (\ln a)x$ and differentiating with base e to obtain

$$\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{(\ln a)x}] = e^u\frac{du}{dx} = e^{(\ln a)x}(\ln a) = (\ln a)a^x.$$

To prove the third rule, you can write

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{1}{\ln a}\ln x\right] = \frac{1}{\ln a}\left(\frac{1}{x}\right) = \frac{1}{(\ln a)x}.$$

The second and fourth rules are simply the Chain Rule versions of the first and third rules. ■

NOTE These differentiation rules are similar to those for the natural exponential function and the natural logarithmic function. In fact, they differ only by the constant factors $\ln a$ and $1/\ln a$. This points out one reason why, for calculus, e is the most convenient base. ■

EXAMPLE 3 Differentiating Functions to Other Bases

Find the derivative of each function.

a. $y = 2^x$

b. $y = 2^{3x}$

c. $y = \log_{10} \cos x$

Solution

a. $y' = \dfrac{d}{dx}[2^x] = (\ln 2)2^x$

b. $y' = \dfrac{d}{dx}[2^{3x}] = (\ln 2)2^{3x}(3) = (3 \ln 2)2^{3x}$

Try writing 2^{3x} as 8^x and differentiating to see that you obtain the same result.

c. $y' = \dfrac{d}{dx}[\log_{10} \cos x] = \dfrac{-\sin x}{(\ln 10)\cos x} = -\dfrac{1}{\ln 10}\tan x$ ■

Occasionally, an integrand involves an exponential function to a base other than e. When this occurs, there are two options: (1) convert to base e using the formula $a^x = e^{(\ln a)x}$ and then integrate, or (2) integrate directly, using the integration formula

$$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$$

(which follows from Theorem 5.13).

EXAMPLE 4 Integrating an Exponential Function to Another Base

Find $\int 2^x\,dx$.

Solution

$$\int 2^x\,dx = \frac{1}{\ln 2}2^x + C$$

■

When the Power Rule, $D_x[x^n] = nx^{n-1}$, was introduced in Chapter 2, the exponent n was required to be a rational number. Now the rule is extended to cover any real value of n. Try to prove this theorem using logarithmic differentiation.

THEOREM 5.14 THE POWER RULE FOR REAL EXPONENTS

Let n be any real number and let u be a differentiable function of x.

1. $\dfrac{d}{dx}[x^n] = nx^{n-1}$
2. $\dfrac{d}{dx}[u^n] = nu^{n-1}\dfrac{du}{dx}$

The next example compares the derivatives of four types of functions. Each function uses a different differentiation formula, depending on whether the base and the exponent are constants or variables.

EXAMPLE 5 Comparing Variables and Constants

a. $\dfrac{d}{dx}[e^e] = 0$ Constant Rule

b. $\dfrac{d}{dx}[e^x] = e^x$ Exponential Rule

c. $\dfrac{d}{dx}[x^e] = ex^{e-1}$ Power Rule

d. $y = x^x$ Logarithmic differentiation

$$\ln y = \ln x^x$$
$$\ln y = x \ln x$$
$$\frac{y'}{y} = x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x$$
$$y' = y(1 + \ln x) = x^x(1 + \ln x)$$

■

NOTE Be sure you see that there is no simple differentiation rule for calculating the derivative of $y = x^x$. In general, if $y = u(x)^{v(x)}$, you need to use logarithmic differentiation.

Applications of Exponential Functions

Suppose P dollars is deposited in an account at an annual interest rate r (in decimal form). If interest accumulates in the account, what is the balance in the account at the end of 1 year? The answer depends on the number of times n the interest is compounded according to the formula

$$A = P\left(1 + \frac{r}{n}\right)^n.$$

For instance, the result for a deposit of \$1000 at 8% interest compounded n times a year is shown in the upper table at the left.

n	A
1	\$1080.00
2	\$1081.60
4	\$1082.43
12	\$1083.00
365	\$1083.28

As n increases, the balance A approaches a limit. To develop this limit, use the following theorem. To test the reasonableness of this theorem, try evaluating $[(x + 1)/x]^x$ for several values of x, as shown in the lower table at the left. (A proof of this theorem is given in Appendix A.)

x	$\left(\frac{x+1}{x}\right)^x$
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

THEOREM 5.15 A LIMIT INVOLVING *e*

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$$

Now, let's take another look at the formula for the balance A in an account in which the interest is compounded n times per year. By taking the limit as n approaches infinity, you obtain

$$\begin{aligned}
A &= \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^n && \text{Take limit as } n \to \infty.\\
&= P \lim_{n\to\infty}\left[\left(1 + \frac{1}{n/r}\right)^{n/r}\right]^r && \text{Rewrite.}\\
&= P\left[\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x\right]^r && \text{Let } x = n/r. \text{ Then } x\to\infty \text{ as } n\to\infty.\\
&= Pe^r. && \text{Apply Theorem 5.15.}
\end{aligned}$$

This limit produces the balance after 1 year of **continuous compounding.** So, for a deposit of \$1000 at 8% interest compounded continuously, the balance at the end of 1 year would be

$$\begin{aligned}
A &= 1000e^{0.08}\\
&\approx \$1083.29.
\end{aligned}$$

These results are summarized below.

SUMMARY OF COMPOUND INTEREST FORMULAS

Let P = amount of deposit, t = number of years, A = balance after t years, r = annual interest rate (decimal form), and n = number of compoundings per year.

1. Compounded n times per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
2. Compounded continuously: $A = Pe^{rt}$

EXAMPLE 6 Comparing Continuous, Quarterly, and Monthly Compounding

A deposit of \$2500 is made in an account that pays an annual interest rate of 5%. Find the balance in the account at the end of 5 years if the interest is compounded (a) quarterly, (b) monthly, and (c) continuously.

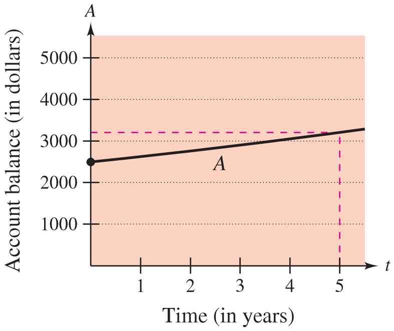
Solution

a. $$A = P\left(1 + \frac{r}{n}\right)^{nt} = 2500\left(1 + \frac{0.05}{4}\right)^{4(5)} \qquad \text{Compounded quarterly}$$
$$= 2500(1.0125)^{20}$$
$$\approx \$3205.09$$

b. $$A = P\left(1 + \frac{r}{n}\right)^{nt} = 2500\left(1 + \frac{0.05}{12}\right)^{12(5)} \qquad \text{Compounded monthly}$$
$$\approx 2500(1.0041667)^{60}$$
$$\approx \$3208.40$$

c. $$A = Pe^{rt} = 2500[e^{0.05(5)}] \qquad \text{Compounded continuously}$$
$$= 2500e^{0.25} \approx \$3210.06$$

Figure 5.26 shows how the balance increases over the five-year period. Notice that the scale used in the figure does not graphically distinguish among the three types of exponential growth in (a), (b), and (c).



The balance in a savings account grows exponentially.
Figure 5.26

EXAMPLE 7 Bacterial Culture Growth

A bacterial culture is growing according to the *logistic growth function*

$$y = \frac{1.25}{1 + 0.25e^{-0.4t}}, \quad t \ge 0$$

where *y* is the weight of the culture in grams and *t* is the time in hours. Find the weight of the culture after (a) 0 hours, (b) 1 hour, and (c) 10 hours. (d) What is the limit as *t* approaches infinity?

Solution

a. When $t = 0$, $$y = \frac{1.25}{1 + 0.25e^{-0.4(0)}}$$
$$= 1 \text{ gram.}$$

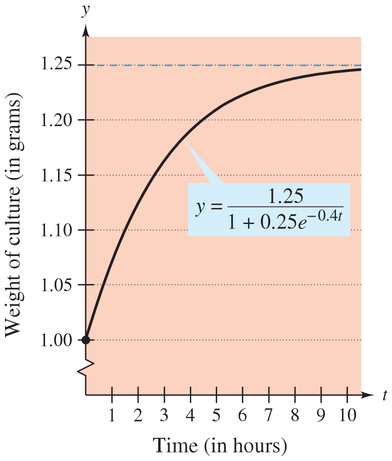
b. When $t = 1$, $$y = \frac{1.25}{1 + 0.25e^{-0.4(1)}}$$
$$\approx 1.071 \text{ grams.}$$

c. When $t = 10$, $$y = \frac{1.25}{1 + 0.25e^{-0.4(10)}}$$
$$\approx 1.244 \text{ grams.}$$

d. Finally, taking the limit as *t* approaches infinity, you obtain

$$\lim_{t\to\infty} \frac{1.25}{1 + 0.25e^{-0.4t}} = \frac{1.25}{1 + 0} = 1.25 \text{ grams.}$$

The graph of the function is shown in Figure 5.27. ■



The limit of the weight of the culture as $t \to \infty$ is 1.25 grams.
Figure 5.27

5.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, evaluate the expression without using a calculator.

1. $\log_2 \frac{1}{8}$

2. $\log_{27} 9$

3. $\log_7 1$

4. $\log_a \dfrac{1}{a}$

In Exercises 5–8, write the exponential equation as a logarithmic equation or vice versa.

5. (a) $2^3 = 8$
(b) $3^{-1} = \frac{1}{3}$

6. (a) $27^{2/3} = 9$
(b) $16^{3/4} = 8$

7. (a) $\log_{10} 0.01 = -2$
(b) $\log_{0.5} 8 = -3$

8. (a) $\log_3 \frac{1}{9} = -2$
(b) $49^{1/2} = 7$

In Exercises 9–14, sketch the graph of the function by hand.

9. $y = 3^x$

10. $y = 3^{x-1}$

11. $y = \left(\frac{1}{3}\right)^x$

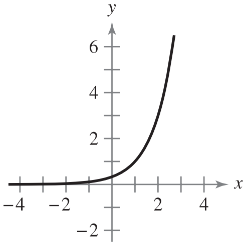
12. $y = 2^{x^2}$

13. $h(x) = 5^{x-2}$

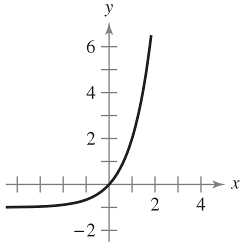
14. $y = 3^{-|x|}$

In Exercises 15–18, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

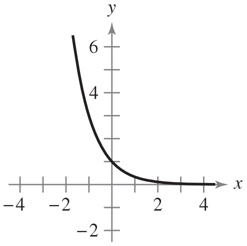
(a)



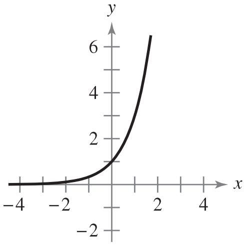
(b)



(c)



(d)



15. $f(x) = 3^x$

16. $f(x) = 3^{-x}$

17. $f(x) = 3^x - 1$

18. $f(x) = 3^{x-1}$

In Exercises 19–24, solve for *x* or *b*.

19. (a) $\log_{10} 1000 = x$
(b) $\log_{10} 0.1 = x$

20. (a) $\log_3 \frac{1}{81} = x$
(b) $\log_6 36 = x$

21. (a) $\log_3 x = -1$
(b) $\log_2 x = -4$

22. (a) $\log_b 27 = 3$
(b) $\log_b 125 = 3$

23. (a) $x^2 - x = \log_5 25$
(b) $3x + 5 = \log_2 64$

24. (a) $\log_3 x + \log_3(x - 2) = 1$
(b) $\log_{10}(x + 3) - \log_{10} x = 1$

In Exercises 25–34, solve the equation accurate to three decimal places.

25. $3^{2x} = 75$

26. $5^{6x} = 8320$

27. $2^{3-z} = 625$

28. $3(5^{x-1}) = 86$

29. $\left(1 + \dfrac{0.09}{12}\right)^{12t} = 3$

30. $\left(1 + \dfrac{0.10}{365}\right)^{365t} = 2$

31. $\log_2(x - 1) = 5$

32. $\log_{10}(t - 3) = 2.6$

33. $\log_3 x^2 = 4.5$

34. $\log_5 \sqrt{x - 4} = 3.2$

In Exercises 35–38, use a graphing utility to graph the function and approximate its zero(s) accurate to three decimal places.

35. $g(x) = 6(2^{1-x}) - 25$

36. $f(t) = 300(1.0075^{12t}) - 735.41$

37. $h(s) = 32 \log_{10}(s - 2) + 15$

38. $g(x) = 1 - 2 \log_{10}[x(x - 3)]$

In Exercises 39 and 40, illustrate that the functions are inverse functions of each other by sketching their graphs on the same set of coordinate axes.

39. $f(x) = 4^x$
$g(x) = \log_4 x$

40. $f(x) = 3^x$
$g(x) = \log_3 x$

In Exercises 41–62, find the derivative of the function. (*Hint:* In some exercises, you may find it helpful to apply logarithmic properties *before* differentiating.)

41. $f(x) = 4^x$

42. $f(x) = 3^{2x}$

43. $y = 5^{-4x}$

44. $y = 7^{2x-1}$

45. $f(x) = x\,9^x$

46. $y = x(6^{-2x})$

47. $g(t) = t^2 2^t$

48. $f(t) = \dfrac{3^{2t}}{t}$

49. $h(\theta) = 2^{-\theta} \cos \pi\theta$

50. $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$

51. $y = \log_4(5x + 1)$

52. $y = \log_3(x^2 - 3x)$

53. $h(t) = \log_5(4 - t)^2$

54. $g(t) = \log_2(t^2 + 7)^3$

55. $y = \log_5 \sqrt{x^2 - 1}$

56. $f(x) = \log_2 \sqrt[3]{2x + 1}$

57. $f(x) = \log_2 \dfrac{x^2}{x - 1}$

58. $y = \log_{10} \dfrac{x^2 - 1}{x}$

59. $h(x) = \log_3 \dfrac{x\sqrt{x - 1}}{2}$

60. $g(x) = \log_5 \dfrac{4}{x^2\sqrt{1 - x}}$

61. $g(t) = \dfrac{10 \log_4 t}{t}$

62. $f(t) = t^{3/2} \log_2 \sqrt{t + 1}$

In Exercises 63–66, find an equation of the tangent line to the graph of the function at the given point.

63. $y = 2^{-x}$, $(-1, 2)$

64. $y = 5^{x-2}$, $(2, 1)$

65. $y = \log_3 x$, $(27, 3)$

66. $y = \log_{10} 2x$, $(5, 1)$

In Exercises 67–70, use logarithmic differentiation to find dy/dx.

67. $y = x^{2/x}$

68. $y = x^{x-1}$

69. $y = (x-2)^{x+1}$

70. $y = (1+x)^{1/x}$

In Exercises 71–74, find an equation of the tangent line to the graph of the function at the given point.

71. $y = x^{\sin x}, \quad \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

72. $y = (\sin x)^{2x}, \quad \left(\frac{\pi}{2}, 1\right)$

73. $y = (\ln x)^{\cos x}, \quad (e, 1)$

74. $y = x^{1/x}, \quad (1, 1)$

In Exercises 75–82, find the integral.

75. $\int 3^x\,dx$

76. $\int 5^{-x}\,dx$

77. $\int (x^2 + 2^{-x})\,dx$

78. $\int (x^3 + 3^{-x})\,dx$

79. $\int x(5^{-x^2})\,dx$

80. $\int (3-x)7^{(3-x)^2}\,dx$

81. $\int \frac{3^{2x}}{1+3^{2x}}\,dx$

82. $\int 2^{\sin x}\cos x\,dx$

In Exercises 83–86, evaluate the integral.

83. $\int_{-1}^{2} 2^x\,dx$

84. $\int_{-2}^{2} 4^{x/2}\,dx$

85. $\int_{0}^{1} (5^x - 3^x)\,dx$

86. $\int_{1}^{e} (6^x - 2^x)\,dx$

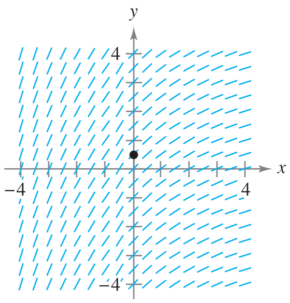
Area **In Exercises 87 and 88, find the area of the region bounded by the graphs of the equations.**

87. $y = 3^x, y = 0, x = 0, x = 3$

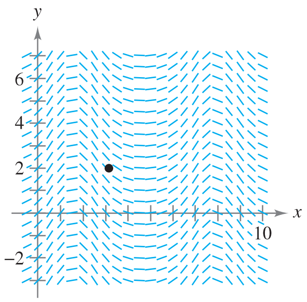
88. $y = 3^{\cos x}\sin x, y = 0, x = 0, x = \pi$

Slope Fields **In Exercises 89 and 90, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

89. $\frac{dy}{dx} = 0.4^{x/3}, \quad \left(0, \frac{1}{2}\right)$



90. $\frac{dy}{dx} = e^{\sin x}\cos x, \quad (\pi, 2)$



WRITING ABOUT CONCEPTS

91. Consider the function $f(x) = \log_{10} x$.

(a) What is the domain of f?

(b) Find f^{-1}.

(c) If x is a real number between 1000 and 10,000, determine the interval in which $f(x)$ will be found.

(d) Determine the interval in which x will be found if $f(x)$ is negative.

(e) If $f(x)$ is increased by one unit, x must have been increased by what factor?

(f) Find the ratio of x_1 to x_2 given that $f(x_1) = 3n$ and $f(x_2) = n$.

92. Order the functions

$$f(x) = \log_2 x, \; g(x) = x^x, \; h(x) = x^2, \text{ and } k(x) = 2^x$$

from the one with the greatest rate of growth to the one with the least rate of growth for large values of x.

93. Find the derivative of each function, given that a is constant.

(a) $y = x^a$ (b) $y = a^x$

(c) $y = x^x$ (d) $y = a^a$

CAPSTONE

94. The table of values below was obtained by evaluating a function. Determine which of the statements may be true and which must be false, and explain why.

x	1	2	8
y	0	1	3

(a) y is an exponential function of x.

(b) y is a logarithmic function of x.

(c) x is an exponential function of y.

(d) y is a linear function of x.

95. ***Inflation*** If the annual rate of inflation averages 5% over the next 10 years, the approximate cost C of goods or services during any year in that decade is

$$C(t) = P(1.05)^t$$

where t is the time in years and P is the present cost.

(a) The price of an oil change for your car is presently \$24.95. Estimate the price 10 years from now.

(b) Find the rates of change of C with respect to t when $t = 1$ and $t = 8$.

(c) Verify that the rate of change of C is proportional to C. What is the constant of proportionality?

96. ***Depreciation*** After t years, the value of a car purchased for \$25,000 is

$$V(t) = 25{,}000\left(\tfrac{3}{4}\right)^t.$$

(a) Use a graphing utility to graph the function and determine the value of the car 2 years after it was purchased.

(b) Find the rates of change of V with respect to t when $t = 1$ and $t = 4$.

(c) Use a graphing utility to graph $V'(t)$ and determine the horizontal asymptote of $V'(t)$. Interpret its meaning in the context of the problem.

Compound Interest **In Exercises 97–100, complete the table by determining the balance *A* for *P* dollars invested at rate *r* for *t* years and compounded *n* times per year.**

n	1	2	4	12	365	Continuous compounding
A						

97. $P = \$1000$
$r = 3\frac{1}{2}\%$
$t = 10$ years

98. $P = \$2500$
$r = 6\%$
$t = 20$ years

99. $P = \$1000$
$r = 5\%$
$t = 30$ years

100. $P = \$5000$
$r = 7\%$
$t = 25$ years

Compound Interest **In Exercises 101–104, complete the table by determining the amount of money *P* (present value) that should be invested at rate *r* to produce a balance of \$100,000 in *t* years.**

t	1	10	20	30	40	50
P						

101. $r = 5\%$
Compounded continuously

102. $r = 6\%$
Compounded continuously

103. $r = 5\%$
Compounded monthly

104. $r = 7\%$
Compounded daily

105. ***Compound Interest*** Assume that you can earn 6% on an investment, compounded daily. Which of the following options would yield the greatest balance after 8 years?

(a) \$20,000 now

(b) \$30,000 after 8 years

(c) \$8000 now and \$20,000 after 4 years

(d) \$9000 now, \$9000 after 4 years, and \$9000 after 8 years

106. ***Compound Interest*** Consider a deposit of \$100 placed in an account for 20 years at r% compounded continuously. Use a graphing utility to graph the exponential functions describing the growth of the investment over the 20 years for the following interest rates. Compare the ending balances for the three rates.

(a) $r = 3\%$

(b) $r = 5\%$

(c) $r = 6\%$

107. ***Timber Yield*** The yield V (in millions of cubic feet per acre) for a stand of timber at age t is

$$V = 6.7e^{(-48.1)/t}$$

where t is measured in years.

(a) Find the limiting volume of wood per acre as t approaches infinity.

(b) Find the rates at which the yield is changing when $t = 20$ years and $t = 60$ years.

108. ***Learning Theory*** In a group project in learning theory, a mathematical model for the proportion P of correct responses after n trials was found to be

$$P = \frac{0.86}{1 + e^{-0.25n}}.$$

(a) Find the limiting proportion of correct responses as n approaches infinity.

(b) Find the rates at which P is changing after $n = 3$ trials and $n = 10$ trials.

109. ***Forest Defoliation*** To estimate the amount of defoliation caused by the gypsy moth during a year, a forester counts the number of egg masses on $\frac{1}{40}$ of an acre the preceding fall. The percent of defoliation y is approximated by

$$y = \frac{300}{3 + 17e^{-0.0625x}}$$

where x is the number of egg masses in thousands. *(Source: USDA Forest Service)*

(a) Use a graphing utility to graph the function.

(b) Estimate the percent of defoliation if 2000 egg masses are counted.

(c) Estimate the number of egg masses that existed if you observe that approximately $\frac{2}{3}$ of a forest is defoliated.

(d) Use calculus to estimate the value of x for which y is increasing most rapidly.

110. ***Population Growth*** A lake is stocked with 500 fish, and the population increases according to the logistic curve

$$p(t) = \frac{10{,}000}{1 + 19e^{-t/5}}$$

where t is measured in months.

(a) Use a graphing utility to graph the function.

(b) What is the limiting size of the fish population?

(c) At what rates is the fish population changing at the end of 1 month and at the end of 10 months?

(d) After how many months is the population increasing most rapidly?

111. ***Modeling Data*** The breaking strengths B (in tons) of steel cables of various diameters d (in inches) are shown in the table.

d	0.50	0.75	1.00	1.25	1.50	1.75
B	9.85	21.8	38.3	59.2	84.4	114.0

(a) Use the regression capabilities of a graphing utility to fit an exponential model to the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Find the rates of growth of the model when $d = 0.8$ and $d = 1.5$.

112. ***Comparing Models*** The numbers y (in thousands) of organ transplants in the United States in the years 2001 through 2006 are shown in the table, with $x = 1$ corresponding to 2001. *(Source: Organ Procurement and Transplantation Network)*

x	1	2	3	4	5	6
y	24.2	24.9	25.5	27.0	28.1	28.9

(a) Use the regression capabilities of a graphing utility to find the following models for the data.

$y_1 = ax + b$ $\qquad$ $y_2 = a + b \ln x$

$y_3 = ab^x$ $\qquad$ $y_4 = ax^b$

(b) Use a graphing utility to plot the data and graph each of the models. Which model do you think best fits the data?

(c) Interpret the slope of the linear model in the context of the problem.

(d) Find the rate of change of each of the models for the year 2004. Which model is increasing at the greatest rate in 2004?

113. ***Conjecture***

(a) Use a graphing utility to approximate the integrals of the functions

$$f(t) = 4\left(\frac{3}{8}\right)^{2t/3}, \quad g(t) = 4\left(\frac{\sqrt[3]{9}}{4}\right)^t, \text{ and } h(t) = 4e^{-0.653886t}$$

on the interval $[0, 4]$.

(b) Use a graphing utility to graph the three functions.

(c) Use the results of parts (a) and (b) to make a conjecture about the three functions. Could you make the conjecture using only part (a)? Explain. Prove your conjecture analytically.

114. Complete the table to demonstrate that e can also be defined as $\lim\limits_{x \to 0^+} (1 + x)^{1/x}$.

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1 + x)^{1/x}$					

In Exercises 115 and 116, find an exponential function that fits the experimental data collected over time t.

115.

t	0	1	2	3	4
y	1200.00	720.00	432.00	259.20	155.52

116.

t	0	1	2	3	4
y	600.00	630.00	661.50	694.58	729.30

In Exercises 117–120, find the exact value of the expression.

117. $5^{1/\ln 5}$

118. $6^{\ln 10/\ln 6}$

119. $9^{1/\ln 3}$

120. $32^{1/\ln 2}$

True or False? **In Exercises 121–126, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

121. $e = \dfrac{271{,}801}{99{,}900}$

122. If $f(x) = \ln x$, then $f(e^{n+1}) - f(e^n) = 1$ for any value of n.

123. The functions $f(x) = 2 + e^x$ and $g(x) = \ln(x - 2)$ are inverse functions of each other.

124. The exponential function $y = Ce^x$ is a solution of the differential equation $d^n y/dx^n = y$, $n = 1, 2, 3, \ldots$.

125. The graphs of $f(x) = e^x$ and $g(x) = e^{-x}$ meet at right angles.

126. If $f(x) = g(x)e^x$, then the only zeros of f are the zeros of g.

127. (a) Show that $(2^3)^2 \neq 2^{(3^2)}$.

(b) Are $f(x) = (x^x)^x$ and $g(x) = x^{(x^x)}$ the same function? Why or why not?

(c) Find $f'(x)$ and $g'(x)$.

128. Let $f(x) = \dfrac{a^x - 1}{a^x + 1}$ for $a > 0$, $a \neq 1$. Show that f has an inverse function. Then find f^{-1}.

129. Show that solving the logistic differential equation

$$\frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), \quad y(0) = 1$$

results in the logistic growth function in Example 7.

$$\left[\textit{Hint: } \frac{1}{y\left(\frac{5}{4} - y\right)} = \frac{4}{5}\left(\frac{1}{y} + \frac{1}{\frac{5}{4} - y}\right)\right]$$

130. Given the exponential function $f(x) = a^x$, show that

(a) $f(u + v) = f(u) \cdot f(v)$.

(b) $f(2x) = [f(x)]^2$.

131. (a) Determine y' given $y^x = x^y$.

(b) Find the slope of the tangent line to the graph of $y^x = x^y$ at each of the following points.

(i) (c, c)

(ii) $(2, 4)$

(iii) $(4, 2)$

(c) At what points on the graph of $y^x = x^y$ does the tangent line not exist?

132. Consider the functions $f(x) = 1 + x$ and $g(x) = b^x$, $b > 1$.

(a) Given $b = 2$, use a graphing utility to graph f and g in the same viewing window. Identify the point(s) of intersection.

(b) Repeat part (a) using $b = 3$.

(c) Find all values of b such that $g(x) \geq f(x)$ for all x.

PUTNAM EXAM CHALLENGE

133. Which is greater

$$\left(\sqrt{n}\right)^{\sqrt{n+1}} \quad \text{or} \quad \left(\sqrt{n+1}\right)^{\sqrt{n}}$$

where $n > 8$?

134. Show that if x is positive, then

$$\log_e\left(1 + \frac{1}{x}\right) > \frac{1}{1 + x}.$$

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SECTION PROJECT

Using Graphing Utilities to Estimate Slope

Let $f(x) = \begin{cases} |x|^x, & x \neq 0 \\ 1, & x = 0. \end{cases}$

(a) Use a graphing utility to graph f in the viewing window $-3 \leq x \leq 3$, $-2 \leq y \leq 2$. What is the domain of f?

(b) Use the *zoom* and *trace* features of a graphing utility to estimate

$$\lim_{x \to 0} f(x).$$

(c) Write a short paragraph explaining why the function f is continuous for all real numbers.

(d) Visually estimate the slope of f at the point $(0, 1)$.

(e) Explain why the derivative of a function can be approximated by the formula

$$\frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

for small values of Δx. Use this formula to approximate the slope of f at the point $(0, 1)$.

$$f'(0) \approx \frac{f(0 + \Delta x) - f(0 - \Delta x)}{2\Delta x} = \frac{f(\Delta x) - f(-\Delta x)}{2\Delta x}$$

What do you think the slope of the graph of f is at $(0, 1)$?

(f) Find a formula for the derivative of f and determine $f'(0)$. Write a short paragraph explaining how a graphing utility might lead you to approximate the slope of a graph incorrectly.

(g) Use your formula for the derivative of f to find the relative extrema of f. Verify your answer using a graphing utility.

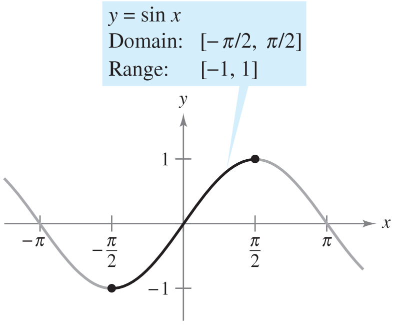
■ **FOR FURTHER INFORMATION** For more information on using graphing utilities to estimate slope, see the article "Computer-Aided Delusions" by Richard L. Hall in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

5.6 Inverse Trigonometric Functions: Differentiation

- **Develop properties of the six inverse trigonometric functions.**
- **Differentiate an inverse trigonometric function.**
- **Review the basic differentiation rules for elementary functions.**

Inverse Trigonometric Functions

This section begins with a rather surprising statement: *None of the six basic trigonometric functions has an inverse function.* This statement is true because all six trigonometric functions are periodic and therefore are not one-to-one. In this section you will examine these six functions to see whether their domains can be redefined in such a way that they will have inverse functions on the *restricted domains.*



The sine function is one-to-one on $[-\pi/2, \pi/2]$.
Figure 5.28

NOTE The term "iff" is used to represent the phrase "if and only if."

In Example 4 of Section 5.3, you saw that the sine function is increasing (and therefore is one-to-one) on the interval $[-\pi/2, \pi/2]$ (see Figure 5.28). On this interval you can define the inverse of the *restricted* sine function as

$$y = \arcsin x \quad \text{if and only if} \quad \sin y = x$$

where $-1 \le x \le 1$ and $-\pi/2 \le \arcsin x \le \pi/2$.

Under suitable restrictions, each of the six trigonometric functions is one-to-one and so has an inverse function, as shown in the following definition.

DEFINITIONS OF INVERSE TRIGONOMETRIC FUNCTIONS

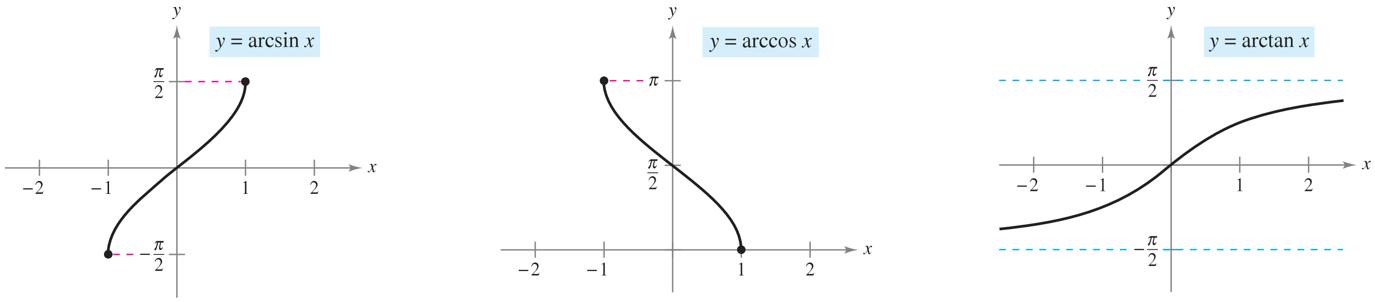
Function	*Domain*	*Range*
$y = \arcsin x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \text{arccot}\ x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \text{arcsec}\ x$ iff $\sec y = x$	$\lvert x \rvert \ge 1$	$0 \le y \le \pi, \quad y \ne \frac{\pi}{2}$
$y = \text{arccsc}\ x$ iff $\csc y = x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \quad y \ne 0$

NOTE The term "arcsin x" is read as "the arcsine of x" or sometimes "the angle whose sine is x." An alternative notation for the inverse sine function is "$\sin^{-1} x$." ■

EXPLORATION

The Inverse Secant Function In the definition above, the inverse secant function is defined by restricting the domain of the secant function to the intervals $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. Most other texts and reference books agree with this, but some disagree. What other domains might make sense? Explain your reasoning graphically. Most calculators do not have a key for the inverse secant function. How can you use a calculator to evaluate the inverse secant function?

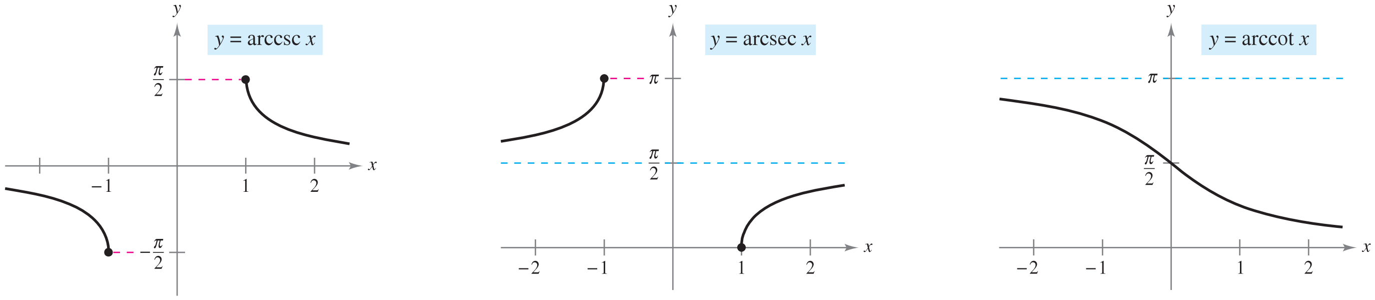
The graphs of the six inverse trigonometric functions are shown in Figure 5.29.



Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$

Domain: $[-1, 1]$
Range: $[0, \pi]$

Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$

Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[0, \pi/2) \cup (\pi/2, \pi]$

Domain: $(-\infty, \infty)$
Range: $(0, \pi)$

Figure 5.29

EXAMPLE 1 Evaluating Inverse Trigonometric Functions

Evaluate each function.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\arccos 0$ **c.** $\arctan \sqrt{3}$ **d.** $\arcsin(0.3)$

NOTE When evaluating inverse trigonometric functions, remember that they denote *angles in radian measure*.

Solution

a. By definition, $y = \arcsin\left(-\frac{1}{2}\right)$ implies that $\sin y = -\frac{1}{2}$. In the interval $[-\pi/2, \pi/2]$, the correct value of y is $-\pi/6$.

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

b. By definition, $y = \arccos 0$ implies that $\cos y = 0$. In the interval $[0, \pi]$, you have $y = \pi/2$.

$$\arccos 0 = \frac{\pi}{2}$$

c. By definition, $y = \arctan \sqrt{3}$ implies that $\tan y = \sqrt{3}$. In the interval $(-\pi/2, \pi/2)$, you have $y = \pi/3$.

$$\arctan \sqrt{3} = \frac{\pi}{3}$$

d. Using a calculator set in *radian* mode produces

$$\arcsin(0.3) \approx 0.305.$$ ■

EXPLORATION

Graph $y = \arccos(\cos x)$ for $-4\pi \le x \le 4\pi$. Why isn't the graph the same as the graph of $y = x$?

Inverse functions have the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

When applying these properties to inverse trigonometric functions, remember that the trigonometric functions have inverse functions only in restricted domains. For x-values outside these domains, these two properties do not hold. For example, $\arcsin(\sin \pi)$ is equal to 0, not π.

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

If $|x| \ge 1$ and $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$, then

$$\sec(\operatorname{arcsec} x) = x \quad \text{and} \quad \operatorname{arcsec}(\sec y) = y.$$

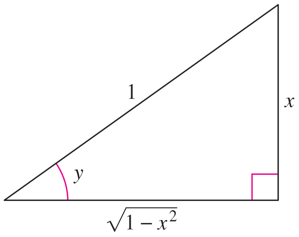
Similar properties hold for the other inverse trigonometric functions.

EXAMPLE 2 Solving an Equation

$$\begin{aligned} \arctan(2x - 3) &= \frac{\pi}{4} && \text{Original equation} \\ \tan[\arctan(2x - 3)] &= \tan\frac{\pi}{4} && \text{Take tangent of each side.} \\ 2x - 3 &= 1 && \tan(\arctan x) = x \\ x &= 2 && \text{Solve for } x. \end{aligned}$$

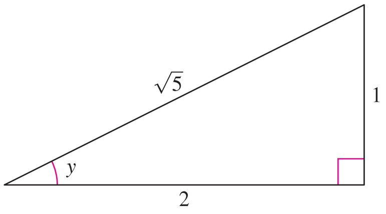
■

Some problems in calculus require that you evaluate expressions such as $\cos(\arcsin x)$, as shown in Example 3.



$y = \arcsin x$

Figure 5.30



$y = \operatorname{arcsec}\dfrac{\sqrt{5}}{2}$

Figure 5.31

EXAMPLE 3 Using Right Triangles

a. Given $y = \arcsin x$, where $0 < y < \pi/2$, find $\cos y$.

b. Given $y = \operatorname{arcsec}(\sqrt{5}/2)$, find $\tan y$.

Solution

a. Because $y = \arcsin x$, you know that $\sin y = x$. This relationship between x and y can be represented by a right triangle, as shown in Figure 5.30.

$$\cos y = \cos(\arcsin x) = \frac{\text{adj.}}{\text{hyp.}} = \sqrt{1 - x^2}$$

(This result is also valid for $-\pi/2 < y < 0$.)

b. Use the right triangle shown in Figure 5.31.

$$\tan y = \tan\left[\operatorname{arcsec}\left(\frac{\sqrt{5}}{2}\right)\right] = \frac{\text{opp.}}{\text{adj.}} = \frac{1}{2}$$

■

NOTE There is no common agreement on the definition of arcsec x (or arccsc x) for negative values of x. When we defined the range of the arcsecant, we chose to preserve the reciprocal identity

$$\operatorname{arcsec} x = \arccos \frac{1}{x}.$$

For example, to evaluate arcsec(-2), you can write

$$\operatorname{arcsec}(-2) = \arccos(-0.5) \approx 2.09.$$

One of the consequences of the definition of the inverse secant function given in this text is that its graph has a positive slope at every x-value in its domain. (See Figure 5.29.) This accounts for the absolute value sign in the formula for the derivative of arcsec x.

Derivatives of Inverse Trigonometric Functions

In Section 5.1 you saw that the derivative of the *transcendental* function $f(x) = \ln x$ is the *algebraic* function $f'(x) = 1/x$. You will now see that the derivatives of the inverse trigonometric functions also are algebraic (even though the inverse trigonometric functions are themselves transcendental).

The following theorem lists the derivatives of the six inverse trigonometric functions. Proofs for arcsin u and arccos u are given in Appendix A, and the rest are left as an exercise. (See Exercise 104.) Note that the derivatives of arccos u, arccot u, and arccsc u are the *negatives* of the derivatives of arcsin u, arctan u, and arcsec u, respectively.

THEOREM 5.16 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

TECHNOLOGY If your graphing utility does not have the arcsecant function, you can obtain its graph using

$$f(x) = \operatorname{arcsec} x = \arccos \frac{1}{x}.$$

EXAMPLE 4 Differentiating Inverse Trigonometric Functions

a. $\displaystyle \frac{d}{dx}[\arcsin(2x)] = \frac{2}{\sqrt{1-(2x)^2}} = \frac{2}{\sqrt{1-4x^2}}$

b. $\displaystyle \frac{d}{dx}[\arctan(3x)] = \frac{3}{1+(3x)^2} = \frac{3}{1+9x^2}$

c. $\displaystyle \frac{d}{dx}[\arcsin \sqrt{x}] = \frac{(1/2)\,x^{-1/2}}{\sqrt{1-x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$

d. $\displaystyle \frac{d}{dx}[\operatorname{arcsec} e^{2x}] = \frac{2e^{2x}}{e^{2x}\sqrt{(e^{2x})^2-1}} = \frac{2e^{2x}}{e^{2x}\sqrt{e^{4x}-1}} = \frac{2}{\sqrt{e^{4x}-1}}$

The absolute value sign is not necessary because $e^{2x} > 0$.

EXAMPLE 5 A Derivative That Can Be Simplified

$$\begin{aligned} y &= \arcsin x + x\sqrt{1-x^2} \\ y' &= \frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{2}\right)(-2x)(1-x^2)^{-1/2} + \sqrt{1-x^2} \\ &= \frac{1}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} \\ &= \sqrt{1-x^2} + \sqrt{1-x^2} = 2\sqrt{1-x^2} \end{aligned}$$

■

NOTE From Example 5, you can see one of the benefits of inverse trigonometric functions—they can be used to integrate common algebraic functions. For instance, from the result shown in the example, it follows that

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right).$$

■

EXAMPLE 6 Analyzing an Inverse Trigonometric Graph

Analyze the graph of $y = (\arctan x)^2$.

Solution From the derivative

$$\begin{aligned} y' &= 2(\arctan x)\left(\frac{1}{1+x^2}\right) \\ &= \frac{2\arctan x}{1+x^2} \end{aligned}$$

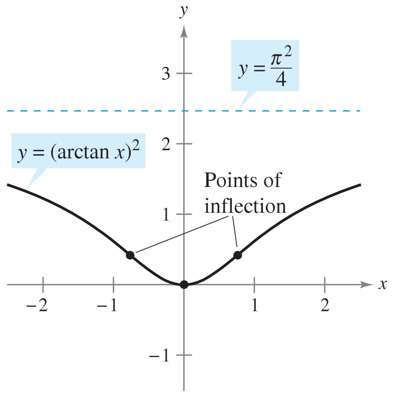
you can see that the only critical number is $x = 0$. By the First Derivative Test, this value corresponds to a relative minimum. From the second derivative

$$\begin{aligned} y'' &= \frac{(1+x^2)\left(\dfrac{2}{1+x^2}\right) - (2\arctan x)(2x)}{(1+x^2)^2} \\ &= \frac{2(1-2x\arctan x)}{(1+x^2)^2} \end{aligned}$$

it follows that points of inflection occur when $2x\arctan x = 1$. Using Newton's Method, these points occur when $x \approx \pm 0.765$. Finally, because

$$\lim_{x\to\pm\infty} (\arctan x)^2 = \frac{\pi^2}{4}$$

it follows that the graph has a horizontal asymptote at $y = \pi^2/4$. The graph is shown in Figure 5.32.



The graph of $y = (\arctan x)^2$ has a horizontal asymptote at $y = \pi^2/4$.
Figure 5.32

EXAMPLE 7 Maximizing an Angle

A photographer is taking a picture of a painting hung in an art gallery. The height of the painting is 4 feet. The camera lens is 1 foot below the lower edge of the painting, as shown in Figure 5.33. How far should the camera be from the painting to maximize the angle subtended by the camera lens?

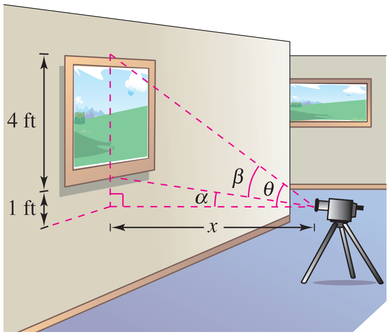
Solution In Figure 5.33, let β be the angle to be maximized.

$$\begin{aligned} \beta &= \theta - \alpha \\ &= \operatorname{arccot}\frac{x}{5} - \operatorname{arccot} x \end{aligned}$$

Differentiating produces

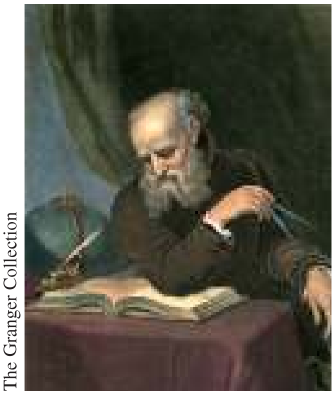
$$\begin{aligned} \frac{d\beta}{dx} &= \frac{-1/5}{1+(x^2/25)} - \frac{-1}{1+x^2} \\ &= \frac{-5}{25+x^2} + \frac{1}{1+x^2} \\ &= \frac{4(5-x^2)}{(25+x^2)(1+x^2)}. \end{aligned}$$

Because $d\beta/dx = 0$ when $x = \sqrt{5}$, you can conclude from the First Derivative Test that this distance yields a maximum value of β. So, the distance is $x \approx 2.236$ feet and the angle is $\beta \approx 0.7297$ radian $\approx 41.81°$. ■



Not drawn to scale

The camera should be 2.236 feet from the painting to maximize the angle β.
Figure 5.33


The Granger Collection

GALILEO GALILEI (1564–1642)

Galileo's approach to science departed from the accepted Aristotelian view that nature had describable *qualities*, such as "fluidity" and "potentiality." He chose to describe the physical world in terms of measurable *quantities*, such as time, distance, force, and mass.

Review of Basic Differentiation Rules

In the 1600s, Europe was ushered into the scientific age by such great thinkers as Descartes, Galileo, Huygens, Newton, and Kepler. These men believed that nature is governed by basic laws—laws that can, for the most part, be written in terms of mathematical equations. One of the most influential publications of this period—*Dialogue on the Great World Systems*, by Galileo Galilei—has become a classic description of modern scientific thought.

As mathematics has developed during the past few hundred years, a small number of elementary functions have proven sufficient for modeling most* phenomena in physics, chemistry, biology, engineering, economics, and a variety of other fields. An **elementary function** is a function from the following list or one that can be formed as the sum, product, quotient, or composition of functions in the list.

Algebraic Functions	*Transcendental Functions*
Polynomial functions	Logarithmic functions
Rational functions	Exponential functions
Functions involving radicals	Trigonometric functions
	Inverse trigonometric functions

With the differentiation rules introduced so far in the text, you can differentiate *any* elementary function. For convenience, these differentiation rules are summarized below.

BASIC DIFFERENTIATION RULES FOR ELEMENTARY FUNCTIONS

1. $\dfrac{d}{dx}[cu] = cu'$
2. $\dfrac{d}{dx}[u \pm v] = u' \pm v'$
3. $\dfrac{d}{dx}[uv] = uv' + vu'$
4. $\dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{vu' - uv'}{v^2}$
5. $\dfrac{d}{dx}[c] = 0$
6. $\dfrac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\dfrac{d}{dx}[x] = 1$
8. $\dfrac{d}{dx}[|u|] = \dfrac{u}{|u|}(u'), \quad u \neq 0$
9. $\dfrac{d}{dx}[\ln u] = \dfrac{u'}{u}$
10. $\dfrac{d}{dx}[e^u] = e^u\, u'$
11. $\dfrac{d}{dx}[\log_a u] = \dfrac{u'}{(\ln a)u}$
12. $\dfrac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\dfrac{d}{dx}[\sin u] = (\cos u)u'$
14. $\dfrac{d}{dx}[\cos u] = -(\sin u)u'$
15. $\dfrac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\dfrac{d}{dx}[\cot u] = -(\csc^2 u)u'$
17. $\dfrac{d}{dx}[\sec u] = (\sec u \tan u)u'$
18. $\dfrac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\dfrac{d}{dx}[\arcsin u] = \dfrac{u'}{\sqrt{1 - u^2}}$
20. $\dfrac{d}{dx}[\arccos u] = \dfrac{-u'}{\sqrt{1 - u^2}}$
21. $\dfrac{d}{dx}[\arctan u] = \dfrac{u'}{1 + u^2}$
22. $\dfrac{d}{dx}[\operatorname{arccot} u] = \dfrac{-u'}{1 + u^2}$
23. $\dfrac{d}{dx}[\operatorname{arcsec} u] = \dfrac{u'}{|u|\sqrt{u^2 - 1}}$
24. $\dfrac{d}{dx}[\operatorname{arccsc} u] = \dfrac{-u'}{|u|\sqrt{u^2 - 1}}$

---

* *Some important functions used in engineering and science (such as Bessel functions and gamma functions) are not elementary functions.*

5.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Numerical and Graphical Analysis **In Exercises 1 and 2, (a) use a graphing utility to complete the table, (b) plot the points in the table and graph the function by hand, (c) use a graphing utility to graph the function and compare the result with your hand-drawn graph in part (b), and (d) determine any intercepts and symmetry of the graph.**

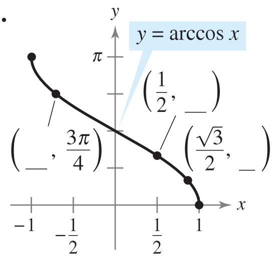
x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y											

1. $y = \arcsin x$

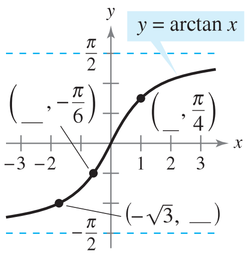
2. $y = \arccos x$

In Exercises 3 and 4, determine the missing coordinates of the points on the graph of the function.

3.



4.



In Exercises 5–12, evaluate the expression without using a calculator.

5. $\arcsin \frac{1}{2}$

6. $\arcsin 0$

7. $\arccos \frac{1}{2}$

8. $\arccos 1$

9. $\arctan \frac{\sqrt{3}}{3}$

10. $\operatorname{arccot}(-\sqrt{3})$

11. $\operatorname{arccsc}(-\sqrt{2})$

12. $\operatorname{arcsec}(-\sqrt{2})$

In Exercises 13–16, use a calculator to approximate the value. Round your answer to two decimal places.

13. $\arccos(-0.8)$

14. $\arcsin(-0.39)$

15. $\operatorname{arcsec} 1.269$

16. $\arctan(-5)$

In Exercises 17–20, evaluate each expression without using a calculator. (*Hint:* See Example 3.)

17. (a) $\sin\left(\arctan \frac{3}{4}\right)$

(b) $\sec\left(\arcsin \frac{4}{5}\right)$

18. (a) $\tan\left(\arccos \frac{\sqrt{2}}{2}\right)$

(b) $\cos\left(\arcsin \frac{5}{13}\right)$

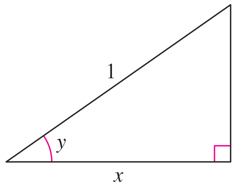
19. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right]$

(b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right]$

20. (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right]$

(b) $\tan\left[\arcsin\left(-\frac{5}{6}\right)\right]$

In Exercises 21–26, use the figure to write the expression in algebraic form given $y = \arccos x$, where $0 < y < \pi/2$.



21. $\cos y$

22. $\sin y$

23. $\tan y$

24. $\cot y$

25. $\sec y$

26. $\csc y$

In Exercises 27–34, write the expression in algebraic form. [*Hint:* Sketch a right triangle, as demonstrated in Example 3.]

27. $\cos(\arcsin 2x)$

28. $\sec(\arctan 4x)$

29. $\sin(\operatorname{arcsec} x)$

30. $\cos(\operatorname{arccot} x)$

31. $\tan\left(\operatorname{arcsec} \frac{x}{3}\right)$

32. $\sec[\arcsin(x - 1)]$

33. $\csc\left(\arctan \frac{x}{\sqrt{2}}\right)$

34. $\cos\left(\arcsin \frac{x - h}{r}\right)$

In Exercises 35 and 36, (a) use a graphing utility to graph f and g in the same viewing window to verify that they are equal, (b) use algebra to verify that f and g are equal, and (c) identify any horizontal asymptotes of the graphs.

35. $f(x) = \sin(\arctan 2x), \quad g(x) = \dfrac{2x}{\sqrt{1 + 4x^2}}$

36. $f(x) = \tan\left(\arccos \dfrac{x}{2}\right), \quad g(x) = \dfrac{\sqrt{4 - x^2}}{x}$

In Exercises 37–40, solve the equation for x.

37. $\arcsin(3x - \pi) = \frac{1}{2}$

38. $\arctan(2x - 5) = -1$

39. $\arcsin\sqrt{2x} = \arccos\sqrt{x}$

40. $\arccos x = \operatorname{arcsec} x$

In Exercises 41 and 42, verify each identity.

41. (a) $\operatorname{arccsc} x = \arcsin \dfrac{1}{x}, \quad x \geq 1$

(b) $\arctan x + \arctan \dfrac{1}{x} = \dfrac{\pi}{2}, \quad x > 0$

42. (a) $\arcsin(-x) = -\arcsin x, \quad |x| \leq 1$

(b) $\arccos(-x) = \pi - \arccos x, \quad |x| \leq 1$

In Exercises 43–62, find the derivative of the function.

43. $f(x) = 2 \arcsin(x - 1)$

44. $f(t) = \arcsin t^2$

45. $g(x) = 3 \arccos \dfrac{x}{2}$

46. $f(x) = \operatorname{arcsec} 2x$

47. $f(x) = \arctan e^x$

48. $f(x) = \arctan \sqrt{x}$

49. $g(x) = \dfrac{\arcsin 3x}{x}$

50. $h(x) = x^2 \arctan 5x$

51. $h(t) = \sin(\arccos t)$

52. $f(x) = \arcsin x + \arccos x$

53. $y = 2x \arccos x - 2\sqrt{1-x^2}$

54. $y = \ln(t^2+4) - \dfrac{1}{2}\arctan\dfrac{t}{2}$

55. $y = \dfrac{1}{2}\left(\dfrac{1}{2}\ln\dfrac{x+1}{x-1} + \arctan x\right)$

56. $y = \dfrac{1}{2}\left[x\sqrt{4-x^2} + 4\arcsin\left(\dfrac{x}{2}\right)\right]$

57. $y = x \arcsin x + \sqrt{1-x^2}$

58. $y = x \arctan 2x - \dfrac{1}{4}\ln(1+4x^2)$

59. $y = 8\arcsin\dfrac{x}{4} - \dfrac{x\sqrt{16-x^2}}{2}$

60. $y = 25\arcsin\dfrac{x}{5} - x\sqrt{25-x^2}$

61. $y = \arctan x + \dfrac{x}{1+x^2}$ **62.** $y = \arctan\dfrac{x}{2} - \dfrac{1}{2(x^2+4)}$

In Exercises 63–68, find an equation of the tangent line to the graph of the function at the given point.

63. $y = 2\arcsin x, \quad \left(\dfrac{1}{2}, \dfrac{\pi}{3}\right)$

64. $y = \dfrac{1}{2}\arccos x, \quad \left(-\dfrac{\sqrt{2}}{2}, \dfrac{3\pi}{8}\right)$

65. $y = \arctan\dfrac{x}{2}, \quad \left(2, \dfrac{\pi}{4}\right)$ **66.** $y = \operatorname{arcsec} 4x, \quad \left(\dfrac{\sqrt{2}}{4}, \dfrac{\pi}{4}\right)$

67. $y = 4x\arccos(x-1), \quad (1, 2\pi)$

68. $y = 3x\arcsin x, \quad \left(\dfrac{1}{2}, \dfrac{\pi}{4}\right)$

CAS ***Linear and Quadratic Approximations*** **In Exercises 69–72, use a computer algebra system to find the linear approximation $P_1(x) = f(a) + f'(a)(x-a)$ and the quadratic approximation $P_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$ of the function f at $x = a$. Sketch the graph of the function and its linear and quadratic approximations.**

69. $f(x) = \arctan x, \quad a = 0$ **70.** $f(x) = \arccos x, \quad a = 0$

71. $f(x) = \arcsin x, \quad a = \frac{1}{2}$ **72.** $f(x) = \arctan x, \quad a = 1$

In Exercises 73–76, find any relative extrema of the function.

73. $f(x) = \operatorname{arcsec} x - x$ **74.** $f(x) = \arcsin x - 2x$

75. $f(x) = \arctan x - \arctan(x-4)$

76. $h(x) = \arcsin x - 2\arctan x$

In Exercises 77–80, analyze and sketch a graph of the function. Identify any relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.

77. $f(x) = \arcsin(x-1)$ **78.** $f(x) = \arctan x + \dfrac{\pi}{2}$

79. $f(x) = \operatorname{arcsec} 2x$ **80.** $f(x) = \arccos\dfrac{x}{4}$

Implicit Differentiation **In Exercises 81–84, find an equation of the tangent line to the graph of the equation at the given point.**

81. $x^2 + x\arctan y = y - 1, \quad \left(-\dfrac{\pi}{4}, 1\right)$

82. $\arctan(xy) = \arcsin(x+y), \quad (0, 0)$

83. $\arcsin x + \arcsin y = \dfrac{\pi}{2}, \quad \left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$

84. $\arctan(x+y) = y^2 + \dfrac{\pi}{4}, \quad (1, 0)$

WRITING ABOUT CONCEPTS

85. Explain why the domains of the trigonometric functions are restricted when finding the inverse trigonometric functions.

86. Explain why $\tan \pi = 0$ does not imply that $\arctan 0 = \pi$.

87. Explain how to graph $y = \operatorname{arccot} x$ on a graphing utility that does not have the arccotangent function.

88. Are the derivatives of the inverse trigonometric functions algebraic or transcendental functions? List the derivatives of the inverse trigonometric functions.

89. (a) Use a graphing utility to evaluate $\arcsin(\arcsin 0.5)$ and $\arcsin(\arcsin 1)$.

(b) Let $f(x) = \arcsin(\arcsin x)$. Find the values of x in the interval $-1 \le x \le 1$ such that $f(x)$ is a real number.

CAPSTONE

90. The point $\left(\dfrac{3\pi}{2}, 0\right)$ is on the graph of $y = \cos x$. Does $\left(0, \dfrac{3\pi}{2}\right)$ lie on the graph of $y = \arccos x$? If not, does this contradict the definition of inverse function?

True or False? **In Exercises 91–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. Because $\cos\left(-\dfrac{\pi}{3}\right) = \dfrac{1}{2}$, it follows that $\arccos\dfrac{1}{2} = -\dfrac{\pi}{3}$.

92. $\arcsin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$

93. The slope of the graph of the inverse tangent function is positive for all x.

94. The range of $y = \arcsin x$ is $[0, \pi]$.

95. $\dfrac{d}{dx}[\arctan(\tan x)] = 1$ for all x in the domain.

96. $\arcsin^2 x + \arccos^2 x = 1$

97. ***Angular Rate of Change*** An airplane flies at an altitude of 5 miles toward a point directly over an observer. Consider θ and x as shown in the figure on the next page.

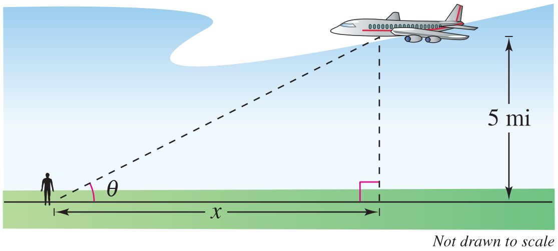


Figure for 97

(a) Write θ as a function of x.

(b) The speed of the plane is 400 miles per hour. Find $d\theta/dt$ when $x = 10$ miles and $x = 3$ miles.

98. ***Writing*** Repeat Exercise 97 for an altitude of 3 miles and describe how the altitude affects the rate of change of θ.

99. ***Angular Rate of Change*** In a free-fall experiment, an object is dropped from a height of 256 feet. A camera on the ground 500 feet from the point of impact records the fall of the object (see figure).

(a) Find the position function that yields the height of the object at time t assuming the object is released at time $t = 0$. At what time will the object reach ground level?

(b) Find the rates of change of the angle of elevation of the camera when $t = 1$ and $t = 2$.

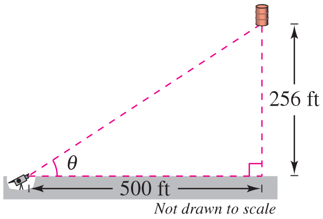


Figure for 99

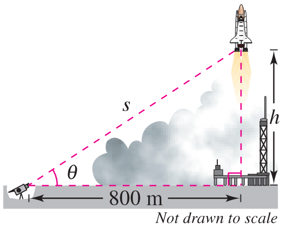


Figure for 100

100. ***Angular Rate of Change*** A television camera at ground level is filming the lift-off of a space shuttle at a point 800 meters from the launch pad. Let θ be the angle of elevation of the shuttle and let s be the distance between the camera and the shuttle (see figure). Write θ as a function of s for the period of time when the shuttle is moving vertically. Differentiate the result to find $d\theta/dt$ in terms of s and ds/dt.

101. ***Maximizing an Angle*** A billboard 85 feet wide is perpendicular to a straight road and is 40 feet from the road (see figure). Find the point on the road at which the angle θ subtended by the billboard is a maximum.

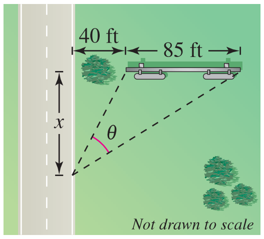


Figure for 101

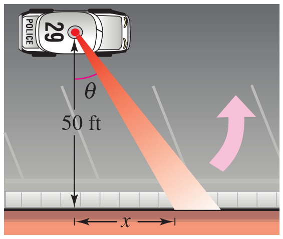


Figure for 102

102. ***Angular Speed*** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. Write θ as a function of x. How fast is the light beam moving along the wall when the beam makes an angle of $\theta = 45°$ with the line perpendicular from the light to the wall?

103. (a) Prove that $\arctan x + \arctan y = \arctan \dfrac{x+y}{1-xy}$, $xy \neq 1$.

(b) Use the formula in part (a) to show that

$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}.$$

104. Verify each differentiation formula.

(a) $\dfrac{d}{dx}[\arctan u] = \dfrac{u'}{1+u^2}$

(b) $\dfrac{d}{dx}[\operatorname{arccot} u] = \dfrac{-u'}{1+u^2}$

(c) $\dfrac{d}{dx}[\operatorname{arcsec} u] = \dfrac{u'}{|u|\sqrt{u^2-1}}$

(d) $\dfrac{d}{dx}[\operatorname{arccsc} u] = \dfrac{-u'}{|u|\sqrt{u^2-1}}$

105. ***Existence of an Inverse*** Determine the values of k such that the function $f(x) = kx + \sin x$ has an inverse function.

106. ***Think About It*** Use a graphing utility to graph $f(x) = \sin x$ and $g(x) = \arcsin(\sin x)$.

(a) Why isn't the graph of g the line $y = x$?

(b) Determine the extrema of g.

107. (a) Graph the function $f(x) = \arccos x + \arcsin x$ on the interval $[-1, 1]$.

(b) Describe the graph of f.

(c) Verify the result of part (b) analytically.

108. Prove that $\arcsin x = \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right)$, $|x| < 1$.

109. In the figure find the value of c in the interval $[0, 4]$ on the x-axis thatmaximizes angle θ.

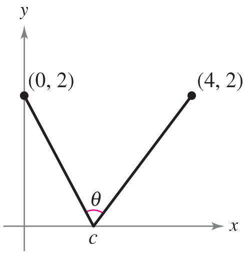


Figure for 109

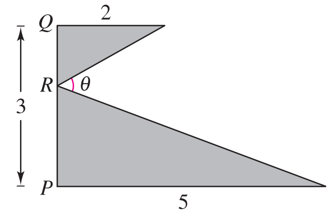


Figure for 110

110. In the figure find PR such that $0 \le PR \le 3$ and $m\angle\theta$ is a maximum.

111. Some calculus textbooks define the inverse secant function using the range $[0, \pi/2) \cup [\pi, 3\pi/2)$.

(a) Sketch the graph $y = \operatorname{arcsec} x$ using this range.

(b) Show that $y' = \dfrac{1}{x\sqrt{x^2-1}}$.

5.7 Inverse Trigonometric Functions: Integration

- **Integrate functions whose antiderivatives involve inverse trigonometric functions.**
- **Use the method of completing the square to integrate a function.**
- **Review the basic integration rules involving elementary functions.**

Integrals Involving Inverse Trigonometric Functions

The derivatives of the six inverse trigonometric functions fall into three pairs. In each pair, the derivative of one function is the negative of the other. For example,

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

and

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}.$$

When listing the *antiderivative* that corresponds to each of the inverse trigonometric functions, you need to use only one member from each pair. It is conventional to use $\arcsin x$ as the antiderivative of $1/\sqrt{1-x^2}$, rather than $-\arccos x$. The next theorem gives one antiderivative formula for each of the three pairs. The proofs of these integration rules are left to you (see Exercises 87–89).

FOR FURTHER INFORMATION For a detailed proof of rule 2 of Theorem 5.17, see the article "A Direct Proof of the Integral Formula for Arctangent" by Arnold J. Insel in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

THEOREM 5.17 INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x, and let $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$ **2.** $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

3. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\,\text{arcsec}\,\frac{|u|}{a} + C$

EXAMPLE 1 Integration with Inverse Trigonometric Functions

a. $$\int \frac{dx}{\sqrt{4-x^2}} = \arcsin\frac{x}{2} + C$$

b. $$\int \frac{dx}{2+9x^2} = \frac{1}{3}\int \frac{3\,dx}{(\sqrt{2})^2+(3x)^2} \qquad u = 3x,\ a = \sqrt{2}$$
$$= \frac{1}{3\sqrt{2}}\arctan\frac{3x}{\sqrt{2}} + C$$

c. $$\int \frac{dx}{x\sqrt{4x^2-9}} = \int \frac{2\,dx}{2x\sqrt{(2x)^2-3^2}} \qquad u = 2x,\ a = 3$$
$$= \frac{1}{3}\,\text{arcsec}\,\frac{|2x|}{3} + C$$

■

The integrals in Example 1 are fairly straightforward applications of integration formulas. Unfortunately, this is not typical. The integration formulas for inverse trigonometric functions can be disguised in many ways.

TECHNOLOGY PITFALL

Computer software that can perform symbolic integration is useful for integrating functions such as the one in Example 2. When using such software, however, you must remember that it can fail to find an antiderivative for two reasons. First, some elementary functions simply do not have antiderivatives that are elementary functions. Second, every symbolic integration utility has limitations—you might have entered a function that the software was not programmed to handle. You should also remember that antiderivatives involving trigonometric functions or logarithmic functions can be written in many different forms. For instance, one symbolic integration utility found the integral in Example 2 to be

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \arctan\sqrt{e^{2x}-1} + C.$$

Try showing that this antiderivative is equivalent to that obtained in Example 2.

EXAMPLE 2 Integration by Substitution

Find $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$.

Solution As it stands, this integral doesn't fit any of the three inverse trigonometric formulas. Using the substitution $u = e^x$, however, produces

$$u = e^x \quad \Rightarrow \quad du = e^x\,dx \quad \Rightarrow \quad dx = \frac{du}{e^x} = \frac{du}{u}.$$

With this substitution, you can integrate as follows.

$$\begin{aligned}
\int \frac{dx}{\sqrt{e^{2x}-1}} &= \int \frac{dx}{\sqrt{(e^x)^2-1}} && \text{Write } e^{2x} \text{ as } (e^x)^2.\\
&= \int \frac{du/u}{\sqrt{u^2-1}} && \text{Substitute.}\\
&= \int \frac{du}{u\sqrt{u^2-1}} && \text{Rewrite to fit Arcsecant Rule.}\\
&= \operatorname{arcsec}\frac{|u|}{1} + C && \text{Apply Arcsecant Rule.}\\
&= \operatorname{arcsec} e^x + C && \text{Back-substitute.}
\end{aligned}$$

EXAMPLE 3 Rewriting as the Sum of Two Quotients

Find $\displaystyle\int \frac{x+2}{\sqrt{4-x^2}}\,dx$.

Solution This integral does not appear to fit any of the basic integration formulas. By splitting the integrand into two parts, however, you can see that the first part can be found with the Power Rule and the second part yields an inverse sine function.

$$\begin{aligned}
\int \frac{x+2}{\sqrt{4-x^2}}\,dx &= \int \frac{x}{\sqrt{4-x^2}}\,dx + \int \frac{2}{\sqrt{4-x^2}}\,dx\\
&= -\frac{1}{2}\int (4-x^2)^{-1/2}(-2x)\,dx + 2\int \frac{1}{\sqrt{4-x^2}}\,dx\\
&= -\frac{1}{2}\left[\frac{(4-x^2)^{1/2}}{1/2}\right] + 2\arcsin\frac{x}{2} + C\\
&= -\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + C
\end{aligned}$$

■

Completing the Square

Completing the square helps when quadratic functions are involved in the integrand. For example, the quadratic $x^2 + bx + c$ can be written as the difference of two squares by adding and subtracting $(b/2)^2$.

$$\begin{aligned}
x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c\\
&= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c
\end{aligned}$$

EXAMPLE 4 Completing the Square

Find $\displaystyle\int \frac{dx}{x^2 - 4x + 7}$.

Solution You can write the denominator as the sum of two squares, as follows.

$$\begin{aligned} x^2 - 4x + 7 &= (x^2 - 4x + 4) - 4 + 7 \\ &= (x - 2)^2 + 3 = u^2 + a^2 \end{aligned}$$

Now, in this completed square form, let $u = x - 2$ and $a = \sqrt{3}$.

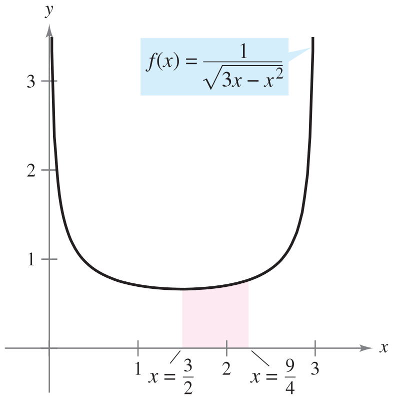
$$\int \frac{dx}{x^2 - 4x + 7} = \int \frac{dx}{(x - 2)^2 + 3} = \frac{1}{\sqrt{3}} \arctan \frac{x - 2}{\sqrt{3}} + C$$ ■

If the leading coefficient is not 1, it helps to factor before completing the square. For instance, you can complete the square of $2x^2 - 8x + 10$ by factoring first.

$$\begin{aligned} 2x^2 - 8x + 10 &= 2(x^2 - 4x + 5) \\ &= 2(x^2 - 4x + 4 - 4 + 5) \\ &= 2[(x - 2)^2 + 1] \end{aligned}$$

To complete the square when the coefficient of x^2 is negative, use the same factoring process shown above. For instance, you can complete the square for $3x - x^2$ as shown.

$$\begin{aligned} 3x - x^2 &= -(x^2 - 3x) \\ &= -\left[x^2 - 3x + \left(\tfrac{3}{2}\right)^2 - \left(\tfrac{3}{2}\right)^2\right] \\ &= \left(\tfrac{3}{2}\right)^2 - \left(x - \tfrac{3}{2}\right)^2 \end{aligned}$$



The area of the region bounded by the graph of f, the x-axis, $x = \frac{3}{2}$, and $x = \frac{9}{4}$ is $\pi/6$.
Figure 5.34

EXAMPLE 5 Completing the Square (Negative Leading Coefficient)

Find the area of the region bounded by the graph of

$$f(x) = \frac{1}{\sqrt{3x - x^2}}$$

the x-axis, and the lines $x = \frac{3}{2}$ and $x = \frac{9}{4}$.

Solution In Figure 5.34, you can see that the area is given by

$$\text{Area} = \int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}}\, dx.$$

Using the completed square form derived above, you can integrate as shown.

$$\begin{aligned} \int_{3/2}^{9/4} \frac{dx}{\sqrt{3x - x^2}} &= \int_{3/2}^{9/4} \frac{dx}{\sqrt{(3/2)^2 - [x - (3/2)]^2}} \\ &= \arcsin \frac{x - (3/2)}{3/2} \bigg]_{3/2}^{9/4} \\ &= \arcsin \frac{1}{2} - \arcsin 0 \\ &= \frac{\pi}{6} \\ &\approx 0.524 \end{aligned}$$ ■

TECHNOLOGY With definite integrals such as the one given in Example 5, remember that you can resort to a numerical solution. For instance, applying Simpson's Rule (with $n = 12$) to the integral in the example, you obtain

$$\int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}}\, dx \approx 0.523599.$$

This differs from the exact value of the integral ($\pi/6 \approx 0.5235988$) by less than one millionth.

Review of Basic Integration Rules

You have now completed the introduction of the **basic integration rules.** To be efficient at applying these rules, you should have practiced enough so that each rule is committed to memory.

BASIC INTEGRATION RULES ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$

2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$

3. $\int du = u + C$

4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

5. $\int \frac{du}{u} = \ln|u| + C$

6. $\int e^u\,du = e^u + C$

7. $\int a^u\,du = \left(\frac{1}{\ln a}\right)a^u + C$

8. $\int \sin u\,du = -\cos u + C$

9. $\int \cos u\,du = \sin u + C$

10. $\int \tan u\,du = -\ln|\cos u| + C$

11. $\int \cot u\,du = \ln|\sin u| + C$

12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$

14. $\int \sec^2 u\,du = \tan u + C$

15. $\int \csc^2 u\,du = -\cot u + C$

16. $\int \sec u \tan u\,du = \sec u + C$

17. $\int \csc u \cot u\,du = -\csc u + C$

18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$

19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

You can learn a lot about the nature of integration by comparing this list with the summary of differentiation rules given in the preceding section. For differentiation, you now have rules that allow you to differentiate *any* elementary function. For integration, this is far from true.

The integration rules listed above are primarily those that were happened on during the development of differentiation rules. So far, you have not learned any rules or techniques for finding the antiderivative of a general product or quotient, the natural logarithmic function, or the inverse trigonometric functions. More importantly, you cannot apply any of the rules in this list unless you can create the proper *du* corresponding to the *u* in the formula. The point is that you need to work more on integration techniques, which you will do in Chapter 8. The next two examples should give you a better feeling for the integration problems that you *can* and *cannot* do with the techniques and rules you now know.

EXAMPLE 6 Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$ **b.** $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-1}}$ **c.** $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$

Solution

a. You *can* find this integral (it fits the Arcsecant Rule).

$$\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arcsec}|x| + C$$

b. You *can* find this integral (it fits the Power Rule).

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{x^2-1}} &= \frac{1}{2}\int (x^2-1)^{-1/2}(2x)\,dx\\ &= \frac{1}{2}\left[\frac{(x^2-1)^{1/2}}{1/2}\right] + C\\ &= \sqrt{x^2-1} + C\end{aligned}$$

c. You *cannot* find this integral using the techniques you have studied so far. (You should scan the list of basic integration rules to verify this conclusion.)

EXAMPLE 7 Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\displaystyle\int \frac{dx}{x\ln x}$ **b.** $\displaystyle\int \frac{\ln x\,dx}{x}$ **c.** $\displaystyle\int \ln x\,dx$

Solution

a. You *can* find this integral (it fits the Log Rule).

$$\begin{aligned}\int \frac{dx}{x\ln x} &= \int \frac{1/x}{\ln x}\,dx\\ &= \ln|\ln x| + C\end{aligned}$$

b. You *can* find this integral (it fits the Power Rule).

$$\begin{aligned}\int \frac{\ln x\,dx}{x} &= \int \left(\frac{1}{x}\right)(\ln x)^1\,dx\\ &= \frac{(\ln x)^2}{2} + C\end{aligned}$$

c. You *cannot* find this integral using the techniques you have studied so far. ■

NOTE Note in Examples 6 and 7 that the *simplest* functions are the ones that you cannot yet integrate. ■

5.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–24, find the integral.

1. $\int \frac{dx}{\sqrt{9-x^2}}$

2. $\int \frac{dx}{\sqrt{1-4x^2}}$

3. $\int \frac{7}{16+x^2}\,dx$

4. $\int \frac{12}{1+9x^2}\,dx$

5. $\int \frac{1}{x\sqrt{4x^2-1}}\,dx$

6. $\int \frac{1}{4+(x-3)^2}\,dx$

7. $\int \frac{1}{\sqrt{1-(x+1)^2}}\,dx$

8. $\int \frac{t}{t^4+16}\,dt$

9. $\int \frac{t}{\sqrt{1-t^4}}\,dt$

10. $\int \frac{1}{x\sqrt{x^4-4}}\,dx$

11. $\int \frac{t}{t^4+25}\,dt$

12. $\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx$

13. $\int \frac{e^{2x}}{4+e^{4x}}\,dx$

14. $\int \frac{1}{3+(x-2)^2}\,dx$

15. $\int \frac{\sec^2 x}{\sqrt{25-\tan^2 x}}\,dx$

16. $\int \frac{\sin x}{7+\cos^2 x}\,dx$

17. $\int \frac{x^3}{x^2+1}\,dx$

18. $\int \frac{x^4-1}{x^2+1}\,dx$

19. $\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$

20. $\int \frac{3}{2\sqrt{x}(1+x)}\,dx$

21. $\int \frac{x-3}{x^2+1}\,dx$

22. $\int \frac{4x+3}{\sqrt{1-x^2}}\,dx$

23. $\int \frac{x+5}{\sqrt{9-(x-3)^2}}\,dx$

24. $\int \frac{x-2}{(x+1)^2+4}\,dx$

In Exercises 25–38, evaluate the integral.

25. $\int_0^{1/6} \frac{3}{\sqrt{1-9x^2}}\,dx$

26. $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

27. $\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2}\,dx$

28. $\int_{\sqrt{3}}^3 \frac{6}{9+x^2}\,dx$

29. $\int_{-1/2}^0 \frac{x}{\sqrt{1-x^2}}\,dx$

30. $\int_{-\sqrt{3}}^0 \frac{x}{1+x^2}\,dx$

31. $\int_3^6 \frac{1}{25+(x-3)^2}\,dx$

32. $\int_1^4 \frac{1}{x\sqrt{16x^2-5}}\,dx$

33. $\int_0^{\ln 5} \frac{e^x}{1+e^{2x}}\,dx$

34. $\int_{\ln 2}^{\ln 4} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx$

35. $\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$

36. $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx$

37. $\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

38. $\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}}\,dx$

In Exercises 39–50, find or evaluate the integral. (Complete the square, if necessary.)

39. $\int_0^2 \frac{dx}{x^2-2x+2}$

40. $\int_{-2}^2 \frac{dx}{x^2+4x+13}$

41. $\int \frac{2x}{x^2+6x+13}\,dx$

42. $\int \frac{2x-5}{x^2+2x+2}\,dx$

43. $\int \frac{1}{\sqrt{-x^2-4x}}\,dx$

44. $\int \frac{2}{\sqrt{-x^2+4x}}\,dx$

45. $\int \frac{x+2}{\sqrt{-x^2-4x}}\,dx$

46. $\int \frac{x-1}{\sqrt{x^2-2x}}\,dx$

47. $\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\,dx$

48. $\int \frac{1}{(x-1)\sqrt{x^2-2x}}\,dx$

49. $\int \frac{x}{x^4+2x^2+2}\,dx$

50. $\int \frac{x}{\sqrt{9+8x^2-x^4}}\,dx$

In Exercises 51–54, use the specified substitution to find or evaluate the integral.

51. $\int \sqrt{e^t-3}\,dt$

$u = \sqrt{e^t-3}$

52. $\int \frac{\sqrt{x-2}}{x+1}\,dx$

$u = \sqrt{x-2}$

53. $\int_1^3 \frac{dx}{\sqrt{x}(1+x)}$

$u = \sqrt{x}$

54. $\int_0^1 \frac{dx}{2\sqrt{3-x}\sqrt{x+1}}$

$u = \sqrt{x+1}$

WRITING ABOUT CONCEPTS

In Exercises 55–57, determine which of the integrals can be found using the basic integration formulas you have studied so far in the text.

55. (a) $\int \frac{1}{\sqrt{1-x^2}}\,dx$ (b) $\int \frac{x}{\sqrt{1-x^2}}\,dx$ (c) $\int \frac{1}{x\sqrt{1-x^2}}\,dx$

56. (a) $\int e^{x^2}\,dx$ (b) $\int xe^{x^2}\,dx$ (c) $\int \frac{1}{x^2}e^{1/x}\,dx$

57. (a) $\int \sqrt{x-1}\,dx$ (b) $\int x\sqrt{x-1}\,dx$ (c) $\int \frac{x}{\sqrt{x-1}}\,dx$

58. Determine which value best approximates the area of the region between the x-axis and the function

$$f(x) = \frac{1}{\sqrt{1-x^2}}$$

over the interval $[-0.5, 0.5]$. (Make your selection on the basis of a sketch of the region and *not* by performing any calculations.)

(a) 4 (b) -3 (c) 1 (d) 2 (e) 3

59. Decide whether you can find the integral

$$\int \frac{2\,dx}{\sqrt{x^2+4}}$$

using the formulas and techniques you have studied so far. Explain your reasoning.

CAPSTONE

60. Determine which of the integrals can be found using the basic integration formulas you have studied so far in the text.

(a) $\int \frac{1}{1+x^4}\,dx$ (b) $\int \frac{x}{1+x^4}\,dx$ (c) $\int \frac{x^3}{1+x^4}\,dx$

Differential Equations **In Exercises 61 and 62, use the differential equation and the specified initial condition to find *y*.**

61. $\frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$

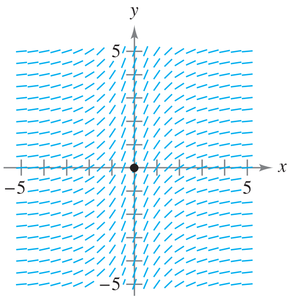
$y(0) = \pi$

62. $\frac{dy}{dx} = \frac{1}{4+x^2}$

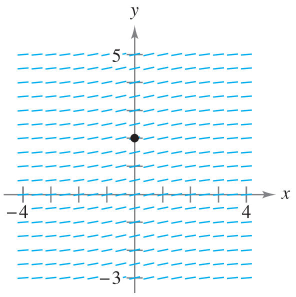
$y(2) = \pi$

Slope Fields **In Exercises 63–66, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

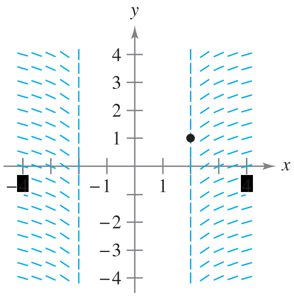
63. $\frac{dy}{dx} = \frac{3}{1+x^2}$, $(0, 0)$



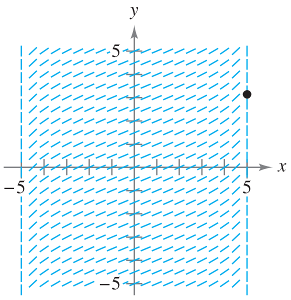
64. $\frac{dy}{dx} = \frac{2}{9+x^2}$, $(0, 2)$



65. $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-4}}$, $(2, 1)$



66. $\frac{dy}{dx} = \frac{2}{\sqrt{25-x^2}}$, $(5, \pi)$



Slope Fields **In Exercises 67–70, use a computer algebra system to graph the slope field for the differential equation and graph the solution satisfying the specified initial condition.**

67. $\frac{dy}{dx} = \frac{10}{x\sqrt{x^2-1}}$, $y(3) = 0$

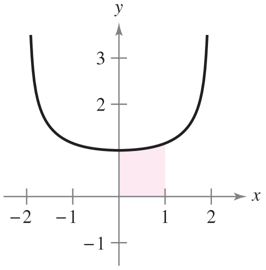
68. $\frac{dy}{dx} = \frac{1}{12+x^2}$, $y(4) = 2$

69. $\frac{dy}{dx} = \frac{2y}{\sqrt{16-x^2}}$, $y(0) = 2$

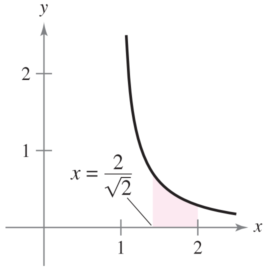
70. $\frac{dy}{dx} = \frac{\sqrt{y}}{1+x^2}$, $y(0) = 4$

Area **In Exercises 71–76, find the area of the region.**

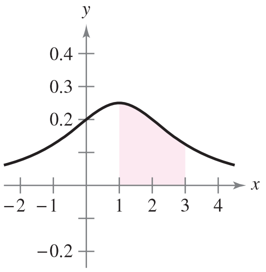
71. $y = \frac{2}{\sqrt{4-x^2}}$



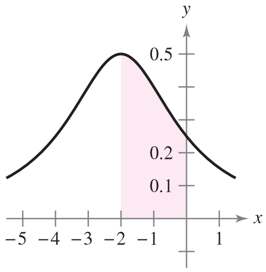
72. $y = \frac{1}{x\sqrt{x^2-1}}$



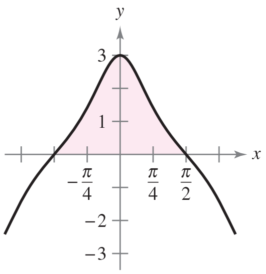
73. $y = \frac{1}{x^2-2x+5}$



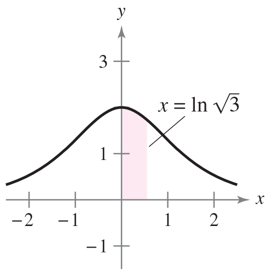
74. $y = \frac{2}{x^2+4x+8}$



75. $y = \frac{3\cos x}{1+\sin^2 x}$



76. $y = \frac{4e^x}{1+e^{2x}}$



In Exercises 77 and 78, (a) verify the integration formula, then (b) use it to find the area of the region.

77. $\int \frac{\arctan x}{x^2}\,dx = \ln x - \frac{1}{2}\ln(1+x^2) - \frac{\arctan x}{x} + C$

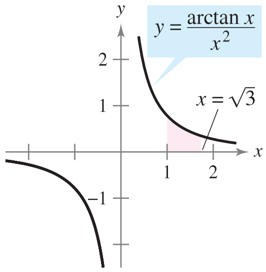


Figure for 77

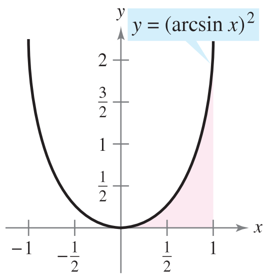


Figure for 78

78. $\int (\arcsin x)^2\,dx$

$= x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\arcsin x + C$

79. (a) Sketch the region whose area is represented by

$$\int_0^1 \arcsin x\, dx.$$

(b) Use the integration capabilities of a graphing utility to approximate the area.

(c) Find the exact area analytically.

80. (a) Show that $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx = \pi$.

(b) Approximate the number π using Simpson's Rule (with $n = 6$) and the integral in part (a).

(c) Approximate the number π by using the integration capabilities of a graphing utility.

81. ***Investigation*** Consider the function $F(x) = \dfrac{1}{2}\displaystyle\int_x^{x+2} \frac{2}{t^2+1}\,dt$.

(a) Write a short paragraph giving a geometric interpretation of the function $F(x)$ relative to the function $f(x) = \dfrac{2}{x^2+1}$. Use what you have written to guess the value of x that will make F maximum.

(b) Perform the specified integration to find an alternative form of $F(x)$. Use calculus to locate the value of x that will make F maximum and compare the result with your guess in part (a).

82. Consider the integral $\displaystyle\int \frac{1}{\sqrt{6x-x^2}}\,dx$.

(a) Find the integral by completing the square of the radicand.

(b) Find the integral by making the substitution $u = \sqrt{x}$.

(c) The antiderivatives in parts (a) and (b) appear to be significantly different. Use a graphing utility to graph each antiderivative in the same viewing window and determine the relationship between them. Find the domain of each.

True or False? **In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

83. $\displaystyle\int \frac{dx}{3x\sqrt{9x^2-16}} = \frac{1}{4}\operatorname{arcsec}\frac{3x}{4} + C$

84. $\displaystyle\int \frac{dx}{25+x^2} = \frac{1}{25}\arctan\frac{x}{25} + C$

85. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = -\arccos\frac{x}{2} + C$

86. One way to find $\displaystyle\int \frac{2e^{2x}}{\sqrt{9-e^{2x}}}\,dx$ is to use the Arcsine Rule.

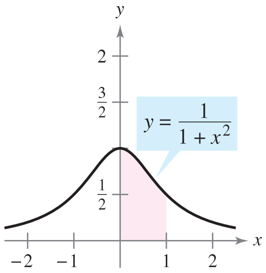
Verifying Integration Rules **In Exercises 87–89, verify each rule by differentiating. Let $a > 0$.**

87. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$

88. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

89. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

90. ***Numerical Integration*** (a) Write an integral that represents the area of the region in the figure. (b) Then use the Trapezoidal Rule with $n = 8$ to estimate the area of the region. (c) Explain how you can use the results of parts (a) and (b) to estimate π.



91. ***Vertical Motion*** An object is projected upward from ground level with an initial velocity of 500 feet per second. In this exercise, the goal is to analyze the motion of the object during its upward flight.

(a) If air resistance is neglected, find the velocity of the object as a function of time. Use a graphing utility to graph this function.

(b) Use the result of part (a) to find the position function and determine the maximum height attained by the object.

(c) If the air resistance is proportional to the square of the velocity, you obtain the equation

$$\frac{dv}{dt} = -(32 + kv^2)$$

where -32 feet per second per second is the acceleration due to gravity and k is a constant. Find the velocity as a function of time by solving the equation

$$\int \frac{dv}{32+kv^2} = -\int dt.$$

(d) Use a graphing utility to graph the velocity function $v(t)$ in part (c) for $k = 0.001$. Use the graph to approximate the time t_0 at which the object reaches its maximum height.

(e) Use the integration capabilities of a graphing utility to approximate the integral

$$\int_0^{t_0} v(t)\,dt$$

where $v(t)$ and t_0 are those found in part (d). This is the approximation of the maximum height of the object.

(f) Explain the difference between the results in parts (b) and (e).

■ **FOR FURTHER INFORMATION** For more information on this topic, see "What Goes Up Must Come Down; Will Air Resistance Make It Return Sooner, or Later?" by John Lekner in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

92. Graph $y_1 = \dfrac{x}{1+x^2}$, $y_2 = \arctan x$, and $y_3 = x$ on $[0, 10]$.

Prove that $\dfrac{x}{1+x^2} < \arctan x < x$ for $x > 0$.

5.8 Hyperbolic Functions

- **Develop properties of hyperbolic functions.**
- **Differentiate and integrate hyperbolic functions.**
- **Develop properties of inverse hyperbolic functions.**
- **Differentiate and integrate functions involving inverse hyperbolic functions.**

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JOHANN HEINRICH LAMBERT (1728–1777)

The first person to publish a comprehensive study on hyperbolic functions was Johann Heinrich Lambert, a Swiss-German mathematician and colleague of Euler.

Hyperbolic Functions

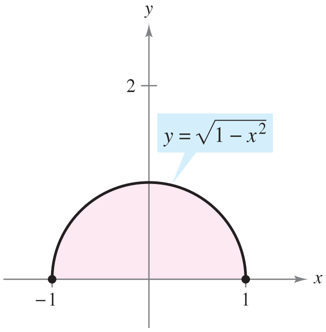
In this section you will look briefly at a special class of exponential functions called **hyperbolic functions.** The name *hyperbolic function* arose from comparison of the area of a semicircular region, as shown in Figure 5.35, with the area of a region under a hyperbola, as shown in Figure 5.36. The integral for the semicircular region involves an inverse trigonometric (circular) function:

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{1}{2}\left[x\sqrt{1-x^2} + \arcsin x\right]_{-1}^{1} = \frac{\pi}{2} \approx 1.571.$$

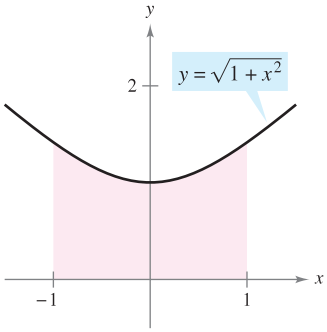
The integral for the hyperbolic region involves an inverse hyperbolic function:

$$\int_{-1}^{1} \sqrt{1+x^2}\,dx = \frac{1}{2}\left[x\sqrt{1+x^2} + \sinh^{-1}x\right]_{-1}^{1} \approx 2.296.$$

This is only one of many ways in which the hyperbolic functions are similar to the trigonometric functions.



Circle: $x^2 + y^2 = 1$
Figure 5.35



Hyperbola: $-x^2 + y^2 = 1$
Figure 5.36

FOR FURTHER INFORMATION For more information on the development of hyperbolic functions, see the article "An Introduction to Hyperbolic Functions in Elementary Calculus" by Jerome Rosenthal in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

DEFINITIONS OF THE HYPERBOLIC FUNCTIONS

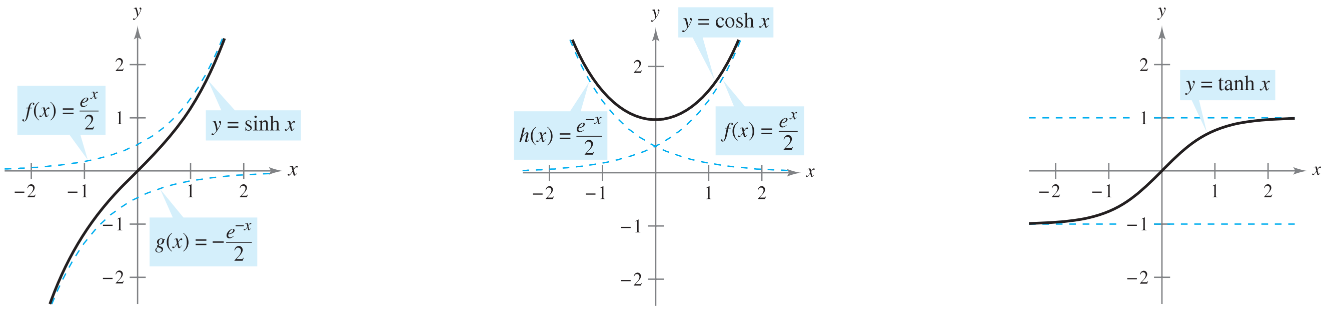
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}, \quad x \neq 0$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}, \quad x \neq 0$$

NOTE sinh x is read as "the hyperbolic sine of x," cosh x as "the hyperbolic cosine of x," and so on.

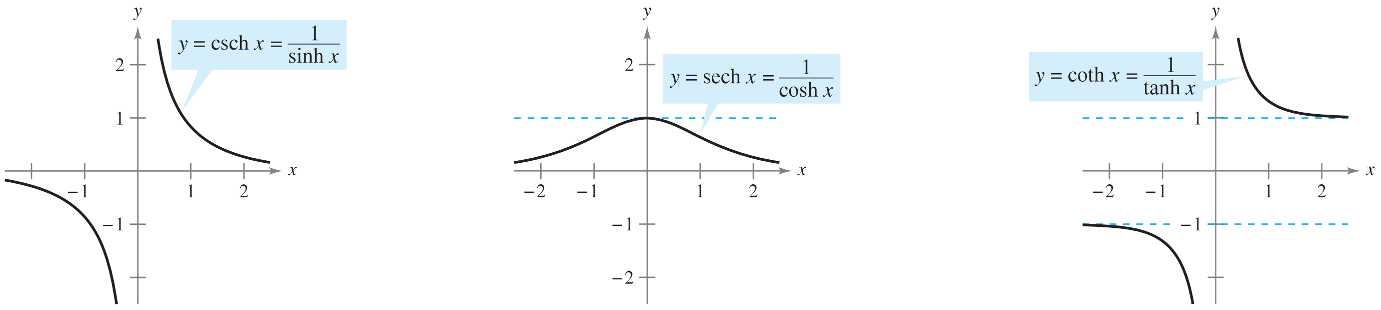
The graphs of the six hyperbolic functions and their domains and ranges are shown in Figure 5.37. Note that the graph of sinh x can be obtained by adding the corresponding y-coordinates of the exponential functions $f(x) = \frac{1}{2}e^x$ and $g(x) = -\frac{1}{2}e^{-x}$. Likewise, the graph of cosh x can be obtained by adding the corresponding y-coordinates of the exponential functions $f(x) = \frac{1}{2}e^x$ and $h(x) = \frac{1}{2}e^{-x}$.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Domain: $(-\infty, \infty)$
Range: $[1, \infty)$

Domain: $(-\infty, \infty)$
Range: $(-1, 1)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Domain: $(-\infty, \infty)$
Range: $(0, 1]$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, -1) \cup (1, \infty)$

Figure 5.37

Many of the trigonometric identities have corresponding *hyperbolic identities*. For instance,

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}\\ &= \frac{4}{4}\\ &= 1\end{aligned}$$

and

$$\begin{aligned}2 \sinh x \cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)\\ &= \frac{e^{2x} - e^{-2x}}{2}\\ &= \sinh 2x.\end{aligned}$$

■ **FOR FURTHER INFORMATION** To understand geometrically the relationship between the hyperbolic and exponential functions, see the article "A Short Proof Linking the Hyperbolic and Exponential Functions" by Michael J. Seery in *The AMATYC Review*.

HYPERBOLIC IDENTITIES

$$\cosh^2 x - \sinh^2 x = 1 \qquad \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1 \qquad \sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$$

$$\coth^2 x - \operatorname{csch}^2 x = 1 \qquad \cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\sinh^2 x = \frac{-1 + \cosh 2x}{2} \qquad \cosh^2 x = \frac{1 + \cosh 2x}{2}$$

$$\sinh 2x = 2 \sinh x \cosh x \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$

Differentiation and Integration of Hyperbolic Functions

Because the hyperbolic functions are written in terms of e^x and e^{-x}, you can easily derive rules for their derivatives. The following theorem lists these derivatives with the corresponding integration rules.

THEOREM 5.18 DERIVATIVES AND INTEGRALS OF HYPERBOLIC FUNCTIONS

Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh u] = (\cosh u)u' \qquad \int \cosh u \, du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = (\sinh u)u' \qquad \int \sinh u \, du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = (\operatorname{sech}^2 u)u' \qquad \int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -(\operatorname{csch}^2 u)u' \qquad \int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -(\operatorname{sech} u \tanh u)u' \qquad \int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -(\operatorname{csch} u \coth u)u' \qquad \int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

PROOF

$$\begin{aligned}\frac{d}{dx}[\sinh x] &= \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right]\\ &= \frac{e^x + e^{-x}}{2} = \cosh x\\ \frac{d}{dx}[\tanh x] &= \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right]\\ &= \frac{\cosh x(\cosh x) - \sinh x(\sinh x)}{\cosh^2 x}\\ &= \frac{1}{\cosh^2 x}\\ &= \operatorname{sech}^2 x\end{aligned}$$

■

In Exercises 122–124, you are asked to prove some of the other differentiation rules.

EXAMPLE 1 Differentiation of Hyperbolic Functions

a. $\frac{d}{dx}[\sinh(x^2 - 3)] = 2x\cosh(x^2 - 3)$ **b.** $\frac{d}{dx}[\ln(\cosh x)] = \frac{\sinh x}{\cosh x} = \tanh x$

c. $\frac{d}{dx}[x\sinh x - \cosh x] = x\cosh x + \sinh x - \sinh x = x\cosh x$

EXAMPLE 2 Finding Relative Extrema

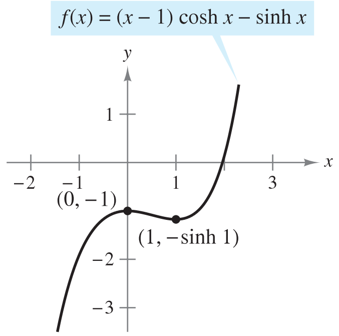
Find the relative extrema of $f(x) = (x - 1)\cosh x - \sinh x$.

Solution Begin by setting the first derivative of f equal to 0.

$$f'(x) = (x - 1)\sinh x + \cosh x - \cosh x = 0$$
$$(x - 1)\sinh x = 0$$

So, the critical numbers are $x = 1$ and $x = 0$. Using the Second Derivative Test, you can verify that the point $(0, -1)$ yields a relative maximum and the point $(1, -\sinh 1)$ yields a relative minimum, as shown in Figure 5.38. Try using a graphing utility to confirm this result. If your graphing utility does not have hyperbolic functions, you can use exponential functions, as follows.

$$\begin{aligned} f(x) &= (x - 1)\left(\tfrac{1}{2}\right)(e^x + e^{-x}) - \tfrac{1}{2}(e^x - e^{-x}) \\ &= \tfrac{1}{2}(xe^x + xe^{-x} - e^x - e^{-x} - e^x + e^{-x}) \\ &= \tfrac{1}{2}(xe^x + xe^{-x} - 2e^x) \end{aligned}$$ ■



$f''(0) < 0$, so $(0, -1)$ is a relative maximum. $f''(1) > 0$, so $(1, -\sinh 1)$ is a relative minimum.
Figure 5.38

When a uniform flexible cable, such as a telephone wire, is suspended from two points, it takes the shape of a *catenary*, as discussed in Example 3.

EXAMPLE 3 Hanging Power Cables

Power cables are suspended between two towers, forming the catenary shown in Figure 5.39. The equation for this catenary is

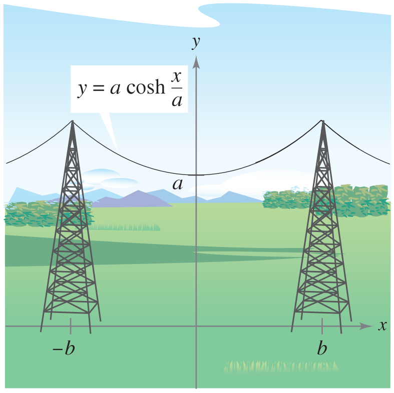
$$y = a\cosh\frac{x}{a}.$$

The distance between the two towers is $2b$. Find the slope of the catenary at the point where the cable meets the right-hand tower.

Solution Differentiating produces

$$y' = a\left(\frac{1}{a}\right)\sinh\frac{x}{a} = \sinh\frac{x}{a}.$$

At the point $(b, a\cosh(b/a))$, the slope (from the left) is given by $m = \sinh\frac{b}{a}$. ■



Catenary
Figure 5.39

FOR FURTHER INFORMATION In Example 3, the cable is a catenary between two supports at the same height. To learn about the shape of a cable hanging between supports of different heights, see the article "Reexamining the Catenary" by Paul Cella in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

EXAMPLE 4 Integrating a Hyperbolic Function

Find $\int \cosh 2x \sinh^2 2x \, dx$.

Solution

$$\begin{aligned}\int \cosh 2x \sinh^2 2x \, dx &= \frac{1}{2}\int (\sinh 2x)^2 (2 \cosh 2x) \, dx \qquad u = \sinh 2x \\ &= \frac{1}{2}\left[\frac{(\sinh 2x)^3}{3}\right] + C \\ &= \frac{\sinh^3 2x}{6} + C\end{aligned}$$

■

Inverse Hyperbolic Functions

Unlike trigonometric functions, hyperbolic functions are not periodic. In fact, by looking back at Figure 5.37, you can see that four of the six hyperbolic functions are actually one-to-one (the hyperbolic sine, tangent, cosecant, and cotangent). So, you can apply Theorem 5.7 to conclude that these four functions have inverse functions. The other two (the hyperbolic cosine and secant) are one-to-one if their domains are restricted to the positive real numbers, and for this restricted domain they also have inverse functions. Because the hyperbolic functions are defined in terms of exponential functions, it is not surprising to find that the inverse hyperbolic functions can be written in terms of logarithmic functions, as shown in Theorem 5.19.

THEOREM 5.19 INVERSE HYPERBOLIC FUNCTIONS

Function	*Domain*
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$	$(-\infty, \infty)$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$	$[1, \infty)$
$\tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x}$	$(-1, 1)$
$\coth^{-1} x = \frac{1}{2} \ln \frac{x + 1}{x - 1}$	$(-\infty, -1) \cup (1, \infty)$
$\operatorname{sech}^{-1} x = \ln \frac{1 + \sqrt{1 - x^2}}{x}$	$(0, 1]$
$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{\lvert x \rvert}\right)$	$(-\infty, 0) \cup (0, \infty)$

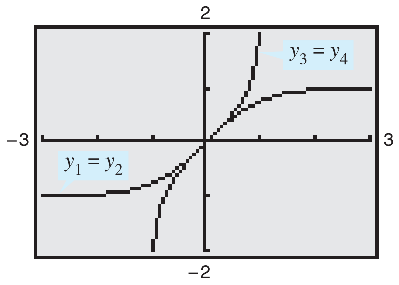
PROOF The proof of this theorem is a straightforward application of the properties of the exponential and logarithmic functions. For example, if

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

and

$$g(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

you can show that $f(g(x)) = x$ and $g(f(x)) = x$, which implies that g is the inverse function of f. ■



Graphs of the hyperbolic tangent function and the inverse hyperbolic tangent function
Figure 5.40

TECHNOLOGY You can use a graphing utility to confirm graphically the results of Theorem 5.19. For instance, graph the following functions.

$$y_1 = \tanh x \qquad \text{Hyperbolic tangent}$$

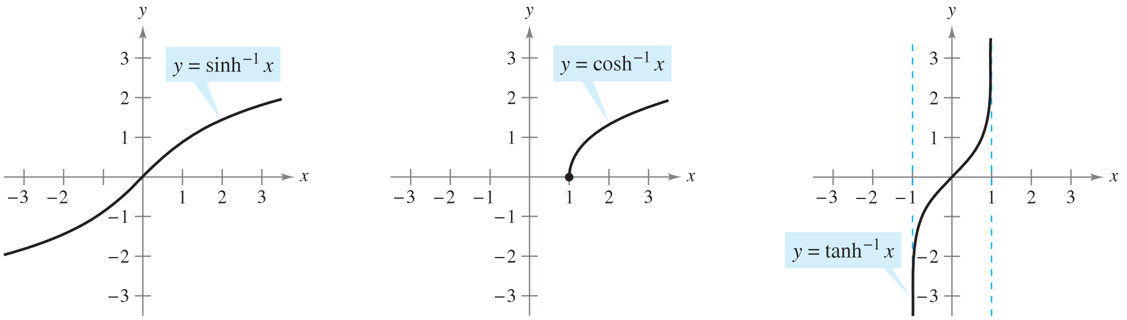
$$y_2 = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \text{Definition of hyperbolic tangent}$$

$$y_3 = \tanh^{-1} x \qquad \text{Inverse hyperbolic tangent}$$

$$y_4 = \frac{1}{2} \ln \frac{1 + x}{1 - x} \qquad \text{Definition of inverse hyperbolic tangent}$$

The resulting display is shown in Figure 5.40. As you watch the graphs being traced out, notice that $y_1 = y_2$ and $y_3 = y_4$. Also notice that the graph of y_1 is the reflection of the graph of y_3 in the line $y = x$.

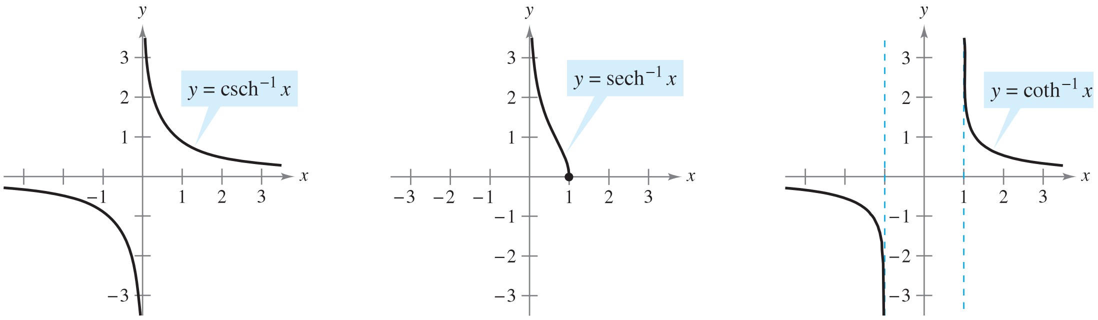
The graphs of the inverse hyperbolic functions are shown in Figure 5.41.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Domain: $[1, \infty)$
Range: $[0, \infty)$

Domain: $(-1, 1)$
Range: $(-\infty, \infty)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$
Figure 5.41

Domain: $(0, 1]$
Range: $[0, \infty)$

Domain: $(-\infty, -1) \cup (1, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

The inverse hyperbolic secant can be used to define a curve called a *tractrix* or *pursuit curve*, as discussed in Example 5.

EXAMPLE 5 A Tractrix

A person is holding a rope that is tied to a boat, as shown in Figure 5.42. As the person walks along the dock, the boat travels along a **tractrix,** given by the equation

$$y = a \operatorname{sech}^{-1} \frac{x}{a} - \sqrt{a^2 - x^2}$$

where a is the length of the rope. If $a = 20$ feet, find the distance the person must walk to bring the boat to a position 5 feet from the dock.

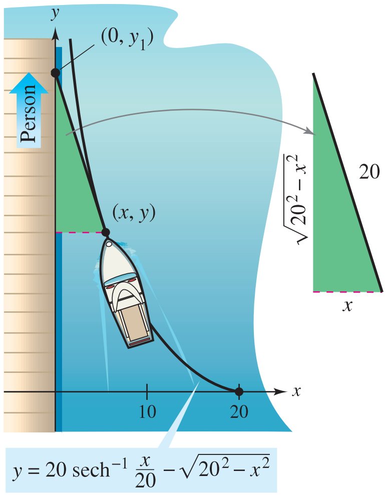
Solution In Figure 5.42, notice that the distance the person has walked is given by

$$\begin{aligned} y_1 = y + \sqrt{20^2 - x^2} &= \left(20 \operatorname{sech}^{-1} \frac{x}{20} - \sqrt{20^2 - x^2}\right) + \sqrt{20^2 - x^2} \\ &= 20 \operatorname{sech}^{-1} \frac{x}{20}. \end{aligned}$$

When $x = 5$, this distance is

$$\begin{aligned} y_1 = 20 \operatorname{sech}^{-1} \frac{5}{20} &= 20 \ln \frac{1 + \sqrt{1 - (1/4)^2}}{1/4} \\ &= 20 \ln\left(4 + \sqrt{15}\right) \\ &\approx 41.27 \text{ feet.} \end{aligned}$$

■


y
(0, y_1)
Person
(x, y)
$\sqrt{20^2 - x^2}$
20
x
10
20
x
$y = 20 \operatorname{sech}^{-1} \frac{x}{20} - \sqrt{20^2 - x^2}$

A person must walk 41.27 feet to bring the boat to a position 5 feet from the dock.
Figure 5.42

Differentiation and Integration of Inverse Hyperbolic Functions

The derivatives of the inverse hyperbolic functions, which resemble the derivatives of the inverse trigonometric functions, are listed in Theorem 5.20 with the corresponding integration formulas (in logarithmic form). You can verify each of these formulas by applying the logarithmic definitions of the inverse hyperbolic functions. (See Exercises 119–121.)

THEOREM 5.20 DIFFERENTIATION AND INTEGRATION INVOLVING INVERSE HYPERBOLIC FUNCTIONS

Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh^{-1} u] = \frac{u'}{\sqrt{u^2 + 1}} \qquad \frac{d}{dx}[\cosh^{-1} u] = \frac{u'}{\sqrt{u^2 - 1}}$$

$$\frac{d}{dx}[\tanh^{-1} u] = \frac{u'}{1 - u^2} \qquad \frac{d}{dx}[\coth^{-1} u] = \frac{u'}{1 - u^2}$$

$$\frac{d}{dx}[\operatorname{sech}^{-1} u] = \frac{-u'}{u\sqrt{1 - u^2}} \qquad \frac{d}{dx}[\operatorname{csch}^{-1} u] = \frac{-u'}{|u|\sqrt{1 + u^2}}$$

$$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$$

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln\left|\frac{a + u}{a - u}\right| + C$$

$$\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a} \ln \frac{a + \sqrt{a^2 \pm u^2}}{|u|} + C$$

EXAMPLE 6 More About a Tractrix

For the tractrix given in Example 5, show that the boat is always pointing toward the person.

Solution For a point (x, y) on a tractrix, the slope of the graph gives the direction of the boat, as shown in Figure 5.42.

$$\begin{aligned}
y' &= \frac{d}{dx}\left[20\operatorname{sech}^{-1}\frac{x}{20} - \sqrt{20^2 - x^2}\right]\\
&= -20\left(\frac{1}{20}\right)\left[\frac{1}{(x/20)\sqrt{1-(x/20)^2}}\right] - \left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{20^2 - x^2}}\right)\\
&= \frac{-20^2}{x\sqrt{20^2 - x^2}} + \frac{x}{\sqrt{20^2 - x^2}}\\
&= -\frac{\sqrt{20^2 - x^2}}{x}
\end{aligned}$$

However, from Figure 5.42, you can see that the slope of the line segment connecting the point $(0, y_1)$ with the point (x, y) is also

$$m = -\frac{\sqrt{20^2 - x^2}}{x}.$$

So, the boat is always pointing toward the person. (It is because of this property that a tractrix is called a *pursuit curve*.)

EXAMPLE 7 Integration Using Inverse Hyperbolic Functions

Find $\displaystyle\int \frac{dx}{x\sqrt{4 - 9x^2}}$.

Solution Let $a = 2$ and $u = 3x$.

$$\begin{aligned}
\int \frac{dx}{x\sqrt{4 - 9x^2}} &= \int \frac{3\,dx}{(3x)\sqrt{4 - 9x^2}}\\
&= -\frac{1}{2}\ln\frac{2 + \sqrt{4 - 9x^2}}{|3x|} + C
\end{aligned}$$

$$\int \frac{du}{u\sqrt{a^2 - u^2}}$$

$$-\frac{1}{a}\ln\frac{a + \sqrt{a^2 - u^2}}{|u|} + C$$

EXAMPLE 8 Integration Using Inverse Hyperbolic Functions

Find $\displaystyle\int \frac{dx}{5 - 4x^2}$.

Solution Let $a = \sqrt{5}$ and $u = 2x$.

$$\begin{aligned}
\int \frac{dx}{5 - 4x^2} &= \frac{1}{2}\int \frac{2\,dx}{(\sqrt{5})^2 - (2x)^2}\\
&= \frac{1}{2}\left(\frac{1}{2\sqrt{5}}\ln\left|\frac{\sqrt{5} + 2x}{\sqrt{5} - 2x}\right|\right) + C\\
&= \frac{1}{4\sqrt{5}}\ln\left|\frac{\sqrt{5} + 2x}{\sqrt{5} - 2x}\right| + C
\end{aligned}$$

$$\int \frac{du}{a^2 - u^2}$$

$$\frac{1}{2a}\ln\left|\frac{a + u}{a - u}\right| + C$$

■

5.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, evaluate the function. If the value is not a rational number, give the answer to three-decimal-place accuracy.

1. (a) $\sinh 3$ **2.** (a) $\cosh 0$

(b) $\tanh(-2)$ (b) $\operatorname{sech} 1$

3. (a) $\operatorname{csch}(\ln 2)$ **4.** (a) $\sinh^{-1} 0$

(b) $\coth(\ln 5)$ (b) $\tanh^{-1} 0$

5. (a) $\cosh^{-1} 2$ **6.** (a) $\operatorname{csch}^{-1} 2$

(b) $\operatorname{sech}^{-1} \frac{2}{3}$ (b) $\coth^{-1} 3$

In Exercises 7–16, verify the identity.

7. $e^x = \sinh x + \cosh x$ **8.** $e^{2x} = \sinh 2x + \cosh 2x$

9. $\tanh^2 x + \operatorname{sech}^2 x = 1$ **10.** $\coth^2 x - \operatorname{csch}^2 x = 1$

11. $\cosh^2 x = \dfrac{1 + \cosh 2x}{2}$ **12.** $\sinh^2 x = \dfrac{-1 + \cosh 2x}{2}$

13. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

14. $\sinh 2x = 2 \sinh x \cosh x$

15. $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$

16. $\cosh x + \cosh y = 2 \cosh \dfrac{x + y}{2} \cosh \dfrac{x - y}{2}$

In Exercises 17 and 18, use the value of the given hyperbolic function to find the values of the other hyperbolic functions at *x*.

17. $\sinh x = \dfrac{3}{2}$ **18.** $\tanh x = \dfrac{1}{2}$

In Exercises 19–30, find the derivative of the function.

19. $f(x) = \sinh 3x$ **20.** $f(x) = \cosh(x - 2)$

21. $y = \operatorname{sech}(5x^2)$ **22.** $y = \tanh(3x^2 - 1)$

23. $f(x) = \ln(\sinh x)$ **24.** $g(x) = \ln(\cosh x)$

25. $y = \ln\left(\tanh \dfrac{x}{2}\right)$ **26.** $y = x \cosh x - \sinh x$

27. $h(x) = \dfrac{1}{4} \sinh 2x - \dfrac{x}{2}$ **28.** $h(t) = t - \coth t$

29. $f(t) = \arctan(\sinh t)$ **30.** $g(x) = \operatorname{sech}^2 3x$

In Exercises 31–34, find an equation of the tangent line to the graph of the function at the given point.

31. $y = \sinh(1 - x^2)$, $(1, 0)$ **32.** $y = x^{\cosh x}$, $(1, 1)$

33. $y = (\cosh x - \sinh x)^2$, $(0, 1)$ **34.** $y = e^{\sinh x}$, $(0, 1)$

In Exercises 35–38, find any relative extrema of the function. Use a graphing utility to confirm your result.

35. $f(x) = \sin x \sinh x - \cos x \cosh x$, $-4 \le x \le 4$

36. $f(x) = x \sinh(x - 1) - \cosh(x - 1)$

37. $g(x) = x \operatorname{sech} x$ **38.** $h(x) = 2 \tanh x - x$

In Exercises 39 and 40, show that the function satisfies the differential equation.

	Function	*Differential Equation*
39.	$y = a \sinh x$	$y''' - y' = 0$
40.	$y = a \cosh x$	$y'' - y = 0$

CAS ***Linear and Quadratic Approximations*** **In Exercises 41 and 42, use a computer algebra system to find the linear approximation**

$$P_1(x) = f(a) + f'(a)(x - a)$$

and the quadratic approximation

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2} f''(a)(x - a)^2$$

of the function *f* at $x = a$. Use a graphing utility to graph the function and its linear and quadratic approximations.

41. $f(x) = \tanh x$, $a = 0$ **42.** $f(x) = \cosh x$, $a = 0$

Catenary **In Exercises 43 and 44, a model for a power cable suspended between two towers is given. (a) Graph the model, (b) find the heights of the cable at the towers and at the midpoint between the towers, and (c) find the slope of the model at the point where the cable meets the right-hand tower.**

43. $y = 10 + 15 \cosh \dfrac{x}{15}$, $-15 \le x \le 15$

44. $y = 18 + 25 \cosh \dfrac{x}{25}$, $-25 \le x \le 25$

In Exercises 45–58, find the integral.

45. $\displaystyle\int \cosh 2x\, dx$ **46.** $\displaystyle\int \operatorname{sech}^2 (-x)\, dx$

47. $\displaystyle\int \sinh(1 - 2x)\, dx$ **48.** $\displaystyle\int \frac{\cosh \sqrt{x}}{\sqrt{x}}\, dx$

49. $\displaystyle\int \cosh^2(x - 1) \sinh(x - 1)\, dx$ **50.** $\displaystyle\int \frac{\sinh x}{1 + \sinh^2 x}\, dx$

51. $\displaystyle\int \frac{\cosh x}{\sinh x}\, dx$ **52.** $\displaystyle\int \operatorname{sech}^2(2x - 1)\, dx$

53. $\displaystyle\int x \operatorname{csch}^2 \frac{x^2}{2}\, dx$ **54.** $\displaystyle\int \operatorname{sech}^3 x \tanh x\, dx$

55. $\displaystyle\int \frac{\operatorname{csch}(1/x) \coth(1/x)}{x^2}\, dx$ **56.** $\displaystyle\int \frac{\cosh x}{\sqrt{9 - \sinh^2 x}}\, dx$

57. $\displaystyle\int \frac{x}{x^4 + 1}\, dx$ **58.** $\displaystyle\int \frac{2}{x\sqrt{1 + 4x^2}}\, dx$

In Exercises 59–64, evaluate the integral.

59. $\displaystyle\int_0^{\ln 2} \tanh x\, dx$ **60.** $\displaystyle\int_0^1 \cosh^2 x\, dx$

61. $\displaystyle\int_0^4 \frac{1}{25 - x^2}\, dx$ **62.** $\displaystyle\int_0^4 \frac{1}{\sqrt{25 - x^2}}\, dx$

63. $\displaystyle\int_0^{\sqrt{2}/4} \frac{2}{\sqrt{1 - 4x^2}}\, dx$ **64.** $\displaystyle\int_0^{\ln 2} 2e^{-x} \cosh x\, dx$

In Exercises 65–74, find the derivative of the function.

65. $y = \cosh^{-1}(3x)$

66. $y = \tanh^{-1}\dfrac{x}{2}$

67. $y = \tanh^{-1}\sqrt{x}$

68. $f(x) = \coth^{-1}(x^2)$

69. $y = \sinh^{-1}(\tan x)$

70. $y = \tanh^{-1}(\sin 2x)$

71. $y = (\operatorname{csch}^{-1} x)^2$

72. $y = \operatorname{sech}^{-1}(\cos 2x), \quad 0 < x < \pi/4$

73. $y = 2x \sinh^{-1}(2x) - \sqrt{1 + 4x^2}$

74. $y = x \tanh^{-1} x + \ln\sqrt{1 - x^2}$

WRITING ABOUT CONCEPTS

75. Discuss several ways in which the hyperbolic functions are similar to the trigonometric functions.

76. Sketch the graph of each hyperbolic function. Then identify the domain and range of each function.

77. Which hyperbolic derivative formulas differ from their trigonometric counterparts by a minus sign?

CAPSTONE

78. Which hyperbolic functions take on only positive values? Which hyperbolic functions are increasing on their domains?

Limits **In Exercises 79–86, find the limit.**

79. $\lim\limits_{x\to\infty} \sinh x$

80. $\lim\limits_{x\to-\infty} \sinh x$

81. $\lim\limits_{x\to\infty} \tanh x$

82. $\lim\limits_{x\to-\infty} \tanh x$

83. $\lim\limits_{x\to\infty} \operatorname{sech} x$

84. $\lim\limits_{x\to-\infty} \operatorname{csch} x$

85. $\lim\limits_{x\to0} \dfrac{\sinh x}{x}$

86. $\lim\limits_{x\to0^-} \coth x$

In Exercises 87–96, find the indefinite integral using the formulas from Theorem 5.20.

87. $\displaystyle\int \frac{1}{3 - 9x^2}\,dx$

88. $\displaystyle\int \frac{1}{2x\sqrt{1 - 4x^2}}\,dx$

89. $\displaystyle\int \frac{1}{\sqrt{1 + e^{2x}}}\,dx$

90. $\displaystyle\int \frac{x}{9 - x^4}\,dx$

91. $\displaystyle\int \frac{1}{\sqrt{x}\sqrt{1 + x}}\,dx$

92. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1 + x^3}}\,dx$

93. $\displaystyle\int \frac{-1}{4x - x^2}\,dx$

94. $\displaystyle\int \frac{dx}{(x + 2)\sqrt{x^2 + 4x + 8}}$

95. $\displaystyle\int \frac{1}{1 - 4x - 2x^2}\,dx$

96. $\displaystyle\int \frac{dx}{(x + 1)\sqrt{2x^2 + 4x + 8}}$

In Exercises 97–100, evaluate the integral using the formulas from Theorem 5.20.

97. $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2 - 4}}\,dx$

98. $\displaystyle\int_1^3 \frac{1}{x\sqrt{4 + x^2}}\,dx$

99. $\displaystyle\int_{-1}^1 \frac{1}{16 - 9x^2}\,dx$

100. $\displaystyle\int_0^1 \frac{1}{\sqrt{25x^2 + 1}}\,dx$

In Exercises 101–104, solve the differential equation.

101. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{80 + 8x - 16x^2}}$

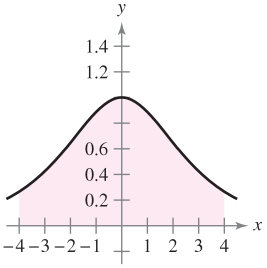
102. $\dfrac{dy}{dx} = \dfrac{1}{(x - 1)\sqrt{-4x^2 + 8x - 1}}$

103. $\dfrac{dy}{dx} = \dfrac{x^3 - 21x}{5 + 4x - x^2}$

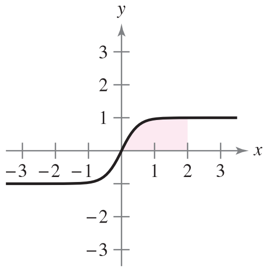
104. $\dfrac{dy}{dx} = \dfrac{1 - 2x}{4x - x^2}$

Area **In Exercises 105–108, find the area of the region.**

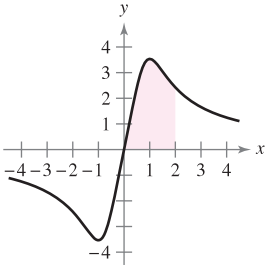
105. $y = \operatorname{sech}\dfrac{x}{2}$



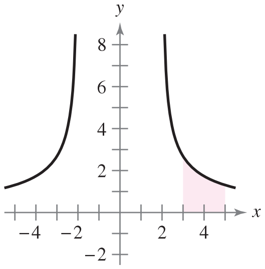
106. $y = \tanh 2x$



107. $y = \dfrac{5x}{\sqrt{x^4 + 1}}$



108. $y = \dfrac{6}{\sqrt{x^2 - 4}}$



In Exercises 109 and 110, evaluate the integral in terms of (a) natural logarithms and (b) inverse hyperbolic functions.

109. $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{x^2 + 1}}$

110. $\displaystyle\int_{-1/2}^{1/2} \frac{dx}{1 - x^2}$

111. ***Chemical Reactions*** Chemicals A and B combine in a 3-to-1 ratio to form a compound. The amount of compound x being produced at any time t is proportional to the unchanged amounts of A and B remaining in the solution. So, if 3 kilograms of A is mixed with 2 kilograms of B, you have

$$\frac{dx}{dt} = k\left(3 - \frac{3x}{4}\right)\left(2 - \frac{x}{4}\right) = \frac{3k}{16}(x^2 - 12x + 32).$$

One kilogram of the compound is formed after 10 minutes. Find the amount formed after 20 minutes by solving the equation

$$\int \frac{3k}{16}\,dt = \int \frac{dx}{x^2 - 12x + 32}.$$

112. ***Vertical Motion*** An object is dropped from a height of 400 feet.

(a) Find the velocity of the object as a function of time (neglect air resistance on the object).

(b) Use the result in part (a) to find the position function.

(c) If the air resistance is proportional to the square of the velocity, then $dv/dt = -32 + kv^2$, where -32 feet per second per second is the acceleration due to gravity and k is a constant. Show that the velocity v as a function of time is $v(t) = -\sqrt{32/k}\tanh\left(\sqrt{32k}\,t\right)$ by performing $\int dv/(32 - kv^2) = -\int dt$ and simplifying the result.

(d) Use the result of part (c) to find $\lim_{t\to\infty} v(t)$ and give its interpretation.

(e) Integrate the velocity function in part (c) and find the position s of the object as a function of t. Use a graphing utility to graph the position function when $k = 0.01$ and the position function in part (b) in the same viewing window. Estimate the additional time required for the object to reach ground level when air resistance is not neglected.

(f) Give a written description of what you believe would happen if k were increased. Then test your assertion with a particular value of k.

Tractrix **In Exercises 113 and 114, use the equation of the tractrix $y = a\,\mathrm{sech}^{-1}(x/a) - \sqrt{a^2 - x^2},\quad a > 0$.**

113. Find dy/dx.

114. Let L be the tangent line to the tractrix at the point P. If L intersects the y-axis at the point Q, show that the distance between P and Q is a.

115. Prove that $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), \quad -1 < x < 1.$

116. Prove that $\sinh^{-1} t = \ln\left(t + \sqrt{t^2+1}\right)$.

117. Show that $\arctan(\sinh x) = \arcsin(\tanh x)$.

118. Let $x > 0$ and $b > 0$. Show that $\displaystyle\int_{-b}^{b} e^{xt}\,dt = \frac{2\sinh bx}{x}$.

In Exercises 119–124, verify the differentiation formula.

119. $\dfrac{d}{dx}[\mathrm{sech}^{-1} x] = \dfrac{-1}{x\sqrt{1-x^2}}$

120. $\dfrac{d}{dx}[\cosh^{-1} x] = \dfrac{1}{\sqrt{x^2-1}}$

121. $\dfrac{d}{dx}[\sinh^{-1} x] = \dfrac{1}{\sqrt{x^2+1}}$

122. $\dfrac{d}{dx}[\cosh x] = \sinh x$

123. $\dfrac{d}{dx}[\coth x] = -\mathrm{csch}^2\, x$

124. $\dfrac{d}{dx}[\mathrm{sech}\, x] = -\mathrm{sech}\, x \tanh x$

PUTNAM EXAM CHALLENGE

125. From the vertex $(0, c)$ of the catenary $y = c\cosh(x/c)$ a line L is drawn perpendicular to the tangent to the catenary at a point P. Prove that the length of L intercepted by the axes is equal to the ordinate y of the point P.

126. Prove or disprove that there is at least one straight line normal to the graph of $y = \cosh x$ at a point $(a, \cosh a)$ and also normal to the graph of $y = \sinh x$ at a point $(c, \sinh c)$.

[At a point on a graph, the normal line is the perpendicular to the tangent at that point. Also, $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.]

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SECTION PROJECT

St. Louis Arch

National Geographic/Getty Images

The Gateway Arch in St. Louis, Missouri was constructed using the hyperbolic cosine function. The equation used for construction was

$y = 693.8597 - 68.7672\cosh 0.0100333x,$

$-299.2239 \le x \le 299.2239$

where x and y are measured in feet. Cross sections of the arch are equilateral triangles, and (x, y) traces the path of the centers of mass of the cross-sectional triangles. For each value of x, the area of the cross-sectional triangle is $A = 125.1406\cosh 0.0100333x$. *(Source: Owner's Manual for the Gateway Arch, Saint Louis, MO, by William Thayer)*

(a) How high above the ground is the center of the highest triangle? (At ground level, $y = 0$.)

(b) What is the height of the arch? (*Hint:* For an equilateral triangle, $A = \sqrt{3}c^2$, where c is one-half the base of the triangle, and the center of mass of the triangle is located at two-thirds the height of the triangle.)

(c) How wide is the arch at ground level?

5 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, sketch the graph of the function by hand. Identify any asymptotes of the graph.

1. $f(x) = \ln x - 3$

2. $f(x) = \ln(x + 3)$

In Exercises 3 and 4, use the properties of logarithms to expand the logarithmic function.

3. $\ln \sqrt[5]{\dfrac{4x^2 - 1}{4x^2 + 1}}$

4. $\ln[(x^2 + 1)(x - 1)]$

In Exercises 5 and 6, write the expression as the logarithm of a single quantity.

5. $\ln 3 + \frac{1}{3}\ln(4 - x^2) - \ln x$

6. $3[\ln x - 2\ln(x^2 + 1)] + 2\ln 5$

In Exercises 7 and 8, solve the equation for *x*.

7. $\ln\sqrt{x + 1} = 2$

8. $\ln x + \ln(x - 3) = 0$

In Exercises 9–14, find the derivative of the function.

9. $g(x) = \ln\sqrt{2x}$

10. $h(x) = \ln \dfrac{x(x - 1)}{x - 2}$

11. $f(x) = x\sqrt{\ln x}$

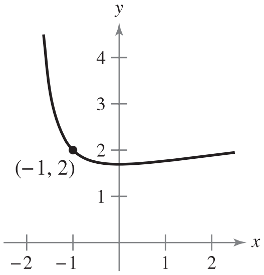
12. $f(x) = \ln[x(x^2 - 2)^{2/3}]$

13. $y = \dfrac{1}{b^2}[a + bx - a\ln(a + bx)]$

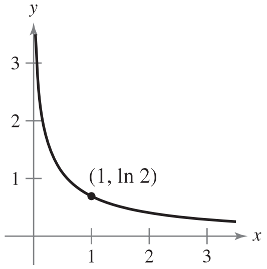
14. $y = -\dfrac{1}{ax} + \dfrac{b}{a^2}\ln\dfrac{a + bx}{x}$

In Exercises 15 and 16, find an equation of the tangent line to the graph of the function at the given point.

15. $y = \ln(2 + x) + \dfrac{2}{2 + x}$



16. $y = \ln\dfrac{1 + x}{x}$



In Exercises 17–24, find or evaluate the integral.

17. $\displaystyle\int \frac{1}{7x - 2}\,dx$

18. $\displaystyle\int \frac{x}{x^2 - 1}\,dx$

19. $\displaystyle\int \frac{\sin x}{1 + \cos x}\,dx$

20. $\displaystyle\int \frac{\ln\sqrt{x}}{x}\,dx$

21. $\displaystyle\int_1^4 \frac{2x + 1}{2x}\,dx$

22. $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

23. $\displaystyle\int_0^{\pi/3} \sec\theta\,d\theta$

24. $\displaystyle\int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right)dx$

In Exercises 25–30, (a) find the inverse function of *f*, (b) use a graphing utility to graph *f* and f^{-1} in the same viewing window, (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$, and (d) state the domains and ranges of *f* and f^{-1}.

25. $f(x) = \frac{1}{2}x - 3$

26. $f(x) = 5x - 7$

27. $f(x) = \sqrt{x + 1}$

28. $f(x) = x^3 + 2$

29. $f(x) = \sqrt[3]{x + 1}$

30. $f(x) = x^2 - 5, \quad x \ge 0$

In Exercises 31–34, verify that *f* has an inverse. Then use the function *f* and the given real number *a* to find $(f^{-1})'(a)$. (*Hint:* Use Theorem 5.9.)

31. $f(x) = x^3 + 2, \quad a = -1$

32. $f(x) = x\sqrt{x - 3}, \quad a = 4$

33. $f(x) = \tan x, \quad -\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}, \quad a = \dfrac{\sqrt{3}}{3}$

34. $f(x) = \cos x, \quad 0 \le x \le \pi, \quad a = 0$

In Exercises 35 and 36, (a) find the inverse function of *f*, (b) use a graphing utility to graph *f* and f^{-1} in the same viewing window, (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$, and (d) state the domains and ranges of *f* and f^{-1}.

35. $f(x) = \ln\sqrt{x}$

36. $f(x) = e^{1-x}$

In Exercises 37 and 38, graph the function without the aid of a graphing utility.

37. $y = e^{-x/2}$

38. $y = e^{-x^2}$

In Exercises 39–44, find the derivative of the function.

39. $g(t) = t^2e^t$

40. $g(x) = \ln\dfrac{e^x}{1 + e^x}$

41. $y = \sqrt{e^{2x} + e^{-2x}}$

42. $h(z) = e^{-z^2/2}$

43. $g(x) = \dfrac{x^2}{e^x}$

44. $y = 3e^{-3/t}$

In Exercises 45 and 46, find an equation of the tangent line to the graph of the function at the given point.

45. $f(x) = \ln(e^{-x^2}), \quad (2, -4)$

46. $f(\theta) = \dfrac{1}{2}e^{\sin 2\theta}, \quad \left(0, \dfrac{1}{2}\right)$

In Exercises 47 and 48, use implicit differentiation to find *dy*/*dx*.

47. $y\ln x + y^2 = 0$

48. $\cos x^2 = xe^y$

In Exercises 49–56, find or evaluate the integral.

49. $\displaystyle\int_0^1 xe^{-3x^2}\,dx$

50. $\displaystyle\int_{1/2}^2 \frac{e^{1/x}}{x^2}\,dx$

51. $\displaystyle\int \frac{e^{4x} - e^{2x} + 1}{e^x}\,dx$

52. $\displaystyle\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx$

53. $\int xe^{1-x^2}\,dx$

54. $\int x^2e^{x^3+1}\,dx$

55. $\int_1^3 \frac{e^x}{e^x-1}\,dx$

56. $\int_0^2 \frac{e^{2x}}{e^{2x}+1}\,dx$

57. Show that $y = e^x(a\cos 3x + b\sin 3x)$ satisfies the differential equation $y'' - 2y' + 10y = 0$.

58. ***Depreciation*** The value V of an item t years after it is purchased is $V = 9000e^{-0.6t}$, $0 \le t \le 5$.

(a) Use a graphing utility to graph the function.

(b) Find the rates of change of V with respect to t when $t = 1$ and $t = 4$.

(c) Use a graphing utility to graph the tangent lines to the function when $t = 1$ and $t = 4$.

In Exercises 59 and 60, find the area of the region bounded by the graphs of the equations.

59. $y = xe^{-x^2}$, $y = 0$, $x = 0$, $x = 4$

60. $y = 2e^{-x}$, $y = 0$, $x = 0$, $x = 2$

In Exercises 61–64, sketch the graph of the function by hand.

61. $y = 3^{x/2}$

62. $y = 6(2^{-x^2})$

63. $y = \log_2(x - 1)$

64. $y = \log_4 x^2$

In Exercises 65–70, find the derivative of the function.

65. $f(x) = 3^{x-1}$

66. $f(x) = (4e)^x$

67. $y = x^{2x+1}$

68. $y = x(4^{-x})$

69. $g(x) = \log_3 \sqrt{1-x}$

70. $h(x) = \log_5 \frac{x}{x-1}$

In Exercises 71 and 72, find the indefinite integral.

71. $\int (x+1)5^{(x+1)^2}\,dx$

72. $\int \frac{2^{-1/t}}{t^2}\,dt$

73. ***Climb Rate*** The time t (in minutes) for a small plane to climb to an altitude of h feet is

$$t = 50\log_{10}\frac{18{,}000}{18{,}000 - h}$$

where 18,000 feet is the plane's absolute ceiling.

(a) Determine the domain of the function appropriate for the context of the problem.

(b) Use a graphing utility to graph the time function and identify any asymptotes.

(c) Find the time when the altitude is increasing at the greatest rate.

74. ***Compound Interest***

(a) How large a deposit, at 5% interest compounded continuously, must be made to obtain a balance of \$10,000 in 15 years?

(b) A deposit earns interest at a rate of r percent compounded continuously and doubles in value in 10 years. Find r.

In Exercises 75 and 76, sketch the graph of the function.

75. $f(x) = 2\arctan(x+3)$

76. $h(x) = -3\arcsin 2x$

In Exercises 77 and 78, evaluate the expression without using a calculator. (*Hint:* Make a sketch of a right triangle.)

77. (a) $\sin\left(\arcsin \frac{1}{2}\right)$
(b) $\cos\left(\arcsin \frac{1}{2}\right)$

78. (a) $\tan(\operatorname{arccot} 2)$
(b) $\cos\left(\operatorname{arcsec} \sqrt{5}\right)$

In Exercises 79–84, find the derivative of the function.

79. $y = \tan(\arcsin x)$

80. $y = \arctan(x^2 - 1)$

81. $y = x \operatorname{arcsec} x$

82. $y = \frac{1}{2}\arctan e^{2x}$

83. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\arcsin x$

84. $y = \sqrt{x^2-4} - 2\operatorname{arcsec}\frac{x}{2}$, $2 < x < 4$

In Exercises 85–90, find the indefinite integral.

85. $\int \frac{1}{e^{2x}+e^{-2x}}\,dx$

86. $\int \frac{1}{3+25x^2}\,dx$

87. $\int \frac{x}{\sqrt{1-x^4}}\,dx$

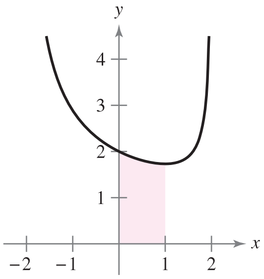
88. $\int \frac{1}{16+x^2}\,dx$

89. $\int \frac{\arctan(x/2)}{4+x^2}\,dx$

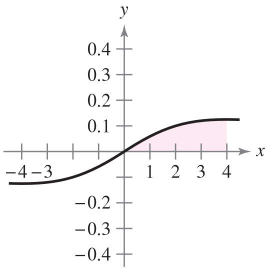
90. $\int \frac{\arcsin 2x}{\sqrt{1-4x^2}}\,dx$

In Exercises 91 and 92, find the area of the region.

91. $y = \frac{4-x}{\sqrt{4-x^2}}$



92. $y = \frac{x}{16+x^2}$



93. ***Harmonic Motion*** A weight of mass m is attached to a spring and oscillates with simple harmonic motion. By Hooke's Law, you can determine that

$$\int \frac{dy}{\sqrt{A^2-y^2}} = \int \sqrt{\frac{k}{m}}\,dt$$

where A is the maximum displacement, t is the time, and k is a constant. Find y as a function of t, given that $y = 0$ when $t = 0$.

In Exercises 94 and 95, find the derivative of the function.

94. $y = 2x - \cosh\sqrt{x}$

95. $y = x\tanh^{-1} 2x$

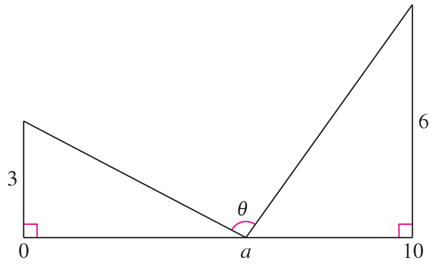
In Exercises 96 and 97, find the indefinite integral.

96. $\int \frac{x}{\sqrt{x^4-1}}\,dx$

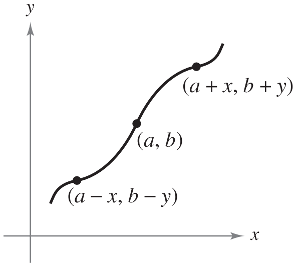
97. $\int x^2 \operatorname{sech}^2 x^3\,dx$

P.S. PROBLEM SOLVING

1. Find the value of a that maximizes the angle θ shown in the figure. What is the approximate measure of this angle?



2. Recall that the graph of a function $y = f(x)$ is symmetric with respect to the origin if, whenever (x, y) is a point on the graph, $(-x, -y)$ is also a point on the graph. The graph of the function $y = f(x)$ is **symmetric with respect to the point (a, b)** if, whenever $(a - x, b - y)$ is a point on the graph, $(a + x, b + y)$ is also a point on the graph, as shown in the figure.



(a) Sketch the graph of $y = \sin x$ on the interval $[0, 2\pi]$. Write a short paragraph explaining how the symmetry of the graph with respect to the point $(0, \pi)$ allows you to conclude that

$$\int_0^{2\pi} \sin x\, dx = 0.$$

(b) Sketch the graph of $y = \sin x + 2$ on the interval $[0, 2\pi]$. Use the symmetry of the graph with respect to the point $(\pi, 2)$ to evaluate the integral

$$\int_0^{2\pi} (\sin x + 2)\, dx.$$

(c) Sketch the graph of $y = \arccos x$ on the interval $[-1, 1]$. Use the symmetry of the graph to evaluate the integral

$$\int_{-1}^{1} \arccos x\, dx.$$

(d) Evaluate the integral $\displaystyle\int_0^{\pi/2} \frac{1}{1 + (\tan x)^{\sqrt{2}}}\, dx.$

3. (a) Use a graphing utility to graph $f(x) = \dfrac{\ln(x + 1)}{x}$ on the interval $[-1, 1]$.

(b) Use the graph to estimate $\lim_{x \to 0} f(x)$.

(c) Use the definition of derivative to prove your answer to part (b).

4. Let $f(x) = \sin(\ln x)$.

(a) Determine the domain of the function f.

(b) Find two values of x satisfying $f(x) = 1$.

(c) Find two values of x satisfying $f(x) = -1$.

(d) What is the range of the function f?

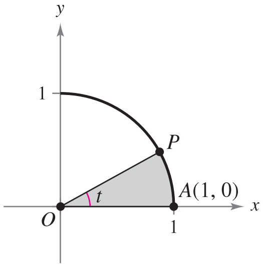
(e) Calculate $f'(x)$ and use calculus to find the maximum value of f on the interval $[1, 10]$.

(f) Use a graphing utility to graph f in the viewing window $[0, 5] \times [-2, 2]$ and estimate $\lim_{x \to 0^+} f(x)$, if it exists.

(g) Determine $\lim_{x \to 0^+} f(x)$ analytically, if it exists.

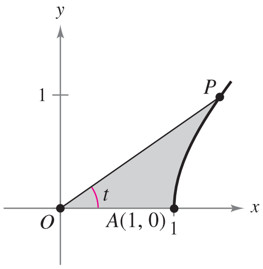
5. Graph the exponential function $y = a^x$ for $a = 0.5$, 1.2, and 2.0. Which of these curves intersects the line $y = x$? Determine all positive numbers a for which the curve $y = a^x$ intersects the line $y = x$.

6. (a) Let $P(\cos t, \sin t)$ be a point on the unit circle $x^2 + y^2 = 1$ in the first quadrant (see figure). Show that t is equal to twice the area of the shaded circular sector AOP.



(b) Let $P(\cosh t, \sinh t)$ be a point on the unit hyperbola $x^2 - y^2 = 1$ in the first quadrant (see figure). Show that t is equal to twice the area of the shaded region AOP. Begin by showing that the area of the shaded region AOP is given by the formula

$$A(t) = \frac{1}{2}\cosh t \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\, dx.$$

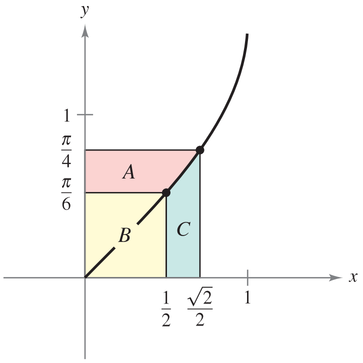


7. Apply the Mean Value Theorem to the function $f(x) = \ln x$ on the closed interval $[1, e]$. Find the value of c in the open interval $(1, e)$ such that

$$f'(c) = \frac{f(e) - f(1)}{e - 1}.$$

8. Show that $f(x) = \dfrac{\ln x^n}{x}$ is a decreasing function for $x > e$ and $n > 0$.

9. Consider the three regions A, B, and C determined by the graph of $f(x) = \arcsin x$, as shown in the figure.



(a) Calculate the areas of regions A and B.

(b) Use your answers in part (a) to evaluate the integral

$$\int_{1/2}^{\sqrt{2}/2} \arcsin x\, dx.$$

(c) Use your answers in part (a) to evaluate the integral

$$\int_1^3 \ln x\, dx.$$

(d) Use your answers in part (a) to evaluate the integral

$$\int_1^{\sqrt{3}} \arctan x\, dx.$$

10. Let L be the tangent line to the graph of the function $y = \ln x$ at the point (a, b). Show that the distance between b and c is always equal to 1.

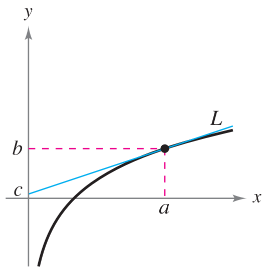


Figure for 10

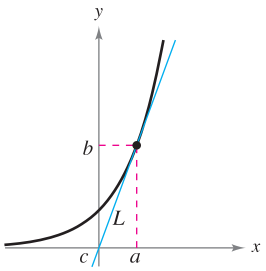


Figure for 11

11. Let L be the tangent line to the graph of the function $y = e^x$ at the point (a, b). Show that the distance between a and c is always equal to 1.

12. The **Gudermannian function** of x is $\text{gd}(x) = \arctan(\sinh x)$.

(a) Graph gd using a graphing utility.

(b) Show that gd is an odd function.

(c) Show that gd is monotonic and therefore has an inverse.

(d) Find the inflection point of gd.

(e) Verify that $\text{gd}(x) = \arcsin(\tanh x)$.

(f) Verify that $\text{gd}(x) = \displaystyle\int_0^x \frac{dx}{\cosh t}$.

13. Use integration by substitution to find the area under the curve

$$y = \frac{1}{\sqrt{x} + x}$$

between $x = 1$ and $x = 4$.

14. Use integration by substitution to find the area under the curve

$$y = \frac{1}{\sin^2 x + 4\cos^2 x}$$

between $x = 0$ and $x = \pi/4$.

15. (a) Use a graphing utility to compare the graph of the function $y = e^x$ with the graph of each given function.

(i) $y_1 = 1 + \dfrac{x}{1!}$

(ii) $y_2 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!}$

(iii) $y_3 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$

(b) Identify the pattern of successive polynomials in part (a) and extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = e^x$.

(c) What do you think this pattern implies?

16. A \$120,000 home mortgage for 35 years at $9\frac{1}{2}\%$ has a monthly payment of \$985.93. Part of the monthly payment goes for the interest charge on the unpaid balance and the remainder of the payment is used to reduce the principal. The amount that goes for interest is

$$u = M - \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}$$

and the amount that goes toward reduction of the principal is

$$v = \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}.$$

In these formulas, P is the amount of the mortgage, r is the interest rate, M is the monthly payment, and t is the time in years.

(a) Use a graphing utility to graph each function in the same viewing window. (The viewing window should show all 35 years of mortgage payments.)

(b) In the early years of the mortgage, the larger part of the monthly payment goes for what purpose? Approximate the time when the monthly payment is evenly divided between interest and principal reduction.

(c) Use the graphs in part (a) to make a conjecture about the relationship between the slopes of the tangent lines to the two curves for a specified value of t. Give an analytical argument to verify your conjecture. Find $u'(15)$ and $v'(15)$.

(d) Repeat parts (a) and (b) for a repayment period of 20 years ($M = \$1118.56$). What can you conclude?

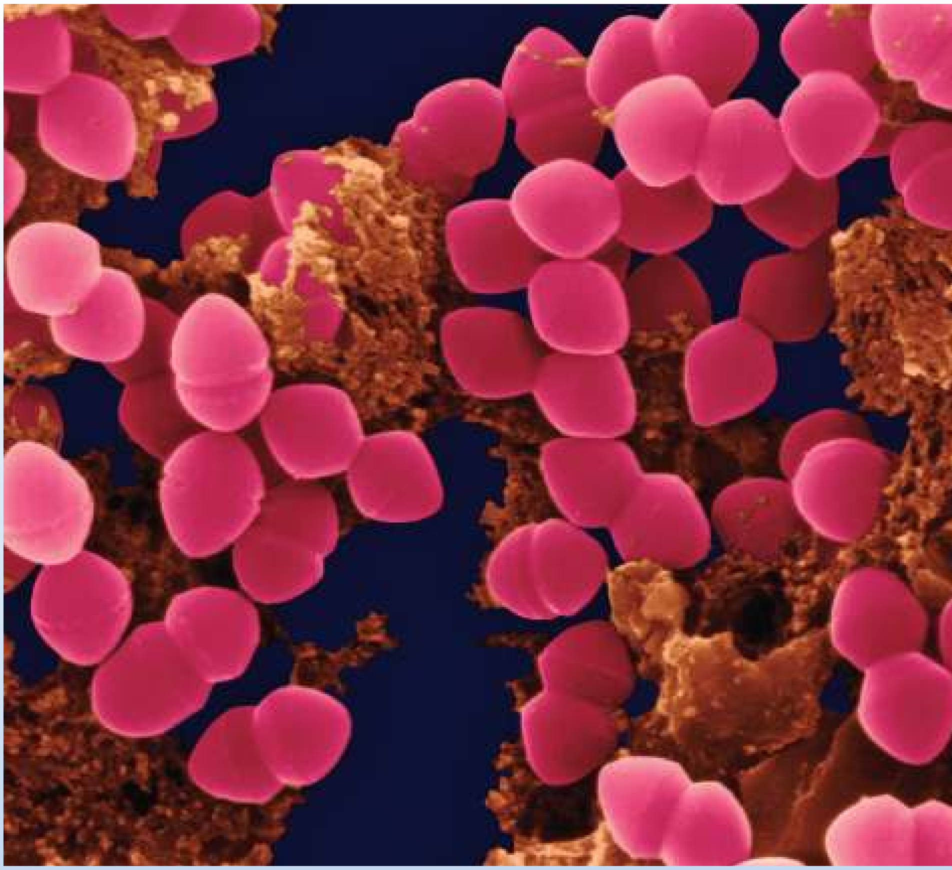
6

Differential Equations

In this chapter, you will study one of the most important applications of calculus—*differential equations*. You will learn several methods for solving different types of differential equations, such as homogeneous, first-order linear, and Bernoulli. Then you will apply these methods to solve differential equations in applied problems.

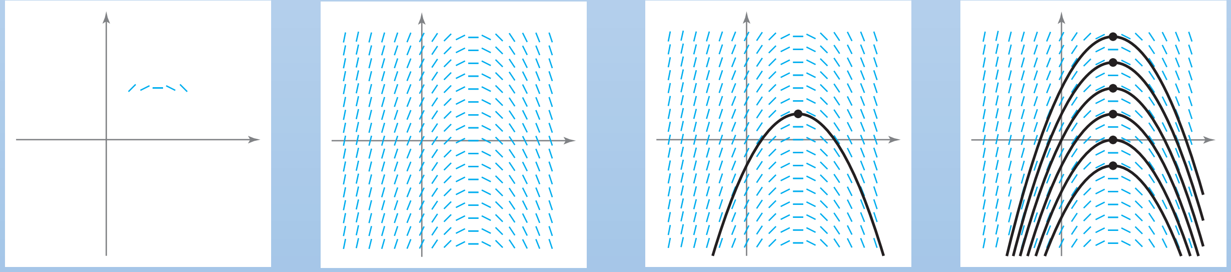
In this chapter, you should learn the following.

- How to sketch a slope field of a differential equation, and find a particular solution. (**6.1**)
- How to use an exponential function to model growth and decay. (**6.2**)
- How to use separation of variables to solve a differential equation. (**6.3**)
- How to solve a first-order linear differential equation and a Bernoulli differential equation. (**6.4**)



Dr. Dennis Kunkel/Getty Images

Depending on the type of bacteria, the time it takes for a culture's weight to double can vary greatly from several minutes to several days. How could you use a differential equation to model the growth rate of a bacteria culture's weight? (See Section 6.3, Exercise 84.)



A function $y = f(x)$ is a solution of a differential equation if the equation is satisfied when y and its derivatives are replaced by $f(x)$ and its derivatives. One way to solve a differential equation is to use slope fields, which show the general shape of all solutions of a differential equation. (See Section 6.1.)

6.1 Slope Fields and Euler's Method

- **Use initial conditions to find particular solutions of differential equations.**
- **Use slope fields to approximate solutions of differential equations.**
- **Use Euler's Method to approximate solutions of differential equations.**

General and Particular Solutions

In this text, you will learn that physical phenomena can be described by differential equations. Recall that a **differential equation** in x and y is an equation that involves x, y, and derivatives of y. In Section 6.2, you will see that problems involving radioactive decay, population growth, and Newton's Law of Cooling can be formulated in terms of differential equations.

A function $y = f(x)$ is called a **solution** of a differential equation if the equation is satisfied when y and its derivatives are replaced by $f(x)$ and its derivatives. For example, differentiation and substitution would show that $y = e^{-2x}$ is a solution of the differential equation $y' + 2y = 0$. It can be shown that every solution of this differential equation is of the form

$$y = Ce^{-2x} \qquad \text{General solution of } y' + 2y = 0$$

where C is any real number. This solution is called the **general solution.** Some differential equations have **singular solutions** that cannot be written as special cases of the general solution. However, such solutions are not considered in this text. The **order** of a differential equation is determined by the highest-order derivative in the equation. For instance, $y' = 4y$ is a first-order differential equation. First-order linear differential equations are discussed in Section 6.4.

In Section 4.1, Example 8, you saw that the second-order differential equation $s''(t) = -32$ has the general solution

$$s(t) = -16t^2 + C_1t + C_2 \qquad \text{General solution of } s''(t) = -32$$

which contains two arbitrary constants. It can be shown that a differential equation of order n has a general solution with n arbitrary constants.

EXAMPLE 1 Verifying Solutions

Determine whether the function is a solution of the differential equation $y'' - y = 0$.

a. $y = \sin x$ **b.** $y = 4e^{-x}$ **c.** $y = Ce^x$

Solution

a. Because $y = \sin x$, $y' = \cos x$, and $y'' = -\sin x$, it follows that

$$y'' - y = -\sin x - \sin x = -2 \sin x \neq 0.$$

So, $y = \sin x$ is *not* a solution.

b. Because $y = 4e^{-x}$, $y' = -4e^{-x}$, and $y'' = 4e^{-x}$, it follows that

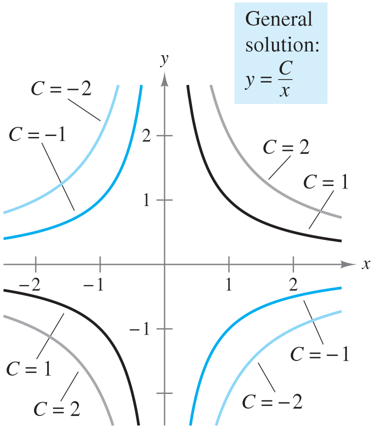
$$y'' - y = 4e^{-x} - 4e^{-x} = 0.$$

So, $y = 4e^{-x}$ is a solution.

c. Because $y = Ce^x$, $y' = Ce^x$, and $y'' = Ce^x$, it follows that

$$y'' - y = Ce^x - Ce^x = 0.$$

So, $y = Ce^x$ is a solution for any value of C. ■



Solution curves for $xy' + y = 0$
Figure 6.1

Geometrically, the general solution of a first-order differential equation represents a family of curves known as **solution curves,** one for each value assigned to the arbitrary constant. For instance, you can verify that every function of the form

$$y = \frac{C}{x} \qquad \text{General solution of } xy' + y = 0$$

is a solution of the differential equation $xy' + y = 0$. Figure 6.1 shows four of the solution curves corresponding to different values of C.

As discussed in Section 4.1, **particular solutions** of a differential equation are obtained from **initial conditions** that give the values of the dependent variable or one of its derivatives for particular values of the independent variable. The term "initial condition" stems from the fact that, often in problems involving time, the value of the dependent variable or one of its derivatives is known at the *initial* time $t = 0$. For instance, the second-order differential equation $s''(t) = -32$ having the general solution

$$s(t) = -16t^2 + C_1t + C_2 \qquad \text{General solution of } s''(t) = -32$$

might have the following initial conditions.

$$s(0) = 80, \qquad s'(0) = 64 \qquad \text{Initial conditions}$$

In this case, the initial conditions yield the particular solution

$$s(t) = -16t^2 + 64t + 80. \qquad \text{Particular solution}$$

EXAMPLE 2 Finding a Particular Solution

For the differential equation $xy' - 3y = 0$, verify that $y = Cx^3$ is a solution, and find the particular solution determined by the initial condition $y = 2$ when $x = -3$.

Solution You know that $y = Cx^3$ is a solution because $y' = 3Cx^2$ and

$$\begin{aligned} xy' - 3y &= x(3Cx^2) - 3(Cx^3) \\ &= 0. \end{aligned}$$

Furthermore, the initial condition $y = 2$ when $x = -3$ yields

$$\begin{aligned} y &= Cx^3 && \text{General solution} \\ 2 &= C(-3)^3 && \text{Substitute initial condition.} \\ -\frac{2}{27} &= C && \text{Solve for } C. \end{aligned}$$

and you can conclude that the particular solution is

$$y = -\frac{2x^3}{27}. \qquad \text{Particular solution}$$

Try checking this solution by substituting for y and y' in the original differential equation. ■

NOTE To determine a particular solution, the number of initial conditions must match the number of constants in the general solution. ■

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

Slope Fields

Solving a differential equation analytically can be difficult or even impossible. However, there is a graphical approach you can use to learn a lot about the solution of a differential equation. Consider a differential equation of the form

$$y' = F(x, y) \qquad \text{Differential equation}$$

where $F(x, y)$ is some expression in x and y. At each point (x, y) in the xy-plane where F is defined, the differential equation determines the slope $y' = F(x, y)$ of the solution at that point. If you draw short line segments with slope $F(x, y)$ at selected points (x, y) in the domain of F, then these line segments form a **slope field,** or a *direction field*, for the differential equation $y' = F(x, y)$. Each line segment has the same slope as the solution curve through that point. A slope field shows the general shape of all the solutions and can be helpful in getting a visual perspective of the directions of the solutions of a differential equation.

EXAMPLE 3 Sketching a Slope Field

Sketch a slope field for the differential equation $y' = x - y$ for the points $(-1, 1)$, $(0, 1)$, and $(1, 1)$.

Solution The slope of the solution curve at any point (x, y) is $F(x, y) = x - y$. So, the slope at $(-1, 1)$ is $y' = -1 - 1 = -2$, the slope at $(0, 1)$ is $y' = 0 - 1 = -1$, and the slope at $(1, 1)$ is $y' = 1 - 1 = 0$. Draw short line segments at the three points with their respective slopes, as shown in Figure 6.2.

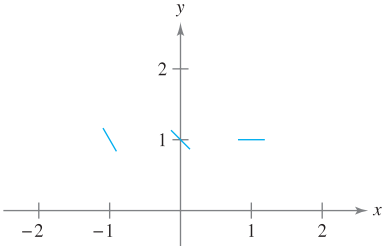


Figure 6.2

EXAMPLE 4 Identifying Slope Fields for Differential Equations

Match each slope field with its differential equation.

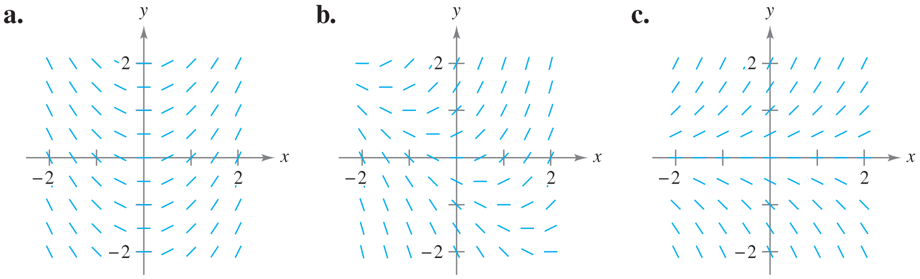


Figure 6.3

i. $y' = x + y$ **ii.** $y' = x$ **iii.** $y' = y$

Solution

a. In Figure 6.3(a), you can see that the slope at any point along the y-axis is 0. The only equation that satisfies this condition is $y' = x$. So, the graph matches equation (ii).

b. In Figure 6.3(b), you can see that the slope at the point $(1, -1)$ is 0. The only equation that satisfies this condition is $y' = x + y$. So, the graph matches equation (i).

c. In Figure 6.3(c), you can see that the slope at any point along the x-axis is 0. The only equation that satisfies this condition is $y' = y$. So, the graph matches equation (iii).

■

A solution curve of a differential equation $y' = F(x, y)$ is simply a curve in the xy-plane whose tangent line at each point (x, y) has slope equal to $F(x, y)$. This is illustrated in Example 5.

EXAMPLE 5 Sketching a Solution Using a Slope Field

Sketch a slope field for the differential equation

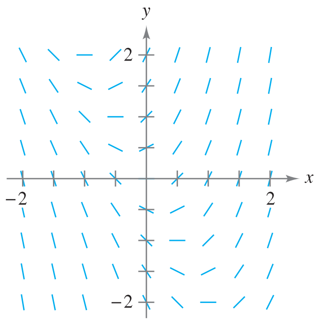
$$y' = 2x + y.$$

Use the slope field to sketch the solution that passes through the point $(1, 1)$.

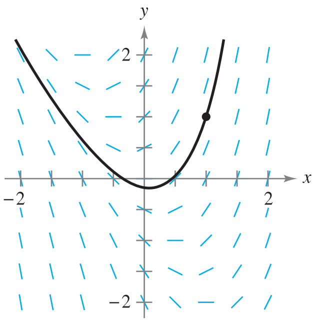
Solution Make a table showing the slopes at several points. The table shown is a small sample. The slopes at many other points should be calculated to get a representative slope field.

x	-2	-2	-1	-1	0	0	1	1	2	2
y	-1	1	-1	1	-1	1	-1	1	-1	1
$y' = 2x + y$	-5	-3	-3	-1	-1	1	1	3	3	5

Next draw line segments at the points with their respective slopes, as shown in Figure 6.4.



Slope field for $y' = 2x + y$
Figure 6.4



Particular solution for $y' = 2x + y$ passing through $(1, 1)$
Figure 6.5

After the slope field is drawn, start at the initial point $(1, 1)$ and move to the right in the direction of the line segment. Continue to draw the solution curve so that it moves parallel to the nearby line segments. Do the same to the left of $(1, 1)$. The resulting solution is shown in Figure 6.5. ■

In Example 5, note that the slope field shows that y' increases to infinity as x increases.

NOTE Drawing a slope field by hand is tedious. In practice, slope fields are usually drawn using a graphing utility. ■

Euler's Method

Euler's Method is a numerical approach to approximating the particular solution of the differential equation

$$y' = F(x, y)$$

that passes through the point (x_0, y_0). From the given information, you know that the graph of the solution passes through the point (x_0, y_0) and has a slope of $F(x_0, y_0)$ at this point. This gives you a "starting point" for approximating the solution.

From this starting point, you can proceed in the direction indicated by the slope. Using a small step h, move along the tangent line until you arrive at the point (x_1, y_1), where

$$x_1 = x_0 + h \quad \text{and} \quad y_1 = y_0 + hF(x_0, y_0)$$

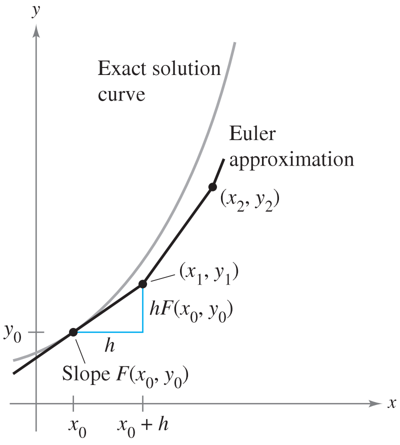


Figure 6.6

as shown in Figure 6.6. If you think of (x_1, y_1) as a new starting point, you can repeat the process to obtain a second point (x_2, y_2). The values of x_i and y_i are as follows.

$$\begin{array}{ll} x_1 = x_0 + h & y_1 = y_0 + hF(x_0, y_0) \\ x_2 = x_1 + h & y_2 = y_1 + hF(x_1, y_1) \\ \vdots & \vdots \\ x_n = x_{n-1} + h & y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \end{array}$$

NOTE You can obtain better approximations of the exact solution by choosing smaller and smaller step sizes.

EXAMPLE 6 Approximating a Solution Using Euler's Method

Use Euler's Method to approximate the particular solution of the differential equation

$$y' = x - y$$

passing through the point (0, 1). Use a step of $h = 0.1$.

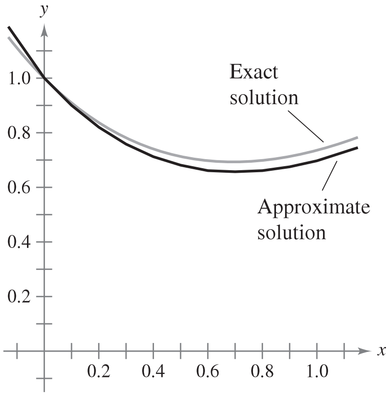


Figure 6.7

Solution Using $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x - y$, you have $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3, \ldots$, and

$$y_1 = y_0 + hF(x_0, y_0) = 1 + (0.1)(0 - 1) = 0.9$$
$$y_2 = y_1 + hF(x_1, y_1) = 0.9 + (0.1)(0.1 - 0.9) = 0.82$$
$$y_3 = y_2 + hF(x_2, y_2) = 0.82 + (0.1)(0.2 - 0.82) = 0.758.$$

The first ten approximations are shown in the table. You can plot these values to see a graph of the approximate solution, as shown in Figure 6.7.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	1	0.900	0.820	0.758	0.712	0.681	0.663	0.657	0.661	0.675	0.697

NOTE For the differential equation in Example 6, you can verify the exact solution to be $y = x - 1 + 2e^{-x}$. Figure 6.7 compares this exact solution with the approximate solution obtained in Example 6.

6.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, verify the solution of the differential equation.

	Solution	*Differential Equation*
1.	$y = Ce^{4x}$	$y' = 4y$
2.	$y = e^{-2x}$	$3y' + 5y = -e^{-2x}$
3.	$x^2 + y^2 = Cy$	$y' = 2xy/(x^2 - y^2)$
4.	$y^2 - 2\ln y = x^2$	$\dfrac{dy}{dx} = \dfrac{xy}{y^2 - 1}$
5.	$y = C_1 \sin x - C_2 \cos x$	$y'' + y = 0$
6.	$y = C_1e^{-x}\cos x + C_2e^{-x}\sin x$	$y'' + 2y' + 2y = 0$
7.	$y = -\cos x \ln\lvert\sec x + \tan x\rvert$	$y'' + y = \tan x$
8.	$y = \frac{2}{5}(e^{-4x} + e^x)$	$y'' + 4y' = 2e^x$

In Exercises 9–12, verify the particular solution of the differential equation.

	Solution	*Differential Equation and Initial Condition*
9.	$y = \sin x \cos x - \cos^2 x$	$2y + y' = 2\sin(2x) - 1$ $y\left(\dfrac{\pi}{4}\right) = 0$
10.	$y = \frac{1}{2}x^2 - 2\cos x - 3$	$y' = x + 2\sin x$ $y(0) = -5$
11.	$y = 4e^{-6x^2}$	$y' = -12xy$ $y(0) = 4$
12.	$y = e^{-\cos x}$	$y' = y \sin x$ $y\left(\dfrac{\pi}{2}\right) = 1$

In Exercises 13–20, determine whether the function is a solution of the differential equation $y^{(4)} - 16y = 0$.

13. $y = 3\cos x$

14. $y = 2\sin x$

15. $y = 3\cos 2x$

16. $y = 3\sin 2x$

17. $y = e^{-2x}$

18. $y = 5\ln x$

19. $y = C_1e^{2x} + C_2e^{-2x} + C_3\sin 2x + C_4\cos 2x$

20. $y = 3e^{2x} - 4\sin 2x$

In Exercises 21–28, determine whether the function is a solution of the differential equation $xy' - 2y = x^3e^x$.

21. $y = x^2$

22. $y = x^3$

23. $y = x^2e^x$

24. $y = x^2(2 + e^x)$

25. $y = \sin x$

26. $y = \cos x$

27. $y = \ln x$

28. $y = x^2e^x - 5x^2$

In Exercises 29–32, some of the curves corresponding to different values of C in the general solution of the differential equation are given. Find the particular solution that passes through the point shown on the graph.

	Solution	*Differential Equation*
29.	$y = Ce^{-x/2}$	$2y' + y = 0$
30.	$y(x^2 + y) = C$	$2xy + (x^2 + 2y)y' = 0$
31.	$y^2 = Cx^3$	$2xy' - 3y = 0$
32.	$2x^2 - y^2 = C$	$yy' - 2x = 0$

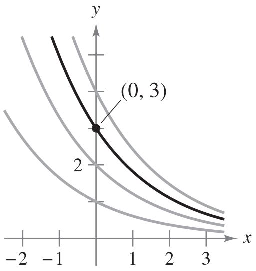


Figure for 29

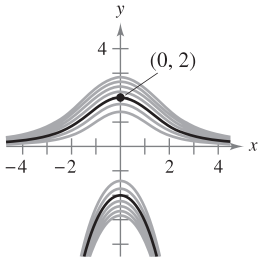


Figure for 30

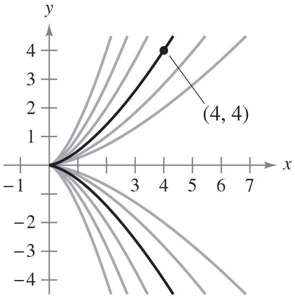


Figure for 31

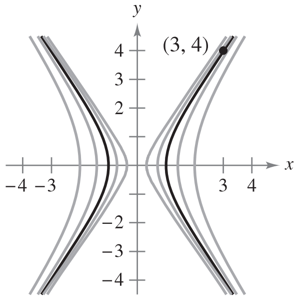


Figure for 32

In Exercises 33 and 34, the general solution of the differential equation is given. Use a graphing utility to graph the particular solutions for the given values of C.

33. $4yy' - x = 0$
$4y^2 - x^2 = C$
$C = 0, C = \pm 1, C = \pm 4$

34. $yy' + x = 0$
$x^2 + y^2 = C$
$C = 0, C = 1, C = 4$

In Exercises 35–40, verify that the general solution satisfies the differential equation. Then find the particular solution that satisfies the initial condition.

35. $y = Ce^{-2x}$
$y' + 2y = 0$
$y = 3$ when $x = 0$

36. $3x^2 + 2y^2 = C$
$3x + 2yy' = 0$
$y = 3$ when $x = 1$

37. $y = C_1\sin 3x + C_2\cos 3x$
$y'' + 9y = 0$
$y = 2$ when $x = \pi/6$
$y' = 1$ when $x = \pi/6$

38. $y = C_1 + C_2\ln x$
$xy'' + y' = 0$
$y = 0$ when $x = 2$
$y' = \frac{1}{2}$ when $x = 2$

39. $y = C_1x + C_2x^3$

$x^2y'' - 3xy' + 3y = 0$

$y = 0$ when $x = 2$

$y' = 4$ when $x = 2$

40. $y = e^{2x/3}(C_1 + C_2x)$

$9y'' - 12y' + 4y = 0$

$y = 4$ when $x = 0$

$y = 0$ when $x = 3$

In Exercises 41–52, use integration to find a general solution of the differential equation.

41. $\dfrac{dy}{dx} = 6x^2$

42. $\dfrac{dy}{dx} = 2x^3 - 3x$

43. $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2}$

44. $\dfrac{dy}{dx} = \dfrac{e^x}{4 + e^x}$

45. $\dfrac{dy}{dx} = \dfrac{x - 2}{x}$

46. $\dfrac{dy}{dx} = x \cos x^2$

47. $\dfrac{dy}{dx} = \sin 2x$

48. $\dfrac{dy}{dx} = \tan^2 x$

49. $\dfrac{dy}{dx} = x\sqrt{x - 6}$

50. $\dfrac{dy}{dx} = 2x\sqrt{3 - x}$

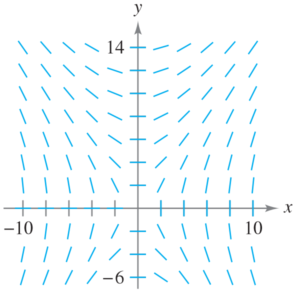
51. $\dfrac{dy}{dx} = xe^{x^2}$

52. $\dfrac{dy}{dx} = 5e^{-x/2}$

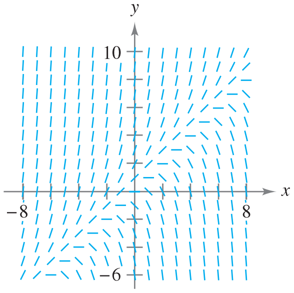
Slope Fields **In Exercises 53–56, a differential equation and its slope field are given. Complete the table by determining the slopes (if possible) in the slope field at the given points.**

x	−4	−2	0	2	4	8
y	2	0	4	4	6	8
dy/dx						

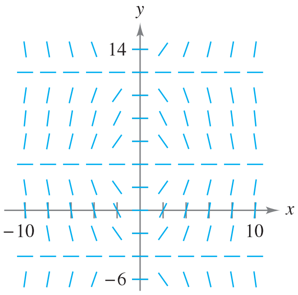
53. $\dfrac{dy}{dx} = \dfrac{2x}{y}$



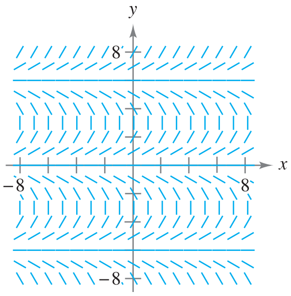
54. $\dfrac{dy}{dx} = y - x$



55. $\dfrac{dy}{dx} = x \cos \dfrac{\pi y}{8}$

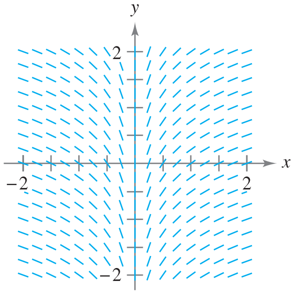


56. $\dfrac{dy}{dx} = \tan\left(\dfrac{\pi y}{6}\right)$

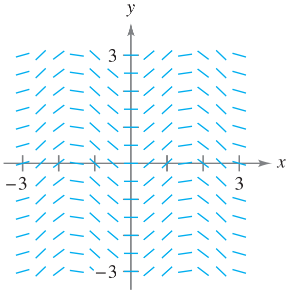


In Exercises 57–60, match the differential equation with its slope field. [The slope fields are labeled (a), (b), (c), and (d).]

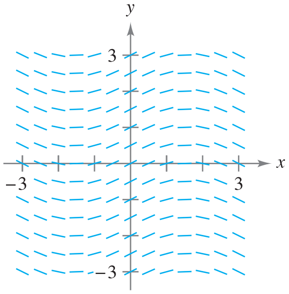
(a)



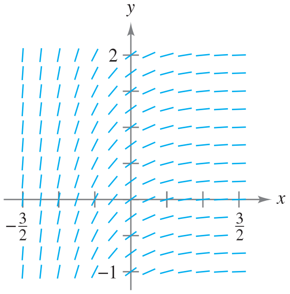
(b)



(c)



(d)



57. $\dfrac{dy}{dx} = \sin(2x)$

58. $\dfrac{dy}{dx} = \dfrac{1}{2}\cos x$

59. $\dfrac{dy}{dx} = e^{-2x}$

60. $\dfrac{dy}{dx} = \dfrac{1}{x}$

Slope Fields **In Exercises 61–64, (a) sketch the slope field for the differential equation, (b) use the slope field to sketch the solution that passes through the given point, and (c) discuss the graph of the solution as $x \to \infty$ and $x \to -\infty$. Use a graphing utility to verify your results.**

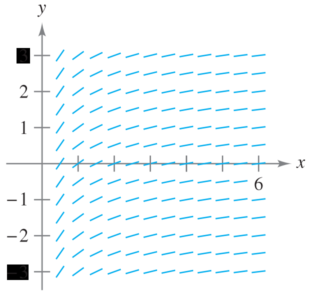
61. $y' = 3 - x, \quad (4, 2)$

62. $y' = \frac{1}{3}x^2 - \frac{1}{2}x, \quad (1, 1)$

63. $y' = y - 4x, \quad (2, 2)$

64. $y' = y + xy, \quad (0, -4)$

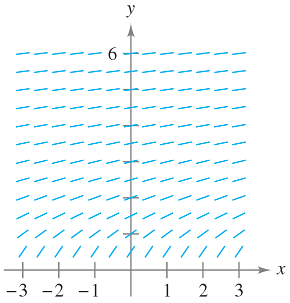
65. ***Slope Field*** Use the slope field for the differential equation $y' = 1/x$, where $x > 0$, to sketch the graph of the solution that satisfies each given initial condition. Then make a conjecture about the behavior of a particular solution of $y' = 1/x$ as $x \to \infty$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) $(1, 0)$

(b) $(2, -1)$

66. ***Slope Field*** Use the slope field for the differential equation $y' = 1/y$, where $y > 0$, to sketch the graph of the solution that satisfies each initial condition. Then make a conjecture about the behavior of a particular solution of $y' = 1/y$ as $x \to \infty$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) $(0, 1)$ (b) $(1, 1)$

CAS ***Slope Fields*** **In Exercises 67–72, use a computer algebra system to (a) graph the slope field for the differential equation and (b) graph the solution satisfying the specified initial condition.**

67. $\dfrac{dy}{dx} = 0.25y, \quad y(0) = 4$

68. $\dfrac{dy}{dx} = 4 - y, \quad y(0) = 6$

69. $\dfrac{dy}{dx} = 0.02y(10 - y), \quad y(0) = 2$

70. $\dfrac{dy}{dx} = 0.2x(2 - y), \quad y(0) = 9$

71. $\dfrac{dy}{dx} = 0.4y(3 - x), \quad y(0) = 1$

72. $\dfrac{dy}{dx} = \dfrac{1}{2}e^{-x/8}\sin\dfrac{\pi y}{4}, \quad y(0) = 2$

Euler's Method **In Exercises 73–78, use Euler's Method to make a table of values for the approximate solution of the differential equation with the specified initial value. Use *n* steps of size *h*.**

73. $y' = x + y, \quad y(0) = 2, \quad n = 10, \quad h = 0.1$

74. $y' = x + y, \quad y(0) = 2, \quad n = 20, \quad h = 0.05$

75. $y' = 3x - 2y, \quad y(0) = 3, \quad n = 10, \quad h = 0.05$

76. $y' = 0.5x(3 - y), \quad y(0) = 1, \quad n = 5, \quad h = 0.4$

77. $y' = e^{xy}, \quad y(0) = 1, \quad n = 10, \quad h = 0.1$

78. $y' = \cos x + \sin y, \quad y(0) = 5, \quad n = 10, \quad h = 0.1$

In Exercises 79–81, complete the table using the exact solution of the differential equation and two approximations obtained using Euler's Method to approximate the particular solution of the differential equation. Use $h = 0.2$ and $h = 0.1$ and compute each approximation to four decimal places.

x	0	0.2	0.4	0.6	0.8	1
$y(x)$ (exact)						
$y(x)$ ($h = 0.2$)						
$y(x)$ ($h = 0.1$)						

Table for 79–81

	Differential Equation	*Initial Condition*	*Exact Solution*
79.	$\dfrac{dy}{dx} = y$	$(0, 3)$	$y = 3e^x$
80.	$\dfrac{dy}{dx} = \dfrac{2x}{y}$	$(0, 2)$	$y = \sqrt{2x^2 + 4}$
81.	$\dfrac{dy}{dx} = y + \cos(x)$	$(0, 0)$	$y = \frac{1}{2}(\sin x - \cos x + e^x)$

82. Compare the values of the approximations in Exercises 79–81 with the values given by the exact solution. How does the error change as h increases?

83. ***Temperature*** At time $t = 0$ minutes, the temperature of an object is 140°F. The temperature of the object is changing at the rate given by the differential equation

$$\frac{dy}{dt} = -\frac{1}{2}(y - 72).$$

(a) Use a graphing utility and Euler's Method to approximate the particular solutions of this differential equation at $t = 1$, 2, and 3. Use a step size of $h = 0.1$. (A graphing utility program for Euler's Method is available at the website *college.hmco.com*.)

(b) Compare your results with the exact solution

$$y = 72 + 68e^{-t/2}.$$

(c) Repeat parts (a) and (b) using a step size of $h = 0.05$. Compare the results.

CAPSTONE

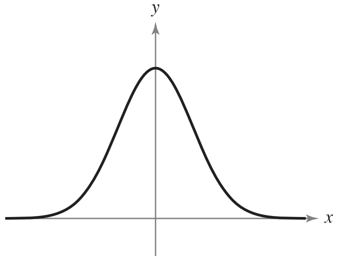
84. The graph shows a solution of one of the following differential equations. Determine the correct equation. Explain your reasoning.

(a) $y' = xy$

(b) $y' = \dfrac{4x}{y}$

(c) $y' = -4xy$

(d) $y' = 4 - xy$



WRITING ABOUT CONCEPTS

85. In your own words, describe the difference between a general solution of a differential equation and a particular solution.

86. Explain how to interpret a slope field.

87. Describe how to use Euler's Method to approximate a particular solution of a differential equation.

88. It is known that $y = Ce^{kx}$ is a solution of the differential equation $y' = 0.07y$. Is it possible to determine C or k from the information given? If so, find its value.

True or False? **In Exercises 89–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

89. If $y = f(x)$ is a solution of a first-order differential equation, then $y = f(x) + C$ is also a solution.

90. The general solution of a differential equation is $y = -4.9x^2 + C_1x + C_2$. To find a particular solution, you must be given two initial conditions.

91. Slope fields represent the general solutions of differential equations.

92. A slope field shows that the slope at the point $(1, 1)$ is 6. This slope field represents the family of solutions for the differential equation $y' = 4x + 2y$.

93. ***Errors and Euler's Method*** The exact solution of the differential equation

$$\frac{dy}{dx} = -2y$$

where $y(0) = 4$, is $y = 4e^{-2x}$.

(a) Use a graphing utility to complete the table, where y is the exact value of the solution, y_1 is the approximate solution using Euler's Method with $h = 0.1$, y_2 is the approximate solution using Euler's Method with $h = 0.2$, e_1 is the absolute error $|y - y_1|$, e_2 is the absolute error $|y - y_2|$, and r is the ratio e_1/e_2.

x	0	0.2	0.4	0.6	0.8	1
y						
y_1						
y_2						
e_1						
e_2						
r						

(b) What can you conclude about the ratio r as h changes?

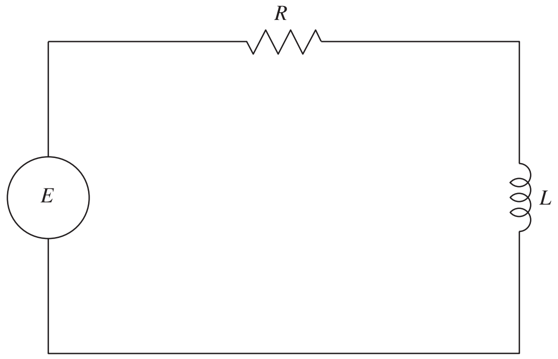
(c) Predict the absolute error when $h = 0.05$.

94. ***Errors and Euler's Method*** Repeat Exercise 93 for which the exact solution of the differential equation

$$\frac{dy}{dx} = x - y$$

where $y(0) = 1$, is $y = x - 1 + 2e^{-x}$.

95. ***Electric Circuits*** The diagram shows a simple electric circuit consisting of a power source, a resistor, and an inductor.



A model of the current I, in amperes (A), at time t is given by the first-order differential equation

$$L\frac{dI}{dt} + RI = E(t)$$

where $E(t)$ is the voltage (V) produced by the power source, R is the resistance, in ohms (Ω), and L is the inductance, in henrys (H). Suppose the electric circuit consists of a 24-V power source, a 12-Ω resistor, and a 4-H inductor.

(a) Sketch a slope field for the differential equation.

(b) What is the limiting value of the current? Explain.

96. ***Think About It*** It is known that $y = e^{kt}$ is a solution of the differential equation $y'' - 16y = 0$. Find the values of k.

97. ***Think About It*** It is known that $y = A \sin \omega t$ is a solution of the differential equation $y'' + 16y = 0$. Find the values of ω.

PUTNAM EXAM CHALLENGE

98. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x)$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

99. Prove that if the family of integral curves of the differential equation

$$\frac{dy}{dx} + p(x)y = q(x), \qquad p(x) \cdot q(x) \neq 0$$

is cut by the line $x = k$, the tangents at the points of intersection are concurrent.

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6.2 Differential Equations: Growth and Decay

- **Use separation of variables to solve a simple differential equation.**
- **Use exponential functions to model growth and decay in applied problems.**

Differential Equations

In the preceding section, you learned to analyze visually the solutions of differential equations using slope fields and to approximate solutions numerically using Euler's Method. Analytically, you have learned to solve only two types of differential equations—those of the forms $y' = f(x)$ and $y'' = f(x)$. In this section, you will learn how to solve a more general type of differential equation. The strategy is to rewrite the equation so that each variable occurs on only one side of the equation. This strategy is called *separation of variables*. (You will study this strategy in detail in Section 6.3.)

EXAMPLE 1 Solving a Differential Equation

$$
\begin{aligned}
y' &= \frac{2x}{y} && \text{Original equation}\\
yy' &= 2x && \text{Multiply both sides by } y.\\
\int yy'\,dx &= \int 2x\,dx && \text{Integrate with respect to } x.\\
\int y\,dy &= \int 2x\,dx && dy = y'\,dx\\
\frac{1}{2}y^2 &= x^2 + C_1 && \text{Apply Power Rule.}\\
y^2 - 2x^2 &= C && \text{Rewrite, letting } C = 2C_1.
\end{aligned}
$$

So, the general solution is given by $y^2 - 2x^2 = C$. ■

STUDY TIP You can use implicit differentiation to check the solution in Example 1.

EXPLORATION

In Example 1, the general solution of the differential equation is

$$y^2 - 2x^2 = C.$$

Use a graphing utility to sketch the particular solutions for $C = \pm 2$, $C = \pm 1$, and $C = 0$. Describe the solutions graphically. Is the following statement true of each solution?

The slope of the graph at the point (x, y) is equal to twice the ratio of x and y.

Explain your reasoning. Are all curves for which this statement is true represented by the general solution?

Notice that when you integrate both sides of the equation in Example 1, you don't need to add a constant of integration to both sides. If you did, you would obtain the same result.

$$
\begin{aligned}
\int y\,dy &= \int 2x\,dx\\
\tfrac{1}{2}y^2 + C_2 &= x^2 + C_3\\
\tfrac{1}{2}y^2 &= x^2 + (C_3 - C_2)\\
\tfrac{1}{2}y^2 &= x^2 + C_1
\end{aligned}
$$

Some people prefer to use Leibniz notation and differentials when applying separation of variables. The solution of Example 1 is shown below using this notation.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{2x}{y}\\
y\,dy &= 2x\,dx\\
\int y\,dy &= \int 2x\,dx\\
\frac{1}{2}y^2 &= x^2 + C_1\\
y^2 - 2x^2 &= C
\end{aligned}
$$

Growth and Decay Models

In many applications, the rate of change of a variable y is proportional to the value of y. If y is a function of time t, the proportion can be written as follows.

$$\overset{\text{Rate of change of } y}{\frac{dy}{dt}} \overset{\text{is}}{=} \overset{\text{proportional to } y.}{ky}$$

The general solution of this differential equation is given in the following theorem.

THEOREM 6.1 EXPONENTIAL GROWTH AND DECAY MODEL

If y is a differentiable function of t such that $y > 0$ and $y' = ky$ for some constant k, then

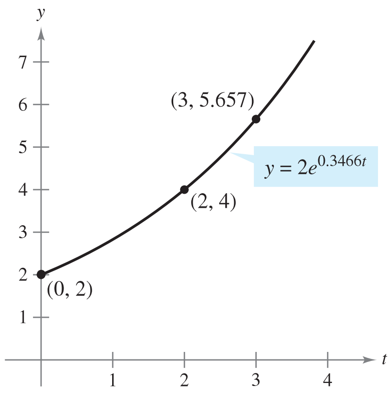
$$y = Ce^{kt}.$$

C is the **initial value** of y, and k is the **proportionality constant. Exponential growth** occurs when $k > 0$, and **exponential decay** occurs when $k < 0$.

PROOF

$$\begin{aligned} y' &= ky && \text{Write original equation.} \\ \frac{y'}{y} &= k && \text{Separate variables.} \\ \int \frac{y'}{y}\,dt &= \int k\,dt && \text{Integrate with respect to } t. \\ \int \frac{1}{y}\,dy &= \int k\,dt && dy = y'\,dt \\ \ln y &= kt + C_1 && \text{Find antiderivative of each side.} \\ y &= e^{kt}e^{C_1} && \text{Solve for } y. \\ y &= Ce^{kt} && \text{Let } C = e^{C_1}. \end{aligned}$$

So, all solutions of $y' = ky$ are of the form $y = Ce^{kt}$. Remember that you can differentiate the function $y = Ce^{kt}$ with respect to t to verify that $y' = ky$. ■



If the rate of change of y is proportional to y, then y follows an exponential model.
Figure 6.8

EXAMPLE 2 Using an Exponential Growth Model

The rate of change of y is proportional to y. When $t = 0$, $y = 2$, and when $t = 2$, $y = 4$. What is the value of y when $t = 3$?

Solution Because $y' = ky$, you know that y and t are related by the equation $y = Ce^{kt}$. You can find the values of the constants C and k by applying the initial conditions.

$$\begin{aligned} 2 &= Ce^{0} \quad \Rightarrow \quad C = 2 && \text{When } t = 0,\ y = 2. \\ 4 &= 2e^{2k} \quad \Rightarrow \quad k = \frac{1}{2}\ln 2 \approx 0.3466 && \text{When } t = 2,\ y = 4. \end{aligned}$$

So, the model is $y \approx 2e^{0.3466t}$. When $t = 3$, the value of y is $2e^{0.3466(3)} \approx 5.657$ (see Figure 6.8). ■

STUDY TIP Using logarithmic properties, note that the value of k in Example 2 can also be written as $\ln(\sqrt{2})$. So, the model becomes $y = 2e^{(\ln\sqrt{2})t}$, which can then be rewritten as $y = 2(\sqrt{2})^t$.

TECHNOLOGY Most graphing utilities have curve-fitting capabilities that can be used to find models that represent data. Use the *exponential regression* feature of a graphing utility and the information in Example 2 to find a model for the data. How does your model compare with the given model?

Radioactive decay is measured in terms of *half-life*—the number of years required for half of the atoms in a sample of radioactive material to decay. The rate of decay is proportional to the amount present. The half-lives of some common radioactive isotopes are shown below.

Uranium (^{238}U)	4,470,000,000 years
Plutonium (^{239}Pu)	24,100 years
Carbon (^{14}C)	5715 years
Radium (^{226}Ra)	1599 years
Einsteinium (^{254}Es)	276 days
Nobelium (^{257}No)	25 seconds

EXAMPLE 3 Radioactive Decay

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Suppose that 10 grams of the plutonium isotope ^{239}Pu was released in the Chernobyl nuclear accident. How long will it take for the 10 grams to decay to 1 gram?

Solution Let y represent the mass (in grams) of the plutonium. Because the rate of decay is proportional to y, you know that

$$y = Ce^{kt}$$

where t is the time in years. To find the values of the constants C and k, apply the initial conditions. Using the fact that $y = 10$ when $t = 0$, you can write

$$10 = Ce^{k(0)} = Ce^0$$

which implies that $C = 10$. Next, using the fact that the half-life of ^{239}Pu is 24,100 years, you have $y = 10/2 = 5$ when $t = 24{,}100$, so you can write

$$5 = 10e^{k(24{,}100)}$$

$$\frac{1}{2} = e^{24{,}100k}$$

$$\frac{1}{24{,}100}\ln\frac{1}{2} = k$$

$$-0.000028761 \approx k.$$

So, the model is

$$y = 10e^{-0.000028761t}. \qquad \text{Half-life model}$$

To find the time it would take for 10 grams to decay to 1 gram, you can solve for t in the equation

$$1 = 10e^{-0.000028761t}.$$

The solution is approximately 80,059 years. ■

NOTE The exponential decay model in Example 3 could also be written as $y = 10\left(\frac{1}{2}\right)^{t/24{,}100}$. This model is much easier to derive, but for some applications it is not as convenient to use.

From Example 3, notice that in an exponential growth or decay problem, it is easy to solve for C when you are given the value of y at $t = 0$. The next example demonstrates a procedure for solving for C and k when you do not know the value of y at $t = 0$.

EXAMPLE 4 Population Growth

Suppose an experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

Solution Let $y = Ce^{kt}$ be the number of flies at time t, where t is measured in days. Note that y is continuous whereas the number of flies is discrete. Because $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$, you can write

$$100 = Ce^{2k} \quad \text{and} \quad 300 = Ce^{4k}.$$

From the first equation, you know that $C = 100e^{-2k}$. Substituting this value into the second equation produces the following.

$$\begin{aligned} 300 &= 100e^{-2k}e^{4k} \\ 300 &= 100e^{2k} \\ \ln 3 &= 2k \\ \frac{1}{2}\ln 3 &= k \\ 0.5493 &\approx k \end{aligned}$$

So, the exponential growth model is

$$y = Ce^{0.5493t}.$$

To solve for C, reapply the condition $y = 100$ when $t = 2$ and obtain

$$\begin{aligned} 100 &= Ce^{0.5493(2)} \\ C &= 100e^{-1.0986} \approx 33. \end{aligned}$$

So, the original population (when $t = 0$) consisted of approximately $y = C = 33$ flies, as shown in Figure 6.9.

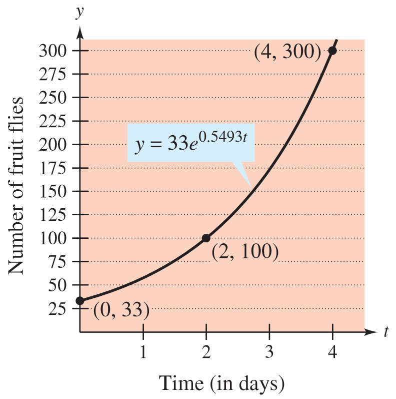


Figure 6.9

EXAMPLE 5 Declining Sales

Four months after it stops advertising, a manufacturing company notices that its sales have dropped from 100,000 units per month to 80,000 units per month. If the sales follow an exponential pattern of decline, what will they be after another 2 months?

Solution Use the exponential decay model $y = Ce^{kt}$, where t is measured in months. From the initial condition $(t = 0)$, you know that $C = 100{,}000$. Moreover, because $y = 80{,}000$ when $t = 4$, you have

$$\begin{aligned} 80{,}000 &= 100{,}000e^{4k} \\ 0.8 &= e^{4k} \\ \ln(0.8) &= 4k \\ -0.0558 &\approx k. \end{aligned}$$

So, after 2 more months $(t = 6)$, you can expect the monthly sales rate to be

$$\begin{aligned} y &\approx 100{,}000e^{-0.0558(6)} \\ &\approx 71{,}500 \text{ units.} \end{aligned}$$

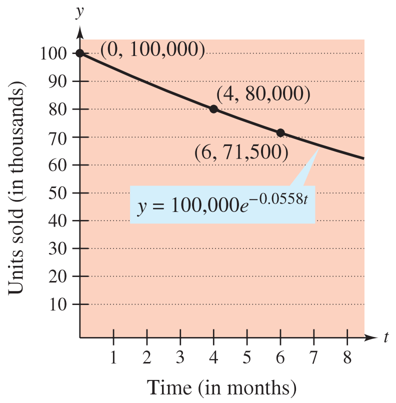


Figure 6.10

See Figure 6.10. ■

In Examples 2 through 5, you did not actually have to solve the differential equation

$$y' = ky.$$

(This was done once in the proof of Theorem 6.1.) The next example demonstrates a problem whose solution involves the separation of variables technique. The example concerns **Newton's Law of Cooling,** which states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium.

EXAMPLE 6 Newton's Law of Cooling

Let y represent the temperature (in °F) of an object in a room whose temperature is kept at a constant 60°. If the object cools from 100° to 90° in 10 minutes, how much longer will it take for its temperature to decrease to 80°?

Solution From Newton's Law of Cooling, you know that the rate of change in y is proportional to the difference between y and 60. This can be written as

$$y' = k(y - 60), \quad 80 \le y \le 100.$$

To solve this differential equation, use separation of variables, as follows.

$$\frac{dy}{dt} = k(y - 60) \qquad \text{Differential equation}$$

$$\left(\frac{1}{y - 60}\right) dy = k\,dt \qquad \text{Separate variables.}$$

$$\int \frac{1}{y - 60}\,dy = \int k\,dt \qquad \text{Integrate each side.}$$

$$\ln|y - 60| = kt + C_1 \qquad \text{Find antiderivative of each side.}$$

Because $y > 60$, $|y - 60| = y - 60$, and you can omit the absolute value signs. Using exponential notation, you have

$$y - 60 = e^{kt + C_1} \quad \Rightarrow \quad y = 60 + Ce^{kt}. \qquad C = e^{C_1}$$

Using $y = 100$ when $t = 0$, you obtain $100 = 60 + Ce^{k(0)} = 60 + C$, which implies that $C = 40$. Because $y = 90$ when $t = 10$,

$$90 = 60 + 40e^{k(10)}$$
$$30 = 40e^{10k}$$
$$k = \tfrac{1}{10}\ln\tfrac{3}{4} \approx -0.02877.$$

So, the model is

$$y = 60 + 40e^{-0.02877t} \qquad \text{Cooling model}$$

and finally, when $y = 80$, you obtain

$$80 = 60 + 40e^{-0.02877t}$$
$$20 = 40e^{-0.02877t}$$
$$\tfrac{1}{2} = e^{-0.02877t}$$
$$\ln\tfrac{1}{2} = -0.02877t$$
$$t \approx 24.09 \text{ minutes.}$$

So, it will require about 14.09 *more* minutes for the object to cool to a temperature of 80° (see Figure 6.11). ■

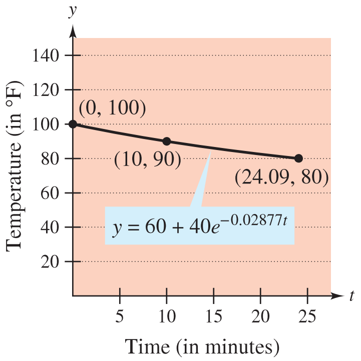


Figure 6.11

6.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, solve the differential equation.

1. $\dfrac{dy}{dx} = x + 3$

2. $\dfrac{dy}{dx} = 6 - x$

3. $\dfrac{dy}{dx} = y + 3$

4. $\dfrac{dy}{dx} = 6 - y$

5. $y' = \dfrac{5x}{y}$

6. $y' = \dfrac{\sqrt{x}}{7y}$

7. $y' = \sqrt{x}\,y$

8. $y' = x(1 + y)$

9. $(1 + x^2)y' - 2xy = 0$

10. $xy + y' = 100x$

In Exercises 11–14, write and solve the differential equation that models the verbal statement.

11. The rate of change of Q with respect to t is inversely proportional to the square of t.

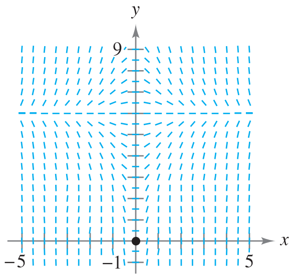
12. The rate of change of P with respect to t is proportional to $25 - t$.

13. The rate of change of N with respect to s is proportional to $500 - s$.

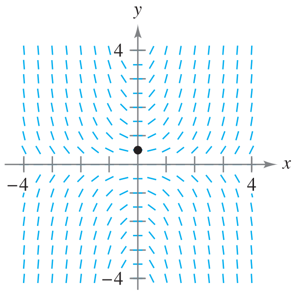
14. The rate of change of y with respect to x varies jointly as x and $L - y$.

Slope Fields **In Exercises 15 and 16, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

15. $\dfrac{dy}{dx} = x(6 - y), \quad (0, 0)$



16. $\dfrac{dy}{dx} = xy, \quad \left(0, \frac{1}{2}\right)$



In Exercises 17–20, find the function $y = f(t)$ passing through the point $(0, 10)$ with the given first derivative. Use a graphing utility to graph the solution.

17. $\dfrac{dy}{dt} = \dfrac{1}{2}t$

18. $\dfrac{dy}{dt} = -\dfrac{3}{4}\sqrt{t}$

19. $\dfrac{dy}{dt} = -\dfrac{1}{2}y$

20. $\dfrac{dy}{dt} = \dfrac{3}{4}y$

In Exercises 21–24, write and solve the differential equation that models the verbal statement. Evaluate the solution at the specified value of the independent variable.

21. The rate of change of y is proportional to y. When $x = 0$, $y = 6$, and when $x = 4$, $y = 15$. What is the value of y when $x = 8$?

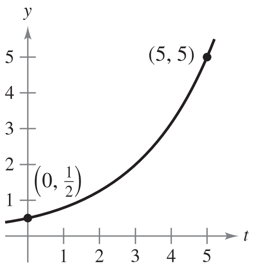
22. The rate of change of N is proportional to N. When $t = 0$, $N = 250$, and when $t = 1$, $N = 400$. What is the value of N when $t = 4$?

23. The rate of change of V is proportional to V. When $t = 0$, $V = 20{,}000$, and when $t = 4$, $V = 12{,}500$. What is the value of V when $t = 6$?

24. The rate of change of P is proportional to P. When $t = 0$, $P = 5000$, and when $t = 1$, $P = 4750$. What is the value of P when $t = 5$?

In Exercises 25–28, find the exponential function $y = Ce^{kt}$ that passes through the two given points.

25.



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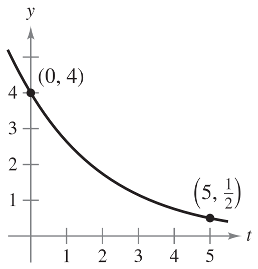

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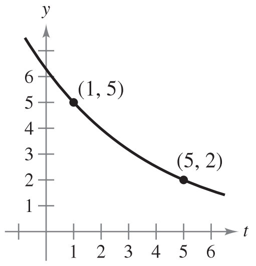

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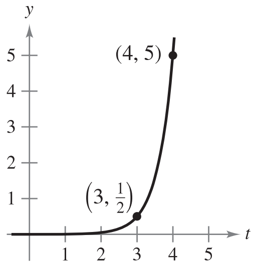


WRITING ABOUT CONCEPTS

29. Describe what the values of C and k represent in the exponential growth and decay model, $y = Ce^{kt}$.

30. Give the differential equation that models exponential growth and decay.

In Exercises 31 and 32, determine the quadrants in which the solution of the differential equation is an increasing function. Explain. (Do not solve the differential equation.)

31. $\dfrac{dy}{dx} = \dfrac{1}{2}xy$

32. $\dfrac{dy}{dx} = \dfrac{1}{2}x^2y$

Radioactive Decay **In Exercises 33–40, complete the table for the radioactive isotope.**

	Isotope	*Half-Life (in years)*	*Initial Quantity*	*Amount After 1000 Years*	*Amount After 10,000 Years*
33.	^{226}Ra	1599	20 g		
34.	^{226}Ra	1599		1.5 g	
35.	^{226}Ra	1599			0.1 g
36.	^{14}C	5715			3 g
37.	^{14}C	5715	5 g		
38.	^{14}C	5715		1.6 g	
39.	^{239}Pu	24,100		2.1 g	
40.	^{239}Pu	24,100			0.4 g

41. ***Radioactive Decay*** Radioactive radium has a half-life of approximately 1599 years. What percent of a given amount remains after 100 years?

42. ***Carbon Dating*** Carbon-14 dating assumes that the carbon dioxide on Earth today has the same radioactive content as it did centuries ago. If this is true, the amount of ^{14}C absorbed by a tree that grew several centuries ago should be the same as the amount of ^{14}C absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much of the radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal? (The half-life of ^{14}C is 5715 years.)

Compound Interest **In Exercises 43–48, complete the table for a savings account in which interest is compounded continuously.**

	Initial Investment	*Annual Rate*	*Time to Double*	*Amount After 10 Years*
43.	\$4000	6%		
44.	\$18,000	$5\frac{1}{2}\%$		
45.	\$750		$7\frac{3}{4}$ yr	
46.	\$12,500		5 yr	
47.	\$500			\$1292.85
48.	\$2000			\$5436.56

Compound Interest **In Exercises 49–52, find the principal P that must be invested at rate r, compounded monthly, so that \$1,000,000 will be available for retirement in t years.**

49. $r = 7\frac{1}{2}\%, \quad t = 20$

50. $r = 6\%, \quad t = 40$

51. $r = 8\%, \quad t = 35$

52. $r = 9\%, \quad t = 25$

Compound Interest **In Exercises 53–56, find the time necessary for \$1000 to double if it is invested at a rate of r compounded (a) annually, (b) monthly, (c) daily, and (d) continuously.**

53. $r = 7\%$

54. $r = 6\%$

55. $r = 8.5\%$

56. $r = 5.5\%$

Population **In Exercises 57–61, the population (in millions) of a country in 2007 and the expected continuous annual rate of change k of the population are given. (*Source: U.S. Census Bureau, International Data Base*)**

(a) Find the exponential growth model $P = Ce^{kt}$ for the population by letting $t = 0$ correspond to 2000.

(b) Use the model to predict the population of the country in 2015.

(c) Discuss the relationship between the sign of k and the change in population for the country.

	Country	*2007 Population*	*k*
57.	Latvia	2.3	−0.006
58.	Egypt	80.3	0.017
59.	Paraguay	6.7	0.024
60.	Hungary	10.0	−0.003
61.	Uganda	30.3	0.036

CAPSTONE

62. (a) Suppose an insect population increases by a constant number each month. Explain why the number of insects can be represented by a linear function.

(b) Suppose an insect population increases by a constant percentage each month. Explain why the number of insects can be represented by an exponential function.

63. ***Modeling Data*** One hundred bacteria are started in a culture and the number N of bacteria is counted each hour for 5 hours. The results are shown in the table, where t is the time in hours.

t	0	1	2	3	4	5
N	100	126	151	198	243	297

(a) Use the regression capabilities of a graphing utility to find an exponential model for the data.

(b) Use the model to estimate the time required for the population to quadruple in size.

64. ***Bacteria Growth*** The number of bacteria in a culture is increasing according to the law of exponential growth. There are 125 bacteria in the culture after 2 hours and 350 bacteria after 4 hours.

(a) Find the initial population.

(b) Write an exponential growth model for the bacteria population. Let t represent time in hours.

(c) Use the model to determine the number of bacteria after 8 hours.

(d) After how many hours will the bacteria count be 25,000?

65. ***Learning Curve*** The management at a certain factory has found that a worker can produce at most 30 units in a day. The learning curve for the number of units N produced per day after a new employee has worked t days is $N = 30(1 - e^{kt})$. After 20 days on the job, a particular worker produces 19 units.

(a) Find the learning curve for this worker.

(b) How many days should pass before this worker is producing 25 units per day?

66. ***Learning Curve*** If the management in Exercise 65 requires a new employee to produce at least 20 units per day after 30 days on the job, find (a) the learning curve that describes this minimum requirement and (b) the number of days before a minimal achiever is producing 25 units per day.

67. ***Modeling Data*** The table shows the populations P (in millions) of the United States from 1960 to 2000. *(Source: U.S. Census Bureau)*

Year	1960	1970	1980	1990	2000
Population, *P*	181	205	228	250	282

(a) Use the 1960 and 1970 data to find an exponential model P_1 for the data. Let $t = 0$ represent 1960.

(b) Use a graphing utility to find an exponential model P_2 for all the data. Let $t = 0$ represent 1960.

(c) Use a graphing utility to plot the data and graph models P_1 and P_2 in the same viewing window. Compare the actual data with the predictions. Which model better fits the data?

(d) Estimate when the population will be 320 million.

68. ***Modeling Data*** The table shows the net receipts and the amounts required to service the national debt (interest on Treasury debt securities) of the United States from 2001 through 2010. The years 2007 through 2010 are estimated, and the monetary amounts are given in billions of dollars. *(Source: U.S. Office of Management and Budget)*

Year	2001	2002	2003	2004	2005
Receipts	1991.4	1853.4	1782.5	1880.3	2153.9
Interest	359.5	332.5	318.1	321.7	352.3

Year	2006	2007	2008	2009	2010
Receipts	2407.3	2540.1	2662.5	2798.3	2954.7
Interest	405.9	433.0	469.9	498.0	523.2

(a) Use the regression capabilities of a graphing utility to find an exponential model R for the receipts and a quartic model I for the amount required to service the debt. Let t represent the time in years, with $t = 1$ corresponding to 2001.

(b) Use a graphing utility to plot the points corresponding to the receipts, and graph the exponential model. Based on the model, what is the continuous rate of growth of the receipts?

(c) Use a graphing utility to plot the points corresponding to the amounts required to service the debt, and graph the quartic model.

(d) Find a function $P(t)$ that approximates the percent of the receipts that is required to service the national debt. Use a graphing utility to graph this function.

69. ***Sound Intensity*** The level of sound β (in decibels) with an intensity of I is $\beta(I) = 10 \log_{10} (I/I_0)$, where I_0 is an intensity of 10^{-16} watt per square centimeter, corresponding roughly to the faintest sound that can be heard. Determine $\beta(I)$ for the following.

(a) $I = 10^{-14}$ watt per square centimeter (whisper)

(b) $I = 10^{-9}$ watt per square centimeter (busy street corner)

(c) $I = 10^{-6.5}$ watt per square centimeter (air hammer)

(d) $I = 10^{-4}$ watt per square centimeter (threshold of pain)

70. ***Noise Level*** With the installation of noise suppression materials, the noise level in an auditorium was reduced from 93 to 80 decibels. Use the function in Exercise 69 to find the percent decrease in the intensity level of the noise as a result of the installation of these materials.

71. ***Forestry*** The value of a tract of timber is $V(t) = 100{,}000e^{0.8\sqrt{t}}$, where t is the time in years, with $t = 0$ corresponding to 2008. If money earns interest continuously at 10%, the present value of the timber at any time t is $A(t) = V(t)e^{-0.10t}$. Find the year in which the timber should be harvested to maximize the present value function.

72. ***Earthquake Intensity*** On the Richter scale, the magnitude R of an earthquake of intensity I is

$$R = \frac{\ln I - \ln I_0}{\ln 10}$$

where I_0 is the minimum intensity used for comparison. Assume that $I_0 = 1$.

(a) Find the intensity of the 1906 San Francisco earthquake $(R = 8.3)$.

(b) Find the factor by which the intensity is increased if the Richter scale measurement is doubled.

(c) Find dR/dI.

73. ***Newton's Law of Cooling*** When an object is removed from a furnace and placed in an environment with a constant temperature of 80°F, its core temperature is 1500°F. One hour after it is removed, the core temperature is 1120°F. Find the core temperature 5 hours after the object is removed from the furnace.

74. ***Newton's Law of Cooling*** A container of hot liquid is placed in a freezer that is kept at a constant temperature of 20°F. The initial temperature of the liquid is 160°F. After 5 minutes, the liquid's temperature is 60°F. How much longer will it take for its temperature to decrease to 30°F?

True or False? **In Exercises 75–78, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

75. In exponential growth, the rate of growth is constant.

76. In linear growth, the rate of growth is constant.

77. If prices are rising at a rate of 0.5% per month, then they are rising at a rate of 6% per year.

78. The differential equation modeling exponential growth is $dy/dx = ky$, where k is a constant.

6.3 Separation of Variables and the Logistic Equation

- **Recognize and solve differential equations that can be solved by separation of variables.**
- **Recognize and solve homogeneous differential equations.**
- **Use differential equations to model and solve applied problems.**
- **Solve and analyze logistic differential equations.**

Separation of Variables

Consider a differential equation that can be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

where M is a continuous function of x alone and N is a continuous function of y alone. As you saw in the preceding section, for this type of equation, all x terms can be collected with dx and all y terms with dy, and a solution can be obtained by integration. Such equations are said to be **separable,** and the solution procedure is called **separation of variables.** Below are some examples of differential equations that are separable.

Original Differential Equation	*Rewritten with Variables Separated*
$x^2 + 3y\dfrac{dy}{dx} = 0$	$3y\,dy = -x^2\,dx$
$(\sin x)y' = \cos x$	$dy = \cot x\,dx$
$\dfrac{xy'}{e^y + 1} = 2$	$\dfrac{1}{e^y + 1}\,dy = \dfrac{2}{x}\,dx$

EXAMPLE 1 Separation of Variables

Find the general solution of $(x^2 + 4)\dfrac{dy}{dx} = xy$.

Solution To begin, note that $y = 0$ is a solution. To find other solutions, assume that $y \neq 0$ and separate variables as shown.

$$(x^2 + 4)\,dy = xy\,dx \qquad \text{Differential form}$$

$$\frac{dy}{y} = \frac{x}{x^2 + 4}\,dx \qquad \text{Separate variables.}$$

Now, integrate to obtain

$$\int \frac{dy}{y} = \int \frac{x}{x^2 + 4}\,dx \qquad \text{Integrate.}$$

$$\ln|y| = \frac{1}{2}\ln(x^2 + 4) + C_1$$

$$\ln|y| = \ln\sqrt{x^2 + 4} + C_1$$

$$|y| = e^{C_1}\sqrt{x^2 + 4}$$

$$y = \pm e^{C_1}\sqrt{x^2 + 4}.$$

Because $y = 0$ is also a solution, you can write the general solution as

$$y = C\sqrt{x^2 + 4}. \qquad \text{General solution } (C = \pm e^{C_1})$$

■

NOTE Be sure to check your solutions throughout this chapter. In Example 1, you can check the solution $y = C\sqrt{x^2 + 4}$ by differentiating and substituting into the original equation.

$$(x^2 + 4)\frac{dy}{dx} = xy$$

$$(x^2 + 4)\frac{Cx}{\sqrt{x^2 + 4}} \overset{?}{=} x\left(C\sqrt{x^2 + 4}\right)$$

$$Cx\sqrt{x^2 + 4} = Cx\sqrt{x^2 + 4}$$

So, the solution checks.

In some cases it is not feasible to write the general solution in the explicit form $y = f(x)$. The next example illustrates such a solution. Implicit differentiation can be used to verify this solution.

■ **FOR FURTHER INFORMATION** For an example (from engineering) of a differential equation that is separable, see the article "Designing a Rose Cutter" by J. S. Hartzler in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

EXAMPLE 2 Finding a Particular Solution

Given the initial condition $y(0) = 1$, find the particular solution of the equation

$$xy\,dx + e^{-x^2}(y^2 - 1)\,dy = 0.$$

Solution Note that $y = 0$ is a solution of the differential equation—but this solution does not satisfy the initial condition. So, you can assume that $y \neq 0$. To separate variables, you must rid the first term of y and the second term of e^{-x^2}. So, you should multiply by e^{x^2}/y and obtain the following.

$$\begin{aligned} xy\,dx + e^{-x^2}(y^2 - 1)\,dy &= 0 \\ e^{-x^2}(y^2 - 1)\,dy &= -xy\,dx \\ \int\left(y - \frac{1}{y}\right)dy &= \int -xe^{x^2}\,dx \\ \frac{y^2}{2} - \ln|y| &= -\frac{1}{2}e^{x^2} + C \end{aligned}$$

From the initial condition $y(0) = 1$, you have $\frac{1}{2} - 0 = -\frac{1}{2} + C$, which implies that $C = 1$. So, the particular solution has the implicit form

$$\begin{aligned} \frac{y^2}{2} - \ln|y| &= -\frac{1}{2}e^{x^2} + 1 \\ y^2 - \ln y^2 + e^{x^2} &= 2. \end{aligned}$$

You can check this by differentiating and rewriting to get the original equation.

EXAMPLE 3 Finding a Particular Solution Curve

Find the equation of the curve that passes through the point $(1, 3)$ and has a slope of y/x^2 at any point (x, y).

Solution Because the slope of the curve is given by y/x^2, you have

$$\frac{dy}{dx} = \frac{y}{x^2}$$

with the initial condition $y(1) = 3$. Separating variables and integrating produces

$$\begin{aligned} \int\frac{dy}{y} &= \int\frac{dx}{x^2}, \quad y \neq 0 \\ \ln|y| &= -\frac{1}{x} + C_1 \\ y &= e^{-(1/x)+C_1} = Ce^{-1/x}. \end{aligned}$$

Because $y = 3$ when $x = 1$, it follows that $3 = Ce^{-1}$ and $C = 3e$. So, the equation of the specified curve is

$$y = (3e)e^{-1/x} = 3e^{(x-1)/x}, \quad x > 0.$$

Because the solution is not defined at $x = 0$ and the initial condition is given at $x = 1$, x is restricted to positive values. See Figure 6.12. ■

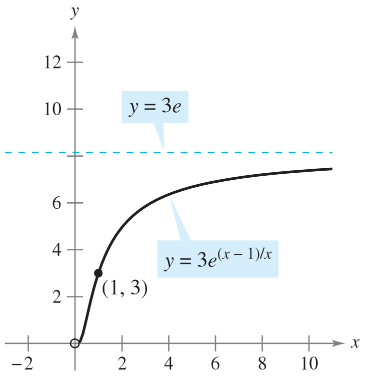


Figure 6.12

Homogeneous Differential Equations

Some differential equations that are not separable in x and y can be made separable by a change of variables. This is true for differential equations of the form $y' = f(x, y)$, where f is a **homogeneous function.** The function given by $f(x, y)$ is **homogeneous of degree n** if

NOTE The notation $f(x, y)$ is used to denote a function of two variables in much the same way as $f(x)$ denotes a function of one variable. You will study functions of two variables in detail in Chapter 13.

$$f(tx, ty) = t^n f(x, y)$$

Homogeneous function of degree n

where n is an integer.

EXAMPLE 4 Verifying Homogeneous Functions

a. $f(x, y) = x^2y - 4x^3 + 3xy^2$ is a homogeneous function of degree 3 because

$$\begin{aligned} f(tx, ty) &= (tx)^2(ty) - 4(tx)^3 + 3(tx)(ty)^2 \\ &= t^3(x^2y) - t^3(4x^3) + t^3(3xy^2) \\ &= t^3(x^2y - 4x^3 + 3xy^2) \\ &= t^3 f(x, y). \end{aligned}$$

b. $f(x, y) = xe^{x/y} + y \sin(y/x)$ is a homogeneous function of degree 1 because

$$\begin{aligned} f(tx, ty) &= txe^{tx/ty} + ty \sin \frac{ty}{tx} \\ &= t\left(xe^{x/y} + y \sin \frac{y}{x}\right) \\ &= tf(x, y). \end{aligned}$$

c. $f(x, y) = x + y^2$ is *not* a homogeneous function because

$$f(tx, ty) = tx + t^2y^2 = t(x + ty^2) \neq t^n(x + y^2).$$

d. $f(x, y) = x/y$ is a homogeneous function of degree 0 because

$$f(tx, ty) = \frac{tx}{ty} = t^0 \frac{x}{y}.$$

■

DEFINITION OF HOMOGENEOUS DIFFERENTIAL EQUATION

A **homogeneous differential equation** is an equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

where M and N are homogeneous functions of the same degree.

EXAMPLE 5 Testing for Homogeneous Differential Equations

a. $(x^2 + xy)\,dx + y^2\,dy = 0$ is homogeneous of degree 2.

b. $x^3\,dx = y^3\,dy$ is homogeneous of degree 3.

c. $(x^2 + 1)\,dx + y^2\,dy = 0$ is *not* a homogeneous differential equation. ■

To solve a homogeneous differential equation by the method of separation of variables, use the following change of variables theorem.

THEOREM 6.2 CHANGE OF VARIABLES FOR HOMOGENEOUS EQUATIONS

If $M(x, y)\,dx + N(x, y)\,dy = 0$ is homogeneous, then it can be transformed into a differential equation whose variables are separable by the substitution

$$y = vx$$

where v is a differentiable function of x.

EXAMPLE 6 Solving a Homogeneous Differential Equation

Find the general solution of

$$(x^2 - y^2)\,dx + 3xy\,dy = 0.$$

STUDY TIP The substitution $y = vx$ will yield a differential equation that is separable with respect to the variables x and v. You must write your final solution, however, in terms of x and y.

Solution Because $(x^2 - y^2)$ and $3xy$ are both homogeneous of degree 2, let $y = vx$ to obtain $dy = x\,dv + v\,dx$. Then, by substitution, you have

$$\begin{aligned}
(x^2 - v^2x^2)\,dx + 3x(vx)\overbrace{(x\,dv + v\,dx)}^{dy} &= 0\\
(x^2 + 2v^2x^2)\,dx + 3x^3v\,dv &= 0\\
x^2(1 + 2v^2)\,dx + x^2(3vx)\,dv &= 0.
\end{aligned}$$

Dividing by x^2 and separating variables produces

$$\begin{aligned}
(1 + 2v^2)\,dx &= -3vx\,dv\\
\int \frac{dx}{x} &= \int \frac{-3v}{1 + 2v^2}\,dv\\
\ln|x| &= -\frac{3}{4}\ln(1 + 2v^2) + C_1\\
4\ln|x| &= -3\ln(1 + 2v^2) + \ln|C|\\
\ln x^4 &= \ln|C(1 + 2v^2)^{-3}|\\
x^4 &= C(1 + 2v^2)^{-3}.
\end{aligned}$$

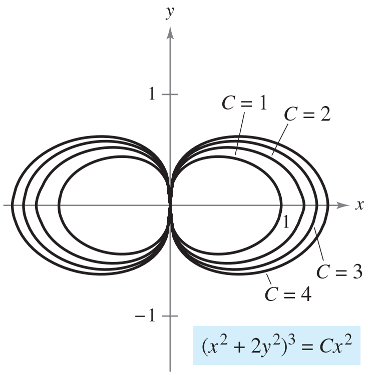
Substituting for v produces the following general solution.

$$x^4 = C\left[1 + 2\left(\frac{y}{x}\right)^2\right]^{-3}$$

$$\left(1 + \frac{2y^2}{x^2}\right)^3 x^4 = C$$

$$(x^2 + 2y^2)^3 = Cx^2 \qquad \text{General solution}$$

You can check this by differentiating and rewriting to get the original equation. ■



General solution of
$(x^2 - y^2)\,dx + 3xy\,dy = 0$
Figure 6.13

TECHNOLOGY If you have access to a graphing utility, try using it to graph several solutions of the equation in Example 6. For instance, Figure 6.13 shows the graphs of

$$(x^2 + 2y^2)^3 = Cx^2$$

for $C = 1, 2, 3$, and 4.

Applications

EXAMPLE 7 Wildlife Population

© franzfoto.com/Alamy

The rate of change of the number of coyotes $N(t)$ in a population is directly proportional to $650 - N(t)$, where t is the time in years. When $t = 0$, the population is 300, and when $t = 2$, the population has increased to 500. Find the population when $t = 3$.

Solution Because the rate of change of the population is proportional to $650 - N(t)$, you can write the following differential equation.

$$\frac{dN}{dt} = k(650 - N)$$

You can solve this equation using separation of variables.

$$dN = k(650 - N)\,dt \qquad \text{Differential form}$$

$$\frac{dN}{650 - N} = k\,dt \qquad \text{Separate variables.}$$

$$-\ln|650 - N| = kt + C_1 \qquad \text{Integrate.}$$

$$\ln|650 - N| = -kt - C_1$$

$$650 - N = e^{-kt - C_1} \qquad \text{Assume } N < 650.$$

$$N = 650 - Ce^{-kt} \qquad \text{General solution}$$

Using $N = 300$ when $t = 0$, you can conclude that $C = 350$, which produces

$$N = 650 - 350e^{-kt}.$$

Then, using $N = 500$ when $t = 2$, it follows that

$$500 = 650 - 350e^{-2k} \quad \Rightarrow \quad e^{-2k} = \frac{3}{7} \quad \Rightarrow \quad k \approx 0.4236.$$

So, the model for the coyote population is

$$N = 650 - 350e^{-0.4236t}. \qquad \text{Model for population}$$

When $t = 3$, you can approximate the population to be

$$N = 650 - 350e^{-0.4236(3)} \approx 552 \text{ coyotes.}$$

The model for the population is shown in Figure 6.14. Note that $N = 650$ is the horizontal asymptote of the graph and is the *carrying capacity* of the model. You will learn more about carrying capacity later in this section.

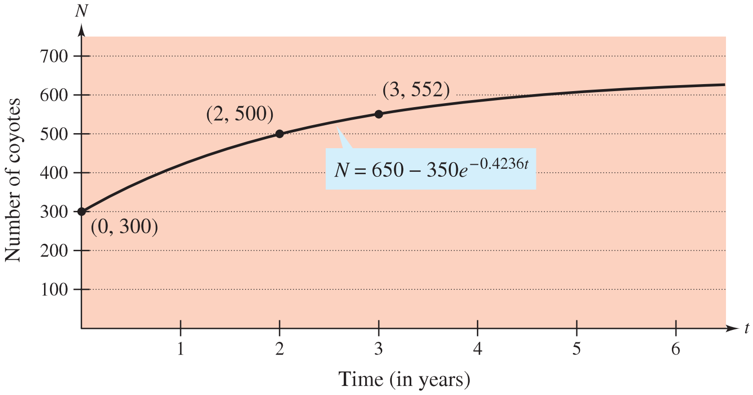
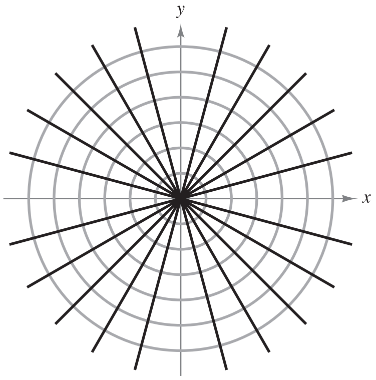


Figure 6.14 ■



Each line $y = Kx$ is an orthogonal trajectory of the family of circles.
Figure 6.15

A common problem in electrostatics, thermodynamics, and hydrodynamics involves finding a family of curves, each of which is orthogonal to all members of a given family of curves. For example, Figure 6.15 shows a family of circles

$$x^2 + y^2 = C \qquad \text{Family of circles}$$

each of which intersects the lines in the family

$$y = Kx \qquad \text{Family of lines}$$

at right angles. Two such families of curves are said to be **mutually orthogonal,** and each curve in one of the families is called an **orthogonal trajectory** of the other family. In electrostatics, lines of force are orthogonal to the *equipotential curves*. In thermodynamics, the flow of heat across a plane surface is orthogonal to the *isothermal curves*. In hydrodynamics, the flow (stream) lines are orthogonal trajectories of the *velocity potential curves*.

EXAMPLE 8 Finding Orthogonal Trajectories

Describe the orthogonal trajectories for the family of curves given by

$$y = \frac{C}{x}$$

for $C \neq 0$. Sketch several members of each family.

Solution First, solve the given equation for C and write $xy = C$. Then, by differentiating implicitly with respect to x, you obtain the differential equation

$$xy' + y = 0 \qquad \text{Differential equation}$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}. \qquad \text{Slope of given family}$$

Because y' represents the slope of the given family of curves at (x, y), it follows that the orthogonal family has the negative reciprocal slope x/y. So,

$$\frac{dy}{dx} = \frac{x}{y}. \qquad \text{Slope of orthogonal family}$$

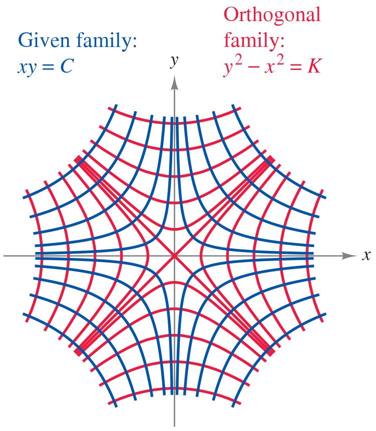
Now you can find the orthogonal family by separating variables and integrating.

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$y^2 - x^2 = K$$

The centers are at the origin, and the transverse axes are vertical for $K > 0$ and horizontal for $K < 0$. If $K = 0$, the orthogonal trajectories are the lines $y = \pm x$. If $K \neq 0$, the orthogonal trajectories are hyperbolas. Several trajectories are shown in Figure 6.16. ■



Orthogonal trajectories
Figure 6.16

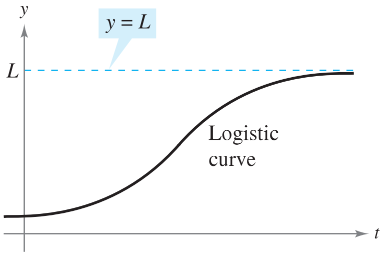
Logistic Differential Equation

In Section 6.2, the exponential growth model was derived from the fact that the rate of change of a variable y is proportional to the value of y. You observed that the differential equation $dy/dt = ky$ has the general solution $y = Ce^{kt}$. Exponential growth is unlimited, but when describing a population, there often exists some upper limit L past which growth cannot occur. This upper limit L is called the **carrying capacity,** which is the maximum population $y(t)$ that can be sustained or supported as time t increases. A model that is often used to describe this type of growth is the **logistic differential equation**

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \qquad \text{Logistic differential equation}$$

where k and L are positive constants. A population that satisfies this equation does not grow without bound, but approaches the carrying capacity L as t increases.

From the equation, you can see that if y is between 0 and the carrying capacity L, then $dy/dt > 0$, and the population increases. If y is greater than L, then $dy/dt < 0$, and the population decreases. The graph of the function y is called the *logistic curve*, as shown in Figure 6.17.



Note that as $t \to \infty$, $y \to L$.
Figure 6.17

EXAMPLE 9 Deriving the General Solution

Solve the logistic differential equation $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{L}\right)$.

Solution Begin by separating variables.

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \qquad \text{Write differential equation.}$$

$$\frac{1}{y(1 - y/L)}dy = kdt \qquad \text{Separate variables.}$$

$$\int \frac{1}{y(1 - y/L)}dy = \int kdt \qquad \text{Integrate each side.}$$

$$\int \left(\frac{1}{y} + \frac{1}{L - y}\right)dy = \int kdt \qquad \text{Rewrite left side using partial fractions.}$$

$$\ln|y| - \ln|L - y| = kt + C \qquad \text{Find antiderivative of each side.}$$

$$\ln\left|\frac{L - y}{y}\right| = -kt - C \qquad \text{Multiply each side by } -1 \text{ and simplify.}$$

$$\left|\frac{L - y}{y}\right| = e^{-kt-C} = e^{-C}e^{-kt} \qquad \text{Exponentiate each side.}$$

$$\frac{L - y}{y} = be^{-kt} \qquad \text{Let } \pm e^{-C} = b.$$

Solving this equation for y produces $y = \dfrac{L}{1 + be^{-kt}}$. ■

EXPLORATION

Use a graphing utility to investigate the effects of the values of L, b, and k on the graph of

$$y = \frac{L}{1 + be^{-kt}}.$$

Include some examples to support your results.

From Example 9, you can conclude that all solutions of the logistic differential equation are of the general form

$$y = \frac{L}{1 + be^{-kt}}.$$

EXAMPLE 10 Solving a Logistic Differential Equation

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population p is

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{4000}\right), \quad 40 \le p \le 4000$$

where t is the number of years.

a. Write a model for the elk population in terms of t.

b. Graph the slope field for the differential equation and the solution that passes through the point $(0, 40)$.

c. Use the model to estimate the elk population after 15 years.

d. Find the limit of the model as $t \to \infty$.

Solution

EXPLORATION

Explain what happens if $p(0) = L$.

a. You know that $L = 4000$. So, the solution of the equation is of the form

$$p = \frac{4000}{1 + be^{-kt}}.$$

Because $p(0) = 40$, you can solve for b as follows.

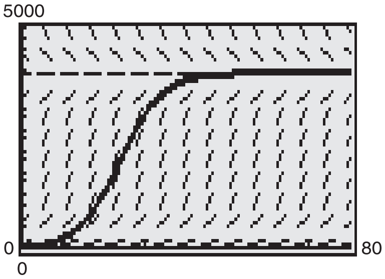
$$40 = \frac{4000}{1 + be^{-k(0)}}$$

$$40 = \frac{4000}{1 + b} \quad \Rightarrow \quad b = 99$$

Then, because $p = 104$ when $t = 5$, you can solve for k.

$$104 = \frac{4000}{1 + 99e^{-k(5)}} \quad \Rightarrow \quad k \approx 0.194$$

So, a model for the elk population is given by $p = \dfrac{4000}{1 + 99e^{-0.194t}}$.



Slope field for

$$\frac{dp}{dt} = 0.194p\left(1 - \frac{p}{4000}\right)$$

and the solution passing through $(0, 40)$

Figure 6.18

b. Using a graphing utility, you can graph the slope field for

$$\frac{dp}{dt} = 0.194p\left(1 - \frac{p}{4000}\right)$$

and the solution that passes through $(0, 40)$, as shown in Figure 6.18.

c. To estimate the elk population after 15 years, substitute 15 for t in the model.

$$p = \frac{4000}{1 + 99e^{-0.194(15)}} \qquad \text{Substitute 15 for } t.$$

$$= \frac{4000}{1 + 99e^{-2.91}} \approx 626 \qquad \text{Simplify.}$$

d. As t increases without bound, the denominator of $\dfrac{4000}{1 + 99e^{-0.194t}}$ gets closer and closer to 1.

So, $\displaystyle\lim_{t\to\infty} \frac{4000}{1 + 99e^{-0.194t}} = 4000.$ ■

6.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–14, find the general solution of the differential equation.

1. $\frac{dy}{dx} = \frac{x}{y}$ **2.** $\frac{dy}{dx} = \frac{3x^2}{y^2}$

3. $x^2 + 5y\frac{dy}{dx} = 0$ **4.** $\frac{dy}{dx} = \frac{x^2 - 3}{6y^2}$

5. $\frac{dr}{ds} = 0.75r$ **6.** $\frac{dr}{ds} = 0.75s$

7. $(2 + x)y' = 3y$ **8.** $xy' = y$

9. $yy' = 4 \sin x$ **10.** $yy' = -8 \cos(\pi x)$

11. $\sqrt{1 - 4x^2}\, y' = x$ **12.** $\sqrt{x^2 - 16}\, y' = 11x$

13. $y \ln x - xy' = 0$ **14.** $12yy' - 7e^x = 0$

In Exercises 15–24, find the particular solution that satisfies the initial condition.

	Differential Equation	*Initial Condition*
15.	$yy' - 2e^x = 0$	$y(0) = 3$
16.	$\sqrt{x} + \sqrt{y}\, y' = 0$	$y(1) = 9$
17.	$y(x + 1) + y' = 0$	$y(-2) = 1$
18.	$2xy' - \ln x^2 = 0$	$y(1) = 2$
19.	$y(1 + x^2)y' - x(1 + y^2) = 0$	$y(0) = \sqrt{3}$
20.	$y\sqrt{1 - x^2}\, y' - x\sqrt{1 - y^2} = 0$	$y(0) = 1$
21.	$\frac{du}{dv} = uv \sin v^2$	$u(0) = 1$
22.	$\frac{dr}{ds} = e^{r - 2s}$	$r(0) = 0$
23.	$dP - kP\, dt = 0$	$P(0) = P_0$
24.	$dT + k(T - 70)\, dt = 0$	$T(0) = 140$

In Exercises 25–28, find an equation of the graph that passes through the point and has the given slope.

25. $(0, 2), \quad y' = \frac{x}{4y}$ **26.** $(1, 1), \quad y' = -\frac{9x}{16y}$

27. $(9, 1), \quad y' = \frac{y}{2x}$ **28.** $(8, 2), \quad y' = \frac{2y}{3x}$

In Exercises 29 and 30, find all functions *f* having the indicated property.

29. The tangent to the graph of f at the point (x, y) intersects the x-axis at $(x + 2, 0)$.

30. All tangents to the graph of f pass through the origin.

In Exercises 31–38, determine whether the function is homogeneous, and if it is, determine its degree.

31. $f(x, y) = x^3 - 4xy^2 + y^3$ **32.** $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

33. $f(x, y) = \frac{x^2y^2}{\sqrt{x^2 + y^2}}$ **34.** $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$

35. $f(x, y) = 2 \ln xy$ **36.** $f(x, y) = \tan(x + y)$

37. $f(x, y) = 2 \ln \frac{x}{y}$ **38.** $f(x, y) = \tan \frac{y}{x}$

In Exercises 39–44, solve the homogeneous differential equation.

39. $y' = \frac{x + y}{2x}$ **40.** $y' = \frac{x^3 + y^3}{xy^2}$

41. $y' = \frac{x - y}{x + y}$ **42.** $y' = \frac{x^2 + y^2}{2xy}$

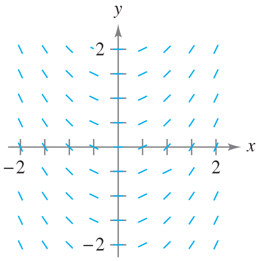
43. $y' = \frac{xy}{x^2 - y^2}$ **44.** $y' = \frac{2x + 3y}{x}$

In Exercises 45–48, find the particular solution that satisfies the initial condition.

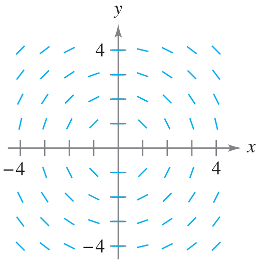
	Differential Equation	*Initial Condition*
45.	$x\, dy - (2xe^{-y/x} + y)\, dx = 0$	$y(1) = 0$
46.	$-y^2\, dx + x(x + y)\, dy = 0$	$y(1) = 1$
47.	$\left(x \sec \frac{y}{x} + y\right) dx - x\, dy = 0$	$y(1) = 0$
48.	$(2x^2 + y^2)\, dx + xy\, dy = 0$	$y(1) = 0$

Slope Fields **In Exercises 49–52, sketch a few solutions of the differential equation on the slope field and then find the general solution analytically. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

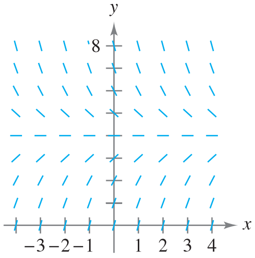
49. $\frac{dy}{dx} = x$



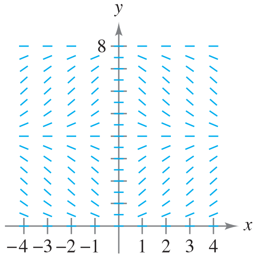
50. $\frac{dy}{dx} = -\frac{x}{y}$



51. $\frac{dy}{dx} = 4 - y$



52. $\frac{dy}{dx} = 0.25x(4 - y)$



Euler's Method **In Exercises 53–56, (a) use Euler's Method with a step size of $h = 0.1$ to approximate the particular solution of the initial value problem at the given x-value, (b) find the exact solution of the differential equation analytically, and (c) compare the solutions at the given x-value.**

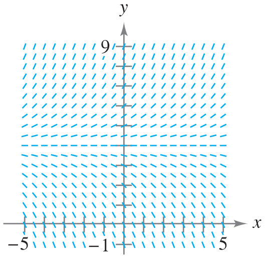
	Differential Equation	*Initial Condition*	*x-value*
53.	$\frac{dy}{dx} = -6xy$	$(0, 5)$	$x = 1$
54.	$\frac{dy}{dx} + 6xy^2 = 0$	$(0, 3)$	$x = 1$
55.	$\frac{dy}{dx} = \frac{2x + 12}{3y^2 - 4}$	$(1, 2)$	$x = 2$
56.	$\frac{dy}{dx} = 2x(1 + y^2)$	$(1, 0)$	$x = 1.5$

57. ***Radioactive Decay*** The rate of decomposition of radioactive radium is proportional to the amount present at any time. The half-life of radioactive radium is 1599 years. What percent of a present amount will remain after 50 years?

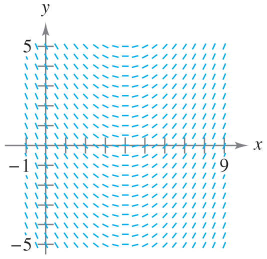
58. ***Chemical Reaction*** In a chemical reaction, a certain compound changes into another compound at a rate proportional to the unchanged amount. If initially there is 40 grams of the original compound, and there is 35 grams after 1 hour, when will 75 percent of the compound be changed?

Slope Fields **In Exercises 59–62, (a) write a differential equation for the statement, (b) match the differential equation with a possible slope field, and (c) verify your result by using a graphing utility to graph a slope field for the differential equation. [The slope fields are labeled (a), (b), (c), and (d).] To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

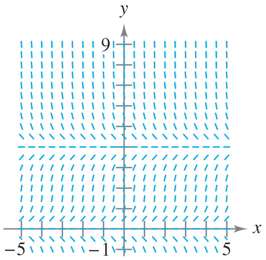
(a)



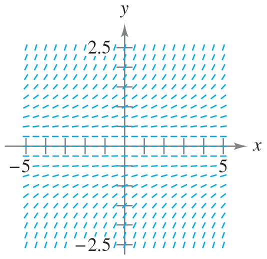
(b)



(c)



(d)



59. The rate of change of y with respect to x is proportional to the difference between y and 4.

60. The rate of change of y with respect to x is proportional to the difference between x and 4.

61. The rate of change of y with respect to x is proportional to the product of y and the difference between y and 4.

62. The rate of change of y with respect to x is proportional to y^2.

CAS **63.** ***Weight Gain*** A calf that weighs 60 pounds at birth gains weight at the rate $dw/dt = k(1200 - w)$, where w is weight in pounds and t is time in years. Solve the differential equation.

(a) Use a computer algebra system to solve the differential equation for $k = 0.8$, 0.9, and 1. Graph the three solutions.

(b) If the animal is sold when its weight reaches 800 pounds, find the time of sale for each of the models in part (a).

(c) What is the maximum weight of the animal for each of the models?

64. ***Weight Gain*** A calf that weighs w_0 pounds at birth gains weight at the rate $dw/dt = 1200 - w$, where w is weight in pounds and t is time in years. Solve the differential equation.

In Exercises 65–70, find the orthogonal trajectories of the family. Use a graphing utility to graph several members of each family.

65. $x^2 + y^2 = C$

66. $x^2 - 2y^2 = C$

67. $x^2 = Cy$

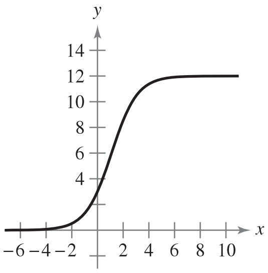
68. $y^2 = 2Cx$

69. $y^2 = Cx^3$

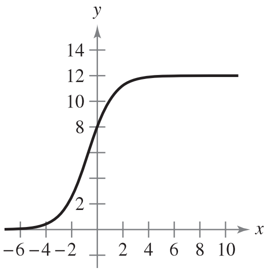
70. $y = Ce^x$

In Exercises 71–74, match the logistic equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]

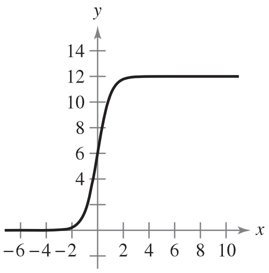
(a)



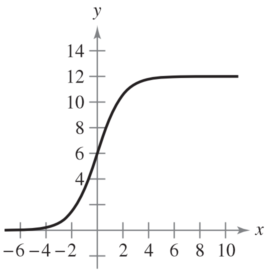
(b)



(c)



(d)



71. $y = \dfrac{12}{1 + e^{-x}}$

72. $y = \dfrac{12}{1 + 3e^{-x}}$

73. $y = \dfrac{12}{1 + \frac{1}{2}e^{-x}}$

74. $y = \dfrac{12}{1 + e^{-2x}}$

In Exercises 75 and 76, the logistic equation models the growth of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) find the initial population, (d) determine when the population will reach 50% of its carrying capacity, and (e) write a logistic differential equation that has the solution $P(t)$.

75. $P(t) = \dfrac{2100}{1 + 29e^{-0.75t}}$

76. $P(t) = \dfrac{5000}{1 + 39e^{-0.2t}}$

CAS **In Exercises 77 and 78, the logistic differential equation models the growth rate of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) graph a slope field using a computer algebra system, and (d) determine the value of P at which the population growth rate is the greatest.**

77. $\dfrac{dP}{dt} = 3P\left(1 - \dfrac{P}{100}\right)$

78. $\dfrac{dP}{dt} = 0.1P - 0.0004P^2$

In Exercises 79–82, find the logistic equation that satisfies the initial condition.

	Logistic Differential Equation	*Initial Condition*
79.	$\dfrac{dy}{dt} = y\left(1 - \dfrac{y}{36}\right)$	$(0, 4)$
80.	$\dfrac{dy}{dt} = 2.8y\left(1 - \dfrac{y}{10}\right)$	$(0, 7)$
81.	$\dfrac{dy}{dt} = \dfrac{4y}{5} - \dfrac{y^2}{150}$	$(0, 8)$
82.	$\dfrac{dy}{dt} = \dfrac{3y}{20} - \dfrac{y^2}{1600}$	$(0, 15)$

83. ***Endangered Species*** A conservation organization releases 25 Florida panthers into a game preserve. After 2 years, there are 39 panthers in the preserve. The Florida preserve has a carrying capacity of 200 panthers.

(a) Write a logistic equation that models the population of panthers in the preserve.

(b) Find the population after 5 years.

(c) When will the population reach 100?

(d) Write a logistic differential equation that models the growth rate of the panther population. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answers.

(e) At what time is the panther population growing most rapidly? Explain.

84. ***Bacteria Growth*** At time $t = 0$, a bacterial culture weighs 1 gram. Two hours later, the culture weighs 4 grams. The maximum weight of the culture is 20 grams.

(a) Write a logistic equation that models the weight of the bacterial culture.

(b) Find the culture's weight after 5 hours.

(c) When will the culture's weight reach 18 grams?

(d) Write a logistic differential equation that models the growth rate of the culture's weight. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answers.

(e) At what time is the culture's weight increasing most rapidly? Explain.

WRITING ABOUT CONCEPTS

85. In your own words, describe how to recognize and solve differential equations that can be solved by separation of variables.

86. State the test for determining if a differential equation is homogeneous. Give an example.

87. In your own words, describe the relationship between two families of curves that are mutually orthogonal.

CAPSTONE

88. Suppose the growth of a population is modeled by a logistic equation. As the population increases, its rate of growth decreases. What do you think causes this to occur in real-life situations such as animal or human populations?

89. Show that if $y = \dfrac{1}{1 + be^{-kt}}$, then $\dfrac{dy}{dt} = ky(1 - y)$.

90. ***Sailing*** Ignoring resistance, a sailboat starting from rest accelerates (dv/dt) at a rate proportional to the difference between the velocities of the wind and the boat.

(a) The wind is blowing at 20 knots, and after 1 half-hour the boat is moving at 10 knots. Write the velocity v as a function of time t.

(b) Use the result of part (a) to write the distance traveled by the boat as a function of time.

True or False? **In Exercises 91–94, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. The function $y = 0$ is always a solution of a differential equation that can be solved by separation of variables.

92. The differential equation $y' = xy - 2y + x - 2$ can be written in separated variables form.

93. The function $f(x, y) = x^2 - 4xy + 6y^2 + 1$ is homogeneous.

94. The families $x^2 + y^2 = 2Cy$ and $x^2 + y^2 = 2Kx$ are mutually orthogonal.

PUTNAM EXAM CHALLENGE

95. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

This problem was composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.

6.4 First-Order Linear Differential Equations

- **Solve a first-order linear differential equation.**
- **Use linear differential equations to solve applied problems.**
- **Solve a Bernoulli differential equation.**

First-Order Linear Differential Equations

In this section, you will see how to solve a very important class of first-order differential equations—first-order linear differential equations.

DEFINITION OF FIRST-ORDER LINEAR DIFFERENTIAL EQUATION

A **first-order linear differential equation** is an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions of x. This first-order linear differential equation is said to be in **standard form.**

NOTE It is instructive to see why the integrating factor helps solve a linear differential equation of the form $y' + P(x)y = Q(x)$. When both sides of the equation are multiplied by the integrating factor $u(x) = e^{\int P(x)\,dx}$, the left-hand side becomes the derivative of a product.

$$y'e^{\int P(x)\,dx} + P(x)ye^{\int P(x)\,dx} = Q(x)e^{\int P(x)\,dx}$$
$$\left[ye^{\int P(x)\,dx}\right]' = Q(x)e^{\int P(x)\,dx}$$

Integrating both sides of this second equation and dividing by $u(x)$ produces the general solution.

To solve a linear differential equation, write it in standard form to identify the functions $P(x)$ and $Q(x)$. Then integrate $P(x)$ and form the expression

$$u(x) = e^{\int P(x)\,dx} \qquad \text{Integrating factor}$$

which is called an **integrating factor.** The general solution of the equation is

$$y = \frac{1}{u(x)}\int Q(x)u(x)\,dx. \qquad \text{General solution}$$

EXAMPLE 1 Solving a Linear Differential Equation

Find the general solution of

$$y' + y = e^x.$$

Solution For this equation, $P(x) = 1$ and $Q(x) = e^x$. So, the integrating factor is

$$\begin{aligned} u(x) &= e^{\int P(x)\,dx} && \text{Integrating factor} \\ &= e^{\int dx} \\ &= e^x. \end{aligned}$$

This implies that the general solution is

$$\begin{aligned} y &= \frac{1}{u(x)}\int Q(x)u(x)\,dx \\ &= \frac{1}{e^x}\int e^x(e^x)\,dx \\ &= e^{-x}\left(\frac{1}{2}e^{2x} + C\right) \\ &= \frac{1}{2}e^x + Ce^{-x}. && \text{General solution} \end{aligned}$$

■

ANNA JOHNSON PELL WHEELER (1883–1966)

Anna Johnson Pell Wheeler was awarded a master's degree from the University of Iowa for her thesis *The Extension of Galois Theory to Linear Differential Equations* in 1904. Influenced by David Hilbert, she worked on integral equations while studying infinite linear spaces.

THEOREM 6.3 SOLUTION OF A FIRST-ORDER LINEAR DIFFERENTIAL EQUATION

An integrating factor for the first-order linear differential equation

$$y' + P(x)y = Q(x)$$

is $u(x) = e^{\int P(x)\,dx}$. The solution of the differential equation is

$$ye^{\int P(x)\,dx} = \int Q(x)e^{\int P(x)\,dx}\,dx + C.$$

STUDY TIP Rather than memorizing the formula in Theorem 6.3, just remember that multiplication by the integrating factor $e^{\int P(x)\,dx}$ converts the left side of the differential equation into the derivative of the product $ye^{\int P(x)\,dx}$.

EXAMPLE 2 Solving a First-Order Linear Differential Equation

Find the general solution of

$$xy' - 2y = x^2.$$

Solution The standard form of the given equation is

$$y' + P(x)y = Q(x)$$

$$y' - \left(\frac{2}{x}\right)y = x. \qquad \text{Standard form}$$

So, $P(x) = -2/x$, and you have

$$\int P(x)\,dx = -\int \frac{2}{x}\,dx$$

$$= -\ln x^2$$

$$e^{\int P(x)\,dx} = e^{-\ln x^2}$$

$$= \frac{1}{e^{\ln x^2}}$$

$$= \frac{1}{x^2}. \qquad \text{Integrating factor}$$

So, multiplying each side of the standard form by $1/x^2$ yields

$$\frac{y'}{x^2} - \frac{2y}{x^3} = \frac{1}{x}$$

$$\frac{d}{dx}\left[\frac{y}{x^2}\right] = \frac{1}{x}$$

$$\frac{y}{x^2} = \int \frac{1}{x}\,dx$$

$$\frac{y}{x^2} = \ln|x| + C$$

$$y = x^2(\ln|x| + C). \qquad \text{General solution}$$

Several solution curves (for $C = -2, -1, 0, 1, 2, 3,$ and 4) are shown in Figure 6.19.

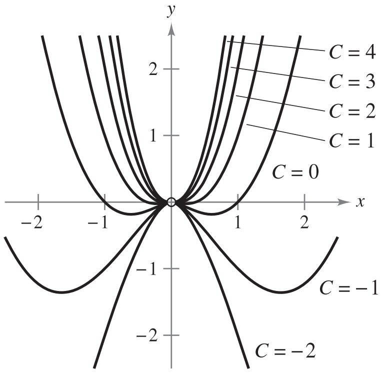


Figure 6.19

EXAMPLE 3 Solving a First-Order Linear Differential Equation

Find the general solution of $y' - y\tan t = 1, \quad -\pi/2 < t < \pi/2$.

Solution The equation is already in the standard form $y' + P(t)y = Q(t)$. So, $P(t) = -\tan t$, and

$$\int P(t)\,dt = -\int \tan t\,dt = \ln|\cos t|.$$

Because $-\pi/2 < t < \pi/2$, you can drop the absolute value signs and conclude that the integrating factor is

$$e^{\int P(t)\,dt} = e^{\ln(\cos t)} = \cos t. \qquad \text{Integrating factor}$$

So, multiplying $y' - y\tan t = 1$ by $\cos t$ produces

$$\frac{d}{dt}[y\cos t] = \cos t$$

$$y\cos t = \int \cos t\,dt$$

$$y\cos t = \sin t + C$$

$$y = \tan t + C\sec t. \qquad \text{General solution}$$

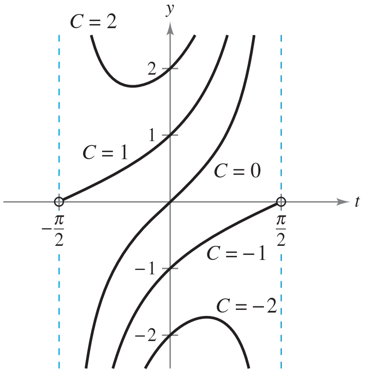


Figure 6.20

Several solution curves are shown in Figure 6.20. ■

Applications

One type of problem that can be described in terms of a differential equation involves chemical mixtures, as illustrated in the next example.

EXAMPLE 4 A Mixture Problem

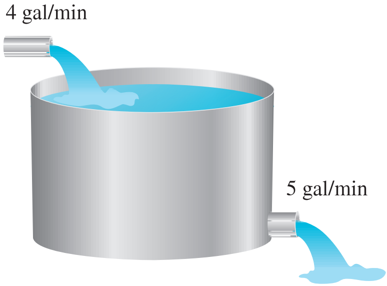


Figure 6.21

A tank contains 50 gallons of a solution composed of 90% water and 10% alcohol. A second solution containing 50% water and 50% alcohol is added to the tank at the rate of 4 gallons per minute. As the second solution is being added, the tank is being drained at a rate of 5 gallons per minute, as shown in Figure 6.21. Assuming the solution in the tank is stirred constantly, how much alcohol is in the tank after 10 minutes?

Solution Let y be the number of gallons of alcohol in the tank at any time t. You know that $y = 5$ when $t = 0$. Because the number of gallons of solution in the tank at any time is $50 - t$, and the tank loses 5 gallons of solution per minute, it must lose $[5/(50 - t)]y$ gallons of alcohol per minute. Furthermore, because the tank is gaining 2 gallons of alcohol per minute, the rate of change of alcohol in the tank is given by

$$\frac{dy}{dt} = 2 - \left(\frac{5}{50 - t}\right)y \quad \Rightarrow \quad \frac{dy}{dt} + \left(\frac{5}{50 - t}\right)y = 2.$$

To solve this linear equation, let $P(t) = 5/(50 - t)$ and obtain

$$\int P(t)\,dt = \int \frac{5}{50 - t}\,dt = -5\ln|50 - t|.$$

Because $t < 50$, you can drop the absolute value signs and conclude that

$$e^{\int P(t)\,dt} = e^{-5\ln(50 - t)} = \frac{1}{(50 - t)^5}.$$

So, the general solution is

$$\frac{y}{(50-t)^5} = \int \frac{2}{(50-t)^5}\,dt = \frac{1}{2(50-t)^4} + C$$

$$y = \frac{50-t}{2} + C(50-t)^5.$$

Because $y = 5$ when $t = 0$, you have

$$5 = \frac{50}{2} + C(50)^5 \quad \Rightarrow \quad -\frac{20}{50^5} = C$$

which means that the particular solution is

$$y = \frac{50-t}{2} - 20\left(\frac{50-t}{50}\right)^5.$$

Finally, when $t = 10$, the amount of alcohol in the tank is

$$y = \frac{50-10}{2} - 20\left(\frac{50-10}{50}\right)^5 \approx 13.45 \text{ gal}$$

which represents a solution containing 33.6% alcohol. ■

In most falling-body problems discussed so far in the text, air resistance has been neglected. The next example includes this factor. In the example, the air resistance on the falling object is assumed to be proportional to its velocity v. If g is the gravitational constant, the downward force F on a falling object of mass m is given by the difference $mg - kv$. But by Newton's Second Law of Motion, you know that

$$F = ma = m(dv/dt) \qquad a = \text{acceleration}$$

which yields the following differential equation.

$$m\frac{dv}{dt} = mg - kv \quad \Rightarrow \quad \frac{dv}{dt} + \frac{kv}{m} = g$$

EXAMPLE 5 A Falling Object with Air Resistance

An object of mass m is dropped from a hovering helicopter. Find its velocity as a function of time t. Assume that the air resistance is proportional to the object's velocity.

Solution The velocity v satisfies the equation

$$\frac{dv}{dt} + \frac{kv}{m} = g. \qquad g = \text{gravitational constant},\ k = \text{constant of proportionality}$$

Letting $b = k/m$, you can *separate variables* to obtain

$$dv = (g - bv)\,dt$$

$$\int \frac{dv}{g - bv} = \int dt$$

$$-\frac{1}{b}\ln|g - bv| = t + C_1$$

$$\ln|g - bv| = -bt - bC_1$$

$$g - bv = Ce^{-bt}. \qquad C = e^{-bC_1}$$

Because the object was dropped, $v = 0$ when $t = 0$; so $g = C$, and it follows that

$$-bv = -g + ge^{-bt} \quad \Rightarrow \quad v = \frac{g - ge^{-bt}}{b} = \frac{mg}{k}\left(1 - e^{-kt/m}\right).$$ ■

NOTE Notice in Example 5 that the velocity approaches a limit of mg/k as a result of the air resistance. For falling-body problems in which air resistance is neglected, the velocity increases without bound.

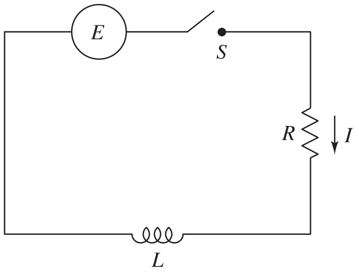


Figure 6.22

A simple electric circuit consists of electric current I (in amperes), a resistance R (in ohms), an inductance L (in henrys), and a constant electromotive force E (in volts), as shown in Figure 6.22. According to Kirchhoff's Second Law, if the switch S is closed when $t = 0$, the applied electromotive force (voltage) is equal to the sum of the voltage drops in the rest of the circuit. This in turn means that the current I satisfies the differential equation

$$L\frac{dI}{dt} + RI = E.$$

EXAMPLE 6 An Electric Circuit Problem

Find the current I as a function of time t (in seconds), given that I satisfies the differential equation

$$L(dI/dt) + RI = \sin 2t$$

where R and L are nonzero constants.

Solution In standard form, the given linear equation is

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}\sin 2t.$$

Let $P(t) = R/L$, so that $e^{\int P(t)\,dt} = e^{(R/L)t}$, and, by Theorem 6.3,

$$\begin{aligned} Ie^{(R/L)t} &= \frac{1}{L}\int e^{(R/L)t}\sin 2t\,dt \\ &= \frac{1}{4L^2 + R^2}e^{(R/L)t}(R\sin 2t - 2L\cos 2t) + C. \end{aligned}$$

So the general solution is

$$I = e^{-(R/L)t}\left[\frac{1}{4L^2 + R^2}e^{(R/L)t}(R\sin 2t - 2L\cos 2t) + C\right]$$

$$I = \frac{1}{4L^2 + R^2}(R\sin 2t - 2L\cos 2t) + Ce^{-(R/L)t}.$$ ■

TECHNOLOGY The integral in Example 6 was found using symbolic algebra software. If you have access to *Maple*, *Mathematica*, or the *TI-89*, try using it to integrate

$$\frac{1}{L}\int e^{(R/L)t}\sin 2t\,dt.$$

In Chapter 8 you will learn how to integrate functions of this type using integration by parts.

Bernoulli Equation

A well-known nonlinear equation that reduces to a linear one with an appropriate substitution is the **Bernoulli equation,** named after James Bernoulli (1654–1705).

$$y' + P(x)y = Q(x)y^n$$

Bernoulli equation

This equation is linear if $n = 0$, and has separable variables if $n = 1$. So, in the following development, assume that $n \neq 0$ and $n \neq 1$. Begin by multiplying by y^{-n} and $(1 - n)$ to obtain

$$\begin{aligned} y^{-n}y' + P(x)y^{1-n} &= Q(x) \\ (1-n)y^{-n}y' + (1-n)P(x)y^{1-n} &= (1-n)Q(x) \\ \frac{d}{dx}\left[y^{1-n}\right] + (1-n)P(x)y^{1-n} &= (1-n)Q(x) \end{aligned}$$

which is a linear equation in the variable y^{1-n}. Letting $z = y^{1-n}$ produces the linear equation

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x).$$

Finally, by Theorem 6.3, the *general solution of the Bernoulli equation* is

$$y^{1-n}e^{\int(1-n)P(x)\,dx} = \int (1-n)Q(x)e^{\int(1-n)P(x)\,dx}\,dx + C.$$

EXAMPLE 7 Solving a Bernoulli Equation

Find the general solution of $y' + xy = xe^{-x^2}y^{-3}$.

Solution For this Bernoulli equation, let $n = -3$, and use the substitution

$z = y^4$	Let $z = y^{1-n} = y^{1-(-3)}$.
$z' = 4y^3y'$.	Differentiate.

Multiplying the original equation by $4y^3$ produces

$y' + xy = xe^{-x^2}y^{-3}$	Write original equation.
$4y^3y' + 4xy^4 = 4xe^{-x^2}$	Multiply each side by $4y^3$.
$z' + 4xz = 4xe^{-x^2}$.	Linear equation: $z' + P(x)z = Q(x)$

This equation is linear in z. Using $P(x) = 4x$ produces

$$\begin{aligned} \int P(x)\,dx &= \int 4x\,dx \\ &= 2x^2 \end{aligned}$$

which implies that e^{2x^2} is an integrating factor. Multiplying the linear equation by this factor produces

$z' + 4xz = 4xe^{-x^2}$	Linear equation
$z'e^{2x^2} + 4xze^{2x^2} = 4xe^{x^2}$	Multiply by integrating factor.
$\frac{d}{dx}\left[ze^{2x^2}\right] = 4xe^{x^2}$	Write left side as derivative.
$ze^{2x^2} = \int 4xe^{x^2}\,dx$	Integrate each side.
$ze^{2x^2} = 2e^{x^2} + C$	
$z = 2e^{-x^2} + Ce^{-2x^2}$.	Divide each side by e^{2x^2}.

Finally, substituting $z = y^4$, the general solution is

$y^4 = 2e^{-x^2} + Ce^{-2x^2}$.	General solution ■

So far you have studied several types of first-order differential equations. Of these, the separable variables case is usually the simplest, and solution by an integrating factor is ordinarily used only as a last resort.

SUMMARY OF FIRST-ORDER DIFFERENTIAL EQUATIONS

Method	*Form of Equation*
1. Separable variables:	$M(x)\,dx + N(y)\,dy = 0$
2. Homogeneous:	$M(x, y)\,dx + N(x, y)\,dy = 0$, where M and N are nth-degree homogeneous functions
3. Linear:	$y' + P(x)y = Q(x)$
4. Bernoulli equation:	$y' + P(x)y = Q(x)y^n$

6.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, determine whether the differential equation is linear. Explain your reasoning.

1. $x^3y' + xy = e^x + 1$

2. $2xy - y'\ln x = y$

3. $y' - y\sin x = xy^2$

4. $\dfrac{2 - y'}{y} = 5x$

In Exercises 5–14, solve the first-order linear differential equation.

5. $\dfrac{dy}{dx} + \left(\dfrac{1}{x}\right)y = 6x + 2$

6. $\dfrac{dy}{dx} + \left(\dfrac{2}{x}\right)y = 3x - 5$

7. $y' - y = 16$

8. $y' + 2xy = 10x$

9. $(y + 1)\cos x\,dx - dy = 0$

10. $(y - 1)\sin x\,dx - dy = 0$

11. $(x - 1)y' + y = x^2 - 1$

12. $y' + 3y = e^{3x}$

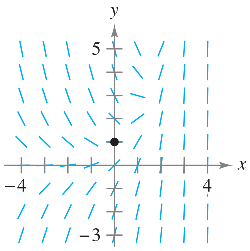
13. $y' - 3x^2y = e^{x^3}$

14. $y' + y\tan x = \sec x$

Slope Fields **In Exercises 15 and 16, (a) sketch an approximate solution of the differential equation satisfying the given initial condition by hand on the slope field, (b) find the particular solution that satisfies the given initial condition, and (c) use a graphing utility to graph the particular solution. Compare the graph with the hand-drawn graph in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

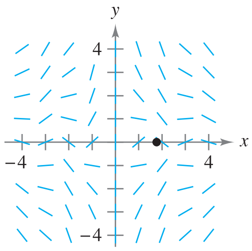
15. $\dfrac{dy}{dx} = e^x - y,$

$(0, 1)$



16. $y' + \left(\dfrac{1}{x}\right)y = \sin x^2,$

$\left(\sqrt{\pi}, 0\right)$



In Exercises 17–24, find the particular solution of the differential equation that satisfies the boundary condition.

	Differential Equation	*Boundary Condition*
17.	$y'\cos^2 x + y - 1 = 0$	$y(0) = 5$
18.	$x^3y' + 2y = e^{1/x^2}$	$y(1) = e$
19.	$y' + y\tan x = \sec x + \cos x$	$y(0) = 1$
20.	$y' + y\sec x = \sec x$	$y(0) = 4$
21.	$y' + \left(\dfrac{1}{x}\right)y = 0$	$y(2) = 2$
22.	$y' + (2x - 1)y = 0$	$y(1) = 2$
23.	$x\,dy = (x + y + 2)\,dx$	$y(1) = 10$
24.	$2x\,y' - y = x^3 - x$	$y(4) = 2$

25. ***Population Growth*** When predicting population growth, demographers must consider birth and death rates as well as the net change caused by the difference between the rates of immigration and emigration. Let P be the population at time t and let N be the net increase per unit time resulting from the difference between immigration and emigration. So, the rate of growth of the population is given by

$$\frac{dP}{dt} = kP + N, \qquad N \text{ is constant.}$$

Solve this differential equation to find P as a function of time if at time $t = 0$ the size of the population is P_0.

26. ***Investment Growth*** A large corporation starts at time $t = 0$ to invest part of its receipts continuously at a rate of P dollars per year in a fund for future corporate expansion. Assume that the fund earns r percent interest per year compounded continuously. So, the rate of growth of the amount A in the fund is given by

$$\frac{dA}{dt} = rA + P$$

where $A = 0$ when $t = 0$. Solve this differential equation for A as a function of t.

Investment Growth **In Exercises 27 and 28, use the result of Exercise 26.**

27. Find A for the following.

(a) $P = \$275{,}000$, $r = 8\%$, and $t = 10$ years

(b) $P = \$550{,}000$, $r = 5.9\%$, and $t = 25$ years

28. Find t if the corporation needs \$1,000,000 and it can invest \$125,000 per year in a fund earning 8% interest compounded continuously.

29. ***Intravenous Feeding*** Glucose is added intravenously to the bloodstream at the rate of q units per minute, and the body removes glucose from the bloodstream at a rate proportional to the amount present. Assume that $Q(t)$ is the amount of glucose in the bloodstream at time t.

(a) Determine the differential equation describing the rate of change of glucose in the bloodstream with respect to time.

(b) Solve the differential equation from part (a), letting $Q = Q_0$ when $t = 0$.

(c) Find the limit of $Q(t)$ as $t \to \infty$.

30. ***Learning Curve*** The management at a certain factory has found that the maximum number of units a worker can produce in a day is 75. The rate of increase in the number of units N produced with respect to time t in days by a new employee is proportional to $75 - N$.

(a) Determine the differential equation describing the rate of change of performance with respect to time.

(b) Solve the differential equation from part (a).

(c) Find the particular solution for a new employee who produced 20 units on the first day at the factory and 35 units on the twentieth day.

Mixture **In Exercises 31–35, consider a tank that at time $t = 0$ contains v_0 gallons of a solution of which, by weight, q_0 pounds is soluble concentrate. Another solution containing q_1 pounds of the concentrate per gallon is running into the tank at the rate of r_1 gallons per minute. The solution in the tank is kept well stirred and is withdrawn at the rate of r_2 gallons per minute.**

31. If Q is the amount of concentrate in the solution at any time t, show that

$$\frac{dQ}{dt} + \frac{r_2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1.$$

32. If Q is the amount of concentrate in the solution at any time t, write the differential equation for the rate of change of Q with respect to t if $r_1 = r_2 = r$.

33. A 200-gallon tank is full of a solution containing 25 pounds of concentrate. Starting at time $t = 0$, distilled water is admitted to the tank at a rate of 10 gallons per minute, and the well-stirred solution is withdrawn at the same rate.

(a) Find the amount of concentrate Q in the solution as a function of t.

(b) Find the time at which the amount of concentrate in the tank reaches 15 pounds.

(c) Find the quantity of the concentrate in the solution as $t \to \infty$.

34. Repeat Exercise 33, assuming that the solution entering the tank contains 0.04 pound of concentrate per gallon.

35. A 200-gallon tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 pound of concentrate per gallon enters the tank at the rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at the rate of 3 gallons per minute.

(a) At what time will the tank be full?

(b) At the time the tank is full, how many pounds of concentrate will it contain?

(c) Repeat parts (a) and (b), assuming that the solution entering the tank contains 1 pound of concentrate per gallon.

CAPSTONE

36. Suppose the expression $u(x)$ is an integrating factor for $y' + P(x)y = Q(x)$. Which of the following is equal to $u'(x)$? Verify your answer.

(a) $P(x)\,u(x)$

(b) $P'(x)\,u(x)$

(c) $Q(x)\,u(x)$

(d) $Q'(x)\,u(x)$

Falling Object **In Exercises 37 and 38, consider an eight-pound object dropped from a height of 5000 feet, where the air resistance is proportional to the velocity.**

37. Write the velocity of the object as a function of time if the velocity after 5 seconds is approximately -101 feet per second. What is the limiting value of the velocity function?

38. Use the result of Exercise 37 to write the position of the object as a function of time. Approximate the velocity of the object when it reaches ground level.

Electric Circuits **In Exercises 39 and 40, use the differential equation for electric circuits given by**

$$L\frac{dI}{dt} + RI = E.$$

In this equation, I is the current, R is the resistance, L is the inductance, and E is the electromotive force (voltage).

39. Solve the differential equation for the current given a constant voltage E_0.

40. Use the result of Exercise 39 to find the equation for the current if $I(0) = 0$, $E_0 = 120$ volts, $R = 600$ ohms, and $L = 4$ henrys. When does the current reach 90% of its limiting value?

WRITING ABOUT CONCEPTS

41. Give the standard form of a first-order linear differential equation. What is its integrating factor?

42. Give the standard form of the Bernoulli equation. Describe how one reduces it to a linear equation.

In Exercises 43–46, match the differential equation with its solution.

Differential Equation	*Solution*
43. $y' - 2x = 0$	(a) $y = Ce^{x^2}$
44. $y' - 2y = 0$	(b) $y = -\frac{1}{2} + Ce^{x^2}$
45. $y' - 2xy = 0$	(c) $y = x^2 + C$
46. $y' - 2xy = x$	(d) $y = Ce^{2x}$

In Exercises 47–54, solve the Bernoulli differential equation.

47. $y' + 3x^2y = x^2y^3$

48. $y' + xy = xy^{-1}$

49. $y' + \left(\frac{1}{x}\right)y = xy^2$

50. $y' + \left(\frac{1}{x}\right)y = x\sqrt{y}$

51. $xy' + y = xy^3$

52. $y' - y = y^3$

53. $y' - y = e^x\sqrt[3]{y}$

54. $yy' - 2y^2 = e^x$

Slope Fields **In Exercises 55–58, (a) use a graphing utility to graph the slope field for the differential equation, (b) find the particular solutions of the differential equation passing through the given points, and (c) use a graphing utility to graph the particular solutions on the slope field.**

Differential Equation	*Points*
55. $\frac{dy}{dx} - \frac{1}{x}y = x^2$	$(-2, 4),\ (2, 8)$
56. $\frac{dy}{dx} + 4x^3y = x^3$	$\left(0, \frac{7}{2}\right),\ \left(0, -\frac{1}{2}\right)$
57. $\frac{dy}{dx} + (\cot x)y = 2$	$(1, 1),\ (3, -1)$
58. $\frac{dy}{dx} + 2xy = xy^2$	$(0, 3),\ (0, 1)$

In Exercises 59–70, solve the first-order differential equation by any appropriate method.

59. $\frac{dy}{dx} = \frac{e^{2x+y}}{e^{x-y}}$

60. $\frac{dy}{dx} = \frac{x-3}{y(y+4)}$

61. $y\cos x - \cos x + \frac{dy}{dx} = 0$

62. $y' = 2x\sqrt{1-y^2}$

63. $(3y^2 + 4xy)\,dx + (2xy + x^2)\,dy = 0$

64. $(x + y)\,dx - x\,dy = 0$

65. $(2y - e^x)\,dx + x\,dy = 0$

66. $(y^2 + xy)\,dx - x^2\,dy = 0$

67. $(x^2y^4 - 1)\,dx + x^3y^3\,dy = 0$

68. $y\,dx + (3x + 4y)\,dy = 0$

69. $3(y - 4x^2)\,dx + x\,dy = 0$

70. $x\,dx + (y + e^y)(x^2 + 1)\,dy = 0$

True or False? **In Exercises 71 and 72, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

71. $y' + x\sqrt{y} = x^2$ is a first-order linear differential equation.

72. $y' + xy = e^xy$ is a first-order linear differential equation.

SECTION PROJECT

Weight Loss

A person's weight depends on both the number of calories consumed and the energy used. Moreover, the amount of energy used depends on a person's weight—the average amount of energy used by a person is 17.5 calories per pound per day. So, the more weight a person loses, the less energy a person uses (assuming that the person maintains a constant level of activity). An equation that can be used to model weight loss is

$$\left(\frac{dw}{dt}\right) = \frac{C}{3500} - \frac{17.5}{3500}w$$

where w is the person's weight (in pounds), t is the time in days, and C is the constant daily calorie consumption.

(a) Find the general solution of the differential equation.

(b) Consider a person who weighs 180 pounds and begins a diet of 2500 calories per day. How long will it take the person to lose 10 pounds? How long will it take the person to lose 35 pounds?

(c) Use a graphing utility to graph the solution. What is the "limiting" weight of the person?

(d) Repeat parts (b) and (c) for a person who weighs 200 pounds when the diet is started.

■ **FOR FURTHER INFORMATION** For more information on modeling weight loss, see the article "A Linear Diet Model" by Arthur C. Segal in *The College Mathematics Journal*.

6 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. Determine whether the function $y = x^3$ is a solution of the differential equation $2xy' + 4y = 10x^3$.

2. Determine whether the function $y = 2 \sin 2x$ is a solution of the differential equation $y''' - 8y = 0$.

In Exercises 3–10, use integration to find a general solution of the differential equation.

3. $\dfrac{dy}{dx} = 4x^2 + 7$

4. $\dfrac{dy}{dx} = 3x^3 - 8x$

5. $\dfrac{dy}{dx} = \cos 2x$

6. $\dfrac{dy}{dx} = 2 \sin x$

7. $\dfrac{dy}{dx} = x\sqrt{x - 5}$

8. $\dfrac{dy}{dx} = 2x\sqrt{x - 7}$

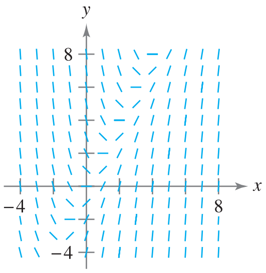
9. $\dfrac{dy}{dx} = e^{2-x}$

10. $\dfrac{dy}{dx} = 3e^{-x/3}$

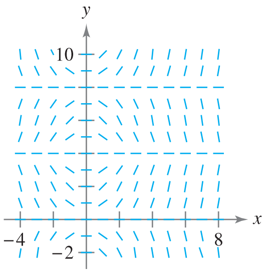
Slope Fields **In Exercises 11 and 12, a differential equation and its slope field are given. Determine the slopes (if possible) in the slope field at the points given in the table.**

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx						

11. $\dfrac{dy}{dx} = 2x - y$



12. $\dfrac{dy}{dx} = x \sin\left(\dfrac{\pi y}{4}\right)$



Slope Fields **In Exercises 13–18, (a) sketch the slope field for the differential equation, and (b) use the slope field to sketch the solution that passes through the given point. Use a graphing utility to verify your results.**

Differential Equation	*Point*
13. $y' = 3 - x$	$(2, 1)$
14. $y' = 2x^2 - x$	$(0, 2)$
15. $y' = \dfrac{1}{4}x^2 - \dfrac{1}{3}x$	$(0, 3)$
16. $y' = y + 4x$	$(-1, 1)$
17. $y' = \dfrac{xy}{x^2 + 4}$	$(0, 1)$
18. $y' = \dfrac{y}{x^2 + 1}$	$(0, -2)$

In Exercises 19–24, solve the differential equation.

19. $\dfrac{dy}{dx} = 8 - x$

20. $\dfrac{dy}{dx} = y + 8$

21. $\dfrac{dy}{dx} = (3 + y)^2$

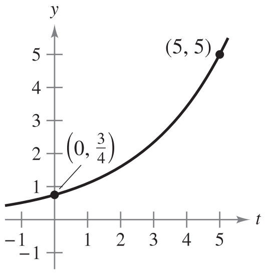
22. $\dfrac{dy}{dx} = 10\sqrt{y}$

23. $(2 + x)y' - xy = 0$

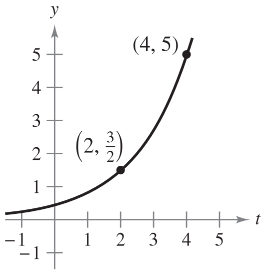
24. $xy' - (x + 1)y = 0$

In Exercises 25–28, find the exponential function $y = Ce^{kt}$ that passes through the two points.

25.



26.



27. $(0, 5), \left(5, \dfrac{1}{6}\right)$

28. $(1, 9), (6, 2)$

29. ***Air Pressure*** Under ideal conditions, air pressure decreases continuously with the height above sea level at a rate proportional to the pressure at that height. The barometer reads 30 inches at sea level and 15 inches at 18,000 feet. Find the barometric pressure at 35,000 feet.

30. ***Radioactive Decay*** Radioactive radium has a half-life of approximately 1599 years. The initial quantity is 15 grams. How much remains after 750 years?

31. ***Sales*** The sales S (in thousands of units) of a new product after it has been on the market for t years is given by

$$S = Ce^{k/t}.$$

(a) Find S as a function of t if 5000 units have been sold after 1 year and the saturation point for the market is 30,000 units (that is, $\lim\limits_{t \to \infty} S = 30$).

(b) How many units will have been sold after 5 years?

(c) Use a graphing utility to graph this sales function.

32. ***Sales*** The sales S (in thousands of units) of a new product after it has been on the market for t years is given by

$$S = 25(1 - e^{kt}).$$

(a) Find S as a function of t if 4000 units have been sold after 1 year.

(b) How many units will saturate this market?

(c) How many units will have been sold after 5 years?

(d) Use a graphing utility to graph this sales function.

33. ***Population Growth*** A population grows continuously at the rate of 1.85%. How long will it take the population to double?

34. ***Fuel Economy*** An automobile gets 28 miles per gallon of gasoline for speeds up to 50 miles per hour. Over 50 miles per hour, the number of miles per gallon drops at the rate of 12 percent for each 10 miles per hour.

(a) s is the speed and y is the number of miles per gallon. Find y as a function of s by solving the differential equation

$$\frac{dy}{ds} = -0.012y, \quad s > 50.$$

(b) Use the function in part (a) to complete the table.

Speed	50	55	60	65	70
Miles per Gallon					

In Exercises 35–40, solve the differential equation.

35. $\dfrac{dy}{dx} = \dfrac{x^5 + 7}{x}$

36. $\dfrac{dy}{dx} = \dfrac{e^{-2x}}{1 + e^{-2x}}$

37. $y' - 16xy = 0$

38. $y' - e^y \sin x = 0$

39. $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$

40. $\dfrac{dy}{dx} = \dfrac{3(x + y)}{x}$

41. Verify that the general solution $y = C_1x + C_2x^3$ satisfies the differential equation $x^2y'' - 3xy' + 3y = 0$. Then find the particular solution that satisfies the initial condition $y = 0$ and $y' = 4$ when $x = 2$.

42. ***Vertical Motion*** A falling object encounters air resistance that is proportional to its velocity. The acceleration due to gravity is -9.8 meters per second per second. The net change in velocity is $dv/dt = kv - 9.8$.

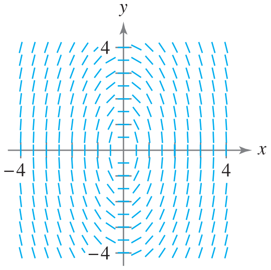
(a) Find the velocity of the object as a function of time if the initial velocity is v_0.

(b) Use the result of part (a) to find the limit of the velocity as t approaches infinity.

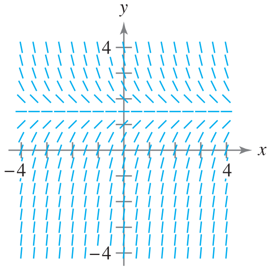
(c) Integrate the velocity function found in part (a) to find the position function s.

Slope Fields **In Exercises 43 and 44, sketch a few solutions of the differential equation on the slope field and then find the general solution analytically. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

43. $\dfrac{dy}{dx} = -\dfrac{4x}{y}$



44. $\dfrac{dy}{dx} = 3 - 2y$



In Exercises 45 and 46, the logistic equation models the growth of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) find the initial population, (d) determine when the population will reach 50% of its carrying capacity, and (e) write a logistic differential equation that has the solution $P(t)$.

45. $P(t) = \dfrac{5250}{1 + 34e^{-0.55t}}$

46. $P(t) = \dfrac{4800}{1 + 14e^{-0.15t}}$

In Exercises 47 and 48, find the logistic equation that satisfies the initial condition.

	Logistic Differential Equation	*Initial Condition*
47.	$\dfrac{dy}{dt} = y\left(1 - \dfrac{y}{80}\right)$	$(0, 8)$
48.	$\dfrac{dy}{dt} = 1.76y\left(1 - \dfrac{y}{8}\right)$	$(0, 3)$

49. ***Environment*** A conservation department releases 1200 brook trout into a lake. It is estimated that the carrying capacity of the lake for the species is 20,400. After the first year, there are 2000 brook trout in the lake.

(a) Write a logistic equation that models the number of brook trout in the lake.

(b) Find the number of brook trout in the lake after 8 years.

(c) When will the number of brook trout reach 10,000?

50. ***Environment*** Write a logistic differential equation that models the growth rate of the brook trout population in Exercise 49. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answers.

In Exercises 51–60, solve the first-order linear differential equation.

51. $y' - y = 10$

52. $e^x y' + 4e^x y = 1$

53. $4y' = e^{x/4} + y$

54. $\dfrac{dy}{dx} - \dfrac{5y}{x^2} = \dfrac{1}{x^2}$

55. $(x - 2)y' + y = 1$

56. $(x + 3)y' + 2y = 2(x + 3)^2$

57. $(3y + \sin 2x)\,dx - dy = 0$

58. $dy = (y \tan x + 2e^x)\,dx$

59. $y' + 5y = e^{5x}$

60. $xy' - ay = bx^4$

In Exercises 61–64, solve the Bernoulli differential equation.

61. $y' + y = xy^2 \quad \left[\text{Hint: } \int xe^{-x}\,dx = (-x - 1)e^{-x}\right]$

62. $y' + 2xy = xy^2$

63. $y' + \left(\dfrac{1}{x}\right)y = \dfrac{y^3}{x^2}$

64. $xy' + y = xy^2$

In Exercises 65–68, write an example of the given differential equation. Then solve your equation.

65. Homogeneous

66. Logistic

67. First-order linear

68. Bernoulli

P.S. PROBLEM SOLVING

1. The differential equation

$$\frac{dy}{dt} = ky^{1+\varepsilon}$$

where k and ε are positive constants, is called the **doomsday equation.**

(a) Solve the doomsday equation

$$\frac{dy}{dt} = y^{1.01}$$

given that $y(0) = 1$. Find the time T at which

$$\lim_{t\to T^-} y(t) = \infty.$$

(b) Solve the doomsday equation

$$\frac{dy}{dt} = ky^{1+\varepsilon}$$

given that $y(0) = y_0$. Explain why this equation is called the doomsday equation.

2. A thermometer is taken from a room at 72°F to the outdoors, where the temperature is 20°F. The reading drops to 48°F after 1 minute. Determine the reading on the thermometer after 5 minutes.

3. Let S represent sales of a new product (in thousands of units), let L represent the maximum level of sales (in thousands of units), and let t represent time (in months). The rate of change of S with respect to t varies jointly as the product of S and $L - S$.

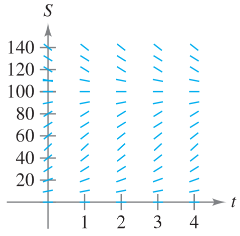
(a) Write the differential equation for the sales model if $L = 100$, $S = 10$ when $t = 0$, and $S = 20$ when $t = 1$. Verify that

$$S = \frac{L}{1 + Ce^{-kt}}.$$

(b) At what time is the growth in sales increasing most rapidly?

(c) Use a graphing utility to graph the sales function.

(d) Sketch the solution from part (a) on the slope field shown in the figure below. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(e) If the estimated maximum level of sales is correct, use the slope field to describe the shape of the solution curves for sales if, at some period of time, sales exceed L.

4. Another model that can be used to represent population growth is the **Gompertz equation,** which is the solution of the differential equation

$$\frac{dy}{dt} = k \ln\left(\frac{L}{y}\right)y$$

where k is a constant and L is the carrying capacity.

(a) Solve the differential equation.

(b) Use a graphing utility to graph the slope field for the differential equation when $k = 0.05$ and $L = 1000$.

(c) Describe the behavior of the graph as $t \to \infty$.

(d) Graph the equation you found in part (a) for $L = 5000$, $y_0 = 500$, and $k = 0.02$. Determine the concavity of the graph and how it compares with the general solution of the logistic differential equation.

5. Show that the logistic equation $y = L/(1 + be^{-kt})$ can be written as

$$y = \frac{1}{2}L\left[1 + \tanh\left(\frac{1}{2}k\left(t - \frac{\ln b}{k}\right)\right)\right].$$

What can you conclude about the graph of the logistic equation?

6. Although it is true for some functions f and g, a common mistake in calculus is to believe that the Product Rule for derivatives is $(fg)' = f'g'$.

(a) Given $g(x) = x$, find f such that $(fg)' = f'g'$.

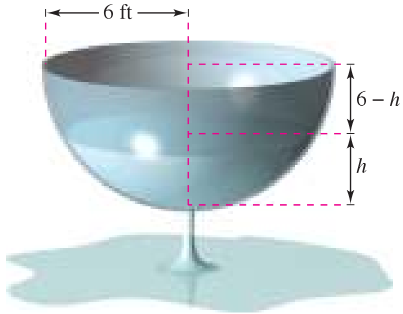
(b) Given an arbitrary function g, find a function f such that $(fg)' = f'g'$.

(c) Describe what happens if $g(x) = e^x$.

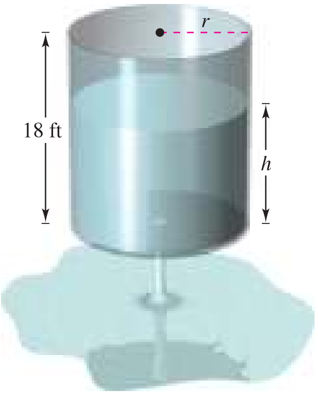
7. **Torricelli's Law** states that water will flow from an opening at the bottom of a tank with the same speed that it would attain falling from the surface of the water to the opening. One of the forms of Torricelli's Law is

$$A(h)\frac{dh}{dt} = -k\sqrt{2gh}$$

where h is the height of the water in the tank, k is the area of the opening at the bottom of the tank, $A(h)$ is the horizontal cross-sectional area at height h, and g is the acceleration due to gravity ($g \approx 32$ feet per second per second). A hemispherical water tank has a radius of 6 feet. When the tank is full, a circular valve with a radius of 1 inch is opened at the bottom, as shown in the figure. How long will it take for the tank to drain completely?



8. The cylindrical water tank shown in the figure has a height of 18 feet. When the tank is full, a circular valve is opened at the bottom of the tank. After 30 minutes, the depth of the water is 12 feet.



(a) How long will it take for the tank to drain completely?

(b) What is the depth of the water in the tank after 1 hour?

9. Suppose the tank in Exercise 8 has a height of 20 feet and a radius of 8 feet, and the valve is circular with a radius of 2 inches. The tank is full when the valve is opened. How long will it take for the tank to drain completely?

10. In hilly areas, radio reception may be poor. Consider a situation in which an FM transmitter is located at the point $(-1, 1)$ behind a hill modeled by the graph of

$$y = x - x^2$$

and a radio receiver is on the opposite side of the hill. (Assume that the x-axis represents ground level at the base of the hill.)

(a) What is the closest position $(x, 0)$ the radio can be to the hill so that reception is unobstructed?

(b) Write the closest position $(x, 0)$ of the radio with x represented as a function of h if the transmitter is located at $(-1, h)$.

(c) Use a graphing utility to graph the function for x in part (b). Determine the vertical asymptote of the function and interpret its meaning.

11. Biomass is a measure of the amount of living matter in an ecosystem. Suppose the biomass $s(t)$ in a given ecosystem increases at a rate of about 3.5 tons per year, and decreases by about 1.9% per year. This situation can be modeled by the differential equation

$$\frac{ds}{dt} = 3.5 - 0.019s.$$

(a) Solve the differential equation.

(b) Use a graphing utility to graph the slope field for the differential equation. What do you notice?

(c) Explain what happens as $t \to \infty$.

In Exercises 12–14, a medical researcher wants to determine the concentration C (in moles per liter) of a tracer drug injected into a moving fluid. Solve this problem by considering a single-compartment dilution model (see figure). Assume that the fluid is continuously mixed and that the volume of the fluid in the compartment is constant.

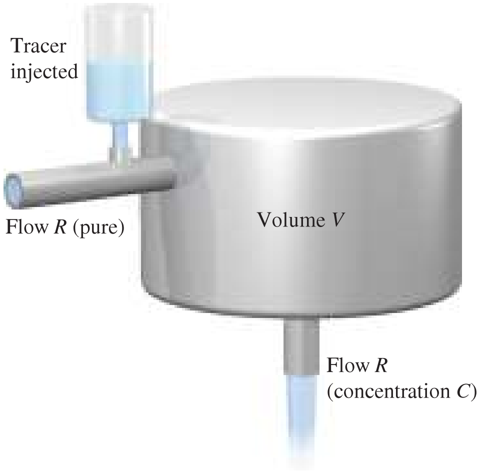


Figure for 12–14

12. If the tracer is injected instantaneously at time $t = 0$, then the concentration of the fluid in the compartment begins diluting according to the differential equation

$$\frac{dC}{dt} = \left(-\frac{R}{V}\right)C, \quad C = C_0 \text{ when } t = 0.$$

(a) Solve this differential equation to find the concentration C as a function of time t.

(b) Find the limit of C as $t \to \infty$.

13. Use the solution of the differential equation in Exercise 12 to find the concentration C as a function of time t, and use a graphing utility to graph the function.

(a) $V = 2$ liters, $R = 0.5$ liter per minute, and $C_0 = 0.6$ mole per liter

(b) $V = 2$ liters, $R = 1.5$ liters per minute, and $C_0 = 0.6$ mole per liter

14. In Exercises 12 and 13, it was assumed that there was a single initial injection of the tracer drug into the compartment. Now consider the case in which the tracer is continuously injected (beginning at $t = 0$) at the rate of Q moles per minute. Considering Q to be negligible compared with R, use the differential equation

$$\frac{dC}{dt} = \frac{Q}{V} - \left(\frac{R}{V}\right)C, \quad C = 0 \text{ when } t = 0.$$

(a) Solve this differential equation to find the concentration C as a function of time t.

(b) Find the limit of C as $t \to \infty$.

7 Applications of Integration

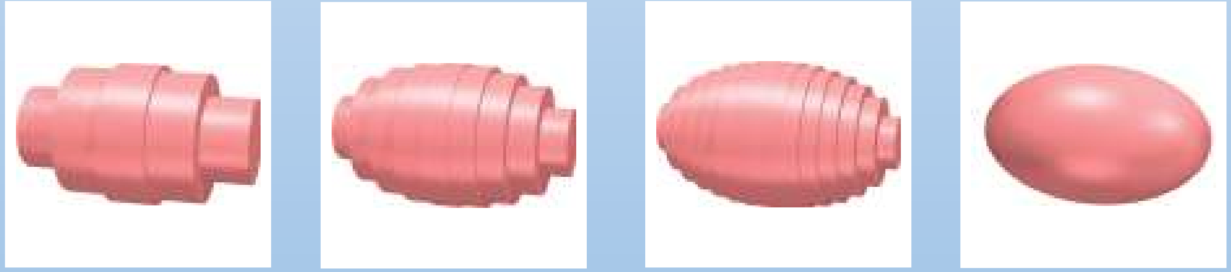
Integration has a wide variety of applications. For each of the applications presented in this chapter, you will begin with a known formula, such as the area of a rectangular region, the volume of a circular disk, or the work done by a constant force. Then you will learn how the limit of a sum gives rise to new formulas that involve integration.

In this chapter, you should learn the following.

- How to use a definite integral to find the area of a region bounded by two curves. (7.1)
- How to find the volume of a solid of revolution by the disk and shell methods. (7.2 and 7.3)
- How to find the length of a curve and the surface area of a surface of revolution. (7.4)
- How to find the work done by a constant force and by a variable force. (7.5)
- How to find centers of mass and centroids. (7.6)
- How to find fluid pressure and fluid force. (7.7)

Eddie Hironaka/Getty Images

An electric cable is hung between two towers that are 200 feet apart. The cable takes the shape of a catenary. What is the length of the cable between the two towers? (See Section 7.4, Example 5.)



The *disk method* is one method that is used to find the volume of a solid. This method requires finding the sum of the volumes of representative disks to approximate the volume of the solid. As you increase the number of disks, the approximation tends to become more accurate. In Section 7.2, you will use limits to write the exact volume of the solid as a definite integral.

7.1 Area of a Region Between Two Curves

- **Find the area of a region between two curves using integration.**
- **Find the area of a region between intersecting curves using integration.**
- **Describe integration as an accumulation process.**

Area of a Region Between Two Curves

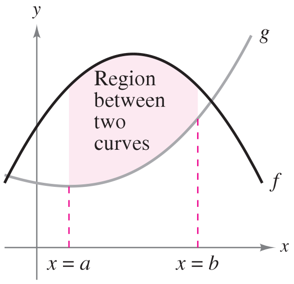
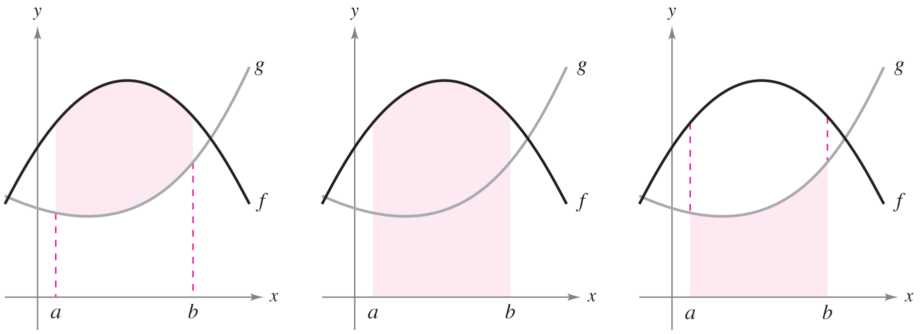


Figure 7.1

With a few modifications, you can extend the application of definite integrals from the area of a region *under* a curve to the area of a region *between* two curves. Consider two functions f and g that are continuous on the interval $[a, b]$. If, as in Figure 7.1, the graphs of both f and g lie above the x-axis, and the graph of g lies below the graph of f, you can geometrically interpret the area of the region between the graphs as the area of the region under the graph of g subtracted from the area of the region under the graph of f, as shown in Figure 7.2.



Area of region between f and g	=	Area of region under f	−	Area of region under g

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Figure 7.2

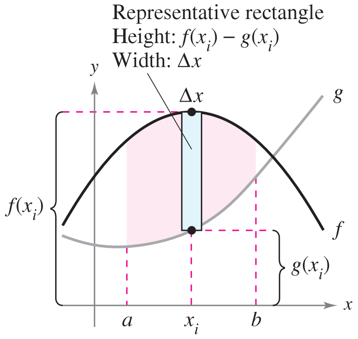


Figure 7.3

To verify the reasonableness of the result shown in Figure 7.2, you can partition the interval $[a, b]$ into n subintervals, each of width Δx. Then, as shown in Figure 7.3, sketch a **representative rectangle** of width Δx and height $f(x_i) - g(x_i)$, where x_i is in the ith subinterval. The area of this representative rectangle is

$$\Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)]\Delta x.$$

By adding the areas of the n rectangles and taking the limit as $\|\Delta\| \to 0$ $(n \to \infty)$, you obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x.$$

Because f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$ and the limit exists. So, the area of the given region is

$$\begin{aligned}\text{Area} &= \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x \\ &= \int_a^b [f(x) - g(x)]\,dx.\end{aligned}$$

AREA OF A REGION BETWEEN TWO CURVES

If f and g are continuous on $[a, b]$ and $g(x) \le f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

In Figure 7.1, the graphs of f and g are shown above the x-axis. This, however, is not necessary. The same integrand $[f(x) - g(x)]$ can be used as long as f and g are continuous and $g(x) \le f(x)$ for all x in the interval $[a, b]$. This is summarized graphically in Figure 7.4. Notice in Figure 7.4 that the height of a representative rectangle is $f(x) - g(x)$ regardless of the relative position of the x-axis.

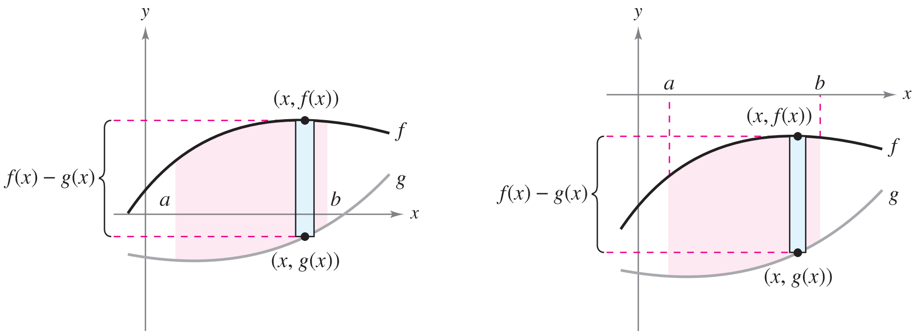
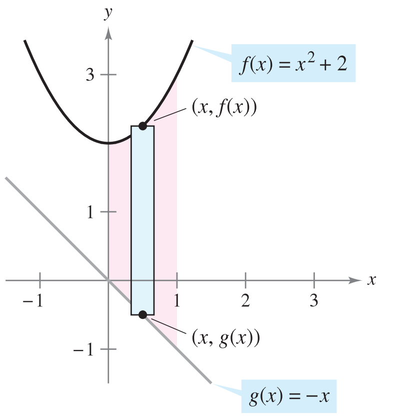


Figure 7.4

Representative rectangles are used throughout this chapter in various applications of integration. A vertical rectangle (of width Δx) implies integration with respect to x, whereas a horizontal rectangle (of width Δy) implies integration with respect to y.

EXAMPLE 1 Finding the Area of a Region Between Two Curves

Find the area of the region bounded by the graphs of $y = x^2 + 2$, $y = -x$, $x = 0$, and $x = 1$.



Region bounded by the graph of f, the graph of g, $x = 0$, and $x = 1$
Figure 7.5

Solution Let $g(x) = -x$ and $f(x) = x^2 + 2$. Then $g(x) \le f(x)$ for all x in $[0, 1]$, as shown in Figure 7.5. So, the area of the representative rectangle is

$$\begin{aligned}\Delta A &= [f(x) - g(x)]\,\Delta x\\ &= [(x^2 + 2) - (-x)]\,\Delta x\end{aligned}$$

and the area of the region is

$$\begin{aligned}A = \int_a^b [f(x) - g(x)]\,dx &= \int_0^1 [(x^2 + 2) - (-x)]\,dx\\ &= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_0^1\\ &= \frac{1}{3} + \frac{1}{2} + 2\\ &= \frac{17}{6}.\end{aligned}$$

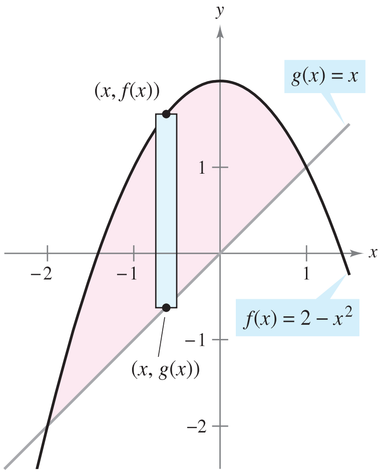
■

Area of a Region Between Intersecting Curves

In Example 1, the graphs of $f(x) = x^2 + 2$ and $g(x) = -x$ do not intersect, and the values of a and b are given explicitly. A more common problem involves the area of a region bounded by two *intersecting* graphs, where the values of a and b must be calculated.

EXAMPLE 2 A Region Lying Between Two Intersecting Graphs

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.



Region bounded by the graph of *f* and the graph of *g*
Figure 7.6

Solution In Figure 7.6, notice that the graphs of f and g have two points of intersection. To find the x-coordinates of these points, set $f(x)$ and $g(x)$ equal to each other and solve for x.

$$\begin{aligned} 2 - x^2 &= x && \text{Set } f(x) \text{ equal to } g(x). \\ -x^2 - x + 2 &= 0 && \text{Write in general form.} \\ -(x+2)(x-1) &= 0 && \text{Factor.} \\ x &= -2 \text{ or } 1 && \text{Solve for } x. \end{aligned}$$

So, $a = -2$ and $b = 1$. Because $g(x) \le f(x)$ for all x in the interval $[-2, 1]$, the representative rectangle has an area of

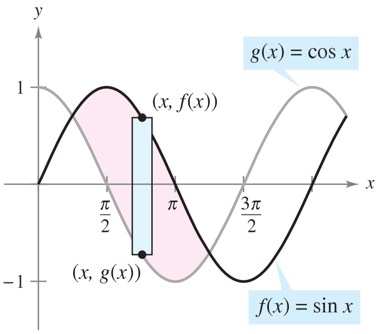
$$\begin{aligned} \Delta A &= [f(x) - g(x)]\,\Delta x \\ &= [(2 - x^2) - x]\,\Delta x \end{aligned}$$

and the area of the region is

$$\begin{aligned} A = \int_{-2}^{1} [(2 - x^2) - x]\,dx &= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1} \\ &= \frac{9}{2}. \end{aligned}$$

EXAMPLE 3 A Region Lying Between Two Intersecting Graphs

The sine and cosine curves intersect infinitely many times, bounding regions of equal areas, as shown in Figure 7.7. Find the area of one of these regions.



One of the regions bounded by the graphs of the sine and cosine functions
Figure 7.7

Solution

$$\begin{aligned} \sin x &= \cos x && \text{Set } f(x) \text{ equal to } g(x). \\ \frac{\sin x}{\cos x} &= 1 && \text{Divide each side by } \cos x. \\ \tan x &= 1 && \text{Trigonometric identity} \\ x &= \frac{\pi}{4} \text{ or } \frac{5\pi}{4}, \quad 0 \le x \le 2\pi && \text{Solve for } x. \end{aligned}$$

So, $a = \pi/4$ and $b = 5\pi/4$. Because $\sin x \ge \cos x$ for all x in the interval $[\pi/4, 5\pi/4]$, the area of the region is

$$\begin{aligned} A = \int_{\pi/4}^{5\pi/4} [\sin x - \cos x]\,dx &= \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4} \\ &= 2\sqrt{2}. \end{aligned}$$

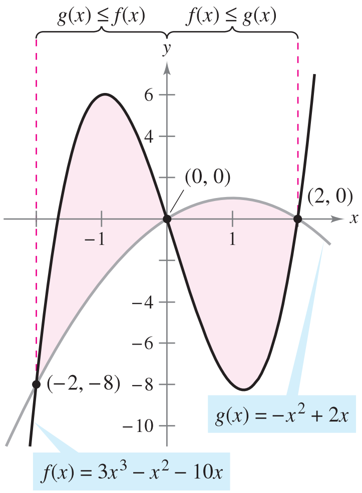
■

If two curves intersect at more than two points, then to find the area of the region between the curves, you must find all points of intersection and check to see which curve is above the other in each interval determined by these points.

EXAMPLE 4 Curves That Intersect at More Than Two Points

Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

Solution Begin by setting $f(x)$ and $g(x)$ equal to each other and solving for x. This yields the x-values at all points of intersection of the two graphs.



On $[-2, 0]$, $g(x) \le f(x)$, and on $[0, 2]$, $f(x) \le g(x)$
Figure 7.8

$$
\begin{aligned}
3x^3 - x^2 - 10x &= -x^2 + 2x && \text{Set } f(x) \text{ equal to } g(x).\\
3x^3 - 12x &= 0 && \text{Write in general form.}\\
3x(x-2)(x+2) &= 0 && \text{Factor.}\\
x &= -2, 0, 2 && \text{Solve for } x.
\end{aligned}
$$

So, the two graphs intersect when $x = -2, 0,$ and 2. In Figure 7.8, notice that $g(x) \le f(x)$ on the interval $[-2, 0]$. However, the two graphs switch at the origin, and $f(x) \le g(x)$ on the interval $[0, 2]$. So, you need two integrals—one for the interval $[-2, 0]$ and one for the interval $[0, 2]$.

$$
\begin{aligned}
A &= \int_{-2}^{0} [f(x) - g(x)]\,dx + \int_{0}^{2} [g(x) - f(x)]\,dx\\
&= \int_{-2}^{0} (3x^3 - 12x)\,dx + \int_{0}^{2} (-3x^3 + 12x)\,dx\\
&= \left[\frac{3x^4}{4} - 6x^2\right]_{-2}^{0} + \left[\frac{-3x^4}{4} + 6x^2\right]_{0}^{2}\\
&= -(12 - 24) + (-12 + 24) = 24
\end{aligned}
$$

■

NOTE In Example 4, notice that you obtain an incorrect result if you integrate from -2 to 2. Such integration produces

$$\int_{-2}^{2} [f(x) - g(x)]\,dx = \int_{-2}^{2} (3x^3 - 12x)\,dx = 0.$$

■

If the graph of a function of y is a boundary of a region, it is often convenient to use representative rectangles that are *horizontal* and find the area by integrating with respect to y. In general, to determine the area between two curves, you can use

$$A = \int_{x_1}^{x_2} \underbrace{[(\text{top curve}) - (\text{bottom curve})]}_{\text{in variable } x}\,dx \qquad \text{Vertical rectangles}$$

$$A = \int_{y_1}^{y_2} \underbrace{[(\text{right curve}) - (\text{left curve})]}_{\text{in variable } y}\,dy \qquad \text{Horizontal rectangles}$$

where (x_1, y_1) and (x_2, y_2) are either adjacent points of intersection of the two curves involved or points on the specified boundary lines.

The icon ↻ *indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems* Maple *and* Mathematica.

EXAMPLE 5 Horizontal Representative Rectangles

Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

Solution Consider

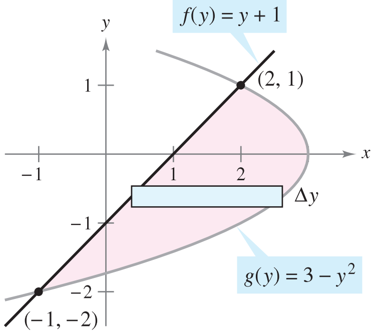
$$g(y) = 3 - y^2 \quad \text{and} \quad f(y) = y + 1.$$

These two curves intersect when $y = -2$ and $y = 1$, as shown in Figure 7.9. Because $f(y) \le g(y)$ on this interval, you have

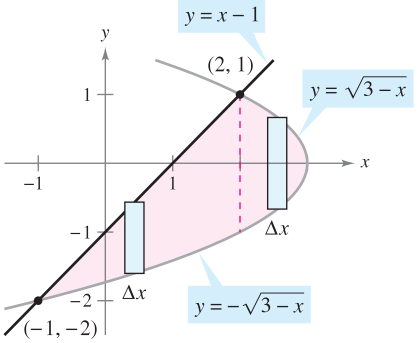
$$\Delta A = [g(y) - f(y)]\,\Delta y = [(3 - y^2) - (y + 1)]\,\Delta y.$$

So, the area is

$$\begin{aligned} A &= \int_{-2}^{1} [(3 - y^2) - (y + 1)]\,dy \\ &= \int_{-2}^{1} (-y^2 - y + 2)\,dy \\ &= \left[\frac{-y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^{1} \\ &= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) \\ &= \frac{9}{2}. \end{aligned}$$



Horizontal rectangles (integration with respect to y)
Figure 7.9



Vertical rectangles (integration with respect to x)
Figure 7.10

In Example 5, notice that by integrating with respect to y you need only one integral. If you had integrated with respect to x, you would have needed two integrals because the upper boundary would have changed at $x = 2$, as shown in Figure 7.10.

$$\begin{aligned} A &= \int_{-1}^{2} \left[(x - 1) + \sqrt{3 - x}\right] dx + \int_{2}^{3} \left(\sqrt{3 - x} + \sqrt{3 - x}\right) dx \\ &= \int_{-1}^{2} \left[x - 1 + (3 - x)^{1/2}\right] dx + 2\int_{2}^{3} (3 - x)^{1/2}\,dx \\ &= \left[\frac{x^2}{2} - x - \frac{(3 - x)^{3/2}}{3/2}\right]_{-1}^{2} - 2\left[\frac{(3 - x)^{3/2}}{3/2}\right]_{2}^{3} \\ &= \left(2 - 2 - \frac{2}{3}\right) - \left(\frac{1}{2} + 1 - \frac{16}{3}\right) - 2(0) + 2\left(\frac{2}{3}\right) \\ &= \frac{9}{2} \end{aligned}$$

Integration as an Accumulation Process

In this section, the integration formula for the area between two curves was developed by using a rectangle as the *representative element*. For each new application in the remaining sections of this chapter, an appropriate representative element will be constructed using precalculus formulas you already know. Each integration formula will then be obtained by summing or accumulating these representative elements.

Known precalculus formula ⇨ Representative element ⇨ New integration formula

For example, the area formula in this section was developed as follows.

$$A = (\text{height})(\text{width}) \;\Rightarrow\; \Delta A = [f(x) - g(x)]\,\Delta x \;\Rightarrow\; A = \int_a^b [f(x) - g(x)]\,dx$$

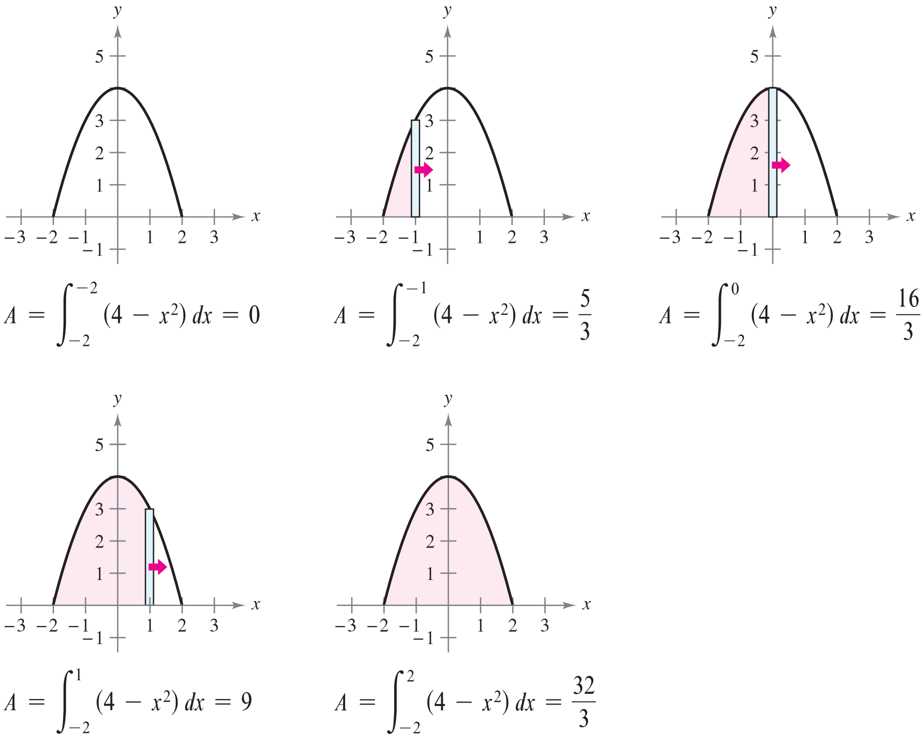
EXAMPLE 6 Describing Integration as an Accumulation Process

Find the area of the region bounded by the graph of $y = 4 - x^2$ and the x-axis. Describe the integration as an accumulation process.

Solution The area of the region is given by

$$A = \int_{-2}^{2} (4 - x^2)\,dx.$$

You can think of the integration as an accumulation of the areas of the rectangles formed as the representative rectangle slides from $x = -2$ to $x = 2$, as shown in Figure 7.11.



$A = \int_{-2}^{-2} (4 - x^2)\,dx = 0$

$A = \int_{-2}^{-1} (4 - x^2)\,dx = \frac{5}{3}$

$A = \int_{-2}^{0} (4 - x^2)\,dx = \frac{16}{3}$

$A = \int_{-2}^{1} (4 - x^2)\,dx = 9$

$A = \int_{-2}^{2} (4 - x^2)\,dx = \frac{32}{3}$

Figure 7.11

■

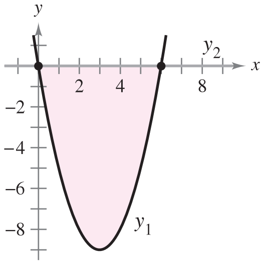
7.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, set up the definite integral that gives the area of the region.

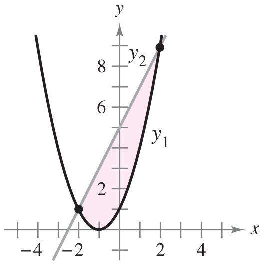
1. $y_1 = x^2 - 6x$

$y_2 = 0$



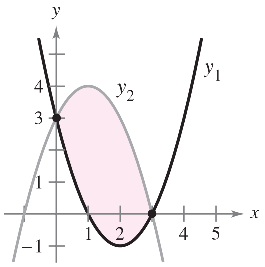
2. $y_1 = x^2 + 2x + 1$

$y_2 = 2x + 5$



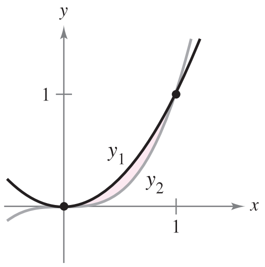
3. $y_1 = x^2 - 4x + 3$

$y_2 = -x^2 + 2x + 3$



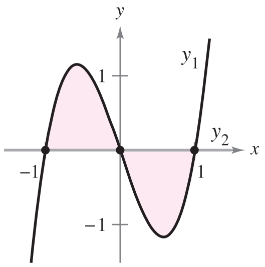
4. $y_1 = x^2$

$y_2 = x^3$



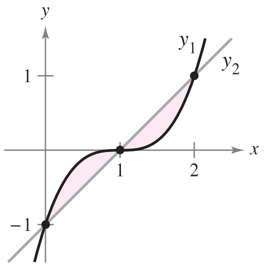
5. $y_1 = 3(x^3 - x)$

$y_2 = 0$



6. $y_1 = (x - 1)^3$

$y_2 = x - 1$



In Exercises 7–14, the integrand of the definite integral is a difference of two functions. Sketch the graph of each function and shade the region whose area is represented by the integral.

7. $\int_0^4 \left[(x + 1) - \frac{x}{2}\right] dx$

8. $\int_{-1}^1 [(2 - x^2) - x^2]\, dx$

9. $\int_0^6 \left[4(2^{-x/3}) - \frac{x}{6}\right] dx$

10. $\int_2^3 \left[\left(\frac{x^3}{3} - x\right) - \frac{x}{3}\right] dx$

11. $\int_{-\pi/3}^{\pi/3} (2 - \sec x)\, dx$

12. $\int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x)\, dx$

13. $\int_{-2}^1 [(2 - y) - y^2]\, dy$

14. $\int_0^4 \left(2\sqrt{y} - y\right) dy$

Think About It **In Exercises 15 and 16, determine which value best approximates the area of the region bounded by the graphs of *f* and *g*. (Make your selection on the basis of a sketch of the region and not by performing any calculations.)**

15. $f(x) = x + 1, \quad g(x) = (x - 1)^2$

(a) -2 (b) 2 (c) 10 (d) 4 (e) 8

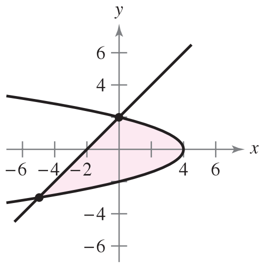
16. $f(x) = 2 - \frac{1}{2}x, \quad g(x) = 2 - \sqrt{x}$

(a) 1 (b) 6 (c) -3 (d) 3 (e) 4

In Exercises 17 and 18, find the area of the region by integrating (a) with respect to *x* and (b) with respect to *y*. (c) Compare your results. Which method is simpler? In general, will this method always be simpler than the other one? Why or why not?

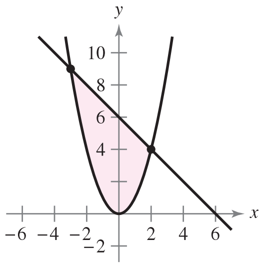
17. $x = 4 - y^2$

$x = y - 2$



18. $y = x^2$

$y = 6 - x$



In Exercises 19–36, sketch the region bounded by the graphs of the algebraic functions and find the area of the region.

19. $y = x^2 - 1, y = -x + 2, x = 0, x = 1$

20. $y = -x^3 + 3, y = x, x = -1, x = 1$

21. $y = \frac{1}{2}x^3 + 2, \ y = x + 1, \ x = 0, \ x = 2$

22. $y = -\frac{3}{8}x(x - 8), \ y = 10 - \frac{1}{2}x, \ x = 2, \ x = 8$

23. $f(x) = x^2 - 4x, \quad g(x) = 0$

24. $f(x) = -x^2 + 4x + 1, \quad g(x) = x + 1$

25. $f(x) = x^2 + 2x, g(x) = x + 2$

26. $f(x) = -x^2 + \frac{9}{2}x + 1, g(x) = \frac{1}{2}x + 1$

27. $y = x, \quad y = 2 - x, \quad y = 0$

28. $y = 1/x^2, \quad y = 0, \quad x = 1, \quad x = 5$

29. $f(x) = \sqrt{x} + 3, g(x) = \frac{1}{2}x + 3$

30. $f(x) = \sqrt[3]{x - 1}, \ g(x) = x - 1$

31. $f(y) = y^2, \quad g(y) = y + 2$

32. $f(y) = y(2 - y), \quad g(y) = -y$

33. $f(y) = y^2 + 1, \quad g(y) = 0, \quad y = -1, \quad y = 2$

34. $f(y) = \frac{y}{\sqrt{16 - y^2}}, \quad g(y) = 0, \quad y = 3$

35. $f(x) = \frac{10}{x}, \ x = 0, \ y = 2, \ y = 10$

36. $g(x) = \frac{4}{2 - x}, \quad y = 4, \quad x = 0$

In Exercises 37–46, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) find the area of the region, and (c) use the integration capabilities of the graphing utility to verify your results.

37. $f(x) = x(x^2 - 3x + 3), \quad g(x) = x^2$

38. $f(x) = x^3 - 2x + 1, \quad g(x) = -2x, \quad x = 1$

39. $y = x^2 - 4x + 3, \quad y = 3 + 4x - x^2$

40. $y = x^4 - 2x^2, \quad y = 2x^2$

41. $f(x) = x^4 - 4x^2, \quad g(x) = x^2 - 4$

42. $f(x) = x^4 - 4x^2, \quad g(x) = x^3 - 4x$

43. $f(x) = 1/(1 + x^2), \quad g(x) = \frac{1}{2}x^2$

44. $f(x) = 6x/(x^2 + 1), \quad y = 0, \quad 0 \le x \le 3$

45. $y = \sqrt{1 + x^3}, \quad y = \frac{1}{2}x + 2, \quad x = 0$

46. $y = x\sqrt{\dfrac{4 - x}{4 + x}}, \quad y = 0, \quad x = 4$

In Exercises 47–52, sketch the region bounded by the graphs of the functions, and find the area of the region.

47. $f(x) = \cos x, \; g(x) = 2 - \cos x, \; 0 \le x \le 2\pi$

48. $f(x) = \sin x, \quad g(x) = \cos 2x, \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{6}$

49. $f(x) = 2 \sin x, \quad g(x) = \tan x, \quad -\dfrac{\pi}{3} \le x \le \dfrac{\pi}{3}$

50. $f(x) = \sec \dfrac{\pi x}{4} \tan \dfrac{\pi x}{4}, \; g(x) = \left(\sqrt{2} - 4\right)x + 4, \quad x = 0$

51. $f(x) = xe^{-x^2}, \quad y = 0, \quad 0 \le x \le 1$

52. $f(x) = 3^x, \quad g(x) = 2x + 1$

In Exercises 53–56, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) find the area of the region, and (c) use the integration capabilities of the graphing utility to verify your results.

53. $f(x) = 2 \sin x + \sin 2x, \quad y = 0, \quad 0 \le x \le \pi$

54. $f(x) = 2 \sin x + \cos 2x, \quad y = 0, \quad 0 < x \le \pi$

55. $f(x) = \dfrac{1}{x^2}e^{1/x}, \quad y = 0, \quad 1 \le x \le 3$

56. $g(x) = \dfrac{4 \ln x}{x}, \quad y = 0, \quad x = 5$

In Exercises 57–60, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) explain why the area of the region is difficult to find by hand, and (c) use the integration capabilities of the graphing utility to approximate the area to four decimal places.

57. $y = \sqrt{\dfrac{x^3}{4 - x}}, \quad y = 0, \quad x = 3$

58. $y = \sqrt{x}\, e^x, \quad y = 0, \quad x = 0, \quad x = 1$

59. $y = x^2, \quad y = 4 \cos x$

60. $y = x^2, \quad y = \sqrt{3 + x}$

In Exercises 61–64, find the accumulation function *F*. Then evaluate *F* at each value of the independent variable and graphically show the area given by each value of *F*.

61. $F(x) = \displaystyle\int_0^x \left(\tfrac{1}{2}t + 1\right) dt$ (a) $F(0)$ (b) $F(2)$ (c) $F(6)$

62. $F(x) = \displaystyle\int_0^x \left(\tfrac{1}{2}t^2 + 2\right) dt$ (a) $F(0)$ (b) $F(4)$ (c) $F(6)$

63. $F(\alpha) = \displaystyle\int_{-1}^{\alpha} \cos \frac{\pi\theta}{2}\, d\theta$ (a) $F(-1)$ (b) $F(0)$ (c) $F\left(\tfrac{1}{2}\right)$

64. $F(y) = \displaystyle\int_{-1}^{y} 4e^{x/2}\, dx$ (a) $F(-1)$ (b) $F(0)$ (c) $F(4)$

In Exercises 65–68, use integration to find the area of the figure having the given vertices.

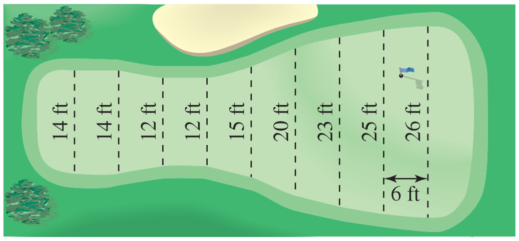
65. $(2, -3), (4, 6), (6, 1)$

66. $(0, 0), (a, 0), (b, c)$

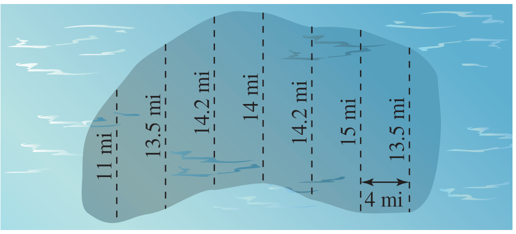
67. $(0, 2), (4, 2), (0, -2), (-4, -2)$

68. $(0, 0), (1, 2), (3, -2), (1, -3)$

69. ***Numerical Integration*** Estimate the surface area of the golf green using (a) the Trapezoidal Rule and (b) Simpson's Rule.



70. ***Numerical Integration*** Estimate the surface area of the oil spill using (a) the Trapezoidal Rule and (b) Simpson's Rule.



In Exercises 71 and 72, evaluate the integral and interpret it as the area of a region. Then use a graphing utility to graph the region.

71. $\displaystyle\int_0^{\pi/4} |\sin 2x - \cos 4x|\, dx$

72. $\displaystyle\int_0^2 \left|\sqrt{x + 3} - 2x\right| dx$

In Exercises 73–76, set up and evaluate the definite integral that gives the area of the region bounded by the graph of the function and the tangent line to the graph at the given point.

73. $f(x) = x^3, \quad (1, 1)$

74. $y = x^3 - 2x, \quad (-1, 1)$

75. $f(x) = \dfrac{1}{x^2 + 1}, \quad \left(1, \dfrac{1}{2}\right)$

76. $y = \dfrac{2}{1 + 4x^2}, \quad \left(\dfrac{1}{2}, 1\right)$

WRITING ABOUT CONCEPTS

77. The graphs of $y = x^4 - 2x^2 + 1$ and $y = 1 - x^2$ intersect at three points. However, the area between the curves *can* be found by a single integral. Explain why this is so, and write an integral for this area.

78. The area of the region bounded by the graphs of $y = x^3$ and $y = x$ *cannot* be found by the single integral $\int_{-1}^{1}(x^3 - x)\,dx$. Explain why this is so. Use symmetry to write a single integral that does represent the area.

79. A college graduate has two job offers. The starting salary for each is \$32,000, and after 8 years of service each will pay \$54,000. The salary increase for each offer is shown in the figure. From a strictly monetary viewpoint, which is the better offer? Explain.

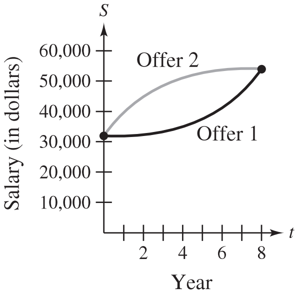


Figure for 79

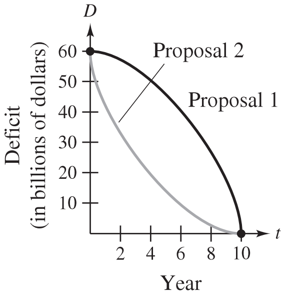


Figure for 80

80. A state legislature is debating two proposals for eliminating the annual budget deficits after 10 years. The rate of decrease of the deficits for each proposal is shown in the figure. From the viewpoint of minimizing the cumulative state deficit, which is the better proposal? Explain.

81. Two cars are tested on a straight track with velocities v_1 and v_2 (in meters per second). Consider the following.

$$\int_0^5 [v_1(t) - v_2(t)]\,dt = 10 \qquad \int_0^{10} [v_1(t) - v_2(t)]\,dt = 30$$

$$\int_{20}^{30} [v_1(t) - v_2(t)]\,dt = -5$$

(a) Write a verbal interpretation of each integral.

(b) Is it possible to determine the distance between the two cars when $t = 5$ seconds? Why or why not?

(c) Assume both cars start at the same time and place. Which car is ahead when $t = 10$ seconds? How far ahead is the car?

(d) Suppose Car 1 has velocity v_1 and is ahead of Car 2 by 13 meters when $t = 20$ seconds. How far ahead or behind is Car 1 when $t = 30$ seconds?

CAPSTONE

82. Let f and g be continuous functions on $[a, b]$ and let $g(x) \le f(x)$ for all x in $[a, b]$. Write in words the area given by $\int_a^b [f(x) - g(x)]\,dx$. Does the area interpretation of this integral change when $f(x) \ge 0$ and $g(x) \le 0$?

In Exercises 83 and 84, find *b* such that the line *y* = *b* divides the region bounded by the graphs of the two equations into two regions of equal area.

83. $y = 9 - x^2, \quad y = 0$ **84.** $y = 9 - |x|, \quad y = 0$

In Exercises 85 and 86, find *a* such that the line *x* = *a* divides the region bounded by the graphs of the equations into two regions of equal area.

85. $y = x, \quad y = 4, \quad x = 0$ **86.** $y^2 = 4 - x, \quad x = 0$

In Exercises 87 and 88, evaluate the limit and sketch the graph of the region whose area is represented by the limit.

87. $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(x_i - x_i^2)\,\Delta x$, where $x_i = i/n$ and $\Delta x = 1/n$

88. $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(4 - x_i^2)\,\Delta x$, where $x_i = -2 + (4i/n)$ and $\Delta x = 4/n$

In Exercises 89 and 90, (a) find the two points of inflection of the graph of *f*, (b) determine the equation of the line that intersects both points, and (c) calculate the areas of the three regions bounded by the graphs of *f* and the line. What do you observe?

89. $f(x) = x^4 + 4x^3 + x + 7$ **90.** $f(x) = 2x^4 - 12x^2 + 3x - 1$

Revenue **In Exercises 91 and 92, two models R_1 and R_2 are given for revenue (in billions of dollars per year) for a large corporation. The model R_1 gives projected annual revenues from 2008 through 2013, with $t = 8$ corresponding to 2008, and R_2 gives projected revenues if there is a decrease in the rate of growth of corporate sales over the period. Approximate the total reduction in revenue if corporate sales are actually closer to the model R_2.**

91. $R_1 = 7.21 + 0.58t$
$R_2 = 7.21 + 0.45t$

92. $R_1 = 7.21 + 0.26t + 0.02t^2$
$R_2 = 7.21 + 0.1t + 0.01t^2$

93. ***Lorenz Curve*** Economists use *Lorenz curves* to illustrate the distribution of income in a country. A Lorenz curve, $y = f(x)$, represents the actual income distribution in the country. In this model, x represents percents of families in the country and y represents percents of total income. The model $y = x$ represents a country in which each family has the same income. The area between these two models, where $0 \le x \le 100$, indicates a country's "income inequality." The table lists percents of income y for selected percents of families x in a country.

x	10	20	30	40	50
y	3.35	6.07	9.17	13.39	19.45

x	60	70	80	90
y	28.03	39.77	55.28	75.12

(a) Use a graphing utility to find a quadratic model for the Lorenz curve.

(b) Plot the data and graph the model.

(c) Graph the model $y = x$. How does this model compare with the model in part (a)?

(d) Use the integration capabilities of a graphing utility to approximate the "income inequality."

94. ***Profit*** The chief financial officer of a company reports that profits for the past fiscal year were \$15.9 million. The officer predicts that profits for the next 5 years will grow at a continuous annual rate somewhere between $3\frac{1}{2}\%$ and 5%. Estimate the cumulative difference in total profit over the 5 years based on the predicted range of growth rates.

95. ***Area*** The shaded region in the figure consists of all points whose distances from the center of the square are less than their distances from the edges of the square. Find the area of the region.

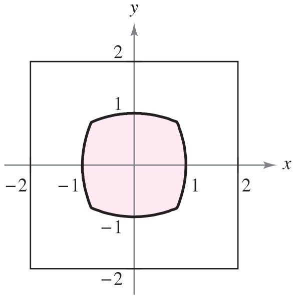


Figure for 95

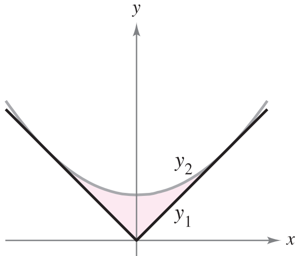


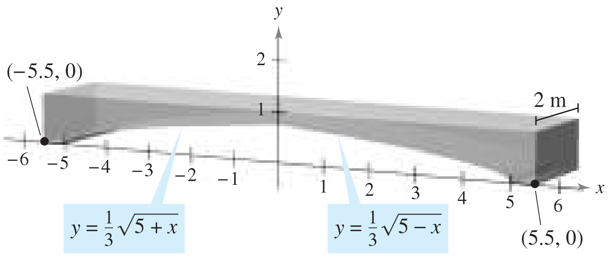
Figure for 96

96. ***Mechanical Design*** The surface of a machine part is the region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 + k$ (see figure).

(a) Find k where the parabola is tangent to the graph of y_1.

(b) Find the area of the surface of the machine part.

97. ***Building Design*** Concrete sections for a new building have the dimensions (in meters) and shape shown in the figure.



(a) Find the area of the face of the section superimposed on the rectangular coordinate system.

(b) Find the volume of concrete in one of the sections by multiplying the area in part (a) by 2 meters.

(c) One cubic meter of concrete weighs 5000 pounds. Find the weight of the section.

98. ***Building Design*** To decrease the weight and to aid in the hardening process, the concrete sections in Exercise 97 often are not completely solid. Rework Exercise 97 to allow for cylindrical openings such as those shown in the figure.

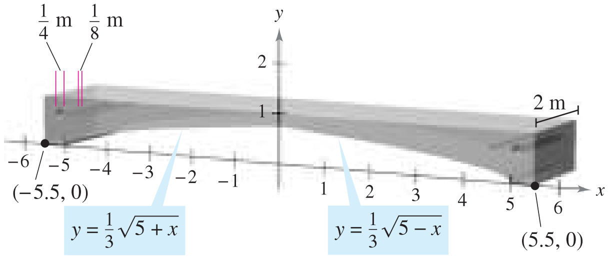


Figure for 98

True or False? **In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

99. If the area of the region bounded by the graphs of f and g is 1, then the area of the region bounded by the graphs of $h(x) = f(x) + C$ and $k(x) = g(x) + C$ is also 1.

100. If $\int_a^b [f(x) - g(x)]\,dx = A$, then $\int_a^b [g(x) - f(x)]\,dx = -A$.

101. If the graphs of f and g intersect midway between $x = a$ and $x = b$, then $\int_a^b [f(x) - g(x)]\,dx = 0$.

102. The line $y = \left(1 - \sqrt[3]{0.5}\right)x$ divides the region under the curve $f(x) = x(1 - x)$ on $[0, 1]$ into two regions of equal area.

103. ***Area*** Find the area between the graph of $y = \sin x$ and the line segment joining the points $(0, 0)$ and $\left(\frac{7\pi}{6}, -\frac{1}{2}\right)$, as shown in the figure.

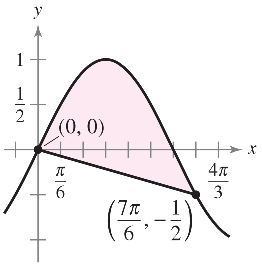


Figure for 103

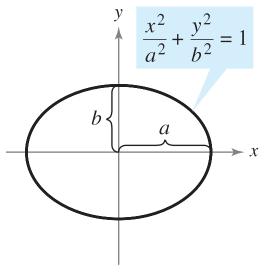
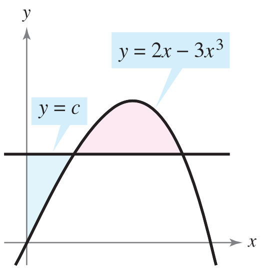


Figure for 104

104. ***Area*** Let $a > 0$ and $b > 0$. Show that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab (see figure).

PUTNAM EXAM CHALLENGE

105. The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as shown in the figure. Find c so that the areas of the two shaded regions are equal.



This problem was composed by the Committee on the Putnam Prize Competition.
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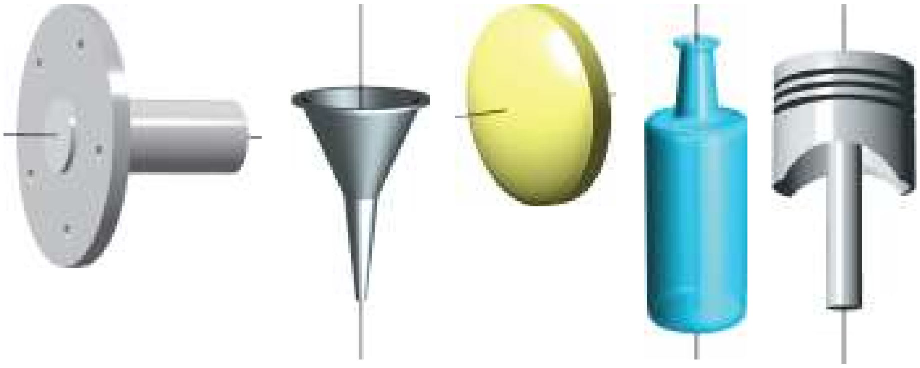
7.2 Volume: The Disk Method

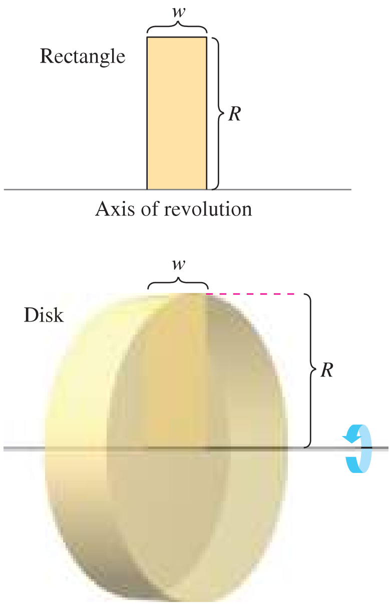
- **Find the volume of a solid of revolution using the disk method.**
- **Find the volume of a solid of revolution using the washer method.**
- **Find the volume of a solid with known cross sections.**

The Disk Method

You have already learned that area is only one of the *many* applications of the definite integral. Another important application is its use in finding the volume of a three-dimensional solid. In this section you will study a particular type of three-dimensional solid—one whose cross sections are similar. Solids of revolution are used commonly in engineering and manufacturing. Some examples are axles, funnels, pills, bottles, and pistons, as shown in Figure 7.12.



Solids of revolution
Figure 7.12



Volume of a disk: $\pi R^2 w$
Figure 7.13

If a region in the plane is revolved about a line, the resulting solid is a **solid of revolution,** and the line is called the **axis of revolution.** The simplest such solid is a right circular cylinder or **disk,** which is formed by revolving a rectangle about an axis adjacent to one side of the rectangle, as shown in Figure 7.13. The volume of such a disk is

$$\begin{aligned}\text{Volume of disk} &= (\text{area of disk})(\text{width of disk})\\ &= \pi R^2 w\end{aligned}$$

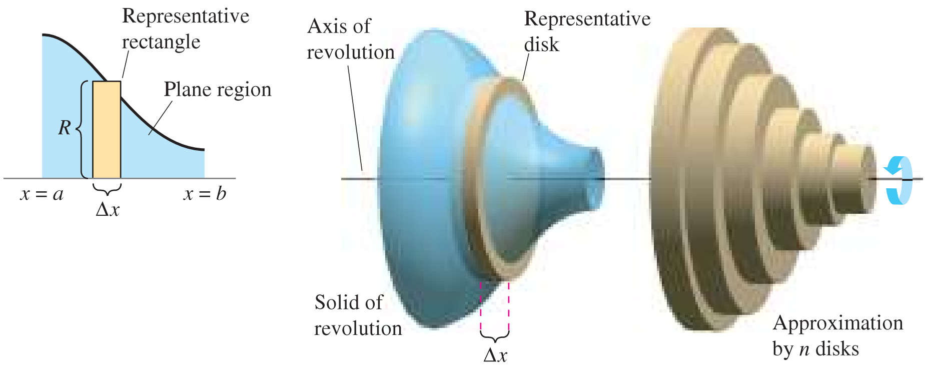
where R is the radius of the disk and w is the width.

To see how to use the volume of a disk to find the volume of a general solid of revolution, consider a solid of revolution formed by revolving the plane region in Figure 7.14 about the indicated axis. To determine the volume of this solid, consider a representative rectangle in the plane region. When this rectangle is revolved about the axis of revolution, it generates a representative disk whose volume is

$$\Delta V = \pi R^2 \Delta x.$$

Approximating the volume of the solid by n such disks of width Δx and radius $R(x_i)$ produces

$$\begin{aligned}\text{Volume of solid} &\approx \sum_{i=1}^{n} \pi[R(x_i)]^2\,\Delta x\\ &= \pi\sum_{i=1}^{n}[R(x_i)]^2\,\Delta x.\end{aligned}$$


Representative rectangle
Plane region
R
$x = a$
Δx
$x = b$
Axis of revolution
Representative disk
Solid of revolution
Δx
Approximation by n disks

Disk method
Figure 7.14

This approximation appears to become better and better as $\|\Delta\| \to 0$ $(n \to \infty)$. So, you can define the volume of the solid as

$$\text{Volume of solid} = \lim_{\|\Delta\| \to 0} \pi \sum_{i=1}^{n} [R(x_i)]^2 \, \Delta x = \pi \int_a^b [R(x)]^2 \, dx.$$

Schematically, the disk method looks like this.

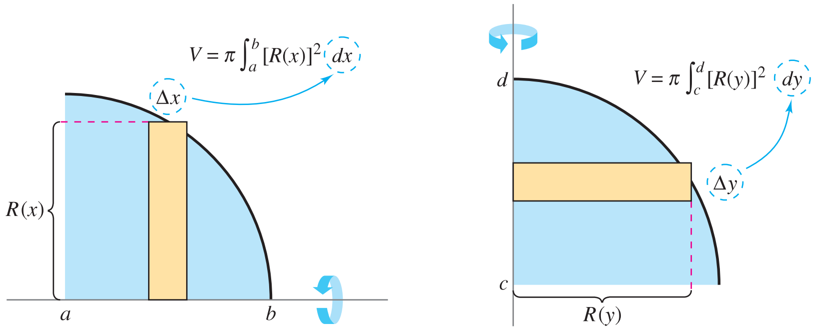
Known Precalculus Formula		*Representative Element*		*New Integration Formula*
Volume of disk $V = \pi R^2 w$	⇨	$\Delta V = \pi[R(x_i)]^2 \, \Delta x$	⇨	Solid of revolution $V = \pi \int_a^b [R(x)]^2 \, dx$

A similar formula can be derived if the axis of revolution is vertical.

THE DISK METHOD

To find the volume of a solid of revolution with the **disk method,** use one of the following, as shown in Figure 7.15.

Horizontal Axis of Revolution	*Vertical Axis of Revolution*
$\text{Volume} = V = \pi \int_a^b [R(x)]^2 \, dx$	$\text{Volume} = V = \pi \int_c^d [R(y)]^2 \, dy$


$V = \pi \int_a^b [R(x)]^2 \, dx$
Δx
$R(x)$
a
b
d
$V = \pi \int_c^d [R(y)]^2 \, dy$
Δy
c
$R(y)$

Horizontal axis of revolution

Vertical axis of revolution

Figure 7.15

NOTE In Figure 7.15, note that you can determine the variable of integration by placing a representative rectangle in the plane region "perpendicular" to the axis of revolution. If the width of the rectangle is Δx, integrate with respect to x, and if the width of the rectangle is Δy, integrate with respect to y.

The simplest application of the disk method involves a plane region bounded by the graph of f and the x-axis. If the axis of revolution is the x-axis, the radius $R(x)$ is simply $f(x)$.

EXAMPLE 1 Using the Disk Method

Find the volume of the solid formed by revolving the region bounded by the graph of

$$f(x) = \sqrt{\sin x}$$

and the x-axis $(0 \le x \le \pi)$ about the x-axis.

Solution From the representative rectangle in the upper graph in Figure 7.16, you can see that the radius of this solid is

$$\begin{aligned} R(x) &= f(x) \\ &= \sqrt{\sin x}. \end{aligned}$$

So, the volume of the solid of revolution is

$$\begin{aligned} V = \pi\int_a^b [R(x)]^2\,dx &= \pi\int_0^{\pi} \left(\sqrt{\sin x}\right)^2 dx && \text{Apply disk method.} \\ &= \pi\int_0^{\pi} \sin x\,dx && \text{Simplify.} \\ &= \pi\Big[-\cos x\Big]_0^{\pi} && \text{Integrate.} \\ &= \pi(1 + 1) \\ &= 2\pi. \end{aligned}$$

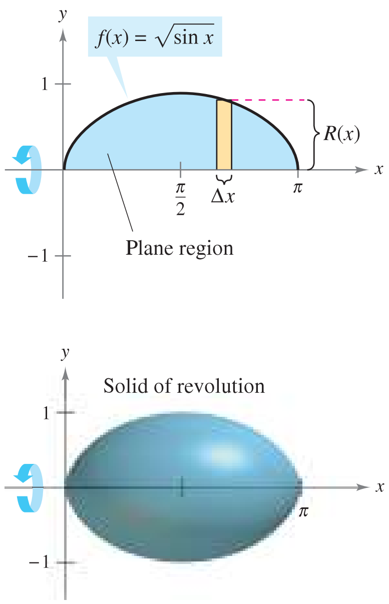

y
$f(x) = \sqrt{\sin x}$
1
$R(x)$
x
$\frac{\pi}{2}$
Δx
π
−1
Plane region
y
Solid of revolution
1
x
π
−1

Figure 7.16

EXAMPLE 2 Revolving About a Line That Is Not a Coordinate Axis

Find the volume of the solid formed by revolving the region bounded by

$$f(x) = 2 - x^2$$

and $g(x) = 1$ about the line $y = 1$, as shown in Figure 7.17.

Solution By equating $f(x)$ and $g(x)$, you can determine that the two graphs intersect when $x = \pm 1$. To find the radius, subtract $g(x)$ from $f(x)$.

$$\begin{aligned} R(x) &= f(x) - g(x) \\ &= (2 - x^2) - 1 \\ &= 1 - x^2 \end{aligned}$$

Finally, integrate between -1 and 1 to find the volume.

$$\begin{aligned} V = \pi\int_a^b [R(x)]^2\,dx &= \pi\int_{-1}^{1} (1 - x^2)^2\,dx && \text{Apply disk method.} \\ &= \pi\int_{-1}^{1} (1 - 2x^2 + x^4)\,dx && \text{Simplify.} \\ &= \pi\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1} && \text{Integrate.} \\ &= \frac{16\pi}{15} \end{aligned}$$

■

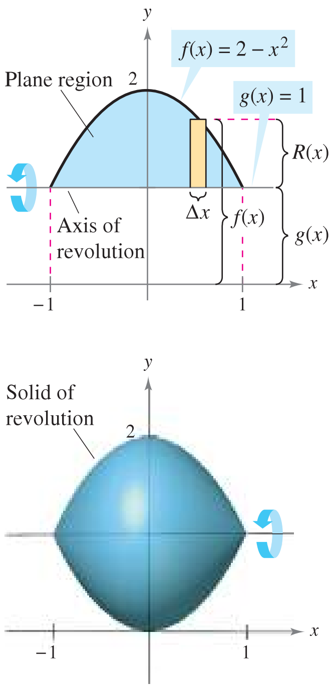

y
$f(x) = 2 - x^2$
Plane region
2
$g(x) = 1$
$R(x)$
Δx
$f(x)$
Axis of revolution
$g(x)$
x
−1
1
y
Solid of revolution
2
x
−1
1

Figure 7.17

The Washer Method

The disk method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative **washer.** The washer is formed by revolving a rectangle about an axis, as shown in Figure 7.18. If r and R are the inner and outer radii of the washer and w is the width of the washer, the volume is given by

$$\text{Volume of washer} = \pi(R^2 - r^2)w.$$

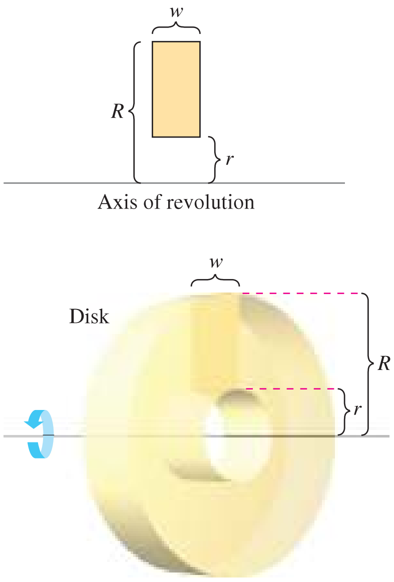

w
R
r
Axis of revolution
w
Disk
R
r
Solid of revolution

Figure 7.18

To see how this concept can be used to find the volume of a solid of revolution, consider a region bounded by an **outer radius** $R(x)$ and an **inner radius** $r(x)$, as shown in Figure 7.19. If the region is revolved about its axis of revolution, the volume of the resulting solid is given by

$$\boxed{V = \pi\int_a^b ([R(x)]^2 - [r(x)]^2)\,dx.}$$

Washer method

Note that the integral involving the inner radius represents the volume of the hole and is *subtracted* from the integral involving the outer radius.

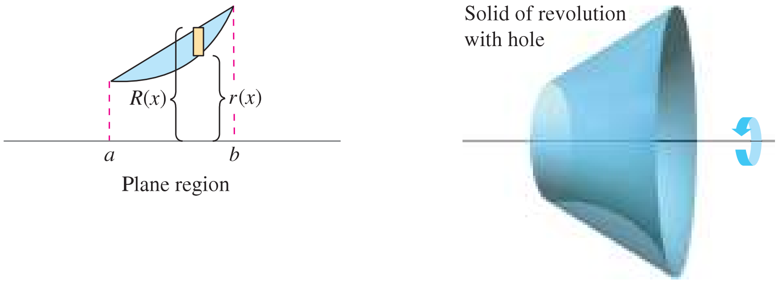

R(x)
r(x)
a
b
Plane region
Solid of revolution with hole

Figure 7.19

EXAMPLE 3 Using the Washer Method

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis, as shown in Figure 7.20.

Solution In Figure 7.20, you can see that the outer and inner radii are as follows.

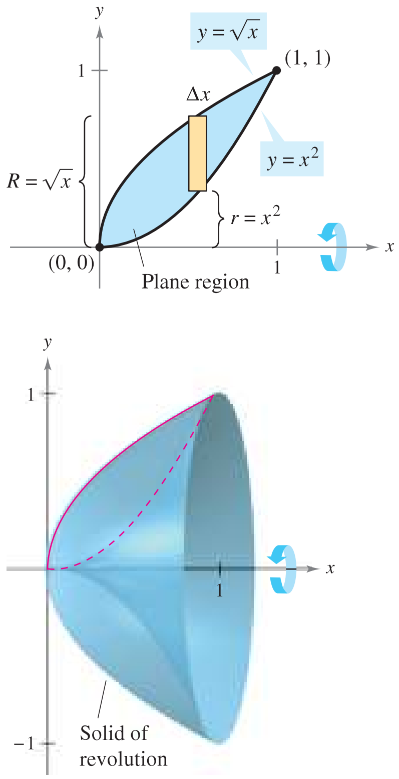
$$R(x) = \sqrt{x} \qquad \text{Outer radius}$$

$$r(x) = x^2 \qquad \text{Inner radius}$$

Integrating between 0 and 1 produces

$$\begin{aligned} V &= \pi\int_a^b ([R(x)]^2 - [r(x)]^2)\,dx && \text{Apply washer method.} \\ &= \pi\int_0^1 \left[\left(\sqrt{x}\right)^2 - (x^2)^2\right] dx \\ &= \pi\int_0^1 (x - x^4)\,dx && \text{Simplify.} \\ &= \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 && \text{Integrate.} \\ &= \frac{3\pi}{10}. \end{aligned}$$

■


y
$y = \sqrt{x}$
(1, 1)
1
Δx
$R = \sqrt{x}$
$y = x^2$
$r = x^2$
(0, 0)
Plane region
1
x
y
1
1
x
−1
Solid of revolution

Solid of revolution
Figure 7.20

In each example so far, the axis of revolution has been *horizontal* and you have integrated with respect to x. In the next example, the axis of revolution is *vertical* and you integrate with respect to y. In this example, you need two separate integrals to compute the volume.

EXAMPLE 4 Integrating with Respect to y, Two-Integral Case

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis, as shown in Figure 7.21.

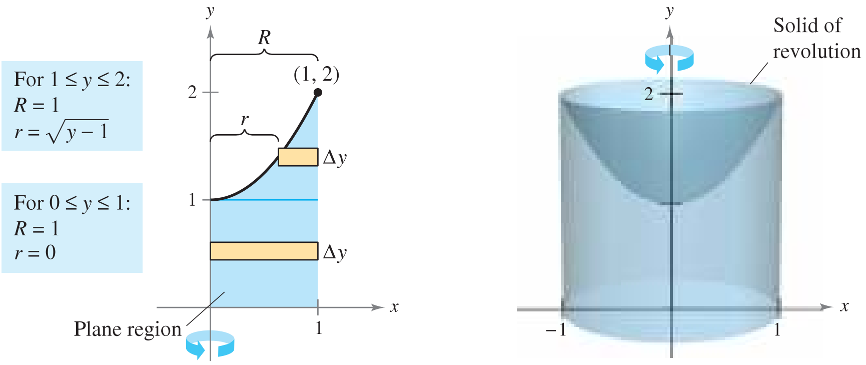


Figure 7.21

Solution For the region shown in Figure 7.21, the outer radius is simply $R = 1$. There is, however, no convenient formula that represents the inner radius. When $0 \le y \le 1$, $r = 0$, but when $1 \le y \le 2$, r is determined by the equation $y = x^2 + 1$, which implies that $r = \sqrt{y-1}$.

$$r(y) = \begin{cases} 0, & 0 \le y \le 1 \\ \sqrt{y-1}, & 1 \le y \le 2 \end{cases}$$

Using this definition of the inner radius, you can use two integrals to find the volume.

$$\begin{aligned} V &= \pi \int_0^1 (1^2 - 0^2)\, dy + \pi \int_1^2 \left[1^2 - \left(\sqrt{y-1}\right)^2\right] dy && \text{Apply washer method.} \\ &= \pi \int_0^1 1\, dy + \pi \int_1^2 (2 - y)\, dy && \text{Simplify.} \\ &= \pi \Big[y\Big]_0^1 + \pi \left[2y - \frac{y^2}{2}\right]_1^2 && \text{Integrate.} \\ &= \pi + \pi\left(4 - 2 - 2 + \frac{1}{2}\right) = \frac{3\pi}{2} \end{aligned}$$

Note that the first integral $\pi \int_0^1 1\, dy$ represents the volume of a right circular cylinder of radius 1 and height 1. This portion of the volume could have been determined without using calculus. ■

TECHNOLOGY Some graphing utilities have the capability of generating (or have built-in software capable of generating) a solid of revolution. If you have access to such a utility, use it to graph some of the solids of revolution described in this section. For instance, the solid in Example 4 might appear like that shown in Figure 7.22.

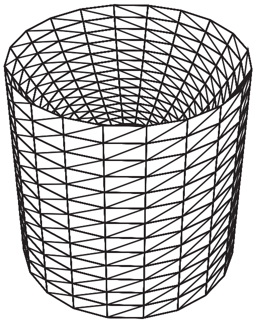
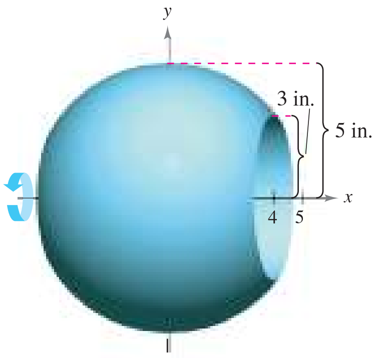
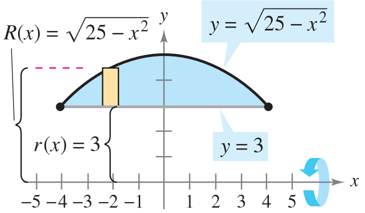

Generated by Mathematica

Figure 7.22



Solid of revolution

(a)



Plane region

(b)

Figure 7.23

EXAMPLE 5 Manufacturing

A manufacturer drills a hole through the center of a metal sphere of radius 5 inches, as shown in Figure 7.23(a). The hole has a radius of 3 inches. What is the volume of the resulting metal ring?

Solution You can imagine the ring to be generated by a segment of the circle whose equation is $x^2 + y^2 = 25$, as shown in Figure 7.23(b). Because the radius of the hole is 3 inches, you can let $y = 3$ and solve the equation $x^2 + y^2 = 25$ to determine that the limits of integration are $x = \pm 4$. So, the inner and outer radii are $r(x) = 3$ and $R(x) = \sqrt{25 - x^2}$ and the volume is given by

$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-4}^{4} \left[\left(\sqrt{25 - x^2}\right)^2 - (3)^2\right] dx$$

$$= \pi \int_{-4}^{4} (16 - x^2)\, dx$$

$$= \pi \left[16x - \frac{x^3}{3}\right]_{-4}^{4}$$

$$= \frac{256\pi}{3} \text{ cubic inches.}$$

■

Solids with Known Cross Sections

With the disk method, you can find the volume of a solid having a circular cross section whose area is $A = \pi R^2$. This method can be generalized to solids of any shape, as long as you know a formula for the area of an arbitrary cross section. Some common cross sections are squares, rectangles, triangles, semicircles, and trapezoids.

VOLUMES OF SOLIDS WITH KNOWN CROSS SECTIONS

1. For cross sections of area $A(x)$ taken perpendicular to the x-axis,

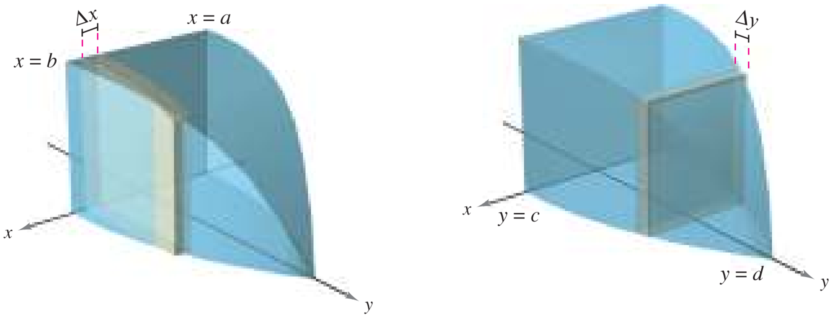
$$\text{Volume} = \int_a^b A(x)\, dx.$$

See Figure 7.24(a).

2. For cross sections of area $A(y)$ taken perpendicular to the y-axis,

$$\text{Volume} = \int_c^d A(y)\, dy.$$

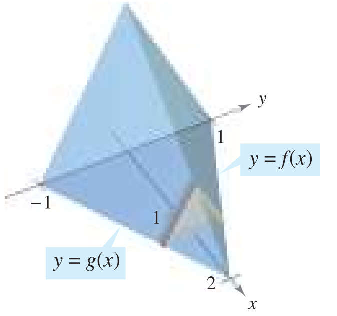
See Figure 7.24(b).



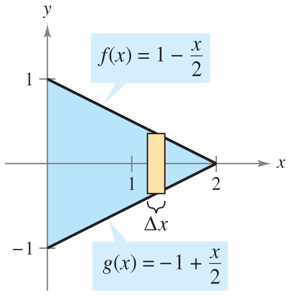
(a) Cross sections perpendicular to x-axis

(b) Cross sections perpendicular to y-axis

Figure 7.24



Cross sections are equilateral triangles.



Triangular base in *xy*-plane
Figure 7.25

EXAMPLE 6 Triangular Cross Sections

Find the volume of the solid shown in Figure 7.25. The base of the solid is the region bounded by the lines

$$f(x) = 1 - \frac{x}{2}, \qquad g(x) = -1 + \frac{x}{2}, \qquad \text{and} \qquad x = 0.$$

The cross sections perpendicular to the x-axis are equilateral triangles.

Solution The base and area of each triangular cross section are as follows.

$$\text{Base} = \left(1 - \frac{x}{2}\right) - \left(-1 + \frac{x}{2}\right) = 2 - x \qquad \text{Length of base}$$

$$\text{Area} = \frac{\sqrt{3}}{4}(\text{base})^2 \qquad \text{Area of equilateral triangle}$$

$$A(x) = \frac{\sqrt{3}}{4}(2 - x)^2 \qquad \text{Area of cross section}$$

Because x ranges from 0 to 2, the volume of the solid is

$$\begin{aligned} V &= \int_a^b A(x)\,dx = \int_0^2 \frac{\sqrt{3}}{4}(2 - x)^2\,dx \\ &= -\frac{\sqrt{3}}{4}\left[\frac{(2 - x)^3}{3}\right]_0^2 = \frac{2\sqrt{3}}{3}. \end{aligned}$$

EXAMPLE 7 An Application to Geometry

Prove that the volume of a pyramid with a square base is $V = \frac{1}{3}hB$, where h is the height of the pyramid and B is the area of the base.

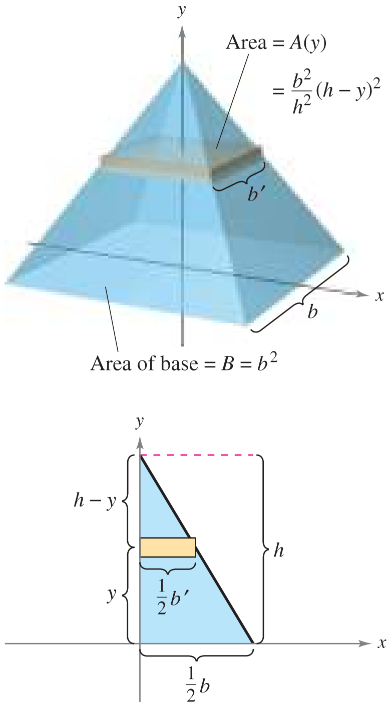


Figure 7.26

Solution As shown in Figure 7.26, you can intersect the pyramid with a plane parallel to the base at height y to form a square cross section whose sides are of length b'. Using similar triangles, you can show that

$$\frac{b'}{b} = \frac{h - y}{h} \qquad \text{or} \qquad b' = \frac{b}{h}(h - y)$$

where b is the length of the sides of the base of the pyramid. So,

$$A(y) = (b')^2 = \frac{b^2}{h^2}(h - y)^2.$$

Integrating between 0 and h produces

$$\begin{aligned} V &= \int_0^h A(y)\,dy = \int_0^h \frac{b^2}{h^2}(h - y)^2\,dy \\ &= \frac{b^2}{h^2}\int_0^h (h - y)^2\,dy \\ &= -\left(\frac{b^2}{h^2}\right)\left[\frac{(h - y)^3}{3}\right]_0^h \\ &= \frac{b^2}{h^2}\left(\frac{h^3}{3}\right) \\ &= \frac{1}{3}hB. \qquad B = b^2 \end{aligned}$$

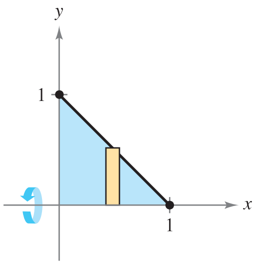
■

7.2 Exercises

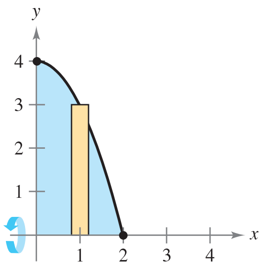
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the *x*-axis.

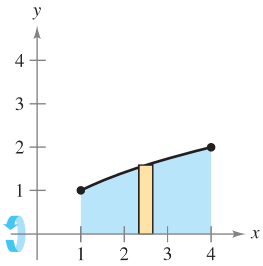
1. $y = -x + 1$



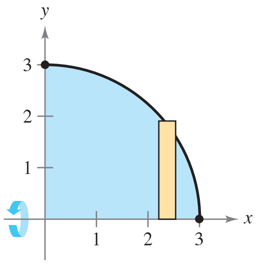
2. $y = 4 - x^2$



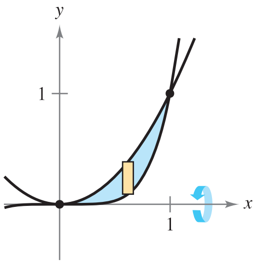
3. $y = \sqrt{x}$



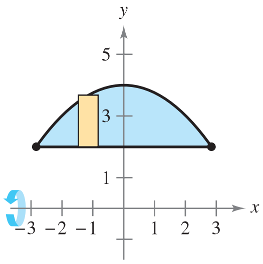
4. $y = \sqrt{9 - x^2}$



5. $y = x^2, \;\; y = x^5$

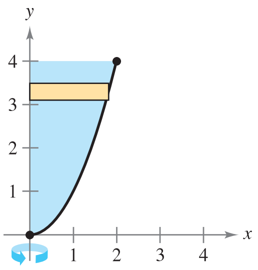


6. $y = 2, \;\; y = 4 - \dfrac{x^2}{4}$

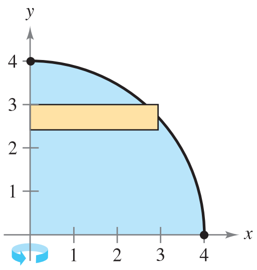


In Exercises 7–10, set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the *y*-axis.

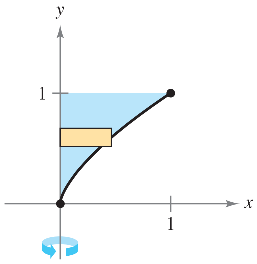
7. $y = x^2$



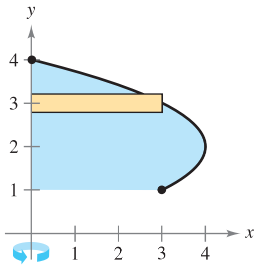
8. $y = \sqrt{16 - x^2}$



9. $y = x^{2/3}$



10. $x = -y^2 + 4y$



In Exercises 11–14, find the volumes of the solids generated by revolving the regions bounded by the graphs of the equations about the given lines.

11. $y = \sqrt{x}, \;\; y = 0, \;\; x = 3$

(a) the x-axis (b) the y-axis

(c) the line $x = 3$ (d) the line $x = 6$

12. $y = 2x^2, \;\; y = 0, \;\; x = 2$

(a) the y-axis (b) the x-axis

(c) the line $y = 8$ (d) the line $x = 2$

13. $y = x^2, \;\; y = 4x - x^2$

(a) the x-axis (b) the line $y = 6$

14. $y = 6 - 2x - x^2, \;\; y = x + 6$

(a) the x-axis (b) the line $y = 3$

In Exercises 15–18, find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the line $y = 4$.

15. $y = x, \;\; y = 3, \;\; x = 0$

16. $y = \frac{1}{2}x^3, \;\; y = 4, \;\; x = 0$

17. $y = \dfrac{3}{1 + x}, \;\; y = 0, \;\; x = 0, \;\; x = 3$

18. $y = \sec x, \;\; y = 0, \;\; 0 \le x \le \dfrac{\pi}{3}$

In Exercises 19–22, find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the line $x = 5$.

19. $y = x, \;\; y = 0, \;\; y = 4, \;\; x = 5$

20. $y = 5 - x, \;\; y = 0, \;\; y = 4, \;\; x = 0$

21. $x = y^2, \;\; x = 4$

22. $xy = 5, \;\; y = 2, \;\; y = 5, \;\; x = 5$

In Exercises 23–30, find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *x*-axis.

23. $y = \dfrac{1}{\sqrt{x + 1}}, \;\; y = 0, \;\; x = 0, \;\; x = 4$

24. $y = x\sqrt{9 - x^2}, \;\; y = 0$

25. $y = \dfrac{1}{x},\quad y = 0,\quad x = 1,\quad x = 3$

26. $y = \dfrac{2}{x+1},\quad y = 0,\quad x = 0,\quad x = 6$

27. $y = e^{-x},\quad y = 0,\quad x = 0,\quad x = 1$

28. $y = e^{x/2},\quad y = 0,\quad x = 0,\quad x = 4$

29. $y = x^2 + 1,\quad y = -x^2 + 2x + 5,\quad x = 0,\quad x = 3$

30. $y = \sqrt{x},\quad y = -\frac{1}{2}x + 4,\quad x = 0,\quad x = 8$

In Exercises 31 and 32, find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *y*-axis.

31. $y = 3(2 - x),\quad y = 0,\quad x = 0$

32. $y = 9 - x^2,\quad y = 0,\quad x = 2,\quad x = 3$

In Exercises 33–36, find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *x*-axis. Verify your results using the integration capabilities of a graphing utility.

33. $y = \sin x,\quad y = 0,\quad x = 0,\quad x = \pi$

34. $y = \cos 2x,\quad y = 0,\quad x = 0,\quad x = \dfrac{\pi}{4}$

35. $y = e^{x-1},\quad y = 0,\quad x = 1,\quad x = 2$

36. $y = e^{x/2} + e^{-x/2},\quad y = 0,\quad x = -1,\quad x = 2$

In Exercises 37–40, use the integration capabilities of a graphing utility to approximate the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *x*-axis.

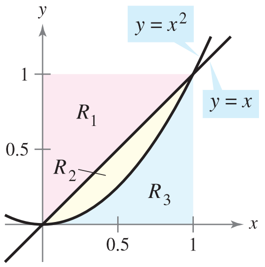
37. $y = e^{-x^2},\quad y = 0,\quad x = 0,\quad x = 2$

38. $y = \ln x,\quad y = 0,\quad x = 1,\quad x = 3$

39. $y = 2\arctan(0.2x),\quad y = 0,\quad x = 0,\quad x = 5$

40. $y = \sqrt{2x},\quad y = x^2$

In Exercises 41–48, find the volume generated by rotating the given region about the specified line.



41. R_1 about $x = 0$

42. R_1 about $x = 1$

43. R_2 about $y = 0$

44. R_2 about $y = 1$

45. R_3 about $x = 0$

46. R_3 about $x = 1$

47. R_2 about $x = 0$

48. R_2 about $x = 1$

Think About It **In Exercises 49 and 50, determine which value best approximates the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *x*-axis. (Make your selection on the basis of a sketch of the solid and *not* by performing any calculations.)**

49. $y = e^{-x^2/2},\quad y = 0,\quad x = 0,\quad x = 2$

(a) 3 (b) -5 (c) 10 (d) 7 (e) 20

50. $y = \arctan x,\quad y = 0,\quad x = 0,\quad x = 1$

(a) 10 (b) $\frac{3}{4}$ (c) 5 (d) -6 (e) 15

WRITING ABOUT CONCEPTS

In Exercises 51 and 52, the integral represents the volume of a solid. Describe the solid.

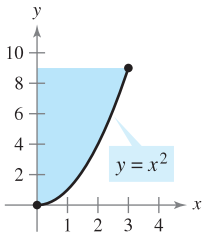
51. $\pi \displaystyle\int_0^{\pi/2} \sin^2 x\,dx$

52. $\pi \displaystyle\int_2^4 y^4\,dy$

53. A region bounded by the parabola $y = 4x - x^2$ and the x-axis is revolved about the x-axis. A second region bounded by the parabola $y = 4 - x^2$ and the x-axis is revolved about the x-axis. Without integrating, how do the volumes of the two solids compare? Explain.

54. The region in the figure is revolved about the indicated axes and line. Order the volumes of the resulting solids from least to greatest. Explain your reasoning.

(a) x-axis (b) y-axis (c) $x = 3$



55. Discuss the validity of the following statements.

(a) For a solid formed by rotating the region under a graph about the x-axis, the cross sections perpendicular to the x-axis are circular disks.

(b) For a solid formed by rotating the region between two graphs about the x-axis, the cross sections perpendicular to the x-axis are circular disks.

CAPSTONE

56. Identify the integral that represents the volume of the solid obtained by rotating the area between $y = f(x)$ and $y = g(x)$, $a \le x \le b$, about the x-axis. [Assume $f(x) \ge g(x) \ge 0$.]

(a) $\pi \displaystyle\int_a^b [f(x) - g(x)]^2\,dx$ (b) $\pi \displaystyle\int_a^b \left([f(x)]^2 - [g(x)]^2\right)dx$

57. If the portion of the line $y = \frac{1}{2}x$ lying in the first quadrant is revolved about the x-axis, a cone is generated. Find the volume of the cone extending from $x = 0$ to $x = 6$.

58. Use the disk method to verify that the volume of a right circular cone is $\frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.

59. Use the disk method to verify that the volume of a sphere is $\frac{4}{3}\pi r^3$.

60. A sphere of radius r is cut by a plane h $(h < r)$ units above the equator. Find the volume of the solid (spherical segment) above the plane.

61. A cone of height H with a base of radius r is cut by a plane parallel to and h units above the base. Find the volume of the solid (frustum of a cone) below the plane.

62. The region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$ is revolved about the x-axis.

(a) Find the value of x in the interval $[0, 4]$ that divides the solid into two parts of equal volume.

(b) Find the values of x in the interval $[0, 4]$ that divide the solid into three parts of equal volume.

63. ***Volume of a Fuel Tank*** A tank on the wing of a jet aircraft is formed by revolving the region bounded by the graph of $y = \frac{1}{8}x^2\sqrt{2 - x}$ and the x-axis $(0 \le x \le 2)$ about the x-axis, where x and y are measured in meters. Use a graphing utility to graph the function and find the volume of the tank.

64. ***Volume of a Lab Glass*** A glass container can be modeled by revolving the graph of

$$y = \begin{cases} \sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}, & 0 \le x \le 11.5 \\ 2.95, & 11.5 < x \le 15 \end{cases}$$

about the x-axis, where x and y are measured in centimeters. Use a graphing utility to graph the function and find the volume of the container.

65. Find the volumes of the solids (see figures) generated if the upper half of the ellipse $9x^2 + 25y^2 = 225$ is revolved about (a) the x-axis to form a prolate spheroid (shaped like a football), and (b) the y-axis to form an oblate spheroid (shaped like half of a candy).

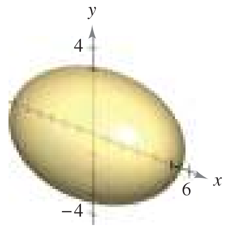


Figure for 65(a)

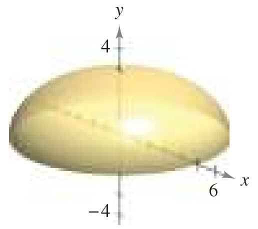


Figure for 65(b)

66. ***Water Depth in a Tank*** A tank on a water tower is a sphere of radius 50 feet. Determine the depths of the water when the tank is filled to one-fourth and three-fourths of its total capacity. (*Note:* Use the *zero* or *root* feature of a graphing utility after evaluating the definite integral.)

67. ***Minimum Volume*** The arc of $y = 4 - (x^2/4)$ on the interval $[0, 4]$ is revolved about the line $y = b$ (see figure).

(a) Find the volume of the resulting solid as a function of b.

(b) Use a graphing utility to graph the function in part (a), and use the graph to approximate the value of b that minimizes the volume of the solid.

(c) Use calculus to find the value of b that minimizes the volume of the solid, and compare the result with the answer to part (b).

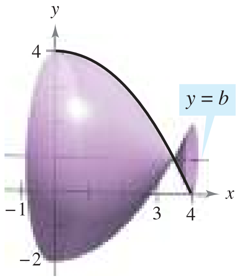


Figure for 67

Figure for 68

68. ***Modeling Data*** A draftsman is asked to determine the amount of material required to produce a machine part (see figure). The diameters d of the part at equally spaced points x are listed in the table. The measurements are listed in centimeters.

x	0	1	2	3	4	5	6	7	8	9	10
d	4.2	3.8	4.2	4.7	5.2	5.7	5.8	5.4	4.9	4.4	4.6

(a) Use these data with Simpson's Rule to approximate the volume of the part.

(b) Use the regression capabilities of a graphing utility to find a fourth-degree polynomial through the points representing the radius of the solid. Plot the data and graph the model.

(c) Use a graphing utility to approximate the definite integral yielding the volume of the part. Compare the result with the answer to part (a).

69. ***Think About It*** Match each integral with the solid whose volume it represents, and give the dimensions of each solid.

(a) Right circular cylinder (b) Ellipsoid

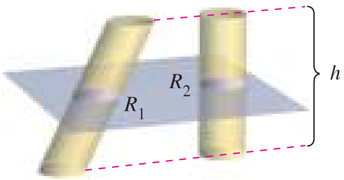
(c) Sphere (d) Right circular cone (e) Torus

(i) $\pi \int_0^h \left(\frac{rx}{h}\right)^2 dx$ (ii) $\pi \int_0^h r^2\, dx$

(iii) $\pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx$ (iv) $\pi \int_{-b}^{b} \left(a\sqrt{1 - \frac{x^2}{b^2}}\right)^2 dx$

(v) $\pi \int_{-r}^{r} \left[\left(R + \sqrt{r^2 - x^2}\right)^2 - \left(R - \sqrt{r^2 - x^2}\right)^2\right] dx$

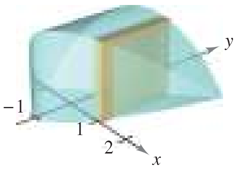
70. ***Cavalieri's Theorem*** Prove that if two solids have equal altitudes and all plane sections parallel to their bases and at equal distances from their bases have equal areas, then the solids have the same volume (see figure).



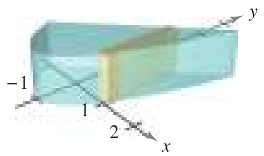
Area of R_1 = area of R_2

71. Find the volumes of the solids whose bases are bounded by the graphs of $y = x + 1$ and $y = x^2 - 1$, with the indicated cross sections taken perpendicular to the x-axis.

(a) Squares

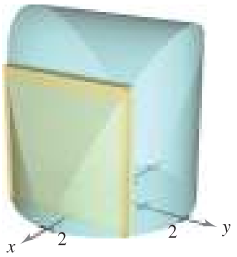


(b) Rectangles of height 1

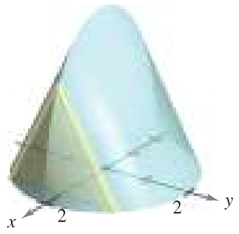


72. Find the volumes of the solids whose bases are bounded by the circle $x^2 + y^2 = 4$, with the indicated cross sections taken perpendicular to the x-axis.

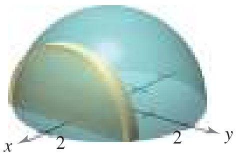
(a) Squares



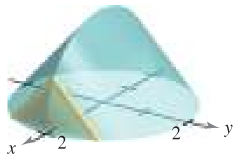
(b) Equilateral triangles



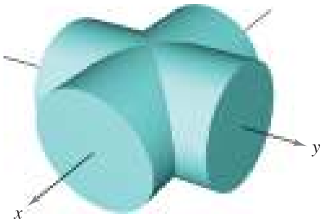
(c) Semicircles



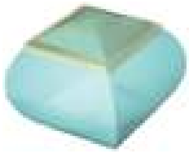
(d) Isosceles right triangles



73. Find the volume of the solid of intersection (the solid common to both) of the two right circular cylinders of radius r whose axes meet at right angles (see figure).



Two intersecting cylinders



Solid of intersection

■ **FOR FURTHER INFORMATION** For more information on this problem, see the article "Estimating the Volumes of Solid Figures with Curved Surfaces" by Donald Cohen in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

74. The base of a solid is bounded by $y = x^3$, $y = 0$, and $x = 1$. Find the volume of the solid for each of the following cross sections (taken perpendicular to the y-axis): (a) squares, (b) semicircles, (c) equilateral triangles, and (d) semiellipses whose heights are twice the lengths of their bases.

75. A manufacturer drills a hole through the center of a metal sphere of radius R. The hole has a radius r. Find the volume of the resulting ring.

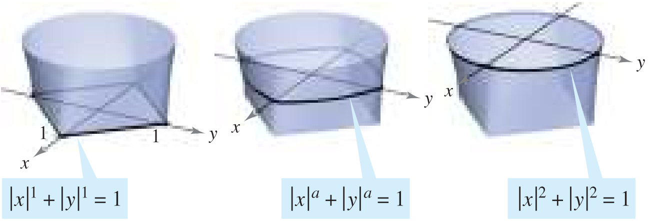
76. For the metal sphere in Exercise 75, let $R = 6$. What value of r will produce a ring whose volume is exactly half the volume of the sphere?

77. The region bounded by the graphs of $y = 8x/(9 + x^2)$, $y = 0$, $x = 0$, and $x = 5$ is revolved about the x-axis. Use a graphing utility and Simpson's Rule (with $n = 10$) to approximate the volume of the solid.

78. The solid shown in the figure has cross sections bounded by the graph of $|x|^a + |y|^a = 1$, where $1 \le a \le 2$.

(a) Describe the cross section when $a = 1$ and $a = 2$.

(b) Describe a procedure for approximating the volume of the solid.



79. Two planes cut a right circular cylinder to form a wedge. One plane is perpendicular to the axis of the cylinder and the second makes an angle of θ degrees with the first (see figure).

(a) Find the volume of the wedge if $\theta = 45°$.

(b) Find the volume of the wedge for an arbitrary angle θ. Assuming that the cylinder has sufficient length, how does the volume of the wedge change as θ increases from 0° to 90°?

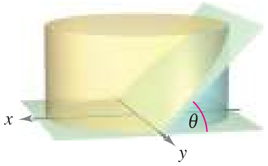


Figure for 79

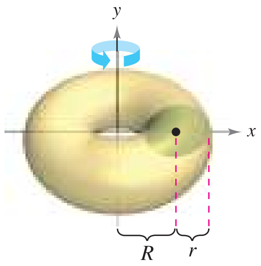


Figure for 80

80. (a) Show that the volume of the torus shown in the figure is given by the integral $8\pi R \int_0^r \sqrt{r^2 - y^2}\, dy$, where $R > r > 0$.

(b) Find the volume of the torus.

7.3 Volume: The Shell Method

- Find the volume of a solid of revolution using the shell method.
- Compare the uses of the disk method and the shell method.

The Shell Method

In this section, you will study an alternative method for finding the volume of a solid of revolution. This method is called the **shell method** because it uses cylindrical shells. A comparison of the advantages of the disk and shell methods is given later in this section.

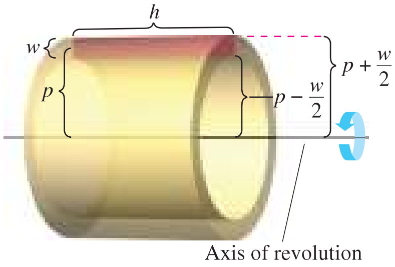


Figure 7.27

To begin, consider a representative rectangle as shown in Figure 7.27, where w is the width of the rectangle, h is the height of the rectangle, and p is the distance between the axis of revolution and the *center* of the rectangle. When this rectangle is revolved about its axis of revolution, it forms a cylindrical shell (or tube) of thickness w. To find the volume of this shell, consider two cylinders. The radius of the larger cylinder corresponds to the outer radius of the shell, and the radius of the smaller cylinder corresponds to the inner radius of the shell. Because p is the average radius of the shell, you know the outer radius is $p + (w/2)$ and the inner radius is $p - (w/2)$.

$$p + \frac{w}{2} \qquad \text{Outer radius}$$

$$p - \frac{w}{2} \qquad \text{Inner radius}$$

So, the volume of the shell is

$$\begin{aligned}\text{Volume of shell} &= (\text{volume of cylinder}) - (\text{volume of hole})\\ &= \pi\left(p + \frac{w}{2}\right)^2 h - \pi\left(p - \frac{w}{2}\right)^2 h\\ &= 2\pi phw\\ &= 2\pi(\text{average radius})(\text{height})(\text{thickness}).\end{aligned}$$

You can use this formula to find the volume of a solid of revolution. Assume that the plane region in Figure 7.28 is revolved about a line to form the indicated solid. If you consider a horizontal rectangle of width Δy, then, as the plane region is revolved about a line parallel to the x-axis, the rectangle generates a representative shell whose volume is

$$\Delta V = 2\pi[p(y)h(y)]\,\Delta y.$$

You can approximate the volume of the solid by n such shells of thickness Δy, height $h(y_i)$, and average radius $p(y_i)$.

$$\text{Volume of solid} \approx \sum_{i=1}^{n} 2\pi[p(y_i)h(y_i)]\,\Delta y = 2\pi\sum_{i=1}^{n}[p(y_i)h(y_i)]\,\Delta y$$

This approximation appears to become better and better as $\|\Delta\| \to 0$ $(n \to \infty)$. So, the volume of the solid is

$$\begin{aligned}\text{Volume of solid} &= \lim_{\|\Delta\|\to 0} 2\pi\sum_{i=1}^{n}[p(y_i)h(y_i)]\,\Delta y\\ &= 2\pi\int_c^d [p(y)h(y)]\,dy.\end{aligned}$$

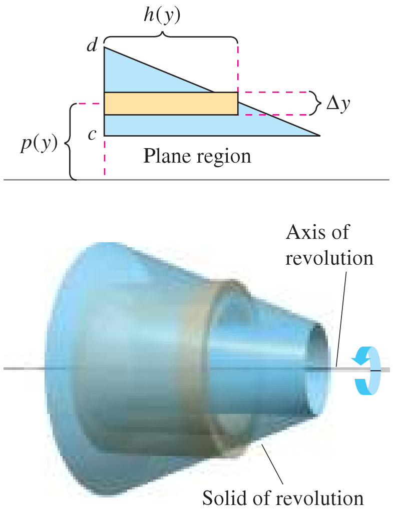
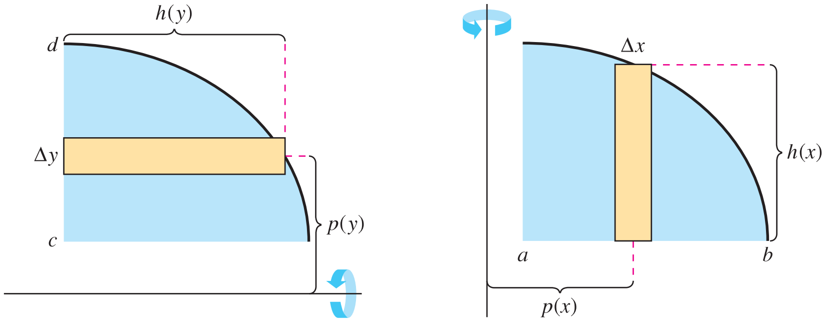


Figure 7.28

THE SHELL METHOD

To find the volume of a solid of revolution with the **shell method,** use one of the following, as shown in Figure 7.29.

Horizontal Axis of Revolution	*Vertical Axis of Revolution*
$\text{Volume} = V = 2\pi \int_c^d p(y)h(y)\,dy$	$\text{Volume} = V = 2\pi \int_a^b p(x)h(x)\,dx$



Horizontal axis of revolution Vertical axis of revolution

Figure 7.29

EXAMPLE 1 Using the Shell Method to Find Volume

Find the volume of the solid of revolution formed by revolving the region bounded by

$$y = x - x^3$$

and the x-axis $(0 \le x \le 1)$ about the y-axis.

Solution Because the axis of revolution is vertical, use a vertical representative rectangle, as shown in Figure 7.30. The width Δx indicates that x is the variable of integration. The distance from the center of the rectangle to the axis of revolution is $p(x) = x$, and the height of the rectangle is

$$h(x) = x - x^3.$$

Because x ranges from 0 to 1, the volume of the solid is

$$\begin{aligned} V = 2\pi \int_a^b p(x)h(x)\,dx &= 2\pi \int_0^1 x(x - x^3)\,dx && \text{Apply shell method.} \\ &= 2\pi \int_0^1 (-x^4 + x^2)\,dx && \text{Simplify.} \\ &= 2\pi \left[-\frac{x^5}{5} + \frac{x^3}{3} \right]_0^1 && \text{Integrate.} \\ &= 2\pi \left(-\frac{1}{5} + \frac{1}{3} \right) \\ &= \frac{4\pi}{15}. \end{aligned}$$

■

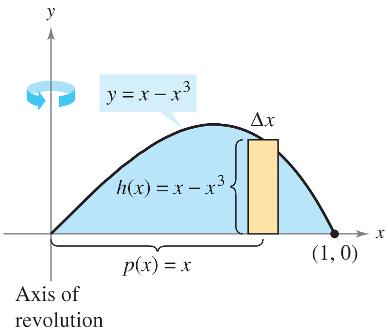


Figure 7.30

EXAMPLE 2 Using the Shell Method to Find Volume

Find the volume of the solid of revolution formed by revolving the region bounded by the graph of

$$x = e^{-y^2}$$

and the y-axis $(0 \le y \le 1)$ about the x-axis.

Solution Because the axis of revolution is horizontal, use a horizontal representative rectangle, as shown in Figure 7.31. The width Δy indicates that y is the variable of integration. The distance from the center of the rectangle to the axis of revolution is $p(y) = y$, and the height of the rectangle is $h(y) = e^{-y^2}$. Because y ranges from 0 to 1, the volume of the solid is

$$\begin{aligned} V &= 2\pi \int_c^d p(y)h(y)\,dy = 2\pi \int_0^1 ye^{-y^2}\,dy && \text{Apply shell method.} \\ &= -\pi \Big[e^{-y^2}\Big]_0^1 && \text{Integrate.} \\ &= \pi\left(1 - \frac{1}{e}\right) \\ &\approx 1.986. \end{aligned}$$

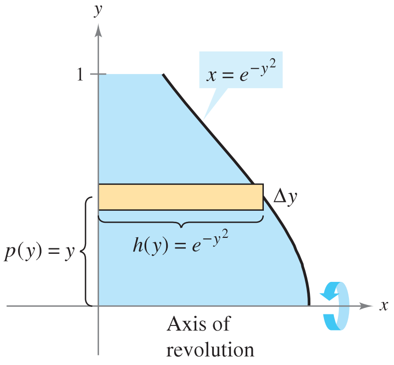

y
1
$x = e^{-y^2}$
Δy
$p(y) = y$
$h(y) = e^{-y^2}$
x
Axis of revolution

Figure 7.31

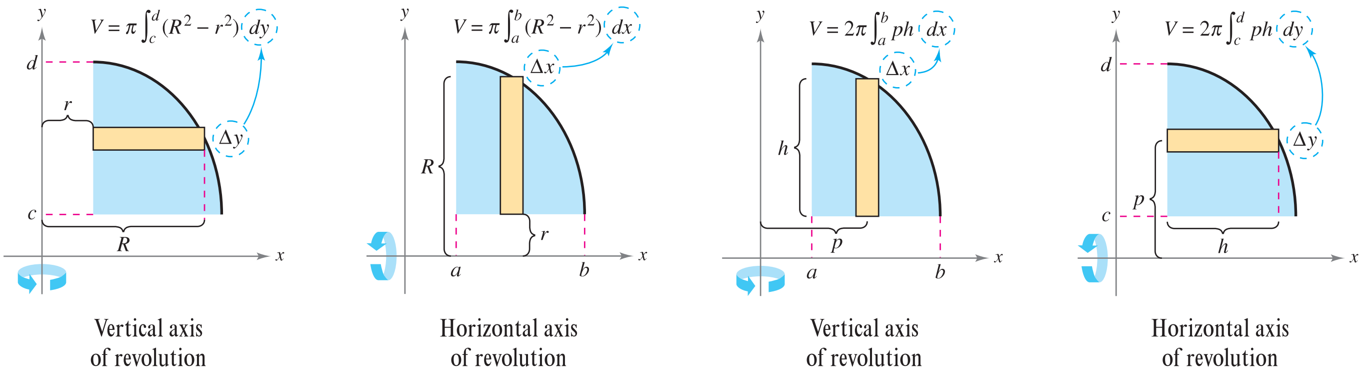
NOTE To see the advantage of using the shell method in Example 2, solve the equation $x = e^{-y^2}$ for y.

$$y = \begin{cases} 1, & 0 \le x \le 1/e \\ \sqrt{-\ln x}, & 1/e < x \le 1 \end{cases}$$

Then use this equation to find the volume using the disk method.

Comparison of Disk and Shell Methods

The disk and shell methods can be distinguished as follows. For the disk method, the representative rectangle is always *perpendicular* to the axis of revolution, whereas for the shell method, the representative rectangle is always *parallel* to the axis of revolution, as shown in Figure 7.32.


$V = \pi \int_c^d (R^2 - r^2)\,dy$
y
d
r
Δy
c
R
x
Vertical axis of revolution
$V = \pi \int_a^b (R^2 - r^2)\,dx$
y
Δx
R
r
a
b
x
Horizontal axis of revolution
$V = 2\pi \int_a^b ph\,dx$
y
Δx
h
p
a
b
x
Vertical axis of revolution
$V = 2\pi \int_c^d ph\,dy$
y
d
Δy
p
c
h
x
Horizontal axis of revolution

Disk method: Representative rectangle is perpendicular to the axis of revolution.

Shell method: Representative rectangle is parallel to the axis of revolution.

Figure 7.32

Often, one method is more convenient to use than the other. The following example illustrates a case in which the shell method is preferable.

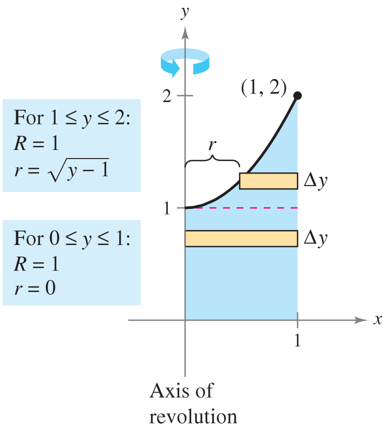
EXAMPLE 3 Shell Method Preferable

Find the volume of the solid formed by revolving the region bounded by the graphs of

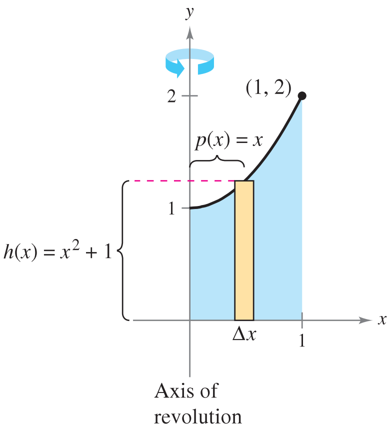
$$y = x^2 + 1, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 1$$

about the y-axis.

Solution In Example 4 in the preceding section, you saw that the washer method requires two integrals to determine the volume of this solid. See Figure 7.33(a).



(a) Disk method



(b) Shell method

Figure 7.33

$$\begin{aligned}
V &= \pi\int_0^1 (1^2 - 0^2)\,dy + \pi\int_1^2 \left[1^2 - \left(\sqrt{y-1}\right)^2\right] dy && \text{Apply washer method.}\\
&= \pi\int_0^1 1\,dy + \pi\int_1^2 (2-y)\,dy && \text{Simplify.}\\
&= \pi\Big[y\Big]_0^1 + \pi\left[2y - \frac{y^2}{2}\right]_1^2 && \text{Integrate.}\\
&= \pi + \pi\left(4 - 2 - 2 + \frac{1}{2}\right)\\
&= \frac{3\pi}{2}
\end{aligned}$$

In Figure 7.33(b), you can see that the shell method requires only one integral to find the volume.

$$\begin{aligned}
V &= 2\pi\int_a^b p(x)h(x)\,dx && \text{Apply shell method.}\\
&= 2\pi\int_0^1 x(x^2+1)\,dx\\
&= 2\pi\left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^1 && \text{Integrate.}\\
&= 2\pi\left(\frac{3}{4}\right)\\
&= \frac{3\pi}{2}
\end{aligned}$$

■

Suppose the region in Example 3 were revolved about the vertical line $x = 1$. Would the resulting solid of revolution have a greater volume or a smaller volume than the solid in Example 3? Without integrating, you should be able to reason that the resulting solid would have a smaller volume because “more” of the revolved region would be closer to the axis of revolution. To confirm this, try solving the following integral, which gives the volume of the solid.

$$V = 2\pi\int_0^1 (1-x)(x^2+1)\,dx \qquad p(x) = 1 - x$$

■ **FOR FURTHER INFORMATION** To learn more about the disk and shell methods, see the article “The Disk and Shell Method” by Charles A. Cable in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*. ■

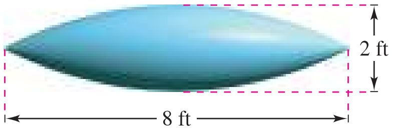


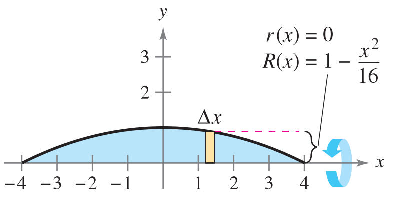
Figure 7.34

EXAMPLE 4 Volume of a Pontoon

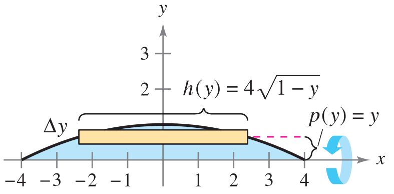
A pontoon is to be made in the shape shown in Figure 7.34. The pontoon is designed by rotating the graph of

$$y = 1 - \frac{x^2}{16}, \quad -4 \le x \le 4$$

about the x-axis, where x and y are measured in feet. Find the volume of the pontoon.



(a) Disk method



(b) Shell method

Figure 7.35

Solution Refer to Figure 7.35(a) and use the disk method as follows.

$$V = \pi\int_{-4}^{4}\left(1 - \frac{x^2}{16}\right)^2 dx \qquad \text{Apply disk method.}$$

$$= \pi\int_{-4}^{4}\left(1 - \frac{x^2}{8} + \frac{x^4}{256}\right) dx \qquad \text{Simplify.}$$

$$= \pi\left[x - \frac{x^3}{24} + \frac{x^5}{1280}\right]_{-4}^{4} \qquad \text{Integrate.}$$

$$= \frac{64\pi}{15} \approx 13.4 \text{ cubic feet}$$

Try using Figure 7.35(b) to set up the integral for the volume using the shell method. Does the integral seem more complicated? ■

To use the shell method in Example 4, you would have to solve for x in terms of y in the equation

$$y = 1 - (x^2/16).$$

Sometimes, solving for x is very difficult (or even impossible). In such cases you must use a vertical rectangle (of width Δx), thus making x the variable of integration. The position (horizontal or vertical) of the axis of revolution then determines the method to be used. This is shown in Example 5.

EXAMPLE 5 Shell Method Necessary

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^3 + x + 1$, $y = 1$, and $x = 1$ about the line $x = 2$, as shown in Figure 7.36.

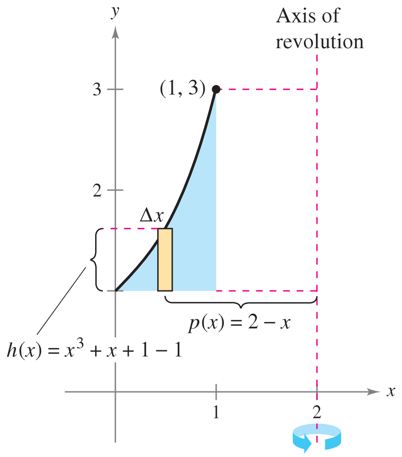


Figure 7.36

Solution In the equation $y = x^3 + x + 1$, you cannot easily solve for x in terms of y. (See Section 3.8 on Newton's Method.) Therefore, the variable of integration must be x, and you should choose a vertical representative rectangle. Because the rectangle is parallel to the axis of revolution, use the shell method and obtain

$$V = 2\pi\int_a^b p(x)h(x)\,dx = 2\pi\int_0^1 (2 - x)(x^3 + x + 1 - 1)\,dx \qquad \text{Apply shell method.}$$

$$= 2\pi\int_0^1 (-x^4 + 2x^3 - x^2 + 2x)\,dx \qquad \text{Simplify.}$$

$$= 2\pi\left[-\frac{x^5}{5} + \frac{x^4}{2} - \frac{x^3}{3} + x^2\right]_0^1 \qquad \text{Integrate.}$$

$$= 2\pi\left(-\frac{1}{5} + \frac{1}{2} - \frac{1}{3} + 1\right)$$

$$= \frac{29\pi}{15}.$$

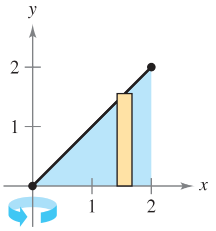
■

7.3 Exercises

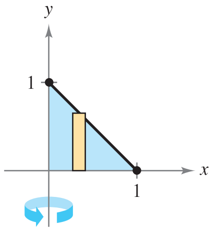
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–14, use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the y-axis.

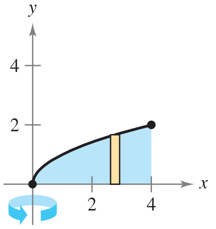
1. $y = x$



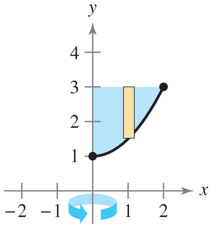
2. $y = 1 - x$



3. $y = \sqrt{x}$



4. $y = \frac{1}{2}x^2 + 1$



5. $y = x^2, \quad y = 0, \quad x = 3$

6. $y = \frac{1}{4}x^2, \quad y = 0, \quad x = 6$

7. $y = x^2, \quad y = 4x - x^2$

8. $y = 4 - x^2, \quad y = 0$

9. $y = 4x - x^2, \quad x = 0, \quad y = 4$

10. $y = 3x, \quad y = 6, \quad x = 0$

11. $y = \sqrt{x - 2}, \quad y = 0, \quad x = 4$

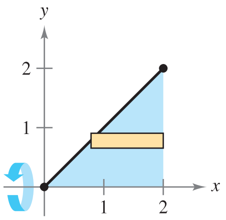
12. $y = -x^2 + 1, \quad y = 0$

13. $y = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad y = 0, \quad x = 0, \quad x = 1$

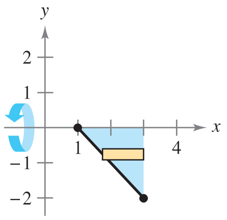
14. $y = \begin{cases} \dfrac{\sin x}{x}, & x > 0 \\ 1, & x = 0 \end{cases}, \quad y = 0, \quad x = 0, \quad x = \pi$

In Exercises 15–22, use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the x-axis.

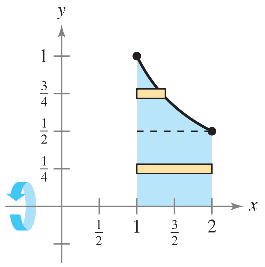
15. $y = x$



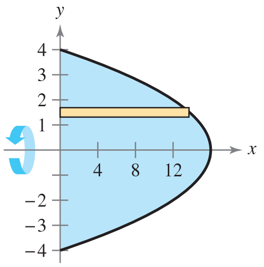
16. $y = 1 - x$



17. $y = \dfrac{1}{x}$



18. $x + y^2 = 16$



19. $y = x^3, \quad x = 0, \quad y = 8$

20. $y = x^2, \quad x = 0, \quad y = 9$

21. $x + y = 4, \quad y = x, \quad y = 0$

22. $y = \sqrt{x + 2}, \quad y = x, \quad y = 0$

In Exercises 23–26, use the shell method to find the volume of the solid generated by revolving the plane region about the given line.

23. $y = 4x - x^2, \quad y = 0,$ about the line $x = 5$

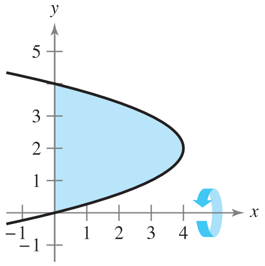
24. $y = \sqrt{x}, \quad y = 0, \quad x = 4,$ about the line $x = 6$

25. $y = x^2, \quad y = 4x - x^2,$ about the line $x = 4$

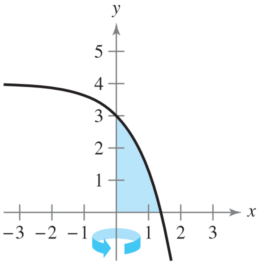
26. $y = x^2, \quad y = 4x - x^2,$ about the line $x = 2$

In Exercises 27 and 28, decide whether it is more convenient to use the disk method or the shell method to find the volume of the solid of revolution. Explain your reasoning. (Do not find the volume.)

27. $(y - 2)^2 = 4 - x$



28. $y = 4 - e^x$



In Exercises 29–32, use the disk *or* the shell method to find the volume of the solid generated by revolving the region bounded by the graphs of the equations about each given line.

29. $y = x^3, \quad y = 0, \quad x = 2$

(a) the x-axis (b) the y-axis (c) the line $x = 4$

30. $y = \dfrac{10}{x^2}, \quad y = 0, \quad x = 1, \quad x = 5$

(a) the x-axis (b) the y-axis (c) the line $y = 10$

31. $x^{1/2} + y^{1/2} = a^{1/2}, \quad x = 0, \quad y = 0$

(a) the x-axis (b) the y-axis (c) the line $x = a$

32. $x^{2/3} + y^{2/3} = a^{2/3}, \quad a > 0$ (hypocycloid)

(a) the x-axis (b) the y-axis

In Exercises 33–36, (a) use a graphing utility to graph the plane region bounded by the graphs of the equations, and (b) use the integration capabilities of the graphing utility to approximate the volume of the solid generated by revolving the region about the y-axis.

33. $x^{4/3} + y^{4/3} = 1, x = 0, y = 0$, first quadrant

34. $y = \sqrt{1 - x^3}, y = 0, x = 0$

35. $y = \sqrt[3]{(x-2)^2(x-6)^2}, y = 0, x = 2, x = 6$

36. $y = \dfrac{2}{1 + e^{1/x}}, y = 0, x = 1, x = 3$

Think About It **In Exercises 37 and 38, determine which value best approximates the volume of the solid generated by revolving the region bounded by the graphs of the equations about the y-axis. (Make your selection on the basis of a sketch of the solid and *not* by performing any calculations.)**

37. $y = 2e^{-x}, y = 0, x = 0, x = 2$

(a) $\frac{3}{2}$ (b) -2 (c) 4 (d) 7.5 (e) 15

38. $y = \tan x, y = 0, x = 0, x = \dfrac{\pi}{4}$

(a) 3.5 (b) $-\frac{9}{4}$ (c) 8 (d) 10 (e) 1

WRITING ABOUT CONCEPTS

39. The region in the figure is revolved about the indicated axes and line. Order the volumes of the resulting solids from least to greatest. Explain your reasoning.

(a) x-axis (b) y-axis (c) $x = 4$

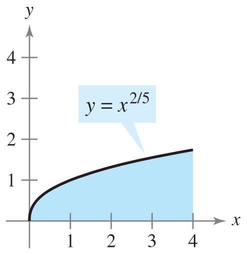


Figure for 39

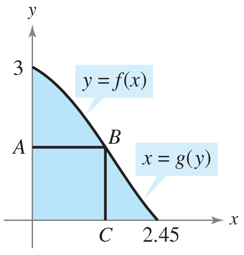


Figure for 40

40. (a) Describe the figure generated by revolving segment AB about the y-axis (see figure).

(b) Describe the figure generated by revolving segment BC about the y-axis.

(c) Assume the curve in the figure can be described as $y = f(x)$ or $x = g(y)$. A solid is generated by revolving the region bounded by the curve, $y = 0$, and $x = 0$ about the y-axis. Set up integrals to find the volume of this solid using the disk method and the shell method. (Do not integrate.)

WRITING ABOUT CONCEPTS (continued)

In Exercises 41 and 42, give a geometric argument that explains why the integrals have equal values.

41. $\pi \displaystyle\int_1^5 (x - 1)\,dx = 2\pi \int_0^2 y[5 - (y^2 + 1)]\,dy$

42. $\pi \displaystyle\int_0^2 [16 - (2y)^2]\,dy = 2\pi \int_0^4 x\left(\frac{x}{2}\right)dx$

43. Consider a solid that is generated by revolving a plane region about the y-axis. Describe the position of a representative rectangle when using (a) the shell method and (b) the disk method to find the volume of the solid.

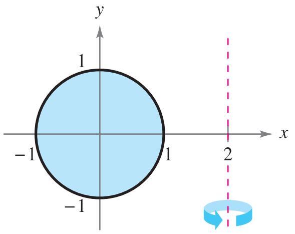
CAPSTONE

44. Consider the plane region bounded by the graphs of $y = k$, $y = 0, x = 0$, and $x = b$, where $k > 0$ and $b > 0$. What are the heights and radii of the cylinders generated when this region is revolved about (a) the x-axis and (b) the y-axis?

45. ***Machine Part*** A solid is generated by revolving the region bounded by $y = \frac{1}{2}x^2$ and $y = 2$ about the y-axis. A hole, centered along the axis of revolution, is drilled through this solid so that one-fourth of the volume is removed. Find the diameter of the hole.

46. ***Machine Part*** A solid is generated by revolving the region bounded by $y = \sqrt{9 - x^2}$ and $y = 0$ about the y-axis. A hole, centered along the axis of revolution, is drilled through this solid so that one-third of the volume is removed. Find the diameter of the hole.

47. ***Volume of a Torus*** A torus is formed by revolving the region bounded by the circle $x^2 + y^2 = 1$ about the line $x = 2$ (see figure). Find the volume of this "doughnut-shaped" solid. (*Hint:* The integral $\int_{-1}^{1} \sqrt{1 - x^2}\,dx$ represents the area of a semicircle.)



48. ***Volume of a Torus*** Repeat Exercise 47 for a torus formed by revolving the region bounded by the circle $x^2 + y^2 = r^2$ about the line $x = R$, where $r < R$.

In Exercises 49–52, the integral represents the volume of a solid of revolution. Identify (a) the plane region that is revolved and (b) the axis of revolution.

49. $2\pi \displaystyle\int_0^2 x^3\,dx$

50. $2\pi \displaystyle\int_0^1 y - y^{3/2}\,dy$

51. $2\pi \displaystyle\int_0^6 (y + 2)\sqrt{6 - y}\,dy$

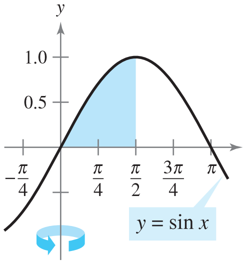
52. $2\pi \displaystyle\int_0^1 (4 - x)e^x\,dx$

53. (a) Use differentiation to verify that

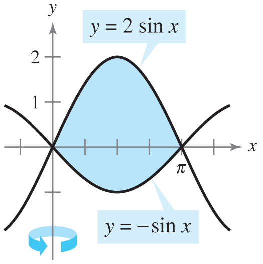
$$\int x \sin x\,dx = \sin x - x\cos x + C.$$

(b) Use the result of part (a) to find the volume of the solid generated by revolving each plane region about the y-axis.

(i)



(ii)

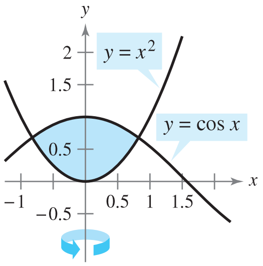


54. (a) Use differentiation to verify that

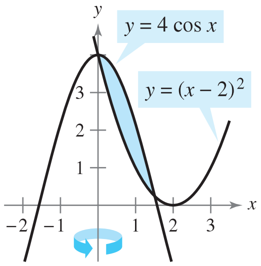
$$\int x \cos x\,dx = \cos x + x\sin x + C.$$

(b) Use the result of part (a) to find the volume of the solid generated by revolving each plane region about the y-axis. (*Hint:* Begin by approximating the points of intersection.)

(i)



(ii)



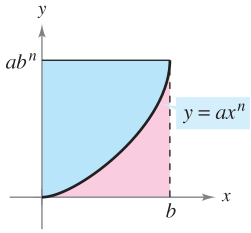
55. ***Volume of a Segment of a Sphere*** Let a sphere of radius r be cut by a plane, thereby forming a segment of height h. Show that the volume of this segment is $\frac{1}{3}\pi h^2(3r - h)$.

56. ***Volume of an Ellipsoid*** Consider the plane region bounded by the graph of

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

where $a > 0$ and $b > 0$. Show that the volume of the ellipsoid formed when this region is revolved about the y-axis is $\frac{4}{3}\pi a^2 b$. What is the volume when the region is revolved about the x-axis?

57. ***Exploration*** Consider the region bounded by the graphs of $y = ax^n$, $y = ab^n$, and $x = 0$ (see figure).



(a) Find the ratio $R_1(n)$ of the area of the region to the area of the circumscribed rectangle.

(b) Find $\lim\limits_{n\to\infty} R_1(n)$ and compare the result with the area of the circumscribed rectangle.

(c) Find the volume of the solid of revolution formed by revolving the region about the y-axis. Find the ratio $R_2(n)$ of this volume to the volume of the circumscribed right circular cylinder.

(d) Find $\lim\limits_{n\to\infty} R_2(n)$ and compare the result with the volume of the circumscribed cylinder.

(e) Use the results of parts (b) and (d) to make a conjecture about the shape of the graph of $y = ax^n$ $(0 \le x \le b)$ as $n \to \infty$.

58. ***Think About It*** Match each integral with the solid whose volume it represents, and give the dimensions of each solid.

(a) Right circular cone (b) Torus (c) Sphere

(d) Right circular cylinder (e) Ellipsoid

(i) $2\pi \int_0^r hx\,dx$ (ii) $2\pi \int_0^r hx\left(1 - \frac{x}{r}\right)dx$

(iii) $2\pi \int_0^r 2x\sqrt{r^2 - x^2}\,dx$ (iv) $2\pi \int_0^b 2ax\sqrt{1 - \frac{x^2}{b^2}}\,dx$

(v) $2\pi \int_{-r}^r (R - x)\left(2\sqrt{r^2 - x^2}\right)dx$

59. ***Volume of a Storage Shed*** A storage shed has a circular base of diameter 80 feet. Starting at the center, the interior height is measured every 10 feet and recorded in the table (see figure).

x	0	10	20	30	40
Height	50	45	40	20	0

(a) Use Simpson's Rule to approximate the volume of the shed.

(b) Note that the roof line consists of two line segments. Find the equations of the line segments and use integration to find the volume of the shed.

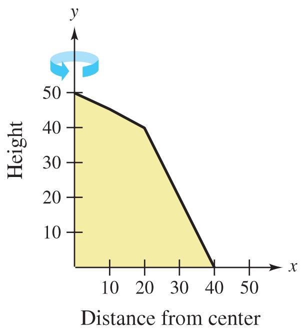


Figure for 59

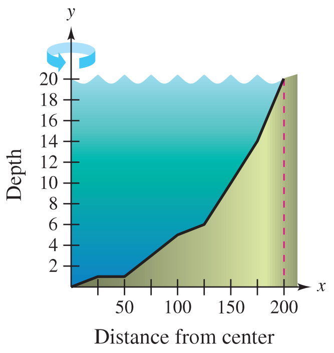


Figure for 60

60. ***Modeling Data*** A pond is approximately circular, with a diameter of 400 feet. Starting at the center, the depth of the water is measured every 25 feet and recorded in the table (see figure).

x	0	25	50	75	100	125	150	175	200
Depth	20	19	19	17	15	14	10	6	0

(a) Use Simpson's Rule to approximate the volume of water in the pond.

(b) Use the regression capabilities of a graphing utility to find a quadratic model for the depths recorded in the table. Use the graphing utility to plot the depths and graph the model.

(c) Use the integration capabilities of a graphing utility and the model in part (b) to approximate the volume of water in the pond.

(d) Use the result of part (c) to approximate the number of gallons of water in the pond if 1 cubic foot of water is approximately 7.48 gallons.

61. Let V_1 and V_2 be the volumes of the solids that result when the plane region bounded by $y = 1/x$, $y = 0$, $x = \frac{1}{4}$, and $x = c\left(\text{where } c > \frac{1}{4}\right)$ is revolved about the x-axis and the y-axis, respectively. Find the value of c for which $V_1 = V_2$.

62. The region bounded by $y = r^2 - x^2$, $y = 0$, and $x = 0$ is revolved about the y-axis to form a paraboloid. A hole, centered along the axis of revolution, is drilled through this solid. The hole has a radius k, $0 < k < r$. Find the volume of the resulting ring (a) by integrating with respect to x and (b) by integrating with respect to y.

63. Consider the graph of $y^2 = x(4 - x)^2$ (see figure). Find the volumes of the solids that are generated when the loop of this graph is revolved about (a) the x-axis, (b) the y-axis, and (c) the line $x = 4$.

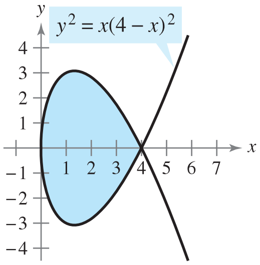


Figure for 63

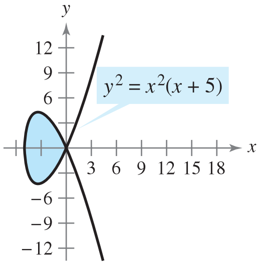


Figure for 64

64. Consider the graph of $y^2 = x^2(x + 5)$ (see figure). Find the volumes of the solids that are generated when the loop of this graph is revolved about (a) the x-axis, (b) the y-axis, and (c) the line $x = -5$.

SECTION PROJECT

Saturn

The Oblateness of Saturn Saturn is the most oblate of the nine planets in our solar system. Its equatorial radius is 60,268 kilometers and its polar radius is 54,364 kilometers. The color-enhanced photograph of Saturn was taken by Voyager 1. In the photograph, the oblateness of Saturn is clearly visible.

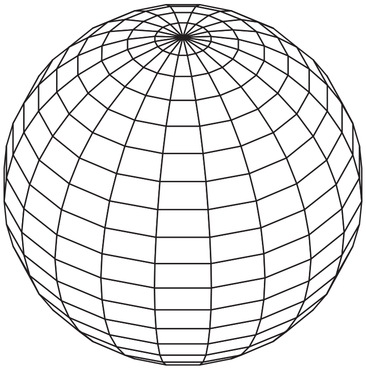
(a) Find the ratio of the volumes of the sphere and the oblate ellipsoid shown below.

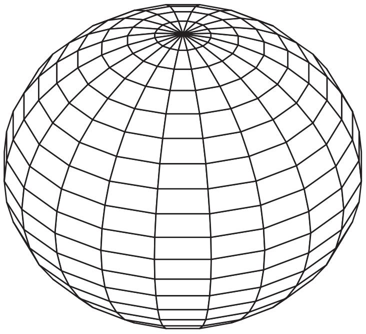
(b) If a planet were spherical and had the same volume as Saturn, what would its radius be?

NSSDC

Computer model of "spherical Saturn," whose equatorial radius is equal to its polar radius. The equation of the cross section passing through the pole is

$x^2 + y^2 = 60{,}268^2.$





Computer model of "oblate Saturn," whose equatorial radius is greater than its polar radius. The equation of the cross section passing through the pole is

$$\frac{x^2}{60{,}268^2} + \frac{y^2}{54{,}364^2} = 1.$$

7.4 Arc Length and Surfaces of Revolution

- **Find the arc length of a smooth curve.**
- **Find the area of a surface of revolution.**

Arc Length

In this section, definite integrals are used to find the arc lengths of curves and the areas of surfaces of revolution. In either case, an arc (a segment of a curve) is approximated by straight line segments whose lengths are given by the familiar Distance Formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

A **rectifiable** curve is one that has a finite arc length. You will see that a sufficient condition for the graph of a function f to be rectifiable between $(a, f(a))$ and $(b, f(b))$ is that f' be continuous on $[a, b]$. Such a function is **continuously differentiable** on $[a, b]$, and its graph on the interval $[a, b]$ is a **smooth curve.**

Bettmann/Corbis

CHRISTIAN HUYGENS (1629–1695)

The Dutch mathematician Christian Huygens, who invented the pendulum clock, and James Gregory (1638–1675), a Scottish mathematician, both made early contributions to the problem of finding the length of a rectifiable curve.

Consider a function $y = f(x)$ that is continuously differentiable on the interval $[a, b]$. You can approximate the graph of f by n line segments whose endpoints are determined by the partition

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_n = b$$

as shown in Figure 7.37. By letting $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, you can approximate the length of the graph by

$$\begin{aligned} s &\approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2 (\Delta x_i)^2} \\ &= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,(\Delta x_i). \end{aligned}$$

This approximation appears to become better and better as $\|\Delta\| \to 0$ $(n \to \infty)$. So, the length of the graph is

$$s = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,(\Delta x_i).$$

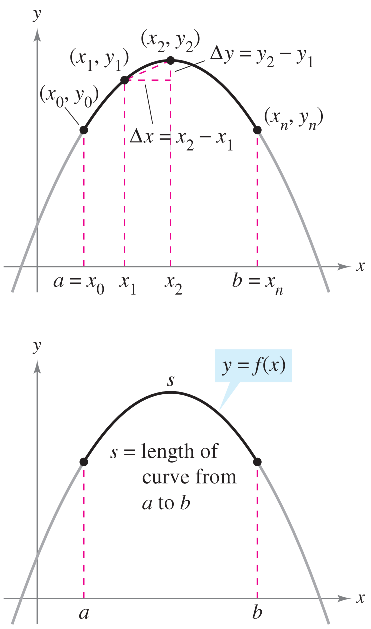


Figure 7.37

Because $f'(x)$ exists for each x in (x_{i-1}, x_i), the Mean Value Theorem guarantees the existence of c_i in (x_{i-1}, x_i) such that

$$\begin{aligned} f(x_i) - f(x_{i-1}) &= f'(c_i)(x_i - x_{i-1}) \\ \frac{\Delta y_i}{\Delta x_i} &= f'(c_i). \end{aligned}$$

Because f' is continuous on $[a, b]$, it follows that $\sqrt{1 + [f'(x)]^2}$ is also continuous (and therefore integrable) on $[a, b]$, which implies that

$$\begin{aligned} s &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\,(\Delta x_i) \\ &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \end{aligned}$$

where s is called the **arc length** of f between a and b.

FOR FURTHER INFORMATION To see how arc length can be used to define trigonometric functions, see the article "Trigonometry Requires Calculus, Not Vice Versa" by Yves Nievergelt in *UMAP Modules*.

DEFINITION OF ARC LENGTH

Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

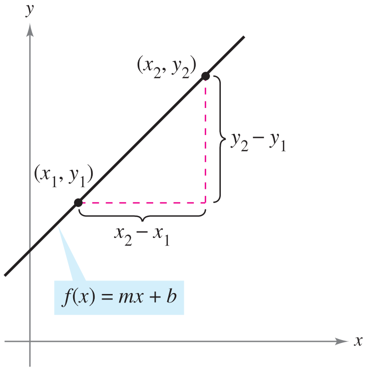
Similarly, for a smooth curve given by $x = g(y)$, the **arc length** of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy.$$

Because the definition of arc length can be applied to a linear function, you can check to see that this new definition agrees with the standard Distance Formula for the length of a line segment. This is shown in Example 1.

EXAMPLE 1 The Length of a Line Segment

Find the arc length from (x_1, y_1) to (x_2, y_2) on the graph of $f(x) = mx + b$, as shown in Figure 7.38.



The formula for the arc length of the graph of f from (x_1, y_1) to (x_2, y_2) is the same as the standard Distance Formula.
Figure 7.38

Solution Because

$$m = f'(x) = \frac{y_2 - y_1}{x_2 - x_1}$$

it follows that

$$\begin{aligned} s &= \int_{x_1}^{x_2} \sqrt{1 + [f'(x)]^2}\,dx && \text{Formula for arc length} \\ &= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2}\,dx \\ &= \sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{(x_2 - x_1)^2}}\,(x)\Big]_{x_1}^{x_2} && \text{Integrate and simplify.} \\ &= \sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{(x_2 - x_1)^2}}\,(x_2 - x_1) \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$

which is the formula for the distance between two points in the plane. ■

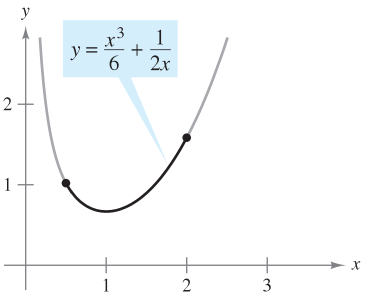
TECHNOLOGY Definite integrals representing arc length often are very difficult to evaluate. In this section, a few examples are presented. In the next chapter, with more advanced integration techniques, you will be able to tackle more difficult arc length problems. In the meantime, remember that you can always use a numerical integration program to approximate an arc length. For instance, use the *numerical integration* feature of a graphing utility to approximate the arc lengths in Examples 2 and 3.

EXAMPLE 2 Finding Arc Length

Find the arc length of the graph of

$$y = \frac{x^3}{6} + \frac{1}{2x}$$

on the interval $\left[\frac{1}{2}, 2\right]$, as shown in Figure 7.39.



The arc length of the graph of y on $\left[\frac{1}{2}, 2\right]$
Figure 7.39

Solution Using

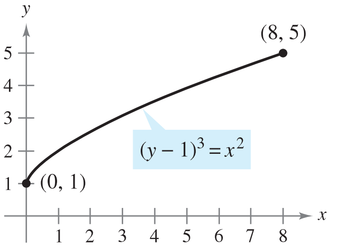
$$\frac{dy}{dx} = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$$

yields an arc length of

$$\begin{aligned} s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx &= \int_{1/2}^{2} \sqrt{1 + \left[\frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)\right]^2}\, dx && \text{Formula for arc length} \\ &= \int_{1/2}^{2} \sqrt{\frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right)}\, dx \\ &= \int_{1/2}^{2} \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right) dx && \text{Simplify.} \\ &= \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_{1/2}^{2} && \text{Integrate.} \\ &= \frac{1}{2}\left(\frac{13}{6} + \frac{47}{24}\right) \\ &= \frac{33}{16}. \end{aligned}$$

EXAMPLE 3 Finding Arc Length

Find the arc length of the graph of $(y - 1)^3 = x^2$ on the interval $[0, 8]$, as shown in Figure 7.40.



The arc length of the graph of y on $[0, 8]$
Figure 7.40

Solution Begin by solving for x in terms of y: $x = \pm(y - 1)^{3/2}$. Choosing the positive value of x produces

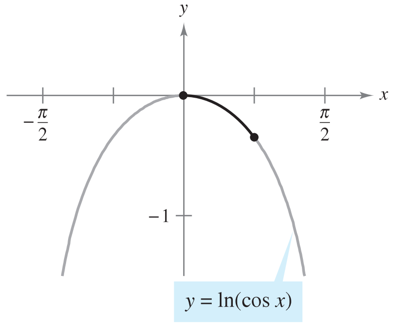
$$\frac{dx}{dy} = \frac{3}{2}(y - 1)^{1/2}.$$

The x-interval $[0, 8]$ corresponds to the y-interval $[1, 5]$, and the arc length is

$$\begin{aligned} s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy &= \int_{1}^{5} \sqrt{1 + \left[\frac{3}{2}(y - 1)^{1/2}\right]^2}\, dy && \text{Formula for arc length} \\ &= \int_{1}^{5} \sqrt{\frac{9}{4}y - \frac{5}{4}}\, dy \\ &= \frac{1}{2}\int_{1}^{5} \sqrt{9y - 5}\, dy && \text{Simplify.} \\ &= \frac{1}{18}\left[\frac{(9y - 5)^{3/2}}{3/2}\right]_{1}^{5} && \text{Integrate.} \\ &= \frac{1}{27}\left(40^{3/2} - 4^{3/2}\right) \\ &\approx 9.073. \end{aligned}$$

■

EXAMPLE 4 Finding Arc Length



The arc length of the graph of y on $\left[0, \frac{\pi}{4}\right]$

Figure 7.41

Find the arc length of the graph of $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$, as shown in Figure 7.41.

Solution Using

$$\frac{dy}{dx} = -\frac{\sin x}{\cos x} = -\tan x$$

yields an arc length of

$$\begin{aligned} s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx &= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx && \text{Formula for arc length} \\ &= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx && \text{Trigonometric identity} \\ &= \int_0^{\pi/4} \sec x\, dx && \text{Simplify.} \\ &= \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4} && \text{Integrate.} \\ &= \ln\left(\sqrt{2} + 1\right) - \ln 1 \\ &\approx 0.881. \end{aligned}$$

EXAMPLE 5 Length of a Cable

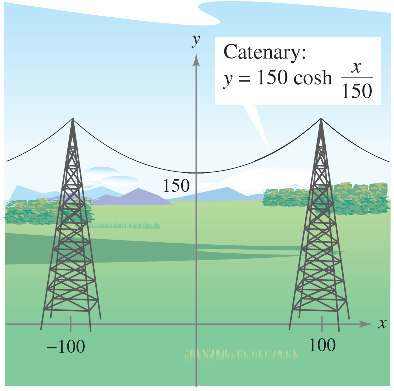


Figure 7.42

An electric cable is hung between two towers that are 200 feet apart, as shown in Figure 7.42. The cable takes the shape of a catenary whose equation is

$$y = 75\left(e^{x/150} + e^{-x/150}\right) = 150 \cosh \frac{x}{150}.$$

Find the arc length of the cable between the two towers.

Solution Because $y' = \frac{1}{2}\left(e^{x/150} - e^{-x/150}\right)$, you can write

$$(y')^2 = \frac{1}{4}\left(e^{x/75} - 2 + e^{-x/75}\right)$$

and

$$1 + (y')^2 = \frac{1}{4}\left(e^{x/75} + 2 + e^{-x/75}\right) = \left[\frac{1}{2}\left(e^{x/150} + e^{-x/150}\right)\right]^2.$$

Therefore, the arc length of the cable is

$$\begin{aligned} s = \int_a^b \sqrt{1 + (y')^2}\, dx &= \frac{1}{2}\int_{-100}^{100} \left(e^{x/150} + e^{-x/150}\right) dx && \text{Formula for arc length} \\ &= 75\Big[e^{x/150} - e^{-x/150}\Big]_{-100}^{100} && \text{Integrate.} \\ &= 150\left(e^{2/3} - e^{-2/3}\right) \\ &\approx 215 \text{ feet.} \end{aligned}$$

■

Area of a Surface of Revolution

In Sections 7.2 and 7.3, integration was used to calculate the volume of a solid of revolution. You will now look at a procedure for finding the area of a surface of revolution.

DEFINITION OF SURFACE OF REVOLUTION

If the graph of a continuous function is revolved about a line, the resulting surface is a **surface of revolution.**

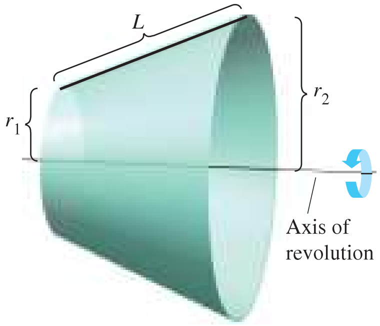


Figure 7.43

The area of a surface of revolution is derived from the formula for the lateral surface area of the frustum of a right circular cone. Consider the line segment in Figure 7.43, where L is the length of the line segment, r_1 is the radius at the left end of the line segment, and r_2 is the radius at the right end of the line segment. When the line segment is revolved about its axis of revolution, it forms a frustum of a right circular cone, with

$$S = 2\pi r L \qquad \text{Lateral surface area of frustum}$$

where

$$r = \frac{1}{2}(r_1 + r_2). \qquad \text{Average radius of frustum}$$

(In Exercise 62, you are asked to verify the formula for S.)

Suppose the graph of a function f, having a continuous derivative on the interval $[a, b]$, is revolved about the x-axis to form a surface of revolution, as shown in Figure 7.44. Let Δ be a partition of $[a, b]$, with subintervals of width Δx_i. Then the line segment of length

$$\Delta L_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

generates a frustum of a cone. Let r_i be the average radius of this frustum. By the Intermediate Value Theorem, a point d_i exists (in the ith subinterval) such that $r_i = f(d_i)$. The lateral surface area ΔS_i of the frustum is

$$\begin{aligned}\Delta S_i &= 2\pi r_i \,\Delta L_i\\ &= 2\pi f(d_i)\sqrt{\Delta x_i^2 + \Delta y_i^2}\\ &= 2\pi f(d_i)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i.\end{aligned}$$

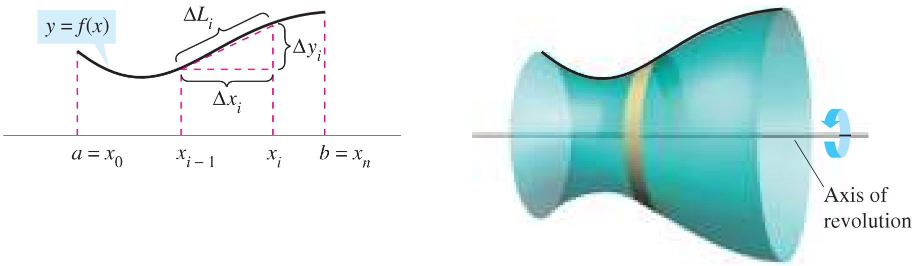


Figure 7.44

By the Mean Value Theorem, a point c_i exists in (x_{i-1}, x_i) such that

$$\begin{aligned} f'(c_i) &= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \\ &= \frac{\Delta y_i}{\Delta x_i}. \end{aligned}$$

So, $\Delta S_i = 2\pi f(d_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x_i$, and the total surface area can be approximated by

$$S \approx 2\pi \sum_{i=1}^{n} f(d_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x_i.$$

It can be shown that the limit of the right side as $\|\Delta\| \to 0$ $(n \to \infty)$ is

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

In a similar manner, if the graph of f is revolved about the y-axis, then S is

$$S = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}\,dx.$$

In these two formulas for S, you can regard the products $2\pi f(x)$ and $2\pi x$ as the circumferences of the circles traced by a point (x, y) on the graph of f as it is revolved about the x-axis and the y-axis (Figure 7.45). In one case the radius is $r = f(x)$, and in the other case the radius is $r = x$. Moreover, by appropriately adjusting r, you can generalize the formula for surface area to cover *any* horizontal or vertical axis of revolution, as indicated in the following definition.

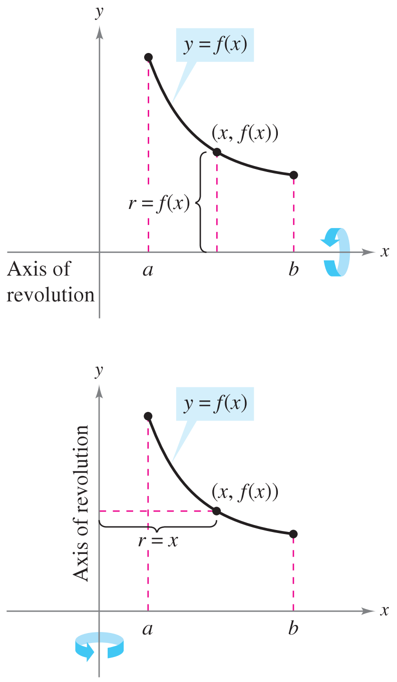


Figure 7.45

DEFINITION OF THE AREA OF A SURFACE OF REVOLUTION

Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,dx \qquad \text{y is a function of x.}$$

where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_c^d r(y)\sqrt{1 + [g'(y)]^2}\,dy \qquad \text{x is a function of y.}$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.

The formulas in this definition are sometimes written as

$$S = 2\pi \int_a^b r(x)\,ds \qquad \text{y is a function of x.}$$

and

$$S = 2\pi \int_c^d r(y)\,ds \qquad \text{x is a function of y.}$$

where $ds = \sqrt{1 + [f'(x)]^2}\,dx$ and $ds = \sqrt{1 + [g'(y)]^2}\,dy$, respectively.

EXAMPLE 6 The Area of a Surface of Revolution

Find the area of the surface formed by revolving the graph of

$$f(x) = x^3$$

on the interval $[0, 1]$ about the x-axis, as shown in Figure 7.46.

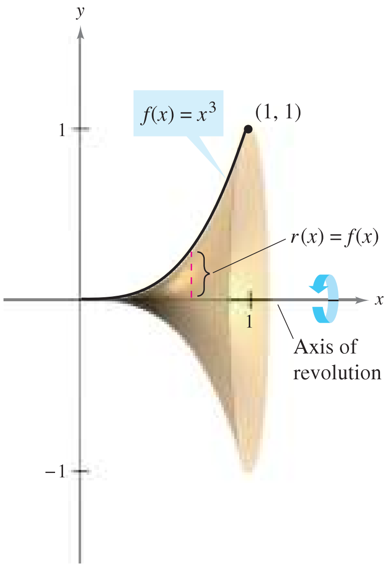


Figure 7.46

Solution The distance between the x-axis and the graph of f is $r(x) = f(x)$, and because $f'(x) = 3x^2$, the surface area is

$$\begin{aligned}
S &= 2\pi\int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,dx && \text{Formula for surface area}\\
&= 2\pi\int_0^1 x^3\sqrt{1 + (3x^2)^2}\,dx\\
&= \frac{2\pi}{36}\int_0^1 (36x^3)(1 + 9x^4)^{1/2}\,dx && \text{Simplify.}\\
&= \frac{\pi}{18}\left[\frac{(1 + 9x^4)^{3/2}}{3/2}\right]_0^1 && \text{Integrate.}\\
&= \frac{\pi}{27}(10^{3/2} - 1)\\
&\approx 3.563.
\end{aligned}$$

EXAMPLE 7 The Area of a Surface of Revolution

Find the area of the surface formed by revolving the graph of

$$f(x) = x^2$$

on the interval $[0, \sqrt{2}]$ about the y-axis, as shown in Figure 7.47.

Solution In this case, the distance between the graph of f and the y-axis is $r(x) = x$. Using $f'(x) = 2x$, you can determine that the surface area is

$$\begin{aligned}
S &= 2\pi\int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,dx && \text{Formula for surface area}\\
&= 2\pi\int_0^{\sqrt{2}} x\sqrt{1 + (2x)^2}\,dx\\
&= \frac{2\pi}{8}\int_0^{\sqrt{2}} (1 + 4x^2)^{1/2}(8x)\,dx && \text{Simplify.}\\
&= \frac{\pi}{4}\left[\frac{(1 + 4x^2)^{3/2}}{3/2}\right]_0^{\sqrt{2}} && \text{Integrate.}\\
&= \frac{\pi}{6}[(1 + 8)^{3/2} - 1]\\
&= \frac{13\pi}{3}\\
&\approx 13.614.
\end{aligned}$$

■

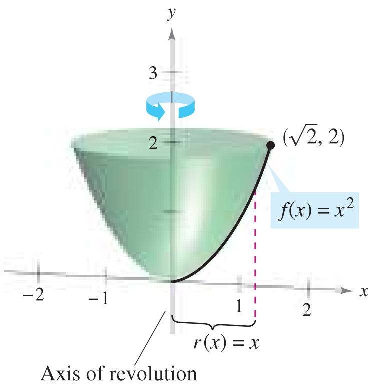


Figure 7.47

7.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

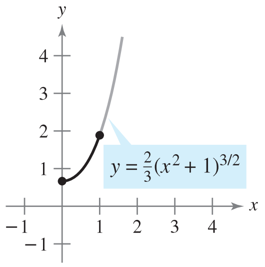
In Exercises 1 and 2, find the distance between the points using (a) the Distance Formula and (b) integration.

1. $(0, 0), \quad (8, 15)$

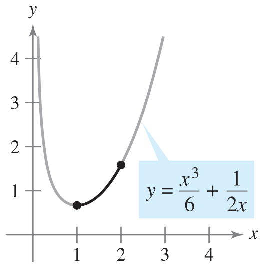
2. $(1, 2), \quad (7, 10)$

In Exercises 3–16, find the arc length of the graph of the function over the indicated interval.

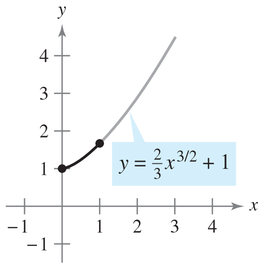
3. $y = \frac{2}{3}(x^2 + 1)^{3/2}$



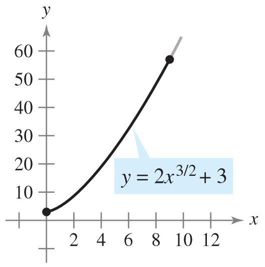
4. $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$



5. $y = \frac{2}{3}x^{3/2} + 1$



6. $y = 2x^{3/2} + 3$



7. $y = \dfrac{3}{2}x^{2/3}, \quad [1, 8]$

8. $y = \dfrac{x^4}{8} + \dfrac{1}{4x^2}, \quad [1, 3]$

9. $y = \dfrac{x^5}{10} + \dfrac{1}{6x^3}, \quad [2, 5]$

10. $y = \dfrac{3}{2}x^{2/3} + 4, \quad [1, 27]$

11. $y = \ln(\sin x), \quad \left[\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right]$

12. $y = \ln(\cos x), \quad \left[0, \dfrac{\pi}{3}\right]$

13. $y = \frac{1}{2}(e^x + e^{-x}), \quad [0, 2]$

14. $y = \ln\left(\dfrac{e^x + 1}{e^x - 1}\right), \quad [\ln 2, \ln 3]$

15. $x = \frac{1}{3}(y^2 + 2)^{3/2}, \quad 0 \le y \le 4$

16. $x = \frac{1}{3}\sqrt{y}(y - 3), \quad 1 \le y \le 4$

In Exercises 17–26, (a) sketch the graph of the function, highlighting the part indicated by the given interval, (b) find a definite integral that represents the arc length of the curve over the indicated interval and observe that the integral cannot be evaluated with the techniques studied so far, and (c) use the integration capabilities of a graphing utility to approximate the arc length.

17. $y = 4 - x^2, \quad 0 \le x \le 2$

18. $y = x^2 + x - 2, \quad -2 \le x \le 1$

19. $y = \dfrac{1}{x}, \quad 1 \le x \le 3$

20. $y = \dfrac{1}{x + 1}, \quad 0 \le x \le 1$

21. $y = \sin x, \quad 0 \le x \le \pi$

22. $y = \cos x, \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

23. $x = e^{-y}, \quad 0 \le y \le 2$

24. $y = \ln x, \quad 1 \le x \le 5$

25. $y = 2 \arctan x, \quad 0 \le x \le 1$

26. $x = \sqrt{36 - y^2}, \quad 0 \le y \le 3$

Approximation **In Exercises 27 and 28, determine which value best approximates the length of the arc represented by the integral. (Make your selection on the basis of a sketch of the arc and *not* by performing any calculations.)**

27. $\displaystyle\int_0^2 \sqrt{1 + \left[\frac{d}{dx}\left(\frac{5}{x^2 + 1}\right)\right]^2}\, dx$

(a) 25 (b) 5 (c) 2 (d) -4 (e) 3

28. $\displaystyle\int_0^{\pi/4} \sqrt{1 + \left[\frac{d}{dx}(\tan x)\right]^2}\, dx$

(a) 3 (b) -2 (c) 4 (d) $\dfrac{4\pi}{3}$ (e) 1

Approximation **In Exercises 29 and 30, approximate the arc length of the graph of the function over the interval [0, 4] in four ways. (a) Use the Distance Formula to find the distance between the endpoints of the arc. (b) Use the Distance Formula to find the lengths of the four line segments connecting the points on the arc when $x = 0$, $x = 1$, $x = 2$, $x = 3$, and $x = 4$. Find the sum of the four lengths. (c) Use Simpson's Rule with $n = 10$ to approximate the integral yielding the indicated arc length. (d) Use the integration capabilities of a graphing utility to approximate the integral yielding the indicated arc length.**

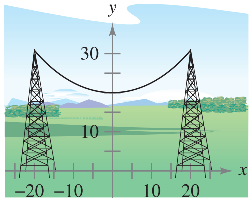
29. $f(x) = x^3$

30. $f(x) = (x^2 - 4)^2$

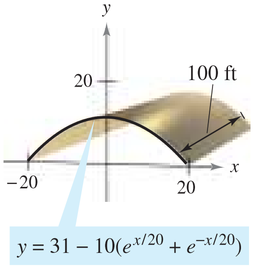
31. ***Length of a Catenary*** Electrical wires suspended between two towers form a catenary (see figure) modeled by the equation

$$y = 20 \cosh \frac{x}{20}, \quad -20 \le x \le 20$$

where x and y are measured in meters. The towers are 40 meters apart. Find the length of the suspended cable.



32. ***Roof Area*** A barn is 100 feet long and 40 feet wide (see figure). A cross section of the roof is the inverted catenary $y = 31 - 10(e^{x/20} + e^{-x/20})$. Find the number of square feet of roofing on the barn.



33. ***Length of Gateway Arch*** The Gateway Arch in St. Louis, Missouri is modeled by

$$y = 693.8597 - 68.7672 \cosh 0.0100333x,$$
$$-299.2239 \le x \le 299.2239.$$

(See Section 5.8, Section Project: St. Louis Arch.) Find the length of this curve (see figure).

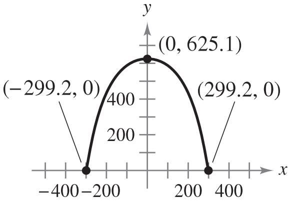


Figure for 33

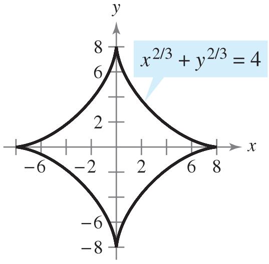


Figure for 34

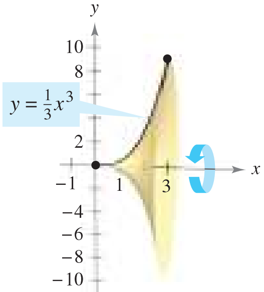
34. ***Astroid*** Find the total length of the graph of the astroid $x^{2/3} + y^{2/3} = 4$.

35. Find the arc length from $(0, 3)$ clockwise to $(2, \sqrt{5})$ along the circle $x^2 + y^2 = 9$.

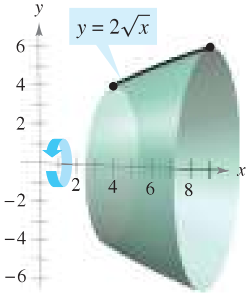
36. Find the arc length from $(-3, 4)$ clockwise to $(4, 3)$ along the circle $x^2 + y^2 = 25$. Show that the result is one-fourth the circumference of the circle.

In Exercises 37–42, set up and evaluate the definite integral for the area of the surface generated by revolving the curve about the *x*-axis.

37. $y = \frac{1}{3}x^3$



38. $y = 2\sqrt{x}$



39. $y = \frac{x^3}{6} + \frac{1}{2x}, \quad 1 \le x \le 2$

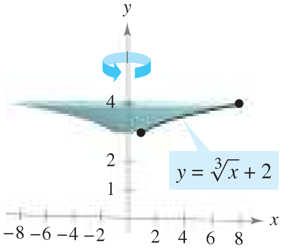
40. $y = \frac{x}{2}, \quad 0 \le x \le 6$

41. $y = \sqrt{4 - x^2}, \quad -1 \le x \le 1$

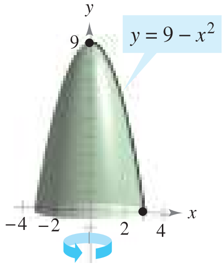
42. $y = \sqrt{9 - x^2}, \quad -2 \le x \le 2$

In Exercises 43–46, set up and evaluate the definite integral for the area of the surface generated by revolving the curve about the *y*-axis.

43. $y = \sqrt[3]{x} + 2$



44. $y = 9 - x^2$



45. $y = 1 - \frac{x^2}{4}, \quad 0 \le x \le 2$

46. $y = 2x + 5, \quad 1 \le x \le 4$

In Exercises 47 and 48, use the integration capabilities of a graphing utility to approximate the surface area of the solid of revolution.

	Function	*Interval*
47.	$y = \sin x$ revolved about the x-axis	$[0, \pi]$
48.	$y = \ln x$ revolved about the y-axis	$[1, e]$

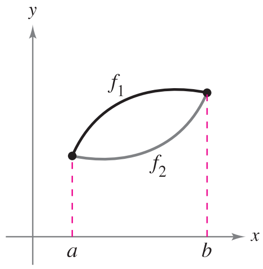
WRITING ABOUT CONCEPTS

49. Define a rectifiable curve.

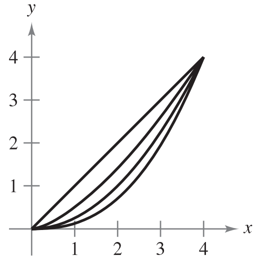
50. What precalculus formula and representative element are used to develop the integration formula for arc length?

51. What precalculus formula and representative element are used to develop the integration formula for the area of a surface of revolution?

52. The graphs of the functions f_1 and f_2 on the interval $[a, b]$ are shown in the figure. The graph of each function is revolved about the x-axis. Which surface of revolution has the greater surface area? Explain.



53. ***Think About It*** The figure shows the graphs of the functions $y_1 = x$, $y_2 = \frac{1}{2}x^{3/2}$, $y_3 = \frac{1}{4}x^2$, and $y_4 = \frac{1}{8}x^{5/2}$ on the interval $[0, 4]$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) Label the functions.

(b) List the functions in order of increasing arc length.

(c) Verify your answer in part (b) by approximating each arc length accurate to three decimal places.

CAPSTONE

54. ***Think About It*** Explain why the two integrals are equal.

$$\int_1^e \sqrt{1 + \frac{1}{x^2}}\,dx = \int_0^1 \sqrt{1 + e^{2x}}\,dx$$

Use the integration capabilities of a graphing utility to verify that the integrals are equal.

55. A right circular cone is generated by revolving the region bounded by $y = 3x/4$, $y = 3$, and $x = 0$ about the y-axis. Find the lateral surface area of the cone.

56. A right circular cone is generated by revolving the region bounded by $y = hx/r$, $y = h$, and $x = 0$ about the y-axis. Verify that the lateral surface area of the cone is

$$S = \pi r\sqrt{r^2 + h^2}.$$

57. Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{9 - x^2}$, $0 \le x \le 2$, about the y-axis.

58. Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{r^2 - x^2}$, $0 \le x \le a$, about the y-axis. Assume that $a < r$.

59. ***Modeling Data*** The circumference C (in inches) of a vase is measured at three-inch intervals starting at its base. The measurements are shown in the table, where y is the vertical distance in inches from the base.

y	0	3	6	9	12	15	18
C	50	65.5	70	66	58	51	48

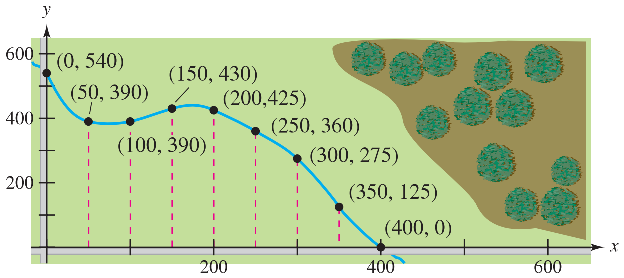
(a) Use the data to approximate the volume of the vase by summing the volumes of approximating disks.

(b) Use the data to approximate the outside surface area (excluding the base) of the vase by summing the outside surface areas of approximating frustums of right circular cones.

(c) Use the regression capabilities of a graphing utility to find a cubic model for the points (y, r) where $r = C/(2\pi)$. Use the graphing utility to plot the points and graph the model.

(d) Use the model in part (c) and the integration capabilities of a graphing utility to approximate the volume and outside surface area of the vase. Compare the results with your answers in parts (a) and (b).

60. ***Modeling Data*** Property bounded by two perpendicular roads and a stream is shown in the figure. All distances are measured in feet.



(a) Use the regression capabilities of a graphing utility to fit a fourth-degree polynomial to the path of the stream.

(b) Use the model in part (a) to approximate the area of the property in acres.

(c) Use the integration capabilities of a graphing utility to find the length of the stream that bounds the property.

61. Let R be the region bounded by $y = 1/x$, the x-axis, $x = 1$, and $x = b$, where $b > 1$. Let D be the solid formed when R is revolved about the x-axis.

(a) Find the volume V of D.

(b) Write the surface area S as an integral.

(c) Show that V approaches a finite limit as $b \to \infty$.

(d) Show that $S \to \infty$ as $b \to \infty$.

62. (a) Given a circular sector with radius L and central angle θ (see figure), show that the area of the sector is given by

$$S = \frac{1}{2}L^2\theta.$$

(b) By joining the straight-line edges of the sector in part (a), a right circular cone is formed (see figure) and the lateral surface area of the cone is the same as the area of the sector. Show that the area is $S = \pi rL$, where r is the radius of the base of the cone. (*Hint:* The arc length of the sector equals the circumference of the base of the cone.)

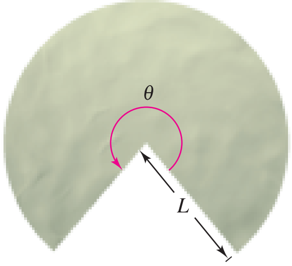


Figure for 62(a)

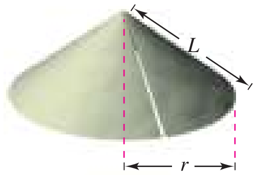
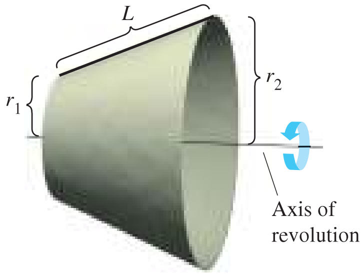


Figure for 62(b)

(c) Use the result of part (b) to verify that the formula for the lateral surface area of the frustum of a cone with slant height L and radii r_1 and r_2 (see figure) is $S = \pi(r_1 + r_2)L$. (*Note:* This formula was used to develop the integral for finding the surface area of a surface of revolution.)



63. ***Think About It*** Consider the equation $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

(a) Use a graphing utility to graph the equation.

(b) Set up the definite integral for finding the first-quadrant arc length of the graph in part (a).

(c) Compare the interval of integration in part (b) and the domain of the integrand. Is it possible to evaluate the definite integral? Is it possible to use Simpson's Rule to evaluate the definite integral? Explain. (You will learn how to evaluate this type of integral in Section 8.8.)

64. ***Writing*** Read the article "Arc Length, Area and the Arcsine Function" by Andrew M. Rockett in *Mathematics Magazine*. Then write a paragraph explaining how the arcsine function can be defined in terms of an arc length. (To view this article, go to the website *www.matharticles.com*.)

In Exercises 65–68, set up the definite integral for finding the indicated arc length or surface area. Then use the integration capabilities of a graphing utility to approximate the arc length or surface area. (You will learn how to evaluate this type of integral in Section 8.8.)

65. ***Length of Pursuit*** A fleeing object leaves the origin and moves up the y-axis (see figure). At the same time, a pursuer leaves the point $(1, 0)$ and always moves toward the fleeing object. The pursuer's speed is twice that of the fleeing object. The equation of the path is modeled by

$$y = \frac{1}{3}(x^{3/2} - 3x^{1/2} + 2).$$

How far has the fleeing object traveled when it is caught? Show that the pursuer has traveled twice as far.

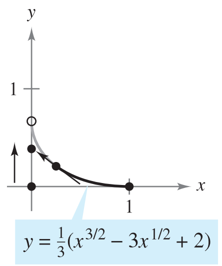


Figure for 65

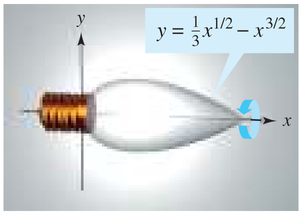


Figure for 66

66. ***Bulb Design*** An ornamental light bulb is designed by revolving the graph of $y = \frac{1}{3}x^{1/2} - x^{3/2}$, $0 \le x \le \frac{1}{3}$, about the x-axis, where x and y are measured in feet (see figure). Find the surface area of the bulb and use the result to approximate the amount of glass needed to make the bulb. (Assume that the glass is 0.015 inch thick.)

67. ***Astroid*** Find the area of the surface formed by revolving the portion in the first quadrant of the graph of $x^{2/3} + y^{2/3} = 4$, $0 \le y \le 8$ about the y-axis.

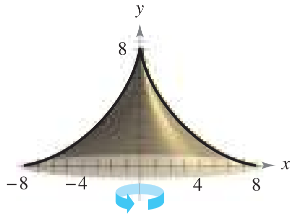


Figure for 67

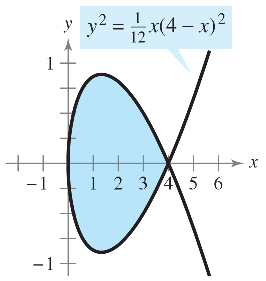
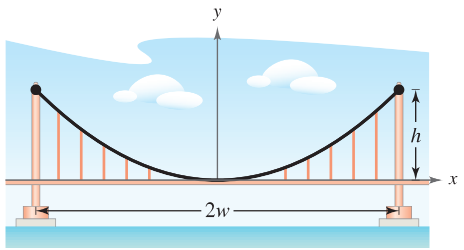


Figure for 68

68. Consider the graph of $y^2 = \frac{1}{12}x(4 - x)^2$ (see figure). Find the area of the surface formed when the loop of this graph is revolved about the x-axis.

69. ***Suspension Bridge*** A cable for a suspension bridge has the shape of a parabola with equation $y = kx^2$. Let h represent the height of the cable from its lowest point to its highest point and let $2w$ represent the total span of the bridge (see figure). Show that the length C of the cable is given by $C = 2\int_0^w \sqrt{1 + (4h^2/w^4)x^2}\,dx$.



70. ***Suspension Bridge*** The Humber Bridge, located in the United Kingdom and opened in 1981, has a main span of about 1400 meters. Each of its towers has a height of about 155 meters. Use these dimensions, the integral in Exercise 69, and the integration capabilities of a graphing utility to approximate the length of a parabolic cable along the main span.

71. Let C be the curve given by $f(x) = \cosh x$ for $0 \le x \le t$, where $t > 0$. Show that the arc length of C is equal to the area bounded by C and the x-axis. Identify another curve on the interval $0 \le x \le t$ with this property.

PUTNAM EXAM CHALLENGE

72. Find the length of the curve $y^2 = x^3$ from the origin to the point where the tangent makes an angle of 45° with the x-axis.

This problem was composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.

7.5 Work

- **Find the work done by a constant force.**
- **Find the work done by a variable force.**

Work Done by a Constant Force

The concept of work is important to scientists and engineers for determining the energy needed to perform various jobs. For instance, it is useful to know the amount of work done when a crane lifts a steel girder, when a spring is compressed, when a rocket is propelled into the air, or when a truck pulls a load along a highway.

In general, **work** is done by a force when it moves an object. If the force applied to the object is *constant*, then the definition of work is as follows.

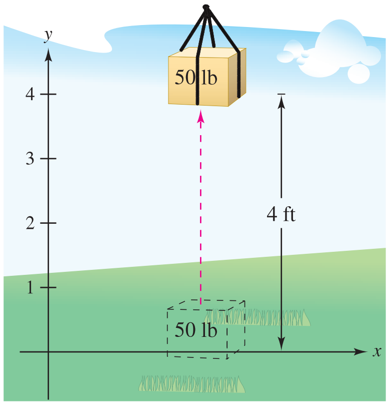
DEFINITION OF WORK DONE BY A CONSTANT FORCE

If an object is moved a distance D in the direction of an applied constant force F, then the **work** W done by the force is defined as $W = FD$.

There are many types of forces—centrifugal, electromotive, and gravitational, to name a few. A **force** can be thought of as a *push* or a *pull*; a force changes the state of rest or state of motion of a body. For gravitational forces on Earth, it is common to use units of measure corresponding to the weight of an object.

EXAMPLE 1 Lifting an Object

Determine the work done in lifting a 50-pound object 4 feet.



The work done in lifting a 50-pound object 4 feet is 200 foot-pounds.
Figure 7.48

Solution The magnitude of the required force F is the weight of the object, as shown in Figure 7.48. So, the work done in lifting the object 4 feet is

$$\begin{aligned} W &= FD && \text{Work} = \text{(force)(distance)} \\ &= 50(4) && \text{Force} = 50 \text{ pounds, distance} = 4 \text{ feet} \\ &= 200 \text{ foot-pounds.} \end{aligned}$$

■

In the U.S. measurement system, work is typically expressed in foot-pounds (ft-lb), inch-pounds, or foot-tons. In the centimeter-gram-second (C-G-S) system, the basic unit of force is the **dyne**—the force required to produce an acceleration of 1 centimeter per second per second on a mass of 1 gram. In this system, work is typically expressed in dyne-centimeters (ergs) or newton-meters (joules), where 1 joule $= 10^7$ ergs.

EXPLORATION

How Much Work? In Example 1, 200 foot-pounds of work was needed to lift the 50-pound object 4 feet vertically off the ground. Suppose that once you lifted the object, you held it and walked a horizontal distance of 4 feet. Would this require an additional 200 foot-pounds of work? Explain your reasoning.

Work Done by a Variable Force

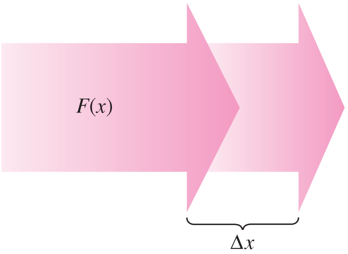
In Example 1, the force involved was *constant*. If a *variable* force is applied to an object, calculus is needed to determine the work done, because the amount of force changes as the object changes position. For instance, the force required to compress a spring increases as the spring is compressed.

Suppose that an object is moved along a straight line from $x = a$ to $x = b$ by a continuously varying force $F(x)$. Let Δ be a partition that divides the interval $[a, b]$ into n subintervals determined by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_n = b$$

and let $\Delta x_i = x_i - x_{i-1}$. For each i, choose c_i such that $x_{i-1} \leq c_i \leq x_i$. Then at c_i the force is given by $F(c_i)$. Because F is continuous, you can approximate the work done in moving the object through the ith subinterval by the increment

$$\Delta W_i = F(c_i)\,\Delta x_i$$



The amount of force changes as an object changes position (Δx).
Figure 7.49

as shown in Figure 7.49. So, the total work done as the object moves from a to b is approximated by

$$\begin{aligned} W &\approx \sum_{i=1}^{n} \Delta W_i \\ &= \sum_{i=1}^{n} F(c_i)\,\Delta x_i. \end{aligned}$$

This approximation appears to become better and better as $\|\Delta\| \to 0$ $(n \to \infty)$. So, the work done is

$$\begin{aligned} W &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} F(c_i)\,\Delta x_i \\ &= \int_a^b F(x)\,dx. \end{aligned}$$

DEFINITION OF WORK DONE BY A VARIABLE FORCE

If an object is moved along a straight line by a continuously varying force $F(x)$, then the **work** W done by the force as the object is moved from $x = a$ to $x = b$ is

$$\begin{aligned} W &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \Delta W_i \\ &= \int_a^b F(x)\,dx. \end{aligned}$$

Bettmann/Corbis

EMILIE DE BRETEUIL (1706–1749)

Another major work by Breteuil was the translation of Newton's "Philosophiae Naturalis Principia Mathematica" into French. Her translation and commentary greatly contributed to the acceptance of Newtonian science in Europe.

The remaining examples in this section use some well-known physical laws. The discoveries of many of these laws occurred during the same period in which calculus was being developed. In fact, during the seventeenth and eighteenth centuries, there was little difference between physicists and mathematicians. One such physicist-mathematician was Emilie de Breteuil. Breteuil was instrumental in synthesizing the work of many other scientists, including Newton, Leibniz, Huygens, Kepler, and Descartes. Her physics text *Institutions* was widely used for many years.

The following three laws of physics were developed by Robert Hooke (1635–1703), Isaac Newton (1642–1727), and Charles Coulomb (1736–1806).

1. **Hooke's Law:** The force F required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed or stretched from its original length. That is,

$$F = kd$$

where the constant of proportionality k (the spring constant) depends on the specific nature of the spring.

2. **Newton's Law of Universal Gravitation:** The force F of attraction between two particles of masses m_1 and m_2 is proportional to the product of the masses and inversely proportional to the square of the distance d between the two particles. That is,

$$F = k\frac{m_1 m_2}{d^2}.$$

EXPLORATION

The work done in compressing the spring in Example 2 from $x = 3$ inches to $x = 6$ inches is 3375 inch-pounds. Should the work done in compressing the spring from $x = 0$ inches to $x = 3$ inches be more than, the same as, or less than this? Explain.

If m_1 and m_2 are given in grams and d in centimeters, F will be in dynes for a value of $k = 6.670 \times 10^{-8}$ cubic centimeter per gram-second squared.

3. **Coulomb's Law:** The force F between two charges q_1 and q_2 in a vacuum is proportional to the product of the charges and inversely proportional to the square of the distance d between the two charges. That is,

$$F = k\frac{q_1 q_2}{d^2}.$$

If q_1 and q_2 are given in electrostatic units and d in centimeters, F will be in dynes for a value of $k = 1$.

EXAMPLE 2 Compressing a Spring

A force of 750 pounds compresses a spring 3 inches from its natural length of 15 inches. Find the work done in compressing the spring an additional 3 inches.

Solution By Hooke's Law, the force $F(x)$ required to compress the spring x units (from its natural length) is $F(x) = kx$. Using the given data, it follows that $F(3) = 750 = (k)(3)$ and so $k = 250$ and $F(x) = 250x$, as shown in Figure 7.50. To find the increment of work, assume that the force required to compress the spring over a small increment Δx is nearly constant. So, the increment of work is

$$\Delta W = (\text{force})(\text{distance increment}) = (250x)\,\Delta x.$$

Because the spring is compressed from $x = 3$ to $x = 6$ inches less than its natural length, the work required is

$$W = \int_a^b F(x)\,dx = \int_3^6 250x\,dx \qquad \text{Formula for work}$$

$$= 125x^2\Big]_3^6 = 4500 - 1125 = 3375 \text{ inch-pounds.}$$

Note that you do *not* integrate from $x = 0$ to $x = 6$ because you were asked to determine the work done in compressing the spring an *additional* 3 inches (not including the first 3 inches). ■

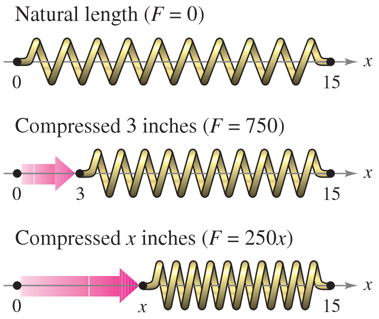


Figure 7.50

EXAMPLE 3 Moving a Space Module into Orbit

A space module weighs 15 metric tons on the surface of Earth. How much work is done in propelling the module to a height of 800 miles above Earth, as shown in Figure 7.51? (Use 4000 miles as the radius of Earth. Do not consider the effect of air resistance or the weight of the propellant.)

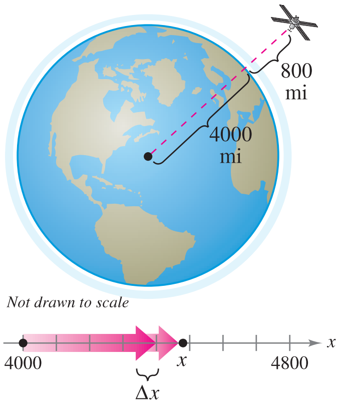


Figure 7.51

Solution Because the weight of a body varies inversely as the square of its distance from the center of Earth, the force $F(x)$ exerted by gravity is

$$F(x) = \frac{C}{x^2}.$$

C is the constant of proportionality.

Because the module weighs 15 metric tons on the surface of Earth and the radius of Earth is approximately 4000 miles, you have

$$\begin{aligned} 15 &= \frac{C}{(4000)^2} \\ 240{,}000{,}000 &= C. \end{aligned}$$

So, the increment of work is

$$\begin{aligned} \Delta W &= \text{(force)(distance increment)} \\ &= \frac{240{,}000{,}000}{x^2}\,\Delta x. \end{aligned}$$

Finally, because the module is propelled from $x = 4000$ to $x = 4800$ miles, the total work done is

$$\begin{aligned} W &= \int_a^b F(x)\,dx = \int_{4000}^{4800} \frac{240{,}000{,}000}{x^2}\,dx && \text{Formula for work} \\ &= \frac{-240{,}000{,}000}{x}\Bigg]_{4000}^{4800} && \text{Integrate.} \\ &= -50{,}000 + 60{,}000 \\ &= 10{,}000 \text{ mile-tons} \\ &\approx 1.164 \times 10^{11} \text{ foot-pounds.} \end{aligned}$$

In the C-G-S system, using a conversion factor of 1 foot-pound ≈ 1.35582 joules, the work done is

$$W \approx 1.578 \times 10^{11} \text{ joules.}$$

■

The solutions to Examples 2 and 3 conform to our development of work as the summation of increments in the form

$$\Delta W = \text{(force)(distance increment)} = (F)(\Delta x).$$

Another way to formulate the increment of work is

$$\Delta W = \text{(force increment)(distance)} = (\Delta F)(x).$$

This second interpretation of ΔW is useful in problems involving the movement of nonrigid substances such as fluids and chains.

EXAMPLE 4 Emptying a Tank of Oil

A spherical tank of radius 8 feet is half full of oil that weighs 50 pounds per cubic foot. Find the work required to pump oil out through a hole in the top of the tank.

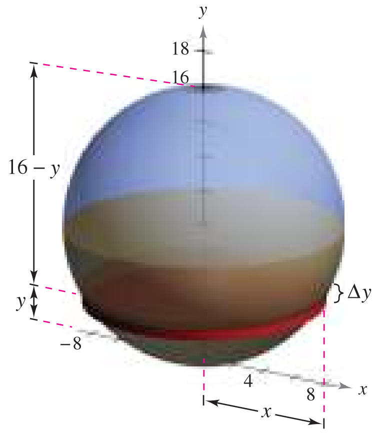


Figure 7.52

Solution Consider the oil to be subdivided into disks of thickness Δy and radius x, as shown in Figure 7.52. Because the increment of force for each disk is given by its weight, you have

$$\begin{aligned}\Delta F &= \text{weight}\\ &= \left(\frac{50 \text{ pounds}}{\text{cubic foot}}\right)(\text{volume})\\ &= 50(\pi x^2 \Delta y) \text{ pounds.}\end{aligned}$$

For a circle of radius 8 and center at $(0, 8)$, you have

$$\begin{aligned}x^2 + (y - 8)^2 &= 8^2\\ x^2 &= 16y - y^2\end{aligned}$$

and you can write the force increment as

$$\begin{aligned}\Delta F &= 50(\pi x^2 \Delta y)\\ &= 50\pi(16y - y^2)\,\Delta y.\end{aligned}$$

In Figure 7.52, note that a disk y feet from the bottom of the tank must be moved a distance of $(16 - y)$ feet. So, the increment of work is

$$\begin{aligned}\Delta W &= \Delta F(16 - y)\\ &= 50\pi(16y - y^2)\,\Delta y(16 - y)\\ &= 50\pi(256y - 32y^2 + y^3)\,\Delta y.\end{aligned}$$

Because the tank is half full, y ranges from 0 to 8, and the work required to empty the tank is

$$\begin{aligned}W &= \int_0^8 50\pi(256y - 32y^2 + y^3)\,dy\\ &= 50\pi\left[128y^2 - \frac{32}{3}y^3 + \frac{y^4}{4}\right]_0^8\\ &= 50\pi\left(\frac{11{,}264}{3}\right)\\ &\approx 589{,}782 \text{ foot-pounds.}\end{aligned}$$

■

To estimate the reasonableness of the result in Example 4, consider that the weight of the oil in the tank is

$$\begin{aligned}\left(\frac{1}{2}\right)(\text{volume})(\text{density}) &= \frac{1}{2}\left(\frac{4}{3}\pi 8^3\right)(50)\\ &\approx 53{,}616.5 \text{ pounds.}\end{aligned}$$

Lifting the entire half-tank of oil 8 feet would involve work of $8(53{,}616.5) \approx 428{,}932$ foot-pounds. Because the oil is actually lifted between 8 and 16 feet, it seems reasonable that the work done is 589,782 foot-pounds.

EXAMPLE 5 Lifting a Chain

A 20-foot chain weighing 5 pounds per foot is lying coiled on the ground. How much work is required to raise one end of the chain to a height of 20 feet so that it is fully extended, as shown in Figure 7.53?

Solution Imagine that the chain is divided into small sections, each of length Δy. Then the weight of each section is the increment of force

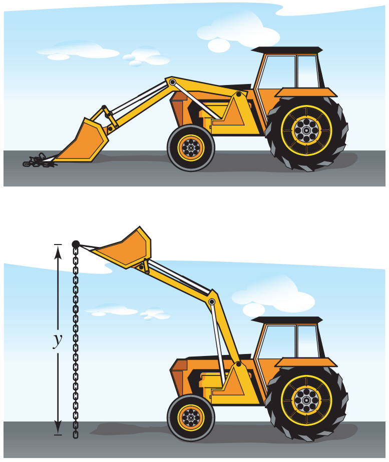
$$\Delta F = (\text{weight}) = \left(\frac{5 \text{ pounds}}{\text{foot}}\right)(\text{length}) = 5\Delta y.$$

Because a typical section (initially on the ground) is raised to a height of y, the increment of work is

$$\Delta W = (\text{force increment})(\text{distance}) = (5\,\Delta y)y = 5y\,\Delta y.$$

Because y ranges from 0 to 20, the total work is

$$W = \int_0^{20} 5y\,dy = \frac{5y^2}{2}\bigg]_0^{20} = \frac{5(400)}{2} = 1000 \text{ foot-pounds.} \qquad ■$$



Work required to raise one end of the chain
Figure 7.53

In the next example you will consider a piston of radius r in a cylindrical casing, as shown in Figure 7.54. As the gas in the cylinder expands, the piston moves and work is done. If p represents the pressure of the gas (in pounds per square foot) against the piston head and V represents the volume of the gas (in cubic feet), the work increment involved in moving the piston Δx feet is

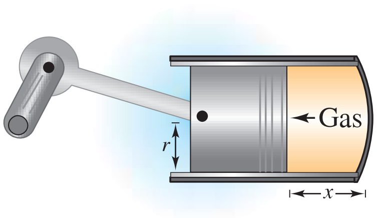
$$\Delta W = (\text{force})(\text{distance increment}) = F(\Delta x) = p(\pi r^2)\,\Delta x = p\,\Delta V.$$

So, as the volume of the gas expands from V_0 to V_1, the work done in moving the piston is

$$W = \int_{V_0}^{V_1} p\,dV.$$

Assuming the pressure of the gas to be inversely proportional to its volume, you have $p = k/V$ and the integral for work becomes

$$W = \int_{V_0}^{V_1} \frac{k}{V}\,dV.$$



Work done by expanding gas
Figure 7.54

EXAMPLE 6 Work Done by an Expanding Gas

A quantity of gas with an initial volume of 1 cubic foot and a pressure of 500 pounds per square foot expands to a volume of 2 cubic feet. Find the work done by the gas. (Assume that the pressure is inversely proportional to the volume.)

Solution Because $p = k/V$ and $p = 500$ when $V = 1$, you have $k = 500$. So, the work is

$$\begin{aligned} W &= \int_{V_0}^{V_1} \frac{k}{V}\,dV \\ &= \int_1^2 \frac{500}{V}\,dV \\ &= 500 \ln|V|\bigg]_1^2 \approx 346.6 \text{ foot-pounds.} \qquad ■ \end{aligned}$$

7.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Constant Force **In Exercises 1–4, determine the work done by the constant force.**

1. A 100-pound bag of sugar is lifted 20 feet.
2. An electric hoist lifts a 3500-pound car 4 feet.
3. A force of 112 newtons is required to slide a cement block 8 meters in a construction project.
4. The locomotive of a freight train pulls its cars with a constant force of 9 tons a distance of one-half mile.

Hooke's Law **In Exercises 5–12, use Hooke's Law to determine the variable force in the spring problem.**

5. A force of 5 pounds compresses a 15-inch spring a total of 3 inches. How much work is done in compressing the spring 7 inches?
6. How much work is done in compressing the spring in Exercise 5 from a length of 10 inches to a length of 6 inches?
7. A force of 250 newtons stretches a spring 30 centimeters. How much work is done in stretching the spring from 20 centimeters to 50 centimeters?
8. A force of 800 newtons stretches a spring 70 centimeters on a mechanical device for driving fence posts. Find the work done in stretching the spring the required 70 centimeters.
9. A force of 20 pounds stretches a spring 9 inches in an exercise machine. Find the work done in stretching the spring 1 foot from its natural position.
10. An overhead garage door has two springs, one on each side of the door. A force of 15 pounds is required to stretch each spring 1 foot. Because of the pulley system, the springs stretch only one-half the distance the door travels. The door moves a total of 8 feet and the springs are at their natural length when the door is open. Find the work done by the pair of springs.
11. Eighteen foot-pounds of work is required to stretch a spring 4 inches from its natural length. Find the work required to stretch the spring an additional 3 inches.
12. Seven and one-half foot-pounds of work is required to compress a spring 2 inches from its natural length. Find the work required to compress the spring an additional one-half inch.

13. ***Propulsion*** Neglecting air resistance and the weight of the propellant, determine the work done in propelling a five-ton satellite to a height of

    (a) 100 miles above Earth.

    (b) 300 miles above Earth.

14. ***Propulsion*** Use the information in Exercise 13 to write the work W of the propulsion system as a function of the height h of the satellite above Earth. Find the limit (if it exists) of W as h approaches infinity.

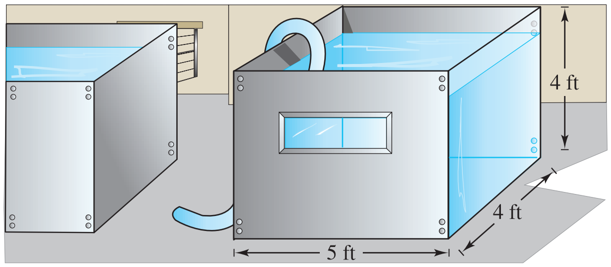
15. ***Propulsion*** Neglecting air resistance and the weight of the propellant, determine the work done in propelling a 10-ton satellite to a height of

    (a) 11,000 miles above Earth.

    (b) 22,000 miles above Earth.

16. ***Propulsion*** A lunar module weighs 12 tons on the surface of Earth. How much work is done in propelling the module from the surface of the moon to a height of 50 miles? Consider the radius of the moon to be 1100 miles and its force of gravity to be one-sixth that of Earth.

17. ***Pumping Water*** A rectangular tank with a base 4 feet by 5 feet and a height of 4 feet is full of water (see figure). The water weighs 62.4 pounds per cubic foot. How much work is done in pumping water out over the top edge in order to empty (a) half of the tank? (b) all of the tank?



18. ***Think About It*** Explain why the answer in part (b) of Exercise 17 is not twice the answer in part (a).

19. ***Pumping Water*** A cylindrical water tank 4 meters high with a radius of 2 meters is buried so that the top of the tank is 1 meter below ground level (see figure). How much work is done in pumping a full tank of water up to ground level? (The water weighs 9800 newtons per cubic meter.)

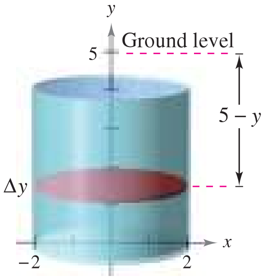


Figure for 19

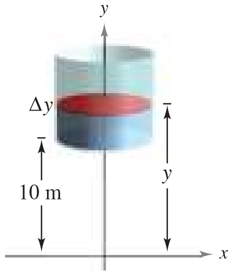


Figure for 20

20. ***Pumping Water*** Suppose the tank in Exercise 19 is located on a tower so that the bottom of the tank is 10 meters above the level of a stream (see figure). How much work is done in filling the tank half full of water through a hole in the bottom, using water from the stream?

21. ***Pumping Water*** An open tank has the shape of a right circular cone (see figure on the next page). The tank is 8 feet across the top and 6 feet high. How much work is done in emptying the tank by pumping the water over the top edge?

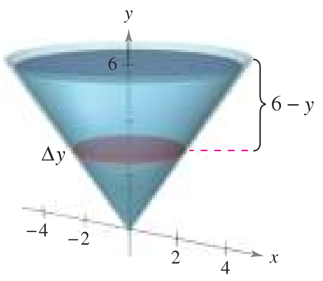


Figure for 21

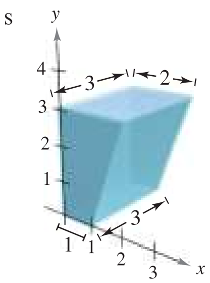


Figure for 24

22. ***Pumping Water*** Water is pumped in through the bottom of the tank in Exercise 21. How much work is done to fill the tank

(a) to a depth of 2 feet?

(b) from a depth of 4 feet to a depth of 6 feet?

23. ***Pumping Water*** A hemispherical tank of radius 6 feet is positioned so that its base is circular. How much work is required to fill the tank with water through a hole in the base if the water source is at the base?

24. ***Pumping Diesel Fuel*** The fuel tank on a truck has trapezoidal cross sections with the dimensions (in feet) shown in the figure. Assume that the engine is approximately 3 feet above the top of the fuel tank and that diesel fuel weighs approximately 53.1 pounds per cubic foot. Find the work done by the fuel pump in raising a full tank of fuel to the level of the engine.

Pumping Gasoline **In Exercises 25 and 26, find the work done in pumping gasoline that weighs 42 pounds per cubic foot. (*Hint:* Evaluate one integral by a geometric formula and the other by observing that the integrand is an odd function.)**

25. A cylindrical gasoline tank 3 feet in diameter and 4 feet long is carried on the back of a truck and is used to fuel tractors. The axis of the tank is horizontal. The opening on the tractor tank is 5 feet above the top of the tank in the truck. Find the work done in pumping the entire contents of the fuel tank into the tractor.

26. The top of a cylindrical gasoline storage tank at a service station is 4 feet below ground level. The axis of the tank is horizontal and its diameter and length are 5 feet and 12 feet, respectively. Find the work done in pumping the entire contents of the full tank to a height of 3 feet above ground level.

Lifting a Chain **In Exercises 27–30, consider a 20-foot chain that weighs 3 pounds per foot hanging from a winch 20 feet above ground level. Find the work done by the winch in winding up the specified amount of chain.**

27. Wind up the entire chain.

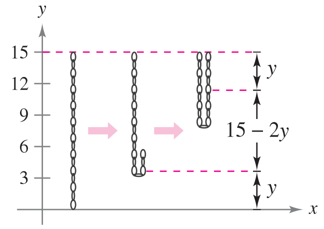
28. Wind up one-third of the chain.

29. Run the winch until the bottom of the chain is at the 10-foot level.

30. Wind up the entire chain with a 500-pound load attached to it.

Lifting a Chain **In Exercises 31 and 32, consider a 15-foot hanging chain that weighs 3 pounds per foot. Find the work done in lifting the chain vertically to the indicated position.**

31. Take the bottom of the chain and raise it to the 15-foot level, leaving the chain doubled and still hanging vertically (see figure).



32. Repeat Exercise 31 raising the bottom of the chain to the 12-foot level.

WRITING ABOUT CONCEPTS

33. State the definition of work done by a constant force.

34. State the definition of work done by a variable force.

35. Which of the following requires more work? Explain your reasoning.

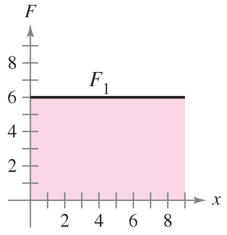
(a) A 60-pound box of books is lifted 3 feet.

(b) A 60-pound box of books is held 3 feet in the air for 2 minutes.

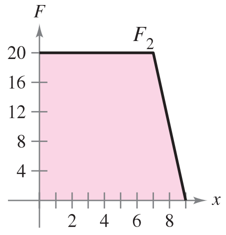
CAPSTONE

36. The graphs show the force F_i (in pounds) required to move an object 9 feet along the x-axis. Order the force functions from the one that yields the least work to the one that yields the most work without doing any calculations. Explain your reasoning.

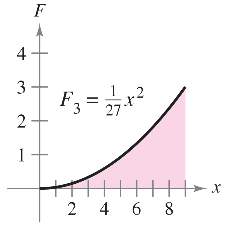
(a)



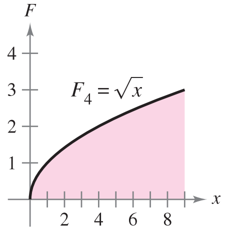
(b)



(c)



(d)



37. Verify your answer to Exercise 36 by calculating the work for each force function.

38. ***Demolition Crane*** Consider a demolition crane with a 50-pound ball suspended from a 40-foot cable that weighs 2 pounds per foot.

(a) Find the work required to wind up 15 feet of the apparatus.

(b) Find the work required to wind up all 40 feet of the apparatus.

Boyle's Law **In Exercises 39 and 40, find the work done by the gas for the given volume and pressure. Assume that the pressure is inversely proportional to the volume. (See Example 6.)**

39. A quantity of gas with an initial volume of 2 cubic feet and a pressure of 1000 pounds per square foot expands to a volume of 3 cubic feet.

40. A quantity of gas with an initial volume of 1 cubic foot and a pressure of 2500 pounds per square foot expands to a volume of 3 cubic feet.

41. ***Electric Force*** Two electrons repel each other with a force that varies inversely as the square of the distance between them. One electron is fixed at the point $(2, 4)$. Find the work done in moving the second electron from $(-2, 4)$ to $(1, 4)$.

42. ***Modeling Data*** The hydraulic cylinder on a woodsplitter has a four-inch bore (diameter) and a stroke of 2 feet. The hydraulic pump creates a maximum pressure of 2000 pounds per square inch. Therefore, the maximum force created by the cylinder is $2000(\pi 2^2) = 8000\pi$ pounds.

(a) Find the work done through one extension of the cylinder given that the maximum force is required.

(b) The force exerted in splitting a piece of wood is variable. Measurements of the force obtained in splitting a piece of wood are shown in the table. The variable x measures the extension of the cylinder in feet, and F is the force in pounds. Use Simpson's Rule to approximate the work done in splitting the piece of wood.

x	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2
$F(x)$	0	20,000	22,000	15,000	10,000	5000	0

Table for 42(b)

(c) Use the regression capabilities of a graphing utility to find a fourth-degree polynomial model for the data. Plot the data and graph the model.

(d) Use the model in part (c) to approximate the extension of the cylinder when the force is maximum.

(e) Use the model in part (c) to approximate the work done in splitting the piece of wood.

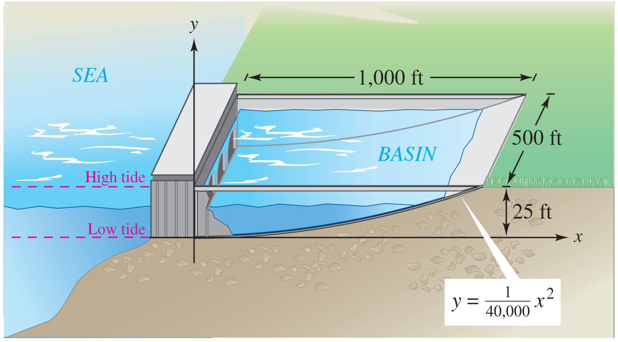
Hydraulic Press **In Exercises 43–46, use the integration capabilities of a graphing utility to approximate the work done by a press in a manufacturing process. A model for the variable force F (in pounds) and the distance x (in feet) the press moves is given.**

	Force	*Interval*
43.	$F(x) = 1000[1.8 - \ln(x + 1)]$	$0 \le x \le 5$
44.	$F(x) = \dfrac{e^{x^2} - 1}{100}$	$0 \le x \le 4$
45.	$F(x) = 100x\sqrt{125 - x^3}$	$0 \le x \le 5$
46.	$F(x) = 1000 \sinh x$	$0 \le x \le 2$

SECTION PROJECT

Tidal Energy

Tidal power plants use "tidal energy" to produce electrical energy. To construct a tidal power plant, a dam is built to separate a basin from the sea. Electrical energy is produced as the water flows back and forth between the basin and the sea. The amount of "natural energy" produced depends on the volume of the basin and the tidal range—the vertical distance between high and low tides. (Several natural basins have tidal ranges in excess of 15 feet; the Bay of Fundy in Nova Scotia has a tidal range of 53 feet.)



(a) Consider a basin with a rectangular base, as shown in the figure. The basin has a tidal range of 25 feet, with low tide corresponding to $y = 0$. How much water does the basin hold at high tide?

(b) The amount of energy produced during the filling (or the emptying) of the basin is proportional to the amount of work required to fill (or empty) the basin. How much work is required to fill the basin with seawater? (Use a seawater density of 64 pounds per cubic foot.)

Andrew J. Martinez/Photo Researchers, Inc.

Andrew J. Martinez/Photo Researchers, Inc.

The Bay of Fundy in Nova Scotia has an extreme tidal range, as displayed in the greatly contrasting photos above.

■ **FOR FURTHER INFORMATION** For more information on tidal power, see the article "LaRance: Six Years of Operating a Tidal Power Plant in France" by J. Cotillon in *Water Power Magazine*.

7.6 Moments, Centers of Mass, and Centroids

- **Understand the definition of mass.**
- **Find the center of mass in a one-dimensional system.**
- **Find the center of mass in a two-dimensional system.**
- **Find the center of mass of a planar lamina.**
- **Use the Theorem of Pappus to find the volume of a solid of revolution.**

Mass

In this section you will study several important applications of integration that are related to **mass.** Mass is a measure of a body's resistance to changes in motion, and is independent of the particular gravitational system in which the body is located. However, because so many applications involving mass occur on Earth's surface, an object's mass is sometimes equated with its weight. This is not technically correct. Weight is a type of force and as such is dependent on gravity. Force and mass are related by the equation

$$\text{Force} = (\text{mass})(\text{acceleration}).$$

The table below lists some commonly used measures of mass and force, together with their conversion factors.

System of Measurement	Measure of Mass	Measure of Force
U.S.	Slug	Pound $=$ (slug)(ft/sec^2)
International	Kilogram	Newton $=$ (kilogram)(m/sec^2)
C-G-S	Gram	Dyne $=$ (gram)(cm/sec^2)

Conversions:

1 pound $=$ 4.448 newtons	1 slug $=$ 14.59 kilograms
1 newton $=$ 0.2248 pound	1 kilogram $=$ 0.06852 slug
1 dyne $=$ 0.000002248 pound	1 gram $=$ 0.00006852 slug
1 dyne $=$ 0.00001 newton	1 foot $=$ 0.3048 meter

EXAMPLE 1 Mass on the Surface of Earth

Find the mass (in slugs) of an object whose weight at sea level is 1 pound.

Solution Using 32 feet per second per second as the acceleration due to gravity produces

$$\begin{aligned}\text{Mass} &= \frac{\text{force}}{\text{acceleration}} && \text{Force} = (\text{mass})(\text{acceleration})\\ &= \frac{1\text{ pound}}{32\text{ feet per second per second}}\\ &= 0.03125\,\frac{\text{pound}}{\text{foot per second per second}}\\ &= 0.03125\text{ slug.}\end{aligned}$$

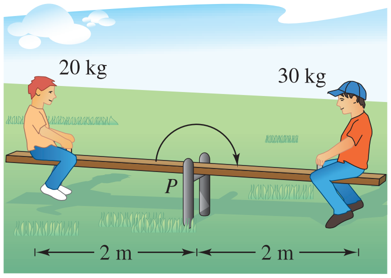
Because many applications involving mass occur on Earth's surface, this amount of mass is called a **pound mass.** ■

Center of Mass in a One-Dimensional System

You will now consider two types of moments of a mass—the **moment about a point** and the **moment about a line.** To define these two moments, consider an idealized situation in which a mass m is concentrated at a point. If x is the distance between this point mass and another point P, the **moment of m about the point P is**

$$\text{Moment} = mx$$

and x is the **length of the moment arm.**



The seesaw will balance when the left and the right moments are equal.
Figure 7.55

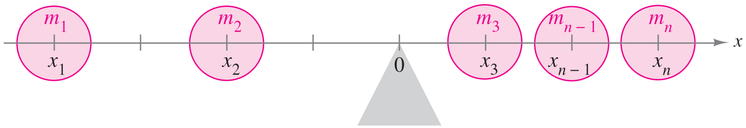
The concept of moment can be demonstrated simply by a seesaw, as shown in Figure 7.55. A child of mass 20 kilograms sits 2 meters to the left of fulcrum P, and an older child of mass 30 kilograms sits 2 meters to the right of P. From experience, you know that the seesaw will begin to rotate clockwise, moving the larger child down. This rotation occurs because the moment produced by the child on the left is less than the moment produced by the child on the right.

$$\text{Left moment} = (20)(2) = 40 \text{ kilogram-meters}$$
$$\text{Right moment} = (30)(2) = 60 \text{ kilogram-meters}$$

To balance the seesaw, the two moments must be equal. For example, if the larger child moved to a position $\frac{4}{3}$ meters from the fulcrum, the seesaw would balance, because each child would produce a moment of 40 kilogram-meters.

To generalize this, you can introduce a coordinate line on which the origin corresponds to the fulcrum, as shown in Figure 7.56. Suppose several point masses are located on the x-axis. The measure of the tendency of this system to rotate about the origin is the **moment about the origin,** and it is defined as the sum of the n products $m_i x_i$.

$$M_0 = m_1x_1 + m_2x_2 + \cdot\cdot\cdot + m_nx_n$$



If $m_1x_1 + m_2x_2 + \cdot\cdot\cdot + m_nx_n = 0$, the system is in equilibrium.
Figure 7.56

If M_0 is 0, the system is said to be in **equilibrium.**

For a system that is not in equilibrium, the **center of mass** is defined as the point $\bar{x}$ at which the fulcrum could be relocated to attain equilibrium. If the system were translated $\bar{x}$ units, each coordinate x_i would become $(x_i - \bar{x})$, and because the moment of the translated system is 0, you have

$$\sum_{i=1}^{n} m_i(x_i - \bar{x}) = \sum_{i=1}^{n} m_ix_i - \sum_{i=1}^{n} m_i\bar{x} = 0.$$

Solving for $\bar{x}$ produces

$$\bar{x} = \frac{\displaystyle\sum_{i=1}^{n} m_ix_i}{\displaystyle\sum_{i=1}^{n} m_i} = \frac{\text{moment of system about origin}}{\text{total mass of system}}.$$

If $m_1x_1 + m_2x_2 + \cdot\cdot\cdot + m_nx_n = 0$, the system is in equilibrium.

MOMENTS AND CENTER OF MASS: ONE-DIMENSIONAL SYSTEM

Let the point masses $m_1, m_2, \ldots, m_n$ be located at $x_1, x_2, \ldots, x_n$.

1. The **moment about the origin** is $M_0 = m_1x_1 + m_2x_2 + \cdots + m_nx_n$.
2. The **center of mass** is $\bar{x} = \dfrac{M_0}{m}$, where $m = m_1 + m_2 + \cdots + m_n$ is the **total mass** of the system.

EXAMPLE 2 The Center of Mass of a Linear System

Find the center of mass of the linear system shown in Figure 7.57.

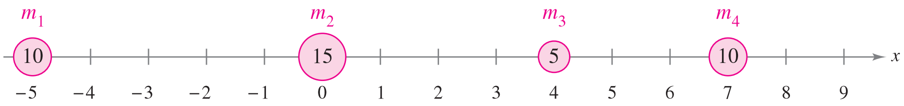


Figure 7.57

Solution The moment about the origin is

$$\begin{aligned} M_0 &= m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 \\ &= 10(-5) + 15(0) + 5(4) + 10(7) \\ &= -50 + 0 + 20 + 70 \\ &= 40. \end{aligned}$$

Because the total mass of the system is $m = 10 + 15 + 5 + 10 = 40$, the center of mass is

$$\bar{x} = \frac{M_0}{m} = \frac{40}{40} = 1.$$

■

NOTE In Example 2, where should you locate the fulcrum so that the point masses will be in equilibrium? ■

Rather than define the moment of a mass, you could define the moment of a *force*. In this context, the center of mass is called the **center of gravity.** Suppose that a system of point masses $m_1, m_2, \ldots, m_n$ is located at $x_1, x_2, \ldots, x_n$. Then, because force = (mass)(acceleration), the total force of the system is

$$\begin{aligned} F &= m_1a + m_2a + \cdots + m_na \\ &= ma. \end{aligned}$$

The **torque** (moment) about the origin is

$$\begin{aligned} T_0 &= (m_1a)x_1 + (m_2a)x_2 + \cdots + (m_na)x_n \\ &= M_0a \end{aligned}$$

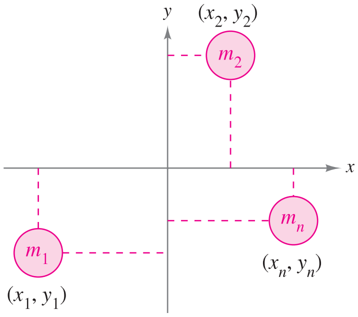
and the **center of gravity** is

$$\frac{T_0}{F} = \frac{M_0a}{ma} = \frac{M_0}{m} = \bar{x}.$$

So, the center of gravity and the center of mass have the same location.

Center of Mass in a Two-Dimensional System

You can extend the concept of moment to two dimensions by considering a system of masses located in the xy-plane at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, as shown in Figure 7.58. Rather than defining a single moment (with respect to the origin), two moments are defined—one with respect to the x-axis and one with respect to the y-axis.



In a two-dimensional system, there is a moment about the y-axis, M_y, and a moment about the x-axis, M_x.

Figure 7.58

MOMENT AND CENTER OF MASS: TWO-DIMENSIONAL SYSTEM

Let the point masses $m_1, m_2, \ldots, m_n$ be located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

1. The **moment about the y-axis** is $M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n$.
2. The **moment about the x-axis** is $M_x = m_1y_1 + m_2y_2 + \cdots + m_ny_n$.
3. The **center of mass** $(\bar{x}, \bar{y})$ (or **center of gravity**) is

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

where $m = m_1 + m_2 + \cdots + m_n$ is the **total mass** of the system.

The moment of a system of masses in the plane can be taken about any horizontal or vertical line. In general, the moment about a line is the sum of the product of the masses and the *directed distances* from the points to the line.

$$\text{Moment} = m_1(y_1 - b) + m_2(y_2 - b) + \cdots + m_n(y_n - b) \quad \text{Horizontal line } y = b$$
$$\text{Moment} = m_1(x_1 - a) + m_2(x_2 - a) + \cdots + m_n(x_n - a) \quad \text{Vertical line } x = a$$

EXAMPLE 3 The Center of Mass of a Two-Dimensional System

Find the center of mass of a system of point masses $m_1 = 6$, $m_2 = 3$, $m_3 = 2$, and $m_4 = 9$, located at

$$(3, -2), (0, 0), (-5, 3), \text{ and } (4, 2)$$

as shown in Figure 7.59.

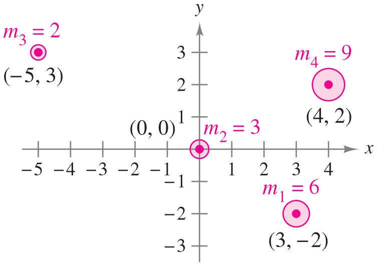


Figure 7.59

Solution

$$\begin{aligned} m &= 6 + 3 + 2 + 9 = 20 && \text{Mass} \\ M_y &= 6(3) + 3(0) + 2(-5) + 9(4) = 44 && \text{Moment about } y\text{-axis} \\ M_x &= 6(-2) + 3(0) + 2(3) + 9(2) = 12 && \text{Moment about } x\text{-axis} \end{aligned}$$

So,

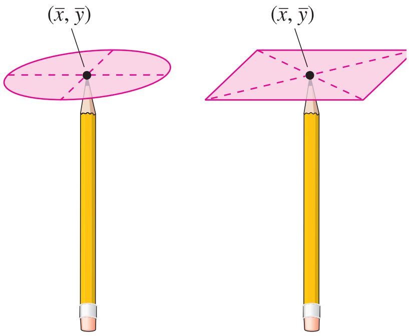
$$\bar{x} = \frac{M_y}{m} = \frac{44}{20} = \frac{11}{5}$$

and

$$\bar{y} = \frac{M_x}{m} = \frac{12}{20} = \frac{3}{5}$$

and so the center of mass is $\left(\frac{11}{5}, \frac{3}{5}\right)$. ■

Center of Mass of a Planar Lamina



You can think of the center of mass $(\bar{x}, \bar{y})$ of a lamina as its balancing point. For a circular lamina, the center of mass is the center of the circle. For a rectangular lamina, the center of mass is the center of the rectangle.
Figure 7.60

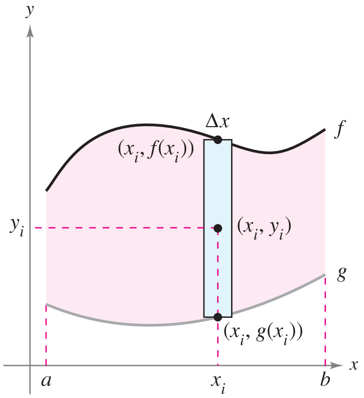
So far in this section you have assumed the total mass of a system to be distributed at discrete points in a plane or on a line. Now consider a thin, flat plate of material of constant density called a **planar lamina** (see Figure 7.60). **Density** is a measure of mass per unit of volume, such as grams per cubic centimeter. For planar laminas, however, density is considered to be a measure of mass per unit of area. Density is denoted by ρ, the lowercase Greek letter rho.

Consider an irregularly shaped planar lamina of uniform density ρ, bounded by the graphs of $y = f(x)$, $y = g(x)$, and $a \leq x \leq b$, as shown in Figure 7.61. The mass of this region is given by

$$\begin{aligned} m &= (\text{density})(\text{area}) \\ &= \rho \int_a^b [f(x) - g(x)]\, dx \\ &= \rho A \end{aligned}$$

where A is the area of the region. To find the center of mass of this lamina, partition the interval $[a, b]$ into n subintervals of equal width Δx. Let x_i be the center of the ith subinterval. You can approximate the portion of the lamina lying in the ith subinterval by a rectangle whose height is $h = f(x_i) - g(x_i)$. Because the density of the rectangle is ρ, its mass is

$$\begin{aligned} m_i &= (\text{density})(\text{area}) \\ &= \underbrace{\rho}_{\text{Density}}\underbrace{[f(x_i) - g(x_i)]}_{\text{Height}}\underbrace{\Delta x}_{\text{Width}}. \end{aligned}$$



Planar lamina of uniform density ρ
Figure 7.61

Now, considering this mass to be located at the center (x_i, y_i) of the rectangle, the directed distance from the x-axis to (x_i, y_i) is $y_i = [f(x_i) + g(x_i)]/2$. So, the moment of m_i about the x-axis is

$$\begin{aligned} \text{Moment} &= (\text{mass})(\text{distance}) \\ &= m_i y_i \\ &= \rho[f(x_i) - g(x_i)]\, \Delta x\left[\frac{f(x_i) + g(x_i)}{2}\right]. \end{aligned}$$

Summing the moments and taking the limit as $n \to \infty$ suggest the definitions below.

MOMENTS AND CENTER OF MASS OF A PLANAR LAMINA

Let f and g be continuous functions such that $f(x) \geq g(x)$ on $[a, b]$, and consider the planar lamina of uniform density ρ bounded by the graphs of $y = f(x)$, $y = g(x)$, and $a \leq x \leq b$.

1. The **moments about the x- and y-axes** are

$$M_x = \rho \int_a^b \left[\frac{f(x) + g(x)}{2}\right][f(x) - g(x)]\, dx$$

$$M_y = \rho \int_a^b x[f(x) - g(x)]\, dx.$$

2. The **center of mass** $(\bar{x}, \bar{y})$ is given by $\bar{x} = \dfrac{M_y}{m}$ and $\bar{y} = \dfrac{M_x}{m}$, where $m = \rho \int_a^b [f(x) - g(x)]\, dx$ is the mass of the lamina.

EXAMPLE 4 The Center of Mass of a Planar Lamina

Find the center of mass of the lamina of uniform density ρ bounded by the graph of $f(x) = 4 - x^2$ and the x-axis.

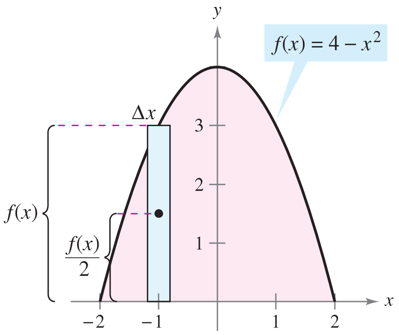


Figure 7.62

Solution Because the center of mass lies on the axis of symmetry, you know that $\bar{x} = 0$. Moreover, the mass of the lamina is

$$m = \rho \int_{-2}^{2} (4 - x^2)\,dx$$

$$= \rho\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{32\rho}{3}.$$

To find the moment about the x-axis, place a representative rectangle in the region, as shown in Figure 7.62. The distance from the x-axis to the center of this rectangle is

$$y_i = \frac{f(x)}{2} = \frac{4 - x^2}{2}.$$

Because the mass of the representative rectangle is

$$\rho f(x)\,\Delta x = \rho(4 - x^2)\,\Delta x$$

you have

$$M_x = \rho \int_{-2}^{2} \frac{4 - x^2}{2}(4 - x^2)\,dx$$

$$= \frac{\rho}{2}\int_{-2}^{2} (16 - 8x^2 + x^4)\,dx$$

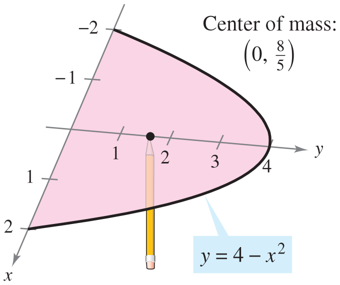
$$= \frac{\rho}{2}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2}$$

$$= \frac{256\rho}{15}$$

and $\bar{y}$ is given by

$$\bar{y} = \frac{M_x}{m} = \frac{256\rho/15}{32\rho/3} = \frac{8}{5}.$$

So, the center of mass (the balancing point) of the lamina is $\left(0, \frac{8}{5}\right)$, as shown in Figure 7.63. ■



The center of mass is the balancing point.
Figure 7.63

The density ρ in Example 4 is a common factor of both the moments and the mass, and as such divides out of the quotients representing the coordinates of the center of mass. So, the center of mass of a lamina of *uniform* density depends only on the shape of the lamina and not on its density. For this reason, the point

$(\bar{x}, \bar{y})$ Center of mass or centroid

is sometimes called the center of mass of a *region* in the plane, or the **centroid** of the region. In other words, to find the centroid of a region in the plane, you simply assume that the region has a constant density of $\rho = 1$ and compute the corresponding center of mass.

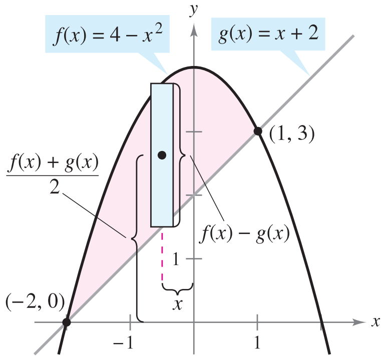


Figure 7.64

EXPLORATION

Cut an irregular shape from a piece of cardboard.

a. Hold a pencil vertically and move the object on the pencil point until the centroid is located.

b. Divide the object into representative elements. Make the necessary measurements and numerically approximate the centroid. Compare your result with the result in part (a).

EXAMPLE 5 The Centroid of a Plane Region

Find the centroid of the region bounded by the graphs of $f(x) = 4 - x^2$ and $g(x) = x + 2$.

Solution The two graphs intersect at the points $(-2, 0)$ and $(1, 3)$, as shown in Figure 7.64. So, the area of the region is

$$A = \int_{-2}^{1} [f(x) - g(x)]\,dx = \int_{-2}^{1} (2 - x - x^2)\,dx = \frac{9}{2}.$$

The centroid $(\bar{x}, \bar{y})$ of the region has the following coordinates.

$$\begin{aligned}
\bar{x} &= \frac{1}{A}\int_{-2}^{1} x[(4 - x^2) - (x + 2)]\,dx = \frac{2}{9}\int_{-2}^{1} (-x^3 - x^2 + 2x)\,dx \\
&= \frac{2}{9}\left[-\frac{x^4}{4} - \frac{x^3}{3} + x^2\right]_{-2}^{1} = -\frac{1}{2} \\
\bar{y} &= \frac{1}{A}\int_{-2}^{1}\left[\frac{(4 - x^2) + (x + 2)}{2}\right][(4 - x^2) - (x + 2)]\,dx \\
&= \frac{2}{9}\left(\frac{1}{2}\right)\int_{-2}^{1} (-x^2 + x + 6)(-x^2 - x + 2)\,dx \\
&= \frac{1}{9}\int_{-2}^{1} (x^4 - 9x^2 - 4x + 12)\,dx \\
&= \frac{1}{9}\left[\frac{x^5}{5} - 3x^3 - 2x^2 + 12x\right]_{-2}^{1} = \frac{12}{5}
\end{aligned}$$

So, the centroid of the region is $(\bar{x}, \bar{y}) = \left(-\frac{1}{2}, \frac{12}{5}\right)$.

For simple plane regions, you may be able to find the centroids without resorting to integration.

EXAMPLE 6 The Centroid of a Simple Plane Region

Find the centroid of the region shown in Figure 7.65(a).

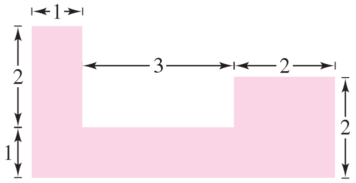
Solution By superimposing a coordinate system on the region, as shown in Figure 7.65(b), you can locate the centroids of the three rectangles at

$$\left(\frac{1}{2}, \frac{3}{2}\right), \quad \left(\frac{5}{2}, \frac{1}{2}\right), \quad \text{and} \quad (5, 1).$$

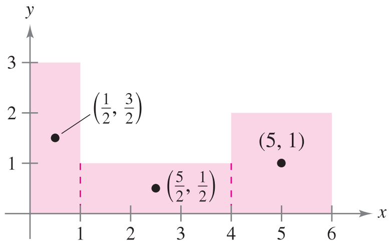
Using these three points, you can find the centroid of the region.

$$\begin{aligned}
A &= \text{area of region} = 3 + 3 + 4 = 10 \\
\bar{x} &= \frac{(1/2)(3) + (5/2)(3) + (5)(4)}{10} = \frac{29}{10} = 2.9 \\
\bar{y} &= \frac{(3/2)(3) + (1/2)(3) + (1)(4)}{10} = \frac{10}{10} = 1
\end{aligned}$$

So, the centroid of the region is $(2.9, 1)$.



(a) Original region

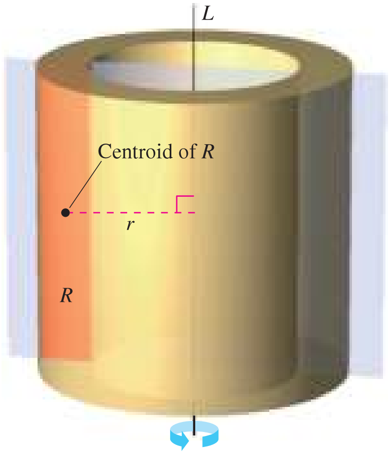

y
3
2
1
$\left(\frac{1}{2}, \frac{3}{2}\right)$
$(5, 1)$
$\left(\frac{5}{2}, \frac{1}{2}\right)$
1
2
3
4
5
6
x

(b) The centroids of the three rectangles
Figure 7.65

NOTE In Example 6, notice that $(2.9, 1)$ is not the "average" of $\left(\frac{1}{2}, \frac{3}{2}\right)$, $\left(\frac{5}{2}, \frac{1}{2}\right)$, and $(5, 1)$.

Theorem of Pappus

The final topic in this section is a useful theorem credited to Pappus of Alexandria (ca. 300 A.D.), a Greek mathematician whose eight-volume *Mathematical Collection* is a record of much of classical Greek mathematics. You are asked to prove this theorem in Section 14.4.



The volume V is $2\pi rA$, where A is the area of region R.
Figure 7.66

THEOREM 7.1 THE THEOREM OF PAPPUS

Let R be a region in a plane and let L be a line in the same plane such that L does not intersect the interior of R, as shown in Figure 7.66. If r is the distance between the centroid of R and the line, then the volume V of the solid of revolution formed by revolving R about the line is

$$V = 2\pi rA$$

where A is the area of R. (Note that $2\pi r$ is the distance traveled by the centroid as the region is revolved about the line.)

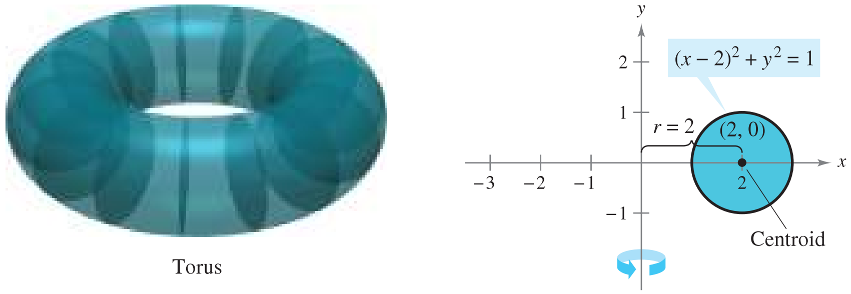
The Theorem of Pappus can be used to find the volume of a torus, as shown in the following example. Recall that a torus is a doughnut-shaped solid formed by revolving a circular region about a line that lies in the same plane as the circle (but does not intersect the circle).

EXAMPLE 7 Finding Volume by the Theorem of Pappus

Find the volume of the torus shown in Figure 7.67(a), which was formed by revolving the circular region bounded by

$$(x-2)^2 + y^2 = 1$$

about the y-axis, as shown in Figure 7.67(b).



(a) **(b)**
Figure 7.67

Solution In Figure 7.67(b), you can see that the centroid of the circular region is (2, 0). So, the distance between the centroid and the axis of revolution is $r = 2$. Because the area of the circular region is $A = \pi$, the volume of the torus is

$$\begin{aligned} V &= 2\pi rA \\ &= 2\pi(2)(\pi) \\ &= 4\pi^2 \\ &\approx 39.5. \end{aligned}$$

■

EXPLORATION

Use the shell method to show that the volume of the torus in Example 7 is given by

$$V = \int_1^3 4\pi x\sqrt{1-(x-2)^2}\,dx.$$

Evaluate this integral using a graphing utility. Does your answer agree with the one in Example 7?

7.6 Exercises

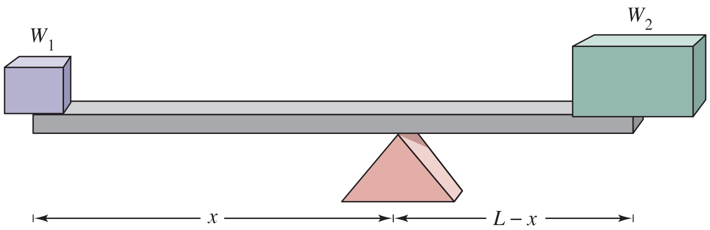
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find the center of mass of the point masses lying on the x-axis.

1. $m_1 = 7, m_2 = 3, m_3 = 5$
   $x_1 = -5, x_2 = 0, x_3 = 3$
2. $m_1 = 7, m_2 = 4, m_3 = 3, m_4 = 8$
   $x_1 = -3, x_2 = -2, x_3 = 5, x_4 = 4$
3. $m_1 = 1, m_2 = 1, m_3 = 1, m_4 = 1, m_5 = 1$
   $x_1 = 7, x_2 = 8, x_3 = 12, x_4 = 15, x_5 = 18$
4. $m_1 = 12, m_2 = 1, m_3 = 6, m_4 = 3, m_5 = 11$
   $x_1 = -6, x_2 = -4, x_3 = -2, x_4 = 0, x_5 = 8$

5. ***Graphical Reasoning***
   (a) Translate each point mass in Exercise 3 to the right four units and determine the resulting center of mass.
   (b) Translate each point mass in Exercise 4 to the left two units and determine the resulting center of mass.
6. ***Conjecture*** Use the result of Exercise 5 to make a conjecture about the change in the center of mass that results when each point mass is translated k units horizontally.

Statics Problems **In Exercises 7 and 8, consider a beam of length L with a fulcrum x feet from one end (see figure). There are objects with weights W_1 and W_2 placed on opposite ends of the beam. Find x such that the system is in equilibrium.**



7. Two children weighing 48 pounds and 72 pounds are going to play on a seesaw that is 10 feet long.
8. In order to move a 600-pound rock, a person weighing 200 pounds wants to balance it on a beam that is 5 feet long.

In Exercises 9–12, find the center of mass of the given system of point masses.

9.

m_i	5	1	3
(x_1, y_1)	(2, 2)	(−3, 1)	(1, −4)

10.

m_i	10	2	5
(x_1, y_1)	(1, −1)	(5, 5)	(−4, 0)

11.

m_i	12	6	4.5	15
(x_1, y_1)	(2, 3)	(−1, 5)	(6, 8)	(2, −2)

12.

m_i	3	4
(x_1, y_1)	(−2, −3)	(5, 5)

m_i	2	1	6
(x_1, y_1)	(7, 1)	(0, 0)	(−3, 0)

In Exercises 13–26, find M_x, M_y, and $(\bar{x}, \bar{y})$ for the laminas of uniform density ρ bounded by the graphs of the equations.

13. $y = \frac{1}{2}x, y = 0, x = 2$
14. $y = -x + 3, y = 0, x = 0$
15. $y = \sqrt{x}, y = 0, x = 4$
16. $y = \frac{1}{3}x^2, y = 0, x = 3$
17. $y = x^2, y = x^3$
18. $y = \sqrt{x}, y = \frac{1}{2}x$
19. $y = -x^2 + 4x + 2, y = x + 2$
20. $y = \sqrt{x} + 1, y = \frac{1}{3}x + 1$
21. $y = x^{2/3}, y = 0, x = 8$
22. $y = x^{2/3}, y = 4$
23. $x = 4 - y^2, x = 0$
24. $x = 2y - y^2, x = 0$
25. $x = -y, x = 2y - y^2$
26. $x = y + 2, x = y^2$

In Exercises 27–30, set up and evaluate the integrals for finding the area and moments about the x- and y-axes for the region bounded by the graphs of the equations. (Assume $\rho = 1$.)

27. $y = x^2, y = 2x$
28. $y = \dfrac{1}{x}, y = 0, 1 \le x \le 4$
29. $y = 2x + 4, y = 0, 0 \le x \le 3$
30. $y = x^2 - 4, y = 0$

In Exercises 31–34, use a graphing utility to graph the region bounded by the graphs of the equations. Use the integration capabilities of the graphing utility to approximate the centroid of the region.

31. $y = 10x\sqrt{125 - x^3}, y = 0$
32. $y = xe^{-x/2}, y = 0, x = 0, x = 4$
33. ***Prefabricated End Section of a Building***
    $y = 5\sqrt[3]{400 - x^2}, y = 0$
34. ***Witch of Agnesi***
    $y = \dfrac{8}{x^2 + 4}, y = 0, x = -2, x = 2$

In Exercises 35–40, find and/or verify the centroid of the common region used in engineering.

35. ***Triangle*** Show that the centroid of the triangle with vertices $(-a, 0)$, $(a, 0)$, and (b, c) is the point of intersection of the medians (see figure).

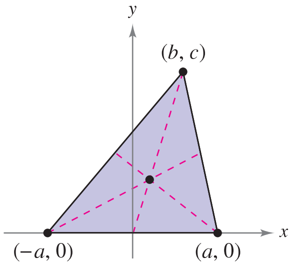


Figure for 35

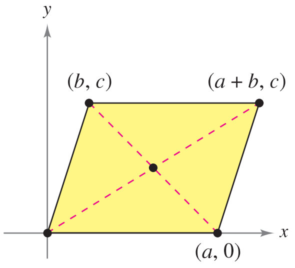


Figure for 36

36. ***Parallelogram*** Show that the centroid of the parallelogram with vertices $(0, 0)$, $(a, 0)$, (b, c), and $(a + b, c)$ is the point of intersection of the diagonals (see figure).

37. ***Trapezoid*** Find the centroid of the trapezoid with vertices $(0, 0)$, $(0, a)$, (c, b), and $(c, 0)$. Show that it is the intersection of the line connecting the midpoints of the parallel sides and the line connecting the extended parallel sides, as shown in the figure.

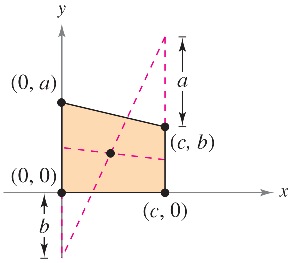


Figure for 37

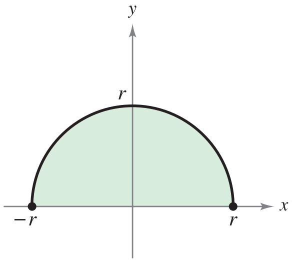


Figure for 38

38. ***Semicircle*** Find the centroid of the region bounded by the graphs of $y = \sqrt{r^2 - x^2}$ and $y = 0$ (see figure).

39. ***Semiellipse*** Find the centroid of the region bounded by the graphs of $y = \frac{b}{a}\sqrt{a^2 - x^2}$ and $y = 0$ (see figure).

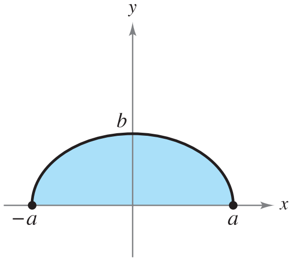


Figure for 39

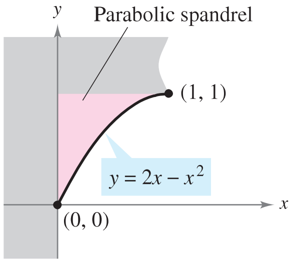


Figure for 40

40. ***Parabolic Spandrel*** Find the centroid of the **parabolic spandrel** shown in the figure.

41. ***Graphical Reasoning*** Consider the region bounded by the graphs of $y = x^2$ and $y = b$, where $b > 0$.

(a) Sketch a graph of the region.

(b) Use the graph in part (a) to determine $\bar{x}$. Explain.

(c) Set up the integral for finding M_y. Because of the form of the integrand, the value of the integral can be obtained without integrating. What is the form of the integrand and what is the value of the integral? Compare with the result in part (b).

(d) Use the graph in part (a) to determine whether $\bar{y} > \frac{b}{2}$ or $\bar{y} < \frac{b}{2}$. Explain.

(e) Use integration to verify your answer in part (d).

42. ***Graphical and Numerical Reasoning*** Consider the region bounded by the graphs of $y = x^{2n}$ and $y = b$, where $b > 0$ and n is a positive integer.

(a) Set up the integral for finding M_y. Because of the form of the integrand, the value of the integral can be obtained without integrating. What is the form of the integrand and what is the value of the integral? Compare with the result in part (b).

(b) Is $\bar{y} > \frac{b}{2}$ or $\bar{y} < \frac{b}{2}$? Explain.

(c) Use integration to find $\bar{y}$ as a function of n.

(d) Use the result of part (c) to complete the table.

n	1	2	3	4
$\bar{y}$				

(e) Find $\lim_{n\to\infty} \bar{y}$.

(f) Give a geometric explanation of the result in part (e).

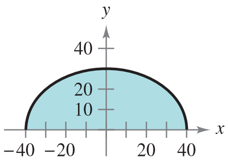
43. ***Modeling Data*** The manufacturer of glass for a window in a conversion van needs to approximate its center of mass. A coordinate system is superimposed on a prototype of the glass (see figure). The measurements (in centimeters) for the right half of the symmetric piece of glass are shown in the table.

x	0	10	20	30	40
y	30	29	26	20	0

(a) Use Simpson's Rule to approximate the center of mass of the glass.

(b) Use the regression capabilities of a graphing utility to find a fourth-degree polynomial model for the data.

(c) Use the integration capabilities of a graphing utility and the model to approximate the center of mass of the glass. Compare with the result in part (a).



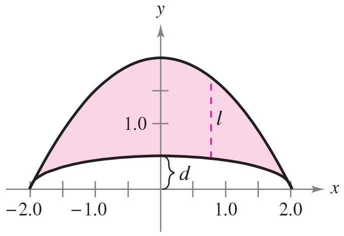
44. ***Modeling Data*** The manufacturer of a boat needs to approximate the center of mass of a section of the hull. A coordinate system is superimposed on a prototype (see figure). The measurements (in feet) for the right half of the symmetric prototype are listed in the table.

x	0	0.5	1.0	1.5	2
l	1.50	1.45	1.30	0.99	0
d	0.50	0.48	0.43	0.33	0

(a) Use Simpson's Rule to approximate the center of mass of the hull section.

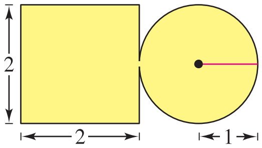
(b) Use the regression capabilities of a graphing utility to find fourth-degree polynomial models for both curves shown in the figure. Plot the data and graph the models.

(c) Use the integration capabilities of a graphing utility and the model to approximate the center of mass of the hull section. Compare with the result in part (a).

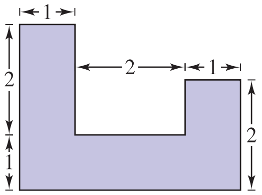


In Exercises 45–48, introduce an appropriate coordinate system and find the coordinates of the center of mass of the planar lamina. (The answer depends on the position of the coordinate system.)

45.



46.



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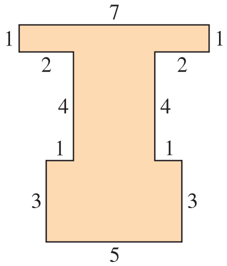
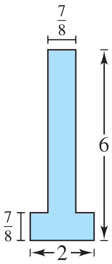


48.



49. Find the center of mass of the lamina in Exercise 45 if the circular portion of the lamina has twice the density of the square portion of the lamina.

50. Find the center of mass of the lamina in Exercise 45 if the square portion of the lamina has twice the density of the circular portion of the lamina.

In Exercises 51–54, use the Theorem of Pappus to find the volume of the solid of revolution.

51. The torus formed by revolving the circle $(x - 5)^2 + y^2 = 16$ about the y-axis

52. The torus formed by revolving the circle $x^2 + (y - 3)^2 = 4$ about the x-axis

53. The solid formed by revolving the region bounded by the graphs of $y = x$, $y = 4$, and $x = 0$ about the x-axis

54. The solid formed by revolving the region bounded by the graphs of $y = 2\sqrt{x - 2}$, $y = 0$, and $x = 6$ about the y-axis

WRITING ABOUT CONCEPTS

55. Let the point masses $m_1, m_2, \ldots, m_n$ be located at (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$. Define the center of mass $(\bar{x}, \bar{y})$.

56. What is a planar lamina? Describe what is meant by the center of mass $(\bar{x}, \bar{y})$ of a planar lamina.

57. State the Theorem of Pappus.

CAPSTONE

58. The centroid of the plane region bounded by the graphs of $y = f(x)$, $y = 0$, $x = 0$, and $x = 1$ is $\left(\frac{5}{6}, \frac{5}{18}\right)$. Is it possible to find the centroid of each of the regions bounded by the graphs of the following sets of equations? If so, identify the centroid and explain your answer.

(a) $y = f(x) + 2$, $y = 2$, $x = 0$, and $x = 1$

(b) $y = f(x - 2)$, $y = 0$, $x = 2$, and $x = 3$

(c) $y = -f(x)$, $y = 0$, $x = 0$, and $x = 1$

(d) $y = f(x)$, $y = 0$, $x = -1$, and $x = 1$

In Exercises 59 and 60, use the *Second Theorem of Pappus*, which is stated as follows. If a segment of a plane curve C is revolved about an axis that does not intersect the curve (except possibly at its endpoints), the area S of the resulting surface of revolution is equal to the product of the length of C times the distance d traveled by the centroid of C.

59. A sphere is formed by revolving the graph of $y = \sqrt{r^2 - x^2}$ about the x-axis. Use the formula for surface area, $S = 4\pi r^2$, to find the centroid of the semicircle $y = \sqrt{r^2 - x^2}$.

60. A torus is formed by revolving the graph of $(x - 1)^2 + y^2 = 1$ about the y-axis. Find the surface area of the torus.

61. Let $n \geq 1$ be constant, and consider the region bounded by $f(x) = x^n$, the x-axis, and $x = 1$. Find the centroid of this region. As $n \to \infty$, what does the region look like, and where is its centroid?

PUTNAM EXAM CHALLENGE

62. Let V be the region in the cartesian plane consisting of all points (x, y) satisfying the simultaneous conditions $|x| \leq y \leq |x| + 3$ and $y \leq 4$. Find the centroid $(\bar{x}, \bar{y})$ of V.

This problem was composed by the Committee on the Putnam Prize Competition.
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7.7 Fluid Pressure and Fluid Force

- **Find fluid pressure and fluid force.**

Fluid Pressure and Fluid Force

Swimmers know that the deeper an object is submerged in a fluid, the greater the pressure on the object. **Pressure** is defined as the force per unit of area over the surface of a body. For example, because a column of water that is 10 feet in height and 1 inch square weighs 4.3 pounds, the *fluid pressure* at a depth of 10 feet of water is 4.3 pounds per square inch.* At 20 feet, this would increase to 8.6 pounds per square inch, and in general the pressure is proportional to the depth of the object in the fluid.

The Granger Collection

BLAISE PASCAL (1623–1662)

Pascal is well known for his work in many areas of mathematics and physics, and also for his influence on Leibniz. Although much of Pascal's work in calculus was intuitive and lacked the rigor of modern mathematics, he nevertheless anticipated many important results.

DEFINITION OF FLUID PRESSURE

The **pressure** on an object at depth h in a liquid is

$$\text{Pressure} = P = wh$$

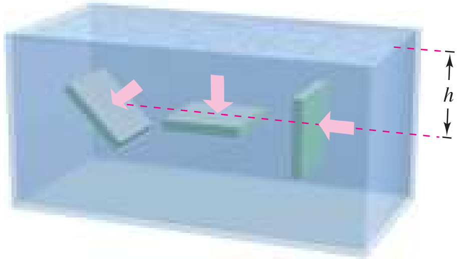
where w is the weight-density of the liquid per unit of volume.

Below are some common weight-densities of fluids in pounds per cubic foot.

Ethyl alcohol	49.4
Gasoline	41.0–43.0
Glycerin	78.6
Kerosene	51.2
Mercury	849.0
Seawater	64.0
Water	62.4

When calculating fluid pressure, you can use an important (and rather surprising) physical law called **Pascal's Principle,** named after the French mathematician Blaise Pascal. Pascal's Principle states that the pressure exerted by a fluid at a depth h is transmitted equally *in all directions*. For example, in Figure 7.68, the pressure at the indicated depth is the same for all three objects. Because fluid pressure is given in terms of force per unit area $(P = F/A)$, the fluid force on a *submerged horizontal* surface of area A is

$$\text{Fluid force} = F = PA = (\text{pressure})(\text{area}).$$



The pressure at h is the same for all three objects.
Figure 7.68

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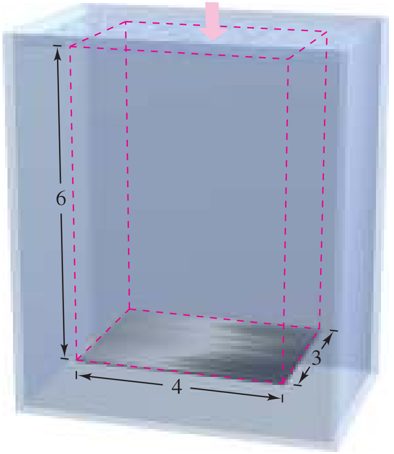
* *The total pressure on an object in 10 feet of water would also include the pressure due to Earth's atmosphere. At sea level, atmospheric pressure is approximately 14.7 pounds per square inch.*

EXAMPLE 1 Fluid Force on a Submerged Sheet



The fluid force on a horizontal metal sheet is equal to the fluid pressure times the area.
Figure 7.69

Find the fluid force on a rectangular metal sheet measuring 3 feet by 4 feet that is submerged in 6 feet of water, as shown in Figure 7.69.

Solution Because the weight-density of water is 62.4 pounds per cubic foot and the sheet is submerged in 6 feet of water, the fluid pressure is

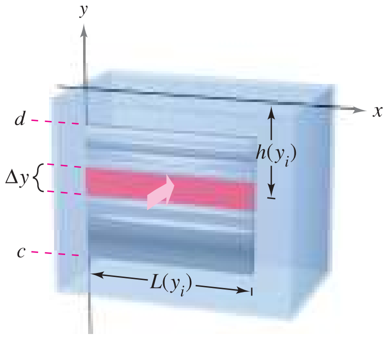
$$\begin{aligned} P &= (62.4)(6) \qquad\qquad P = wh \\ &= 374.4 \text{ pounds per square foot.} \end{aligned}$$

Because the total area of the sheet is $A = (3)(4) = 12$ square feet, the fluid force is

$$\begin{aligned} F = PA &= \left(374.4 \frac{\text{pounds}}{\text{square foot}}\right)(12 \text{ square feet}) \\ &= 4492.8 \text{ pounds.} \end{aligned}$$

This result is independent of the size of the body of water. The fluid force would be the same in a swimming pool or lake. ■

In Example 1, the fact that the sheet is rectangular and horizontal means that you do not need the methods of calculus to solve the problem. Consider a surface that is submerged vertically in a fluid. This problem is more difficult because the pressure is not constant over the surface.



Calculus methods must be used to find the fluid force on a vertical metal plate.
Figure 7.70

Suppose a vertical plate is submerged in a fluid of weight-density w (per unit of volume), as shown in Figure 7.70. To determine the total force against *one side* of the region from depth c to depth d, you can subdivide the interval $[c, d]$ into n subintervals, each of width Δy. Next, consider the representative rectangle of width Δy and length $L(y_i)$, where y_i is in the ith subinterval. The force against this representative rectangle is

$$\begin{aligned} \Delta F_i &= w(\text{depth})(\text{area}) \\ &= wh(y_i)L(y_i)\,\Delta y. \end{aligned}$$

The force against n such rectangles is

$$\sum_{i=1}^{n} \Delta F_i = w \sum_{i=1}^{n} h(y_i)L(y_i)\,\Delta y.$$

Note that w is considered to be constant and is factored out of the summation. Therefore, taking the limit as $\|\Delta\| \to 0$ $(n \to \infty)$ suggests the following definition.

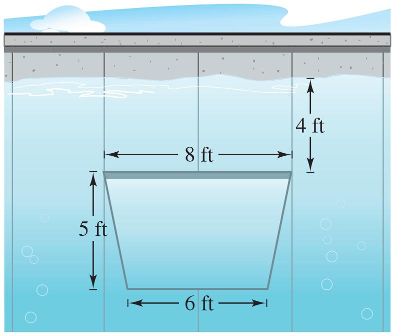
DEFINITION OF FORCE EXERTED BY A FLUID

The **force F exerted by a fluid** of constant weight-density w (per unit of volume) against a submerged vertical plane region from $y = c$ to $y = d$ is

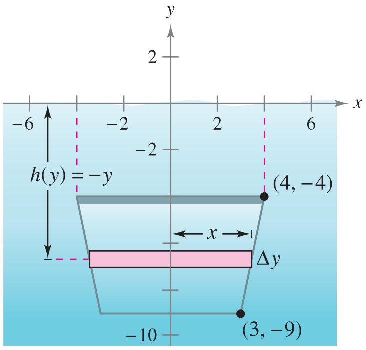
$$\begin{aligned} F &= w \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} h(y_i)L(y_i)\,\Delta y \\ &= w \int_c^d h(y)L(y)\,dy \end{aligned}$$

where $h(y)$ is the depth of the fluid at y and $L(y)$ is the horizontal length of the region at y.

EXAMPLE 2 Fluid Force on a Vertical Surface



(a) Water gate in a dam



(b) The fluid force against the gate

Figure 7.71

A vertical gate in a dam has the shape of an isosceles trapezoid 8 feet across the top and 6 feet across the bottom, with a height of 5 feet, as shown in Figure 7.71(a). What is the fluid force on the gate when the top of the gate is 4 feet below the surface of the water?

Solution In setting up a mathematical model for this problem, you are at liberty to locate the x- and y-axes in several different ways. A convenient approach is to let the y-axis bisect the gate and place the x-axis at the surface of the water, as shown in Figure 7.71(b). So, the depth of the water at y in feet is

$$\text{Depth} = h(y) = -y.$$

To find the length $L(y)$ of the region at y, find the equation of the line forming the right side of the gate. Because this line passes through the points $(3, -9)$ and $(4, -4)$, its equation is

$$\begin{aligned} y - (-9) &= \frac{-4 - (-9)}{4 - 3}(x - 3) \\ y + 9 &= 5(x - 3) \\ y &= 5x - 24 \\ x &= \frac{y + 24}{5}. \end{aligned}$$

In Figure 7.71(b) you can see that the length of the region at y is

$$\begin{aligned} \text{Length} &= 2x \\ &= \frac{2}{5}(y + 24) \\ &= L(y). \end{aligned}$$

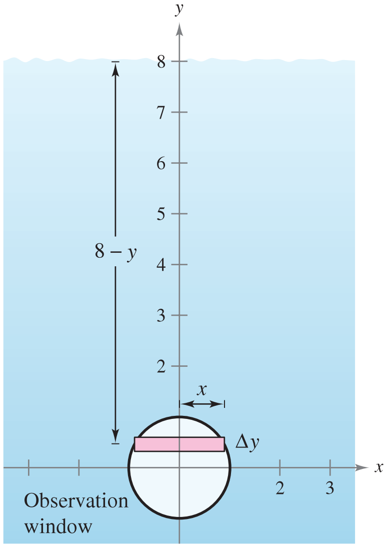
Finally, by integrating from $y = -9$ to $y = -4$, you can calculate the fluid force to be

$$\begin{aligned} F &= w\int_c^d h(y)L(y)\,dy \\ &= 62.4\int_{-9}^{-4} (-y)\left(\frac{2}{5}\right)(y + 24)\,dy \\ &= -62.4\left(\frac{2}{5}\right)\int_{-9}^{-4} (y^2 + 24y)\,dy \\ &= -62.4\left(\frac{2}{5}\right)\left[\frac{y^3}{3} + 12y^2\right]_{-9}^{-4} \\ &= -62.4\left(\frac{2}{5}\right)\left(\frac{-1675}{3}\right) \\ &= 13{,}936 \text{ pounds.} \end{aligned}$$

■

NOTE In Example 2, the x-axis coincided with the surface of the water. This was convenient, but arbitrary. In choosing a coordinate system to represent a physical situation, you should consider various possibilities. Often you can simplify the calculations in a problem by locating the coordinate system to take advantage of special characteristics of the problem, such as symmetry. ■

EXAMPLE 3 Fluid Force on a Vertical Surface



The fluid force on the window
Figure 7.72

A circular observation window on a marine science ship has a radius of 1 foot, and the center of the window is 8 feet below water level, as shown in Figure 7.72. What is the fluid force on the window?

Solution To take advantage of symmetry, locate a coordinate system such that the origin coincides with the center of the window, as shown in Figure 7.72. The depth at y is then

$$\text{Depth} = h(y) = 8 - y.$$

The horizontal length of the window is $2x$, and you can use the equation for the circle, $x^2 + y^2 = 1$, to solve for x as follows.

$$\begin{aligned}\text{Length} &= 2x\\ &= 2\sqrt{1-y^2} = L(y)\end{aligned}$$

Finally, because y ranges from -1 to 1, and using 64 pounds per cubic foot as the weight-density of seawater, you have

$$\begin{aligned}F &= w\int_c^d h(y)L(y)\,dy\\ &= 64\int_{-1}^{1}(8-y)(2)\sqrt{1-y^2}\,dy.\end{aligned}$$

Initially it looks as if this integral would be difficult to solve. However, if you break the integral into two parts and apply symmetry, the solution is simple.

$$F = 64(16)\int_{-1}^{1}\sqrt{1-y^2}\,dy - 64(2)\int_{-1}^{1}y\sqrt{1-y^2}\,dy$$

The second integral is 0 (because the integrand is odd and the limits of integration are symmetric with respect to the origin). Moreover, by recognizing that the first integral represents the area of a semicircle of radius 1, you obtain

$$\begin{aligned}F &= 64(16)\left(\frac{\pi}{2}\right) - 64(2)(0)\\ &= 512\pi\\ &\approx 1608.5 \text{ pounds.}\end{aligned}$$

So, the fluid force on the window is 1608.5 pounds. ■

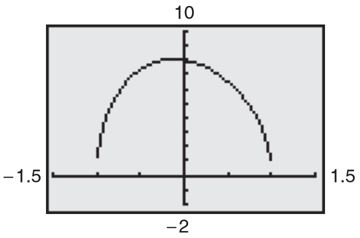
TECHNOLOGY To confirm the result obtained in Example 3, you might have considered using Simpson's Rule to approximate the value of

$$128\int_{-1}^{1}(8-x)\sqrt{1-x^2}\,dx.$$

From the graph of

$$f(x) = (8-x)\sqrt{1-x^2}$$

however, you can see that f is not differentiable when $x = \pm 1$ (see Figure 7.73). This means that you cannot apply Theorem 4.19 from Section 4.6 to determine the potential error in Simpson's Rule. Without knowing the potential error, the approximation is of little value. Use a graphing utility to approximate the integral.



f is not differentiable at $x = \pm 1$.
Figure 7.73

7.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Force on a Submerged Sheet **In Exercises 1–4, the area of the top side of a piece of sheet metal is given. The sheet metal is submerged horizontally in 8 feet of water. Find the fluid force on the top side.**

1. 3 square feet

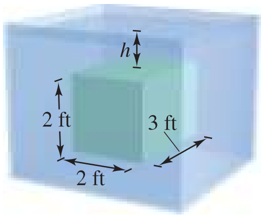
2. 16 square feet

3. 10 square feet

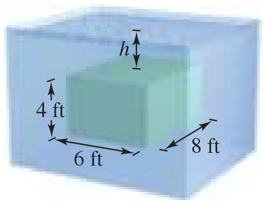
4. 22 square feet

Buoyant Force **In Exercises 5 and 6, find the buoyant force of a rectangular solid of the given dimensions submerged in water so that the top side is parallel to the surface of the water. The buoyant force is the difference between the fluid forces on the top and bottom sides of the solid.**

5.

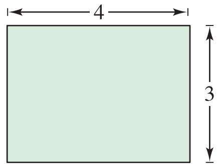


6.

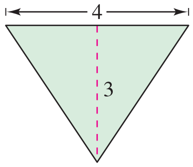


Fluid Force on a Tank Wall **In Exercises 7–12, find the fluid force on the vertical side of the tank, where the dimensions are given in feet. Assume that the tank is full of water.**

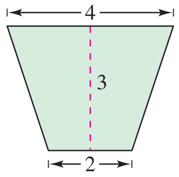
7. Rectangle



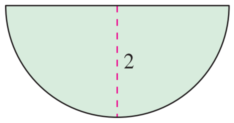
8. Triangle



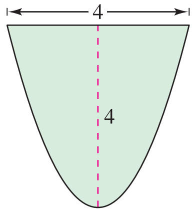
9. Trapezoid



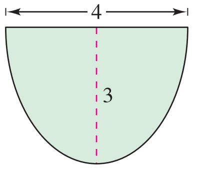
10. Semicircle



11. Parabola, $y = x^2$

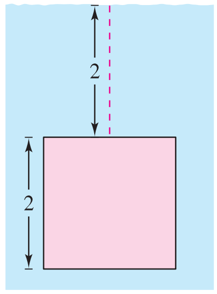


12. Semiellipse, $y = -\frac{1}{2}\sqrt{36 - 9x^2}$

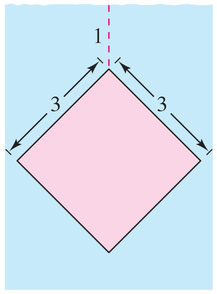


Fluid Force of Water **In Exercises 13–16, find the fluid force on the vertical plate submerged in water, where the dimensions are given in meters and the weight-density of water is 9800 newtons per cubic meter.**

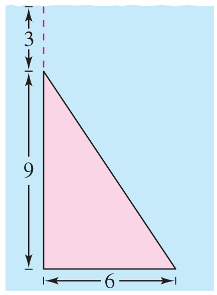
13. Square



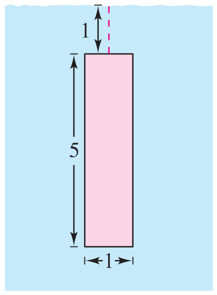
14. Square



15. Triangle

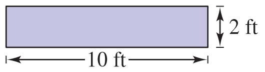


16. Rectangle

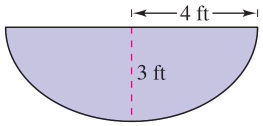


Force on a Concrete Form **In Exercises 17–20, the figure is the vertical side of a form for poured concrete that weighs 140.7 pounds per cubic foot. Determine the force on this part of the concrete form.**

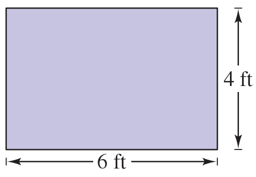
17. Rectangle



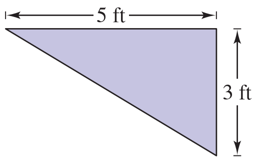
18. Semiellipse, $y = -\frac{3}{4}\sqrt{16 - x^2}$



19. Rectangle



20. Triangle



21. ***Fluid Force of Gasoline*** A cylindrical gasoline tank is placed so that the axis of the cylinder is horizontal. Find the fluid force on a circular end of the tank if the tank is half full, assuming that the diameter is 3 feet and the gasoline weighs 42 pounds per cubic foot.

22. ***Fluid Force of Gasoline*** Repeat Exercise 21 for a tank that is full. (Evaluate one integral by a geometric formula and the other by observing that the integrand is an odd function.)

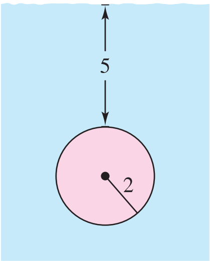
23. ***Fluid Force on a Circular Plate*** A circular plate of radius r feet is submerged vertically in a tank of fluid that weighs w pounds per cubic foot. The center of the circle is k $(k > r)$ feet below the surface of the fluid. Show that the fluid force on the surface of the plate is

$$F = wk(\pi r^2).$$

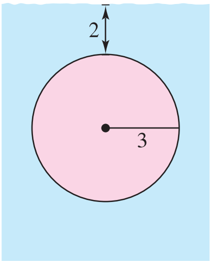
(Evaluate one integral by a geometric formula and the other by observing that the integrand is an odd function.)

24. ***Fluid Force on a Circular Plate*** Use the result of Exercise 23 to find the fluid force on the circular plate shown in each figure. Assume the plates are in the wall of a tank filled with water and the measurements are given in feet.

(a)



(b)

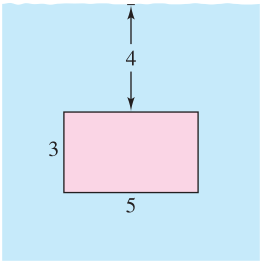


25. ***Fluid Force on a Rectangular Plate*** A rectangular plate of height h feet and base b feet is submerged vertically in a tank of fluid that weighs w pounds per cubic foot. The center is k feet below the surface of the fluid, where $k > h/2$. Show that the fluid force on the surface of the plate is

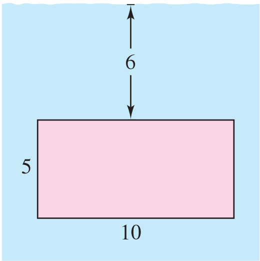
$$F = wkhb.$$

26. ***Fluid Force on a Rectangular Plate*** Use the result of Exercise 25 to find the fluid force on the rectangular plate shown in each figure. Assume the plates are in the wall of a tank filled with water and the measurements are given in feet.

(a)



(b)

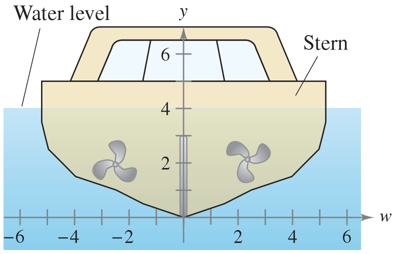


27. ***Submarine Porthole*** A square porthole on a vertical side of a submarine (submerged in seawater) has an area of 1 square foot. Find the fluid force on the porthole, assuming that the center of the square is 15 feet below the surface.

28. ***Submarine Porthole*** Repeat Exercise 27 for a circular porthole that has a diameter of 1 foot. The center is 15 feet below the surface.

29. ***Modeling Data*** The vertical stern of a boat with a superimposed coordinate system is shown in the figure. The table shows the widths w of the stern at indicated values of y. Find the fluid force against the stern if the measurements are given in feet.

y	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
w	0	3	5	8	9	10	10.25	10.5	10.5



30. ***Irrigation Canal Gate*** The vertical cross section of an irrigation canal is modeled by $f(x) = 5x^2/(x^2 + 4)$, where x is measured in feet and $x = 0$ corresponds to the center of the canal. Use the integration capabilities of a graphing utility to approximate the fluid force against a vertical gate used to stop the flow of water if the water is 3 feet deep.

In Exercises 31 and 32, use the integration capabilities of a graphing utility to approximate the fluid force on the vertical plate bounded by the x-axis and the top half of the graph of the equation. Assume that the base of the plate is 15 feet beneath the surface of the water.

31. $x^{2/3} + y^{2/3} = 4^{2/3}$

32. $\dfrac{x^2}{28} + \dfrac{y^2}{16} = 1$

WRITING ABOUT CONCEPTS

33. ***Think About It*** Approximate the depth of the water in the tank in Exercise 7 if the fluid force is one-half as great as when the tank is full. Explain why the answer is not $\frac{3}{2}$.

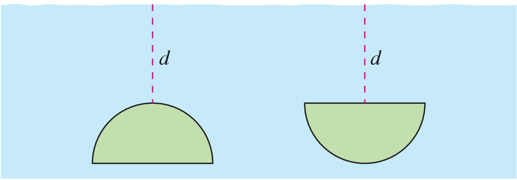
34. (a) Define fluid pressure.

(b) Define fluid force against a submerged vertical plane region.

35. Explain why fluid pressure on a surface is calculated using horizontal representative rectangles instead of vertical representative rectangles.

CAPSTONE

36. Two identical semicircular windows are placed at the same depth in the vertical wall of an aquarium (see figure). Which is subjected to the greater fluid force? Explain.



7 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, sketch the region bounded by the graphs of the equations, and determine the area of the region.

1. $y = \dfrac{1}{x^2},\ y = 0,\ x = 1,\ x = 5$

2. $y = \dfrac{1}{x^2},\ y = 4,\ x = 5$

3. $y = \dfrac{1}{x^2 + 1},\ y = 0,\ x = -1,\ x = 1$

4. $x = y^2 - 2y,\ x = -1,\ y = 0$

5. $y = x,\ y = x^3$

6. $x = y^2 + 1,\ x = y + 3$

7. $y = e^x,\ y = e^2,\ x = 0$

8. $y = \csc x,\ y = 2$ (one region)

9. $y = \sin x,\ y = \cos x,\ \dfrac{\pi}{4} \le x \le \dfrac{5\pi}{4}$

10. $x = \cos y,\ x = \dfrac{1}{2},\ \dfrac{\pi}{3} \le y \le \dfrac{7\pi}{3}$

In Exercises 11–14, use a graphing utility to graph the region bounded by the graphs of the functions, and use the integration capabilities of the graphing utility to find the area of the region.

11. $y = x^2 - 8x + 3,\ y = 3 + 8x - x^2$

12. $y = x^2 - 4x + 3,\ y = x^3,\ x = 0$

13. $\sqrt{x} + \sqrt{y} = 1,\ y = 0,\ x = 0$

14. $y = x^4 - 2x^2,\ y = 2x^2$

In Exercises 15–18, use vertical and horizontal representative rectangles to set up integrals for finding the area of the region bounded by the graphs of the equations. Find the area of the region by evaluating the easier of the two integrals.

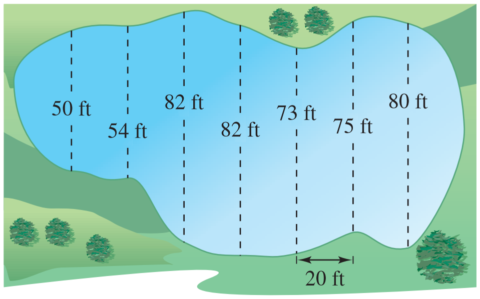
15. $x = y^2 - 2y,\ x = 0$

16. $y = \sqrt{x - 1},\ y = \dfrac{x - 1}{2}$

17. $y = 1 - \dfrac{x}{2},\ y = x - 2,\ y = 1$

18. $y = \sqrt{x - 1},\ y = 2,\ y = 0,\ x = 0$

19. Estimate the surface area of the pond using (a) the Trapezoidal Rule and (b) Simpson's Rule.



20. ***Modeling Data*** The table shows the annual service revenues R_1 in billions of dollars for the cellular telephone industry for the years 2000 through 2006. *(Source: Cellular Telecommunications & Internet Association)*

Year	2000	2001	2002	2003	2004	2005	2006
R_1	52.5	65.3	76.5	87.6	102.1	113.5	125.5

(a) Use the regression capabilities of a graphing utility to find an exponential model for the data. Let t represent the year, with $t = 10$ corresponding to 2000. Use the graphing utility to plot the data and graph the model in the same viewing window.

(b) A financial consultant believes that a model for service revenues for the years 2010 through 2015 is $R_2 = 6 + 13.9e^{0.14t}$. What is the difference in total service revenues between the two models for the years 2010 through 2015?

In Exercises 21–28, find the volume of the solid generated by revolving the plane region bounded by the equations about the indicated line(s).

21. $y = x,\ y = 0,\ x = 3$

(a) the x-axis (b) the y-axis
(c) the line $x = 3$ (d) the line $x = 6$

22. $y = \sqrt{x},\ y = 2,\ x = 0$

(a) the x-axis (b) the line $y = 2$
(c) the y-axis (d) the line $x = -1$

23. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ (a) the y-axis (oblate spheroid)
(b) the x-axis (prolate spheroid)

24. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (a) the y-axis (oblate spheroid)
(b) the x-axis (prolate spheroid)

25. $y = 1/(x^4 + 1),\ y = 0,\ x = 0,\ x = 1$
revolved about the y-axis

26. $y = 1/\sqrt{1 + x^2},\ y = 0,\ x = -1,\ x = 1$
revolved about the x-axis

27. $y = 1/(1 + \sqrt{x - 2}),\ y = 0,\ x = 2,\ x = 6$
revolved about the y-axis

28. $y = e^{-x},\ y = 0,\ x = 0,\ x = 1$
revolved about the x-axis

29. ***Area and Volume*** Consider the region bounded by the graphs of the equations $y = x\sqrt{x + 1}$ and $y = 0$.

(a) Find the area of the region.

(b) Find the volume of the solid generated by revolving the region about the x-axis.

(c) Find the volume of the solid generated by revolving the region about the y-axis.

30. ***Think About It*** A solid is generated by revolving the region bounded by $y = x^2 + 4$, $y = 0$, $x = 0$, and $x = 3$ about the x-axis. Set up the integral that gives the volume of this solid using (a) the disk method and (b) the shell method. (Do not integrate.) (c) Does each method lead to an integral with respect to x?

31. ***Depth of Gasoline in a Tank*** A gasoline tank is an oblate spheroid generated by revolving the region bounded by the graph of $(x^2/16) + (y^2/9) = 1$ about the y-axis, where x and y are measured in feet. Find the depth of the gasoline in the tank when it is filled to one-fourth its capacity.

32. ***Magnitude of a Base*** The base of a solid is a circle of radius a, and its vertical cross sections are equilateral triangles. The volume of the solid is 10 cubic meters. Find the radius of the circle.

In Exercises 33 and 34, find the arc length of the graph of the function over the given interval.

33. $f(x) = \frac{4}{5}x^{5/4}, \quad [0, 4]$

34. $y = \frac{1}{6}x^3 + \frac{1}{2x}, \quad [1, 3]$

35. ***Length of a Catenary*** A cable of a suspension bridge forms a catenary modeled by the equation

$$y = 300 \cosh\left(\frac{x}{2000}\right) - 280, \quad -2000 \le x \le 2000$$

where x and y are measured in feet. Use a graphing utility to approximate the length of the cable.

36. ***Approximation*** Determine which value best approximates the length of the arc represented by the integral

$$\int_0^{\pi/4} \sqrt{1 + (\sec^2 x)^2}\, dx.$$

(Make your selection on the basis of a sketch of the arc and *not* by performing any calculations.)

(a) -2 (b) 1 (c) π (d) 4 (e) 3

37. ***Surface Area*** Use integration to find the lateral surface area of a right circular cone of height 4 and radius 3.

38. ***Surface Area*** The region bounded by the graphs of $y = 2\sqrt{x}$, $y = 0$, $x = 3$, and $x = 8$ is revolved about the x-axis. Find the surface area of the solid generated.

39. ***Work*** A force of 5 pounds is needed to stretch a spring 1 inch from its natural position. Find the work done in stretching the spring from its natural length of 10 inches to a length of 15 inches.

40. ***Work*** A force of 50 pounds is needed to stretch a spring 1 inch from its natural position. Find the work done in stretching the spring from its natural length of 10 inches to double that length.

41. ***Work*** A water well has an eight-inch casing (diameter) and is 190 feet deep. The water is 25 feet from the top of the well. Determine the amount of work done in pumping the well dry, assuming that no water enters it while it is being pumped.

42. ***Work*** Repeat Exercise 41, assuming that water enters the well at a rate of 4 gallons per minute and the pump works at a rate of 12 gallons per minute. How many gallons are pumped in this case?

43. ***Work*** A chain 10 feet long weighs 4 pounds per foot and is hung from a platform 20 feet above the ground. How much work is required to raise the entire chain to the 20-foot level?

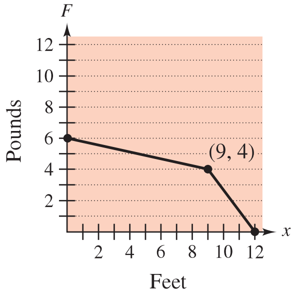
44. ***Work*** A windlass, 200 feet above ground level on the top of a building, uses a cable weighing 5 pounds per foot. Find the work done in winding up the cable if

(a) one end is at ground level.

(b) there is a 300-pound load attached to the end of the cable.

45. ***Work*** The work done by a variable force in a press is 80 foot-pounds. The press moves a distance of 4 feet and the force is a quadratic of the form $F = ax^2$. Find a.

46. ***Work*** Find the work done by the force F shown in the figure.



In Exercises 47–50, find the centroid of the region bounded by the graphs of the equations.

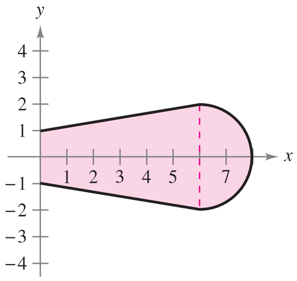
47. $\sqrt{x} + \sqrt{y} = \sqrt{a}, \; x = 0, \; y = 0$

48. $y = x^2, \; y = 2x + 3$

49. $y = a^2 - x^2, \; y = 0$

50. $y = x^{2/3}, \; y = \frac{1}{2}x$

51. ***Centroid*** A blade on an industrial fan has the configuration of a semicircle attached to a trapezoid (see figure). Find the centroid of the blade.



52. ***Fluid Force*** A swimming pool is 5 feet deep at one end and 10 feet deep at the other, and the bottom is an inclined plane. The length and width of the pool are 40 feet and 20 feet. If the pool is full of water, what is the fluid force on each of the vertical walls?

53. ***Fluid Force*** Show that the fluid force against any vertical region in a liquid is the product of the weight per cubic volume of the liquid, the area of the region, and the depth of the centroid of the region.

54. ***Fluid Force*** Using the result of Exercise 53, find the fluid force on one side of a vertical circular plate of radius 4 feet that is submerged in water so that its center is 10 feet below the surface.

P.S. PROBLEM SOLVING

1. Let R be the area of the region in the first quadrant bounded by the parabola $y = x^2$ and the line $y = cx$, $c > 0$. Let T be the area of the triangle AOB. Calculate the limit

$$\lim_{c \to 0^+} \frac{T}{R}.$$

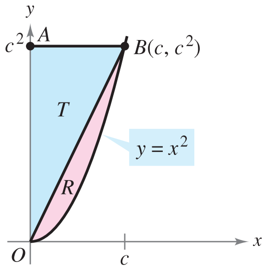


Figure for 1

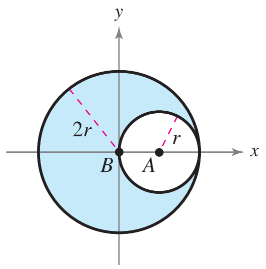


Figure for 2

2. Let L be the lamina of uniform density $\rho = 1$ obtained by removing circle A of radius r from circle B of radius $2r$ (see figure).

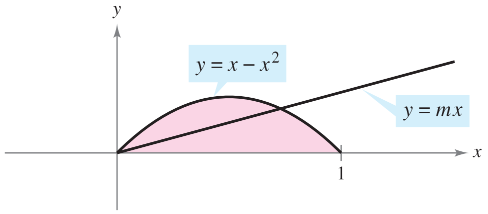
(a) Show that $M_x = 0$ for L.

(b) Show that M_y for L is equal to $(M_y \text{ for } B) - (M_y \text{ for } A)$.

(c) Find M_y for B and M_y for A. Then use part (b) to compute M_y for L.

(d) What is the center of mass of L?

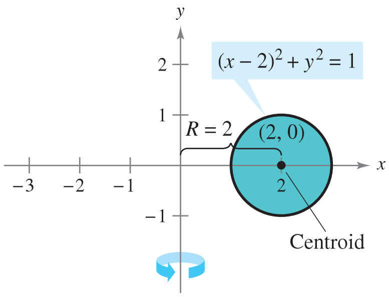
3. Let R be the region bounded by the parabola $y = x - x^2$ and the x-axis. Find the equation of the line $y = mx$ that divides this region into two regions of equal area.



4. (a) A torus is formed by revolving the region bounded by the circle

$$(x - 2)^2 + y^2 = 1$$

about the y-axis (see figure). Use the disk method to calculate the volume of the torus.



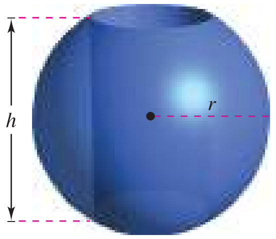
(b) Use the disk method to find the volume of the general torus if the circle has radius r and its center is R units from the axis of rotation.

CAS **5.** Graph the curve

$$8y^2 = x^2(1 - x^2).$$

Use a computer algebra system to find the surface area of the solid of revolution obtained by revolving the curve about the y-axis.

6. A hole is cut through the center of a sphere of radius r (see figure). The height of the remaining spherical ring is h. Find the volume of the ring and show that it is independent of the radius of the sphere.



7. A rectangle R of length l and width w is revolved about the line L (see figure). Find the volume of the resulting solid of revolution.

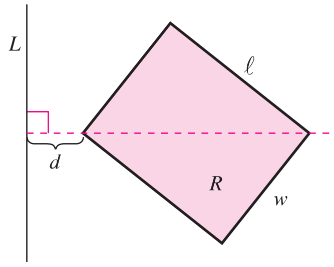


Figure for 7

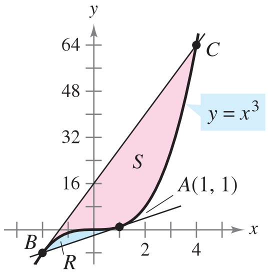


Figure for 8

8. (a) The tangent line to the curve $y = x^3$ at the point $A(1, 1)$ intersects the curve at another point B. Let R be the area of the region bounded by the curve and the tangent line. The tangent line at B intersects the curve at another point C (see figure). Let S be the area of the region bounded by the curve and this second tangent line. How are the areas R and S related?

(b) Repeat the construction in part (a) by selecting an arbitrary point A on the curve $y = x^3$. Show that the two areas R and S are always related in the same way.

9. The graph of $y = f(x)$ passes through the origin. The arc length of the curve from $(0, 0)$ to $(x, f(x))$ is given by

$$s(x) = \int_0^x \sqrt{1 + e^t}\, dt.$$

Identify the function f.

10. Let f be rectifiable on the interval $[a, b]$, and let

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

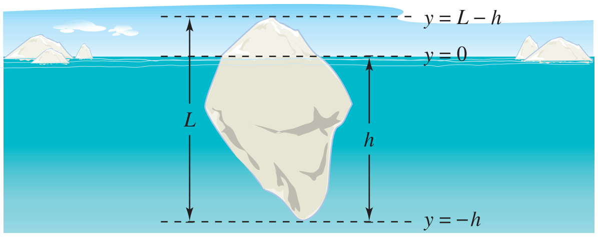
(a) Find $\dfrac{ds}{dx}$.

(b) Find ds and $(ds)^2$.

(c) If $f(t) = t^{3/2}$, find $s(x)$ on $[1, 3]$.

(d) Calculate $s(2)$ and describe what it signifies.

11. Archimedes' Principle states that the upward or buoyant force on an object within a fluid is equal to the weight of the fluid that the object displaces. For a partially submerged object, you can obtain information about the relative densities of the floating object and the fluid by observing how much of the object is above and below the surface. You can also determine the size of a floating object if you know the amount that is above the surface and the relative densities. You can see the top of a floating iceberg (see figure). The density of ocean water is 1.03×10^3 kilograms per cubic meter, and that of ice is 0.92×10^3 kilograms per cubic meter. What percent of the total iceberg is below the surface?



12. Sketch the region bounded on the left by $x = 1$, bounded above by $y = 1/x^3$, and bounded below by $y = -1/x^3$.

(a) Find the centroid of the region for $1 \le x \le 6$.

(b) Find the centroid of the region for $1 \le x \le b$.

(c) Where is the centroid as $b \to \infty$?

13. Sketch the region to the right of the y-axis, bounded above by $y = 1/x^4$ and bounded below by $y = -1/x^4$.

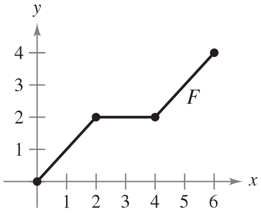
(a) Find the centroid of the region for $1 \le x \le 6$.

(b) Find the centroid of the region for $1 \le x \le b$.

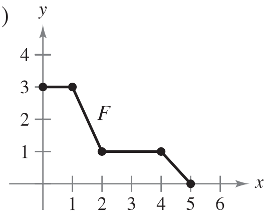
(c) Where is the centroid as $b \to \infty$?

14. Find the work done by each force F.

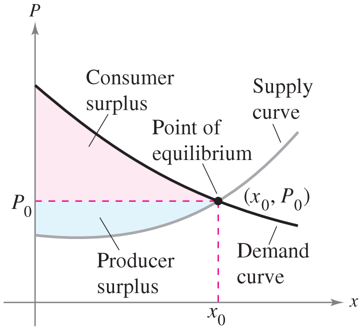
(a)



(b)



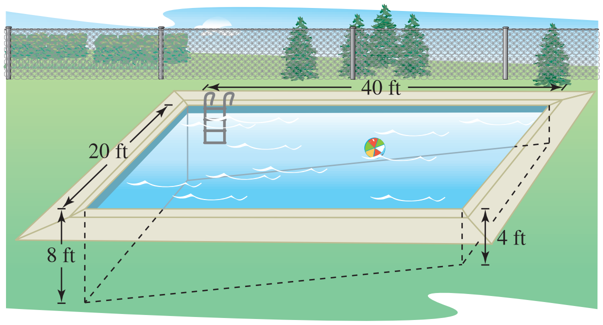
In Exercises 15 and 16, find the consumer surplus and producer surplus for the given demand $[p_1(x)]$ and supply $[p_2(x)]$ curves. The consumer surplus and producer surplus are represented by the areas shown in the figure.

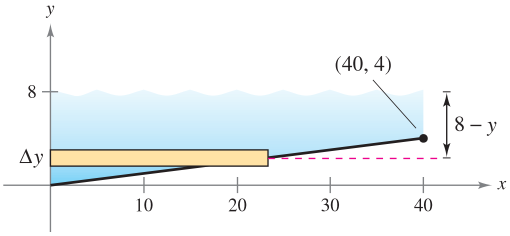


15. $p_1(x) = 50 - 0.5x, \quad p_2(x) = 0.125x$

16. $p_1(x) = 1000 - 0.4x^2, \quad p_2(x) = 42x$

17. A swimming pool is 20 feet wide, 40 feet long, 4 feet deep at one end, and 8 feet deep at the other end (see figure). The bottom is an inclined plane. Find the fluid force on each vertical wall.





18. (a) Find at least two continuous functions f that satisfy each condition.

(i) $f(x) \ge 0$ on $[0, 1]$

(ii) $f(0) = 0$ and $f(1) = 0$

(iii) The area bounded by the graph of f and the x-axis for $0 \le x \le 1$ equals 1.

(b) For each function found in part (a), approximate the arc length of the graph of the function on the interval $[0, 1]$. (Use a graphing utility if necessary.)

(c) Can you find a function f that satisfies each condition in part (a) and whose graph has an arc length of less than 3 on the interval $[0, 1]$?

8

Integration Techniques, L'Hôpital's Rule, and Improper Integrals

In previous chapters, you studied several basic techniques for evaluating simple integrals. In this chapter, you will study other integration techniques, such as integration by parts, that are used to evaluate more complicated integrals. You will also learn an important rule for evaluating limits called L'Hôpital's Rule. This rule can also help you evaluate improper integrals.

In this chapter, you should learn the following.

- How to fit an integrand to one of the basic integration rules. (8.1)
- How to find an antiderivative using integration by parts. (8.2)
- How to evaluate trigonometric integrals. (8.3)
- How to use trigonometric substitution to evaluate an integral. (8.4)
- How to use partial fraction decomposition to integrate rational functions. (8.5)
- How to evaluate an indefinite integral using a table of integrals and using reduction formulas. (8.6)
- How to apply L'Hôpital's Rule to evaluate a limit. (8.7)
- How to evaluate an improper integral. (8.8)

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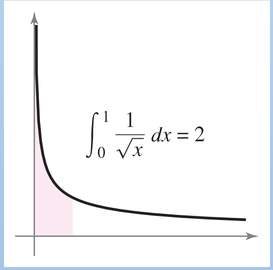
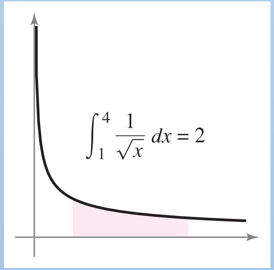
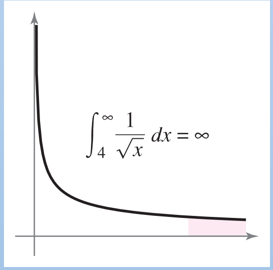
Partial fraction decomposition is an integration technique that can be used to evaluate integrals involving rational functions. How can partial fraction decomposition be used to evaluate an integral that gives the average cost of removing a certain percent of a chemical from a company's waste water? (See Section 8.5, Exercise 63.)







From your studies of calculus thus far, you know that a definite integral has finite limits of integration and a continuous integrand. In Section 8.8, you will study *improper integrals*. Improper integrals have at least one infinite limit of integration or have an integrand with an infinite discontinuity. You will see that improper integrals either converge or diverge.

8.1 Basic Integration Rules

- **Review procedures for fitting an integrand to one of the basic integration rules.**

Fitting Integrands to Basic Integration Rules

In this chapter, you will study several integration techniques that greatly expand the set of integrals to which the basic integration rules can be applied. These rules are reviewed on page 522. A major step in solving any integration problem is recognizing which basic integration rule to use. As shown in Example 1, slight differences in the integrand can lead to very different solution techniques.

↻ EXAMPLE 1 A Comparison of Three Similar Integrals

Find each integral.

a. $\displaystyle\int \frac{4}{x^2+9}\,dx$ **b.** $\displaystyle\int \frac{4x}{x^2+9}\,dx$ **c.** $\displaystyle\int \frac{4x^2}{x^2+9}\,dx$

Solution

a. Use the Arctangent Rule and let $u = x$ and $a = 3$.

$$\begin{aligned}
\int \frac{4}{x^2+9}\,dx &= 4\int \frac{1}{x^2+3^2}\,dx && \text{Constant Multiple Rule}\\
&= 4\left(\frac{1}{3}\arctan\frac{x}{3}\right) + C && \text{Arctangent Rule}\\
&= \frac{4}{3}\arctan\frac{x}{3} + C && \text{Simplify.}
\end{aligned}$$

b. Here the Arctangent Rule does not apply because the numerator contains a factor of x. Consider the Log Rule and let $u = x^2 + 9$. Then $du = 2x\,dx$, and you have

$$\begin{aligned}
\int \frac{4x}{x^2+9}\,dx &= 2\int \frac{2x\,dx}{x^2+9} && \text{Constant Multiple Rule}\\
&= 2\int \frac{du}{u} && \text{Substitution: } u = x^2+9\\
&= 2\ln|u| + C = 2\ln(x^2+9) + C. && \text{Log Rule}
\end{aligned}$$

c. Because the degree of the numerator is equal to the degree of the denominator, you should first use division to rewrite the improper rational function as the sum of a polynomial and a proper rational function.

$$\begin{aligned}
\int \frac{4x^2}{x^2+9}\,dx &= \int \left(4 - \frac{36}{x^2+9}\right)dx && \text{Rewrite using long division.}\\
&= \int 4\,dx - 36\int \frac{1}{x^2+9}\,dx && \text{Write as two integrals.}\\
&= 4x - 36\left(\frac{1}{3}\arctan\frac{x}{3}\right) + C && \text{Integrate.}\\
&= 4x - 12\arctan\frac{x}{3} + C && \text{Simplify.}
\end{aligned}$$

■

EXPLORATION

A Comparison of Three Similar Integrals Which, if any, of the following integrals can be evaluated using the 20 basic integration rules? For any that can be evaluated, do so. For any that can't, explain why.

a. $\displaystyle\int \frac{3}{\sqrt{1-x^2}}\,dx$

b. $\displaystyle\int \frac{3x}{\sqrt{1-x^2}}\,dx$

c. $\displaystyle\int \frac{3x^2}{\sqrt{1-x^2}}\,dx$

NOTE Notice in Example 1(c) that some preliminary algebra is required before applying the rules for integration, and that subsequently more than one rule is needed to evaluate the resulting integral.

The icon ↻ *indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems* Maple *and* Mathematica.

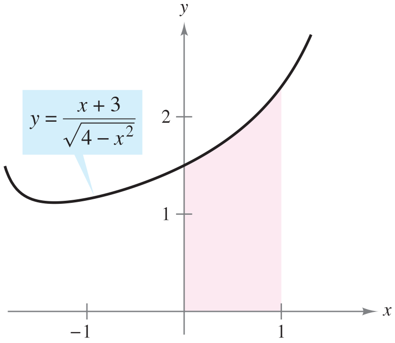
EXAMPLE 2 Using Two Basic Rules to Solve a Single Integral

Evaluate $\displaystyle\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,dx$.

Solution Begin by writing the integral as the sum of two integrals. Then apply the Power Rule and the Arcsine Rule, as follows.

$$\begin{aligned}\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,dx &= \int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx + \int_0^1 \frac{3}{\sqrt{4-x^2}}\,dx\\ &= -\frac{1}{2}\int_0^1 (4-x^2)^{-1/2}(-2x)\,dx + 3\int_0^1 \frac{1}{\sqrt{2^2-x^2}}\,dx\\ &= \left[-(4-x^2)^{1/2} + 3\arcsin\frac{x}{2}\right]_0^1\\ &= \left(-\sqrt{3}+\frac{\pi}{2}\right) - (-2+0)\\ &\approx 1.839\end{aligned}$$

See Figure 8.1. ■



The area of the region is approximately 1.839.

Figure 8.1

TECHNOLOGY Simpson's Rule can be used to give a good approximation of the value of the integral in Example 2 (for $n = 10$, the approximation is 1.839). When using numerical integration, however, you should be aware that Simpson's Rule does not always give good approximations when one or both of the limits of integration are near a vertical asymptote. For instance, using the Fundamental Theorem of Calculus, you can obtain

$$\int_0^{1.99} \frac{x+3}{\sqrt{4-x^2}}\,dx \approx 6.213.$$

Applying Simpson's Rule (with $n = 10$) to this integral produces an approximation of 6.889.

EXAMPLE 3 A Substitution Involving $a^2 - u^2$

Find $\displaystyle\int \frac{x^2}{\sqrt{16-x^6}}\,dx$.

Solution Because the radical in the denominator can be written in the form

$$\sqrt{a^2-u^2} = \sqrt{4^2-(x^3)^2}$$

you can try the substitution $u = x^3$. Then $du = 3x^2\,dx$, and you have

$$\begin{aligned}\int \frac{x^2}{\sqrt{16-x^6}}\,dx &= \frac{1}{3}\int \frac{3x^2\,dx}{\sqrt{16-(x^3)^2}} && \text{Rewrite integral.}\\ &= \frac{1}{3}\int \frac{du}{\sqrt{4^2-u^2}} && \text{Substitution: } u = x^3\\ &= \frac{1}{3}\arcsin\frac{u}{4} + C && \text{Arcsine Rule}\\ &= \frac{1}{3}\arcsin\frac{x^3}{4} + C. && \text{Rewrite as a function of } x.\end{aligned}$$

■

STUDY TIP Rules 18, 19, and 20 of the basic integration rules on the next page all have expressions involving the sum or difference of two squares:

$a^2 - u^2$
$a^2 + u^2$
$u^2 - a^2$

These expressions are often apparent after a u-substitution, as shown in Example 3.

Surprisingly, two of the most commonly overlooked integration rules are the Log Rule and the Power Rule. Notice in the next two examples how these two integration rules can be disguised.

REVIEW OF BASIC INTEGRATION RULES ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \left(\frac{1}{\ln a}\right)a^u + C$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = -\ln|\cos u| + C$
11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$
17. $\int \csc u \cot u\,du = -\csc u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

EXAMPLE 4 A Disguised Form of the Log Rule

Find $\displaystyle\int \frac{1}{1 + e^x}\,dx$.

Solution The integral does not appear to fit any of the basic rules. However, the quotient form suggests the Log Rule. If you let $u = 1 + e^x$, then $du = e^x\,dx$. You can obtain the required du by adding and subtracting e^x in the numerator, as follows.

$$\begin{aligned}
\int \frac{1}{1 + e^x}\,dx &= \int \frac{1 + e^x - e^x}{1 + e^x}\,dx && \text{Add and subtract } e^x \text{ in numerator.}\\
&= \int \left(\frac{1 + e^x}{1 + e^x} - \frac{e^x}{1 + e^x}\right)dx && \text{Rewrite as two fractions.}\\
&= \int dx - \int \frac{e^x\,dx}{1 + e^x} && \text{Rewrite as two integrals.}\\
&= x - \ln(1 + e^x) + C && \text{Integrate.}
\end{aligned}$$

■

NOTE There is usually more than one way to solve an integration problem. For instance, in Example 4, try integrating by multiplying the numerator and denominator by e^{-x} to obtain an integral of the form $-\int du/u$. See if you can get the same answer by this procedure. (Be careful: the answer will appear in a different form.) ■

EXAMPLE 5 A Disguised Form of the Power Rule

Find $\displaystyle\int (\cot x)[\ln(\sin x)]\,dx$.

Solution Again, the integral does not appear to fit any of the basic rules. However, considering the two primary choices for u [$u = \cot x$ and $u = \ln(\sin x)$], you can see that the second choice is the appropriate one because

$$u = \ln(\sin x) \qquad \text{and} \qquad du = \frac{\cos x}{\sin x}\,dx = \cot x\,dx.$$

So,

$$\begin{aligned}
\int (\cot x)[\ln(\sin x)]\,dx &= \int u\,du && \text{Substitution: } u = \ln(\sin x)\\
&= \frac{u^2}{2} + C && \text{Integrate.}\\
&= \frac{1}{2}[\ln(\sin x)]^2 + C. && \text{Rewrite as a function of } x.
\end{aligned}$$

■

NOTE In Example 5, try *checking* that the derivative of

$$\frac{1}{2}[\ln(\sin x)]^2 + C$$

is the integrand of the original integral. ■

Trigonometric identities can often be used to fit integrals to one of the basic integration rules.

EXAMPLE 6 Using Trigonometric Identities

Find $\int \tan^2 2x\,dx$.

Solution Note that $\tan^2 u$ is not in the list of basic integration rules. However, $\sec^2 u$ is in the list. This suggests the trigonometric identity $\tan^2 u = \sec^2 u - 1$. If you let $u = 2x$, then $du = 2\,dx$ and

$$\begin{aligned}
\int \tan^2 2x\,dx &= \frac{1}{2}\int \tan^2 u\,du && \text{Substitution: } u = 2x \\
&= \frac{1}{2}\int (\sec^2 u - 1)\,du && \text{Trigonometric identity} \\
&= \frac{1}{2}\int \sec^2 u\,du - \frac{1}{2}\int du && \text{Rewrite as two integrals.} \\
&= \frac{1}{2}\tan u - \frac{u}{2} + C && \text{Integrate.} \\
&= \frac{1}{2}\tan 2x - x + C. && \text{Rewrite as a function of } x.
\end{aligned}$$

■

TECHNOLOGY If you have access to a computer algebra system, try using it to evaluate the integrals in this section. Compare the *forms* of the antiderivatives given by the software with the forms obtained by hand. Sometimes the forms will be the same, but often they will differ. For instance, why is the antiderivative $\ln 2x + C$ equivalent to the antiderivative $\ln x + C$?

This section concludes with a summary of the common procedures for fitting integrands to the basic integration rules.

PROCEDURES FOR FITTING INTEGRANDS TO BASIC INTEGRATION RULES

Technique	*Example*
Expand (numerator).	$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$
Separate numerator.	$\dfrac{1 + x}{x^2 + 1} = \dfrac{1}{x^2 + 1} + \dfrac{x}{x^2 + 1}$
Complete the square.	$\dfrac{1}{\sqrt{2x - x^2}} = \dfrac{1}{\sqrt{1 - (x - 1)^2}}$
Divide improper rational function.	$\dfrac{x^2}{x^2 + 1} = 1 - \dfrac{1}{x^2 + 1}$
Add and subtract terms in numerator.	$\dfrac{2x}{x^2 + 2x + 1} = \dfrac{2x + 2 - 2}{x^2 + 2x + 1} = \dfrac{2x + 2}{x^2 + 2x + 1} - \dfrac{2}{(x + 1)^2}$
Use trigonometric identities.	$\cot^2 x = \csc^2 x - 1$
Multiply and divide by Pythagorean conjugate.	$\dfrac{1}{1 + \sin x} = \left(\dfrac{1}{1 + \sin x}\right)\left(\dfrac{1 - \sin x}{1 - \sin x}\right) = \dfrac{1 - \sin x}{1 - \sin^2 x}$ $= \dfrac{1 - \sin x}{\cos^2 x} = \sec^2 x - \dfrac{\sin x}{\cos^2 x}$

NOTE Remember that you can separate numerators but not denominators. Watch out for this common error when fitting integrands to basic rules.

$$\frac{1}{x^2 + 1} \neq \frac{1}{x^2} + \frac{1}{1} \qquad \text{Do not separate denominators.}$$

■

8.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, select the correct antiderivative.

1. $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2+1}}$

(a) $2\sqrt{x^2+1} + C$ (b) $\sqrt{x^2+1} + C$

(c) $\frac{1}{2}\sqrt{x^2+1} + C$ (d) $\ln(x^2+1) + C$

2. $\dfrac{dy}{dx} = \dfrac{x}{x^2+1}$

(a) $\ln\sqrt{x^2+1} + C$ (b) $\dfrac{2x}{(x^2+1)^2} + C$

(c) $\arctan x + C$ (d) $\ln(x^2+1) + C$

3. $\dfrac{dy}{dx} = \dfrac{1}{x^2+1}$

(a) $\ln\sqrt{x^2+1} + C$ (b) $\dfrac{2x}{(x^2+1)^2} + C$

(c) $\arctan x + C$ (d) $\ln(x^2+1) + C$

4. $\dfrac{dy}{dx} = x\cos(x^2+1)$

(a) $2x\sin(x^2+1) + C$ (b) $-\frac{1}{2}\sin(x^2+1) + C$

(c) $\frac{1}{2}\sin(x^2+1) + C$ (d) $-2x\sin(x^2+1) + C$

In Exercises 5–14, select the basic integration formula you can use to find the integral, and identify *u* and *a* when appropriate.

5. $\displaystyle\int (5x-3)^4\,dx$ **6.** $\displaystyle\int \frac{2t+1}{t^2+t-4}\,dt$

7. $\displaystyle\int \frac{1}{\sqrt{x}\,(1-2\sqrt{x})}\,dx$ **8.** $\displaystyle\int \frac{2}{(2t-1)^2+4}\,dt$

9. $\displaystyle\int \frac{3}{\sqrt{1-t^2}}\,dt$ **10.** $\displaystyle\int \frac{-2x}{\sqrt{x^2-4}}\,dx$

11. $\displaystyle\int t\sin t^2\,dt$ **12.** $\displaystyle\int \sec 5x\tan 5x\,dx$

13. $\displaystyle\int (\cos x)e^{\sin x}\,dx$ **14.** $\displaystyle\int \frac{1}{x\sqrt{x^2-4}}\,dx$

In Exercises 15–52, find the indefinite integral.

15. $\displaystyle\int 14(x-5)^6\,dx$ **16.** $\displaystyle\int \frac{9}{(t-8)^2}\,dt$

17. $\displaystyle\int \frac{7}{(z-10)^7}\,dz$ **18.** $\displaystyle\int t^2\sqrt[3]{t^3-1}\,dt$

19. $\displaystyle\int \left[v + \frac{1}{(3v-1)^3}\right]dv$ **20.** $\displaystyle\int \left[x - \frac{5}{(3x+5)^2}\right]dx$

21. $\displaystyle\int \frac{t^2-3}{-t^3+9t+1}\,dt$ **22.** $\displaystyle\int \frac{x+1}{\sqrt{x^2+2x-4}}\,dx$

23. $\displaystyle\int \frac{x^2}{x-1}\,dx$ **24.** $\displaystyle\int \frac{4x}{x-8}\,dx$

25. $\displaystyle\int \frac{e^x}{1+e^x}\,dx$ **26.** $\displaystyle\int \left(\frac{1}{7x-2} - \frac{1}{7x+2}\right)dx$

27. $\displaystyle\int (5+4x^2)^2\,dx$ **28.** $\displaystyle\int x\left(1+\frac{1}{x}\right)^3 dx$

29. $\displaystyle\int x\cos 2\pi x^2\,dx$ **30.** $\displaystyle\int \sec 4x\,dx$

31. $\displaystyle\int \csc \pi x\cot \pi x\,dx$ **32.** $\displaystyle\int \frac{\sin x}{\sqrt{\cos x}}\,dx$

33. $\displaystyle\int e^{11x}\,dx$ **34.** $\displaystyle\int \csc^2 x e^{\cot x}\,dx$

35. $\displaystyle\int \frac{2}{e^{-x}+1}\,dx$ **36.** $\displaystyle\int \frac{5}{3e^x-2}\,dx$

37. $\displaystyle\int \frac{\ln x^2}{x}\,dx$ **38.** $\displaystyle\int (\tan x)[\ln(\cos x)]\,dx$

39. $\displaystyle\int \frac{1+\sin x}{\cos x}\,dx$ **40.** $\displaystyle\int \frac{1+\cos\alpha}{\sin\alpha}\,d\alpha$

41. $\displaystyle\int \frac{1}{\cos\theta - 1}\,d\theta$ **42.** $\displaystyle\int \frac{2}{3(\sec x - 1)}\,dx$

43. $\displaystyle\int \frac{-1}{\sqrt{1-(4t+1)^2}}\,dt$ **44.** $\displaystyle\int \frac{1}{9+5x^2}\,dx$

45. $\displaystyle\int \frac{\tan(2/t)}{t^2}\,dt$ **46.** $\displaystyle\int \frac{e^{1/t}}{t^2}\,dt$

47. $\displaystyle\int \frac{6}{\sqrt{10x-x^2}}\,dx$

48. $\displaystyle\int \frac{1}{(x-1)\sqrt{4x^2-8x+3}}\,dx$

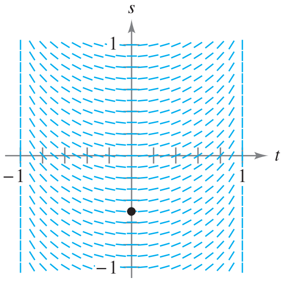
49. $\displaystyle\int \frac{4}{4x^2+4x+65}\,dx$ **50.** $\displaystyle\int \frac{1}{x^2-4x+9}\,dx$

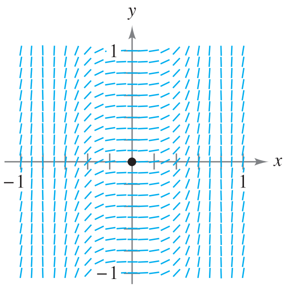
51. $\displaystyle\int \frac{1}{\sqrt{1-4x-x^2}}\,dx$ **52.** $\displaystyle\int \frac{12}{\sqrt{3-8x-x^2}}\,dx$

Slope Fields **In Exercises 53–56, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.***

53. $\dfrac{ds}{dt} = \dfrac{t}{\sqrt{1-t^4}}$ **54.** $\dfrac{dy}{dx} = \tan^2(2x)$

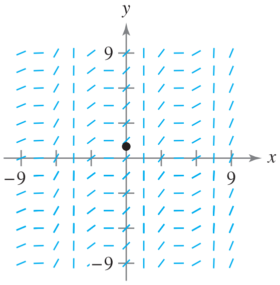
$\left(0, -\dfrac{1}{2}\right)$ $(0, 0)$





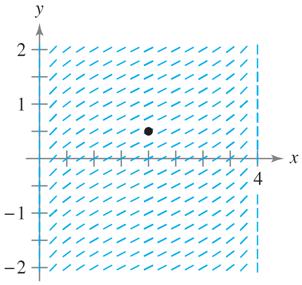
55. $\dfrac{dy}{dx} = (\sec x + \tan x)^2$

$(0, 1)$



56. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{4x - x^2}}$

$\left(2, \dfrac{1}{2}\right)$



CAS ***Slope Fields*** **In Exercises 57 and 58, use a computer algebra system to graph the slope field for the differential equation and graph the solution through the specified initial condition.**

57. $\dfrac{dy}{dx} = 0.8y, \ y(0) = 4$

58. $\dfrac{dy}{dx} = 5 - y, \ y(0) = 1$

In Exercises 59–64, solve the differential equation.

59. $\dfrac{dy}{dx} = (e^x + 5)^2$

60. $\dfrac{dy}{dx} = (3 - e^x)^2$

61. $\dfrac{dr}{dt} = \dfrac{10e^t}{\sqrt{1 - e^{2t}}}$

62. $\dfrac{dr}{dt} = \dfrac{(1 + e^t)^2}{e^t}$

63. $(4 + \tan^2 x)y' = \sec^2 x$

64. $y' = \dfrac{1}{x\sqrt{4x^2 - 1}}$

In Exercises 65–72, evaluate the definite integral. Use the integration capabilities of a graphing utility to verify your result.

65. $\displaystyle\int_0^{\pi/4} \cos 2x \, dx$

66. $\displaystyle\int_0^{\pi} \sin^2 t \cos t \, dt$

67. $\displaystyle\int_0^1 xe^{-x^2} \, dx$

68. $\displaystyle\int_1^e \frac{1 - \ln x}{x} \, dx$

69. $\displaystyle\int_0^8 \frac{2x}{\sqrt{x^2 + 36}} \, dx$

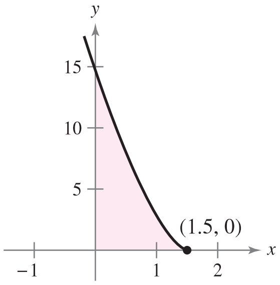
70. $\displaystyle\int_1^2 \frac{x - 2}{x} \, dx$

71. $\displaystyle\int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2} \, dx$

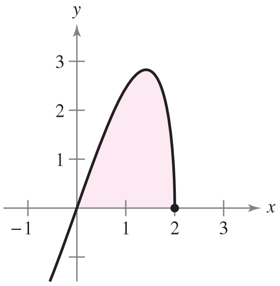
72. $\displaystyle\int_0^7 \frac{1}{\sqrt{100 - x^2}} \, dx$

Area **In Exercises 73–78, find the area of the region.**

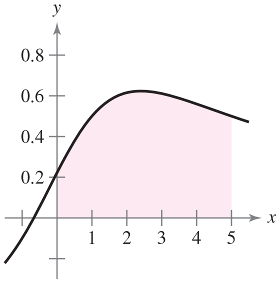
73. $y = (-4x + 6)^{3/2}$



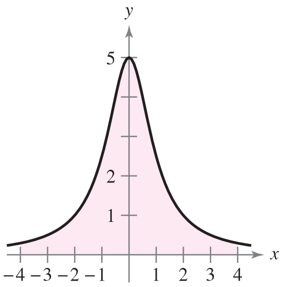
74. $y = x\sqrt{8 - 2x^2}$



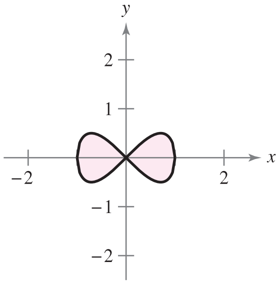
75. $y = \dfrac{3x + 2}{x^2 + 9}$



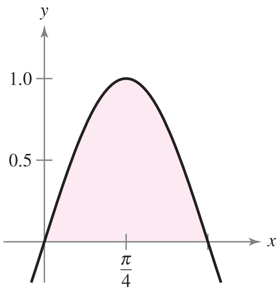
76. $y = \dfrac{5}{x^2 + 1}$



77. $y^2 = x^2(1 - x^2)$



78. $y = \sin 2x$



CAS **In Exercises 79–82, use a computer algebra system to find the integral. Use the computer algebra system to graph two antiderivatives. Describe the relationship between the graphs of the two antiderivatives.**

79. $\displaystyle\int \frac{1}{x^2 + 4x + 13} \, dx$

80. $\displaystyle\int \frac{x - 2}{x^2 + 4x + 13} \, dx$

81. $\displaystyle\int \frac{1}{1 + \sin \theta} \, d\theta$

82. $\displaystyle\int \left(\frac{e^x + e^{-x}}{2}\right)^3 dx$

WRITING ABOUT CONCEPTS

In Exercises 83–86, state the integration formula you would use to perform the integration. Explain why you chose that formula. Do not integrate.

83. $\displaystyle\int x(x^2 + 1)^3 \, dx$

84. $\displaystyle\int x \sec(x^2 + 1) \tan(x^2 + 1) \, dx$

85. $\displaystyle\int \frac{x}{x^2 + 1} \, dx$

86. $\displaystyle\int \frac{1}{x^2 + 1} \, dx$

87. Determine the constants a and b such that

$$\sin x + \cos x = a \sin(x + b).$$

Use this result to integrate $\displaystyle\int \frac{dx}{\sin x + \cos x}$.

88. Show that $\sec x = \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{1 + \sin x}$. Then use this identity to derive the basic integration rule

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C.$$

89. ***Area*** The graphs of $f(x) = x$ and $g(x) = ax^2$ intersect at the points $(0, 0)$ and $(1/a, 1/a)$. Find a $(a > 0)$ such that the area of the region bounded by the graphs of these two functions is $\frac{2}{3}$.

CAPSTONE

90. (a) Explain why the antiderivative $y_1 = e^{x+C_1}$ is equivalent to the antiderivative $y_2 = Ce^x$.

(b) Explain why the antiderivative $y_1 = \sec^2 x + C_1$ is equivalent to the antiderivative $y_2 = \tan^2 x + C$.

91. ***Think About It*** Use a graphing utility to graph the function $f(x) = \frac{1}{5}(x^3 - 7x^2 + 10x)$. Use the graph to determine whether $\int_0^5 f(x)\,dx$ is positive or negative. Explain.

92. ***Think About It*** When evaluating $\int_{-1}^{1} x^2\,dx$, is it appropriate to substitute $u = x^2$, $x = \sqrt{u}$, and $dx = \dfrac{du}{2\sqrt{u}}$ to obtain $\dfrac{1}{2}\displaystyle\int_1^1 \sqrt{u}\,du = 0$? Explain.

Approximation **In Exercises 93 and 94, determine which value best approximates the area of the region between the x-axis and the function over the given interval. (Make your selection on the basis of a sketch of the region and *not* by integrating.)**

93. $f(x) = \dfrac{4x}{x^2 + 1}$, $[0, 2]$

(a) 3 (b) 1 (c) -8 (d) 8 (e) 10

94. $f(x) = \dfrac{4}{x^2 + 1}$, $[0, 2]$

(a) 3 (b) 1 (c) -4 (d) 4 (e) 10

Interpreting Integrals **In Exercises 95 and 96, (a) sketch the region whose area is given by the integral, (b) sketch the solid whose volume is given by the integral if the disk method is used, and (c) sketch the solid whose volume is given by the integral if the shell method is used. (There is more than one correct answer for each part.)**

95. $\displaystyle\int_0^2 2\pi x^2\,dx$

96. $\displaystyle\int_0^4 \pi y\,dy$

97. ***Volume*** The region bounded by $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = b$ $(b > 0)$ is revolved about the y-axis.

(a) Find the volume of the solid generated if $b = 1$.

(b) Find b such that the volume of the generated solid is $\frac{4}{3}$ cubic units.

98. ***Volume*** Consider the region bounded by the graphs of $x = 0$, $y = \cos x^2$, $y = \sin x^2$, and $x = \sqrt{\pi}/2$. Find the volume of the solid generated by revolving the region about the y-axis.

99. ***Arc Length*** Find the arc length of the graph of $y = \ln(\sin x)$ from $x = \pi/4$ to $x = \pi/2$.

100. ***Arc Length*** Find the arc length of the graph of $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/3$.

101. ***Surface Area*** Find the area of the surface formed by revolving the graph of $y = 2\sqrt{x}$ on the interval $[0, 9]$ about the x-axis.

102. ***Centroid*** Find the x-coordinate of the centroid of the region bounded by the graphs of

$$y = \frac{5}{\sqrt{25 - x^2}}, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 4.$$

In Exercises 103 and 104, find the average value of the function over the given interval.

103. $f(x) = \dfrac{1}{1 + x^2}$, $-3 \le x \le 3$

104. $f(x) = \sin nx$, $0 \le x \le \pi/n$, n is a positive integer.

Arc Length **In Exercises 105 and 106, use the integration capabilities of a graphing utility to approximate the arc length of the curve over the given interval.**

105. $y = \tan \pi x$, $\left[0, \frac{1}{4}\right]$

106. $y = x^{2/3}$, $[1, 8]$

107. ***Finding a Pattern***

(a) Find $\displaystyle\int \cos^3 x\,dx$. (b) Find $\displaystyle\int \cos^5 x\,dx$.

(c) Find $\displaystyle\int \cos^7 x\,dx$.

(d) Explain how to find $\int \cos^{15} x\,dx$ without actually integrating.

108. ***Finding a Pattern***

(a) Write $\int \tan^3 x\,dx$ in terms of $\int \tan x\,dx$. Then find $\int \tan^3 x\,dx$.

(b) Write $\int \tan^5 x\,dx$ in terms of $\int \tan^3 x\,dx$.

(c) Write $\int \tan^{2k+1} x\,dx$, where k is a positive integer, in terms of $\int \tan^{2k-1} x\,dx$.

(d) Explain how to find $\int \tan^{15} x\,dx$ without actually integrating.

109. ***Methods of Integration*** Show that the following results are equivalent.

Integration by tables:

$$\int \sqrt{x^2 + 1}\,dx = \frac{1}{2}\left(x\sqrt{x^2 + 1} + \ln\left|x + \sqrt{x^2 + 1}\right|\right) + C$$

Integration by computer algebra system:

$$\int \sqrt{x^2 + 1}\,dx = \frac{1}{2}\left[x\sqrt{x^2 + 1} + \operatorname{arcsinh}(x)\right] + C$$

PUTNAM EXAM CHALLENGE

110. Evaluate $\displaystyle\int_2^4 \frac{\sqrt{\ln(9 - x)}\,dx}{\sqrt{\ln(9 - x)} + \sqrt{\ln(x + 3)}}$.

This problem was composed by the Committee on the Putnam Prize Competition.
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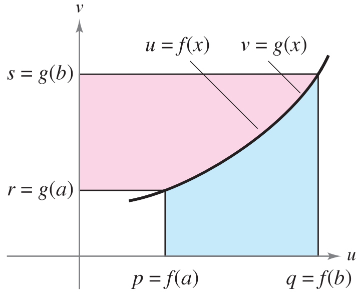
8.2 Integration by Parts

- **Find an antiderivative using integration by parts.**
- **Use a tabular method to perform integration by parts.**

EXPLORATION

Proof Without Words Here is a different approach to proving the formula for integration by parts. Exercise taken from "Proof Without Words: Integration by Parts" by Roger B. Nelsen, *Mathematics Magazine*, 64, No. 2, April 1991, p. 130, by permission of the author.



Area 🟪 + Area 🟦 $= qs - pr$

$$\int_r^s u\,dv + \int_q^p v\,du = \Big[uv\Big]_{(p,r)}^{(q,s)}$$

$$\int_r^s u\,dv = \Big[uv\Big]_{(p,r)}^{(q,s)} - \int_q^p v\,du$$

Explain how this graph proves the theorem. Which notation in this proof is unfamiliar? What do you think it means?

Integration by Parts

In this section you will study an important integration technique called **integration by parts.** This technique can be applied to a wide variety of functions and is particularly useful for integrands involving *products* of algebraic and transcendental functions. For instance, integration by parts works well with integrals such as

$$\int x \ln x\,dx, \quad \int x^2 e^x\,dx, \quad \text{and} \quad \int e^x \sin x\,dx.$$

Integration by parts is based on the formula for the derivative of a product

$$\begin{aligned}\frac{d}{dx}[uv] &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= uv' + vu'\end{aligned}$$

where both u and v are differentiable functions of x. If u' and v' are continuous, you can integrate both sides of this equation to obtain

$$\begin{aligned}uv &= \int uv'\,dx + \int vu'\,dx\\ &= \int u\,dv + \int v\,du.\end{aligned}$$

By rewriting this equation, you obtain the following theorem.

THEOREM 8.1 INTEGRATION BY PARTS

If u and v are functions of x and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du.$$

This formula expresses the original integral in terms of another integral. Depending on the choices of u and dv, it may be easier to evaluate the second integral than the original one. Because the choices of u and dv are critical in the integration by parts process, the following guidelines are provided.

GUIDELINES FOR INTEGRATION BY PARTS

1. Try letting dv be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand.
2. Try letting u be the portion of the integrand whose derivative is a function simpler than u. Then dv will be the remaining factor(s) of the integrand.

Note that dv always includes the dx of the original integrand.

EXAMPLE 1 Integration by Parts

Find $\int xe^x\,dx.$

Solution To apply integration by parts, you need to write the integral in the form $\int u\,dv$. There are several ways to do this.

$$\int \underbrace{(x)}_{u}\,\underbrace{(e^x dx)}_{dv}, \quad \int \underbrace{(e^x)}_{u}\underbrace{(x\,dx)}_{dv}, \quad \int \underbrace{(1)}_{u}\,\underbrace{(xe^x\,dx)}_{dv}, \quad \int \underbrace{(xe^x)}_{u}\underbrace{(dx)}_{dv}$$

The guidelines on page 527 suggest the first option because the derivative of $u = x$ is simpler than x, and $dv = e^x\,dx$ is the most complicated portion of the integrand that fits a basic integration formula.

$$dv = e^x\,dx \quad \Rightarrow \quad v = \int dv = \int e^x\,dx = e^x$$

$$u = x \quad \Rightarrow \quad du = dx$$

Now, integration by parts produces

$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx \qquad \text{Substitute.}$$

$$= xe^x - e^x + C. \qquad \text{Integrate.}$$

To check this, differentiate $xe^x - e^x + C$ to see that you obtain the original integrand.

NOTE In Example 1, note that it is not necessary to include a constant of integration when solving

$$v = \int e^x\,dx = e^x + C_1.$$

To illustrate this, replace $v = e^x$ by $v = e^x + C_1$ and apply integration by parts to see that you obtain the same result.

EXAMPLE 2 Integration by Parts

Find $\int x^2 \ln x\,dx.$

Solution In this case, x^2 is more easily integrated than $\ln x$. Furthermore, the derivative of $\ln x$ is simpler than $\ln x$. So, you should let $dv = x^2\,dx$.

$$dv = x^2\,dx \quad \Rightarrow \quad v = \int x^2\,dx = \frac{x^3}{3}$$

$$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}\,dx$$

Integration by parts produces

$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right)dx \qquad \text{Substitute.}$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx \qquad \text{Simplify.}$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C. \qquad \text{Integrate.}$$

You can check this result by differentiating.

$$\frac{d}{dx}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right] = \frac{x^3}{3}\left(\frac{1}{x}\right) + (\ln x)(x^2) - \frac{x^2}{3} = x^2 \ln x$$ ■

TECHNOLOGY Try graphing

$$\int x^2 \ln x\,dx \quad \text{and} \quad \frac{x^3}{3}\ln x - \frac{x^3}{9}$$

on your graphing utility. Do you get the same graph? (This will take a while, so be patient.)

FOR FURTHER INFORMATION To see how integration by parts is used to prove Stirling's approximation

$$\ln(n!) = n \ln n - n$$

see the article "The Validity of Stirling's Approximation: A Physical Chemistry Project" by A. S. Wallner and K. A. Brandt in *Journal of Chemical Education*.

One surprising application of integration by parts involves integrands consisting of single terms, such as $\int \ln x\, dx$ or $\int \arcsin x\, dx$. In these cases, try letting $dv = dx$, as shown in the next example.

EXAMPLE 3 An Integrand with a Single Term

Evaluate $\displaystyle\int_0^1 \arcsin x\, dx$.

Solution Let $dv = dx$.

$$dv = dx \quad \Longrightarrow \quad v = \int dx = x$$

$$u = \arcsin x \quad \Longrightarrow \quad du = \frac{1}{\sqrt{1-x^2}}\, dx$$

Integration by parts now produces

$$\int u\, dv = uv - \int v\, du \qquad \text{Integration by parts formula}$$

$$\int \arcsin x\, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\, dx \qquad \text{Substitute.}$$

$$= x \arcsin x + \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\, dx \qquad \text{Rewrite.}$$

$$= x \arcsin x + \sqrt{1-x^2} + C. \qquad \text{Integrate.}$$

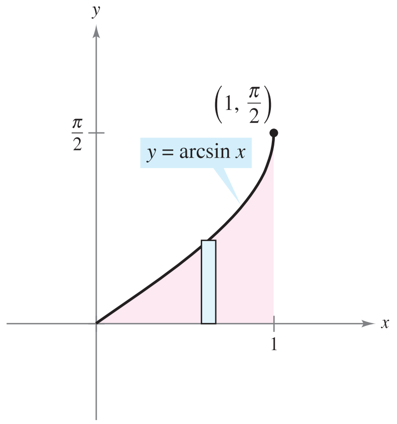
Using this antiderivative, you can evaluate the definite integral as follows.

$$\int_0^1 \arcsin x\, dx = \Big[x \arcsin x + \sqrt{1-x^2}\Big]_0^1$$

$$= \frac{\pi}{2} - 1$$

$$\approx 0.571$$

The area represented by this definite integral is shown in Figure 8.2. ■



The area of the region is approximately 0.571.
Figure 8.2

TECHNOLOGY Remember that there are two ways to use technology to evaluate a definite integral: (1) you can use a numerical approximation such as the Trapezoidal Rule or Simpson's Rule, or (2) you can use a computer algebra system to find the antiderivative and then apply the Fundamental Theorem of Calculus. Both methods have shortcomings. To find the possible error when using a numerical method, the integrand must have a second derivative (Trapezoidal Rule) or a fourth derivative (Simpson's Rule) in the interval of integration: the integrand in Example 3 fails to meet either of these requirements. To apply the Fundamental Theorem of Calculus, the symbolic integration utility must be able to find the antiderivative.

Which method would you use to evaluate

$$\int_0^1 \arctan x\, dx?$$

Which method would you use to evaluate

$$\int_0^1 \arctan x^2\, dx?$$

Some integrals require repeated use of the integration by parts formula.

EXAMPLE 4 Repeated Use of Integration by Parts

Find $\displaystyle\int x^2 \sin x\, dx$.

Solution The factors x^2 and $\sin x$ are equally easy to integrate. However, the derivative of x^2 becomes simpler, whereas the derivative of $\sin x$ does not. So, you should let $u = x^2$.

$$\begin{aligned} dv &= \sin x\, dx \quad \Longrightarrow \quad v = \int \sin x\, dx = -\cos x \\ u &= x^2 \quad \Longrightarrow \quad du = 2x\, dx \end{aligned}$$

Now, integration by parts produces

$$\int x^2 \sin x\, dx = -x^2 \cos x + \int 2x \cos x\, dx. \qquad \text{First use of integration by parts}$$

This first use of integration by parts has succeeded in simplifying the original integral, but the integral on the right still doesn't fit a basic integration rule. To evaluate that integral, you can apply integration by parts again. This time, let $u = 2x$.

$$\begin{aligned} dv &= \cos x\, dx \quad \Longrightarrow \quad v = \int \cos x\, dx = \sin x \\ u &= 2x \quad \Longrightarrow \quad du = 2\, dx \end{aligned}$$

Now, integration by parts produces

$$\begin{aligned} \int 2x \cos x\, dx &= 2x \sin x - \int 2 \sin x\, dx \qquad \text{Second use of integration by parts} \\ &= 2x \sin x + 2 \cos x + C. \end{aligned}$$

Combining these two results, you can write

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C. \qquad \blacksquare$$

When making repeated applications of integration by parts, you need to be careful not to interchange the substitutions in successive applications. For instance, in Example 4, the first substitution was $u = x^2$ and $dv = \sin x\, dx$. If, in the second application, you had switched the substitution to $u = \cos x$ and $dv = 2x$, you would have obtained

$$\begin{aligned} \int x^2 \sin x\, dx &= -x^2 \cos x + \int 2x \cos x\, dx \\ &= -x^2 \cos x + x^2 \cos x + \int x^2 \sin x\, dx = \int x^2 \sin x\, dx \end{aligned}$$

thereby undoing the previous integration and returning to the *original* integral. When making repeated applications of integration by parts, you should also watch for the appearance of a *constant multiple* of the original integral. For instance, this occurs when you use integration by parts to evaluate $\int e^x \cos 2x\, dx$, and also occurs in Example 5 on the next page.

The integral in Example 5 is an important one. In Section 8.4 (Example 5), you will see that it is used to find the arc length of a parabolic segment.

EXPLORATION

Try to find

$$\int e^x \cos 2x\, dx$$

by letting $u = \cos 2x$ and $dv = e^x\, dx$ in the first substitution. For the second substitution, let $u = \sin 2x$ and $dv = e^x\, dx$.

EXAMPLE 5 Integration by Parts

Find $\int \sec^3 x\,dx$.

Solution The most complicated portion of the integrand that can be easily integrated is $\sec^2 x$, so you should let $dv = \sec^2 x\,dx$ and $u = \sec x$.

$$dv = \sec^2 x\,dx \quad \Rightarrow \quad v = \int \sec^2 x\,dx = \tan x$$

$$u = \sec x \quad \Rightarrow \quad du = \sec x \tan x\,dx$$

Integration by parts produces

$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx \qquad \text{Substitute.}$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx \qquad \text{Trigonometric identity}$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \qquad \text{Rewrite.}$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx \qquad \text{Collect like integrals.}$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C \qquad \text{Integrate.}$$

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C. \qquad \text{Divide by 2.}$$

STUDY TIP The trigonometric identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

play an important role in this chapter.

EXAMPLE 6 Finding a Centroid

A machine part is modeled by the region bounded by the graph of $y = \sin x$ and the x-axis, $0 \le x \le \pi/2$, as shown in Figure 8.3. Find the centroid of this region.

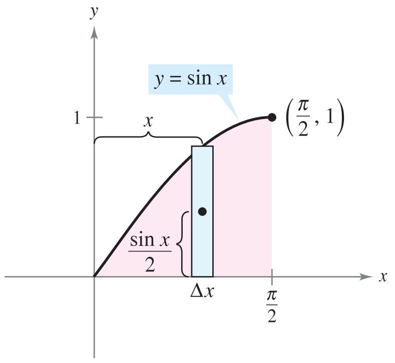


Figure 8.3

Solution Begin by finding the area of the region.

$$A = \int_0^{\pi/2} \sin x\,dx = \Big[-\cos x\Big]_0^{\pi/2} = 1$$

Now, you can find the coordinates of the centroid as follows.

$$\bar{y} = \frac{1}{A}\int_0^{\pi/2} \frac{\sin x}{2}(\sin x)\,dx = \frac{1}{4}\int_0^{\pi/2} (1 - \cos 2x)\,dx = \frac{1}{4}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{8}$$

You can evaluate the integral for $\bar{x}$, $(1/A)\int_0^{\pi/2} x \sin x\,dx$, with integration by parts. To do this, let $dv = \sin x\,dx$ and $u = x$. This produces $v = -\cos x$ and $du = dx$, and you can write

$$\int x \sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

Finally, you can determine $\bar{x}$ to be

$$\bar{x} = \frac{1}{A}\int_0^{\pi/2} x \sin x\,dx = \Big[-x\cos x + \sin x\Big]_0^{\pi/2} = 1.$$

So, the centroid of the region is $(1, \pi/8)$. ■

As you gain experience in using integration by parts, your skill in determining u and dv will increase. The following summary lists several common integrals with suggestions for the choices of u and dv.

STUDY TIP You can use the acronym LIATE as a guideline for choosing u in integration by parts. In order, check the integrand for the following.

Is there a Logarithmic part?
Is there an Inverse trigonometric part?
Is there an Algebraic part?
Is there a Trigonometric part?
Is there an Exponential part?

SUMMARY OF COMMON INTEGRALS USING INTEGRATION BY PARTS

1. For integrals of the form

$$\int x^n e^{ax}\,dx,\qquad \int x^n \sin ax\,dx,\qquad \text{or}\qquad \int x^n \cos ax\,dx$$

let $u = x^n$ and let $dv = e^{ax}\,dx$, $\sin ax\,dx$, or $\cos ax\,dx$.

2. For integrals of the form

$$\int x^n \ln x\,dx,\qquad \int x^n \arcsin ax\,dx,\qquad \text{or}\qquad \int x^n \arctan ax\,dx$$

let $u = \ln x$, $\arcsin ax$, or $\arctan ax$ and let $dv = x^n\,dx$.

3. For integrals of the form

$$\int e^{ax} \sin bx\,dx\qquad \text{or}\qquad \int e^{ax} \cos bx\,dx$$

let $u = \sin bx$ or $\cos bx$ and let $dv = e^{ax}\,dx$.

Tabular Method

In problems involving repeated applications of integration by parts, a tabular method, illustrated in Example 7, can help to organize the work. This method works well for integrals of the form $\int x^n \sin ax\,dx$, $\int x^n \cos ax\,dx$, and $\int x^n e^{ax}\,dx$.

EXAMPLE 7 Using the Tabular Method

Find $\displaystyle\int x^2 \sin 4x\,dx$.

Solution Begin as usual by letting $u = x^2$ and $dv = v'\,dx = \sin 4x\,dx$. Next, create a table consisting of three columns, as shown.

Alternate Signs	*u and Its Derivatives*	*v′ and Its Antiderivatives*
$+$	x^2	$\sin 4x$
$-$	$2x$	$-\frac{1}{4}\cos 4x$
$+$	2	$-\frac{1}{16}\sin 4x$
$-$	0	$\frac{1}{64}\cos 4x$

Differentiate until you obtain 0 as a derivative.

The solution is obtained by adding the signed products of the diagonal entries:

$$\int x^2 \sin 4x\,dx = -\frac{1}{4}x^2\cos 4x + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x + C.$$

■ **FOR FURTHER INFORMATION** For more information on the tabular method, see the article "Tabular Integration by Parts" by David Horowitz in *The College Mathematics Journal*, and the article "More on Tabular Integration by Parts" by Leonard Gillman in *The College Mathematics Journal*. To view these articles, go to the website *www.matharticles.com*.

8.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, identify u and dv for finding the integral using integration by parts. (Do not evaluate the integral.)

1. $\int xe^{2x}\,dx$ **2.** $\int x^2 e^{2x}\,dx$

3. $\int (\ln x)^2\,dx$ **4.** $\int \ln 5x\,dx$

5. $\int x\sec^2 x\,dx$ **6.** $\int x^2 \cos x\,dx$

In Exercises 7–10, evaluate the integral using integration by parts with the given choices of u and dv.

7. $\displaystyle\int x^3 \ln x\,dx;\ u = \ln x,\ dv = x^3\,dx$

8. $\displaystyle\int (4x + 7)e^x\,dx;\ u = 4x + 7,\ dv = e^x\,dx$

9. $\displaystyle\int x\sin 3x\,dx;\ u = x,\ dv = \sin 3x\,dx$

10. $\displaystyle\int x\cos 4x\,dx;\ u = x,\ dv = \cos 4x\,dx$

In Exercises 11–38, find the integral. (*Note:* Solve by the simplest method—not all require integration by parts.)

11. $\displaystyle\int xe^{-4x}\,dx$ **12.** $\displaystyle\int \frac{4x}{e^x}\,dx$

13. $\displaystyle\int x^3e^x\,dx$ **14.** $\displaystyle\int \frac{e^{1/t}}{t^2}\,dt$

15. $\displaystyle\int x^2e^{x^3}\,dx$ **16.** $\displaystyle\int x^4\ln x\,dx$

17. $\displaystyle\int t\ln(t + 1)\,dt$ **18.** $\displaystyle\int \frac{1}{x(\ln x)^3}\,dx$

19. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$ **20.** $\displaystyle\int \frac{\ln x}{x^2}\,dx$

21. $\displaystyle\int \frac{xe^{2x}}{(2x + 1)^2}\,dx$ **22.** $\displaystyle\int \frac{x^3e^{x^2}}{(x^2 + 1)^2}\,dx$

23. $\displaystyle\int (x^2 - 1)e^x\,dx$ **24.** $\displaystyle\int \frac{\ln 2x}{x^2}\,dx$

25. $\displaystyle\int x\sqrt{x - 5}\,dx$ **26.** $\displaystyle\int \frac{x}{\sqrt{5 + 4x}}\,dx$

27. $\displaystyle\int x\cos x\,dx$ **28.** $\displaystyle\int x\sin x\,dx$

29. $\displaystyle\int x^3\sin x\,dx$ **30.** $\displaystyle\int x^2\cos x\,dx$

31. $\displaystyle\int t\csc t\cot t\,dt$ **32.** $\displaystyle\int \theta\sec\theta\tan\theta\,d\theta$

33. $\displaystyle\int \arctan x\,dx$ **34.** $\displaystyle\int 4\arccos x\,dx$

35. $\displaystyle\int e^{2x}\sin x\,dx$ **36.** $\displaystyle\int e^{-3x}\sin 5x\,dx$

37. $\displaystyle\int e^{-x}\cos 2x\,dx$ **38.** $\displaystyle\int e^{3x}\cos 4x\,dx$

In Exercises 39–44, solve the differential equation.

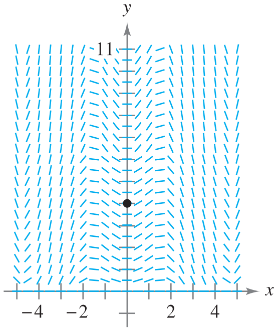
39. $y' = xe^{x^2}$ **40.** $y' = \ln x$

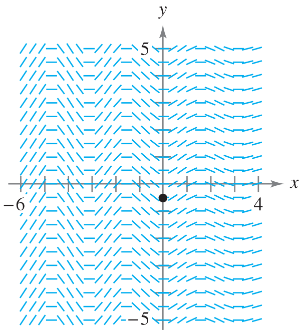
41. $\displaystyle\frac{dy}{dt} = \frac{t^2}{\sqrt{3 + 5t}}$ **42.** $\displaystyle\frac{dy}{dx} = x^2\sqrt{x - 3}$

43. $(\cos y)y' = 2x$ **44.** $\displaystyle y' = \arctan\frac{x}{2}$

Slope Fields **In Exercises 45 and 46, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

45. $\displaystyle\frac{dy}{dx} = x\sqrt{y}\cos x,\ (0, 4)$ **46.** $\displaystyle\frac{dy}{dx} = e^{-x/3}\sin 2x,\ \left(0, -\frac{18}{37}\right)$





CAS ***Slope Fields*** **In Exercises 47 and 48, use a computer algebra system to graph the slope field for the differential equation and graph the solution through the specified initial condition.**

47. $\displaystyle\frac{dy}{dx} = \frac{x}{y}e^{x/8}$ **48.** $\displaystyle\frac{dy}{dx} = \frac{x}{y}\sin x$

$y(0) = 2$ $y(0) = 4$

In Exercises 49–60, evaluate the definite integral. Use a graphing utility to confirm your result.

49. $\displaystyle\int_0^3 xe^{x/2}\,dx$ **50.** $\displaystyle\int_0^2 x^2e^{-2x}\,dx$

51. $\displaystyle\int_0^{\pi/4} x\cos 2x\,dx$ **52.** $\displaystyle\int_0^{\pi} x\sin 2x\,dx$

53. $\displaystyle\int_0^{1/2} \arccos x\,dx$ **54.** $\displaystyle\int_0^1 x\arcsin x^2\,dx$

55. $\displaystyle\int_0^1 e^x\sin x\,dx$ **56.** $\displaystyle\int_0^2 e^{-x}\cos x\,dx$

57. $\displaystyle\int_1^2 \sqrt{x}\ln x\,dx$ **58.** $\displaystyle\int_0^1 \ln(4 + x^2)\,dx$

59. $\displaystyle\int_2^4 x\,\text{arcsec}\,x\,dx$ **60.** $\displaystyle\int_0^{\pi/8} x\sec^2 2x\,dx$

In Exercises 61–66, use the tabular method to find the integral.

61. $\int x^2e^{2x}\,dx$

62. $\int x^3e^{-2x}\,dx$

63. $\int x^3\sin x\,dx$

64. $\int x^3\cos 2x\,dx$

65. $\int x\sec^2 x\,dx$

66. $\int x^2(x-2)^{3/2}\,dx$

In Exercises 67–74, find or evaluate the integral using substitution first, then using integration by parts.

67. $\int \sin\sqrt{x}\,dx$

68. $\int \cos\sqrt{x}\,dx$

69. $\int_0^4 x\sqrt{4-x}\,dx$

70. $\int 2x^3\cos x^2\,dx$

71. $\int x^5e^{x^2}\,dx$

72. $\int_0^2 e^{\sqrt{2x}}\,dx$

73. $\int \cos(\ln x)\,dx$

74. $\int \ln(x^2+1)\,dx$

WRITING ABOUT CONCEPTS

75. Integration by parts is based on what differentiation rule? Explain.

76. In your own words, state how you determine which parts of the integrand should be u and dv.

77. When evaluating $\int x\sin x\,dx$, explain how letting $u=\sin x$ and $dv=x\,dx$ makes the solution more difficult to find.

CAPSTONE

78. State whether you would use integration by parts to evaluate each integral. If so, identify what you would use for u and dv. Explain your reasoning.

(a) $\int \frac{\ln x}{x}\,dx$ (b) $\int x\ln x\,dx$ (c) $\int x^2e^{-3x}\,dx$

(d) $\int 2x\,e^{x^2}\,dx$ (e) $\int \frac{x}{\sqrt{x+1}}\,dx$ (f) $\int \frac{x}{\sqrt{x^2+1}}\,dx$

CAS **In Exercises 79–82, use a computer algebra system to (a) find or evaluate the integral and (b) graph two antiderivatives. (c) Describe the relationship between the graphs of the antiderivatives.**

79. $\int t^3e^{-4t}\,dt$

80. $\int \alpha^4\sin\pi\alpha\,d\alpha$

81. $\int_0^{\pi/2} e^{-2x}\sin 3x\,dx$

82. $\int_0^5 x^4(25-x^2)^{3/2}\,dx$

83. Integrate $\int 2x\sqrt{2x-3}\,dx$

(a) by parts, letting $dv=\sqrt{2x-3}\,dx$.

(b) by substitution, letting $u=2x-3$.

84. Integrate $\int x\sqrt{9+x}\,dx$

(a) by parts, letting $dv=\sqrt{9+x}\,dx$.

(b) by substitution, letting $u=9+x$.

85. Integrate $\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

(a) by parts, letting $dv=\left(x/\sqrt{4+x^2}\right)dx$.

(b) by substitution, letting $u=4+x^2$.

86. Integrate $\int x\sqrt{4-x}\,dx$

(a) by parts, letting $dv=\sqrt{4-x}\,dx$.

(b) by substitution, letting $u=4-x$.

CAS **In Exercises 87 and 88, use a computer algebra system to find the integrals for $n=0$, 1, 2, and 3. Use the result to obtain a general rule for the integrals for any positive integer n and test your results for $n=4$.**

87. $\int x^n\ln x\,dx$

88. $\int x^ne^x\,dx$

In Exercises 89–94, use integration by parts to prove the formula. (For Exercises 89–92, assume that n is a positive integer.)

89. $\int x^n\sin x\,dx = -x^n\cos x + n\int x^{n-1}\cos x\,dx$

90. $\int x^n\cos x\,dx = x^n\sin x - n\int x^{n-1}\sin x\,dx$

91. $\int x^n\ln x\,dx = \frac{x^{n+1}}{(n+1)^2}[-1+(n+1)\ln x]+C$

92. $\int x^n\,e^{ax}\,dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}\,e^{ax}\,dx$

93. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}+C$

94. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}+C$

In Exercises 95–98, find the integral by using the appropriate formula from Exercises 89–94.

95. $\int x^5\ln x\,dx$

96. $\int x^2\cos x\,dx$

97. $\int e^{2x}\cos 3x\,dx$

98. $\int x^3e^{2x}\,dx$

Area **In Exercises 99–102, use a graphing utility to graph the region bounded by the graphs of the equations, and find the area of the region.**

99. $y=2xe^{-x},\ y=0,\ x=3$

100. $y=\frac{1}{16}xe^{-x/4},\ y=0,\ x=0,\ x=4$

101. $y=e^{-x}\sin\pi x,\ y=0,\ x=1$

102. $y=x\sin x,\ y=0,\ x=\pi$

103. ***Area, Volume, and Centroid*** Given the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$, find

(a) the area of the region.

(b) the volume of the solid generated by revolving the region about the x-axis.

(c) the volume of the solid generated by revolving the region about the y-axis.

(d) the centroid of the region.

104. ***Volume and Centroid*** Given the region bounded by the graphs of $y = x \sin x$, $y = 0$, $x = 0$, and $x = \pi$, find

(a) the volume of the solid generated by revolving the region about the x-axis.

(b) the volume of the solid generated by revolving the region about the y-axis.

(c) the centroid of the region.

105. ***Centroid*** Find the centroid of the region bounded by the graphs of $y = \arcsin x$, $x = 0$, and $y = \pi/2$. How is this problem related to Example 6 in this section?

106. ***Centroid*** Find the centroid of the region bounded by the graphs of $f(x) = x^2$, $g(x) = 2^x$, $x = 2$, and $x = 4$.

107. ***Average Displacement*** A damping force affects the vibration of a spring so that the displacement of the spring is given by $y = e^{-4t}(\cos 2t + 5 \sin 2t)$. Find the average value of y on the interval from $t = 0$ to $t = \pi$.

108. ***Memory Model*** A model for the ability M of a child to memorize, measured on a scale from 0 to 10, is given by $M = 1 + 1.6t \ln t$, $0 < t \le 4$, where t is the child's age in years. Find the average value of this model

(a) between the child's first and second birthdays.

(b) between the child's third and fourth birthdays.

Present Value **In Exercises 109 and 110, find the present value P of a continuous income flow of $c(t)$ dollars per year if**

$$P = \int_0^{t_1} c(t)e^{-rt}\,dt$$

where t_1 is the time in years and r is the annual interest rate compounded continuously.

109. $c(t) = 100{,}000 + 4000t$, $r = 5\%$, $t_1 = 10$

110. $c(t) = 30{,}000 + 500t$, $r = 7\%$, $t_1 = 5$

Integrals Used to Find Fourier Coefficients **In Exercises 111 and 112, verify the value of the definite integral, where n is a positive integer.**

111. $$\int_{-\pi}^{\pi} x \sin nx\,dx = \begin{cases} \dfrac{2\pi}{n}, & n \text{ is odd} \\ -\dfrac{2\pi}{n}, & n \text{ is even} \end{cases}$$

112. $$\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{(-1)^n 4\pi}{n^2}$$

113. ***Vibrating String*** A string stretched between the two points $(0, 0)$ and $(2, 0)$ is plucked by displacing the string h units at its midpoint. The motion of the string is modeled by a **Fourier Sine Series** whose coefficients are given by

$$b_n = h\int_0^1 x \sin\frac{n\pi x}{2}\,dx + h\int_1^2 (-x + 2)\sin\frac{n\pi x}{2}\,dx.$$

Find b_n.

114. Find the fallacy in the following argument that $0 = 1$.

$$dv = dx \quad \Rightarrow \quad v = \int dx = x$$

$$u = \frac{1}{x} \quad \Rightarrow \quad du = -\frac{1}{x^2}\,dx$$

$$0 + \int \frac{dx}{x} = \left(\frac{1}{x}\right)(x) - \int \left(-\frac{1}{x^2}\right)(x)\,dx = 1 + \int \frac{dx}{x}$$

So, $0 = 1$.

115. Let $y = f(x)$ be positive and strictly increasing on the interval $0 < a \le x \le b$. Consider the region R bounded by the graphs of $y = f(x)$, $y = 0$, $x = a$, and $x = b$. If R is revolved about the y-axis, show that the disk method and shell method yield the same volume.

116. ***Euler's Method*** Consider the differential equation $f'(x) = xe^{-x}$ with the initial condition $f(0) = 0$.

(a) Use integration to solve the differential equation.

(b) Use a graphing utility to graph the solution of the differential equation.

(c) Use Euler's Method with $h = 0.05$, and the recursive capabilities of a graphing utility, to generate the first 80 points of the graph of the approximate solution. Use the graphing utility to plot the points. Compare the result with the graph in part (b).

(d) Repeat part (c) using $h = 0.1$ and generate the first 40 points.

(e) Why is the result in part (c) a better approximation of the solution than the result in part (d)?

Euler's Method **In Exercises 117 and 118, consider the differential equation and repeat parts (a)–(d) of Exercise 116.**

117. $f'(x) = 3x \sin(2x)$
$f(0) = 0$

118. $f'(x) = \cos\sqrt{x}$
$f(0) = 1$

119. ***Think About It*** Give a geometric explanation of why

$$\int_0^{\pi/2} x \sin x\,dx \le \int_0^{\pi/2} x\,dx.$$

Verify the inequality by evaluating the integrals.

120. ***Finding a Pattern*** Find the area bounded by the graphs of $y = x \sin x$ and $y = 0$ over each interval.

(a) $[0, \pi]$ (b) $[\pi, 2\pi]$ (c) $[2\pi, 3\pi]$

Describe any patterns that you notice. What is the area between the graphs of $y = x \sin x$ and $y = 0$ over the interval $[n\pi, (n + 1)\pi]$, where n is any nonnegative integer? Explain.

8.3 Trigonometric Integrals

- **Solve trigonometric integrals involving powers of sine and cosine.**
- **Solve trigonometric integrals involving powers of secant and tangent.**
- **Solve trigonometric integrals involving sine-cosine products with different angles.**

Integrals Involving Powers of Sine and Cosine

SHEILA SCOTT MACINTYRE (1910–1960)

Sheila Scott Macintyre published her first paper on the asymptotic periods of integral functions in 1935. She completed her doctorate work at Aberdeen University, where she taught. In 1958 she accepted a visiting research fellowship at the University of Cincinnati.

In this section you will study techniques for evaluating integrals of the form

$$\int \sin^m x \cos^n x\,dx \qquad \text{and} \qquad \int \sec^m x \tan^n x\,dx$$

where either m or n is a positive integer. To find antiderivatives for these forms, try to break them into combinations of trigonometric integrals to which you can apply the Power Rule.

For instance, you can evaluate $\int \sin^5 x \cos x\,dx$ with the Power Rule by letting $u = \sin x$. Then, $du = \cos x\,dx$ and you have

$$\int \sin^5 x \cos x\,dx = \int u^5\,du = \frac{u^6}{6} + C = \frac{\sin^6 x}{6} + C.$$

To break up $\int \sin^m x \cos^n x\,dx$ into forms to which you can apply the Power Rule, use the following identities.

$\sin^2 x + \cos^2 x = 1$ Pythagorean identity

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$ Half-angle identity for $\sin^2 x$

$\cos^2 x = \dfrac{1 + \cos 2x}{2}$ Half-angle identity for $\cos^2 x$

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING POWERS OF SINE AND COSINE

1. If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then, expand and integrate.

$$\int \sin^{\overbrace{2k+1}^{\text{Odd}}} x \cos^n x\,dx = \int \overbrace{(\sin^2 x)^k}^{\text{Convert to cosines}} \cos^n x \overbrace{\sin x\,dx}^{\text{Save for } du} = \int (1 - \cos^2 x)^k \cos^n x \sin x\,dx$$

2. If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then, expand and integrate.

$$\int \sin^m x \cos^{\overbrace{2k+1}^{\text{Odd}}} x\,dx = \int \sin^m x \overbrace{(\cos^2 x)^k}^{\text{Convert to sines}} \overbrace{\cos x\,dx}^{\text{Save for } du} = \int \sin^m x\,(1 - \sin^2 x)^k \cos x\,dx$$

3. If the powers of both the sine and cosine are even and nonnegative, make repeated use of the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to convert the integrand to odd powers of the cosine. Then proceed as in guideline 2.

TECHNOLOGY A computer algebra system used to find the integral in Example 1 yielded the following.

$$\int \sin^3 x \cos^4 x \, dx =$$

$$-\cos^5 x\left(\frac{1}{7}\sin^2 x + \frac{2}{35}\right) + C$$

Is this equivalent to the result obtained in Example 1?

EXAMPLE 1 Power of Sine Is Odd and Positive

Find $\int \sin^3 x \cos^4 x \, dx$.

Solution Because you expect to use the Power Rule with $u = \cos x$, *save one sine factor* to form du and convert the remaining sine factors to cosines.

$$\begin{aligned}
\int \sin^3 x \cos^4 x \, dx &= \int \sin^2 x \cos^4 x(\sin x)\, dx && \text{Rewrite.}\\
&= \int (1 - \cos^2 x)\cos^4 x \sin x \, dx && \text{Trigonometric identity}\\
&= \int (\cos^4 x - \cos^6 x)\sin x \, dx && \text{Multiply.}\\
&= \int \cos^4 x \sin x \, dx - \int \cos^6 x \sin x \, dx && \text{Rewrite.}\\
&= -\int \cos^4 x(-\sin x)\, dx + \int \cos^6 x(-\sin x)\, dx\\
&= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C && \text{Integrate.}
\end{aligned}$$

■

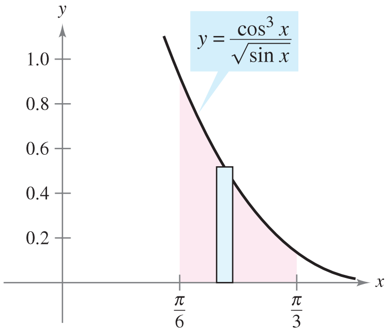
In Example 1, *both* of the powers m and n happened to be positive integers. However, the same strategy will work as long as either m or n is odd and positive. For instance, in the next example the power of the cosine is 3, but the power of the sine is $-\frac{1}{2}$.

EXAMPLE 2 Power of Cosine Is Odd and Positive

Evaluate $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx$.

Solution Because you expect to use the Power Rule with $u = \sin x$, *save one cosine factor* to form du and convert the remaining cosine factors to sines.

$$\begin{aligned}
\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx &= \int_{\pi/6}^{\pi/3} \frac{\cos^2 x \cos x}{\sqrt{\sin x}}\, dx\\
&= \int_{\pi/6}^{\pi/3} \frac{(1 - \sin^2 x)(\cos x)}{\sqrt{\sin x}}\, dx\\
&= \int_{\pi/6}^{\pi/3} \left[(\sin x)^{-1/2}\cos x - (\sin x)^{3/2}\cos x\right] dx\\
&= \left[\frac{(\sin x)^{1/2}}{1/2} - \frac{(\sin x)^{5/2}}{5/2}\right]_{\pi/6}^{\pi/3}\\
&= 2\left(\frac{\sqrt{3}}{2}\right)^{1/2} - \frac{2}{5}\left(\frac{\sqrt{3}}{2}\right)^{5/2} - \sqrt{2} + \frac{\sqrt{32}}{80}\\
&\approx 0.239
\end{aligned}$$



The area of the region is approximately 0.239.

Figure 8.4

Figure 8.4 shows the region whose area is represented by this integral. ■

EXAMPLE 3 Power of Cosine Is Even and Nonnegative

Find $\int \cos^4 x\,dx$.

Solution Because m and n are both even and nonnegative $(m = 0)$, you can replace $\cos^4 x$ by $[(1 + \cos 2x)/2]^2$.

$$\begin{aligned}
\int \cos^4 x\,dx &= \int \left(\frac{1 + \cos 2x}{2}\right)^2 dx && \text{Half-angle identity} \\
&= \int \left(\frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}\right) dx && \text{Expand.} \\
&= \int \left[\frac{1}{4} + \frac{\cos 2x}{2} + \frac{1}{4}\left(\frac{1 + \cos 4x}{2}\right)\right] dx && \text{Half-angle identity} \\
&= \frac{3}{8}\int dx + \frac{1}{4}\int 2\cos 2x\,dx + \frac{1}{32}\int 4\cos 4x\,dx && \text{Rewrite.} \\
&= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C && \text{Integrate.}
\end{aligned}$$

Use a symbolic differentiation utility to verify this. Can you simplify the derivative to obtain the original integrand? ■

In Example 3, if you were to evaluate the definite integral from 0 to $\pi/2$, you would obtain

$$\begin{aligned}
\int_0^{\pi/2} \cos^4 x\,dx &= \left[\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right]_0^{\pi/2} \\
&= \left(\frac{3\pi}{16} + 0 + 0\right) - (0 + 0 + 0) \\
&= \frac{3\pi}{16}.
\end{aligned}$$

Note that the only term that contributes to the solution is $3x/8$. This observation is generalized in the following formulas developed by John Wallis.

Bettmann/Corbis

JOHN WALLIS (1616–1703)

Wallis did much of his work in calculus prior to Newton and Leibniz, and he influenced the thinking of both of these men. Wallis is also credited with introducing the present symbol (∞) for infinity.

WALLIS'S FORMULAS

1. If n is odd $(n \geq 3)$, then

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) \cdots \left(\frac{n-1}{n}\right).$$

2. If n is even $(n \geq 2)$, then

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \cdots \left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right).$$

These formulas are also valid if $\cos^n x$ is replaced by $\sin^n x$. (You are asked to prove both formulas in Exercise 108.)

Integrals Involving Powers of Secant and Tangent

The following guidelines can help you evaluate integrals of the form

$$\int \sec^m x \tan^n x \, dx.$$

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING POWERS OF SECANT AND TANGENT

1. If the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then expand and integrate.

$$\int \sec^{\overbrace{2k}^{\text{Even}}} x \tan^n x \, dx = \int \overbrace{(\sec^2 x)^{k-1}}^{\text{Convert to tangents}} \tan^n x \overbrace{\sec^2 x \, dx}^{\text{Save for } du} = \int (1 + \tan^2 x)^{k-1} \tan^n x \sec^2 x \, dx$$

2. If the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then expand and integrate.

$$\int \sec^m x \tan^{\overbrace{2k+1}^{\text{Odd}}} x \, dx = \int \sec^{m-1} x \overbrace{(\tan^2 x)^k}^{\text{Convert to secants}} \overbrace{\sec x \tan x \, dx}^{\text{Save for } du} = \int \sec^{m-1} x (\sec^2 x - 1)^k \sec x \tan x \, dx$$

3. If there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\int \tan^n x \, dx = \int \tan^{n-2} x \overbrace{(\tan^2 x)}^{\text{Convert to secants}} dx = \int \tan^{n-2} x (\sec^2 x - 1) \, dx$$

4. If the integral is of the form $\int \sec^m x \, dx$, where m is odd and positive, use integration by parts, as illustrated in Example 5 in the preceding section.

5. If none of the first four guidelines applies, try converting to sines and cosines.

EXAMPLE 4 Power of Tangent Is Odd and Positive

Find $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}} \, dx.$

Solution Because you expect to use the Power Rule with $u = \sec x$, *save a factor of* $(\sec x \tan x)$ to form du and convert the remaining tangent factors to secants.

$$\begin{aligned} \int \frac{\tan^3 x}{\sqrt{\sec x}} \, dx &= \int (\sec x)^{-1/2} \tan^3 x \, dx \\ &= \int (\sec x)^{-3/2} (\tan^2 x)(\sec x \tan x) \, dx \\ &= \int (\sec x)^{-3/2} (\sec^2 x - 1)(\sec x \tan x) \, dx \\ &= \int \left[(\sec x)^{1/2} - (\sec x)^{-3/2}\right](\sec x \tan x) \, dx \\ &= \frac{2}{3}(\sec x)^{3/2} + 2(\sec x)^{-1/2} + C \end{aligned}$$

■

NOTE In Example 5, the power of the tangent is odd and positive. So, you could also find the integral using the procedure described in guideline 2 on page 539. In Exercise 89, you are asked to show that the results obtained by these two procedures differ only by a constant.

EXAMPLE 5 Power of Secant Is Even and Positive

Find $\int \sec^4 3x \tan^3 3x\, dx$.

Solution Let $u = \tan 3x$, then $du = 3\sec^2 3x\, dx$ and you can write

$$\begin{aligned}\int \sec^4 3x \tan^3 3x\, dx &= \int \sec^2 3x \tan^3 3x(\sec^2 3x)\, dx\\ &= \int (1 + \tan^2 3x)\tan^3 3x(\sec^2 3x)\, dx\\ &= \frac{1}{3}\int (\tan^3 3x + \tan^5 3x)(3\sec^2 3x)\, dx\\ &= \frac{1}{3}\left(\frac{\tan^4 3x}{4} + \frac{\tan^6 3x}{6}\right) + C\\ &= \frac{\tan^4 3x}{12} + \frac{\tan^6 3x}{18} + C.\end{aligned}$$

EXAMPLE 6 Power of Tangent Is Even

Evaluate $\int_0^{\pi/4} \tan^4 x\, dx$.

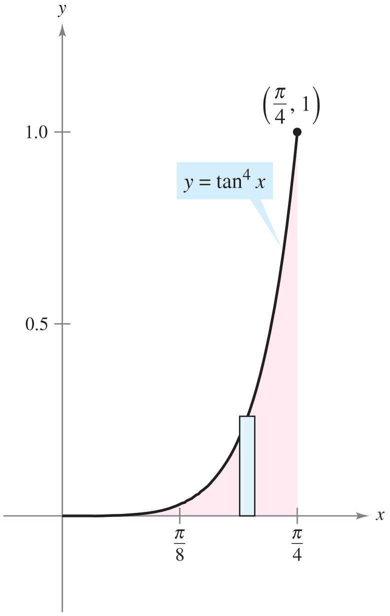
Solution Because there are no secant factors, you can begin by converting a tangent-squared factor to a secant-squared factor.

$$\begin{aligned}\int \tan^4 x\, dx &= \int \tan^2 x(\tan^2 x)\, dx\\ &= \int \tan^2 x(\sec^2 x - 1)\, dx\\ &= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx\\ &= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx\\ &= \frac{\tan^3 x}{3} - \tan x + x + C\end{aligned}$$

You can evaluate the definite integral as follows.

$$\begin{aligned}\int_0^{\pi/4} \tan^4 x\, dx &= \left[\frac{\tan^3 x}{3} - \tan x + x\right]_0^{\pi/4}\\ &= \frac{\pi}{4} - \frac{2}{3}\\ &\approx 0.119\end{aligned}$$

The area represented by the definite integral is shown in Figure 8.5. Try using Simpson's Rule to approximate this integral. With $n = 18$, you should obtain an approximation that is within 0.00001 of the actual value. ■



The area of the region is approximately 0.119.
Figure 8.5

For integrals involving powers of cotangents and cosecants, you can follow a strategy similar to that used for powers of tangents and secants. Also, when integrating trigonometric functions, remember that it sometimes helps to convert the entire integrand to powers of sines and cosines.

EXAMPLE 7 Converting to Sines and Cosines

Find $\displaystyle\int \frac{\sec x}{\tan^2 x}\,dx$.

Solution Because the first four guidelines on page 539 do not apply, try converting the integrand to sines and cosines. In this case, you are able to integrate the resulting powers of sine and cosine as follows.

$$\begin{aligned}\int \frac{\sec x}{\tan^2 x}\,dx &= \int \left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{\sin x}\right)^2 dx\\ &= \int (\sin x)^{-2}(\cos x)\,dx\\ &= -(\sin x)^{-1} + C\\ &= -\csc x + C\end{aligned}$$

■

Integrals Involving Sine-Cosine Products with Different Angles

FOR FURTHER INFORMATION To learn more about integrals involving sine-cosine products with different angles, see the article "Integrals of Products of Sine and Cosine with Different Arguments" by Sherrie J. Nicol in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

Integrals involving the products of sines and cosines of two *different* angles occur in many applications. In such instances you can use the following product-to-sum identities.

$$\sin mx \sin nx = \frac{1}{2}(\cos[(m-n)x] - \cos[(m+n)x])$$

$$\sin mx \cos nx = \frac{1}{2}(\sin[(m-n)x] + \sin[(m+n)x])$$

$$\cos mx \cos nx = \frac{1}{2}(\cos[(m-n)x] + \cos[(m+n)x])$$

EXAMPLE 8 Using Product-to-Sum Identities

Find $\displaystyle\int \sin 5x \cos 4x\,dx$.

Solution Considering the second product-to-sum identity above, you can write

$$\begin{aligned}\int \sin 5x \cos 4x\,dx &= \frac{1}{2}\int (\sin x + \sin 9x)\,dx\\ &= \frac{1}{2}\left(-\cos x - \frac{\cos 9x}{9}\right) + C\\ &= -\frac{\cos x}{2} - \frac{\cos 9x}{18} + C.\end{aligned}$$

■

8.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, use differentiation to match the antiderivative with the correct integral. [Integrals are labeled (a), (b), (c), and (d).]

(a) $\int \sin x \tan^2 x\,dx$ **(b)** $8\int \cos^4 x\,dx$

(c) $\int \sin x \sec^2 x\,dx$ **(d)** $\int \tan^4 x\,dx$

1. $y = \sec x$

2. $y = \cos x + \sec x$

3. $y = x - \tan x + \frac{1}{3}\tan^3 x$

4. $y = 3x + 2\sin x\cos^3 x + 3\sin x\cos x$

In Exercises 5–18, find the integral.

5. $\int \cos^5 x \sin x\,dx$ **6.** $\int \cos^3 x \sin^4 x\,dx$

7. $\int \sin^7 2x \cos 2x\,dx$ **8.** $\int \sin^3 x\,dx$

9. $\int \sin^3 x \cos^2 x\,dx$ **10.** $\int \cos^3 \frac{x}{3}\,dx$

11. $\int \sin^3 2\theta\sqrt{\cos 2\theta}\,d\theta$ **12.** $\int \frac{\cos^5 t}{\sqrt{\sin t}}\,dt$

13. $\int \cos^2 3x\,dx$ **14.** $\int \sin^2 5x\,dx$

15. $\int \cos^4 3\alpha\,d\alpha$ **16.** $\int \sin^4 6\theta\,d\theta$

17. $\int x\sin^2 x\,dx$ **18.** $\int x^2 \sin^2 x\,dx$

In Exercises 19–24, use Wallis's Formulas to evaluate the integral.

19. $\int_0^{\pi/2} \cos^7 x\,dx$ **20.** $\int_0^{\pi/2} \cos^9 x\,dx$

21. $\int_0^{\pi/2} \cos^{10} x\,dx$ **22.** $\int_0^{\pi/2} \sin^5 x\,dx$

23. $\int_0^{\pi/2} \sin^6 x\,dx$ **24.** $\int_0^{\pi/2} \sin^8 x\,dx$

In Exercises 25–42, find the integral involving secant and tangent.

25. $\int \sec 7x\,dx$ **26.** $\int \sec^2(2x - 1)\,dx$

27. $\int \sec^4 5x\,dx$ **28.** $\int \sec^6 3x\,dx$

29. $\int \sec^3 \pi x\,dx$ **30.** $\int \tan^5 x\,dx$

31. $\int \tan^5 \frac{x}{2}\,dx$ **32.** $\int \tan^3 \frac{\pi x}{2}\sec^2 \frac{\pi x}{2}\,dx$

33. $\int \sec^2 x \tan x\,dx$ **34.** $\int \tan^3 2t \sec^3 2t\,dt$

35. $\int \tan^2 x \sec^4 x\,dx$ **36.** $\int \tan^5 2x \sec^4 2x\,dx$

37. $\int \sec^6 4x \tan 4x\,dx$ **38.** $\int \sec^2 \frac{x}{2}\tan \frac{x}{2}\,dx$

39. $\int \sec^5 x \tan^3 x\,dx$ **40.** $\int \tan^3 3x\,dx$

41. $\int \frac{\tan^2 x}{\sec x}\,dx$ **42.** $\int \frac{\tan^2 x}{\sec^5 x}\,dx$

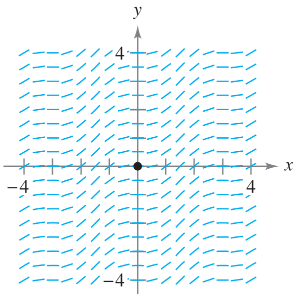
In Exercises 43–46, solve the differential equation.

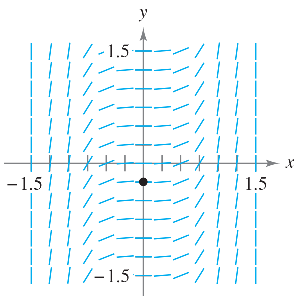
43. $\frac{dr}{d\theta} = \sin^4 \pi\theta$ **44.** $\frac{ds}{d\alpha} = \sin^2 \frac{\alpha}{2}\cos^2 \frac{\alpha}{2}$

45. $y' = \tan^3 3x \sec 3x$ **46.** $y' = \sqrt{\tan x}\sec^4 x$

Slope Fields **In Exercises 47 and 48, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

47. $\frac{dy}{dx} = \sin^2 x,\ (0, 0)$ **48.** $\frac{dy}{dx} = \sec^2 x \tan^2 x,\ \left(0, -\frac{1}{4}\right)$





CAS ***Slope Fields*** **In Exercises 49 and 50, use a computer algebra system to graph the slope field for the differential equation, and graph the solution through the specified initial condition.**

49. $\frac{dy}{dx} = \frac{3\sin x}{y},\ y(0) = 2$ **50.** $\frac{dy}{dx} = 3\sqrt{y}\tan^2 x,\ y(0) = 3$

In Exercises 51–56, find the integral.

51. $\int \cos 2x \cos 6x\,dx$ **52.** $\int \cos 4\theta \cos(-3\theta)\,d\theta$

53. $\int \sin 2x \cos 4x\,dx$ **54.** $\int \sin(-4x)\cos 3x\,dx$

55. $\int \sin\theta \sin 3\theta\,d\theta$ **56.** $\int \sin 5x \sin 4x\,dx$

In Exercises 57–66, find the integral. Use a computer algebra system to confirm your result.

57. $\displaystyle\int \cot^3 2x\,dx$

58. $\displaystyle\int \tan^4 \frac{x}{2} \sec^4 \frac{x}{2}\,dx$

59. $\displaystyle\int \csc^4 2x\,dx$

60. $\displaystyle\int \cot^3 x \csc^3 x\,dx$

61. $\displaystyle\int \frac{\cot^2 t}{\csc t}\,dt$

62. $\displaystyle\int \frac{\cot^3 t}{\csc t}\,dt$

63. $\displaystyle\int \frac{1}{\sec x \tan x}\,dx$

64. $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx$

65. $\displaystyle\int (\tan^4 t - \sec^4 t)\,dt$

66. $\displaystyle\int \frac{1 - \sec t}{\cos t - 1}\,dt$

In Exercises 67–74, evaluate the definite integral.

67. $\displaystyle\int_{-\pi}^{\pi} \sin^2 x\,dx$

68. $\displaystyle\int_{0}^{\pi/3} \tan^2 x\,dx$

69. $\displaystyle\int_{0}^{\pi/4} 6 \tan^3 x\,dx$

70. $\displaystyle\int_{0}^{\pi/4} \sec^2 t\sqrt{\tan t}\,dt$

71. $\displaystyle\int_{0}^{\pi/2} \frac{\cos t}{1 + \sin t}\,dt$

72. $\displaystyle\int_{-\pi}^{\pi} \sin 5x \cos 3x\,dx$

73. $\displaystyle\int_{-\pi/2}^{\pi/2} 3 \cos^3 x\,dx$

74. $\displaystyle\int_{-\pi/2}^{\pi/2} (\sin^2 x + 1)\,dx$

CAS **In Exercises 75–80, use a computer algebra system to find the integral. Graph the antiderivatives for two different values of the constant of integration.**

75. $\displaystyle\int \cos^4 \frac{x}{2}\,dx$

76. $\displaystyle\int \sin^2 x \cos^2 x\,dx$

77. $\displaystyle\int \sec^5 \pi x\,dx$

78. $\displaystyle\int \tan^3(1 - x)\,dx$

79. $\displaystyle\int \sec^5 \pi x \tan \pi x\,dx$

80. $\displaystyle\int \sec^4(1 - x) \tan(1 - x)\,dx$

CAS **In Exercises 81–84, use a computer algebra system to evaluate the definite integral.**

81. $\displaystyle\int_{0}^{\pi/4} \sin 3\theta \sin 4\theta\,d\theta$

82. $\displaystyle\int_{0}^{\pi/2} (1 - \cos\theta)^2\,d\theta$

83. $\displaystyle\int_{0}^{\pi/2} \sin^4 x\,dx$

84. $\displaystyle\int_{0}^{\pi/2} \sin^{12} x\,dx$

WRITING ABOUT CONCEPTS

85. In your own words, describe how you would integrate $\int \sin^m x \cos^n x\,dx$ for each condition.

(a) m is positive and odd. (b) n is positive and odd.

(c) m and n are both positive and even.

86. In your own words, describe how you would integrate $\int \sec^m x \tan^n x\,dx$ for each condition.

(a) m is positive and even. (b) n is positive and odd.

(c) n is positive and even, and there are no secant factors.

(d) m is positive and odd, and there are no tangent factors.

87. Evaluate $\int \sin x \cos x\,dx$ using the given method. Explain how your answers differ for each method.

(a) Substitution where $u = \sin x$

(b) Substitution where $u = \cos x$

(c) Integration by parts

(d) Using the identity $\sin 2x = 2 \sin x \cos x$

CAPSTONE

88. For each pair of integrals, determine which one is more difficult to evaluate. Explain your reasoning.

(a) $\int \sin^{372} x \cos x\,dx$, $\int \sin^4 x \cos^4 x\,dx$

(b) $\int \tan^{400} x \sec^2 x\,dx$, $\int \tan^{400} x \sec x\,dx$

In Exercises 89 and 90, (a) find the indefinite integral in two different ways. (b) Use a graphing utility to graph the antiderivative (without the constant of integration) obtained by each method to show that the results differ only by a constant. (c) Verify analytically that the results differ only by a constant.

89. $\int \sec^4 3x \tan^3 3x\,dx$

90. $\int \sec^2 x \tan x\,dx$

Area **In Exercises 91–94, find the area of the region bounded by the graphs of the equations.**

91. $y = \sin x, \quad y = \sin^3 x, \quad x = 0, \quad x = \pi/2$

92. $y = \sin^2 \pi x, \quad y = 0, \quad x = 0, \quad x = 1$

93. $y = \cos^2 x, \quad y = \sin^2 x, \quad x = -\pi/4, \quad x = \pi/4$

94. $y = \cos^2 x, \quad y = \sin x \cos x, \quad x = -\pi/2, \quad x = \pi/4$

Volume **In Exercises 95 and 96, find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *x*-axis.**

95. $y = \tan x, \quad y = 0, \quad x = -\pi/4, \quad x = \pi/4$

96. $y = \cos \frac{x}{2}, \quad y = \sin \frac{x}{2}, \quad x = 0, \quad x = \pi/2$

Volume and Centroid **In Exercises 97 and 98, for the region bounded by the graphs of the equations, find (a) the volume of the solid formed by revolving the region about the *x*-axis and (b) the centroid of the region.**

97. $y = \sin x, y = 0, x = 0, x = \pi$

98. $y = \cos x, y = 0, x = 0, x = \pi/2$

In Exercises 99–102, use integration by parts to verify the reduction formula.

99. $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

100. $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

101. $\displaystyle\int \cos^m x \sin^n x\,dx = -\frac{\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2} x\,dx$

102. $\displaystyle\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$

In Exercises 103–106, use the results of Exercises 99–102 to find the integral.

103. $\displaystyle\int \sin^5 x\,dx$

104. $\displaystyle\int \cos^4 x\,dx$

105. $\displaystyle\int \sec^4 (2\pi x/5)\,dx$

106. $\displaystyle\int \sin^4 x\cos^2 x\,dx$

107. ***Modeling Data*** The table shows the normal maximum (high) and minimum (low) temperatures (in degrees Fahrenheit) in Erie, Pennsylvania for each month of the year. *(Source: NOAA)*

Month	Jan	Feb	Mar	Apr	May	Jun
Max	33.5	35.4	44.7	55.6	67.4	76.2
Min	20.3	20.9	28.2	37.9	48.7	58.5

Month	Jul	Aug	Sep	Oct	Nov	Dec
Max	80.4	79.0	72.0	61.0	49.3	38.6
Min	63.7	62.7	55.9	45.5	36.4	26.8

The maximum and minimum temperatures can be modeled by $f(t) = a_0 + a_1\cos(\pi t/6) + b_1\sin(\pi t/6)$, where $t = 0$ corresponds to January 1 and a_0, a_1, and b_1 are as follows.

$$a_0 = \frac{1}{12}\int_0^{12} f(t)\,dt \qquad a_1 = \frac{1}{6}\int_0^{12} f(t)\cos\frac{\pi t}{6}\,dt$$

$$b_1 = \frac{1}{6}\int_0^{12} f(t)\sin\frac{\pi t}{6}\,dt$$

(a) Approximate the model $H(t)$ for the maximum temperatures. (*Hint:* Use Simpson's Rule to approximate the integrals and use the January data twice.)

(b) Repeat part (a) for a model $L(t)$ for the minimum temperature data.

(c) Use a graphing utility to graph each model. During what part of the year is the difference between the maximum and minimum temperatures greatest?

108. ***Wallis's Formulas*** Use the result of Exercise 100 to prove the following versions of Wallis's Formulas.

(a) If n is odd ($n \geq 3$), then

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right).$$

(b) If n is even ($n \geq 2$), then

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right).$$

109. The **inner product** of two functions f and g on $[a, b]$ is given by $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$. Two distinct functions f and g are said to be **orthogonal** if $\langle f, g\rangle = 0$. Show that the following set of functions is orthogonal on $[-\pi, \pi]$.

$\{\sin x, \sin 2x, \sin 3x, \ldots, \cos x, \cos 2x, \cos 3x, \ldots\}$

110. ***Fourier Series*** The following sum is a *finite Fourier series.*

$$f(x) = \sum_{i=1}^{N} a_i \sin ix$$
$$= a_1\sin x + a_2\sin 2x + a_3\sin 3x + \cdots + a_N \sin Nx$$

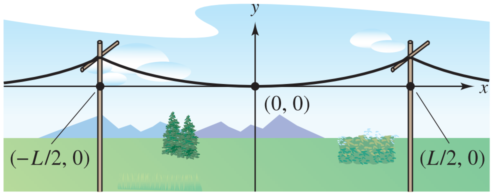
(a) Use Exercise 109 to show that the nth coefficient a_n is given by $a_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\sin nx\,dx$.

(b) Let $f(x) = x$. Find a_1, a_2, and a_3.

SECTION PROJECT

Power Lines

Power lines are constructed by stringing wire between supports and adjusting the tension on each span. The wire hangs between supports in the shape of a catenary, as shown in the figure.



Let T be the tension (in pounds) on a span of wire, let u be the density (in pounds per foot), let $g \approx 32.2$ be the acceleration due to gravity (in feet per second per second), and let L be the distance (in feet) between the supports. Then the equation of the catenary is $y = \dfrac{T}{ug}\left(\cosh\dfrac{ugx}{T} - 1\right)$, where x and y are measured in feet.

(a) Find the length of the wire between two spans.

(b) To measure the tension in a span, power line workers use the *return wave method.* The wire is struck at one support, creating a wave in the line, and the time t (in seconds) it takes for the wave to make a round trip is measured. The velocity v (in feet per second) is given by $v = \sqrt{T/u}$. How long does it take the wave to make a round trip between supports?

(c) The sag s (in inches) can be obtained by evaluating y when $x = L/2$ in the equation for the catenary (and multiplying by 12). In practice, however, power line workers use the "lineman's equation" given by $s \approx 12.075t^2$. Use the fact that $[\cosh(ugL/2T) + 1] \approx 2$ to derive this equation.

FOR FURTHER INFORMATION To learn more about the mathematics of power lines, see the article "Constructing Power Lines" by Thomas O'Neil in *The UMAP Journal.*

8.4 Trigonometric Substitution

- Use trigonometric substitution to solve an integral.
- Use integrals to model and solve real-life applications.

EXPLORATION

Integrating a Radical Function
Up to this point in the text, you have not evaluated the following integral.

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx$$

From geometry, you should be able to find the exact value of this integral—what is it? Using numerical integration, with Simpson's Rule or the Trapezoidal Rule, you can't be sure of the accuracy of the approximation. Why?

Try finding the exact value using the substitution

$$x = \sin\theta \text{ and } dx = \cos\theta\,d\theta.$$

Does your answer agree with the value you obtained using geometry?

Trigonometric Substitution

Now that you can evaluate integrals involving powers of trigonometric functions, you can use **trigonometric substitution** to evaluate integrals involving the radicals

$$\sqrt{a^2-u^2},\qquad \sqrt{a^2+u^2},\qquad \text{and}\qquad \sqrt{u^2-a^2}.$$

The objective with trigonometric substitution is to eliminate the radical in the integrand. You do this by using the Pythagorean identities

$$\cos^2\theta = 1-\sin^2\theta,\quad \sec^2\theta = 1+\tan^2\theta,\quad \text{and}\quad \tan^2\theta = \sec^2\theta - 1.$$

For example, if $a > 0$, let $u = a\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then

$$\begin{aligned}\sqrt{a^2-u^2} &= \sqrt{a^2-a^2\sin^2\theta}\\ &= \sqrt{a^2(1-\sin^2\theta)}\\ &= \sqrt{a^2\cos^2\theta}\\ &= a\cos\theta.\end{aligned}$$

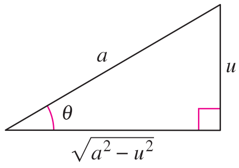
Note that $\cos\theta \ge 0$, because $-\pi/2 \le \theta \le \pi/2$.

TRIGONOMETRIC SUBSTITUTION ($a > 0$)

1. For integrals involving $\sqrt{a^2-u^2}$, let

$$u = a\sin\theta.$$

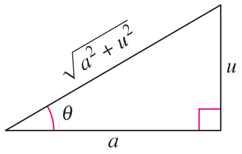
Then $\sqrt{a^2-u^2} = a\cos\theta$, where $-\pi/2 \le \theta \le \pi/2$.



2. For integrals involving $\sqrt{a^2+u^2}$, let

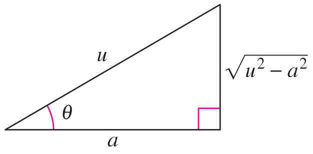
$$u = a\tan\theta.$$

Then $\sqrt{a^2+u^2} = a\sec\theta$, where $-\pi/2 < \theta < \pi/2$.



3. For integrals involving $\sqrt{u^2-a^2}$, let

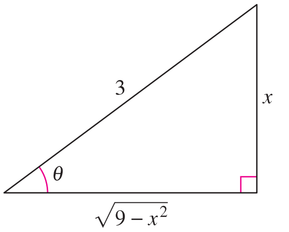
$$u = a\sec\theta.$$



Then

$$\sqrt{u^2-a^2} = \begin{cases} a\tan\theta \text{ if } u > a, \text{ where } 0 \le \theta < \pi/2\\ -a\tan\theta \text{ if } u < -a, \text{ where } \pi/2 < \theta \le \pi\end{cases}$$

NOTE The restrictions on θ ensure that the function that defines the substitution is one-to-one. In fact, these are the same intervals over which the arcsine, arctangent, and arcsecant are defined. ■



$\sin\theta = \dfrac{x}{3}, \cot\theta = \dfrac{\sqrt{9-x^2}}{x}$

Figure 8.6

EXAMPLE 1 Trigonometric Substitution: $u = a\sin\theta$

Find $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$.

Solution First, note that none of the basic integration rules applies. To use trigonometric substitution, you should observe that $\sqrt{9-x^2}$ is of the form $\sqrt{a^2-u^2}$. So, you can use the substitution

$$x = a\sin\theta = 3\sin\theta.$$

Using differentiation and the triangle shown in Figure 8.6, you obtain

$$dx = 3\cos\theta\,d\theta, \qquad \sqrt{9-x^2} = 3\cos\theta, \qquad \text{and} \qquad x^2 = 9\sin^2\theta.$$

So, trigonometric substitution yields

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{9-x^2}} &= \int \frac{3\cos\theta\,d\theta}{(9\sin^2\theta)(3\cos\theta)} && \text{Substitute.}\\
&= \frac{1}{9}\int \frac{d\theta}{\sin^2\theta} && \text{Simplify.}\\
&= \frac{1}{9}\int \csc^2\theta\,d\theta && \text{Trigonometric identity}\\
&= -\frac{1}{9}\cot\theta + C && \text{Apply Cosecant Rule.}\\
&= -\frac{1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right) + C && \text{Substitute for } \cot\theta.\\
&= -\frac{\sqrt{9-x^2}}{9x} + C.
\end{aligned}$$

Note that the triangle in Figure 8.6 can be used to convert the θ's back to x's, as follows.

$$\begin{aligned}
\cot\theta &= \frac{\text{adj.}}{\text{opp.}}\\
&= \frac{\sqrt{9-x^2}}{x}
\end{aligned}$$

■

TECHNOLOGY Use a computer algebra system to find each indefinite integral.

$$\int \frac{dx}{\sqrt{9-x^2}} \qquad \int \frac{dx}{x\sqrt{9-x^2}} \qquad \int \frac{dx}{x^2\sqrt{9-x^2}} \qquad \int \frac{dx}{x^3\sqrt{9-x^2}}$$

Then use trigonometric substitution to duplicate the results obtained with the computer algebra system.

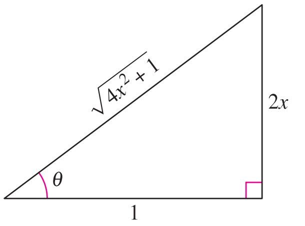
In an earlier chapter, you saw how the inverse hyperbolic functions can be used to evaluate the integrals

$$\int \frac{du}{\sqrt{u^2 \pm a^2}}, \qquad \int \frac{du}{a^2-u^2}, \qquad \text{and} \qquad \int \frac{du}{u\sqrt{a^2 \pm u^2}}.$$

You can also evaluate these integrals using trigonometric substitution. This is shown in the next example.

EXAMPLE 2 Trigonometric Substitution: $u = a \tan \theta$

Find $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$.



$\tan \theta = 2x$, $\sec \theta = \sqrt{4x^2+1}$
Figure 8.7

Solution Let $u = 2x$, $a = 1$, and $2x = \tan \theta$, as shown in Figure 8.7. Then,

$$dx = \frac{1}{2}\sec^2 \theta \, d\theta \qquad \text{and} \qquad \sqrt{4x^2+1} = \sec \theta.$$

Trigonometric substitution produces

$$\begin{aligned}
\int \frac{1}{\sqrt{4x^2+1}}\,dx &= \frac{1}{2}\int \frac{\sec^2 \theta \, d\theta}{\sec \theta} && \text{Substitute.}\\
&= \frac{1}{2}\int \sec \theta \, d\theta && \text{Simplify.}\\
&= \frac{1}{2}\ln|\sec \theta + \tan \theta| + C && \text{Apply Secant Rule.}\\
&= \frac{1}{2}\ln\left|\sqrt{4x^2+1} + 2x\right| + C. && \text{Back-substitute.}
\end{aligned}$$

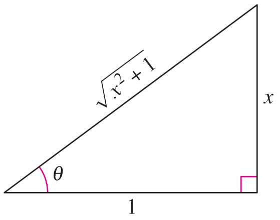
Try checking this result with a computer algebra system. Is the result given in this form or in the form of an inverse hyperbolic function? ■

You can extend the use of trigonometric substitution to cover integrals involving expressions such as $(a^2 - u^2)^{n/2}$ by writing the expression as

$$(a^2 - u^2)^{n/2} = \left(\sqrt{a^2 - u^2}\right)^n.$$

EXAMPLE 3 Trigonometric Substitution: Rational Powers

Find $\displaystyle\int \frac{dx}{(x^2+1)^{3/2}}$.



$\tan \theta = x$, $\sin \theta = \dfrac{x}{\sqrt{x^2+1}}$
Figure 8.8

Solution Begin by writing $(x^2+1)^{3/2}$ as $\left(\sqrt{x^2+1}\right)^3$. Then, let $a = 1$ and $u = x = \tan \theta$, as shown in Figure 8.8. Using

$$dx = \sec^2 \theta \, d\theta \qquad \text{and} \qquad \sqrt{x^2+1} = \sec \theta$$

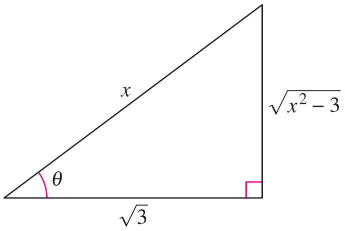
you can apply trigonometric substitution, as follows.

$$\begin{aligned}
\int \frac{dx}{(x^2+1)^{3/2}} &= \int \frac{dx}{\left(\sqrt{x^2+1}\right)^3} && \text{Rewrite denominator.}\\
&= \int \frac{\sec^2 \theta \, d\theta}{\sec^3 \theta} && \text{Substitute.}\\
&= \int \frac{d\theta}{\sec \theta} && \text{Simplify.}\\
&= \int \cos \theta \, d\theta && \text{Trigonometric identity}\\
&= \sin \theta + C && \text{Apply Cosine Rule.}\\
&= \frac{x}{\sqrt{x^2+1}} + C && \text{Back-substitute.}
\end{aligned}$$
■

For definite integrals, it is often convenient to determine the integration limits for θ that avoid converting back to x. You might want to review this procedure in Section 4.5, Examples 8 and 9.

EXAMPLE 4 Converting the Limits of Integration

Evaluate $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$.



$\sec\theta = \dfrac{x}{\sqrt{3}},\ \tan\theta = \dfrac{\sqrt{x^2-3}}{\sqrt{3}}$

Figure 8.9

Solution Because $\sqrt{x^2-3}$ has the form $\sqrt{u^2-a^2}$, you can consider

$$u = x, \quad a = \sqrt{3}, \quad \text{and} \quad x = \sqrt{3}\sec\theta$$

as shown in Figure 8.9. Then,

$$dx = \sqrt{3}\sec\theta\tan\theta\,d\theta \qquad \text{and} \qquad \sqrt{x^2-3} = \sqrt{3}\tan\theta.$$

To determine the upper and lower limits of integration, use the substitution $x = \sqrt{3}\sec\theta$, as follows.

Lower Limit

When $x = \sqrt{3}$, $\sec\theta = 1$ and $\theta = 0$.

Upper Limit

When $x = 2$, $\sec\theta = \dfrac{2}{\sqrt{3}}$ and $\theta = \dfrac{\pi}{6}$.

So, you have

Integration limits for x

Integration limits for θ

$$\begin{aligned}
\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx &= \int_{0}^{\pi/6} \frac{(\sqrt{3}\tan\theta)(\sqrt{3}\sec\theta\tan\theta)\,d\theta}{\sqrt{3}\sec\theta}\\
&= \int_{0}^{\pi/6} \sqrt{3}\tan^2\theta\,d\theta\\
&= \sqrt{3}\int_{0}^{\pi/6} (\sec^2\theta - 1)\,d\theta\\
&= \sqrt{3}\Big[\tan\theta - \theta\Big]_{0}^{\pi/6}\\
&= \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right)\\
&= 1 - \frac{\sqrt{3}\pi}{6}\\
&\approx 0.0931.
\end{aligned}$$

■

In Example 4, try converting back to the variable x and evaluating the antiderivative at the original limits of integration. You should obtain

$$\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx = \sqrt{3}\left[\frac{\sqrt{x^2-3}}{\sqrt{3}} - \operatorname{arcsec}\frac{x}{\sqrt{3}}\right]_{\sqrt{3}}^{2}.$$

When using trigonometric substitution to evaluate definite integrals, you must be careful to check that the values of θ lie in the intervals discussed at the beginning of this section. For instance, if in Example 4 you had been asked to evaluate the definite integral

$$\int_{-2}^{-\sqrt{3}} \frac{\sqrt{x^2-3}}{x}\,dx$$

then using $u = x$ and $a = \sqrt{3}$ in the interval $\left[-2, -\sqrt{3}\right]$ would imply that $u < -a$. So, when determining the upper and lower limits of integration, you would have to choose θ such that $\pi/2 < \theta \leq \pi$. In this case the integral would be evaluated as follows.

$$\begin{aligned}\int_{-2}^{-\sqrt{3}} \frac{\sqrt{x^2-3}}{x}\,dx &= \int_{5\pi/6}^{\pi} \frac{\left(-\sqrt{3}\tan\theta\right)\left(\sqrt{3}\sec\theta\tan\theta\right)d\theta}{\sqrt{3}\sec\theta}\\ &= \int_{5\pi/6}^{\pi} -\sqrt{3}\tan^2\theta\,d\theta\\ &= -\sqrt{3}\int_{5\pi/6}^{\pi}(\sec^2\theta-1)\,d\theta\\ &= -\sqrt{3}\Big[\tan\theta-\theta\Big]_{5\pi/6}^{\pi}\\ &= -\sqrt{3}\left[(0-\pi)-\left(-\frac{1}{\sqrt{3}}-\frac{5\pi}{6}\right)\right]\\ &= -1+\frac{\sqrt{3}\pi}{6}\\ &\approx -0.0931\end{aligned}$$

Trigonometric substitution can be used with completing the square. For instance, try evaluating the following integral.

$$\int \sqrt{x^2-2x}\,dx$$

To begin, you could complete the square and write the integral as

$$\int \sqrt{(x-1)^2-1^2}\,dx.$$

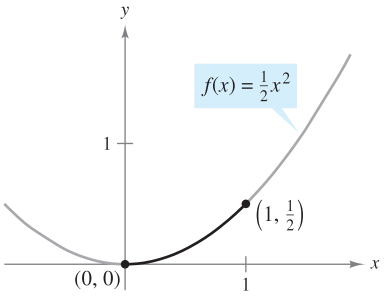
Trigonometric substitution can be used to evaluate the three integrals listed in the following theorem. These integrals will be encountered several times in the remainder of the text. When this happens, we will simply refer to this theorem. (In Exercise 85, you are asked to verify the formulas given in the theorem.)

THEOREM 8.2 SPECIAL INTEGRATION FORMULAS ($a > 0$)

1. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{1}{2}\left(a^2\arcsin\frac{u}{a} + u\sqrt{a^2-u^2}\right) + C$
2. $\displaystyle\int \sqrt{u^2-a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2-a^2} - a^2\ln\left|u+\sqrt{u^2-a^2}\right|\right) + C, \quad u > a$
3. $\displaystyle\int \sqrt{u^2+a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2+a^2} + a^2\ln\left|u+\sqrt{u^2+a^2}\right|\right) + C$

Applications

EXAMPLE 5 Finding Arc Length



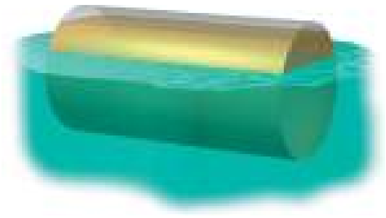
The arc length of the curve from (0, 0) to $\left(1, \frac{1}{2}\right)$
Figure 8.10

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$ (see Figure 8.10).

Solution Refer to the arc length formula in Section 7.4.

$$
\begin{aligned}
s &= \int_0^1 \sqrt{1 + [f'(x)]^2}\, dx && \text{Formula for arc length} \\
&= \int_0^1 \sqrt{1 + x^2}\, dx && f'(x) = x \\
&= \int_0^{\pi/4} \sec^3 \theta \, d\theta && \text{Let } a = 1 \text{ and } x = \tan \theta. \\
&= \frac{1}{2}\Big[\sec \theta \tan \theta + \ln|\sec \theta + \tan \theta|\Big]_0^{\pi/4} && \text{Example 5, Section 8.2} \\
&= \frac{1}{2}\left[\sqrt{2} + \ln\left(\sqrt{2} + 1\right)\right] \approx 1.148
\end{aligned}
$$

EXAMPLE 6 Comparing Two Fluid Forces



The barrel is not quite full of oil—the top 0.2 foot of the barrel is empty.
Figure 8.11

A sealed barrel of oil (weighing 48 pounds per cubic foot) is floating in seawater (weighing 64 pounds per cubic foot), as shown in Figures 8.11 and 8.12. (The barrel is not completely full of oil. With the barrel lying on its side, the top 0.2 foot of the barrel is empty.) Compare the fluid forces against one end of the barrel from the inside and from the outside.

Solution In Figure 8.12, locate the coordinate system with the origin at the center of the circle given by $x^2 + y^2 = 1$. To find the fluid force against an end of the barrel *from the inside*, integrate between -1 and 0.8 (using a weight of $w = 48$).

$$
\begin{aligned}
F &= w \int_c^d h(y)L(y)\, dy && \text{General equation (see Section 7.7)} \\
F_{\text{inside}} &= 48 \int_{-1}^{0.8} (0.8 - y)(2)\sqrt{1 - y^2}\, dy \\
&= 76.8 \int_{-1}^{0.8} \sqrt{1 - y^2}\, dy - 96 \int_{-1}^{0.8} y\sqrt{1 - y^2}\, dy
\end{aligned}
$$

To find the fluid force *from the outside*, integrate between -1 and 0.4 (using a weight of $w = 64$).

$$
\begin{aligned}
F_{\text{outside}} &= 64 \int_{-1}^{0.4} (0.4 - y)(2)\sqrt{1 - y^2}\, dy \\
&= 51.2 \int_{-1}^{0.4} \sqrt{1 - y^2}\, dy - 128 \int_{-1}^{0.4} y\sqrt{1 - y^2}\, dy
\end{aligned}
$$

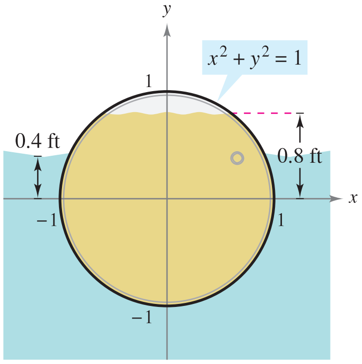


Figure 8.12

The details of integration are left for you to complete in Exercise 84. Intuitively, would you say that the force from the oil (the inside) or the force from the seawater (the outside) is greater? By evaluating these two integrals, you can determine that

$$F_{\text{inside}} \approx 121.3 \text{ pounds} \quad \text{and} \quad F_{\text{outside}} \approx 93.0 \text{ pounds.}$$ ■

8.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, state the trigonometric substitution you would use to find the integral. Do not integrate.

1. $\int (9 + x^2)^{-2}\,dx$

2. $\int \sqrt{4 - x^2}\,dx$

3. $\int \frac{x^2}{\sqrt{16 - x^2}}\,dx$

4. $\int x^2(x^2 - 25)^{3/2}\,dx$

In Exercises 5–8, find the indefinite integral using the substitution $x = 4 \sin \theta$.

5. $\int \frac{1}{(16 - x^2)^{3/2}}\,dx$

6. $\int \frac{4}{x^2\sqrt{16 - x^2}}\,dx$

7. $\int \frac{\sqrt{16 - x^2}}{x}\,dx$

8. $\int \frac{x^2}{\sqrt{16 - x^2}}\,dx$

In Exercises 9–12, find the indefinite integral using the substitution $x = 5 \sec \theta$.

9. $\int \frac{1}{\sqrt{x^2 - 25}}\,dx$

10. $\int \frac{\sqrt{x^2 - 25}}{x}\,dx$

11. $\int x^3\sqrt{x^2 - 25}\,dx$

12. $\int \frac{x^3}{\sqrt{x^2 - 25}}\,dx$

In Exercises 13–16, find the indefinite integral using the substitution $x = \tan \theta$.

13. $\int x\sqrt{1 + x^2}\,dx$

14. $\int \frac{9x^3}{\sqrt{1 + x^2}}\,dx$

15. $\int \frac{1}{(1 + x^2)^2}\,dx$

16. $\int \frac{x^2}{(1 + x^2)^2}\,dx$

In Exercises 17–20, use the Special Integration Formulas (Theorem 8.2) to find the integral.

17. $\int \sqrt{9 + 16x^2}\,dx$

18. $\int \sqrt{4 + x^2}\,dx$

19. $\int \sqrt{25 - 4x^2}\,dx$

20. $\int \sqrt{5x^2 - 1}\,dx$

In Exercises 21–42, find the integral.

21. $\int \frac{x}{\sqrt{x^2 + 36}}\,dx$

22. $\int \frac{x}{\sqrt{36 - x^2}}\,dx$

23. $\int \frac{1}{\sqrt{16 - x^2}}\,dx$

24. $\int \frac{1}{\sqrt{49 - x^2}}\,dx$

25. $\int \sqrt{16 - 4x^2}\,dx$

26. $\int x\sqrt{16 - 4x^2}\,dx$

27. $\int \frac{1}{\sqrt{x^2 - 4}}\,dx$

28. $\int \frac{t}{(4 - t^2)^{3/2}}\,dt$

29. $\int \frac{\sqrt{1 - x^2}}{x^4}\,dx$

30. $\int \frac{\sqrt{4x^2 + 9}}{x^4}\,dx$

31. $\int \frac{1}{x\sqrt{4x^2 + 9}}\,dx$

32. $\int \frac{1}{x\sqrt{4x^2 + 16}}\,dx$

33. $\int \frac{-3x}{(x^2 + 3)^{3/2}}\,dx$

34. $\int \frac{1}{(x^2 + 5)^{3/2}}\,dx$

35. $\int e^{2x}\sqrt{1 + e^{2x}}\,dx$

36. $\int (x + 1)\sqrt{x^2 + 2x + 2}\,dx$

37. $\int e^x\sqrt{1 - e^{2x}}\,dx$

38. $\int \frac{\sqrt{1 - x}}{\sqrt{x}}\,dx$

39. $\int \frac{1}{4 + 4x^2 + x^4}\,dx$

40. $\int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1}\,dx$

41. $\int \operatorname{arcsec} 2x\,dx, \quad x > \frac{1}{2}$

42. $\int x \arcsin x\,dx$

In Exercises 43–46, complete the square and find the integral.

43. $\int \frac{1}{\sqrt{4x - x^2}}\,dx$

44. $\int \frac{x^2}{\sqrt{2x - x^2}}\,dx$

45. $\int \frac{x}{\sqrt{x^2 + 6x + 12}}\,dx$

46. $\int \frac{x}{\sqrt{x^2 - 6x + 5}}\,dx$

In Exercises 47–52, evaluate the integral using (a) the given integration limits and (b) the limits obtained by trigonometric substitution.

47. $\int_0^{\sqrt{3}/2} \frac{t^2}{(1 - t^2)^{3/2}}\,dt$

48. $\int_0^{\sqrt{3}/2} \frac{1}{(1 - t^2)^{5/2}}\,dt$

49. $\int_0^3 \frac{x^3}{\sqrt{x^2 + 9}}\,dx$

50. $\int_0^{3/5} \sqrt{9 - 25x^2}\,dx$

51. $\int_4^6 \frac{x^2}{\sqrt{x^2 - 9}}\,dx$

52. $\int_3^6 \frac{\sqrt{x^2 - 9}}{x^2}\,dx$

In Exercises 53 and 54, find the particular solution of the differential equation.

53. $x\frac{dy}{dx} = \sqrt{x^2 - 9}, \quad x \geq 3, \quad y(3) = 1$

54. $\sqrt{x^2 + 4}\,\frac{dy}{dx} = 1, \quad x \geq -2, \quad y(0) = 4$

CAS **In Exercises 55–58, use a computer algebra system to find the integral. Verify the result by differentiation.**

55. $\int \frac{x^2}{\sqrt{x^2 + 10x + 9}}\,dx$

56. $\int (x^2 + 2x + 11)^{3/2}\,dx$

57. $\int \frac{x^2}{\sqrt{x^2 - 1}}\,dx$

58. $\int x^2\sqrt{x^2 - 4}\,dx$

WRITING ABOUT CONCEPTS

59. State the substitution you would make if you used trigonometric substitution and the integral involving the given radical, where $a > 0$. Explain your reasoning.

(a) $\sqrt{a^2 - u^2}$ (b) $\sqrt{a^2 + u^2}$ (c) $\sqrt{u^2 - a^2}$

WRITING ABOUT CONCEPTS (continued)

In Exercises 60 and 61, state the method of integration you would use to perform each integration. Explain why you chose that method. Do not integrate.

60. $\int x\sqrt{x^2+1}\,dx$ **61.** $\int x^2\sqrt{x^2-1}\,dx$

CAPSTONE

62. (a) Evaluate the integral $\int \frac{x}{x^2+9}\,dx$ using u-substitution. Then evaluate using trigonometric substitution. Discuss the results.

(b) Evaluate the integral $\int \frac{x^2}{x^2+9}\,dx$ algebraically using $x^2 = (x^2+9) - 9$. Then evaluate using trigonometric substitution. Discuss the results.

(c) Evaluate the integral $\int \frac{4}{4-x^2}\,dx$ using trigonometric substitution. Then evaluate using the identity $\frac{4}{4-x^2} = \left(\frac{1}{x+2} - \frac{1}{x-2}\right)$. Discuss the results.

True or False? **In Exercises 63–66, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

63. If $x = \sin\theta$, then $\int \frac{dx}{\sqrt{1-x^2}} = \int d\theta$.

64. If $x = \sec\theta$, then $\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \sec\theta\tan\theta\,d\theta$.

65. If $x = \tan\theta$, then $\int_0^{\sqrt{3}} \frac{dx}{(1+x^2)^{3/2}} = \int_0^{4\pi/3} \cos\theta\,d\theta$.

66. If $x = \sin\theta$, then $\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx = 2\int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta$.

67. ***Area*** Find the area enclosed by the ellipse shown in the figure.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

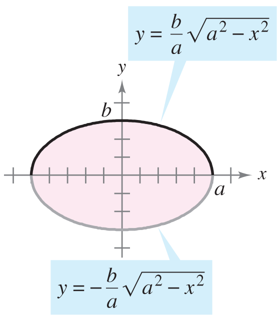

$y = \frac{b}{a}\sqrt{a^2-x^2}$
y
b
a
x
$y = -\frac{b}{a}\sqrt{a^2-x^2}$

Figure for 67

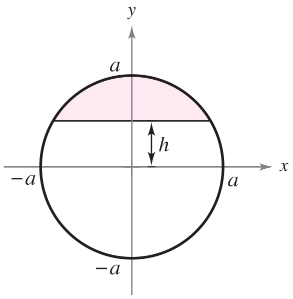


Figure for 68

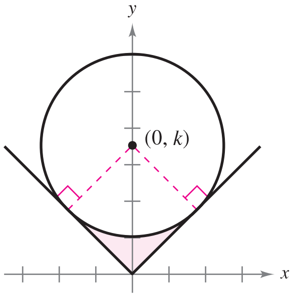
68. ***Area*** Find the area of the shaded region of the circle of radius a, if the chord is h units $(0 < h < a)$ from the center of the circle (see figure).

69. ***Mechanical Design*** The surface of a machine part is the region between the graphs of $y = |x|$ and $x^2 + (y-k)^2 = 25$ (see figure).

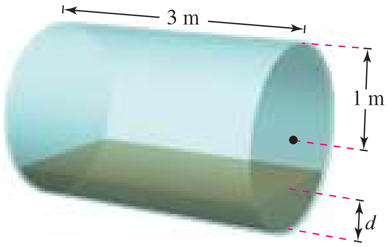
(a) Find k if the circle is tangent to the graph of $y = |x|$.

(b) Find the area of the surface of the machine part.

(c) Find the area of the surface of the machine part as a function of the radius r of the circle.



70. ***Volume*** The axis of a storage tank in the form of a right circular cylinder is horizontal (see figure). The radius and length of the tank are 1 meter and 3 meters, respectively.



(a) Determine the volume of fluid in the tank as a function of its depth d.

(b) Use a graphing utility to graph the function in part (a).

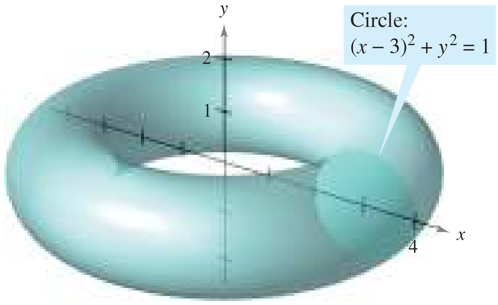
(c) Design a dip stick for the tank with markings of $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.

(d) Fluid is entering the tank at a rate of $\frac{1}{4}$ cubic meter per minute. Determine the rate of change of the depth of the fluid as a function of its depth d.

(e) Use a graphing utility to graph the function in part (d). When will the rate of change of the depth be minimum? Does this agree with your intuition? Explain.

Volume of a Torus **In Exercises 71 and 72, find the volume of the torus generated by revolving the region bounded by the graph of the circle about the y-axis.**

71. $(x-3)^2 + y^2 = 1$ (see figure)



72. $(x-h)^2 + y^2 = r^2,\ h > r$

Arc Length **In Exercises 73 and 74, find the arc length of the curve over the given interval.**

73. $y = \ln x, \quad [1, 5]$

74. $y = \frac{1}{2}x^2, \quad [0, 4]$

75. ***Arc Length*** Show that the length of one arch of the sine curve is equal to the length of one arch of the cosine curve.

76. ***Conjecture***

(a) Find formulas for the distances between $(0, 0)$ and (a, a^2) along the line between these points and along the parabola $y = x^2$.

(b) Use the formulas from part (a) to find the distances for $a = 1$ and $a = 10$.

(c) Make a conjecture about the difference between the two distances as a increases.

Projectile Motion **In Exercises 77 and 78, (a) use a graphing utility to graph the path of a projectile that follows the path given by the graph of the equation, (b) determine the range of the projectile, and (c) use the integration capabilities of a graphing utility to determine the distance the projectile travels.**

77. $y = x - 0.005x^2$

78. $y = x - \dfrac{x^2}{72}$

Centroid **In Exercises 79 and 80, find the centroid of the region determined by the graphs of the inequalities.**

79. $y \le 3/\sqrt{x^2 + 9}, \ y \ge 0, \ x \ge -4, \ x \le 4$

80. $y \le \frac{1}{4}x^2, \ (x - 4)^2 + y^2 \le 16, \ y \ge 0$

81. ***Surface Area*** Find the surface area of the solid generated by revolving the region bounded by the graphs of $y = x^2$, $y = 0$, $x = 0$, and $x = \sqrt{2}$ about the x-axis.

82. ***Field Strength*** The field strength H of a magnet of length $2L$ on a particle r units from the center of the magnet is

$$H = \frac{2mL}{(r^2 + L^2)^{3/2}}$$

where $\pm m$ are the poles of the magnet (see figure). Find the average field strength as the particle moves from 0 to R units from the center by evaluating the integral

$$\frac{1}{R}\int_0^R \frac{2mL}{(r^2 + L^2)^{3/2}}\,dr.$$

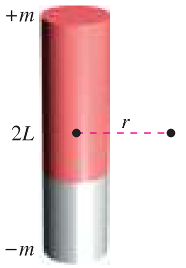


Figure for 82

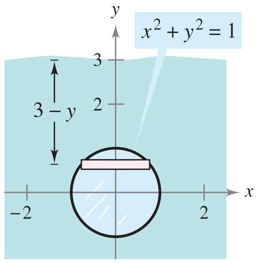


Figure for 83

83. ***Fluid Force*** Find the fluid force on a circular observation window of radius 1 foot in a vertical wall of a large water-filled tank at a fish hatchery when the center of the window is (a) 3 feet and (b) d feet $(d > 1)$ below the water's surface (see figure). Use trigonometric substitution to evaluate the one integral. (Recall that in Section 7.7 in a similar problem, you evaluated one integral by a geometric formula and the other by observing that the integrand was odd.)

84. ***Fluid Force*** Evaluate the following two integrals, which yield the fluid forces given in Example 6.

(a) $F_{\text{inside}} = 48\displaystyle\int_{-1}^{0.8} (0.8 - y)(2)\sqrt{1 - y^2}\,dy$

(b) $F_{\text{outside}} = 64\displaystyle\int_{-1}^{0.4} (0.4 - y)(2)\sqrt{1 - y^2}\,dy$

85. Use trigonometric substitution to verify the integration formulas given in Theorem 8.2.

86. ***Arc Length*** Show that the arc length of the graph of $y = \sin x$ on the interval $[0, 2\pi]$ is equal to the circumference of the ellipse $x^2 + 2y^2 = 2$ (see figure).

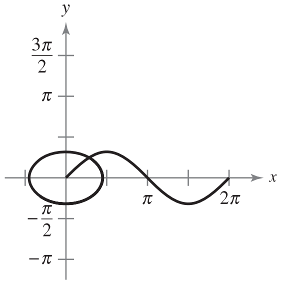


Figure for 86

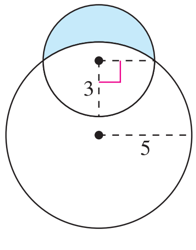
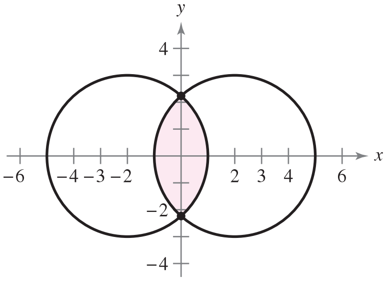


Figure for 87

87. ***Area of a Lune*** The crescent-shaped region bounded by two circles forms a *lune* (see figure). Find the area of the lune given that the radius of the smaller circle is 3 and the radius of the larger circle is 5.

88. ***Area*** Two circles of radius 3, with centers at $(-2, 0)$ and $(2, 0)$, intersect as shown in the figure. Find the area of the shaded region.



PUTNAM EXAM CHALLENGE

89. Evaluate

$$\int_0^1 \frac{\ln(x + 1)}{x^2 + 1}\,dx.$$

This problem was composed by the Committee on the Putnam Prize Competition.
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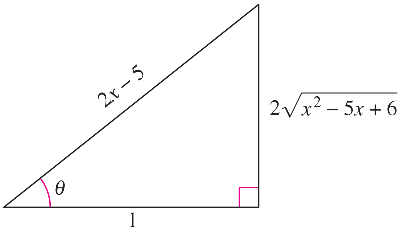
8.5 Partial Fractions

- **Understand the concept of partial fraction decomposition.**
- **Use partial fraction decomposition with linear factors to integrate rational functions.**
- **Use partial fraction decomposition with quadratic factors to integrate rational functions.**

Partial Fractions

This section examines a procedure for decomposing a rational function into simpler rational functions to which you can apply the basic integration formulas. This procedure is called the **method of partial fractions.** To see the benefit of the method of partial fractions, consider the integral

$$\int \frac{1}{x^2 - 5x + 6}\,dx.$$



$\sec\theta = 2x - 5$
Figure 8.13

To evaluate this integral *without* partial fractions, you can complete the square and use trigonometric substitution (see Figure 8.13) to obtain

$$\begin{aligned}
\int \frac{1}{x^2 - 5x + 6}\,dx &= \int \frac{dx}{(x - 5/2)^2 - (1/2)^2} && a = \tfrac{1}{2},\ x - \tfrac{5}{2} = \tfrac{1}{2}\sec\theta\\
&= \int \frac{(1/2)\sec\theta\tan\theta\,d\theta}{(1/4)\tan^2\theta} && dx = \tfrac{1}{2}\sec\theta\tan\theta\,d\theta\\
&= 2\int \csc\theta\,d\theta\\
&= 2\ln\left|\csc\theta - \cot\theta\right| + C\\
&= 2\ln\left|\frac{2x - 5}{2\sqrt{x^2 - 5x + 6}} - \frac{1}{2\sqrt{x^2 - 5x + 6}}\right| + C\\
&= 2\ln\left|\frac{x - 3}{\sqrt{x^2 - 5x + 6}}\right| + C\\
&= 2\ln\left|\frac{\sqrt{x - 3}}{\sqrt{x - 2}}\right| + C\\
&= \ln\left|\frac{x - 3}{x - 2}\right| + C\\
&= \ln|x - 3| - \ln|x - 2| + C.
\end{aligned}$$

Now, suppose you had observed that

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x - 3} - \frac{1}{x - 2}. \qquad \text{Partial fraction decomposition}$$

Then you could evaluate the integral easily, as follows.

$$\begin{aligned}
\int \frac{1}{x^2 - 5x + 6}\,dx &= \int \left(\frac{1}{x - 3} - \frac{1}{x - 2}\right) dx\\
&= \ln|x - 3| - \ln|x - 2| + C
\end{aligned}$$

This method is clearly preferable to trigonometric substitution. However, its use depends on the ability to factor the denominator, $x^2 - 5x + 6$, and to find the **partial fractions**

$$\frac{1}{x - 3} \quad \text{and} \quad -\frac{1}{x - 2}.$$

In this section, you will study techniques for finding partial fraction decompositions.

Mary Evans Picture Library

JOHN BERNOULLI (1667–1748)

The method of partial fractions was introduced by John Bernoulli, a Swiss mathematician who was instrumental in the early development of calculus. John Bernoulli was a professor at the University of Basel and taught many outstanding students, the most famous of whom was Leonhard Euler.

STUDY TIP In precalculus you learned how to combine functions such as

$$\frac{1}{x-2}+\frac{-1}{x+3}=\frac{5}{(x-2)(x+3)}.$$

The method of partial fractions shows you how to reverse this process.

$$\frac{5}{(x-2)(x+3)}=\frac{?}{x-2}+\frac{?}{x+3}$$

Recall from algebra that every polynomial with real coefficients can be factored into linear and irreducible quadratic factors.* For instance, the polynomial

$$x^5+x^4-x-1$$

can be written as

$$\begin{aligned}x^5+x^4-x-1&=x^4(x+1)-(x+1)\\&=(x^4-1)(x+1)\\&=(x^2+1)(x^2-1)(x+1)\\&=(x^2+1)(x+1)(x-1)(x+1)\\&=(x-1)(x+1)^2(x^2+1)\end{aligned}$$

where $(x-1)$ is a linear factor, $(x+1)^2$ is a repeated linear factor, and (x^2+1) is an irreducible quadratic factor. Using this factorization, you can write the partial fraction decomposition of the rational expression

$$\frac{N(x)}{x^5+x^4-x-1}$$

where $N(x)$ is a polynomial of degree less than 5, as follows.

$$\frac{N(x)}{(x-1)(x+1)^2(x^2+1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{Dx+E}{x^2+1}$$

DECOMPOSITION OF $N(x)/D(x)$ INTO PARTIAL FRACTIONS

1. **Divide if improper:** If $N(x)/D(x)$ is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

   $$\frac{N(x)}{D(x)}=(\text{a polynomial})+\frac{N_1(x)}{D(x)}$$

   where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator:** Completely factor the denominator into factors of the form

   $$(px+q)^m \qquad \text{and} \qquad (ax^2+bx+c)^n$$

   where ax^2+bx+c is irreducible.

3. **Linear factors:** For each factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of m fractions.

   $$\frac{A_1}{(px+q)}+\frac{A_2}{(px+q)^2}+\cdots+\frac{A_m}{(px+q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum of n fractions.

   $$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

---

* *For a review of factorization techniques, see* Precalculus, *7th edition, by Larson and Hostetler or* Precalculus: A Graphing Approach, *5th edition, by Larson, Hostetler, and Edwards (Boston, Massachusetts: Houghton Mifflin, 2007 and 2008, respectively).*

Linear Factors

Algebraic techniques for determining the constants in the numerators of a partial fraction decomposition with linear or repeated linear factors are shown in Examples 1 and 2.

EXAMPLE 1 Distinct Linear Factors

Write the partial fraction decomposition for $\dfrac{1}{x^2 - 5x + 6}$.

Solution Because $x^2 - 5x + 6 = (x - 3)(x - 2)$, you should include one partial fraction for each factor and write

$$\frac{1}{x^2 - 5x + 6} = \frac{A}{x - 3} + \frac{B}{x - 2}$$

where A and B are to be determined. Multiplying this equation by the least common denominator $(x - 3)(x - 2)$ yields the **basic equation**

$$1 = A(x - 2) + B(x - 3). \qquad \text{Basic equation}$$

Because this equation is to be true for all x, you can substitute any *convenient* values for x to obtain equations in A and B. The most convenient values are the ones that make particular factors equal to 0.

NOTE Note that the substitutions for x in Example 1 are chosen for their convenience in determining values for A and B; $x = 2$ is chosen to eliminate the term $A(x - 2)$, and $x = 3$ is chosen to eliminate the term $B(x - 3)$. The goal is to make *convenient* substitutions whenever possible.

To solve for A, let $x = 3$ and obtain

$$1 = A(3 - 2) + B(3 - 3) \qquad \text{Let } x = 3 \text{ in basic equation.}$$

$$1 = A(1) + B(0)$$

$$A = 1.$$

To solve for B, let $x = 2$ and obtain

$$1 = A(2 - 2) + B(2 - 3) \qquad \text{Let } x = 2 \text{ in basic equation.}$$

$$1 = A(0) + B(-1)$$

$$B = -1.$$

So, the decomposition is

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x - 3} - \frac{1}{x - 2}$$

as shown at the beginning of this section. ■

■ **FOR FURTHER INFORMATION** To learn a different method for finding partial fraction decompositions, called the Heavyside Method, see the article "Calculus to Algebra Connections in Partial Fraction Decomposition" by Joseph Wiener and Will Watkins in *The AMATYC Review*.

Be sure you see that the method of partial fractions is practical only for integrals of rational functions whose denominators factor "nicely." For instance, if the denominator in Example 1 were changed to $x^2 - 5x + 5$, its factorization as

$$x^2 - 5x + 5 = \left[x + \frac{5 + \sqrt{5}}{2}\right]\left[x - \frac{5 - \sqrt{5}}{2}\right]$$

would be too cumbersome to use with partial fractions. In such cases, you should use completing the square or a computer algebra system to perform the integration. If you do this, you should obtain

$$\int \frac{1}{x^2 - 5x + 5}\,dx = \frac{\sqrt{5}}{5}\ln\left|2x - \sqrt{5} - 5\right| - \frac{\sqrt{5}}{5}\ln\left|2x + \sqrt{5} - 5\right| + C.$$

■ **FOR FURTHER INFORMATION** For an alternative approach to using partial fractions, see the article "A Shortcut in Partial Fractions" by Xun-Cheng Huang in *The College Mathematics Journal*.

EXAMPLE 2 Repeated Linear Factors

Find $\displaystyle\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\,dx.$

Solution Because

$$\begin{aligned} x^3 + 2x^2 + x &= x(x^2 + 2x + 1) \\ &= x(x + 1)^2 \end{aligned}$$

you should include one fraction for *each power* of x and $(x + 1)$ and write

$$\frac{5x^2 + 20x + 6}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}.$$

Multiplying by the least common denominator $x(x + 1)^2$ yields the *basic equation*

$$5x^2 + 20x + 6 = A(x + 1)^2 + Bx(x + 1) + Cx. \qquad \text{Basic equation}$$

To solve for A, let $x = 0$. This eliminates the B and C terms and yields

$$\begin{aligned} 6 &= A(1) + 0 + 0 \\ A &= 6. \end{aligned}$$

To solve for C, let $x = -1$. This eliminates the A and B terms and yields

$$\begin{aligned} 5 - 20 + 6 &= 0 + 0 - C \\ C &= 9. \end{aligned}$$

The most convenient choices for x have been used, so to find the value of B, you can use *any other value* of x along with the calculated values of A and C. Using $x = 1$, $A = 6$, and $C = 9$ produces

$$\begin{aligned} 5 + 20 + 6 &= A(4) + B(2) + C \\ 31 &= 6(4) + 2B + 9 \\ -2 &= 2B \\ B &= -1. \end{aligned}$$

So, it follows that

$$\begin{aligned} \int \frac{5x^2 + 20x + 6}{x(x + 1)^2}\,dx &= \int \left(\frac{6}{x} - \frac{1}{x + 1} + \frac{9}{(x + 1)^2}\right) dx \\ &= 6\ln|x| - \ln|x + 1| + 9\frac{(x + 1)^{-1}}{-1} + C \\ &= \ln\left|\frac{x^6}{x + 1}\right| - \frac{9}{x + 1} + C. \end{aligned}$$

Try checking this result by differentiating. Include algebra in your check, simplifying the derivative until you have obtained the original integrand. ■

TECHNOLOGY Most computer algebra systems, such as *Maple*, *Mathematica*, and the *TI-89*, can be used to convert a rational function to its partial fraction decomposition. For instance, using *Maple*, you obtain the following.

$> \text{convert}\left(\dfrac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}, \text{parfrac}, x\right)$

$\dfrac{6}{x} + \dfrac{9}{(x + 1)^2} - \dfrac{1}{x + 1}$

NOTE It is necessary to make as many substitutions for x as there are unknowns $(A, B, C, \ldots)$ to be determined. For instance, in Example 2, three substitutions $(x = 0$, $x = -1$, and $x = 1)$ were made to solve for A, B, and C. ■

Quadratic Factors

When using the method of partial fractions with *linear* factors, a convenient choice of x immediately yields a value for one of the coefficients. With *quadratic* factors, a system of linear equations usually has to be solved, regardless of the choice of x.

EXAMPLE 3 Distinct Linear and Quadratic Factors

Find $\displaystyle\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)}\,dx$.

Solution Because

$$(x^2 - x)(x^2 + 4) = x(x - 1)(x^2 + 4)$$

you should include one partial fraction for each factor and write

$$\frac{2x^3 - 4x - 8}{x(x - 1)(x^2 + 4)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + 4}.$$

Multiplying by the least common denominator $x(x - 1)(x^2 + 4)$ yields the *basic equation*

$$2x^3 - 4x - 8 = A(x - 1)(x^2 + 4) + Bx(x^2 + 4) + (Cx + D)(x)(x - 1).$$

To solve for A, let $x = 0$ and obtain

$$-8 = A(-1)(4) + 0 + 0 \quad \Rightarrow \quad 2 = A.$$

To solve for B, let $x = 1$ and obtain

$$-10 = 0 + B(5) + 0 \quad \Rightarrow \quad -2 = B.$$

At this point, C and D are yet to be determined. You can find these remaining constants by choosing two other values for x and solving the resulting system of linear equations. If $x = -1$, then, using $A = 2$ and $B = -2$, you can write

$$\begin{aligned} -6 &= (2)(-2)(5) + (-2)(-1)(5) + (-C + D)(-1)(-2) \\ 2 &= -C + D. \end{aligned}$$

If $x = 2$, you have

$$\begin{aligned} 0 &= (2)(1)(8) + (-2)(2)(8) + (2C + D)(2)(1) \\ 8 &= 2C + D. \end{aligned}$$

Solving the linear system by subtracting the first equation from the second

$$\begin{aligned} -C + D &= 2 \\ 2C + D &= 8 \end{aligned}$$

yields $C = 2$. Consequently, $D = 4$, and it follows that

$$\begin{aligned} \int \frac{2x^3 - 4x - 8}{x(x - 1)(x^2 + 4)}\,dx &= \int \left(\frac{2}{x} - \frac{2}{x - 1} + \frac{2x}{x^2 + 4} + \frac{4}{x^2 + 4}\right) dx \\ &= 2\ln|x| - 2\ln|x - 1| + \ln(x^2 + 4) + 2\arctan\frac{x}{2} + C. \end{aligned}$$

■

In Examples 1, 2, and 3, the solution of the basic equation began with substituting values of x that made the linear factors equal to 0. This method works well when the partial fraction decomposition involves linear factors. However, if the decomposition involves only quadratic factors, an alternative procedure is often more convenient.

EXAMPLE 4 Repeated Quadratic Factors

Find $\displaystyle\int \frac{8x^3 + 13x}{(x^2 + 2)^2}\,dx$.

Solution Include one partial fraction for each power of $(x^2 + 2)$ and write

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2}.$$

Multiplying by the least common denominator $(x^2 + 2)^2$ yields the *basic equation*

$$8x^3 + 13x = (Ax + B)(x^2 + 2) + Cx + D.$$

Expanding the basic equation and collecting like terms produces

$$8x^3 + 13x = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$$
$$8x^3 + 13x = Ax^3 + Bx^2 + (2A + C)x + (2B + D).$$

Now, you can equate the coefficients of like terms on opposite sides of the equation.

$$8 = A \qquad\qquad 0 = 2B + D$$
$$8x^3 + 0x^2 + 13x + 0 = Ax^3 + Bx^2 + (2A + C)x + (2B + D)$$
$$0 = B$$
$$13 = 2A + C$$

Using the known values $A = 8$ and $B = 0$, you can write

$$13 = 2A + C = 2(8) + C \quad\Rightarrow\quad C = -3$$
$$0 = 2B + D = 2(0) + D \quad\Rightarrow\quad D = 0.$$

Finally, you can conclude that

$$\int \frac{8x^3 + 13x}{(x^2 + 2)^2}\,dx = \int \left(\frac{8x}{x^2 + 2} + \frac{-3x}{(x^2 + 2)^2}\right) dx$$
$$= 4\ln(x^2 + 2) + \frac{3}{2(x^2 + 2)} + C.$$

■

TECHNOLOGY Use a computer algebra system to evaluate the integral in Example 4—you might find that the form of the antiderivative is different. For instance, when you use a computer algebra system to work Example 4, you obtain

$$\int \frac{8x^3 + 13x}{(x^2 + 2)^2}\,dx = \ln(x^8 + 8x^6 + 24x^4 + 32x^2 + 16) + \frac{3}{2(x^2 + 2)} + C.$$

Is this result equivalent to that obtained in Example 4?

When integrating rational expressions, keep in mind that for *improper* rational expressions such as

$$\frac{N(x)}{D(x)} = \frac{2x^3 + x^2 - 7x + 7}{x^2 + x - 2}$$

you must first divide to obtain

$$\frac{N(x)}{D(x)} = 2x - 1 + \frac{-2x + 5}{x^2 + x - 2}.$$

The proper rational expression is then decomposed into its partial fractions by the usual methods. Here are some guidelines for solving the basic equation that is obtained in a partial fraction decomposition.

GUIDELINES FOR SOLVING THE BASIC EQUATION

Linear Factors

1. Substitute the roots of the distinct linear factors in the basic equation.
2. For repeated linear factors, use the coefficients determined in guideline 1 to rewrite the basic equation. Then substitute other convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.
2. Collect terms according to powers of x.
3. Equate the coefficients of like powers to obtain a system of linear equations involving A, B, C, and so on.
4. Solve the system of linear equations.

Before concluding this section, here are a few things you should remember. First, it is not necessary to use the partial fractions technique on all rational functions. For instance, the following integral is evaluated more easily by the Log Rule.

$$\begin{aligned}\int \frac{x^2 + 1}{x^3 + 3x - 4}\,dx &= \frac{1}{3}\int \frac{3x^2 + 3}{x^3 + 3x - 4}\,dx \\ &= \frac{1}{3}\ln\left|x^3 + 3x - 4\right| + C\end{aligned}$$

Second, if the integrand is not in reduced form, reducing it may eliminate the need for partial fractions, as shown in the following integral.

$$\begin{aligned}\int \frac{x^2 - x - 2}{x^3 - 2x - 4}\,dx &= \int \frac{(x + 1)(x - 2)}{(x - 2)(x^2 + 2x + 2)}\,dx \\ &= \int \frac{x + 1}{x^2 + 2x + 2}\,dx \\ &= \frac{1}{2}\ln\left|x^2 + 2x + 2\right| + C\end{aligned}$$

Finally, partial fractions can be used with some quotients involving transcendental functions. For instance, the substitution $u = \sin x$ allows you to write

$$\int \frac{\cos x}{\sin x(\sin x - 1)}\,dx = \int \frac{du}{u(u - 1)}. \qquad u = \sin x,\ du = \cos x\,dx$$

8.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, write the form of the partial fraction decomposition of the rational expression. Do not solve for the constants.

1. $\dfrac{4}{x^2 - 8x}$ **2.** $\dfrac{2x^2 + 1}{(x - 3)^3}$

3. $\dfrac{2x - 3}{x^3 + 10x}$ **4.** $\dfrac{x - 4}{x^2 + 6x + 5}$

5. $\dfrac{x - 9}{x^2 - 6x}$ **6.** $\dfrac{2x - 1}{x(x^2 + 1)^2}$

In Exercises 7–28, use partial fractions to find the integral.

7. $\displaystyle\int \frac{1}{x^2 - 9}\,dx$ **8.** $\displaystyle\int \frac{1}{4x^2 - 1}\,dx$

9. $\displaystyle\int \frac{5}{x^2 + 3x - 4}\,dx$ **10.** $\displaystyle\int \frac{x + 2}{x^2 + 11x + 18}\,dx$

11. $\displaystyle\int \frac{5 - x}{2x^2 + x - 1}\,dx$ **12.** $\displaystyle\int \frac{5x^2 - 12x - 12}{x^3 - 4x}\,dx$

13. $\displaystyle\int \frac{x^2 + 12x + 12}{x^3 - 4x}\,dx$ **14.** $\displaystyle\int \frac{x^3 - x + 3}{x^2 + x - 2}\,dx$

15. $\displaystyle\int \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8}\,dx$ **16.** $\displaystyle\int \frac{x + 2}{x^2 - 4x}\,dx$

17. $\displaystyle\int \frac{4x^2 + 2x - 1}{x^3 + x^2}\,dx$ **18.** $\displaystyle\int \frac{3x - 4}{(x - 1)^2}\,dx$

19. $\displaystyle\int \frac{x^2 + 3x - 4}{x^3 - 4x^2 + 4x}\,dx$ **20.** $\displaystyle\int \frac{4x^2}{x^3 + x^2 - x - 1}\,dx$

21. $\displaystyle\int \frac{x^2 - 1}{x^3 + x}\,dx$ **22.** $\displaystyle\int \frac{6x}{x^3 - 8}\,dx$

23. $\displaystyle\int \frac{x^2}{x^4 - 2x^2 - 8}\,dx$ **24.** $\displaystyle\int \frac{x^2 - x + 9}{(x^2 + 9)^2}\,dx$

25. $\displaystyle\int \frac{x}{16x^4 - 1}\,dx$ **26.** $\displaystyle\int \frac{x^2 - 4x + 7}{x^3 - x^2 + x + 3}\,dx$

27. $\displaystyle\int \frac{x^2 + 5}{x^3 - x^2 + x + 3}\,dx$ **28.** $\displaystyle\int \frac{x^2 + x + 3}{x^4 + 6x^2 + 9}\,dx$

In Exercises 29–32, evaluate the definite integral. Use a graphing utility to verify your result.

29. $\displaystyle\int_0^2 \frac{3}{4x^2 + 5x + 1}\,dx$ **30.** $\displaystyle\int_1^5 \frac{x - 1}{x^2(x + 1)}\,dx$

31. $\displaystyle\int_1^2 \frac{x + 1}{x(x^2 + 1)}\,dx$ **32.** $\displaystyle\int_0^1 \frac{x^2 - x}{x^2 + x + 1}\,dx$

CAS **In Exercises 33–40, use a computer algebra system to determine the antiderivative that passes through the given point. Use the system to graph the resulting antiderivative.**

33. $\displaystyle\int \frac{5x}{x^2 - 10x + 25}\,dx,\quad (6, 0)$ **34.** $\displaystyle\int \frac{6x^2 + 1}{x^2(x - 1)^3}\,dx,\quad (2, 1)$

35. $\displaystyle\int \frac{x^2 + x + 2}{(x^2 + 2)^2}\,dx,\quad (0, 1)$ **36.** $\displaystyle\int \frac{x^3}{(x^2 - 4)^2}\,dx,\quad (3, 4)$

37. $\displaystyle\int \frac{2x^2 - 2x + 3}{x^3 - x^2 - x - 2}\,dx,\quad (3, 10)$

38. $\displaystyle\int \frac{x(2x - 9)}{x^3 - 6x^2 + 12x - 8}\,dx,\quad (3, 2)$

39. $\displaystyle\int \frac{1}{x^2 - 25}\,dx,\quad (7, 2)$

40. $\displaystyle\int \frac{x^2 - x + 2}{x^3 - x^2 + x - 1}\,dx,\quad (2, 6)$

In Exercises 41–50, use substitution to find the integral.

41. $\displaystyle\int \frac{\sin x}{\cos x(\cos x - 1)}\,dx$ **42.** $\displaystyle\int \frac{\sin x}{\cos x + \cos^2 x}\,dx$

43. $\displaystyle\int \frac{\cos x}{\sin x + \sin^2 x}\,dx$ **44.** $\displaystyle\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\,dx$

45. $\displaystyle\int \frac{\sec^2 x}{\tan^2 x + 5\tan x + 6}\,dx$ **46.** $\displaystyle\int \frac{\sec^2 x}{\tan x(\tan x + 1)}\,dx$

47. $\displaystyle\int \frac{e^x}{(e^x - 1)(e^x + 4)}\,dx$ **48** $\displaystyle\int \frac{e^x}{(e^{2x} + 1)(e^x - 1)}\,dx$

49. $\displaystyle\int \frac{\sqrt{x}}{x - 4}\,dx$ **50.** $\displaystyle\int \frac{1}{\sqrt{x} - \sqrt[3]{x}}\,dx$

In Exercises 51–54, use the method of partial fractions to verify the integration formula.

51. $\displaystyle\int \frac{1}{x(a + bx)}\,dx = \frac{1}{a}\ln\left|\frac{x}{a + bx}\right| + C$

52. $\displaystyle\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| + C$

53. $\displaystyle\int \frac{x}{(a + bx)^2}\,dx = \frac{1}{b^2}\left(\frac{a}{a + bx} + \ln|a + bx|\right) + C$

54. $\displaystyle\int \frac{1}{x^2(a + bx)}\,dx = -\frac{1}{ax} - \frac{b}{a^2}\ln\left|\frac{x}{a + bx}\right| + C$

CAS ***Slope Fields*** **In Exercises 55 and 56, use a computer algebra system to graph the slope field for the differential equation and graph the solution through the given initial condition.**

55. $\dfrac{dy}{dx} = \dfrac{6}{4 - x^2}$ **56.** $\dfrac{dy}{dx} = \dfrac{4}{x^2 - 2x - 3}$

$y(0) = 3$ $y(0) = 5$

WRITING ABOUT CONCEPTS

57. What is the first step when integrating $\displaystyle\int \frac{x^3}{x - 5}\,dx$? Explain.

58. Describe the decomposition of the proper rational function $N(x)/D(x)$ (a) if $D(x) = (px + q)^m$ and (b) if $D(x) = (ax^2 + bx + c)^n$, where $ax^2 + bx + c$ is irreducible. Explain why you chose that method.

59. ***Area*** Find the area of the region bounded by the graphs of $y = 12/(x^2 + 5x + 6)$, $y = 0$, $x = 0$, and $x = 1$.

60. ***Area*** Find the area of the region bounded by the graphs of $y = 15/(x^2 + 7x + 12)$, $y = 0$, $x = 0$, and $x = 2$.

61. ***Area*** Find the area of the region bounded by the graphs of $y = 7/(16 - x^2)$ and $y = 1$.

CAPSTONE

62. State the method you would use to evaluate each integral. Explain why you chose that method. Do not integrate.

(a) $\displaystyle\int \frac{x+1}{x^2 + 2x - 8}\,dx$ (b) $\displaystyle\int \frac{7x+4}{x^2 + 2x - 8}\,dx$

(c) $\displaystyle\int \frac{4}{x^2 + 2x + 5}\,dx$

63. ***Modeling Data*** The predicted cost C (in hundreds of thousands of dollars) for a company to remove $p\%$ of a chemical from its waste water is shown in the table.

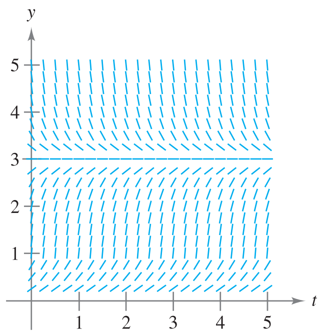
p	0	10	20	30	40	50	60	70	80	90
C	0	0.7	1.0	1.3	1.7	2.0	2.7	3.6	5.5	11.2

A model for the data is given by $C = \dfrac{124p}{(10 + p)(100 - p)}$, $0 \le p < 100$. Use the model to find the average cost of removing between 75% and 80% of the chemical.

64. ***Logistic Growth*** In Chapter 6, the exponential growth equation was derived from the assumption that the rate of growth was proportional to the existing quantity. In practice, there often exists some upper limit L past which growth cannot occur. In such cases, you assume the rate of growth to be proportional not only to the existing quantity, but also to the difference between the existing quantity y and the upper limit L. That is, $dy/dt = ky(L - y)$. In integral form, you can write this relationship as

$$\int \frac{dy}{y(L - y)} = \int k\,dt.$$

(a) A slope field for the differential equation $dy/dt = y(3 - y)$ is shown. Draw a possible solution to the differential equation if $y(0) = 5$, and another if $y(0) = \frac{1}{2}$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*



(b) Where $y(0)$ is greater than 3, what is the sign of the slope of the solution?

(c) For $y > 0$, find $\displaystyle\lim_{t \to \infty} y(t)$.

(d) Evaluate the two given integrals and solve for y as a function of t, where y_0 is the initial quantity.

(e) Use the result of part (d) to find and graph the solutions in part (a). Use a graphing utility to graph the solutions and compare the results with the solutions in part (a).

(f) The graph of the function y is a **logistic curve.** Show that the rate of growth is maximum at the point of inflection, and that this occurs when $y = L/2$.

65. ***Volume and Centroid*** Consider the region bounded by the graphs of $y = 2x/(x^2 + 1)$, $y = 0$, $x = 0$, and $x = 3$. Find the volume of the solid generated by revolving the region about the x-axis. Find the centroid of the region.

66. ***Volume*** Consider the region bounded by the graph of $y^2 = (2 - x)^2/(1 + x)^2$ on the interval $[0, 1]$. Find the volume of the solid generated by revolving this region about the x-axis.

67. ***Epidemic Model*** A single infected individual enters a community of n susceptible individuals. Let x be the number of newly infected individuals at time t. The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So, $dx/dt = k(x + 1)(n - x)$ and you obtain

$$\int \frac{1}{(x + 1)(n - x)}\,dx = \int k\,dt.$$

Solve for x as a function of t.

68. ***Chemical Reactions*** In a chemical reaction, one unit of compound Y and one unit of compound Z are converted into a single unit of compound X. x is the amount of compound X formed, and the rate of formation of X is proportional to the product of the amounts of unconverted compounds Y and Z. So, $dx/dt = k(y_0 - x)(z_0 - x)$, where y_0 and z_0 are the initial amounts of compounds Y and Z. From this equation you obtain

$$\int \frac{1}{(y_0 - x)(z_0 - x)}\,dx = \int k\,dt.$$

(a) Perform the two integrations and solve for x in terms of t.

(b) Use the result of part (a) to find x as $t \to \infty$ if (1) $y_0 < z_0$, (2) $y_0 > z_0$, and (3) $y_0 = z_0$.

69. Evaluate

$$\int_0^1 \frac{x}{1 + x^4}\,dx$$

in two different ways, one of which is partial fractions.

PUTNAM EXAM CHALLENGE

70. Prove $\dfrac{22}{7} - \pi = \displaystyle\int_0^1 \frac{x^4(1 - x)^4}{1 + x^2}\,dx.$

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8.6 Integration by Tables and Other Integration Techniques

- **Evaluate an indefinite integral using a table of integrals.**
- **Evaluate an indefinite integral using reduction formulas.**
- **Evaluate an indefinite integral involving rational functions of sine and cosine.**

Integration by Tables

So far in this chapter you have studied several integration techniques that can be used with the basic integration rules. But merely knowing *how* to use the various techniques is not enough. You also need to know *when* to use them. Integration is first and foremost a problem of recognition. That is, you must recognize which rule or technique to apply to obtain an antiderivative. Frequently, a slight alteration of an integrand will require a different integration technique (or produce a function whose antiderivative is not an elementary function), as shown below.

$$\int x \ln x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C \qquad \text{Integration by parts}$$

$$\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2} + C \qquad \text{Power Rule}$$

$$\int \frac{1}{x \ln x}\,dx = \ln|\ln x| + C \qquad \text{Log Rule}$$

$$\int \frac{x}{\ln x}\,dx = ? \qquad \text{Not an elementary function}$$

TECHNOLOGY A computer algebra system consists, in part, of a database of integration formulas. The primary difference between using a computer algebra system and using tables of integrals is that with a computer algebra system the computer searches through the database to find a fit. With integration tables, *you* must do the searching.

Many people find tables of integrals to be a valuable supplement to the integration techniques discussed in this chapter. Tables of common integrals can be found in Appendix B. **Integration by tables** is not a "cure-all" for all of the difficulties that can accompany integration—using tables of integrals requires considerable thought and insight and often involves substitution.

Each integration formula in Appendix B can be developed using one or more of the techniques in this chapter. You should try to verify several of the formulas. For instance, Formula 4

$$\int \frac{u}{(a+bu)^2}\,du = \frac{1}{b^2}\left(\frac{a}{a+bu} + \ln|a+bu|\right) + C \qquad \text{Formula 4}$$

can be verified using the method of partial fractions, and Formula 19

$$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}} \qquad \text{Formula 19}$$

can be verified using integration by parts. Note that the integrals in Appendix B are classified according to forms involving the following.

u^n	$(a + bu)$
$(a + bu + cu^2)$	$\sqrt{a + bu}$
$(a^2 \pm u^2)$	$\sqrt{u^2 \pm a^2}$
$\sqrt{a^2 - u^2}$	Trigonometric functions
Inverse trigonometric functions	Exponential functions
Logarithmic functions	

EXAMPLE 1 Integration by Tables

Find $\displaystyle\int \frac{dx}{x\sqrt{x-1}}$.

Solution Because the expression inside the radical is linear, you should consider forms involving $\sqrt{a+bu}$.

$$\int \frac{du}{u\sqrt{a+bu}} = \frac{2}{\sqrt{-a}} \arctan \sqrt{\frac{a+bu}{-a}} + C$$

Formula 17 $(a < 0)$

Let $a = -1$, $b = 1$, and $u = x$. Then $du = dx$, and you can write

$$\int \frac{dx}{x\sqrt{x-1}} = 2 \arctan \sqrt{x-1} + C.$$

EXPLORATION

Use the tables of integrals in Appendix B and the substitution

$$u = \sqrt{x-1}$$

to evaluate the integral in Example 1. If you do this, you should obtain

$$\int \frac{dx}{x\sqrt{x-1}} = \int \frac{2\,du}{u^2+1}.$$

Does this produce the same result as that obtained in Example 1?

EXAMPLE 2 Integration by Tables

Find $\displaystyle\int x\sqrt{x^4-9}\,dx$.

Solution Because the radical has the form $\sqrt{u^2-a^2}$, you should consider Formula 26.

$$\int \sqrt{u^2-a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2-a^2} - a^2 \ln\left|u + \sqrt{u^2-a^2}\right|\right) + C$$

Let $u = x^2$ and $a = 3$. Then $du = 2x\,dx$, and you have

$$\begin{aligned}\int x\sqrt{x^4-9}\,dx &= \frac{1}{2}\int \sqrt{(x^2)^2-3^2}\,(2x)\,dx \\ &= \frac{1}{4}\left(x^2\sqrt{x^4-9} - 9\ln\left|x^2+\sqrt{x^4-9}\right|\right) + C.\end{aligned}$$

EXAMPLE 3 Integration by Tables

Find $\displaystyle\int \frac{x}{1+e^{-x^2}}\,dx$.

Solution Of the forms involving e^u, consider the following formula.

$$\int \frac{du}{1+e^u} = u - \ln(1+e^u) + C$$

Formula 84

Let $u = -x^2$. Then $du = -2x\,dx$, and you have

$$\begin{aligned}\int \frac{x}{1+e^{-x^2}}\,dx &= -\frac{1}{2}\int \frac{-2x\,dx}{1+e^{-x^2}} \\ &= -\frac{1}{2}\left[-x^2 - \ln(1+e^{-x^2})\right] + C \\ &= \frac{1}{2}\left[x^2 + \ln(1+e^{-x^2})\right] + C.\end{aligned}$$

■

TECHNOLOGY Example 3 shows the importance of having several solution techniques at your disposal. This integral is not difficult to solve with a table, but when it was entered into a well-known computer algebra system, the utility was unable to find the antiderivative.

Reduction Formulas

Several of the integrals in the integration tables have the form $\int f(x)\,dx = g(x) + \int h(x)\,dx$. Such integration formulas are called **reduction formulas** because they reduce a given integral to the sum of a function and a simpler integral.

EXAMPLE 4 Using a Reduction Formula

Find $\displaystyle\int x^3 \sin x\,dx$.

Solution Consider the following three formulas.

$$\int u \sin u\,du = \sin u - u\cos u + C \qquad \text{Formula 52}$$

$$\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du \qquad \text{Formula 54}$$

$$\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du \qquad \text{Formula 55}$$

Using Formula 54, Formula 55, and then Formula 52 produces

$$\begin{aligned}\int x^3 \sin x\,dx &= -x^3\cos x + 3\int x^2\cos x\,dx\\ &= -x^3\cos x + 3\left(x^2\sin x - 2\int x\sin x\,dx\right)\\ &= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C.\end{aligned}$$

EXAMPLE 5 Using a Reduction Formula

Find $\displaystyle\int \frac{\sqrt{3-5x}}{2x}\,dx$.

Solution Consider the following two formulas.

$$\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C \qquad \text{Formula 17 } (a > 0)$$

$$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}} \qquad \text{Formula 19}$$

Using Formula 19, with $a = 3$, $b = -5$, and $u = x$, produces

$$\begin{aligned}\frac{1}{2}\int \frac{\sqrt{3-5x}}{x}\,dx &= \frac{1}{2}\left(2\sqrt{3-5x} + 3\int \frac{dx}{x\sqrt{3-5x}}\right)\\ &= \sqrt{3-5x} + \frac{3}{2}\int \frac{dx}{x\sqrt{3-5x}}.\end{aligned}$$

Using Formula 17, with $a = 3$, $b = -5$, and $u = x$, produces

$$\begin{aligned}\int \frac{\sqrt{3-5x}}{2x}\,dx &= \sqrt{3-5x} + \frac{3}{2}\left(\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3-5x}-\sqrt{3}}{\sqrt{3-5x}+\sqrt{3}}\right|\right) + C\\ &= \sqrt{3-5x} + \frac{\sqrt{3}}{2}\ln\left|\frac{\sqrt{3-5x}-\sqrt{3}}{\sqrt{3-5x}+\sqrt{3}}\right| + C.\end{aligned}$$ ■

TECHNOLOGY Sometimes when you use computer algebra systems you obtain results that look very different, but are actually equivalent. Here is how two different systems evaluated the integral in Example 5.

Maple

$\sqrt{3-5x}\,-$

$\sqrt{3}\operatorname{arctanh}\left(\frac{1}{3}\sqrt{3-5x}\sqrt{3}\right)$

Mathematica

Sqrt[3 − 5*x*] −

$\text{Sqrt[3] ArcTanh}\left[\dfrac{\text{Sqrt[3 − 5}x\text{]}}{\text{Sqrt[3]}}\right]$

Notice that computer algebra systems do not include a constant of integration.

Rational Functions of Sine and Cosine

EXAMPLE 6 Integration by Tables

Find $\displaystyle\int \frac{\sin 2x}{2 + \cos x}\,dx$.

Solution Substituting $2 \sin x \cos x$ for $\sin 2x$ produces

$$\int \frac{\sin 2x}{2 + \cos x}\,dx = 2\int \frac{\sin x \cos x}{2 + \cos x}\,dx.$$

A check of the forms involving $\sin u$ or $\cos u$ in Appendix B shows that none of those listed applies. So, you can consider forms involving $a + bu$. For example,

$$\int \frac{u\,du}{a + bu} = \frac{1}{b^2}\left(bu - a\ln|a + bu|\right) + C. \qquad \text{Formula 3}$$

Let $a = 2$, $b = 1$, and $u = \cos x$. Then $du = -\sin x\,dx$, and you have

$$\begin{aligned} 2\int \frac{\sin x \cos x}{2 + \cos x}\,dx &= -2\int \frac{\cos x(-\sin x\,dx)}{2 + \cos x} \\ &= -2(\cos x - 2\ln|2 + \cos x|) + C \\ &= -2\cos x + 4\ln|2 + \cos x| + C. \end{aligned}$$

■

Example 6 involves a rational expression of $\sin x$ and $\cos x$. If you are unable to find an integral of this form in the integration tables, try using the following special substitution to convert the trigonometric expression to a standard rational expression.

SUBSTITUTION FOR RATIONAL FUNCTIONS OF SINE AND COSINE

For integrals involving rational functions of sine and cosine, the substitution

$$u = \frac{\sin x}{1 + \cos x} = \tan\frac{x}{2}$$

yields

$$\cos x = \frac{1 - u^2}{1 + u^2}, \quad \sin x = \frac{2u}{1 + u^2}, \quad \text{and} \quad dx = \frac{2\,du}{1 + u^2}.$$

PROOF From the substitution for u, it follows that

$$u^2 = \frac{\sin^2 x}{(1 + \cos x)^2} = \frac{1 - \cos^2 x}{(1 + \cos x)^2} = \frac{1 - \cos x}{1 + \cos x}.$$

Solving for $\cos x$ produces $\cos x = (1 - u^2)/(1 + u^2)$. To find $\sin x$, write $u = \sin x/(1 + \cos x)$ as

$$\sin x = u(1 + \cos x) = u\left(1 + \frac{1 - u^2}{1 + u^2}\right) = \frac{2u}{1 + u^2}.$$

Finally, to find dx, consider $u = \tan(x/2)$. Then you have $\arctan u = x/2$ and $dx = (2\,du)/(1 + u^2)$. ■

8.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use a table of integrals with forms involving $a + bu$ to find the integral.

1. $\displaystyle\int \frac{x^2}{5+x}\,dx$

2. $\displaystyle\int \frac{2}{3x^2(2x-5)^2}\,dx$

In Exercises 3 and 4, use a table of integrals with forms involving $\sqrt{u^2 \pm a^2}$ to find the integral.

3. $\displaystyle\int e^x\sqrt{1+e^{2x}}\,dx$

4. $\displaystyle\int \frac{\sqrt{x^2-36}}{6x}\,dx$

In Exercises 5 and 6, use a table of integrals with forms involving $\sqrt{a^2 - u^2}$ to find the integral.

5. $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$

6. $\displaystyle\int \frac{x}{\sqrt{100-x^4}}\,dx$

In Exercises 7–10, use a table of integrals with forms involving the trigonometric functions to find the integral.

7. $\displaystyle\int \cos^4 3x\,dx$

8. $\displaystyle\int \frac{\sin^3\sqrt{x}}{\sqrt{x}}\,dx$

9. $\displaystyle\int \frac{1}{\sqrt{x}\left(1-\cos\sqrt{x}\right)}\,dx$

10. $\displaystyle\int \frac{1}{1-\tan 5x}\,dx$

In Exercises 11 and 12, use a table of integrals with forms involving e^u to find the integral.

11. $\displaystyle\int \frac{1}{1+e^{2x}}\,dx$

12. $\displaystyle\int e^{-x/2}\sin 2x\,dx$

In Exercises 13 and 14, use a table of integrals with forms involving $\ln u$ to find the integral.

13. $\displaystyle\int x^7\ln x\,dx$

14. $\displaystyle\int (\ln x)^3\,dx$

In Exercises 15–18, find the indefinite integral (a) using integration tables and (b) using the given method.

	Integral	*Method*
15.	$\displaystyle\int x^2 e^{3x}\,dx$	Integration by parts
16.	$\displaystyle\int x^6\ln x\,dx$	Integration by parts
17.	$\displaystyle\int \frac{1}{x^2(x+1)}\,dx$	Partial fractions
18.	$\displaystyle\int \frac{1}{x^2-48}\,dx$	Partial fractions

In Exercises 19–42, use integration tables to find the integral.

19. $\displaystyle\int x\,\operatorname{arccsc}(x^2+1)\,dx$

20. $\displaystyle\int \operatorname{arcsec} 2x\,dx$

21. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$

22. $\displaystyle\int \frac{1}{x^2+4x+8}\,dx$

23. $\displaystyle\int \frac{4x}{(2-5x)^2}\,dx$

24. $\displaystyle\int \frac{\theta^2}{1-\sin\theta^3}\,d\theta$

25. $\displaystyle\int e^x\arccos e^x\,dx$

26. $\displaystyle\int \frac{e^x}{1-\tan e^x}\,dx$

27. $\displaystyle\int \frac{x}{1-\sec x^2}\,dx$

28. $\displaystyle\int \frac{1}{t[1+(\ln t)^2]}\,dt$

29. $\displaystyle\int \frac{\cos\theta}{3+2\sin\theta+\sin^2\theta}\,d\theta$

30. $\displaystyle\int x^2\sqrt{2+9x^2}\,dx$

31. $\displaystyle\int \frac{1}{x^2\sqrt{2+9x^2}}\,dx$

32. $\displaystyle\int \sqrt{x}\arctan x^{3/2}\,dx$

33. $\displaystyle\int \frac{\ln x}{x(3+2\ln x)}\,dx$

34. $\displaystyle\int \frac{e^x}{(1-e^{2x})^{3/2}}\,dx$

35. $\displaystyle\int \frac{x}{(x^2-6x+10)^2}\,dx$

36. $\displaystyle\int (2x-3)^2\sqrt{(2x-3)^2+4}\,dx$

37. $\displaystyle\int \frac{x}{\sqrt{x^4-6x^2+5}}\,dx$

38. $\displaystyle\int \frac{\cos x}{\sqrt{\sin^2 x+1}}\,dx$

39. $\displaystyle\int \frac{x^3}{\sqrt{4-x^2}}\,dx$

40. $\displaystyle\int \sqrt{\frac{5-x}{5+x}}\,dx$

41. $\displaystyle\int \frac{e^{3x}}{(1+e^x)^3}\,dx$

42. $\displaystyle\int \cot^4\theta\,d\theta$

In Exercises 43–50, use integration tables to evaluate the integral.

43. $\displaystyle\int_0^1 xe^{x^2}\,dx$

44. $\displaystyle\int_0^7 \frac{x}{\sqrt{9+x}}\,dx$

45. $\displaystyle\int_1^2 x^4\ln x\,dx$

46. $\displaystyle\int_0^{\pi/2} x\cos x\,dx$

47. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx$

48. $\displaystyle\int_4^6 \frac{x^2}{(2x-7)^2}\,dx$

49. $\displaystyle\int_0^{\pi/2} t^3\cos t\,dt$

50. $\displaystyle\int_0^1 \sqrt{3+x^2}\,dx$

In Exercises 51–56, verify the integration formula.

51. $\displaystyle\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

52. $\displaystyle\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2}{(2n+1)b}\left(u^n\sqrt{a+bu} - na\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du\right)$

53. $\displaystyle\int \frac{1}{(u^2\pm a^2)^{3/2}}\,du = \frac{\pm u}{a^2\sqrt{u^2\pm a^2}} + C$

54. $\displaystyle\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

55. $\displaystyle\int \arctan u\,du = u\arctan u - \ln\sqrt{1+u^2} + C$

56. $\displaystyle\int (\ln u)^n\,du = u(\ln u)^n - n\int (\ln u)^{n-1}\,du$

CAS **In Exercises 57–62, use a computer algebra system to determine the antiderivative that passes through the given point. Use the system to graph the resulting antiderivative.**

57. $\displaystyle\int \frac{1}{x^{3/2}\sqrt{1-x}}\,dx,\ \left(\tfrac{1}{2}, 5\right)$

58. $\displaystyle\int x\sqrt{x^2+2x}\,dx,\ (0, 0)$

59. $\displaystyle\int \frac{1}{(x^2-6x+10)^2}\,dx,\ (3, 0)$

60. $\displaystyle\int \frac{\sqrt{2-2x-x^2}}{x+1}\,dx,\ \left(0, \sqrt{2}\right)$

61. $\displaystyle\int \frac{1}{\sin\theta\tan\theta}\,d\theta,\ \left(\tfrac{\pi}{4}, 2\right)$

62. $\displaystyle\int \frac{\sin\theta}{(\cos\theta)(1+\sin\theta)}\,d\theta,\ (0, 1)$

In Exercises 63–70, find or evaluate the integral.

63. $\displaystyle\int \frac{1}{2-3\sin\theta}\,d\theta$

64. $\displaystyle\int \frac{\sin\theta}{1+\cos^2\theta}\,d\theta$

65. $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sin\theta+\cos\theta}\,d\theta$

66. $\displaystyle\int_0^{\pi/2} \frac{1}{3-2\cos\theta}\,d\theta$

67. $\displaystyle\int \frac{\sin\theta}{3-2\cos\theta}\,d\theta$

68. $\displaystyle\int \frac{\cos\theta}{1+\cos\theta}\,d\theta$

69. $\displaystyle\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$

70. $\displaystyle\int \frac{4}{\csc\theta-\cot\theta}\,d\theta$

Area **In Exercises 71 and 72, find the area of the region bounded by the graphs of the equations.**

71. $y = \dfrac{x}{\sqrt{x+3}},\ y = 0,\ x = 6$

72. $y = \dfrac{x}{1+e^{x^2}},\ y = 0,\ x = 2$

WRITING ABOUT CONCEPTS

73. (a) Evaluate $\int x^n \ln x\,dx$ for $n = 1$, 2, and 3. Describe any patterns you notice.

(b) Write a general rule for evaluating the integral in part (a), for an integer $n \geq 1$.

74. Describe what is meant by a reduction formula. Give an example.

True or False? **In Exercises 75 and 76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

75. To use a table of integrals, the integral you are evaluating must appear in the table.

76. When using a table of integrals, you may have to make substitutions to rewrite your integral in the form in which it appears in the table.

77. ***Volume*** Consider the region bounded by the graphs of $y = x\sqrt{16-x^2}$, $y = 0$, $x = 0$, and $x = 4$. Find the volume of the solid generated by revolving the region about the y-axis.

CAPSTONE

78. State (if possible) the method or integration formula you would use to find the antiderivative. Explain why you chose that method or formula. Do not integrate.

(a) $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$

(b) $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

(c) $\displaystyle\int x\,e^{x^2}\,dx$

(d) $\displaystyle\int x\,e^x\,dx$

(e) $\displaystyle\int e^{x^2}\,dx$

(f) $\displaystyle\int e^{2x}\sqrt{e^{2x}+1}\,dx$

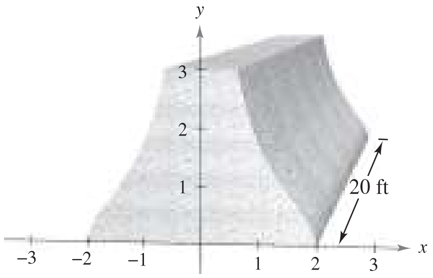
79. ***Work*** A hydraulic cylinder on an industrial machine pushes a steel block a distance of x feet $(0 \leq x \leq 5)$, where the variable force required is $F(x) = 2000xe^{-x}$ pounds. Find the work done in pushing the block the full 5 feet through the machine.

80. ***Work*** Repeat Exercise 79, using $F(x) = \dfrac{500x}{\sqrt{26-x^2}}$ pounds.

81. ***Building Design*** The cross section of a precast concrete beam for a building is bounded by the graphs of the equations

$$x = \frac{2}{\sqrt{1+y^2}},\ x = \frac{-2}{\sqrt{1+y^2}},\ y = 0, \text{ and } y = 3$$

where x and y are measured in feet. The length of the beam is 20 feet (see figure). (a) Find the volume V and the weight W of the beam. Assume the concrete weighs 148 pounds per cubic foot. (b) Then find the centroid of a cross section of the beam.



82. ***Population*** A population is growing according to the logistic model $N = \dfrac{5000}{1+e^{4.8-1.9t}}$, where t is the time in days. Find the average population over the interval $[0, 2]$.

In Exercises 83 and 84, use a graphing utility to (a) solve the integral equation for the constant k and (b) graph the region whose area is given by the integral.

83. $\displaystyle\int_0^4 \frac{k}{2+3x}\,dx = 10$

84. $\displaystyle\int_0^k 6x^2 e^{-x/2}\,dx = 50$

PUTNAM EXAM CHALLENGE

85. Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{2}}}$.

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8.7 Indeterminate Forms and L'Hôpital's Rule

- **Recognize limits that produce indeterminate forms.**
- **Apply L'Hôpital's Rule to evaluate a limit.**

Indeterminate Forms

Recall that the forms $0/0$ and ∞/∞ are called *indeterminate* because they do not guarantee that a limit exists, nor do they indicate what the limit is, if one does exist. When you encountered one of these indeterminate forms earlier in the text, you attempted to rewrite the expression by using various algebraic techniques.

Indeterminate Form	*Limit*	*Algebraic Technique*
$\dfrac{0}{0}$	$\displaystyle\lim_{x\to -1}\frac{2x^2-2}{x+1} = \lim_{x\to -1} 2(x-1) = -4$	Divide numerator and denominator by $(x+1)$.
$\dfrac{\infty}{\infty}$	$\displaystyle\lim_{x\to\infty}\frac{3x^2-1}{2x^2+1} = \lim_{x\to\infty}\frac{3-(1/x^2)}{2+(1/x^2)} = \frac{3}{2}$	Divide numerator and denominator by x^2.

Occasionally, you can extend these algebraic techniques to find limits of transcendental functions. For instance, the limit

$$\lim_{x\to 0}\frac{e^{2x}-1}{e^x-1}$$

produces the indeterminate form $0/0$. Factoring and then dividing produces

$$\lim_{x\to 0}\frac{e^{2x}-1}{e^x-1} = \lim_{x\to 0}\frac{(e^x+1)(e^x-1)}{e^x-1} = \lim_{x\to 0}(e^x+1) = 2.$$

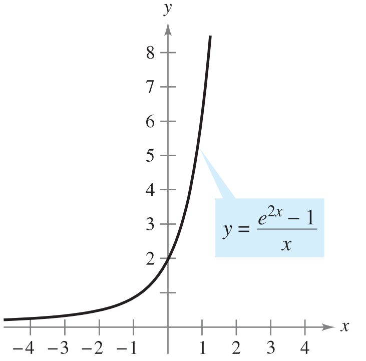
However, not all indeterminate forms can be evaluated by algebraic manipulation. This is often true when *both* algebraic and transcendental functions are involved. For instance, the limit

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}$$

produces the indeterminate form $0/0$. Rewriting the expression to obtain

$$\lim_{x\to 0}\left(\frac{e^{2x}}{x}-\frac{1}{x}\right)$$

merely produces another indeterminate form, $\infty - \infty$. Of course, you could use technology to estimate the limit, as shown in the table and in Figure 8.14. From the table and the graph, the limit appears to be 2. (This limit will be verified in Example 1.)



The limit as x approaches 0 appears to be 2.
Figure 8.14

x	-1	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	1
$\dfrac{e^{2x}-1}{x}$	0.865	1.813	1.980	1.998	?	2.002	2.020	2.214	6.389

The Granger Collection

GUILLAUME L'HÔPITAL (1661–1704)

L'Hôpital's Rule is named after the French mathematician Guillaume François Antoine de L'Hôpital. L'Hôpital is credited with writing the first text on differential calculus (in 1696) in which the rule publicly appeared. It was recently discovered that the rule and its proof were written in a letter from John Bernoulli to L'Hôpital. "… I acknowledge that I owe very much to the bright minds of the Bernoulli brothers. … I have made free use of their discoveries …," said L'Hôpital.

L'Hôpital's Rule

To find the limit illustrated in Figure 8.14, you can use a theorem called **L'Hôpital's Rule.** This theorem states that under certain conditions the limit of the quotient $f(x)/g(x)$ is determined by the limit of the quotient of the derivatives

$$\frac{f'(x)}{g'(x)}.$$

To prove this theorem, you can use a more general result called the **Extended Mean Value Theorem.**

THEOREM 8.3 THE EXTENDED MEAN VALUE THEOREM

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

NOTE To see why this is called the Extended Mean Value Theorem, consider the special case in which $g(x) = x$. For this case, you obtain the "standard" Mean Value Theorem as presented in Section 3.2. ■

The Extended Mean Value Theorem and L'Hôpital's Rule are both proved in Appendix A.

THEOREM 8.4 L'HÔPITAL'S RULE

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

NOTE People occasionally use L'Hôpital's Rule incorrectly by applying the Quotient Rule to $f(x)/g(x)$. Be sure you see that the rule involves $f'(x)/g'(x)$, not the derivative of $f(x)/g(x)$. ■

L'Hôpital's Rule can also be applied to one-sided limits. For instance, if the limit of $f(x)/g(x)$ as x approaches c *from the right* produces the indeterminate form $0/0$, then

$$\lim_{x\to c^+} \frac{f(x)}{g(x)} = \lim_{x\to c^+} \frac{f'(x)}{g'(x)}$$

provided the limit exists (or is infinite).

■ **FOR FURTHER INFORMATION** To enhance your understanding of the necessity of the restriction that $g'(x)$ be nonzero for all x in (a, b), except possibly at c, see the article "Counterexamples to L'Hôpital's Rule" by R. P. Boas in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

TECHNOLOGY *Numerical and Graphical Approaches* Use a numerical or a graphical approach to approximate each limit.

a. $\displaystyle\lim_{x\to0}\frac{2^{2x}-1}{x}$

b. $\displaystyle\lim_{x\to0}\frac{3^{2x}-1}{x}$

c. $\displaystyle\lim_{x\to0}\frac{4^{2x}-1}{x}$

d. $\displaystyle\lim_{x\to0}\frac{5^{2x}-1}{x}$

What pattern do you observe? Does an analytic approach have an advantage for these limits? If so, explain your reasoning.

EXAMPLE 1 Indeterminate Form 0/0

Evaluate $\displaystyle\lim_{x\to0}\frac{e^{2x}-1}{x}$.

Solution Because direct substitution results in the indeterminate form 0/0

$$\begin{tikzcd} & \displaystyle\lim_{x\to0}(e^{2x}-1)=0 \\ \displaystyle\lim_{x\to0}\frac{e^{2x}-1}{x} \arrow[ur] \arrow[dr] & \\ & \displaystyle\lim_{x\to0} x = 0 \end{tikzcd}$$

you can apply L'Hôpital's Rule, as shown below.

$$\begin{aligned}\lim_{x\to0}\frac{e^{2x}-1}{x} &= \lim_{x\to0}\frac{\frac{d}{dx}[e^{2x}-1]}{\frac{d}{dx}[x]} && \text{Apply L'Hôpital's Rule.}\\ &= \lim_{x\to0}\frac{2e^{2x}}{1} && \text{Differentiate numerator and denominator.}\\ &= 2 && \text{Evaluate the limit.}\end{aligned}$$

■

NOTE In writing the string of equations in Example 1, you actually do not know that the first limit is equal to the second until you have shown that the second limit exists. In other words, if the second limit had not existed, it would not have been permissible to apply L'Hôpital's Rule. ■

Another form of L'Hôpital's Rule states that if the limit of $f(x)/g(x)$ as x approaches ∞ (or $-\infty$) produces the indeterminate form 0/0 or ∞/∞, then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

EXAMPLE 2 Indeterminate Form ∞/∞

Evaluate $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x}$.

Solution Because direct substitution results in the indeterminate form ∞/∞, you can apply L'Hôpital's Rule to obtain

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln x}{x} &= \lim_{x\to\infty}\frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[x]} && \text{Apply L'Hôpital's Rule.}\\ &= \lim_{x\to\infty}\frac{1}{x} && \text{Differentiate numerator and denominator.}\\ &= 0. && \text{Evaluate the limit.}\end{aligned}$$

■

NOTE Try graphing $y_1 = \ln x$ and $y_2 = x$ in the same viewing window. Which function grows faster as x approaches ∞? How is this observation related to Example 2?

Occasionally it is necessary to apply L'Hôpital's Rule more than once to remove an indeterminate form, as shown in Example 3.

EXAMPLE 3 Applying L'Hôpital's Rule More Than Once

Evaluate $\lim\limits_{x\to-\infty} \dfrac{x^2}{e^{-x}}$.

Solution Because direct substitution results in the indeterminate form ∞/∞, you can apply L'Hôpital's Rule.

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{\dfrac{d}{dx}[x^2]}{\dfrac{d}{dx}[e^{-x}]} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}}$$

This limit yields the indeterminate form $(-\infty)/(-\infty)$, so you can apply L'Hôpital's Rule again to obtain

$$\lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{\dfrac{d}{dx}[2x]}{\dfrac{d}{dx}[-e^{-x}]} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0. \quad ■$$

In addition to the forms $0/0$ and ∞/∞, there are other indeterminate forms such as $0\cdot\infty$, 1^∞, ∞^0, 0^0, and $\infty-\infty$. For example, consider the following four limits that lead to the indeterminate form $0\cdot\infty$.

$$\underbrace{\lim_{x\to0}(x)\left(\frac{1}{x}\right)}_{\text{Limit is 1.}}, \quad \underbrace{\lim_{x\to0}(x)\left(\frac{2}{x}\right)}_{\text{Limit is 2.}}, \quad \underbrace{\lim_{x\to\infty}(x)\left(\frac{1}{e^x}\right)}_{\text{Limit is 0.}}, \quad \underbrace{\lim_{x\to\infty}(e^x)\left(\frac{1}{x}\right)}_{\text{Limit is }\infty.}$$

Because each limit is different, it is clear that the form $0\cdot\infty$ is indeterminate in the sense that it does not determine the value (or even the existence) of the limit. The following examples indicate methods for evaluating these forms. Basically, you attempt to convert each of these forms to $0/0$ or ∞/∞ so that L'Hôpital's Rule can be applied.

EXAMPLE 4 Indeterminate Form $0\cdot\infty$

Evaluate $\lim\limits_{x\to\infty} e^{-x}\sqrt{x}$.

Solution Because direct substitution produces the indeterminate form $0\cdot\infty$, you should try to rewrite the limit to fit the form $0/0$ or ∞/∞. In this case, you can rewrite the limit to fit the second form.

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x}$$

Now, by L'Hôpital's Rule, you have

$$\lim_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to\infty} \frac{1/(2\sqrt{x})}{e^x} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}\,e^x} = 0. \quad ■$$

If rewriting a limit in one of the forms $0/0$ or ∞/∞ does not seem to work, try the other form. For instance, in Example 4 you can write the limit as

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{e^{-x}}{x^{-1/2}}$$

which yields the indeterminate form $0/0$. As it happens, applying L'Hôpital's Rule to this limit produces

$$\lim_{x\to\infty} \frac{e^{-x}}{x^{-1/2}} = \lim_{x\to\infty} \frac{-e^{-x}}{-1/(2x^{3/2})}$$

which also yields the indeterminate form $0/0$.

The indeterminate forms 1^∞, ∞^0, and 0^0 arise from limits of functions that have variable bases and variable exponents. When you previously encountered this type of function, you used logarithmic differentiation to find the derivative. You can use a similar procedure when taking limits, as shown in the next example.

EXAMPLE 5 Indeterminate Form 1^∞

Evaluate $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Solution Because direct substitution yields the indeterminate form 1^∞, you can proceed as follows. To begin, assume that the limit exists and is equal to y.

$$y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Taking the natural logarithm of each side produces

$$\ln y = \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right].$$

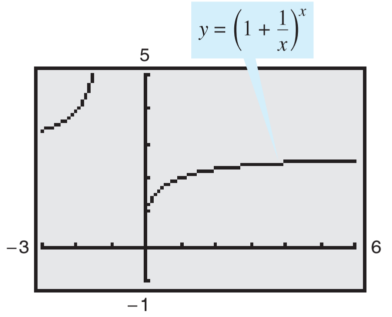
Because the natural logarithmic function is continuous, you can write

$$\begin{aligned}
\ln y &= \lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right] && \text{Indeterminate form } \infty\cdot 0\\
&= \lim_{x\to\infty}\left(\frac{\ln[1+(1/x)]}{1/x}\right) && \text{Indeterminate form } 0/0\\
&= \lim_{x\to\infty}\left(\frac{(-1/x^2)\{1/[1+(1/x)]\}}{-1/x^2}\right) && \text{L'Hôpital's Rule}\\
&= \lim_{x\to\infty}\frac{1}{1+(1/x)}\\
&= 1.
\end{aligned}$$

Now, because you have shown that $\ln y = 1$, you can conclude that $y = e$ and obtain

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

You can use a graphing utility to confirm this result, as shown in Figure 8.15. ■



The limit of $[1 + (1/x)]^x$ as x approaches infinity is e.
Figure 8.15

L'Hôpital's Rule can also be applied to one-sided limits, as demonstrated in Examples 6 and 7.

EXAMPLE 6 Indeterminate Form 0^0

Find $\lim_{x \to 0^+} (\sin x)^x$.

Solution Because direct substitution produces the indeterminate form 0^0, you can proceed as shown below. To begin, assume that the limit exists and is equal to y.

$$
\begin{aligned}
y &= \lim_{x \to 0^+} (\sin x)^x && \text{Indeterminate form } 0^0 \\
\ln y &= \ln\Big[\lim_{x \to 0^+} (\sin x)^x\Big] && \text{Take natural log of each side.} \\
&= \lim_{x \to 0^+} [\ln(\sin x)^x] && \text{Continuity} \\
&= \lim_{x \to 0^+} [x \ln(\sin x)] && \text{Indeterminate form } 0 \cdot (-\infty) \\
&= \lim_{x \to 0^+} \frac{\ln(\sin x)}{1/x} && \text{Indeterminate form } -\infty/\infty \\
&= \lim_{x \to 0^+} \frac{\cot x}{-1/x^2} && \text{L'Hôpital's Rule} \\
&= \lim_{x \to 0^+} \frac{-x^2}{\tan x} && \text{Indeterminate form } 0/0 \\
&= \lim_{x \to 0^+} \frac{-2x}{\sec^2 x} = 0 && \text{L'Hôpital's Rule}
\end{aligned}
$$

Now, because $\ln y = 0$, you can conclude that $y = e^0 = 1$, and it follows that

$$\lim_{x \to 0^+} (\sin x)^x = 1.$$

■

TECHNOLOGY When evaluating complicated limits such as the one in Example 6, it is helpful to check the reasonableness of the solution with a computer or with a graphing utility. For instance, the calculations in the following table and the graph in Figure 8.16 are consistent with the conclusion that $(\sin x)^x$ approaches 1 as x approaches 0 from the right.

x	1.0	0.1	0.01	0.001	0.0001	0.00001
$(\sin x)^x$	0.8415	0.7942	0.9550	0.9931	0.9991	0.9999

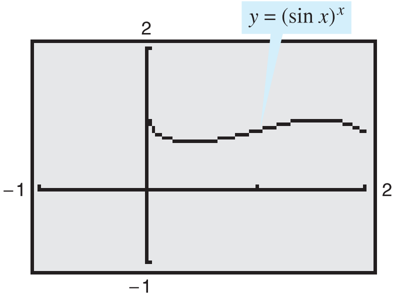
Use a computer algebra system or graphing utility to estimate the following limits:

$$\lim_{x \to 0} (1 - \cos x)^x$$

and

$$\lim_{x \to 0^+} (\tan x)^x.$$

Then see if you can verify your estimates analytically.



The limit of $(\sin x)^x$ is 1 as x approaches 0 from the right.
Figure 8.16

STUDY TIP In each of the examples presented in this section, L'Hôpital's Rule is used to find a limit that exists. It can also be used to conclude that a limit is infinite. For instance, try using L'Hôpital's Rule to show that

$$\lim_{x\to\infty}\frac{e^x}{x} = \infty.$$

EXAMPLE 7 Indeterminate Form $\infty - \infty$

Evaluate $\displaystyle\lim_{x\to1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

Solution Because direct substitution yields the indeterminate form $\infty - \infty$, you should try to rewrite the expression to produce a form to which you can apply L'Hôpital's Rule. In this case, you can combine the two fractions to obtain

$$\lim_{x\to1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to1^+}\left[\frac{x-1-\ln x}{(x-1)\ln x}\right].$$

Now, because direct substitution produces the indeterminate form $0/0$, you can apply L'Hôpital's Rule to obtain

$$\begin{aligned}\lim_{x\to1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) &= \lim_{x\to1^+}\frac{\frac{d}{dx}[x-1-\ln x]}{\frac{d}{dx}[(x-1)\ln x]}\\ &= \lim_{x\to1^+}\left[\frac{1-(1/x)}{(x-1)(1/x)+\ln x}\right]\\ &= \lim_{x\to1^+}\left(\frac{x-1}{x-1+x\ln x}\right).\end{aligned}$$

This limit also yields the indeterminate form $0/0$, so you can apply L'Hôpital's Rule again to obtain

$$\begin{aligned}\lim_{x\to1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) &= \lim_{x\to1^+}\left[\frac{1}{1+x(1/x)+\ln x}\right]\\ &= \frac{1}{2}.\end{aligned}$$

■

The forms $0/0$, ∞/∞, $\infty - \infty$, $0\cdot\infty$, 0^0, 1^∞, and ∞^0 have been identified as *indeterminate*. There are similar forms that you should recognize as "determinate."

$\infty + \infty \to \infty$ Limit is positive infinity.

$-\infty - \infty \to -\infty$ Limit is negative infinity.

$0^\infty \to 0$ Limit is zero.

$0^{-\infty} \to \infty$ Limit is positive infinity.

(You are asked to verify two of these in Exercises 116 and 117.)

As a final comment, remember that L'Hôpital's Rule can be applied only to quotients leading to the indeterminate forms $0/0$ and ∞/∞. For instance, the following application of L'Hôpital's Rule is *incorrect*.

$$\lim_{x\to0}\frac{e^x}{x} \boxed{=} \lim_{x\to0}\frac{e^x}{1} = 1$$

Incorrect use of L'Hôpital's Rule

The reason this application is incorrect is that, even though the limit of the denominator is 0, the limit of the numerator is 1, which means that the hypotheses of L'Hôpital's Rule have not been satisfied.

8.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Numerical and Graphical Analysis **In Exercises 1–4, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to support your result.**

1. $\displaystyle\lim_{x\to 0}\frac{\sin 4x}{\sin 3x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

2. $\displaystyle\lim_{x\to 0}\frac{1-e^x}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

3. $\displaystyle\lim_{x\to\infty} x^5 e^{-x/100}$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$						

4. $\displaystyle\lim_{x\to\infty}\frac{6x}{\sqrt{3x^2-2x}}$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$						

In Exercises 5–10, evaluate the limit (a) using techniques from Chapters 1 and 3 and (b) using L'Hôpital's Rule.

5. $\displaystyle\lim_{x\to 4}\frac{3(x-4)}{x^2-16}$

6. $\displaystyle\lim_{x\to -2}\frac{2x^2+x-6}{x+2}$

7. $\displaystyle\lim_{x\to 6}\frac{\sqrt{x+10}-4}{x-6}$

8. $\displaystyle\lim_{x\to 0}\frac{\sin 6x}{4x}$

9. $\displaystyle\lim_{x\to\infty}\frac{5x^2-3x+1}{3x^2-5}$

10. $\displaystyle\lim_{x\to\infty}\frac{2x+1}{4x^2+x}$

In Exercises 11–44, evaluate the limit, using L'Hôpital's Rule if necessary. (In Exercise 18, *n* is a positive integer.)

11. $\displaystyle\lim_{x\to 3}\frac{x^2-2x-3}{x-3}$

12. $\displaystyle\lim_{x\to -1}\frac{x^2-2x-3}{x+1}$

13. $\displaystyle\lim_{x\to 0}\frac{\sqrt{25-x^2}-5}{x}$

14. $\displaystyle\lim_{x\to 5^-}\frac{\sqrt{25-x^2}}{x-5}$

15. $\displaystyle\lim_{x\to 0}\frac{e^x-(1-x)}{x}$

16. $\displaystyle\lim_{x\to 1}\frac{\ln x^2}{x^2-1}$

17. $\displaystyle\lim_{x\to 0^+}\frac{e^x-(1+x)}{x^3}$

18. $\displaystyle\lim_{x\to 0^+}\frac{e^x-(1+x)}{x^n}$

19. $\displaystyle\lim_{x\to 1}\frac{x^{11}-1}{x^4-1}$

20. $\displaystyle\lim_{x\to 1}\frac{x^a-1}{x^b-1}$, where $a, b \neq 0$

21. $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}$

22. $\displaystyle\lim_{x\to 0}\frac{\sin ax}{\sin bx}$, where $a, b \neq 0$

23. $\displaystyle\lim_{x\to 0}\frac{\arcsin x}{x}$

24. $\displaystyle\lim_{x\to 1}\frac{\arctan x-(\pi/4)}{x-1}$

25. $\displaystyle\lim_{x\to\infty}\frac{5x^2+3x-1}{4x^2+5}$

26. $\displaystyle\lim_{x\to\infty}\frac{x-6}{x^2+4x+7}$

27. $\displaystyle\lim_{x\to\infty}\frac{x^2+4x+7}{x-6}$

28. $\displaystyle\lim_{x\to\infty}\frac{x^3}{x+2}$

29. $\displaystyle\lim_{x\to\infty}\frac{x^3}{e^{x/2}}$

30. $\displaystyle\lim_{x\to\infty}\frac{x^3}{e^{x^2}}$

31. $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}$

32. $\displaystyle\lim_{x\to\infty}\frac{x^2}{\sqrt{x^2+1}}$

33. $\displaystyle\lim_{x\to\infty}\frac{\cos x}{x}$

34. $\displaystyle\lim_{x\to\infty}\frac{\sin x}{x-\pi}$

35. $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x^2}$

36. $\displaystyle\lim_{x\to\infty}\frac{\ln x^4}{x^3}$

37. $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^4}$

38. $\displaystyle\lim_{x\to\infty}\frac{e^{x/2}}{x}$

39. $\displaystyle\lim_{x\to 0}\frac{\sin 5x}{\tan 9x}$

40. $\displaystyle\lim_{x\to 1}\frac{\ln x}{\sin \pi x}$

41. $\displaystyle\lim_{x\to 0}\frac{\arctan x}{\sin x}$

42. $\displaystyle\lim_{x\to 0}\frac{x}{\arctan 2x}$

43. $\displaystyle\lim_{x\to\infty}\frac{\int_1^x \ln(e^{4t-1})\,dt}{x}$

44. $\displaystyle\lim_{x\to 1^+}\frac{\int_1^x \cos\theta\,d\theta}{x-1}$

In Exercises 45–62, (a) describe the type of indeterminate form (if any) that is obtained by direct substitution. (b) Evaluate the limit, using L'Hôpital's Rule if necessary. (c) Use a graphing utility to graph the function and verify the result in part (b).

45. $\displaystyle\lim_{x\to\infty} x\ln x$

46. $\displaystyle\lim_{x\to 0^+} x^3\cot x$

47. $\displaystyle\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$

48. $\displaystyle\lim_{x\to\infty} x\tan\frac{1}{x}$

49. $\displaystyle\lim_{x\to 0^+} x^{1/x}$

50. $\displaystyle\lim_{x\to 0^+}(e^x+x)^{2/x}$

51. $\displaystyle\lim_{x\to\infty} x^{1/x}$

52. $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

53. $\displaystyle\lim_{x\to 0^+}(1+x)^{1/x}$

54. $\displaystyle\lim_{x\to\infty}(1+x)^{1/x}$

55. $\displaystyle\lim_{x\to 0^+}\left[3(x)^{x/2}\right]$

56. $\displaystyle\lim_{x\to 4^+}\left[3(x-4)\right]^{x-4}$

57. $\displaystyle\lim_{x\to 1^+}(\ln x)^{x-1}$

58. $\displaystyle\lim_{x\to 0^+}\left[\cos\left(\frac{\pi}{2}-x\right)\right]^x$

59. $\displaystyle\lim_{x\to 2^+}\left(\frac{8}{x^2-4}-\frac{x}{x-2}\right)$

60. $\displaystyle\lim_{x\to 2^+}\left(\frac{1}{x^2-4}-\frac{\sqrt{x-1}}{x^2-4}\right)$

61. $\displaystyle\lim_{x\to 1^+}\left(\frac{3}{\ln x}-\frac{2}{x-1}\right)$

62. $\displaystyle\lim_{x\to 0^+}\left(\frac{10}{x}-\frac{3}{x^2}\right)$

In Exercises 63–66, use a graphing utility to (a) graph the function and (b) find the required limit (if it exists).

63. $\displaystyle\lim_{x\to3}\frac{x-3}{\ln(2x-5)}$

64. $\displaystyle\lim_{x\to0^+}(\sin x)^x$

65. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+5x+2}-x\right)$

66. $\displaystyle\lim_{x\to\infty}\frac{x^3}{e^{2x}}$

WRITING ABOUT CONCEPTS

67. List six different indeterminate forms.

68. State L'Hôpital's Rule.

69. Find differentiable functions f and g that satisfy the specified condition such that $\displaystyle\lim_{x\to5}f(x)=0$ and $\displaystyle\lim_{x\to5}g(x)=0$.

Explain how you obtained your answers. (*Note:* There are many correct answers.)

(a) $\displaystyle\lim_{x\to5}\frac{f(x)}{g(x)}=10$

(b) $\displaystyle\lim_{x\to5}\frac{f(x)}{g(x)}=0$

(c) $\displaystyle\lim_{x\to5}\frac{f(x)}{g(x)}=\infty$

70. Find differentiable functions f and g such that

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}g(x)=\infty \quad\text{and}$$

$$\lim_{x\to\infty}[f(x)-g(x)]=25.$$

Explain how you obtained your answers. (*Note:* There are many correct answers.)

71. ***Numerical Approach*** Complete the table to show that x eventually "overpowers" $(\ln x)^4$.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\dfrac{(\ln x)^4}{x}$						

72. ***Numerical Approach*** Complete the table to show that e^x eventually "overpowers" x^5.

x	1	5	10	20	30	40	50	100
$\dfrac{e^x}{x^5}$								

Comparing Functions **In Exercises 73–78, use L'Hôpital's Rule to determine the comparative rates of increase of the functions $f(x)=x^m$, $g(x)=e^{nx}$, and $h(x)=(\ln x)^n$, where $n>0$, $m>0$, and $x\to\infty$.**

73. $\displaystyle\lim_{x\to\infty}\frac{x^2}{e^{5x}}$

74. $\displaystyle\lim_{x\to\infty}\frac{x^3}{e^{2x}}$

75. $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^3}{x}$

76. $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^2}{x^3}$

77. $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^n}{x^m}$

78. $\displaystyle\lim_{x\to\infty}\frac{x^m}{e^{nx}}$

In Exercises 79–82, find any asymptotes and relative extrema that may exist and use a graphing utility to graph the function. (*Hint:* Some of the limits required in finding asymptotes have been found in previous exercises.)

79. $y=x^{1/x}, \quad x>0$

80. $y=x^x, \quad x>0$

81. $y=2xe^{-x}$

82. $y=\dfrac{\ln x}{x}$

Think About It **In Exercises 83–87, L'Hôpital's Rule is used incorrectly. Describe the error.**

83. $$\xcancel{\lim_{x\to2}\frac{3x^2+4x+1}{x^2-x-2}=\lim_{x\to2}\frac{6x+4}{2x-1}=\lim_{x\to2}\frac{6}{2}=3}$$

84. $$\xcancel{\lim_{x\to0}\frac{e^{2x}-1}{e^x}=\lim_{x\to0}\frac{2e^{2x}}{e^x}=\lim_{x\to0}2e^x=2}$$

85. $$\xcancel{\lim_{x\to0}\frac{\sin\pi x-1}{x}=\lim_{x\to0}\frac{\pi\cos\pi x}{1}=\pi}$$

86. $$\xcancel{\begin{aligned}\lim_{x\to\infty}x\cos\frac{1}{x}&=\lim_{x\to\infty}\frac{\cos(1/x)}{1/x}\\&=\lim_{x\to\infty}\frac{[-\sin(1/x)](1/x^2)}{-1/x^2}\\&=0\end{aligned}}$$

87. $$\xcancel{\begin{aligned}\lim_{x\to\infty}\frac{e^{-x}}{1+e^{-x}}&=\lim_{x\to\infty}\frac{-e^{-x}}{-e^{-x}}\\&=\lim_{x\to\infty}\\&=1\end{aligned}}$$

CAPSTONE

88. Determine which of the following limits can be evaluated using L'Hôpital's Rule. Explain your reasoning. Do not evaluate the limit.

(a) $\displaystyle\lim_{x\to2}\frac{x-2}{x^3-x-6}$

(b) $\displaystyle\lim_{x\to0}\frac{x^2-4x}{2x-1}$

(c) $\displaystyle\lim_{x\to\infty}\frac{x^3}{e^x}$

(d) $\displaystyle\lim_{x\to3}\frac{e^{x^2}-e^9}{x-3}$

(e) $\displaystyle\lim_{x\to1}\frac{\cos\pi x}{\ln x}$

(f) $\displaystyle\lim_{x\to1}\frac{1+x(\ln x-1)}{\ln x(x-1)}$

Analytical Approach **In Exercises 89 and 90, (a) explain why L'Hôpital's Rule cannot be used to find the limit, (b) find the limit analytically, and (c) use a graphing utility to graph the function and approximate the limit from the graph. Compare the result with that in part (b).**

89. $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}$

90. $\displaystyle\lim_{x\to\pi/2^-}\frac{\tan x}{\sec x}$

Graphical Analysis **In Exercises 91 and 92, graph $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x=0$. What do you notice about these ratios as $x\to0$? How does this illustrate L'Hôpital's Rule?**

91. $f(x)=\sin3x, \quad g(x)=\sin4x$

92. $f(x)=e^{3x}-1, \quad g(x)=x$

93. ***Velocity in a Resisting Medium*** The velocity v of an object falling through a resisting medium such as air or water is given by

$$v = \frac{32}{k}\left(1 - e^{-kt} + \frac{v_0 ke^{-kt}}{32}\right)$$

where v_0 is the initial velocity, t is the time in seconds, and k is the resistance constant of the medium. Use L'Hôpital's Rule to find the formula for the velocity of a falling body in a vacuum by fixing v_0 and t and letting k approach zero. (Assume that the downward direction is positive.)

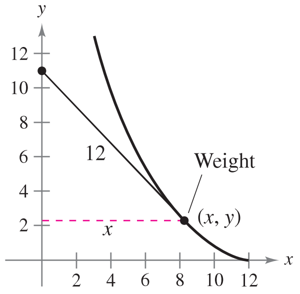
94. ***Compound Interest*** The formula for the amount A in a savings account compounded n times per year for t years at an interest rate r and an initial deposit of P is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Use L'Hôpital's Rule to show that the limiting formula as the number of compoundings per year approaches infinity is given by $A = Pe^{rt}$.

95. ***The Gamma Function*** The Gamma Function $\Gamma(n)$ is defined in terms of the integral of the function given by $f(x) = x^{n-1}e^{-x}$, $n > 0$. Show that for any fixed value of n, the limit of $f(x)$ as x approaches infinity is zero.

96. ***Tractrix*** A person moves from the origin along the positive y-axis pulling a weight at the end of a 12-meter rope (see figure). Initially, the weight is located at the point $(12, 0)$.



(a) Show that the slope of the tangent line of the path of the weight is

$$\frac{dy}{dx} = -\frac{\sqrt{144 - x^2}}{x}.$$

(b) Use the result of part (a) to find the equation of the path of the weight. Use a graphing utility to graph the path and compare it with the figure.

(c) Find any vertical asymptotes of the graph in part (b).

(d) When the person has reached the point $(0, 12)$, how far has the weight moved?

In Exercises 97–100, apply the Extended Mean Value Theorem to the functions f and g on the given interval. Find all values c in the interval (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

	Functions	*Interval*
97.	$f(x) = x^3$, $g(x) = x^2 + 1$	$[0, 1]$
98.	$f(x) = \frac{1}{x}$, $g(x) = x^2 - 4$	$[1, 2]$
99.	$f(x) = \sin x$, $g(x) = \cos x$	$\left[0, \frac{\pi}{2}\right]$
100.	$f(x) = \ln x$, $g(x) = x^3$	$[1, 4]$

True or False? **In Exercises 101–104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

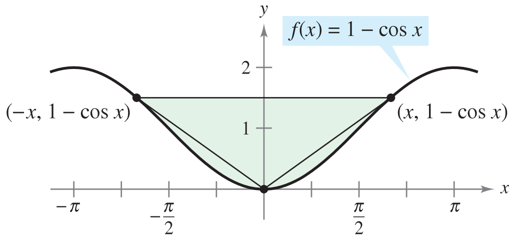
101. $\lim_{x\to 0}\left[\frac{x^2 + x + 1}{x}\right] = \lim_{x\to 0}\left[\frac{2x + 1}{1}\right] = 1$

102. If $y = e^x/x^2$, then $y' = e^x/2x$.

103. If $p(x)$ is a polynomial, then $\lim_{x\to\infty}[p(x)/e^x] = 0$.

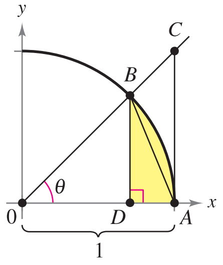
104. If $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$, then $\lim_{x\to\infty}[f(x) - g(x)] = 0$.

105. ***Area*** Find the limit, as x approaches 0, of the ratio of the area of the triangle to the total shaded area in the figure.



106. In Section 1.3, a geometric argument (see figure) was used to prove that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$



(a) Write the area of $\triangle ABD$ in terms of θ.

(b) Write the area of the shaded region in terms of θ.

(c) Write the ratio R of the area of $\triangle ABD$ to that of the shaded region.

(d) Find $\lim_{\theta\to 0} R$.

Continuous Functions **In Exercises 107 and 108, find the value of c that makes the function continuous at $x = 0$.**

107. $f(x) = \begin{cases} \dfrac{4x - 2\sin 2x}{2x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$

108. $f(x) = \begin{cases} (e^x + x)^{1/x}, & x \neq 0 \\ c, & x = 0 \end{cases}$

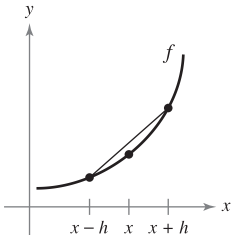
109. Find the values of a and b such that $\displaystyle\lim_{x\to 0} \frac{a - \cos bx}{x^2} = 2$.

110. Show that $\displaystyle\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for any integer $n > 0$.

111. (a) Let $f'(x)$ be continuous. Show that

$$\lim_{h\to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x).$$

(b) Explain the result of part (a) graphically.



112. Let $f''(x)$ be continuous. Show that

$$\lim_{h\to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x).$$

113. Sketch the graph of

$$g(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and determine $g'(0)$.

114. Use a graphing utility to graph

$$f(x) = \frac{x^k - 1}{k}$$

for $k = 1, 0.1$, and 0.01. Then evaluate the limit

$$\lim_{k\to 0^+} \frac{x^k - 1}{k}.$$

115. Consider the limit $\displaystyle\lim_{x\to 0^+} (-x \ln x)$.

(a) Describe the type of indeterminate form that is obtained by direct substitution.

(b) Evaluate the limit. Use a graphing utility to verify the result.

FOR FURTHER INFORMATION For a geometric approach to this exercise, see the article "A Geometric Proof of $\displaystyle\lim_{d\to 0^+} (-d \ln d) = 0$" by John H. Mathews in the *College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

116. Prove that if $f(x) \geq 0$, $\displaystyle\lim_{x\to a} f(x) = 0$, and $\displaystyle\lim_{x\to a} g(x) = \infty$, then $\displaystyle\lim_{x\to a} f(x)^{g(x)} = 0$.

117. Prove that if $f(x) \geq 0$, $\displaystyle\lim_{x\to a} f(x) = 0$, and $\displaystyle\lim_{x\to a} g(x) = -\infty$, then $\displaystyle\lim_{x\to a} f(x)^{g(x)} = \infty$.

118. Prove the following generalization of the Mean Value Theorem. If f is twice differentiable on the closed interval $[a, b]$, then

$$f(b) - f(a) = f'(a)(b - a) - \int_a^b f''(t)(t - b)\, dt.$$

119. ***Indeterminate Forms*** Show that the indeterminate forms 0^0, ∞^0, and 1^∞ do not always have a value of 1 by evaluating each limit.

(a) $\displaystyle\lim_{x\to 0^+} x^{\ln 2/(1 + \ln x)}$

(b) $\displaystyle\lim_{x\to\infty} x^{\ln 2/(1 + \ln x)}$

(c) $\displaystyle\lim_{x\to 0} (x + 1)^{(\ln 2)/x}$

120. ***Calculus History*** In L'Hôpital's 1696 calculus textbook, he illustrated his rule using the limit of the function

$$f(x) = \frac{\sqrt{2a^3 x - x^4} - a\sqrt[3]{a^2 x}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, $a > 0$. Find this limit.

121. Consider the function

$$h(x) = \frac{x + \sin x}{x}.$$

(a) Use a graphing utility to graph the function. Then use the *zoom* and *trace* features to investigate $\displaystyle\lim_{x\to\infty} h(x)$.

(b) Find $\displaystyle\lim_{x\to\infty} h(x)$ analytically by writing

$$h(x) = \frac{x}{x} + \frac{\sin x}{x}.$$

(c) Can you use L'Hôpital's Rule to find $\displaystyle\lim_{x\to\infty} h(x)$? Explain your reasoning.

122. Let $f(x) = x + x \sin x$ and $g(x) = x^2 - 4$.

(a) Show that $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

(b) Show that $\displaystyle\lim_{x\to\infty} f(x) = \infty$ and $\displaystyle\lim_{x\to\infty} g(x) = \infty$.

(c) Evaluate the limit $\displaystyle\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$. What do you notice?

(d) Do your answers to parts (a) through (c) contradict L'Hôpital's Rule? Explain your reasoning.

PUTNAM EXAM CHALLENGE

123. Evaluate $\displaystyle\lim_{x\to\infty} \left[\frac{1}{x} \cdot \frac{a^x - 1}{a - 1}\right]^{1/x}$ where $a > 0$, $a \neq 1$.

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8.8 Improper Integrals

- **Evaluate an improper integral that has an infinite limit of integration.**
- **Evaluate an improper integral that has an infinite discontinuity.**

Improper Integrals with Infinite Limits of Integration

The definition of a definite integral

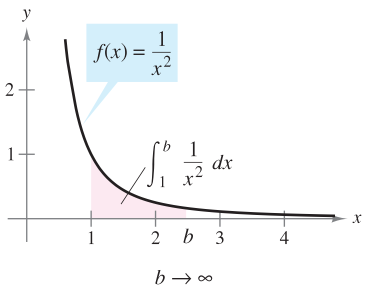
$$\int_a^b f(x)\,dx$$

requires that the interval $[a, b]$ be finite. Furthermore, the Fundamental Theorem of Calculus, by which you have been evaluating definite integrals, requires that f be continuous on $[a, b]$. In this section you will study a procedure for evaluating integrals that do not satisfy these requirements—usually because either one or both of the limits of integration are infinite, or f has a finite number of infinite discontinuities in the interval $[a, b]$. Integrals that possess either property are **improper integrals.** Note that a function f is said to have an **infinite discontinuity** at c if, *from the right or left,*

$$\lim_{x\to c} f(x) = \infty \qquad \text{or} \qquad \lim_{x\to c} f(x) = -\infty.$$

To get an idea of how to evaluate an improper integral, consider the integral

$$\int_1^b \frac{dx}{x^2} = -\frac{1}{x}\bigg]_1^b = -\frac{1}{b} + 1 = 1 - \frac{1}{b}$$



The unbounded region has an area of 1.
Figure 8.17

which can be interpreted as the area of the shaded region shown in Figure 8.17. Taking the limit as $b \to \infty$ produces

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\left(\int_1^b \frac{dx}{x^2}\right) = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1.$$

This improper integral can be interpreted as the area of the *unbounded* region between the graph of $f(x) = 1/x^2$ and the x-axis (to the right of $x = 1$).

DEFINITION OF IMPROPER INTEGRALS WITH INFINITE INTEGRATION LIMITS

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

where c is any real number (see Exercise 120).

In the first two cases, the improper integral **converges** if the limit exists—otherwise, the improper integral **diverges.** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

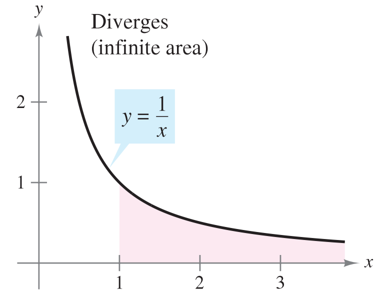
EXAMPLE 1 An Improper Integral That Diverges

Evaluate $\displaystyle\int_1^{\infty} \frac{dx}{x}$.

Solution

$$\begin{aligned}
\int_1^{\infty} \frac{dx}{x} &= \lim_{b\to\infty} \int_1^{b} \frac{dx}{x} && \text{Take limit as } b \to \infty. \\
&= \lim_{b\to\infty} \Big[\ln x\Big]_1^{b} && \text{Apply Log Rule.} \\
&= \lim_{b\to\infty} (\ln b - 0) && \text{Apply Fundamental Theorem of Calculus.} \\
&= \infty && \text{Evaluate limit.}
\end{aligned}$$

See Figure 8.18.



This unbounded region has an infinite area.
Figure 8.18

NOTE Try comparing the regions shown in Figures 8.17 and 8.18. They look similar, yet the region in Figure 8.17 has a finite area of 1 and the region in Figure 8.18 has an infinite area.

EXAMPLE 2 Improper Integrals That Converge

Evaluate each improper integral.

a. $\displaystyle\int_0^{\infty} e^{-x}\,dx$ **b.** $\displaystyle\int_0^{\infty} \frac{1}{x^2+1}\,dx$

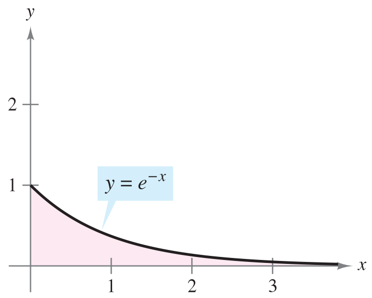
Solution

a.
$$\begin{aligned}
\int_0^{\infty} e^{-x}\,dx &= \lim_{b\to\infty} \int_0^{b} e^{-x}\,dx \\
&= \lim_{b\to\infty} \Big[-e^{-x}\Big]_0^{b} \\
&= \lim_{b\to\infty} (-e^{-b} + 1) \\
&= 1
\end{aligned}$$

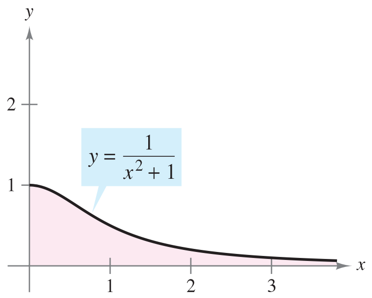
See Figure 8.19.

b.
$$\begin{aligned}
\int_0^{\infty} \frac{1}{x^2+1}\,dx &= \lim_{b\to\infty} \int_0^{b} \frac{1}{x^2+1}\,dx \\
&= \lim_{b\to\infty} \Big[\arctan x\Big]_0^{b} \\
&= \lim_{b\to\infty} \arctan b \\
&= \frac{\pi}{2}
\end{aligned}$$

See Figure 8.20.



The area of the unbounded region is 1.
Figure 8.19



The area of the unbounded region is $\pi/2$.
Figure 8.20

In the following example, note how L'Hôpital's Rule can be used to evaluate an improper integral.

EXAMPLE 3 Using L'Hôpital's Rule with an Improper Integral

Evaluate $\displaystyle\int_1^{\infty} (1 - x)e^{-x}\,dx$.

Solution Use integration by parts, with $dv = e^{-x}\,dx$ and $u = (1 - x)$.

$$\begin{aligned}\int (1 - x)e^{-x}\,dx &= -e^{-x}(1 - x) - \int e^{-x}\,dx\\ &= -e^{-x} + xe^{-x} + e^{-x} + C\\ &= xe^{-x} + C\end{aligned}$$

Now, apply the definition of an improper integral.

$$\begin{aligned}\int_1^{\infty} (1 - x)e^{-x}\,dx &= \lim_{b\to\infty}\Big[xe^{-x}\Big]_1^b\\ &= \left(\lim_{b\to\infty}\frac{b}{e^b}\right) - \frac{1}{e}\end{aligned}$$

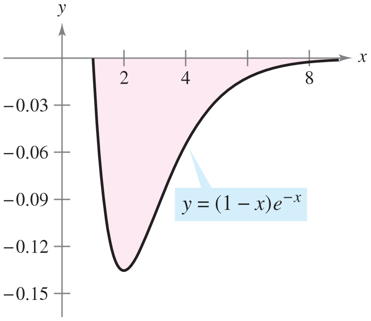
Finally, using L'Hôpital's Rule on the right-hand limit produces

$$\lim_{b\to\infty}\frac{b}{e^b} = \lim_{b\to\infty}\frac{1}{e^b} = 0$$

from which you can conclude that

$$\int_1^{\infty} (1 - x)e^{-x}\,dx = -\frac{1}{e}.$$

See Figure 8.21.



The area of the unbounded region is $|-1/e|$.
Figure 8.21

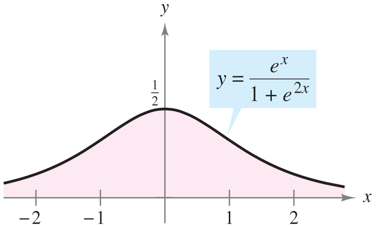
EXAMPLE 4 Infinite Upper and Lower Limits of Integration

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{1 + e^{2x}}\,dx$.

Solution Note that the integrand is continuous on $(-\infty, \infty)$. To evaluate the integral, you can break it into two parts, choosing $c = 0$ as a convenient value.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^x}{1 + e^{2x}}\,dx &= \int_{-\infty}^{0} \frac{e^x}{1 + e^{2x}}\,dx + \int_{0}^{\infty} \frac{e^x}{1 + e^{2x}}\,dx\\ &= \lim_{b\to-\infty}\Big[\arctan e^x\Big]_b^0 + \lim_{b\to\infty}\Big[\arctan e^x\Big]_0^b\\ &= \lim_{b\to-\infty}\left(\frac{\pi}{4} - \arctan e^b\right) + \lim_{b\to\infty}\left(\arctan e^b - \frac{\pi}{4}\right)\\ &= \frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4}\\ &= \frac{\pi}{2}\end{aligned}$$

See Figure 8.22. ■



The area of the unbounded region is $\pi/2$.
Figure 8.22

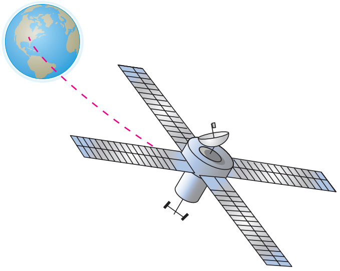
EXAMPLE 5 Sending a Space Module into Orbit

In Example 3 in Section 7.5, you found that it would require 10,000 mile-tons of work to propel a 15-metric-ton space module to a height of 800 miles above Earth. How much work is required to propel the module an unlimited distance away from Earth's surface?

Solution At first you might think that an infinite amount of work would be required. But if this were the case, it would be impossible to send rockets into outer space. Because this has been done, the work required must be finite. You can determine the work in the following manner. Using the integral in Example 3, Section 7.5, replace the upper bound of 4800 miles by ∞ and write

$$\begin{aligned} W &= \int_{4000}^{\infty} \frac{240{,}000{,}000}{x^2}\,dx \\ &= \lim_{b\to\infty}\left[-\frac{240{,}000{,}000}{x}\right]_{4000}^{b} \\ &= \lim_{b\to\infty}\left(-\frac{240{,}000{,}000}{b} + \frac{240{,}000{,}000}{4000}\right) \\ &= 60{,}000 \text{ mile-tons} \\ &\approx 6.984 \times 10^{11} \text{ foot-pounds.} \end{aligned}$$

See Figure 8.23. ■



The work required to move a space module an unlimited distance away from Earth is approximately 6.984×10^{11} foot-pounds.
Figure 8.23

Improper Integrals with Infinite Discontinuities

The second basic type of improper integral is one that has an infinite discontinuity *at or between* the limits of integration.

DEFINITION OF IMPROPER INTEGRALS WITH INFINITE DISCONTINUITIES

1. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$

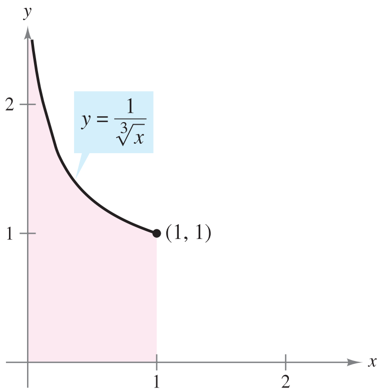
2. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx.$$

3. If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In the first two cases, the improper integral **converges** if the limit exists—otherwise, the improper integral **diverges.** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.



Infinite discontinuity at $x = 0$
Figure 8.24

EXAMPLE 6 An Improper Integral with an Infinite Discontinuity

Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$.

Solution The integrand has an infinite discontinuity at $x = 0$, as shown in Figure 8.24. You can evaluate this integral as shown below.

$$\begin{aligned}\int_0^1 x^{-1/3}\,dx &= \lim_{b\to 0^+}\left[\frac{x^{2/3}}{2/3}\right]_b^1\\ &= \lim_{b\to 0^+}\frac{3}{2}(1 - b^{2/3})\\ &= \frac{3}{2}\end{aligned}$$

EXAMPLE 7 An Improper Integral That Diverges

Evaluate $\displaystyle\int_0^2 \frac{dx}{x^3}$.

Solution Because the integrand has an infinite discontinuity at $x = 0$, you can write

$$\begin{aligned}\int_0^2 \frac{dx}{x^3} &= \lim_{b\to 0^+}\left[-\frac{1}{2x^2}\right]_b^2\\ &= \lim_{b\to 0^+}\left(-\frac{1}{8} + \frac{1}{2b^2}\right)\\ &= \infty.\end{aligned}$$

So, you can conclude that the improper integral diverges.

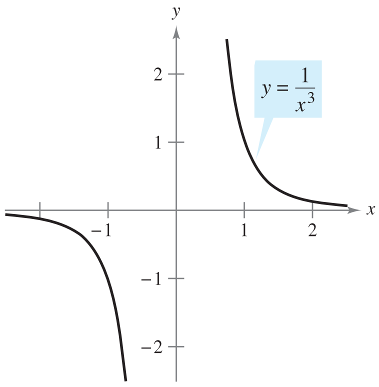
EXAMPLE 8 An Improper Integral with an Interior Discontinuity

Evaluate $\displaystyle\int_{-1}^2 \frac{dx}{x^3}$.

Solution This integral is improper because the integrand has an infinite discontinuity at the interior point $x = 0$, as shown in Figure 8.25. So, you can write

$$\int_{-1}^2 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^2 \frac{dx}{x^3}.$$

From Example 7 you know that the second integral diverges. So, the original improper integral also diverges. ■



The improper integral $\displaystyle\int_{-1}^2 1/x^3\,dx$ diverges.
Figure 8.25

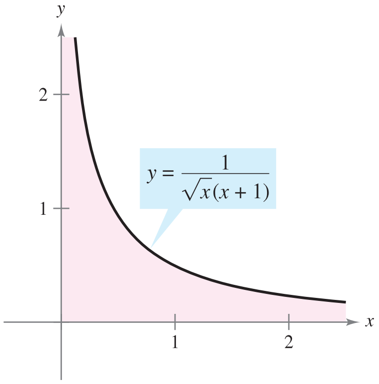
NOTE Remember to check for infinite discontinuities at interior points as well as at endpoints when determining whether an integral is improper. For instance, if you had not recognized that the integral in Example 8 was improper, you would have obtained the *incorrect* result

$$\int_{-1}^2 \frac{dx}{x^3} = \left[\frac{-1}{2x^2}\right]_{-1}^2 = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}.$$

Incorrect evaluation ■

The integral in the next example is improper for *two* reasons. One limit of integration is infinite, and the integrand has an infinite discontinuity at the outer limit of integration.

EXAMPLE 9 A Doubly Improper Integral



The area of the unbounded region is π.
Figure 8.26

Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$.

Solution To evaluate this integral, split it at a convenient point (say, $x = 1$) and write

$$\begin{aligned}\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)} &= \int_0^{1} \frac{dx}{\sqrt{x}(x+1)} + \int_1^{\infty} \frac{dx}{\sqrt{x}(x+1)} \\ &= \lim_{b\to 0^+}\Big[2 \arctan \sqrt{x}\Big]_b^1 + \lim_{c\to\infty}\Big[2 \arctan \sqrt{x}\Big]_1^c \\ &= 2\left(\frac{\pi}{4}\right) - 0 + 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right) \\ &= \pi.\end{aligned}$$

See Figure 8.26.

EXAMPLE 10 An Application Involving Arc Length

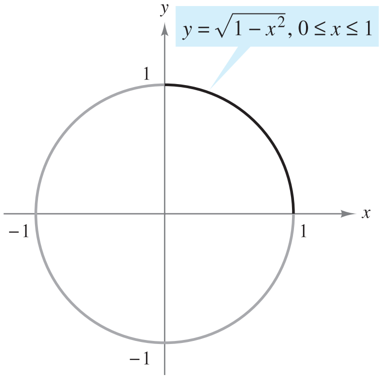
Use the formula for arc length to show that the circumference of the circle $x^2 + y^2 = 1$ is 2π.

Solution To simplify the work, consider the quarter circle given by $y = \sqrt{1 - x^2}$, where $0 \le x \le 1$. The function y is differentiable for any x in this interval except $x = 1$. Therefore, the arc length of the quarter circle is given by the improper integral

$$\begin{aligned} s &= \int_0^1 \sqrt{1 + (y')^2}\, dx \\ &= \int_0^1 \sqrt{1 + \left(\frac{-x}{\sqrt{1 - x^2}}\right)^2}\, dx \\ &= \int_0^1 \frac{dx}{\sqrt{1 - x^2}}.\end{aligned}$$

This integral is improper because it has an infinite discontinuity at $x = 1$. So, you can write

$$\begin{aligned} s &= \int_0^1 \frac{dx}{\sqrt{1 - x^2}} \\ &= \lim_{b\to 1^-}\Big[\arcsin x\Big]_0^b \\ &= \frac{\pi}{2} - 0 \\ &= \frac{\pi}{2}.\end{aligned}$$



The circumference of the circle is 2π.
Figure 8.27

Finally, multiplying by 4, you can conclude that the circumference of the circle is $4s = 2\pi$, as shown in Figure 8.27. ■

This section concludes with a useful theorem describing the convergence or divergence of a common type of improper integral. The proof of this theorem is left as an exercise (see Exercise 55).

THEOREM 8.5 A SPECIAL TYPE OF IMPROPER INTEGRAL

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \le 1 \end{cases}$$

EXAMPLE 11 An Application Involving A Solid of Revolution

The solid formed by revolving (about the x-axis) the *unbounded* region lying between the graph of $f(x) = 1/x$ and the x-axis ($x \ge 1$) is called **Gabriel's Horn.** (See Figure 8.28.) Show that this solid has a finite volume and an infinite surface area.

■ **FOR FURTHER INFORMATION** For further investigation of solids that have finite volumes and infinite surface areas, see the article "Supersolids: Solids Having Finite Volume and Infinite Surfaces" by William P. Love in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

Solution Using the disk method and Theorem 8.5, you can determine the volume to be

$$V = \pi\int_1^{\infty} \left(\frac{1}{x}\right)^2 dx \qquad \text{Theorem 8.5, } p = 2 > 1$$

$$= \pi\left(\frac{1}{2-1}\right) = \pi.$$

The surface area is given by

$$S = 2\pi\int_1^{\infty} f(x)\sqrt{1+[f'(x)]^2}\,dx = 2\pi\int_1^{\infty}\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx.$$

Because

$$\sqrt{1+\frac{1}{x^4}} > 1$$

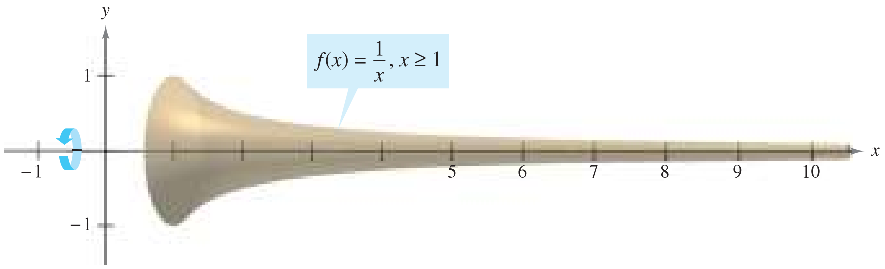
on the interval $[1, \infty)$, and the improper integral

$$\int_1^{\infty}\frac{1}{x}\,dx$$

diverges, you can conclude that the improper integral

$$\int_1^{\infty}\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$$

also diverges. (See Exercise 58.) So, the surface area is infinite.



Gabriel's Horn has a finite volume and an infinite surface area.
Figure 8.28

■ **FOR FURTHER INFORMATION** To learn about another function that has a finite volume and an infinite surface area, see the article "Gabriel's Wedding Cake" by Julian F. Fleron in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

8.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, decide whether the integral is improper. Explain your reasoning.

1. $\displaystyle\int_0^1 \frac{dx}{5x-3}$

2. $\displaystyle\int_1^2 \frac{dx}{x^3}$

3. $\displaystyle\int_0^1 \frac{2x-5}{x^2-5x+6}\,dx$

4. $\displaystyle\int_1^\infty \ln(x^2)\,dx$

5. $\displaystyle\int_0^2 e^{-x}\,dx$

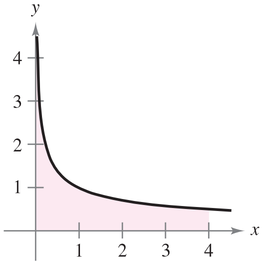
6. $\displaystyle\int_0^\infty \cos x\,dx$

7. $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{4+x^2}\,dx$

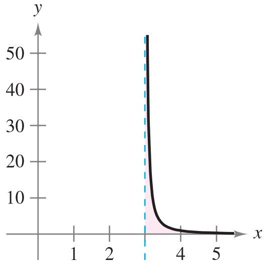
8. $\displaystyle\int_0^{\pi/4} \csc x\,dx$

In Exercises 9–14, explain why the integral is improper and determine whether it diverges or converges. Evaluate the integral if it converges.

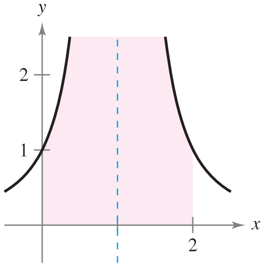
9. $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,dx$



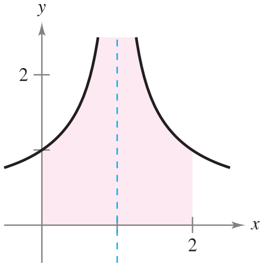
10. $\displaystyle\int_3^4 \frac{1}{(x-3)^{3/2}}\,dx$



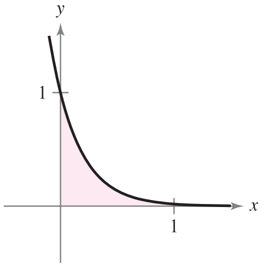
11. $\displaystyle\int_0^2 \frac{1}{(x-1)^2}\,dx$



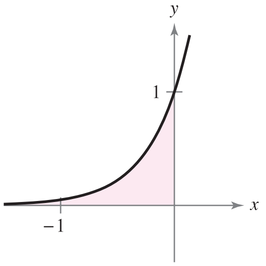
12. $\displaystyle\int_0^2 \frac{1}{(x-1)^{2/3}}\,dx$



13. $\displaystyle\int_0^\infty e^{-4x}\,dx$



14. $\displaystyle\int_{-\infty}^0 e^{3x}\,dx$



Writing **In Exercises 15–18, explain why the evaluation of the integral is *incorrect*. Use the integration capabilities of a graphing utility to attempt to evaluate the integral. Determine whether the utility gives the correct answer.**

15. $\xcancel{\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx = -2}$

16. $\xcancel{\displaystyle\int_{-2}^2 \frac{-2}{(x-1)^3}\,dx = \frac{8}{9}}$

17. $\xcancel{\displaystyle\int_0^\infty e^{-x}\,dx = 0}$

18. $\xcancel{\displaystyle\int_0^\pi \sec x\,dx = 0}$

In Exercises 19–36, determine whether the improper integral diverges or converges. Evaluate the integral if it converges.

19. $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

20. $\displaystyle\int_1^\infty \frac{3}{x^5}\,dx$

21. $\displaystyle\int_1^\infty \frac{3}{\sqrt[3]{x}}\,dx$

22. $\displaystyle\int_1^\infty \frac{4}{\sqrt[4]{x}}\,dx$

23. $\displaystyle\int_{-\infty}^0 xe^{-4x}\,dx$

24. $\displaystyle\int_0^\infty xe^{-x/4}\,dx$

25. $\displaystyle\int_0^\infty x^2e^{-x}\,dx$

26. $\displaystyle\int_0^\infty (x-1)e^{-x}\,dx$

27. $\displaystyle\int_0^\infty e^{-x}\cos x\,dx$

28. $\displaystyle\int_0^\infty e^{-ax}\sin bx\,dx, \quad a>0$

29. $\displaystyle\int_4^\infty \frac{1}{x(\ln x)^3}\,dx$

30. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

31. $\displaystyle\int_{-\infty}^\infty \frac{4}{16+x^2}\,dx$

32. $\displaystyle\int_0^\infty \frac{x^3}{(x^2+1)^2}\,dx$

33. $\displaystyle\int_0^\infty \frac{1}{e^x+e^{-x}}\,dx$

34. $\displaystyle\int_0^\infty \frac{e^x}{1+e^x}\,dx$

35. $\displaystyle\int_0^\infty \cos \pi x\,dx$

36. $\displaystyle\int_0^\infty \sin\frac{x}{2}\,dx$

In Exercises 37–54, determine whether the improper integral diverges or converges. Evaluate the integral if it converges, and check your results with the results obtained by using the integration capabilities of a graphing utility.

37. $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$

38. $\displaystyle\int_0^5 \frac{10}{x}\,dx$

39. $\displaystyle\int_0^8 \frac{1}{\sqrt[3]{8-x}}\,dx$

40. $\displaystyle\int_0^{12} \frac{9}{\sqrt{12-x}}\,dx$

41. $\displaystyle\int_0^1 x\ln x\,dx$

42. $\displaystyle\int_0^e \ln x^2\,dx$

43. $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta$

44. $\displaystyle\int_0^{\pi/2} \sec\theta\,d\theta$

45. $\displaystyle\int_2^4 \frac{2}{x\sqrt{x^2-4}}\,dx$

46. $\displaystyle\int_0^5 \frac{1}{\sqrt{25-x^2}}\,dx$

47. $\displaystyle\int_2^4 \frac{1}{\sqrt{x^2-4}}\,dx$

48. $\displaystyle\int_0^5 \frac{1}{25-x^2}\,dx$

49. $\displaystyle\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx$

50. $\displaystyle\int_1^3 \frac{2}{(x-2)^{8/3}}\,dx$

51. $\displaystyle\int_3^\infty \frac{1}{x\sqrt{x^2-9}}\,dx$ **52.** $\displaystyle\int_5^\infty \frac{1}{x\sqrt{x^2-25}}\,dx$

53. $\displaystyle\int_0^\infty \frac{4}{\sqrt{x}(x+6)}\,dx$ **54.** $\displaystyle\int_1^\infty \frac{1}{x\ln x}\,dx$

In Exercises 55 and 56, determine all values of *p* for which the improper integral converges.

55. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ **56.** $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

57. Use mathematical induction to verify that the following integral converges for any positive integer n.

$$\int_0^\infty x^n e^{-x}\,dx$$

58. ***Comparison Test for Improper Integrals*** In some cases, it is impossible to find the exact value of an improper integral but it is important to determine whether the integral converges or diverges. Suppose the functions f and g are continuous and $0 \le g(x) \le f(x)$ on the interval $[a, \infty)$. It can be shown that if $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ also converges, and if $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ also diverges. This is known as the Comparison Test for improper integrals.

(a) Use the Comparison Test to determine if $\int_1^\infty e^{-x^2}\,dx$ converges or diverges. (*Hint:* Use the fact that $e^{-x^2} \le e^{-x}$ for $x \ge 1$.)

(b) Use the Comparison Test to determine if $\int_1^\infty \frac{1}{x^5+1}\,dx$ converges or diverges. (*Hint:* Use the fact that $\frac{1}{x^5+1} \le \frac{1}{x^5}$ for $x \ge 1$.)

In Exercises 59–70, use the results of Exercises 55–58 to determine whether the improper integral converges or diverges.

59. $\displaystyle\int_0^1 \frac{1}{x^5}\,dx$ **60.** $\displaystyle\int_0^1 \frac{1}{\sqrt[5]{x}}\,dx$

61. $\displaystyle\int_1^\infty \frac{1}{x^5}\,dx$ **62.** $\displaystyle\int_0^\infty x^4e^{-x}\,dx$

63. $\displaystyle\int_1^\infty \frac{1}{x^2+5}\,dx$ **64.** $\displaystyle\int_2^\infty \frac{1}{\sqrt{x-1}}\,dx$

65. $\displaystyle\int_2^\infty \frac{1}{\sqrt[3]{x(x-1)}}\,dx$ **66.** $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}(x+1)}\,dx$

67. $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$ **68.** $\displaystyle\int_0^\infty \frac{1}{e^x+x}\,dx$

69. $\displaystyle\int_1^\infty \frac{1-\sin x}{x^2}\,dx$ **70.** $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}\ln x}\,dx$

WRITING ABOUT CONCEPTS

71. Describe the different types of improper integrals.

72. Define the terms *converges* and *diverges* when working with improper integrals.

WRITING ABOUT CONCEPTS (continued)

73. Explain why $\displaystyle\int_{-1}^1 \frac{1}{x^3}\,dx \ne 0$.

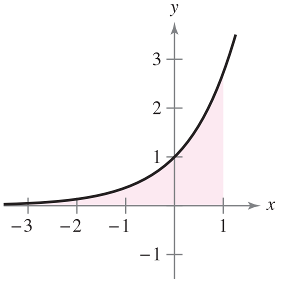
74. Consider the integral

$$\int_0^3 \frac{10}{x^2-2x}\,dx.$$

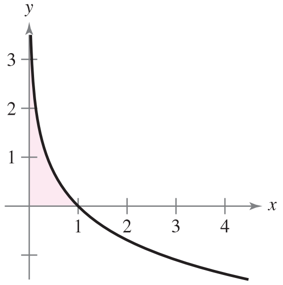
To determine the convergence or divergence of the integral, how many improper integrals must be analyzed? What must be true of each of these integrals if the given integral converges?

Area **In Exercises 75–78, find the area of the unbounded shaded region.**

75. $y = e^x, \quad -\infty < x \le 1$

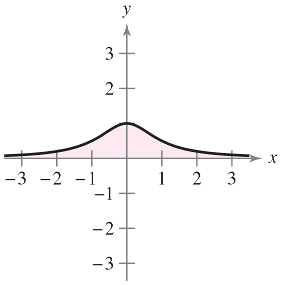


76. $y = -\ln x$



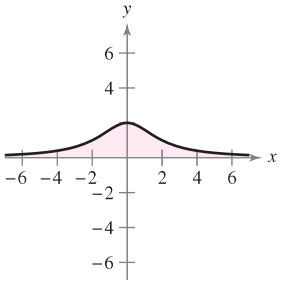
77. Witch of Agnesi:

$$y = \frac{1}{x^2+1}$$



78. Witch of Agnesi:

$$y = \frac{8}{x^2+4}$$



Area and Volume **In Exercises 79 and 80, consider the region satisfying the inequalities. (a) Find the area of the region. (b) Find the volume of the solid generated by revolving the region about the *x*-axis. (c) Find the volume of the solid generated by revolving the region about the *y*-axis.**

79. $y \le e^{-x}, \ y \ge 0, \ x \ge 0$ **80.** $y \le \frac{1}{x^2}, \ y \ge 0, \ x \ge 1$

81. ***Arc Length*** Sketch the graph of the hypocycloid of four cusps $x^{2/3} + y^{2/3} = 4$ and find its perimeter.

82. ***Arc Length*** Find the arc length of the graph of $y = \sqrt{16 - x^2}$ over the interval $[0, 4]$.

83. ***Surface Area*** The region bounded by $(x-2)^2 + y^2 = 1$ is revolved about the y-axis to form a torus. Find the surface area of the torus.

84. ***Surface Area*** Find the area of the surface formed by revolving the graph of $y = 2e^{-x}$ on the interval $[0, \infty)$ about the x-axis.

Propulsion **In Exercises 85 and 86, use the weight of the rocket to answer each question. (Use 4000 miles as the radius of Earth and do not consider the effect of air resistance.)**

(a) How much work is required to propel the rocket an unlimited distance away from Earth's surface?

(b) How far has the rocket traveled when half the total work has occurred?

85. 5-ton rocket

86. 10-ton rocket

Probability **A nonnegative function f is called a *probability density function* if**

$$\int_{-\infty}^{\infty} f(t)\,dt = 1.$$

The probability that x lies between a and b is given by

$$P(a \le x \le b) = \int_a^b f(t)\,dt.$$

The expected value of x is given by

$$E(x) = \int_{-\infty}^{\infty} t f(t)\,dt.$$

In Exercises 87 and 88, (a) show that the nonnegative function is a probability density function, (b) find $P(0 \le x \le 4)$, and (c) find $E(x)$.

87. $f(t) = \begin{cases} \frac{1}{7}e^{-t/7}, & t \ge 0 \\ 0, & t < 0 \end{cases}$

88. $f(t) = \begin{cases} \frac{2}{5}e^{-2t/5}, & t \ge 0 \\ 0, & t < 0 \end{cases}$

Capitalized Cost **In Exercises 89 and 90, find the capitalized cost C of an asset (a) for $n = 5$ years, (b) for $n = 10$ years, and (c) forever. The capitalized cost is given by**

$$C = C_0 + \int_0^n c(t)e^{-rt}\,dt$$

where C_0 is the original investment, t is the time in years, r is the annual interest rate compounded continuously, and $c(t)$ is the annual cost of maintenance.

89. $C_0 = \$650{,}000$
$c(t) = \$25{,}000$
$r = 0.06$

90. $C_0 = \$650{,}000$
$c(t) = \$25{,}000(1 + 0.08t)$
$r = 0.06$

91. ***Electromagnetic Theory*** The magnetic potential P at a point on the axis of a circular coil is given by

$$P = \frac{2\pi NIr}{k}\int_c^{\infty} \frac{1}{(r^2 + x^2)^{3/2}}\,dx$$

where N, I, r, k, and c are constants. Find P.

92. ***Gravitational Force*** A "semi-infinite" uniform rod occupies the nonnegative x-axis. The rod has a linear density δ which means that a segment of length dx has a mass of $\delta\,dx$. A particle of mass M is located at the point $(-a, 0)$. The gravitational force F that the rod exerts on the mass is given by $F = \int_0^{\infty} \frac{GM\delta}{(a + x)^2}\,dx$, where G is the gravitational constant. Find F.

True or False? **In Exercises 93–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

93. If f is continuous on $[0, \infty)$ and $\lim\limits_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f(x)\,dx$ converges.

94. If f is continuous on $[0, \infty)$ and $\int_0^{\infty} f(x)\,dx$ diverges, then $\lim\limits_{x\to\infty} f(x) \ne 0$.

95. If f' is continuous on $[0, \infty)$ and $\lim\limits_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f'(x)\,dx = -f(0)$.

96. If the graph of f is symmetric with respect to the origin or the y-axis, then $\int_0^{\infty} f(x)\,dx$ converges if and only if $\int_{-\infty}^{\infty} f(x)\,dx$ converges.

97. (a) Show that $\int_{-\infty}^{\infty} \sin x\,dx$ diverges.

(b) Show that $\lim\limits_{a\to\infty} \int_{-a}^{a} \sin x\,dx = 0$.

(c) What do parts (a) and (b) show about the definition of improper integrals?

CAPSTONE

98. For each integral, find a nonnegative real number b that makes the integral improper. Explain your reasoning.

(a) $\displaystyle\int_0^b \frac{1}{x^2 - 9}\,dx$

(b) $\displaystyle\int_0^b \frac{1}{\sqrt{4 - x}}\,dx$

(c) $\displaystyle\int_0^b \frac{x}{x^2 - 7x + 12}\,dx$

(d) $\displaystyle\int_b^{10} \ln x\,dx$

(e) $\displaystyle\int_0^b \tan 2x\,dx$

(f) $\displaystyle\int_0^b \frac{\cos x}{1 - \sin x}\,dx$

99. ***Writing***

(a) The improper integrals

$$\int_1^{\infty} \frac{1}{x}\,dx \quad \text{and} \quad \int_1^{\infty} \frac{1}{x^2}\,dx$$

diverge and converge, respectively. Describe the essential differences between the integrands that cause one integral to converge and the other to diverge.

(b) Sketch a graph of the function $y = \sin x/x$ over the interval $(1, \infty)$. Use your knowledge of the definite integral to make an inference as to whether the integral

$$\int_1^{\infty} \frac{\sin x}{x}\,dx$$

converges. Give reasons for your answer.

(c) Use one iteration of integration by parts on the integral in part (b) to determine its divergence or convergence.

100. ***Exploration*** Consider the integral

$$\int_0^{\pi/2} \frac{4}{1+(\tan x)^n}\,dx$$

where n is a positive integer.

(a) Is the integral improper? Explain.

(b) Use a graphing utility to graph the integrand for $n = 2, 4, 8$, and 12.

(c) Use the graphs to approximate the integral as $n \to \infty$.

CAS (d) Use a computer algebra system to evaluate the integral for the values of n in part (b). Make a conjecture about the value of the integral for any positive integer n. Compare your results with your answer in part (c).

101. ***The Gamma Function*** The Gamma Function $\Gamma(n)$ is defined by

$$\Gamma(n) = \int_0^{\infty} x^{n-1}e^{-x}\,dx, \quad n > 0.$$

(a) Find $\Gamma(1)$, $\Gamma(2)$, and $\Gamma(3)$.

(b) Use integration by parts to show that $\Gamma(n+1) = n\Gamma(n)$.

(c) Write $\Gamma(n)$ using factorial notation where n is a positive integer.

102. Prove that $I_n = \left(\dfrac{n-1}{n+2}\right)I_{n-1}$, where

$$I_n = \int_0^{\infty} \frac{x^{2n-1}}{(x^2+1)^{n+3}}\,dx, \quad n \geq 1.$$

Then evaluate each integral.

(a) $\displaystyle\int_0^{\infty} \frac{x}{(x^2+1)^4}\,dx$ (b) $\displaystyle\int_0^{\infty} \frac{x^3}{(x^2+1)^5}\,dx$

(c) $\displaystyle\int_0^{\infty} \frac{x^5}{(x^2+1)^6}\,dx$

Laplace Transforms **Let $f(t)$ be a function defined for all positive values of t. The Laplace Transform of $f(t)$ is defined by**

$$F(s) = \int_0^{\infty} e^{-st}f(t)\,dt$$

if the improper integral exists. Laplace Transforms are used to solve differential equations. In Exercises 103–110, find the Laplace Transform of the function.

103. $f(t) = 1$

104. $f(t) = t$

105. $f(t) = t^2$

106. $f(t) = e^{at}$

107. $f(t) = \cos at$

108. $f(t) = \sin at$

109. $f(t) = \cosh at$

110. $f(t) = \sinh at$

111. ***Normal Probability*** The mean height of American men between 20 and 29 years old is 70 inches, and the standard deviation is 3 inches. A 20- to 29-year-old man is chosen at random from the population. The probability that he is 6 feet tall or taller is

$$P(72 \leq x < \infty) = \int_{72}^{\infty} \frac{1}{3\sqrt{2\pi}}e^{-(x-70)^2/18}\,dx.$$

(Source: National Center for Health Statistics)

(a) Use a graphing utility to graph the integrand. Use the graphing utility to convince yourself that the area between the x-axis and the integrand is 1.

(b) Use a graphing utility to approximate $P(72 \leq x < \infty)$.

(c) Approximate $0.5 - P(70 \leq x \leq 72)$ using a graphing utility. Use the graph in part (a) to explain why this result is the same as the answer in part (b).

112. (a) Sketch the semicircle $y = \sqrt{4-x^2}$.

(b) Explain why

$$\int_{-2}^{2} \frac{2\,dx}{\sqrt{4-x^2}} = \int_{-2}^{2} \sqrt{4-x^2}\,dx$$

without evaluating either integral.

113. For what value of c does the integral

$$\int_0^{\infty} \left(\frac{1}{\sqrt{x^2+1}} - \frac{c}{x+1}\right)dx$$

converge? Evaluate the integral for this value of c.

114. For what value of c does the integral

$$\int_1^{\infty} \left(\frac{cx}{x^2+2} - \frac{1}{3x}\right)dx$$

converge? Evaluate the integral for this value of c.

115. ***Volume*** Find the volume of the solid generated by revolving the region bounded by the graph of f about the x-axis.

$$f(x) = \begin{cases} x \ln x, & 0 < x \leq 2 \\ 0, & x = 0 \end{cases}$$

116. ***Volume*** Find the volume of the solid generated by revolving the unbounded region lying between $y = -\ln x$ and the y-axis $(y \geq 0)$ about the x-axis.

u-Substitution **In Exercises 117 and 118, rewrite the improper integral as a proper integral using the given u-substitution. Then use the Trapezoidal Rule with $n = 5$ to approximate the integral.**

117. $\displaystyle\int_0^1 \frac{\sin x}{\sqrt{x}}\,dx, \quad u = \sqrt{x}$

118. $\displaystyle\int_0^1 \frac{\cos x}{\sqrt{1-x}}\,dx, \quad u = \sqrt{1-x}$

119. (a) Use a graphing utility to graph the function $y = e^{-x^2}$.

(b) Show that $\displaystyle\int_0^{\infty} e^{-x^2}\,dx = \int_0^1 \sqrt{-\ln y}\,dy$.

120. Let $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ be convergent and let a and b be real numbers where $a \neq b$. Show that

$$\int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_b^{\infty} f(x)\,dx.$$

8 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, use the basic integration rules to find or evaluate the integral.

1. $\int x\sqrt{x^2-36}\,dx$

2. $\int xe^{x^2-1}\,dx$

3. $\int \frac{x}{x^2-49}\,dx$

4. $\int \frac{x}{\sqrt{25-x^2}}\,dx$

5. $\int_1^e \frac{\ln(2x)}{x}\,dx$

6. $\int_{3/2}^{2} 2x\sqrt{2x-3}\,dx$

7. $\int \frac{100}{\sqrt{100-x^2}}\,dx$

8. $\int \frac{x^4+2x^2+x+1}{(x^2+1)^2}\,dx$

In Exercises 9–18, use integration by parts to find the integral.

9. $\int xe^{3x}\,dx$

10. $\int x^3e^x\,dx$

11. $\int e^{2x}\sin 3x\,dx$

12. $\int (x^2-3x)e^x\,dx$

13. $\int x\sqrt{x-1}\,dx$

14. $\int \arctan 2x\,dx$

15. $\int x^2\sin 2x\,dx$

16. $\int \ln\sqrt{x^2-4}\,dx$

17. $\int x\arcsin 2x\,dx$

18. $\int e^x\arctan e^x\,dx$

In Exercises 19–24, find the trigonometric integral.

19. $\int \cos^3(\pi x-1)\,dx$

20. $\int \sin^2\frac{\pi x}{2}\,dx$

21. $\int \sec^4\frac{x}{2}\,dx$

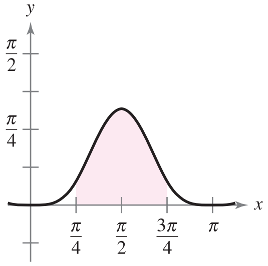
22. $\int \tan\theta\sec^4\theta\,d\theta$

23. $\int \frac{1}{1-\sin\theta}\,d\theta$

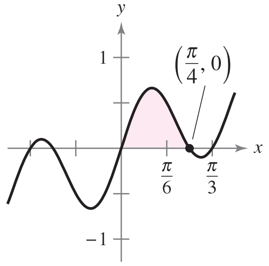
24. $\int \cos 2\theta(\sin\theta+\cos\theta)^2\,d\theta$

Area **In Exercises 25 and 26, find the area of the region.**

25. $y=\sin^4 x$



26. $y=\sin 3x\cos 2x$



In Exercises 27–32, use trigonometric substitution to find or evaluate the integral.

27. $\int \frac{-12}{x^2\sqrt{4-x^2}}\,dx$

28. $\int \frac{\sqrt{x^2-9}}{x}\,dx,\quad x>3$

29. $\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

30. $\int \sqrt{25-9x^2}\,dx$

31. $\int_{-2}^{0}\sqrt{4-x^2}\,dx$

32. $\int_0^{\pi/2}\frac{\sin\theta}{1+2\cos^2\theta}\,d\theta$

In Exercises 33 and 34, find the integral using each method.

33. $\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

(a) Trigonometric substitution

(b) Substitution: $u^2=4+x^2$

(c) Integration by parts: $dv=\left(x/\sqrt{4+x^2}\right)dx$

34. $\int x\sqrt{4+x}\,dx$

(a) Trigonometric substitution

(b) Substitution: $u^2=4+x$

(c) Substitution: $u=4+x$

(d) Integration by parts: $dv=\sqrt{4+x}\,dx$

In Exercises 35–40, use partial fractions to find the integral.

35. $\int \frac{x-39}{x^2-x-12}\,dx$

36. $\int \frac{2x^3-5x^2+4x-4}{x^2-x}\,dx$

37. $\int \frac{x^2+2x}{x^3-x^2+x-1}\,dx$

38. $\int \frac{4x-2}{3(x-1)^2}\,dx$

39. $\int \frac{x^2}{x^2+5x-24}\,dx$

40. $\int \frac{\sec^2\theta}{\tan\theta(\tan\theta-1)}\,d\theta$

In Exercises 41–48, use integration tables to find or evaluate the integral.

41. $\int \frac{x}{(4+5x)^2}\,dx$

42. $\int \frac{x}{\sqrt{4+5x}}\,dx$

43. $\int_0^{\sqrt{\pi}/2}\frac{x}{1+\sin x^2}\,dx$

44. $\int_0^1 \frac{x}{1+e^{x^2}}\,dx$

45. $\int \frac{x}{x^2+4x+8}\,dx$

46. $\int \frac{3}{2x\sqrt{9x^2-1}}\,dx,\quad x>\frac{1}{3}$

47. $\int \frac{1}{\sin\pi x\cos\pi x}\,dx$

48. $\int \frac{1}{1+\tan\pi x}\,dx$

49. Verify the reduction formula

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int(\ln x)^{n-1}\,dx.$$

50. Verify the reduction formula

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx.$$

In Exercises 51–58, find the integral using any method.

51. $\int \theta \sin \theta \cos \theta \, d\theta$

52. $\int \frac{\csc \sqrt{2x}}{\sqrt{x}} \, dx$

53. $\int \frac{x^{1/4}}{1 + x^{1/2}} \, dx$

54. $\int \sqrt{1 + \sqrt{x}} \, dx$

55. $\int \sqrt{1 + \cos x} \, dx$

56. $\int \frac{3x^3 + 4x}{(x^2 + 1)^2} \, dx$

57. $\int \cos x \ln(\sin x) \, dx$

58. $\int (\sin \theta + \cos \theta)^2 \, d\theta$

In Exercises 59–62, solve the differential equation using any method.

59. $\frac{dy}{dx} = \frac{25}{x^2 - 25}$

60. $\frac{dy}{dx} = \frac{\sqrt{4 - x^2}}{2x}$

61. $y' = \ln(x^2 + x)$

62. $y' = \sqrt{1 - \cos \theta}$

In Exercises 63–68, evaluate the definite integral using any method. Use a graphing utility to verify your result.

63. $\int_2^{\sqrt{5}} x(x^2 - 4)^{3/2} \, dx$

64. $\int_0^1 \frac{x}{(x - 2)(x - 4)} \, dx$

65. $\int_1^4 \frac{\ln x}{x} \, dx$

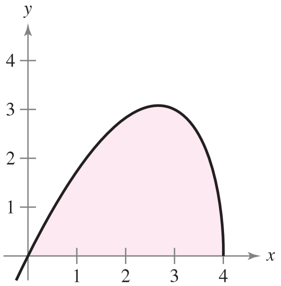
66. $\int_0^2 xe^{3x} \, dx$

67. $\int_0^\pi x \sin x \, dx$

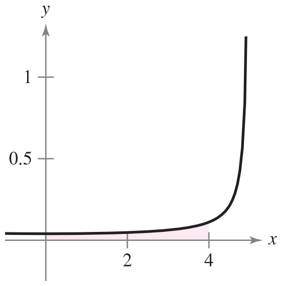
68. $\int_0^5 \frac{x}{\sqrt{4 + x}} \, dx$

Area **In Exercises 69 and 70, find the area of the region.**

69. $y = x\sqrt{4 - x}$



70. $y = \frac{1}{25 - x^2}$



Centroid **In Exercises 71 and 72, find the centroid of the region bounded by the graphs of the equations.**

71. $y = \sqrt{1 - x^2}, \quad y = 0$

72. $(x - 1)^2 + y^2 = 1, \quad (x - 4)^2 + y^2 = 4$

Arc Length **In Exercises 73 and 74, approximate to two decimal places the arc length of the curve over the given interval.**

	Function	*Interval*
73.	$y = \sin x$	$[0, \pi]$
74.	$y = \sin^2 x$	$[0, \pi]$

In Exercises 75–82, use L'Hôpital's Rule to evaluate the limit.

75. $\lim_{x \to 1} \frac{(\ln x)^2}{x - 1}$

76. $\lim_{x \to 0} \frac{\sin \pi x}{\sin 5\pi x}$

77. $\lim_{x \to \infty} \frac{e^{2x}}{x^2}$

78. $\lim_{x \to \infty} xe^{-x^2}$

79. $\lim_{x \to \infty} (\ln x)^{2/x}$

80. $\lim_{x \to 1^+} (x - 1)^{\ln x}$

81. $\lim_{n \to \infty} 1000\left(1 + \frac{0.09}{n}\right)^n$

82. $\lim_{x \to 1^+} \left(\frac{2}{\ln x} - \frac{2}{x - 1}\right)$

In Exercises 83–90, determine whether the improper integral diverges or converges. Evaluate the integral if it converges.

83. $\int_0^{16} \frac{1}{\sqrt[4]{x}} \, dx$

84. $\int_0^2 \frac{7}{x - 2} \, dx$

85. $\int_1^\infty x^2 \ln x \, dx$

86. $\int_0^\infty \frac{e^{-1/x}}{x^2} \, dx$

87. $\int_1^\infty \frac{\ln x}{x^2} \, dx$

88. $\int_1^\infty \frac{1}{\sqrt[4]{x}} \, dx$

89. $\int_2^\infty \frac{1}{x\sqrt{x^2 - 4}} \, dx$

90. $\int_0^\infty \frac{2}{\sqrt{x}(x + 4)} \, dx$

91. ***Present Value*** The board of directors of a corporation is calculating the price to pay for a business that is forecast to yield a continuous flow of profit of \$500,000 per year. If money will earn a nominal rate of 5% per year compounded continuously, what is the present value of the business

(a) for 20 years?

(b) forever (in perpetuity)?

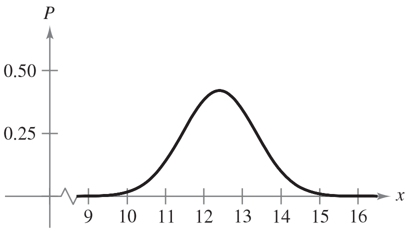
(*Note:* The present value for t_0 years is $\int_0^{t_0} 500{,}000e^{-0.05t} \, dt$.)

92. ***Volume*** Find the volume of the solid generated by revolving the region bounded by the graphs of $y = xe^{-x}$, $y = 0$, and $x = 0$ about the x-axis.

93. ***Probability*** The average lengths (from beak to tail) of different species of warblers in the eastern United States are approximately normally distributed with a mean of 12.9 centimeters and a standard deviation of 0.95 centimeter (see figure). The probability that a randomly selected warbler has a length between a and b centimeters is

$$P(a \le x \le b) = \frac{1}{0.95\sqrt{2\pi}} \int_a^b e^{-(x - 12.9)^2/2(0.95)^2} \, dx.$$

Use a graphing utility to approximate the probability that a randomly selected warbler has a length of (a) 13 centimeters or greater and (b) 15 centimeters or greater. *(Source: Peterson's Field Guide: Eastern Birds)*



P.S. PROBLEM SOLVING

1. (a) Evaluate the integrals

$$\int_{-1}^{1} (1 - x^2)\,dx \quad \text{and} \quad \int_{-1}^{1} (1 - x^2)^2\,dx.$$

(b) Use Wallis's Formulas to prove that

$$\int_{-1}^{1} (1 - x^2)^n\,dx = \frac{2^{2n+1}(n!)^2}{(2n + 1)!}$$

for all positive integers n.

2. (a) Evaluate the integrals $\int_{0}^{1} \ln x\,dx$ and $\int_{0}^{1} (\ln x)^2\,dx$.

(b) Prove that

$$\int_{0}^{1} (\ln x)^n\,dx = (-1)^n n!$$

for all positive integers n.

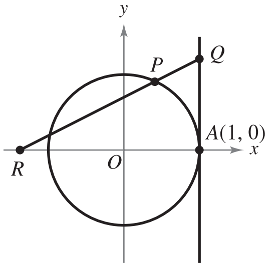
3. Find the value of the positive constant c such that

$$\lim_{x\to\infty} \left(\frac{x + c}{x - c}\right)^x = 9.$$

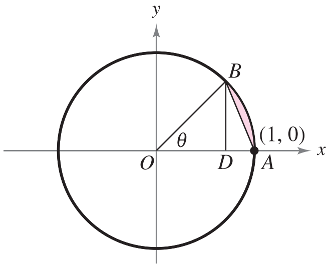
4. Find the value of the positive constant c such that

$$\lim_{x\to\infty} \left(\frac{x - c}{x + c}\right)^x = \frac{1}{4}.$$

5. The line $x = 1$ is tangent to the unit circle at A. The length of segment QA equals the length of the circular arc $\overset{\frown}{PA}$ (see figure). Show that the length of segment OR approaches 2 as P approaches A.



6. The segment BD is the height of $\triangle OAB$. Let R be the ratio of the area of $\triangle DAB$ to that of the shaded region formed by deleting $\triangle OAB$ from the circular sector subtended by angle θ (see figure). Find $\lim_{\theta\to 0^+} R$.



7. Consider the problem of finding the area of the region bounded by the x-axis, the line $x = 4$, and the curve

$$y = \frac{x^2}{(x^2 + 9)^{3/2}}.$$

(a) Use a graphing utility to graph the region and approximate its area.

(b) Use an appropriate trigonometric substitution to find the exact area.

(c) Use the substitution $x = 3 \sinh u$ to find the exact area and verify that you obtain the same answer as in part (b).

8. Use the substitution $u = \tan \dfrac{x}{2}$ to find the area of the shaded region under the graph of $y = \dfrac{1}{2 + \cos x}$, $0 \le x \le \pi/2$ (see figure).

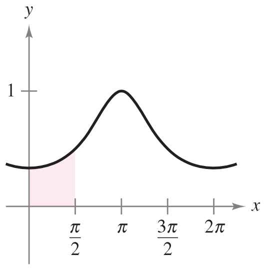


Figure for 8

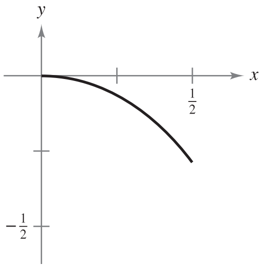
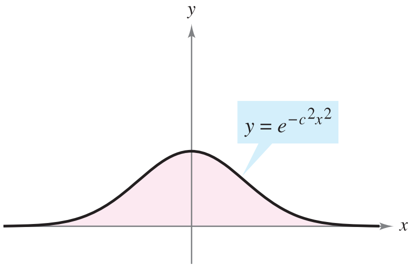


Figure for 9

9. Find the arc length of the graph of the function $y = \ln(1 - x^2)$ on the interval $0 \le x \le \frac{1}{2}$ (see figure).

10. Find the centroid of the region above the x-axis and bounded above by the curve $y = e^{-c^2x^2}$, where c is a positive constant (see figure).

$\left(\textit{Hint:}\ \text{Show that } \int_{0}^{\infty} e^{-c^2x^2}\,dx = \frac{1}{c}\int_{0}^{\infty} e^{-x^2}\,dx.\right)$



11. Some elementary functions, such as $f(x) = \sin(x^2)$, do not have antiderivatives that are elementary functions. Joseph Liouville proved that

$$\int \frac{e^x}{x}\,dx$$

does not have an elementary antiderivative. Use this fact to prove that

$$\int \frac{1}{\ln x}\,dx$$

is not elementary.

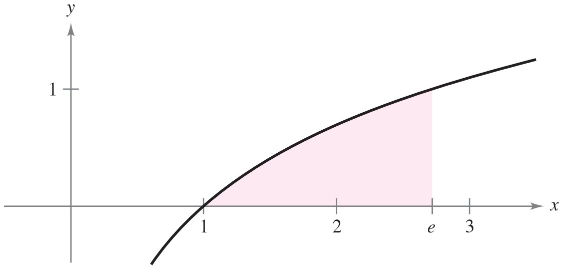
12. (a) Let $y = f^{-1}(x)$ be the inverse function of f. Use integration by parts to derive the formula

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy.$$

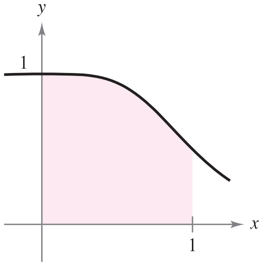
(b) Use the formula in part (a) to find the integral

$$\int \arcsin x\,dx.$$

(c) Use the formula in part (a) to find the area under the graph of $y = \ln x$, $1 \le x \le e$ (see figure).



13. Factor the polynomial $p(x) = x^4 + 1$ and then find the area under the graph of $y = \dfrac{1}{x^4 + 1}$, $0 \le x \le 1$ (see figure).



14. (a) Use the substitution $u = \dfrac{\pi}{2} - x$ to evaluate the integral

$$\int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\,dx.$$

(b) Let n be a positive integer. Evaluate the integral

$$\int_0^{\pi/2} \frac{\sin^n x}{\cos^n x + \sin^n x}\,dx.$$

15. Use a graphing utility to estimate each limit. Then calculate each limit using L'Hôpital's Rule. What can you conclude about the form $0 \cdot \infty$?

(a) $\displaystyle\lim_{x\to0^+}\left(\cot x + \frac{1}{x}\right)$ (b) $\displaystyle\lim_{x\to0^+}\left(\cot x - \frac{1}{x}\right)$

(c) $\displaystyle\lim_{x\to0^+}\left[\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right)\right]$

16. Suppose the denominator of a rational function can be factored into distinct linear factors

$$D(x) = (x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

for a positive integer n and distinct real numbers $c_1, c_2, \ldots, c_n$. If N is a polynomial of degree less than n, show that

$$\frac{N(x)}{D(x)} = \frac{P_1}{x - c_1} + \frac{P_2}{x - c_2} + \cdot \cdot \cdot + \frac{P_n}{x - c_n}$$

where $P_k = N(c_k)/D'(c_k)$ for $k = 1, 2, \ldots, n$. Note that this is the partial fraction decomposition of $N(x)/D(x)$.

17. Use the result of Exercise 16 to find the partial fraction decomposition of

$$\frac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x}.$$

18. The velocity v (in feet per second) of a rocket whose initial mass (including fuel) is m is given by

$$v = gt + u \ln \frac{m}{m - rt}, \quad t < \frac{m}{r}$$

where u is the expulsion speed of the fuel, r is the rate at which the fuel is consumed, and $g = -32$ feet per second per second is the acceleration due to gravity. Find the position equation for a rocket for which $m = 50{,}000$ pounds, $u = 12{,}000$ feet per second, and $r = 400$ pounds per second. What is the height of the rocket when $t = 100$ seconds? (Assume that the rocket was fired from ground level and is moving straight upward.)

19. Suppose that $f(a) = f(b) = g(a) = g(b) = 0$ and the second derivatives of f and g are continuous on the closed interval $[a, b]$. Prove that

$$\int_a^b f(x)g''(x)\,dx = \int_a^b f''(x)g(x)\,dx.$$

20. Suppose that $f(a) = f(b) = 0$ and the second derivatives of f exist on the closed interval $[a, b]$. Prove that

$$\int_a^b (x - a)(x - b)f''(x)\,dx = 2\int_a^b f(x)\,dx.$$

21. Using the inequality

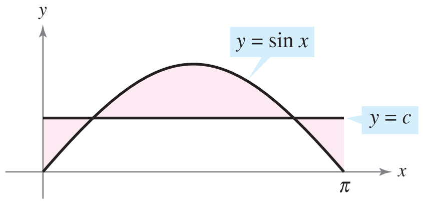
$$\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{1}{x^{15}} < \frac{1}{x^5 - 1} < \frac{1}{x^5} + \frac{1}{x^{10}} + \frac{2}{x^{15}}$$

for $x \ge 2$, approximate $\displaystyle\int_2^\infty \frac{1}{x^5 - 1}\,dx$.

22. Consider the shaded region between the graph of $y = \sin x$, where $0 \le x \le \pi$, and the line $y = c$, where $0 \le c \le 1$ (see figure). A solid is formed by revolving the region about the line $y = c$.

(a) For what value of c does the solid have minimum volume?

(b) For what value of c does the solid have maximum volume?



9

Infinite Series

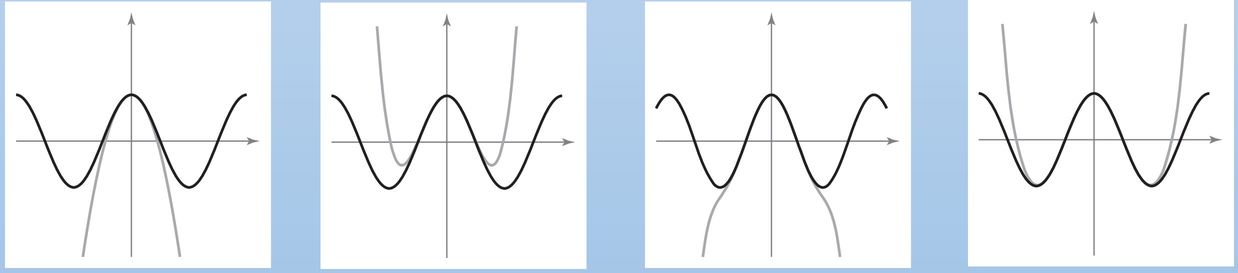
This chapter is divided into two basic parts. The first six sections discuss infinite sequences and infinite series. The last four sections discuss Taylor and Maclaurin polynomials and power series.

In this chapter, you should learn the following.

- How to determine whether a sequence converges or diverges. (9.1)
- How to determine whether an infinite series converges or diverges. (9.2–9.6)
- How to find Taylor or Maclaurin polynomial approximations of elementary functions. (9.7)
- How to find the radius and interval of convergence of a power series and how to differentiate and integrate power series. (9.8)
- How to represent functions by power series. (9.9)
- How to find a Taylor or Maclaurin series for a function. (9.10)

Eric Haines

The sphereflake shown above is a computer-generated fractal that was created by Eric Haines. The radius of the large sphere is 1. To the large sphere, nine spheres of radius $\frac{1}{3}$ are attached. To each of these, nine spheres of radius $\frac{1}{9}$ are attached. This process is continued infinitely. Does the sphereflake have a finite or an infinite surface area? (See Section 9.2, Exercise 114.)



Maclaurin *polynomials* approximate a given function in an interval around $x = 0$. As you add terms to the Maclaurin polynomial, it becomes a better and better approximation of the given function near $x = 0$. In Section 9.10, you will see that a Maclaurin *series* is equivalent to the given function (under suitable conditions).

9.1 Sequences

- **List the terms of a sequence.**
- **Determine whether a sequence converges or diverges.**
- **Write a formula for the *n*th term of a sequence.**
- **Use properties of monotonic sequences and bounded sequences.**

EXPLORATION

Finding Patterns Describe a pattern for each of the following sequences. Then use your description to write a formula for the nth term of each sequence. As n increases, do the terms appear to be approaching a limit? Explain your reasoning.

a. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$
b. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$
c. $10, \frac{10}{3}, \frac{10}{6}, \frac{10}{10}, \frac{10}{15}, \ldots$
d. $\frac{1}{4}, \frac{4}{9}, \frac{9}{16}, \frac{16}{25}, \frac{25}{36}, \ldots$
e. $\frac{3}{7}, \frac{5}{10}, \frac{7}{13}, \frac{9}{16}, \frac{11}{19}, \ldots$

Sequences

In mathematics, the word "sequence" is used in much the same way as in ordinary English. To say that a collection of objects or events is *in sequence* usually means that the collection is ordered so that it has an identified first member, second member, third member, and so on.

Mathematically, a **sequence** is defined as a function whose domain is the set of positive integers. Although a sequence is a function, it is common to represent sequences by subscript notation rather than by the standard function notation. For instance, in the sequence

$$\begin{array}{ccccccc} 1, & 2, & 3, & 4, & \ldots, & n, & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a_1, & a_2, & a_3, & a_4, & \ldots, & a_n, & \ldots \end{array} \qquad \text{Sequence}$$

1 is mapped onto a_1, 2 is mapped onto a_2, and so on. The numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ are the **terms** of the sequence. The number a_n is the ***n*th term** of the sequence, and the entire sequence is denoted by $\{a_n\}$.

NOTE Occasionally, it is convenient to begin a sequence with a_0, so that the terms of the sequence become $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$

EXAMPLE 1 Listing the Terms of a Sequence

a. The terms of the sequence $\{a_n\} = \{3 + (-1)^n\}$ are

$$\begin{array}{ccccc} 3 + (-1)^1, & 3 + (-1)^2, & 3 + (-1)^3, & 3 + (-1)^4, & \ldots \\ 2, & 4, & 2, & 4, & \ldots. \end{array}$$

b. The terms of the sequence $\{b_n\} = \left\{\dfrac{n}{1 - 2n}\right\}$ are

$$\begin{array}{ccccc} \dfrac{1}{1 - 2 \cdot 1}, & \dfrac{2}{1 - 2 \cdot 2}, & \dfrac{3}{1 - 2 \cdot 3}, & \dfrac{4}{1 - 2 \cdot 4}, & \ldots \\ -1, & -\dfrac{2}{3}, & -\dfrac{3}{5}, & -\dfrac{4}{7}, & \ldots. \end{array}$$

c. The terms of the sequence $\{c_n\} = \left\{\dfrac{n^2}{2^n - 1}\right\}$ are

$$\begin{array}{ccccc} \dfrac{1^2}{2^1 - 1}, & \dfrac{2^2}{2^2 - 1}, & \dfrac{3^2}{2^3 - 1}, & \dfrac{4^2}{2^4 - 1}, & \ldots \\ \dfrac{1}{1}, & \dfrac{4}{3}, & \dfrac{9}{7}, & \dfrac{16}{15}, & \ldots. \end{array}$$

d. The terms of the **recursively defined** sequence $\{d_n\}$, where $d_1 = 25$ and $d_{n+1} = d_n - 5$, are

$$25, \quad 25 - 5 = 20, \quad 20 - 5 = 15, \quad 15 - 5 = 10, \ldots.$$

■

STUDY TIP Some sequences are defined recursively. To define a sequence recursively, you need to be given one or more of the first few terms. All other terms of the sequence are then defined using previous terms, as shown in Example 1(d).

Limit of a Sequence

The primary focus of this chapter concerns sequences whose terms approach limiting values. Such sequences are said to **converge.** For instance, the sequence $\{1/2^n\}$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, . . .$$

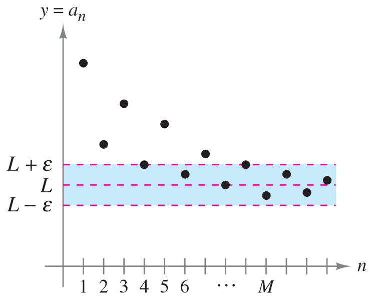
converges to 0, as indicated in the following definition.

DEFINITION OF THE LIMIT OF A SEQUENCE

Let L be a real number. The **limit** of a sequence $\{a_n\}$ is L, written as

$$\lim_{n\to\infty} a_n = L$$

if for each $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. If the limit L of a sequence exists, then the sequence **converges** to L. If the limit of a sequence does not exist, then the sequence **diverges.**



For $n > M$, the terms of the sequence all lie within ε units of L.
Figure 9.1

Graphically, this definition says that eventually (for $n > M$ and $\varepsilon > 0$) the terms of a sequence that converges to L will lie within the band between the lines $y = L + \varepsilon$ and $y = L - \varepsilon$, as shown in Figure 9.1.

If a sequence $\{a_n\}$ agrees with a function f at every positive integer, and if $f(x)$ approaches a limit L as $x \to \infty$, the sequence must converge to the same limit L.

THEOREM 9.1 LIMIT OF A SEQUENCE

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x\to\infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty} a_n = L.$$

NOTE The converse of Theorem 9.1 is not true (see Exercise 138). ■

NOTE There are different ways in which a sequence can fail to have a limit. One way is that the terms of the sequence increase without bound or decrease without bound. These cases are written symbolically as follows.

Terms increase without bound:

$$\lim_{n\to\infty} a_n = \infty$$

Terms decrease without bound:

$$\lim_{n\to\infty} a_n = -\infty$$

EXAMPLE 2 Finding the Limit of a Sequence

Find the limit of the sequence whose nth term is

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

Solution In Theorem 5.15, you learned that

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

So, you can apply Theorem 9.1 to conclude that

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \\ &= e.\end{aligned}$$

■

The following properties of limits of sequences parallel those given for limits of functions of a real variable in Section 1.3.

THEOREM 9.2 PROPERTIES OF LIMITS OF SEQUENCES

Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$.

1. $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$

2. $\lim_{n\to\infty} ca_n = cL$, c is any real number

3. $\lim_{n\to\infty} (a_nb_n) = LK$

4. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{K}$, $b_n \neq 0$ and $K \neq 0$

EXAMPLE 3 Determining Convergence or Divergence

a. Because the sequence $\{a_n\} = \{3 + (-1)^n\}$ has terms

$$2, 4, 2, 4, . . .$$

See Example 1(a), page 596.

that alternate between 2 and 4, the limit

$$\lim_{n\to\infty} a_n$$

does not exist. So, the sequence diverges.

b. For $\{b_n\} = \left\{\dfrac{n}{1-2n}\right\}$, divide the numerator and denominator by n to obtain

$$\lim_{n\to\infty} \frac{n}{1-2n} = \lim_{n\to\infty} \frac{1}{(1/n)-2} = -\frac{1}{2}$$

See Example 1(b), page 596.

which implies that the sequence converges to $-\frac{1}{2}$.

EXAMPLE 4 Using L'Hôpital's Rule to Determine Convergence

Show that the sequence whose nth term is $a_n = \dfrac{n^2}{2^n - 1}$ converges.

Solution Consider the function of a real variable

$$f(x) = \frac{x^2}{2^x - 1}.$$

Applying L'Hôpital's Rule twice produces

$$\lim_{x\to\infty} \frac{x^2}{2^x - 1} = \lim_{x\to\infty} \frac{2x}{(\ln 2)2^x} = \lim_{x\to\infty} \frac{2}{(\ln 2)^2 2^x} = 0.$$

Because $f(n) = a_n$ for every positive integer, you can apply Theorem 9.1 to conclude that

$$\lim_{n\to\infty} \frac{n^2}{2^n - 1} = 0.$$

See Example 1(c), page 596.

So, the sequence converges to 0. ■

TECHNOLOGY Use a graphing utility to graph the function in Example 4. Notice that as x approaches infinity, the value of the function gets closer and closer to 0. If you have access to a graphing utility that can generate terms of a sequence, try using it to calculate the first 20 terms of the sequence in Example 4. Then view the terms to observe numerically that the sequence converges to 0.

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

The symbol $n!$ (read "n factorial") is used to simplify some of the formulas developed in this chapter. Let n be a positive integer; then ***n* factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n.$$

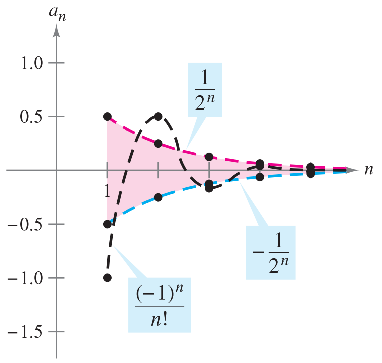
As a special case, **zero factorial** is defined as $0! = 1$. From this definition, you can see that $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, and so on. Factorials follow the same conventions for order of operations as exponents. That is, just as $2x^3$ and $(2x)^3$ imply different orders of operations, $2n!$ and $(2n)!$ imply the following orders.

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdots n)$$

and

$$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n \cdot (n+1) \cdots 2n$$

Another useful limit theorem that can be rewritten for sequences is the Squeeze Theorem from Section 1.3.



For $n \ge 4$, $(-1)^n/n!$ is squeezed between $-1/2^n$ and $1/2^n$.
Figure 9.2

THEOREM 9.3 SQUEEZE THEOREM FOR SEQUENCES

If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$$

and there exists an integer N such that $a_n \le c_n \le b_n$ for all $n > N$, then

$$\lim_{n\to\infty} c_n = L.$$

EXAMPLE 5 Using the Squeeze Theorem

Show that the sequence $\{c_n\} = \left\{(-1)^n \frac{1}{n!}\right\}$ converges, and find its limit.

Solution To apply the Squeeze Theorem, you must find two convergent sequences that can be related to the given sequence. Two possibilities are $a_n = -1/2^n$ and $b_n = 1/2^n$, both of which converge to 0. By comparing the term $n!$ with 2^n, you can see that

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots n = 24 \cdot \underbrace{5 \cdot 6 \cdots n}_{n\,-\,4\text{ factors}} \qquad (n \ge 4)$$

and

$$2^n = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdots 2 = 16 \cdot \underbrace{2 \cdot 2 \cdots 2}_{n\,-\,4\text{ factors}}. \qquad (n \ge 4)$$

This implies that for $n \ge 4$, $2^n < n!$, and you have

$$\frac{-1}{2^n} \le (-1)^n \frac{1}{n!} \le \frac{1}{2^n}, \quad n \ge 4$$

as shown in Figure 9.2. So, by the Squeeze Theorem it follows that

$$\lim_{n\to\infty} (-1)^n \frac{1}{n!} = 0. \qquad ■$$

NOTE Example 5 suggests something about the rate at which $n!$ increases as $n \to \infty$. As Figure 9.2 suggests, both $1/2^n$ and $1/n!$ approach 0 as $n \to \infty$. Yet $1/n!$ approaches 0 so much faster than $1/2^n$ does that

$$\lim_{n\to\infty} \frac{1/n!}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

In fact, it can be shown that for any fixed number k,

$$\lim_{n\to\infty} \frac{k^n}{n!} = 0.$$

This means that *the factorial function grows faster than any exponential function.*

In Example 5, the sequence $\{c_n\}$ has both positive and negative terms. For this sequence, it happens that the sequence of absolute values, $\{|c_n|\}$, also converges to 0. You can show this by the Squeeze Theorem using the inequality

$$0 \le \frac{1}{n!} \le \frac{1}{2^n}, \quad n \ge 4.$$

In such cases, it is often convenient to consider the sequence of absolute values—and then apply Theorem 9.4, which states that if the absolute value sequence converges to 0, the original signed sequence also converges to 0.

THEOREM 9.4 ABSOLUTE VALUE THEOREM

For the sequence $\{a_n\}$, if

$$\lim_{n\to\infty} |a_n| = 0 \qquad \text{then} \qquad \lim_{n\to\infty} a_n = 0.$$

PROOF Consider the two sequences $\{|a_n|\}$ and $\{-|a_n|\}$. Because both of these sequences converge to 0 and

$$-|a_n| \le a_n \le |a_n|$$

you can use the Squeeze Theorem to conclude that $\{a_n\}$ converges to 0. ■

Pattern Recognition for Sequences

Sometimes the terms of a sequence are generated by some rule that does not explicitly identify the nth term of the sequence. In such cases, you may be required to discover a *pattern* in the sequence and to describe the nth term. Once the nth term has been specified, you can investigate the convergence or divergence of the sequence.

EXAMPLE 6 Finding the nth Term of a Sequence

Find a sequence $\{a_n\}$ whose first five terms are

$$\frac{2}{1}, \frac{4}{3}, \frac{8}{5}, \frac{16}{7}, \frac{32}{9}, \ldots$$

and then determine whether the particular sequence you have chosen converges or diverges.

Solution First, note that the numerators are successive powers of 2, and the denominators form the sequence of positive odd integers. By comparing a_n with n, you have the following pattern.

$$\frac{2^1}{1}, \frac{2^2}{3}, \frac{2^3}{5}, \frac{2^4}{7}, \frac{2^5}{9}, \ldots, \frac{2^n}{2n-1}$$

Using L'Hôpital's Rule to evaluate the limit of $f(x) = 2^x/(2x - 1)$, you obtain

$$\lim_{x\to\infty} \frac{2^x}{2x-1} = \lim_{x\to\infty} \frac{2^x(\ln 2)}{2} = \infty \quad \Rightarrow \quad \lim_{n\to\infty} \frac{2^n}{2n-1} = \infty.$$

So, the sequence diverges. ■

Without a specific rule for generating the terms of a sequence or some knowledge of the context in which the terms of the sequence are obtained, it is not possible to determine the convergence or divergence of the sequence merely from its first several terms. For instance, although the first three terms of the following four sequences are identical, the first two sequences converge to 0, the third sequence converges to $\frac{1}{9}$, and the fourth sequence diverges.

$$\{a_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

$$\{b_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \ldots, \frac{6}{(n+1)(n^2-n+6)}, \ldots$$

$$\{c_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{7}{62}, \ldots, \frac{n^2-3n+3}{9n^2-25n+18}, \ldots$$

$$\{d_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 0, \ldots, \frac{-n(n+1)(n-4)}{6(n^2+3n-2)}, \ldots$$

The process of determining an nth term from the pattern observed in the first several terms of a sequence is an example of *inductive reasoning*.

EXAMPLE 7 Finding the *n*th Term of a Sequence

Determine an nth term for a sequence whose first five terms are

$$-\frac{2}{1}, \frac{8}{2}, -\frac{26}{6}, \frac{80}{24}, -\frac{242}{120}, \ldots$$

and then decide whether the sequence converges or diverges.

Solution Note that the numerators are 1 less than 3^n. So, you can reason that the numerators are given by the rule $3^n - 1$. Factoring the denominators produces

$$\begin{aligned} 1 &= 1 \\ 2 &= 1 \cdot 2 \\ 6 &= 1 \cdot 2 \cdot 3 \\ 24 &= 1 \cdot 2 \cdot 3 \cdot 4 \\ 120 &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots. \end{aligned}$$

This suggests that the denominators are represented by $n!$. Finally, because the signs alternate, you can write the nth term as

$$a_n = (-1)^n\left(\frac{3^n-1}{n!}\right).$$

From the discussion about the growth of $n!$, it follows that

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{3^n-1}{n!} = 0.$$

Applying Theorem 9.4, you can conclude that

$$\lim_{n\to\infty} a_n = 0.$$

So, the sequence $\{a_n\}$ converges to 0. ■

Monotonic Sequences and Bounded Sequences

So far you have determined the convergence of a sequence by finding its limit. Even if you cannot determine the limit of a particular sequence, it still may be useful to know whether the sequence converges. Theorem 9.5 (on page 603) provides a test for convergence of sequences without determining the limit. First, some preliminary definitions are given.

DEFINITION OF MONOTONIC SEQUENCE

A sequence $\{a_n\}$ is **monotonic** if its terms are nondecreasing

$$a_1 \le a_2 \le a_3 \le \cdot \cdot \cdot \le a_n \le \cdot \cdot \cdot$$

or if its terms are nonincreasing

$$a_1 \ge a_2 \ge a_3 \ge \cdot \cdot \cdot \ge a_n \ge \cdot \cdot \cdot .$$

EXAMPLE 8 Determining Whether a Sequence Is Monotonic

Determine whether each sequence having the given nth term is monotonic.

a. $a_n = 3 + (-1)^n$ **b.** $b_n = \dfrac{2n}{1 + n}$ **c.** $c_n = \dfrac{n^2}{2^n - 1}$

Solution

a. This sequence alternates between 2 and 4. So, it is not monotonic.

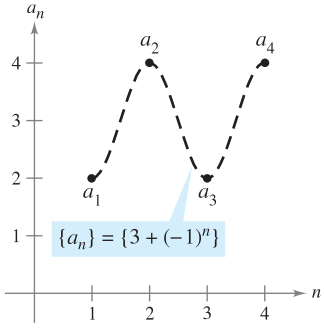
b. This sequence is monotonic because each successive term is larger than its predecessor. To see this, compare the terms b_n and b_{n+1}. [Note that, because n is positive, you can multiply each side of the inequality by $(1 + n)$ and $(2 + n)$ without reversing the inequality sign.]

$$\begin{aligned} b_n = \frac{2n}{1 + n} &\overset{?}{<} \frac{2(n + 1)}{1 + (n + 1)} = b_{n+1} \\ 2n(2 + n) &\overset{?}{<} (1 + n)(2n + 2) \\ 4n + 2n^2 &\overset{?}{<} 2 + 4n + 2n^2 \\ 0 &< 2 \end{aligned}$$

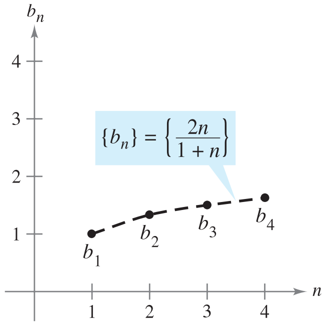
Starting with the final inequality, which is valid, you can reverse the steps to conclude that the original inequality is also valid.

c. This sequence is not monotonic, because the second term is larger than the first term, and larger than the third. (Note that if you drop the first term, the remaining sequence $c_2, c_3, c_4, . . .$ is monotonic.)

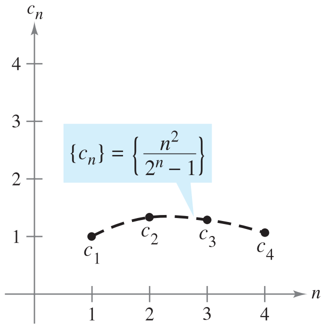
Figure 9.3 graphically illustrates these three sequences. ■



(a) Not monotonic



(b) Monotonic



(c) Not monotonic

Figure 9.3

NOTE In Example 8(b), another way to see that the sequence is monotonic is to argue that the derivative of the corresponding differentiable function $f(x) = 2x/(1 + x)$ is positive for all x. This implies that f is increasing, which in turn implies that $\{a_n\}$ is increasing. ■

NOTE All three sequences shown in Figure 9.3 are bounded. To see this, consider the following.

$$2 \leq a_n \leq 4$$
$$1 \leq b_n \leq 2$$
$$0 \leq c_n \leq \frac{4}{3}$$

DEFINITION OF BOUNDED SEQUENCE

1. A sequence $\{a_n\}$ is **bounded above** if there is a real number M such that $a_n \leq M$ for all n. The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** if there is a real number N such that $N \leq a_n$ for all n. The number N is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is **bounded** if it is bounded above and bounded below.

One important property of the real numbers is that they are **complete.** Informally, this means that there are no holes or gaps on the real number line. (The set of rational numbers does not have the completeness property.) The completeness axiom for real numbers can be used to conclude that if a sequence has an upper bound, it must have a **least upper bound** (an upper bound that is smaller than all other upper bounds for the sequence). For example, the least upper bound of the sequence $\{a_n\} = \{n/(n+1)\}$,

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$$

is 1. The completeness axiom is used in the proof of Theorem 9.5.

THEOREM 9.5 BOUNDED MONOTONIC SEQUENCES

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

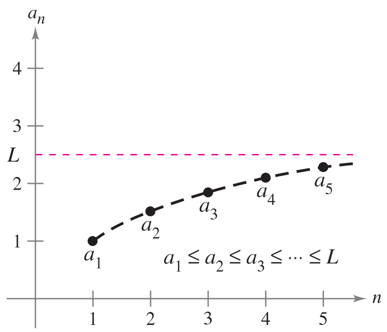
PROOF Assume that the sequence is nondecreasing, as shown in Figure 9.4. For the sake of simplicity, also assume that each term in the sequence is positive. Because the sequence is bounded, there must exist an upper bound M such that

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots \leq M.$$

From the completeness axiom, it follows that there is a least upper bound L such that

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots \leq L.$$

For $\varepsilon > 0$, it follows that $L - \varepsilon < L$, and therefore $L - \varepsilon$ cannot be an upper bound for the sequence. Consequently, at least one term of $\{a_n\}$ is greater than $L - \varepsilon$. That is, $L - \varepsilon < a_N$ for some positive integer N. Because the terms of $\{a_n\}$ are nondecreasing, it follows that $a_N \leq a_n$ for $n > N$. You now know that $L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon$, for every $n > N$. It follows that $|a_n - L| < \varepsilon$ for $n > N$, which by definition means that $\{a_n\}$ converges to L. The proof for a nonincreasing sequence is similar (see Exercise 139). ■



Every bounded nondecreasing sequence converges.
Figure 9.4

EXAMPLE 9 Bounded and Monotonic Sequences

a. The sequence $\{a_n\} = \{1/n\}$ is both bounded and monotonic and so, by Theorem 9.5, must converge.

b. The divergent sequence $\{b_n\} = \{n^2/(n+1)\}$ is monotonic, but not bounded. (It is bounded below.)

c. The divergent sequence $\{c_n\} = \{(-1)^n\}$ is bounded, but not monotonic.

■

9.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, write the first five terms of the sequence.

1. $a_n = 3^n$

2. $a_n = \dfrac{3^n}{n!}$

3. $a_n = \left(-\frac{1}{4}\right)^n$

4. $a_n = \left(-\frac{2}{3}\right)^n$

5. $a_n = \sin\dfrac{n\pi}{2}$

6. $a_n = \dfrac{2n}{n+3}$

7. $a_n = \dfrac{(-1)^{n(n+1)/2}}{n^2}$

8. $a_n = (-1)^{n+1}\left(\dfrac{2}{n}\right)$

9. $a_n = 5 - \dfrac{1}{n} + \dfrac{1}{n^2}$

10. $a_n = 10 + \dfrac{2}{n} + \dfrac{6}{n^2}$

In Exercises 11–14, write the first five terms of the recursively defined sequence.

11. $a_1 = 3,\ a_{k+1} = 2(a_k - 1)$

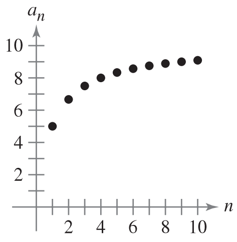
12. $a_1 = 4,\ a_{k+1} = \left(\dfrac{k+1}{2}\right)a_k$

13. $a_1 = 32,\ a_{k+1} = \frac{1}{2}a_k$

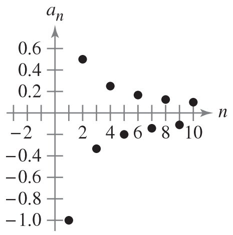
14. $a_1 = 6,\ a_{k+1} = \frac{1}{3}a_k^2$

In Exercises 15–18, match the sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

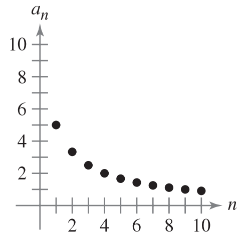
(a)



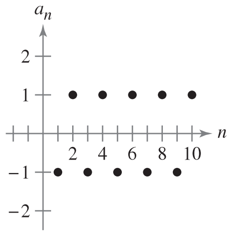
(b)



(c)



(d)



15. $a_n = \dfrac{10}{n+1}$

16. $a_n = \dfrac{10n}{n+1}$

17. $a_n = (-1)^n$

18. $a_n = \dfrac{(-1)^n}{n}$

In Exercises 19–22, match the sequence with the correct expression for its *n*th term. [The *n*th terms are labeled (a), (b), (c), and (d).]

(a) $a_n = \dfrac{2}{3}n$

(b) $a_n = 2 - \dfrac{4}{n}$

(c) $a_n = 16(-0.5)^{n-1}$

(d) $a_n = \dfrac{2n}{n+1}$

19. $-2, 0, \frac{2}{3}, 1, \ldots$

20. $16, -8, 4, -2, \ldots$

21. $\frac{2}{3}, \frac{4}{3}, 2, \frac{8}{3}, \ldots$

22. $1, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \ldots$

In Exercises 23–28, write the next two apparent terms of the sequence. Describe the pattern you used to find these terms.

23. $2, 5, 8, 11, \ldots$

24. $\frac{7}{2}, 4, \frac{9}{2}, 5, \ldots$

25. $5, 10, 20, 40, \ldots$

26. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$

27. $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \ldots$

28. $1, -\frac{3}{2}, \frac{9}{4}, -\frac{27}{8}, \ldots$

In Exercises 29–34, simplify the ratio of factorials.

29. $\dfrac{11!}{8!}$

30. $\dfrac{25!}{20!}$

31. $\dfrac{(n+1)!}{n!}$

32. $\dfrac{(n+2)!}{n!}$

33. $\dfrac{(2n-1)!}{(2n+1)!}$

34. $\dfrac{(2n+2)!}{(2n)!}$

In Exercises 35–40, find the limit (if possible) of the sequence.

35. $a_n = \dfrac{5n^2}{n^2+2}$

36. $a_n = 5 - \dfrac{1}{n^2}$

37. $a_n = \dfrac{2n}{\sqrt{n^2+1}}$

38. $a_n = \dfrac{5n}{\sqrt{n^2+4}}$

39. $a_n = \sin\dfrac{1}{n}$

40. $a_n = \cos\dfrac{2}{n}$

In Exercises 41–44, use a graphing utility to graph the first 10 terms of the sequence. Use the graph to make an inference about the convergence or divergence of the sequence. Verify your inference analytically and, if the sequence converges, find its limit.

41. $a_n = \dfrac{n+1}{n}$

42. $a_n = \dfrac{1}{n^{3/2}}$

43. $a_n = \cos\dfrac{n\pi}{2}$

44. $a_n = 3 - \dfrac{1}{2^n}$

In Exercises 45–72, determine the convergence or divergence of the sequence with the given *n*th term. If the sequence converges, find its limit.

45. $a_n = (0.3)^n - 1$

46. $a_n = 4 - \dfrac{3}{n}$

47. $a_n = \dfrac{5}{n+2}$

48. $a_n = \dfrac{2}{n!}$

49. $a_n = (-1)^n\left(\dfrac{n}{n+1}\right)$

50. $a_n = 1 + (-1)^n$

51. $a_n = \dfrac{3n^2 - n + 4}{2n^2 + 1}$

52. $a_n = \dfrac{\sqrt[3]{n}}{\sqrt[3]{n} + 1}$

53. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n}$

54. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$

55. $a_n = \dfrac{1 + (-1)^n}{n}$

56. $a_n = \dfrac{1 + (-1)^n}{n^2}$

57. $a_n = \dfrac{\ln(n^3)}{2n}$

58. $a_n = \dfrac{\ln \sqrt{n}}{n}$

59. $a_n = \dfrac{3^n}{4^n}$

60. $a_n = (0.5)^n$

61. $a_n = \dfrac{(n+1)!}{n!}$

62. $a_n = \dfrac{(n-2)!}{n!}$

63. $a_n = \dfrac{n-1}{n} - \dfrac{n}{n-1}, \quad n \geq 2$

64. $a_n = \dfrac{n^2}{2n+1} - \dfrac{n^2}{2n-1}$

65. $a_n = \dfrac{n^p}{e^n}, \quad p > 0$

66. $a_n = n \sin \dfrac{1}{n}$

67. $a_n = 2^{1/n}$

68. $a_n = -3^{-n}$

69. $a_n = \left(1 + \dfrac{k}{n}\right)^n$

70. $a_n = \left(1 + \dfrac{1}{n^2}\right)^n$

71. $a_n = \dfrac{\sin n}{n}$

72. $a_n = \dfrac{\cos \pi n}{n^2}$

In Exercises 73–86, write an expression for the *n*th term of the sequence. (There is more than one correct answer.)

73. 1, 4, 7, 10, . . .

74. 3, 7, 11, 15, . . .

75. $-1, 2, 7, 14, 23, \ldots$

76. $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \ldots$

77. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

78. $2, -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \ldots$

79. $2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \ldots$

80. $1 + \frac{1}{2}, 1 + \frac{3}{4}, 1 + \frac{7}{8}, 1 + \frac{15}{16}, 1 + \frac{31}{32}, \ldots$

81. $\dfrac{1}{2 \cdot 3}, \dfrac{2}{3 \cdot 4}, \dfrac{3}{4 \cdot 5}, \dfrac{4}{5 \cdot 6}, \ldots$

82. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$

83. $1, -\dfrac{1}{1 \cdot 3}, \dfrac{1}{1 \cdot 3 \cdot 5}, -\dfrac{1}{1 \cdot 3 \cdot 5 \cdot 7}, \ldots$

84. $1, x, \dfrac{x^2}{2}, \dfrac{x^3}{6}, \dfrac{x^4}{24}, \dfrac{x^5}{120}, \ldots$

85. 2, 24, 720, 40,320, 3,628,800, . . .

86. 1, 6, 120, 5040, 362,880, . . .

In Exercises 87–98, determine whether the sequence with the given *n*th term is monotonic and whether it is bounded. Use a graphing utility to confirm your results.

87. $a_n = 4 - \dfrac{1}{n}$

88. $a_n = \dfrac{3n}{n+2}$

89. $a_n = \dfrac{n}{2^{n+2}}$

90. $a_n = ne^{-n/2}$

91. $a_n = (-1)^n\left(\dfrac{1}{n}\right)$

92. $a_n = \left(-\dfrac{2}{3}\right)^n$

93. $a_n = \left(\dfrac{2}{3}\right)^n$

94. $a_n = \left(\dfrac{3}{2}\right)^n$

95. $a_n = \sin \dfrac{n\pi}{6}$

96. $a_n = \cos \dfrac{n\pi}{2}$

97. $a_n = \dfrac{\cos n}{n}$

98. $a_n = \dfrac{\sin \sqrt{n}}{n}$

In Exercises 99–102, (a) use Theorem 9.5 to show that the sequence with the given *n*th term converges and (b) use a graphing utility to graph the first 10 terms of the sequence and find its limit.

99. $a_n = 5 + \dfrac{1}{n}$

100. $a_n = 4 - \dfrac{3}{n}$

101. $a_n = \dfrac{1}{3}\left(1 - \dfrac{1}{3^n}\right)$

102. $a_n = 4 + \dfrac{1}{2^n}$

103. Let $\{a_n\}$ be an increasing sequence such that $2 \leq a_n \leq 4$. Explain why $\{a_n\}$ has a limit. What can you conclude about the limit?

104. Let $\{a_n\}$ be a monotonic sequence such that $a_n \leq 1$. Discuss the convergence of $\{a_n\}$. If $\{a_n\}$ converges, what can you conclude about its limit?

105. ***Compound Interest*** Consider the sequence $\{A_n\}$ whose *n*th term is given by

$$A_n = P\left(1 + \frac{r}{12}\right)^n$$

where P is the principal, A_n is the account balance after n months, and r is the interest rate compounded annually.

(a) Is $\{A_n\}$ a convergent sequence? Explain.

(b) Find the first 10 terms of the sequence if $P = \$10{,}000$ and $r = 0.055$.

106. ***Compound Interest*** A deposit of $100 is made at the beginning of each month in an account at an annual interest rate of 3% compounded monthly. The balance in the account after n months is $A_n = 100(401)(1.0025^n - 1)$.

(a) Compute the first six terms of the sequence $\{A_n\}$.

(b) Find the balance in the account after 5 years by computing the 60th term of the sequence.

(c) Find the balance in the account after 20 years by computing the 240th term of the sequence.

WRITING ABOUT CONCEPTS

107. Is it possible for a sequence to converge to two different numbers? If so, give an example. If not, explain why not.

108. In your own words, define each of the following.

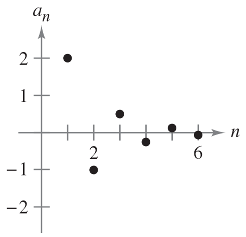
(a) Sequence

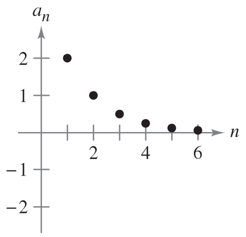
(b) Convergence of a sequence

(c) Monotonic sequence

(d) Bounded sequence

109. The graphs of two sequences are shown in the figures. Which graph represents the sequence with alternating signs? Explain.





CAPSTONE

110. Give an example of a sequence satisfying the condition or explain why no such sequence exists. (Examples are not unique.)

(a) A monotonically increasing sequence that converges to 10

(b) A monotonically increasing bounded sequence that does not converge

(c) A sequence that converges to $\frac{3}{4}$

(d) An unbounded sequence that converges to 100

111. ***Government Expenditures*** A government program that currently costs taxpayers \$4.5 billion per year is cut back by 20 percent per year.

(a) Write an expression for the amount budgeted for this program after n years.

(b) Compute the budgets for the first 4 years.

(c) Determine the convergence or divergence of the sequence of reduced budgets. If the sequence converges, find its limit.

112. ***Inflation*** If the rate of inflation is $4\frac{1}{2}\%$ per year and the average price of a car is currently \$25,000, the average price after n years is

$$P_n = \$25{,}000(1.045)^n.$$

Compute the average prices for the next 5 years.

113. ***Modeling Data*** The federal debts a_n (in billions of dollars) of the United States from 2002 through 2006 are shown in the table, where n represents the year, with $n = 2$ corresponding to 2002. *(Source: U.S. Office of Management and Budget)*

n	2	3	4	5	6
a_n	6198.4	6760.0	7354.7	7905.3	8451.4

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$a_n = bn^2 + cn + d, \quad n = 2, 3, 4, 5, 6$$

for the data. Use the graphing utility to plot the points and graph the model.

(b) Use the model to predict the amount of the federal debt in the year 2012.

114. ***Modeling Data*** The per capita personal incomes a_n in the United States from 1996 through 2006 are given below as ordered pairs of the form (n, a_n), where n represents the year, with $n = 6$ corresponding to 1996. *(Source: U.S. Bureau of Economic Analysis)*

(6, 24,176), (7, 25,334), (8, 26,880), (9, 27,933),
(10, 29,855), (11, 30,572), (12, 30,805), (13, 31,469),
(14, 33,102), (15, 34,493), (16, 36,313)

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$a_n = bn + c, \quad n = 6, 7, \ldots, 16$$

for the data. Graphically compare the points and the model.

(b) Use the model to predict per capita personal income in the year 2012.

115. ***Comparing Exponential and Factorial Growth*** Consider the sequence $a_n = 10^n/n!$.

(a) Find two consecutive terms that are equal in magnitude.

(b) Are the terms following those found in part (a) increasing or decreasing?

(c) In Section 8.7, Exercises 73–78, it was shown that for "large" values of the independent variable an exponential function increases more rapidly than a polynomial function. From the result in part (b), what inference can you make about the rate of growth of an exponential function versus a factorial function for "large" integer values of n?

116. Compute the first six terms of the sequence

$$\{a_n\} = \left\{\left(1 + \frac{1}{n}\right)^n\right\}.$$

If the sequence converges, find its limit.

117. Compute the first six terms of the sequence $\{a_n\} = \left\{\sqrt[n]{n}\right\}$. If the sequence converges, find its limit.

118. Prove that if $\{s_n\}$ converges to L and $L > 0$, then there exists a number N such that $s_n > 0$ for $n > N$.

True or False? **In Exercises 119–124, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

119. If $\{a_n\}$ converges to 3 and $\{b_n\}$ converges to 2, then $\{a_n + b_n\}$ converges to 5.

120. If $\{a_n\}$ converges, then $\lim\limits_{n\to\infty} (a_n - a_{n+1}) = 0$.

121. If $n > 1$, then $n! = n(n - 1)!$.

122. If $\{a_n\}$ converges, then $\{a_n/n\}$ converges to 0.

123. If $\{a_n\}$ converges to 0 and $\{b_n\}$ is bounded, then $\{a_nb_n\}$ converges to 0.

124. If $\{a_n\}$ diverges and $\{b_n\}$ diverges, then $\{a_n + b_n\}$ diverges.

125. ***Fibonacci Sequence*** In a study of the progeny of rabbits, Fibonacci (ca. 1170–ca. 1240) encountered the sequence now bearing his name. The sequence is defined recursively as $a_{n+2} = a_n + a_{n+1}$, where $a_1 = 1$ and $a_2 = 1$.

(a) Write the first 12 terms of the sequence.

(b) Write the first 10 terms of the sequence defined by

$$b_n = \frac{a_{n+1}}{a_n}, \quad n \geq 1.$$

(c) Using the definition in part (b), show that

$$b_n = 1 + \frac{1}{b_{n-1}}.$$

(d) The **golden ratio** ρ can be defined by $\lim\limits_{n\to\infty} b_n = \rho$. Show that $\rho = 1 + 1/\rho$ and solve this equation for ρ.

126. ***Conjecture*** Let $x_0 = 1$ and consider the sequence x_n given by the formula

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \quad n = 1, 2, \ldots .$$

Use a graphing utility to compute the first 10 terms of the sequence and make a conjecture about the limit of the sequence.

127. Consider the sequence

$$\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots$$

(a) Compute the first five terms of this sequence.

(b) Write a recursion formula for a_n, for $n \geq 2$.

(c) Find $\lim\limits_{n\to\infty} a_n$.

128. Consider the sequence

$$\sqrt{6}, \sqrt{6 + \sqrt{6}}, \sqrt{6 + \sqrt{6 + \sqrt{6}}}, \ldots$$

(a) Compute the first five terms of this sequence.

(b) Write a recursion formula for a_n, for $n \geq 2$.

(c) Find $\lim\limits_{n\to\infty} a_n$.

129. Consider the sequence $\{a_n\}$ where $a_1 = \sqrt{k}$, $a_{n+1} = \sqrt{k + a_n}$, and $k > 0$.

(a) Show that $\{a_n\}$ is increasing and bounded.

(b) Prove that $\lim\limits_{n\to\infty} a_n$ exists.

(c) Find $\lim\limits_{n\to\infty} a_n$.

130. ***Arithmetic-Geometric Mean*** Let $a_0 > b_0 > 0$. Let a_1 be the arithmetic mean of a_0 and b_0 and let b_1 be the geometric mean of a_0 and b_0.

$$a_1 = \frac{a_0 + b_0}{2} \qquad \text{Arithmetic mean}$$

$$b_1 = \sqrt{a_0 b_0} \qquad \text{Geometric mean}$$

Now define the sequences $\{a_n\}$ and $\{b_n\}$ as follows.

$$a_n = \frac{a_{n-1} + b_{n-1}}{2} \qquad b_n = \sqrt{a_{n-1}b_{n-1}}$$

(a) Let $a_0 = 10$ and $b_0 = 3$. Write out the first five terms of $\{a_n\}$ and $\{b_n\}$. Compare the terms of $\{b_n\}$. Compare a_n and b_n. What do you notice?

(b) Use induction to show that $a_n > a_{n+1} > b_{n+1} > b_n$, for $a_0 > b_0 > 0$.

(c) Explain why $\{a_n\}$ and $\{b_n\}$ are both convergent.

(d) Show that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

131. (a) Let $f(x) = \sin x$ and $a_n = n \sin 1/n$. Show that $\lim\limits_{n\to\infty} a_n = f'(0) = 1$.

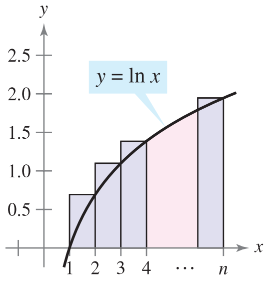
(b) Let $f(x)$ be differentiable on the interval $[0, 1]$ and $f(0) = 0$. Consider the sequence $\{a_n\}$, where $a_n = nf(1/n)$. Show that $\lim\limits_{n\to\infty} a_n = f'(0)$.

132. Consider the sequence $\{a_n\} = \{nr^n\}$. Decide whether $\{a_n\}$ converges for each value of r.

(a) $r = \frac{1}{2}$ (b) $r = 1$ (c) $r = \frac{3}{2}$

(d) For what values of r does the sequence $\{nr^n\}$ converge?

133. (a) Show that $\int_1^n \ln x\,dx < \ln(n!)$ for $n \geq 2$.



(b) Draw a graph similar to the one above that shows

$$\ln(n!) < \int_1^{n+1} \ln x\,dx.$$

(c) Use the results of parts (a) and (b) to show that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}, \text{ for } n > 1.$$

(d) Use the Squeeze Theorem for Sequences and the result of part (c) to show that $\lim\limits_{n\to\infty} \left(\sqrt[n]{n!}/n\right) = 1/e$.

(e) Test the result of part (d) for $n = 20$, 50, and 100.

134. Consider the sequence $\{a_n\} = \left\{\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \frac{1}{1 + (k/n)}\right\}$.

(a) Write the first five terms of $\{a_n\}$.

(b) Show that $\lim\limits_{n\to\infty} a_n = \ln 2$ by interpreting a_n as a Riemann sum of a definite integral.

135. Prove, using the definition of the limit of a sequence, that

$$\lim_{n\to\infty} \frac{1}{n^3} = 0.$$

136. Prove, using the definition of the limit of a sequence, that $\lim\limits_{n\to\infty} r^n = 0$ for $-1 < r < 1$.

137. Find a divergent sequence $\{a_n\}$ such that $\{a_{2n}\}$ converges.

138. Show that the converse of Theorem 9.1 is not true. [*Hint:* Find a function $f(x)$ such that $f(n) = a_n$ converges but $\lim\limits_{x\to\infty} f(x)$ does not exist.]

139. Prove Theorem 9.5 for a nonincreasing sequence.

PUTNAM EXAM CHALLENGE

140. Let $\{x_n\}$, $n \geq 0$, be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove that there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$, for all $n \geq 1$.

141. Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and, for $n \geq 3$,

$$T_n = (n + 4)T_{n-1} - 4nT_{n-2} + (4n - 8)T_{n-3}.$$

The first 10 terms of the sequence are

2, 3, 6, 14, 40, 152, 784, 5168, 40,576, 363,392.

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

These problems were composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.

9.2 Series and Convergence

- **Understand the definition of a convergent infinite series.**
- **Use properties of infinite geometric series.**
- **Use the *n*th-Term Test for Divergence of an infinite series.**

INFINITE SERIES

The study of infinite series was considered a novelty in the fourteenth century. Logician Richard Suiseth, whose nickname was Calculator, solved this problem.

If throughout the first half of a given time interval a variation continues at a certain intensity, throughout the next quarter of the interval at double the intensity, throughout the following eighth at triple the intensity and so ad infinitum; then the average intensity for the whole interval will be the intensity of the variation during the second subinterval (or double the intensity).

This is the same as saying that the sum of the infinite series

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots+\frac{n}{2^n}+\cdots$$

is 2.

Infinite Series

One important application of infinite sequences is in representing "infinite summations." Informally, if $\{a_n\}$ is an infinite sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Infinite series

is an **infinite series** (or simply a **series**). The numbers a_1, a_2, a_3, are the **terms** of the series. For some series it is convenient to begin the index at $n = 0$ (or some other integer). As a typesetting convention, it is common to represent an infinite series as simply $\Sigma\, a_n$. In such cases, the starting value for the index must be taken from the context of the statement.

To find the sum of an infinite series, consider the following **sequence of partial sums.**

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n \end{aligned}$$

If this sequence of partial sums converges, the series is said to converge and has the sum indicated in the following definition.

DEFINITIONS OF CONVERGENT AND DIVERGENT SERIES

For the infinite series $\sum_{n=1}^{\infty} a_n$, the ***n*th partial sum** is given by

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ **converges.** The limit S is called the **sum of the series.**

$$S = a_1 + a_2 + \cdots + a_n + \cdots \qquad S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges, then the series **diverges.**

EXPLORATION

Finding the Sum of an Infinite Series Find the sum of each infinite series. Explain your reasoning.

a. $0.1 + 0.01 + 0.001 + 0.0001 + \cdots$

b. $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \cdots$

c. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

d. $\frac{15}{100} + \frac{15}{10{,}000} + \frac{15}{1{,}000{,}000} + \cdots$

STUDY TIP As you study this chapter, you will see that there are two basic questions involving infinite series. Does a series converge or does it diverge? If a series converges, what is its sum? These questions are not always easy to answer, especially the second one.

TECHNOLOGY Figure 9.5 shows the first 15 partial sums of the infinite series in Example 1(a). Notice how the values appear to approach the line $y = 1$.

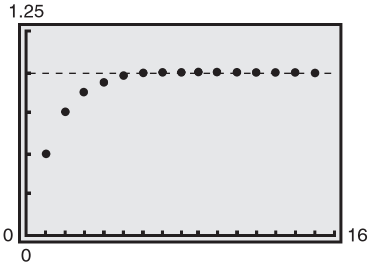


Figure 9.5

EXAMPLE 1 Convergent and Divergent Series

a. The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

has the following partial sums.

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

Because

$$\lim_{n \to \infty} \frac{2^n - 1}{2^n} = 1$$

it follows that the series converges and its sum is 1.

NOTE You can geometrically determine the partial sums of the series in Example 1(a) using Figure 9.6.

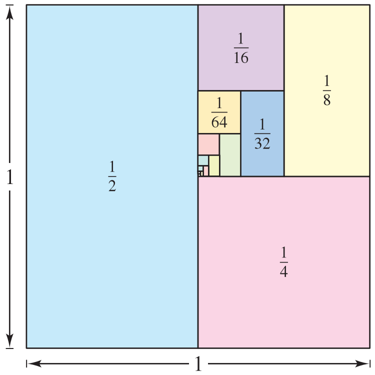


Figure 9.6

b. The nth partial sum of the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

is given by

$$S_n = 1 - \frac{1}{n+1}.$$

Because the limit of S_n is 1, the series converges and its sum is 1.

c. The series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$$

diverges because $S_n = n$ and the sequence of partial sums diverges. ■

The series in Example 1(b) is a **telescoping series** of the form

$$(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots.$$

Telescoping series

FOR FURTHER INFORMATION To learn more about the partial sums of infinite series, see the article "Six Ways to Sum a Series" by Dan Kalman in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

Note that b_2 is canceled by the second term, b_3 is canceled by the third term, and so on. Because the nth partial sum of this series is

$$S_n = b_1 - b_{n+1}$$

it follows that a telescoping series will converge if and only if b_n approaches a finite number as $n \to \infty$. Moreover, if the series converges, its sum is

$$S = b_1 - \lim_{n \to \infty} b_{n+1}.$$

EXAMPLE 2 Writing a Series in Telescoping Form

Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

Solution

Using partial fractions, you can write

$$a_n = \frac{2}{4n^2 - 1} = \frac{2}{(2n - 1)(2n + 1)} = \frac{1}{2n - 1} - \frac{1}{2n + 1}.$$

From this telescoping form, you can see that the nth partial sum is

$$S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdot \cdot \cdot + \left(\frac{1}{2n - 1} - \frac{1}{2n + 1}\right) = 1 - \frac{1}{2n + 1}.$$

So, the series converges and its sum is 1. That is,

$$\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{2n + 1}\right) = 1.$$

■

Geometric Series

The series given in Example 1(a) is a **geometric series.** In general, the series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdot \cdot \cdot + ar^n + \cdot \cdot \cdot, \quad a \neq 0$$

Geometric series

is a **geometric series** with ratio r.

THEOREM 9.6 CONVERGENCE OF A GEOMETRIC SERIES

A geometric series with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}, \quad 0 < |r| < 1.$$

PROOF It is easy to see that the series diverges if $r = \pm 1$. If $r \neq \pm 1$, then $S_n = a + ar + ar^2 + \cdot \cdot \cdot + ar^{n-1}$. Multiplication by r yields

$$rS_n = ar + ar^2 + ar^3 + \cdot \cdot \cdot + ar^n.$$

Subtracting the second equation from the first produces $S_n - rS_n = a - ar^n$. Therefore, $S_n(1 - r) = a(1 - r^n)$, and the nth partial sum is

$$S_n = \frac{a}{1 - r}(1 - r^n).$$

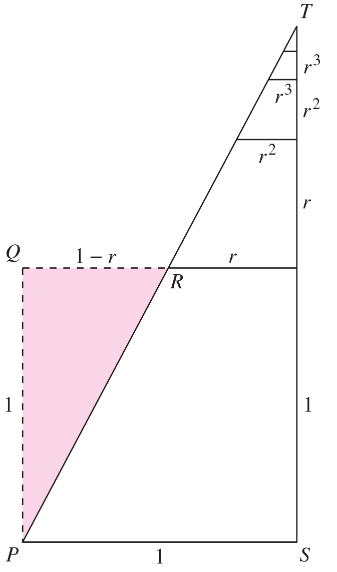
If $0 < |r| < 1$, it follows that $r^n \to 0$ as $n \to \infty$, and you obtain

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[\frac{a}{1 - r}(1 - r^n)\right] = \frac{a}{1 - r}\left[\lim_{n\to\infty} (1 - r^n)\right] = \frac{a}{1 - r}$$

which means that the series *converges* and its sum is $a/(1 - r)$. It is left to you to show that the series diverges if $|r| > 1$. ■

EXPLORATION

In "Proof Without Words," by Benjamin G. Klein and Irl C. Bivens, the authors present the following diagram. Explain why the final statement below the diagram is valid. How is this result related to Theorem 9.6?



$$\Delta PQR \approx \Delta TSP$$

$$1 + r + r^2 + r^3 + \cdot \cdot \cdot = \frac{1}{1 - r}$$

Exercise taken from "Proof Without Words" by Benjamin G. Klein and Irl C. Bivens, *Mathematics Magazine*, 61, No. 4, October 1988, p. 219, by permission of the authors.

TECHNOLOGY Try using a graphing utility or writing a computer program to compute the sum of the first 20 terms of the sequence in Example 3(a). You should obtain a sum of about 5.999994.

EXAMPLE 3 Convergent and Divergent Geometric Series

a. The geometric series

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$$
$$= 3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots$$

has a ratio of $r = \frac{1}{2}$ with $a = 3$. Because $0 < |r| < 1$, the series converges and its sum is

$$S = \frac{a}{1-r} = \frac{3}{1-(1/2)} = 6.$$

b. The geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$$

has a ratio of $r = \frac{3}{2}$. Because $|r| \geq 1$, the series diverges. ■

The formula for the sum of a geometric series can be used to write a repeating decimal as the ratio of two integers, as demonstrated in the next example.

EXAMPLE 4 A Geometric Series for a Repeating Decimal

Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

Solution For the repeating decimal $0.\overline{08}$, you can write

$$0.080808\ldots = \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \frac{8}{10^8} + \cdots$$
$$= \sum_{n=0}^{\infty} \left(\frac{8}{10^2}\right)\left(\frac{1}{10^2}\right)^n.$$

For this series, you have $a = 8/10^2$ and $r = 1/10^2$. So,

$$0.080808\ldots = \frac{a}{1-r} = \frac{8/10^2}{1-(1/10^2)} = \frac{8}{99}.$$

Try dividing 8 by 99 on a calculator to see that it produces $0.\overline{08}$. ■

The convergence of a series is not affected by removal of a finite number of terms from the beginning of the series. For instance, the geometric series

$$\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

both converge. Furthermore, because the sum of the second series is $a/(1-r) = 2$, you can conclude that the sum of the first series is

$$S = 2 - \left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3\right]$$
$$= 2 - \frac{15}{8} = \frac{1}{8}.$$

STUDY TIP As you study this chapter, it is important to distinguish between an infinite series and a sequence. A sequence is an ordered collection of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

whereas a series is an infinite sum of terms from a sequence

$$a_1 + a_2 + \cdots + a_n + \cdots.$$

The following properties are direct consequences of the corresponding properties of limits of sequences.

THEOREM 9.7 PROPERTIES OF INFINITE SERIES

Let $\Sigma\, a_n$ and $\Sigma\, b_n$ be convergent series, and let A, B, and c be real numbers. If $\Sigma\, a_n = A$ and $\Sigma\, b_n = B$, then the following series converge to the indicated sums.

1. $\displaystyle\sum_{n=1}^{\infty} ca_n = cA$
2. $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
3. $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

*n*th-Term Test for Divergence

The following theorem states that if a series converges, the limit of its nth term must be 0.

NOTE Be sure you see that the converse of Theorem 9.8 is generally not true. That is, if the sequence $\{a_n\}$ converges to 0, then the series $\Sigma\, a_n$ may either converge or diverge.

THEOREM 9.8 LIMIT OF THE *n*TH TERM OF A CONVERGENT SERIES

If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

PROOF Assume that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L.$$

Then, because $S_n = S_{n-1} + a_n$ and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = L$$

it follows that

$$\begin{aligned} L = \lim_{n\to\infty} S_n &= \lim_{n\to\infty} (S_{n-1} + a_n) \\ &= \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n \\ &= L + \lim_{n\to\infty} a_n \end{aligned}$$

which implies that $\{a_n\}$ converges to 0. ■

The contrapositive of Theorem 9.8 provides a useful test for *divergence*. This ***n*th-Term Test for Divergence** states that if the limit of the nth term of a series does *not* converge to 0, the series must diverge.

THEOREM 9.9 *n*TH-TERM TEST FOR DIVERGENCE

If $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

EXAMPLE 5 Using the *n*th-Term Test for Divergence

a. For the series $\sum_{n=0}^{\infty} 2^n$, you have

$$\lim_{n\to\infty} 2^n = \infty.$$

So, the limit of the *n*th term is not 0, and the series diverges.

b. For the series $\sum_{n=1}^{\infty} \frac{n!}{2n! + 1}$, you have

$$\lim_{n\to\infty} \frac{n!}{2n! + 1} = \frac{1}{2}.$$

So, the limit of the *n*th term is not 0, and the series diverges.

c. For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, you have

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Because the limit of the *n*th term is 0, the *n*th-Term Test for Divergence does *not* apply and you can draw no conclusions about convergence or divergence. (In the next section, you will see that this particular series diverges.)

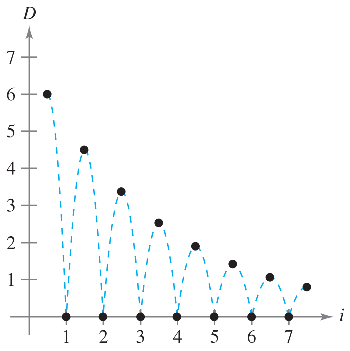
STUDY TIP The series in Example 5(c) will play an important role in this chapter.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdot \cdot \cdot$$

You will see that this series diverges even though the *n*th term approaches 0 as *n* approaches ∞.

EXAMPLE 6 Bouncing Ball Problem

A ball is dropped from a height of 6 feet and begins bouncing, as shown in Figure 9.7. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.



The height of each bounce is three-fourths the height of the preceding bounce.
Figure 9.7

Solution When the ball hits the ground for the first time, it has traveled a distance of $D_1 = 6$ feet. For subsequent bounces, let D_i be the distance traveled up and down. For example, D_2 and D_3 are as follows.

$$D_2 = \underbrace{6\left(\tfrac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\tfrac{3}{4}\right)}_{\text{Down}} = 12\left(\tfrac{3}{4}\right)$$

$$D_3 = \underbrace{6\left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)}_{\text{Down}} = 12\left(\tfrac{3}{4}\right)^2$$

By continuing this process, it can be determined that the total vertical distance is

$$\begin{aligned} D &= 6 + 12\left(\tfrac{3}{4}\right) + 12\left(\tfrac{3}{4}\right)^2 + 12\left(\tfrac{3}{4}\right)^3 + \cdot \cdot \cdot \\ &= 6 + 12\sum_{n=0}^{\infty}\left(\tfrac{3}{4}\right)^{n+1} \\ &= 6 + 12\left(\tfrac{3}{4}\right)\sum_{n=0}^{\infty}\left(\tfrac{3}{4}\right)^{n} \\ &= 6 + 9\left(\frac{1}{1 - \frac{3}{4}}\right) \\ &= 6 + 9(4) \\ &= 42 \text{ feet.} \end{aligned}$$

■

9.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the sequence of partial sums $S_1, S_2, S_3, S_4,$ and S_5.

1. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

2. $\frac{1}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6} + \frac{5}{6 \cdot 7} + \cdots$

3. $3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} - \cdots$

4. $\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2^{n-1}}$ **6.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

In Exercises 7 and 8, determine whether $\{a_n\}$ and $\sum a_n$ are convergent.

7. $a_n = \dfrac{n+1}{n}$ **8.** $a_n = 3\left(\dfrac{4}{5}\right)^n$

In Exercises 9–18, verify that the infinite series diverges.

9. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{7}{6}\right)^n$ **10.** $\displaystyle\sum_{n=0}^{\infty} 5\left(\frac{11}{10}\right)^n$

11. $\displaystyle\sum_{n=0}^{\infty} 1000(1.055)^n$ **12.** $\displaystyle\sum_{n=0}^{\infty} 2(-1.03)^n$

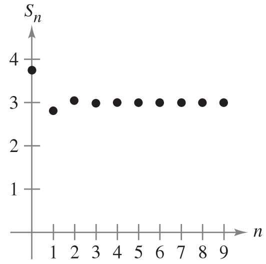
13. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ **14.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+3}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$ **16.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

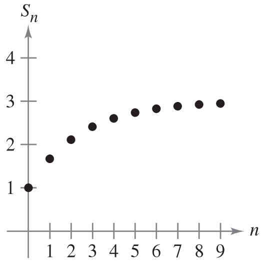
17. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$ **18.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}$

In Exercises 19–24, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).] Use the graph to estimate the sum of the series. Confirm your answer analytically.

(a)



(b)



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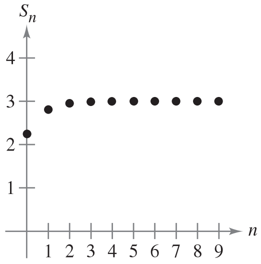
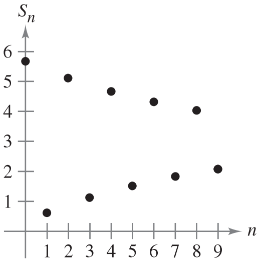
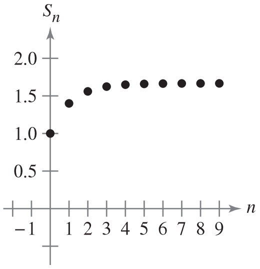


(d)



(e)



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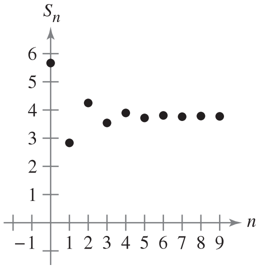


19. $\displaystyle\sum_{n=0}^{\infty} \frac{9}{4}\left(\frac{1}{4}\right)^n$ **20.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

21. $\displaystyle\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n$ **22.** $\displaystyle\sum_{n=0}^{\infty} \frac{17}{3}\left(-\frac{8}{9}\right)^n$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{17}{3}\left(-\frac{1}{2}\right)^n$ **24.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$

In Exercises 25–30, verify that the infinite series converges.

25. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n$ **26.** $\displaystyle\sum_{n=1}^{\infty} 2\left(-\frac{1}{2}\right)^n$

27. $\displaystyle\sum_{n=0}^{\infty} (0.9)^n = 1 + 0.9 + 0.81 + 0.729 + \cdots$

28. $\displaystyle\sum_{n=0}^{\infty} (-0.6)^n = 1 - 0.6 + 0.36 - 0.216 + \cdots$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (Use partial fractions.)

30. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ (Use partial fractions.)

Numerical, Graphical, and Analytic Analysis **In Exercises 31–36, (a) find the sum of the series, (b) use a graphing utility to find the indicated partial sum S_n and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum, and (d) explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.**

n	5	10	20	50	100
S_n					

31. $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n(n+3)}$ **32.** $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n(n+4)}$

33. $\displaystyle\sum_{n=1}^{\infty} 2(0.9)^{n-1}$ **34.** $\displaystyle\sum_{n=1}^{\infty} 3(0.85)^{n-1}$

35. $\displaystyle\sum_{n=1}^{\infty} 10(0.25)^{n-1}$ **36.** $\displaystyle\sum_{n=1}^{\infty} 5\left(-\frac{1}{3}\right)^{n-1}$

In Exercises 37–52, find the sum of the convergent series.

37. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ **38.** $\displaystyle\sum_{n=0}^{\infty} 6\left(\frac{4}{5}\right)^n$

39. $\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n$

40. $\sum_{n=0}^{\infty} 3\left(-\frac{6}{7}\right)^n$

41. $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$

42. $\sum_{n=1}^{\infty} \frac{4}{n(n + 2)}$

43. $\sum_{n=1}^{\infty} \frac{8}{(n + 1)(n + 2)}$

44. $\sum_{n=1}^{\infty} \frac{1}{(2n + 1)(2n + 3)}$

45. $1 + 0.1 + 0.01 + 0.001 + \cdots$

46. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$

47. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \cdots$

48. $4 - 2 + 1 - \frac{1}{2} + \cdots$

49. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n}\right)$

50. $\sum_{n=1}^{\infty} [(0.7)^n + (0.9)^n]$

51. $\sum_{n=1}^{\infty} (\sin 1)^n$

52. $\sum_{n=1}^{\infty} \frac{1}{9n^2 + 3n - 2}$

In Exercises 53–58, (a) write the repeating decimal as a geometric series and (b) write its sum as the ratio of two integers.

53. $0.\overline{4}$

54. $0.\overline{9}$

55. $0.\overline{81}$

56. $0.\overline{01}$

57. $0.0\overline{75}$

58. $0.2\overline{15}$

In Exercises 59–76, determine the convergence or divergence of the series.

59. $\sum_{n=0}^{\infty} (1.075)^n$

60. $\sum_{n=0}^{\infty} \frac{3^n}{1000}$

61. $\sum_{n=1}^{\infty} \frac{n + 10}{10n + 1}$

62. $\sum_{n=1}^{\infty} \frac{4n + 1}{3n - 1}$

63. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n + 2}\right)$

64. $\sum_{n=1}^{\infty} \left(\frac{1}{n + 1} - \frac{1}{n + 2}\right)$

65. $\sum_{n=1}^{\infty} \frac{1}{n(n + 3)}$

66. $\sum_{n=1}^{\infty} \left(\frac{1}{2n(n + 1)}\right)$

67. $\sum_{n=1}^{\infty} \frac{3n - 1}{2n + 1}$

68. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

69. $\sum_{n=0}^{\infty} \frac{4}{2^n}$

70. $\sum_{n=0}^{\infty} \frac{3}{5^n}$

71. $\sum_{n=2}^{\infty} \frac{n}{\ln n}$

72. $\sum_{n=1}^{\infty} \ln \frac{1}{n}$

73. $\sum_{n=1}^{\infty} \left(1 + \frac{k}{n}\right)^n$

74. $\sum_{n=1}^{\infty} e^{-n}$

75. $\sum_{n=1}^{\infty} \arctan n$

76. $\sum_{n=1}^{\infty} \ln\left(\frac{n + 1}{n}\right)$

WRITING ABOUT CONCEPTS

77. State the definitions of convergent and divergent series.

78. Describe the difference between $\lim_{n \to \infty} a_n = 5$ and

$$\sum_{n=1}^{\infty} a_n = 5.$$

79. Define a geometric series, state when it converges, and give the formula for the sum of a convergent geometric series.

80. State the nth-Term Test for Divergence.

WRITING ABOUT CONCEPTS (continued)

81. Explain any differences among the following series.

(a) $\sum_{n=1}^{\infty} a_n$ (b) $\sum_{k=1}^{\infty} a_k$ (c) $\sum_{n=1}^{\infty} a_k$

82. (a) You delete a finite number of terms from a divergent series. Will the new series still diverge? Explain your reasoning.

(b) You add a finite number of terms to a convergent series. Will the new series still converge? Explain your reasoning.

In Exercises 83–90, find all values of x for which the series converges. For these values of x, write the sum of the series as a function of x.

83. $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$

84. $\sum_{n=1}^{\infty} (3x)^n$

85. $\sum_{n=1}^{\infty} (x - 1)^n$

86. $\sum_{n=0}^{\infty} 4\left(\frac{x - 3}{4}\right)^n$

87. $\sum_{n=0}^{\infty} (-1)^n x^n$

88. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

89. $\sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n$

90. $\sum_{n=1}^{\infty} \left(\frac{x^2}{x^2 + 4}\right)^n$

In Exercises 91 and 92, find the value of c for which the series equals the indicated sum.

91. $\sum_{n=2}^{\infty} (1 + c)^{-n} = 2$

92. $\sum_{n=0}^{\infty} e^{cn} = 5$

93. ***Think About It*** Consider the formula

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots.$$

Given $x = -1$ and $x = 2$, can you conclude that either of the following statements is true? Explain your reasoning.

(a) $\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots$

(b) $-1 = 1 + 2 + 4 + 8 + \cdots$

CAPSTONE

94. ***Think About It*** Are the following statements true? Why or why not?

(a) Because $\frac{1}{n^4}$ approaches 0 as n approaches ∞,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 0.$$

(b) Because $\lim_{n \to \infty} \frac{1}{\sqrt[4]{n}} = 0$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ converges.

In Exercises 95 and 96, (a) find the common ratio of the geometric series, (b) write the function that gives the sum of the series, and (c) use a graphing utility to graph the function and the partial sums S_3 and S_5. What do you notice?

95. $1 + x + x^2 + x^3 + \cdots$

96. $1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots$

In Exercises 97 and 98, use a graphing utility to graph the function. Identify the horizontal asymptote of the graph and determine its relationship to the sum of the series.

	Function	*Series*
97.	$f(x) = 3\left[\dfrac{1-(0.5)^x}{1-0.5}\right]$	$\displaystyle\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$
98.	$f(x) = 2\left[\dfrac{1-(0.8)^x}{1-0.8}\right]$	$\displaystyle\sum_{n=0}^{\infty} 2\left(\frac{4}{5}\right)^n$

Writing **In Exercises 99 and 100, use a graphing utility to determine the first term that is less than 0.0001 in each of the convergent series. Note that the answers are very different. Explain how this will affect the rate at which the series converges.**

99. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \quad \sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n$

100. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}, \quad \sum_{n=1}^{\infty} (0.01)^n$

101. ***Marketing*** An electronic games manufacturer producing a new product estimates the annual sales to be 8000 units. Each year 5% of the units that have been sold will become inoperative. So, 8000 units will be in use after 1 year, $[8000 + 0.95(8000)]$ units will be in use after 2 years, and so on. How many units will be in use after n years?

102. ***Depreciation*** A company buys a machine for $475,000 that depreciates at a rate of 30% per year. Find a formula for the value of the machine after n years. What is its value after 5 years?

103. ***Multiplier Effect*** The total annual spending by tourists in a resort city is $200 million. Approximately 75% of that revenue is again spent in the resort city, and of that amount approximately 75% is again spent in the same city, and so on. Write the geometric series that gives the total amount of spending generated by the $200 million and find the sum of the series.

104. ***Multiplier Effect*** Repeat Exercise 103 if the percent of the revenue that is spent again in the city decreases to 60%.

105. ***Distance*** A ball is dropped from a height of 16 feet. Each time it drops h feet, it rebounds $0.81h$ feet. Find the total distance traveled by the ball.

106. ***Time*** The ball in Exercise 105 takes the following times for each fall.

$s_1 = -16t^2 + 16,$ $\quad s_1 = 0$ if $t = 1$

$s_2 = -16t^2 + 16(0.81),$ $\quad s_2 = 0$ if $t = 0.9$

$s_3 = -16t^2 + 16(0.81)^2,$ $\quad s_3 = 0$ if $t = (0.9)^2$

$s_4 = -16t^2 + 16(0.81)^3,$ $\quad s_4 = 0$ if $t = (0.9)^3$

$\vdots \qquad\qquad \vdots$

$s_n = -16t^2 + 16(0.81)^{n-1},$ $\quad s_n = 0$ if $t = (0.9)^{n-1}$

Beginning with s_2, the ball takes the same amount of time to bounce up as it does to fall, and so the total time elapsed before it comes to rest is given by $t = 1 + 2\displaystyle\sum_{n=1}^{\infty} (0.9)^n$. Find this total time.

Probability **In Exercises 107 and 108, the random variable n represents the number of units of a product sold per day in a store. The probability distribution of n is given by $P(n)$. Find the probability that two units are sold in a given day $[P(2)]$ and show that $P(0) + P(1) + P(2) + P(3) + \cdot \cdot \cdot = 1$.**

107. $P(n) = \dfrac{1}{2}\left(\dfrac{1}{2}\right)^n$

108. $P(n) = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^n$

109. ***Probability*** A fair coin is tossed repeatedly. The probability that the first head occurs on the nth toss is given by $P(n) = \left(\frac{1}{2}\right)^n$, where $n \geq 1$.

(a) Show that $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$.

(b) The expected number of tosses required until the first head occurs in the experiment is given by $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n$. Is this series geometric?

CAS (c) Use a computer algebra system to find the sum in part (b).

110. ***Probability*** In an experiment, three people toss a fair coin one at a time until one of them tosses a head. Determine, for each person, the probability that he or she tosses the first head. Verify that the sum of the three probabilities is 1.

111. ***Area*** The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the triangles outside the second square are shaded (see figure). Determine the area of the shaded regions (a) if this process is continued five more times and (b) if this pattern of shading is continued infinitely.

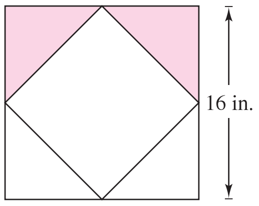


Figure for 111

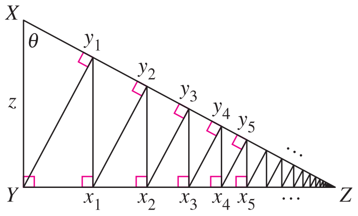


Figure for 112

112. ***Length*** A right triangle XYZ is shown above where $|XY| = z$ and $\angle X = \theta$. Line segments are continually drawn to be perpendicular to the triangle, as shown in the figure.

(a) Find the total length of the perpendicular line segments $|Yy_1| + |x_1y_1| + |x_1y_2| + \cdot \cdot \cdot$ in terms of z and θ.

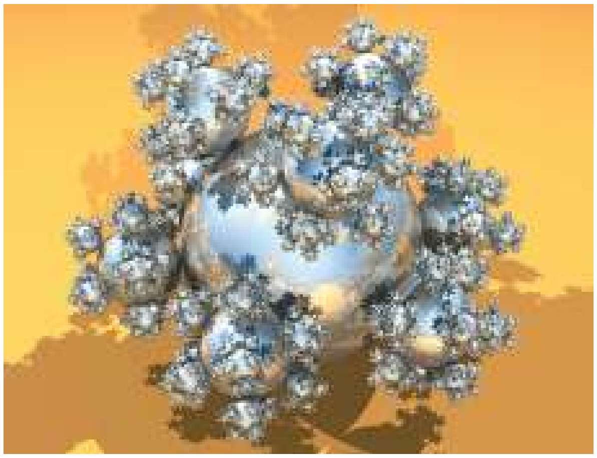
(b) If $z = 1$ and $\theta = \pi/6$, find the total length of the perpendicular line segments.

In Exercises 113–116, use the formula for the nth partial sum of a geometric series

$$\sum_{i=0}^{n-1} ar^i = \frac{a(1-r^n)}{1-r}.$$

113. ***Present Value*** The winner of a $2,000,000 sweepstakes will be paid $100,000 per year for 20 years. The money earns 6% interest per year. The present value of the winnings is $\displaystyle\sum_{n=1}^{20} 100{,}000\left(\frac{1}{1.06}\right)^n$. Compute the present value and interpret its meaning.

114. ***Sphereflake*** The sphereflake shown below is a computer-generated fractal that was created by Eric Haines. The radius of the large sphere is 1. To the large sphere, nine spheres of radius $\frac{1}{3}$ are attached. To each of these, nine spheres of radius $\frac{1}{9}$ are attached. This process is continued infinitely. Prove that the sphereflake has an infinite surface area.



Eric Haines

115. ***Salary*** You go to work at a company that pays \$0.01 for the first day, \$0.02 for the second day, \$0.04 for the third day, and so on. If the daily wage keeps doubling, what would your total income be for working (a) 29 days, (b) 30 days, and (c) 31 days?

116. ***Annuities*** When an employee receives a paycheck at the end of each month, P dollars is invested in a retirement account. These deposits are made each month for t years and the account earns interest at the annual percentage rate r. If the interest is compounded monthly, the amount A in the account at the end of t years is

$$\begin{aligned} A &= P + P\left(1 + \frac{r}{12}\right) + \cdot \cdot \cdot + P\left(1 + \frac{r}{12}\right)^{12t-1} \\ &= P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]. \end{aligned}$$

If the interest is compounded continuously, the amount A in the account after t years is

$$\begin{aligned} A &= P + Pe^{r/12} + Pe^{2r/12} + Pe^{(12t-1)r/12} \\ &= \frac{P(e^{rt} - 1)}{e^{r/12} - 1}. \end{aligned}$$

Verify the formulas for the sums given above.

Annuities **In Exercises 117–120, consider making monthly deposits of *P* dollars in a savings account at an annual interest rate *r*. Use the results of Exercise 116 to find the balance *A* after *t* years if the interest is compounded (a) monthly and (b) continuously.**

117. $P = \$45, \quad r = 3\%, \quad t = 20$ years

118. $P = \$75, \quad r = 5.5\%, \quad t = 25$ years

119. $P = \$100, \quad r = 4\%, \quad t = 35$ years

120. $P = \$30, \quad r = 6\%, \quad t = 50$ years

121. ***Salary*** You accept a job that pays a salary of \$50,000 for the first year. During the next 39 years you receive a 4% raise each year. What would be your total compensation over the 40-year period?

122. ***Salary*** Repeat Exercise 121 if the raise you receive each year is 4.5%. Compare the results.

True or False? **In Exercises 123–128, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

123. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

124. If $\sum_{n=1}^{\infty} a_n = L$, then $\sum_{n=0}^{\infty} a_n = L + a_0$.

125. If $|r| < 1$, then $\sum_{n=1}^{\infty} ar^n = \dfrac{a}{(1 - r)}$.

126. The series $\sum_{n=1}^{\infty} \dfrac{n}{1000(n + 1)}$ diverges.

127. $0.75 = 0.749999 \ldots$

128. Every decimal with a repeating pattern of digits is a rational number.

129. Show that the series $\sum_{n=1}^{\infty} a_n$ can be written in the telescoping form

$$\sum_{n=1}^{\infty} [(c - S_{n-1}) - (c - S_n)]$$

where $S_0 = 0$ and S_n is the nth partial sum.

130. Let $\Sigma\, a_n$ be a convergent series, and let

$$R_N = a_{N+1} + a_{N+2} + \cdot \cdot \cdot$$

be the remainder of the series after the first N terms. Prove that $\lim_{N\to\infty} R_N = 0$.

131. Find two divergent series $\Sigma\, a_n$ and $\Sigma\, b_n$ such that $\Sigma(a_n + b_n)$ converges.

132. Given two infinite series $\Sigma\, a_n$ and $\Sigma\, b_n$ such that $\Sigma\, a_n$ converges and $\Sigma\, b_n$ diverges, prove that $\Sigma(a_n + b_n)$ diverges.

133. Suppose that $\Sigma\, a_n$ diverges and c is a nonzero constant. Prove that $\Sigma\, ca_n$ diverges.

134. If $\sum_{n=1}^{\infty} a_n$ converges where a_n is nonzero, show that $\sum_{n=1}^{\infty} \dfrac{1}{a_n}$ diverges.

135. The Fibonacci sequence is defined recursively by $a_{n+2} = a_n + a_{n+1}$, where $a_1 = 1$ and $a_2 = 1$.

(a) Show that $\dfrac{1}{a_{n+1}\,a_{n+3}} = \dfrac{1}{a_{n+1}\,a_{n+2}} - \dfrac{1}{a_{n+2}\,a_{n+3}}$.

(b) Show that $\sum_{n=0}^{\infty} \dfrac{1}{a_{n+1}\,a_{n+3}} = 1$.

136. Find the values of x for which the infinite series

$$1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + x^6 + \cdot \cdot \cdot$$

converges. What is the sum when the series converges?

137. Prove that $\dfrac{1}{r} + \dfrac{1}{r^2} + \dfrac{1}{r^3} + \cdot \cdot \cdot = \dfrac{1}{r - 1}$, for $|r| > 1$.

138. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$.

Hint: Find the constants A, B, and C such that $\dfrac{1}{n(n+1)(n+2)} = \dfrac{A}{n} + \dfrac{B}{n+1} + \dfrac{C}{n+2}$.

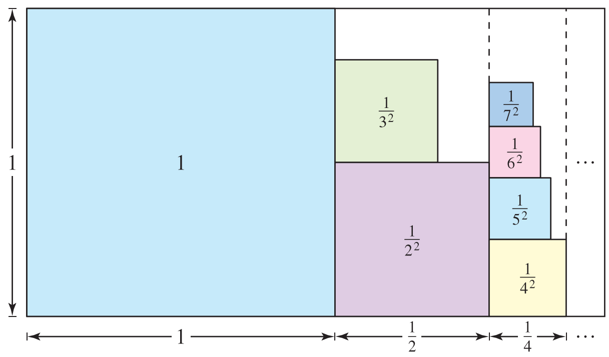
139. (a) The integrand of each definite integral is a difference of two functions. Sketch the graph of each function and shade the region whose area is represented by the integral.

$$\int_0^1 (1-x)\,dx \qquad \int_0^1 (x-x^2)\,dx \qquad \int_0^1 (x^2-x^3)\,dx$$

(b) Find the area of each region in part (a).

(c) Let $a_n = \int_0^1 (x^{n-1} - x^n)\,dx$. Evaluate a_n and $\displaystyle\sum_{n=1}^{\infty} a_n$. What do you observe?

140. ***Writing*** The figure below represents an informal way of showing that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$. Explain how the figure implies this conclusion.



■ **FOR FURTHER INFORMATION** For more on this exercise, see the article "Convergence with Pictures" by P.J. Rippon in *American Mathematical Monthly*.

141. ***Writing*** Read the article "The Exponential-Decay Law Applied to Medical Dosages" by Gerald M. Armstrong and Calvin P. Midgley in *Mathematics Teacher*. (To view this article, go to the website *www.matharticles.com*.) Then write a paragraph on how a geometric sequence can be used to find the total amount of a drug that remains in a patient's system after n equal doses have been administered (at equal time intervals).

PUTNAM EXAM CHALLENGE

142. Write $\displaystyle\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$ as a rational number.

143. Let $f(n)$ be the sum of the first n terms of the sequence 0, 1, 1, 2, 2, 3, 3, 4, . . . , where the nth term is given by

$$a_n = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd} \end{cases}.$$

Show that if x and y are positive integers and $x > y$ then $xy = f(x+y) - f(x-y)$.

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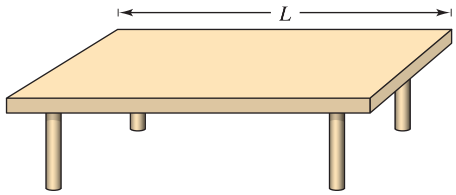
SECTION PROJECT

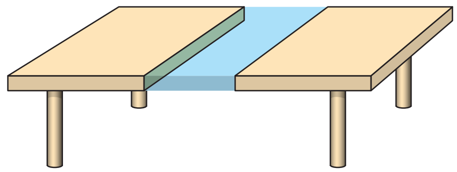
Cantor's Disappearing Table

The following procedure shows how to make a table disappear by removing only half of the table!

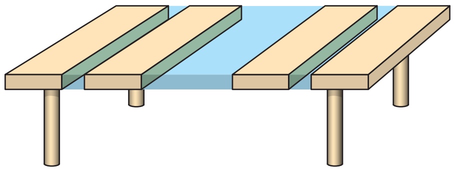
(a) Original table has a length of L.



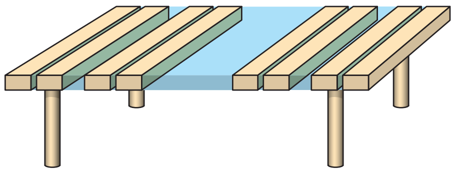
(b) Remove $\frac{1}{4}$ of the table centered at the midpoint. Each remaining piece has a length that is less than $\frac{1}{2}L$.



(c) Remove $\frac{1}{8}$ of the table by taking sections of length $\frac{1}{16}L$ from the centers of each of the two remaining pieces. Now, you have removed $\frac{1}{4} + \frac{1}{8}$ of the table. Each remaining piece has a length that is less than $\frac{1}{4}L$.



(d) Remove $\frac{1}{16}$ of the table by taking sections of length $\frac{1}{64}L$ from the centers of each of the four remaining pieces. Now, you have removed $\frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ of the table. Each remaining piece has a length that is less than $\frac{1}{8}L$.



Will continuing this process cause the table to disappear, even though you have only removed half of the table? Why?

■ **FOR FURTHER INFORMATION** Read the article "Cantor's Disappearing Table" by Larry E. Knop in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

9.3 The Integral Test and *p*-Series

- **Use the Integral Test to determine whether an infinite series converges or diverges.**
- **Use properties of *p*-series and harmonic series.**

The Integral Test

In this and the following section, you will study several convergence tests that apply to series with *positive* terms.

THEOREM 9.10 THE INTEGRAL TEST

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

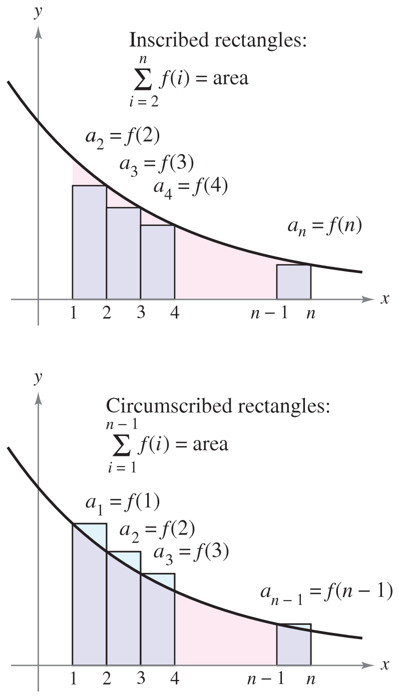


Figure 9.8

PROOF Begin by partitioning the interval $[1, n]$ into $n - 1$ unit intervals, as shown in Figure 9.8. The total areas of the inscribed rectangles and the circumscribed rectangles are as follows.

$$\sum_{i=2}^{n} f(i) = f(2) + f(3) + \cdot \cdot \cdot + f(n) \qquad \text{Inscribed area}$$

$$\sum_{i=1}^{n-1} f(i) = f(1) + f(2) + \cdot \cdot \cdot + f(n-1) \qquad \text{Circumscribed area}$$

The exact area under the graph of f from $x = 1$ to $x = n$ lies between the inscribed and circumscribed areas.

$$\sum_{i=2}^{n} f(i) \leq \int_1^n f(x)\,dx \leq \sum_{i=1}^{n-1} f(i)$$

Using the nth partial sum, $S_n = f(1) + f(2) + \cdot \cdot \cdot + f(n)$, you can write this inequality as

$$S_n - f(1) \leq \int_1^n f(x)\,dx \leq S_{n-1}.$$

Now, assuming that $\int_1^{\infty} f(x)\,dx$ converges to L, it follows that for $n \geq 1$

$$S_n - f(1) \leq L \quad \Longrightarrow \quad S_n \leq L + f(1).$$

Consequently, $\{S_n\}$ is bounded and monotonic, and by Theorem 9.5 it converges. So, $\Sigma\, a_n$ converges. For the other direction of the proof, assume that the improper integral diverges. Then $\int_1^n f(x)\,dx$ approaches infinity as $n \to \infty$, and the inequality $S_{n-1} \geq \int_1^n f(x)\,dx$ implies that $\{S_n\}$ diverges. So, $\Sigma\, a_n$ diverges. ■

NOTE Remember that the convergence or divergence of $\Sigma\, a_n$ is not affected by deleting the first N terms. Similarly, if the conditions for the Integral Test are satisfied for all $x \geq N > 1$, you can simply use the integral $\int_N^{\infty} f(x)\,dx$ to test for convergence or divergence. (This is illustrated in Example 4.) ■

EXAMPLE 1 Using the Integral Test

Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

Solution The function $f(x) = x/(x^2+1)$ is positive and continuous for $x \geq 1$. To determine whether f is decreasing, find the derivative.

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

So, $f'(x) < 0$ for $x > 1$ and it follows that f satisfies the conditions for the Integral Test. You can integrate to obtain

$$\begin{aligned}\int_1^{\infty} \frac{x}{x^2+1}\,dx &= \frac{1}{2}\int_1^{\infty} \frac{2x}{x^2+1}\,dx \\ &= \frac{1}{2}\lim_{b\to\infty}\int_1^{b} \frac{2x}{x^2+1}\,dx \\ &= \frac{1}{2}\lim_{b\to\infty}\Big[\ln(x^2+1)\Big]_1^b \\ &= \frac{1}{2}\lim_{b\to\infty}\left[\ln(b^2+1) - \ln 2\right] \\ &= \infty.\end{aligned}$$

So, the series *diverges*.

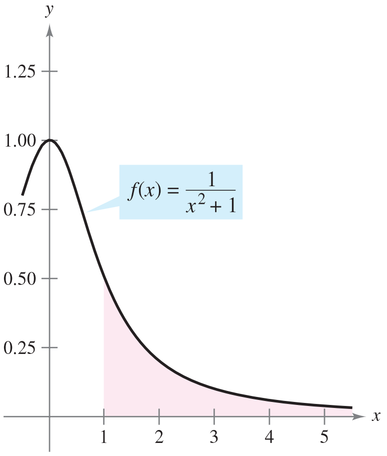
EXAMPLE 2 Using the Integral Test

Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Solution Because $f(x) = 1/(x^2+1)$ satisfies the conditions for the Integral Test (check this), you can integrate to obtain

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+1}\,dx &= \lim_{b\to\infty}\int_1^{b} \frac{1}{x^2+1}\,dx \\ &= \lim_{b\to\infty}\Big[\arctan x\Big]_1^b \\ &= \lim_{b\to\infty}(\arctan b - \arctan 1) \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.\end{aligned}$$

So, the series *converges* (see Figure 9.9). ■



Because the improper integral converges, the infinite series also converges.
Figure 9.9

TECHNOLOGY In Example 2, the fact that the improper integral converges to $\pi/4$ does not imply that the infinite series converges to $\pi/4$. To approximate the sum of the series, you can use the inequality

$$\sum_{n=1}^{N} \frac{1}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \sum_{n=1}^{N} \frac{1}{n^2+1} + \int_N^{\infty} \frac{1}{x^2+1}\,dx.$$

(See Exercise 68.) The larger the value of N, the better the approximation. For instance, using $N = 200$ produces $1.072 \leq \Sigma 1/(n^2+1) \leq 1.077$.

p-Series and Harmonic Series

In the remainder of this section, you will investigate a second type of series that has a simple arithmetic test for convergence or divergence. A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdot \cdot \cdot \qquad p\text{-series}$$

is a ***p*-series,** where p is a positive constant. For $p = 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdot \cdot \cdot \qquad \text{Harmonic series}$$

is the **harmonic** series. A **general harmonic series** is of the form $\Sigma 1/(an + b)$. In music, strings of the same material, diameter, and tension, whose lengths form a harmonic series, produce harmonic tones.

The Integral Test is convenient for establishing the convergence or divergence of p-series. This is shown in the proof of Theorem 9.11.

HARMONIC SERIES

Pythagoras and his students paid close attention to the development of music as an abstract science. This led to the discovery of the relationship between the tone and the length of the vibrating string. It was observed that the most beautiful musical harmonies corresponded to the simplest ratios of whole numbers. Later mathematicians developed this idea into the harmonic series, where the terms in the harmonic series correspond to the nodes on a vibrating string that produce multiples of the fundamental frequency. For example, $\frac{1}{2}$ is twice the fundamental frequency, $\frac{1}{3}$ is three times the fundamental frequency, and so on.

THEOREM 9.11 CONVERGENCE OF *p*-SERIES

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdot \cdot \cdot$$

1. converges if $p > 1$, and
2. diverges if $0 < p \le 1$.

PROOF The proof follows from the Integral Test and from Theorem 8.5, which states that

$$\int_1^{\infty} \frac{1}{x^p}\, dx$$

converges if $p > 1$ and diverges if $0 < p \le 1$. ■

EXAMPLE 3 Convergent and Divergent *p*-Series

Discuss the convergence or divergence of (a) the harmonic series and (b) the p-series with $p = 2$.

Solution

a. From Theorem 9.11, it follows that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdot \cdot \cdot \qquad p = 1$$

diverges.

b. From Theorem 9.11, it follows that the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdot \cdot \cdot \qquad p = 2$$

converges. ■

NOTE The sum of the series in Example 3(b) can be shown to be $\pi^2/6$. (This was proved by Leonhard Euler, but the proof is too difficult to present here.) Be sure you see that the Integral Test does not tell you that the sum of the series is equal to the value of the integral. For instance, the sum of the series in Example 3(b) is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.645$$

but the value of the corresponding improper integral is

$$\int_1^{\infty} \frac{1}{x^2}\, dx = 1.$$

EXAMPLE 4 Testing a Series for Convergence

Determine whether the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Solution This series is similar to the divergent harmonic series. If its terms were larger than those of the harmonic series, you would expect it to diverge. However, because its terms are smaller, you are not sure what to expect. The function $f(x) = 1/(x \ln x)$ is positive and continuous for $x \geq 2$. To determine whether f is decreasing, first rewrite f as $f(x) = (x \ln x)^{-1}$ and then find its derivative.

$$f'(x) = (-1)(x \ln x)^{-2}(1 + \ln x) = -\frac{1 + \ln x}{x^2(\ln x)^2}$$

So, $f'(x) < 0$ for $x > 2$ and it follows that f satisfies the conditions for the Integral Test.

$$\begin{aligned} \int_2^{\infty} \frac{1}{x \ln x}\,dx &= \int_2^{\infty} \frac{1/x}{\ln x}\,dx \\ &= \lim_{b\to\infty} \Big[\ln(\ln x)\Big]_2^b \\ &= \lim_{b\to\infty} [\ln(\ln b) - \ln(\ln 2)] = \infty \end{aligned}$$

The series diverges. ■

NOTE The infinite series in Example 4 diverges very slowly. For instance, the sum of the first 10 terms is approximately 1.6878196, whereas the sum of the first 100 terms is just slightly larger: 2.3250871. In fact, the sum of the first 10,000 terms is approximately 3.015021704. You can see that although the infinite series "adds up to infinity," it does so very slowly. ■

9.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–24, confirm that the Integral Test can be applied to the series. Then use the Integral Test to determine the convergence or divergence of the series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n+5}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

4. $\displaystyle\sum_{n=1}^{\infty} 3^{-n}$

5. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

6. $\displaystyle\sum_{n=1}^{\infty} ne^{-n/2}$

7. $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \cdots$

8. $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots$

9. $\frac{\ln 2}{2} + \frac{\ln 3}{3} + \frac{\ln 4}{4} + \frac{\ln 5}{5} + \frac{\ln 6}{6} + \cdots$

10. $\frac{\ln 2}{\sqrt{2}} + \frac{\ln 3}{\sqrt{3}} + \frac{\ln 4}{\sqrt{4}} + \frac{\ln 5}{\sqrt{5}} + \frac{\ln 6}{\sqrt{6}} + \cdots$

11. $\frac{1}{\sqrt{1}(\sqrt{1}+1)} + \frac{1}{\sqrt{2}(\sqrt{2}+1)} + \frac{1}{\sqrt{3}(\sqrt{3}+1)} + \cdots + \frac{1}{\sqrt{n}(\sqrt{n}+1)} + \cdots$

12. $\frac{1}{4} + \frac{2}{7} + \frac{3}{12} + \cdots + \frac{n}{n^2+3} + \cdots$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$

14. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

16. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$

18. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+3)^3}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n+1}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(4n+5)^{3/2}}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+2n^2+1}$

In Exercises 25 and 26, use the Integral Test to determine the convergence or divergence of the series, where k is a positive integer.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + c}$

26. $\displaystyle\sum_{n=1}^{\infty} n^k e^{-n}$

In Exercises 27–30, explain why the Integral Test does not apply to the series.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

28. $\displaystyle\sum_{n=1}^{\infty} e^{-n} \cos n$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{2 + \sin n}{n}$

30. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sin n}{n}\right)^2$

In Exercises 31–34, use the Integral Test to determine the convergence or divergence of the p-series.

31. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

In Exercises 35–42, use Theorem 9.11 to determine the convergence or divergence of the p-series.

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

37. $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{4}} + \cdots$

38. $1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \dfrac{1}{25} + \cdots$

39. $1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \dfrac{1}{4\sqrt{4}} + \dfrac{1}{5\sqrt{5}} + \cdots$

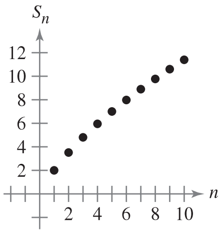
40. $1 + \dfrac{1}{\sqrt[3]{4}} + \dfrac{1}{\sqrt[3]{9}} + \dfrac{1}{\sqrt[3]{16}} + \dfrac{1}{\sqrt[3]{25}} + \cdots$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

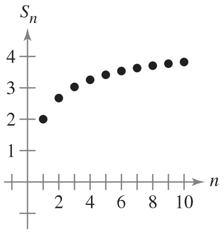
42. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

In Exercises 43–48, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).] Determine the convergence or divergence of the series.

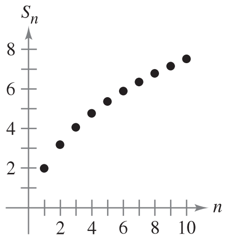
(a)



(b)



(c)



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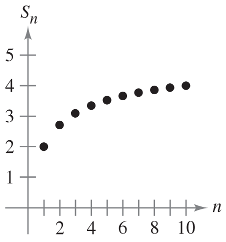

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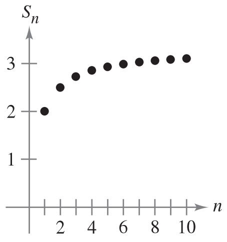

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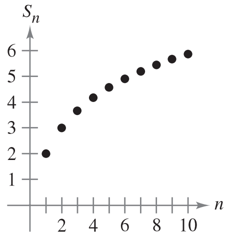


43. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt[4]{n^3}}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^{\pi}}}$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt[5]{n^2}}$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}}$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$

49. ***Numerical and Graphical Analysis*** Use a graphing utility to find the indicated partial sum S_n and complete the table. Then use a graphing utility to graph the first 10 terms of the sequence of partial sums. For each series, compare the rate at which the sequence of partial sums approaches the sum of the series.

n	5	10	20	50	100
S_n					

(a) $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{1}{5}\right)^{n-1} = \frac{15}{4}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

50. ***Numerical Reasoning*** Because the harmonic series diverges, it follows that for any positive real number M there exists a positive integer N such that the partial sum

$$\sum_{n=1}^{N} \frac{1}{n} > M.$$

(a) Use a graphing utility to complete the table.

M	2	4	6	8
N				

(b) As the real number M increases in equal increments, does the number N increase in equal increments? Explain.

WRITING ABOUT CONCEPTS

51. State the Integral Test and give an example of its use.

52. Define a p-series and state the requirements for its convergence.

53. A friend in your calculus class tells you that the following series converges because the terms are very small and approach 0 rapidly. Is your friend correct? Explain.

$$\frac{1}{10{,}000} + \frac{1}{10{,}001} + \frac{1}{10{,}002} + \cdots$$

54. In Exercises 43–48, $\lim\limits_{n\to\infty} a_n = 0$ for each series, but they do not all converge. Is this a contradiction of Theorem 9.9? Why do you think some converge and others diverge? Explain.

55. Let f be a positive, continuous, and decreasing function for $x \geq 1$, such that $a_n = f(n)$. Use a graph to rank the following quantities in decreasing order. Explain your reasoning.

(a) $\displaystyle\sum_{n=2}^{7} a_n$ (b) $\displaystyle\int_1^7 f(x)\,dx$ (c) $\displaystyle\sum_{n=1}^{6} a_n$

CAPSTONE

56. Use a graph to show that the inequality is true. What can you conclude about the convergence or divergence of the series? Explain.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2}\,dx$

In Exercises 57–62, find the positive values of p for which the series converges.

57. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

58. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$

59. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(1+n^2)^p}$

60. $\displaystyle\sum_{n=1}^{\infty} n(1+n^2)^p$

61. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{p^n}$

62. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n[\ln(\ln n)]^p}$

In Exercises 63–66, use the result of Exercise 57 to determine the convergence or divergence of the series.

63. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

64. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt[3]{(\ln n)^2}}$

65. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

66. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln(n^2)}$

67. Let f be a positive, continuous, and decreasing function for $x \geq 1$, such that $a_n = f(n)$. Prove that if the series

$$\sum_{n=1}^{\infty} a_n$$

converges to S, then the remainder $R_N = S - S_N$ is bounded by

$$0 \leq R_N \leq \int_N^{\infty} f(x)\,dx.$$

68. Show that the result of Exercise 67 can be written as

$$\sum_{n=1}^{N} a_n \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{N} a_n + \int_N^{\infty} f(x)\,dx.$$

In Exercises 69–74, use the result of Exercise 67 to approximate the sum of the convergent series using the indicated number of terms. Include an estimate of the maximum error for your approximation.

69. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$, six terms

70. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$, four terms

71. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, ten terms

72. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)[\ln(n+1)]^3}$, ten terms

73. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$, four terms

74. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$, four terms

In Exercises 75–80, use the result of Exercise 67 to find N such that $R_N \leq 0.001$ for the convergent series.

75. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

76. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

77. $\displaystyle\sum_{n=1}^{\infty} e^{-5n}$

78. $\displaystyle\sum_{n=1}^{\infty} e^{-n/2}$

79. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

80. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+5}$

81. (a) Show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.1}}$ converges and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

(b) Compare the first five terms of each series in part (a).

(c) Find $n > 3$ such that

$$\frac{1}{n^{1.1}} < \frac{1}{n \ln n}.$$

82. Ten terms are used to approximate a convergent p-series. Therefore, the remainder is a function of p and is

$$0 \leq R_{10}(p) \leq \int_{10}^{\infty} \frac{1}{x^p}\,dx, \quad p > 1.$$

(a) Perform the integration in the inequality.

(b) Use a graphing utility to represent the inequality graphically.

(c) Identify any asymptotes of the error function and interpret their meaning.

83. ***Euler's Constant*** Let

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdot \cdot \cdot + \frac{1}{n}.$$

(a) Show that $\ln(n + 1) \le S_n \le 1 + \ln n$.

(b) Show that the sequence $\{a_n\} = \{S_n - \ln n\}$ is bounded.

(c) Show that the sequence $\{a_n\}$ is decreasing.

(d) Show that a_n converges to a limit γ (called Euler's constant).

(e) Approximate γ using a_{100}.

84. Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$.

85. Consider the series

$$\sum_{n=2}^{\infty} x^{\ln n}.$$

(a) Determine the convergence or divergence of the series for $x = 1$.

(b) Determine the convergence or divergence of the series for $x = 1/e$.

(c) Find the positive values of x for which the series converges.

86. The **Riemann zeta function** for real numbers is defined for all x for which the series

$$\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$$

converges. Find the domain of the function.

Review **In Exercises 87–98, determine the convergence or divergence of the series.**

87. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n - 2}$

88. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2 - 1}}$

89. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[4]{n}}$

90. $\displaystyle 3\sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$

91. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

92. $\displaystyle\sum_{n=0}^{\infty} (1.042)^n$

93. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 1}}$

94. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n^3}\right)$

95. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

96. $\displaystyle\sum_{n=2}^{\infty} \ln n$

97. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

98. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

SECTION PROJECT

The Harmonic Series

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdot \cdot \cdot + \frac{1}{n} + \cdot \cdot \cdot$$

is one of the most important series in this chapter. Even though its terms tend to zero as n increases,

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

the harmonic series diverges. In other words, even though the terms are getting smaller and smaller, the sum "adds up to infinity."

(a) One way to show that the harmonic series diverges is attributed to Jakob Bernoulli. He grouped the terms of the harmonic series as follows:

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{> \frac{1}{2}} + \underbrace{\frac{1}{5} + \cdot \cdot \cdot + \frac{1}{8}}_{> \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdot \cdot \cdot + \frac{1}{16}}_{> \frac{1}{2}} + \underbrace{\frac{1}{17} + \cdot \cdot \cdot + \frac{1}{32}}_{> \frac{1}{2}} + \cdot \cdot \cdot$$

Write a short paragraph explaining how you can use this grouping to show that the harmonic series diverges.

(b) Use the proof of the Integral Test, Theorem 9.10, to show that

$$\ln(n + 1) \le 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdot \cdot \cdot + \frac{1}{n} \le 1 + \ln n.$$

(c) Use part (b) to determine how many terms M you would need so that

$$\sum_{n=1}^{M} \frac{1}{n} > 50.$$

(d) Show that the sum of the first million terms of the harmonic series is less than 15.

(e) Show that the following inequalities are valid.

$$\ln \tfrac{21}{10} \le \tfrac{1}{10} + \tfrac{1}{11} + \cdot \cdot \cdot + \tfrac{1}{20} \le \ln \tfrac{20}{9}$$

$$\ln \tfrac{201}{100} \le \tfrac{1}{100} + \tfrac{1}{101} + \cdot \cdot \cdot + \tfrac{1}{200} \le \ln \tfrac{200}{99}$$

(f) Use the inequalities in part (e) to find the limit

$$\lim_{m \to \infty} \sum_{n=m}^{2m} \frac{1}{n}.$$

9.4 Comparisons of Series

- **Use the Direct Comparison Test to determine whether a series converges or diverges.**
- **Use the Limit Comparison Test to determine whether a series converges or diverges.**

Direct Comparison Test

For the convergence tests developed so far, the terms of the series have to be fairly simple and the series must have special characteristics in order for the convergence tests to be applied. A slight deviation from these special characteristics can make a test nonapplicable. For example, in the following pairs, the second series cannot be tested by the same convergence test as the first series even though it is similar to the first.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$ is geometric, but $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n}$ is not.
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series, but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is not.
3. $a_n = \dfrac{n}{(n^2+3)^2}$ is easily integrated, but $b_n = \dfrac{n^2}{(n^2+3)^2}$ is not.

In this section you will study two additional tests for positive-term series. These two tests greatly expand the variety of series you are able to test for convergence or divergence. They allow you to *compare* a series having complicated terms with a simpler series whose convergence or divergence is known.

THEOREM 9.12 DIRECT COMPARISON TEST

Let $0 < a_n \le b_n$ for all n.

1. If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.
2. If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

PROOF To prove the first property, let $L = \displaystyle\sum_{n=1}^{\infty} b_n$ and let

$$S_n = a_1 + a_2 + \cdot\cdot\cdot + a_n.$$

Because $0 < a_n \le b_n$, the sequence $S_1, S_2, S_3, . . .$ is nondecreasing and bounded above by L; so, it must converge. Because

$$\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$$

it follows that $\Sigma\, a_n$ converges. The second property is logically equivalent to the first. ■

NOTE As stated, the Direct Comparison Test requires that $0 < a_n \le b_n$ for all n. Because the convergence of a series is not dependent on its first several terms, you could modify the test to require only that $0 < a_n \le b_n$ for all n greater than some integer N. ■

EXAMPLE 1 Using the Direct Comparison Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{2+3^n}.$$

Solution This series resembles

$$\sum_{n=1}^{\infty} \frac{1}{3^n}. \qquad \text{Convergent geometric series}$$

Term-by-term comparison yields

$$a_n = \frac{1}{2+3^n} < \frac{1}{3^n} = b_n, \quad n \geq 1.$$

So, by the Direct Comparison Test, the series converges.

EXAMPLE 2 Using the Direct Comparison Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}.$$

Solution This series resembles

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}. \qquad \text{Divergent } p\text{-series}$$

Term-by-term comparison yields

$$\frac{1}{2+\sqrt{n}} \leq \frac{1}{\sqrt{n}}, \quad n \geq 1$$

which *does not* meet the requirements for divergence. (Remember that if term-by-term comparison reveals a series that is *smaller* than a divergent series, the Direct Comparison Test tells you nothing.) Still expecting the series to diverge, you can compare the given series with

$$\sum_{n=1}^{\infty} \frac{1}{n}. \qquad \text{Divergent harmonic series}$$

In this case, term-by-term comparison yields

$$a_n = \frac{1}{n} \leq \frac{1}{2+\sqrt{n}} = b_n, \quad n \geq 4$$

and, by the Direct Comparison Test, the given series diverges. ■

NOTE To verify the last inequality in Example 2, try showing that $2 + \sqrt{n} \leq n$ whenever $n \geq 4$.

Remember that both parts of the Direct Comparison Test require that $0 < a_n \leq b_n$. Informally, the test says the following about the two series with nonnegative terms.

1. If the "larger" series converges, the "smaller" series must also converge.
2. If the "smaller" series diverges, the "larger" series must also diverge.

Limit Comparison Test

Often a given series closely resembles a p-series or a geometric series, yet you cannot establish the term-by-term comparison necessary to apply the Direct Comparison Test. Under these circumstances you may be able to apply a second comparison test, called the **Limit Comparison Test.**

THEOREM 9.13 LIMIT COMPARISON TEST

Suppose that $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

where L is *finite and positive*. Then the two series $\Sigma\, a_n$ and $\Sigma\, b_n$ either both converge or both diverge.

NOTE As with the Direct Comparison Test, the Limit Comparison Test could be modified to require only that a_n and b_n be positive for all n greater than some integer N.

PROOF Because $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

there exists $N > 0$ such that

$$0 < \frac{a_n}{b_n} < L + 1, \quad \text{for } n \geq N.$$

This implies that

$$0 < a_n < (L + 1)b_n.$$

So, by the Direct Comparison Test, the convergence of $\Sigma\, b_n$ implies the convergence of $\Sigma\, a_n$. Similarly, the fact that

$$\lim_{n\to\infty}\left(\frac{b_n}{a_n}\right) = \frac{1}{L}$$

can be used to show that the convergence of $\Sigma\, a_n$ implies the convergence of $\Sigma\, b_n$. ■

EXAMPLE 3 Using the Limit Comparison Test

Show that the following general harmonic series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{an + b}, \quad a > 0, \quad b > 0$$

Solution By comparison with

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{Divergent harmonic series}$$

you have

$$\lim_{n\to\infty} \frac{1/(an + b)}{1/n} = \lim_{n\to\infty} \frac{n}{an + b} = \frac{1}{a}.$$

Because this limit is greater than 0, you can conclude from the Limit Comparison Test that the given series diverges. ■

The Limit Comparison Test works well for comparing a "messy" algebraic series with a p-series. In choosing an appropriate p-series, you must choose one with an nth term of the same magnitude as the nth term of the given series.

Given Series	*Comparison Series*	*Conclusion*
$\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$	$\sum_{n=1}^{\infty} \frac{1}{n^2}$	Both series converge.
$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n - 2}}$	$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	Both series diverge.
$\sum_{n=1}^{\infty} \frac{n^2 - 10}{4n^5 + n^3}$	$\sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$	Both series converge.

In other words, when choosing a series for comparison, you can disregard all but the *highest powers of n* in both the numerator and the denominator.

EXAMPLE 4 Using the Limit Comparison Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}.$$

Solution Disregarding all but the highest powers of n in the numerator and the denominator, you can compare the series with

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

Convergent p-series

Because

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\sqrt{n}}{n^2 + 1}\right)\left(\frac{n^{3/2}}{1}\right)$$
$$= \lim_{n\to\infty} \frac{n^2}{n^2 + 1} = 1$$

you can conclude by the Limit Comparison Test that the given series converges.

EXAMPLE 5 Using the Limit Comparison Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{n2^n}{4n^3 + 1}.$$

Solution A reasonable comparison would be with the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}.$$

Divergent series

Note that this series diverges by the nth-Term Test. From the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n2^n}{4n^3 + 1}\right)\left(\frac{n^2}{2^n}\right)$$
$$= \lim_{n\to\infty} \frac{1}{4 + (1/n^3)} = \frac{1}{4}$$

you can conclude that the given series diverges. ■

9.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. ***Graphical Analysis*** The figures show the graphs of the first 10 terms, and the graphs of the first 10 terms of the sequence of partial sums, of each series.

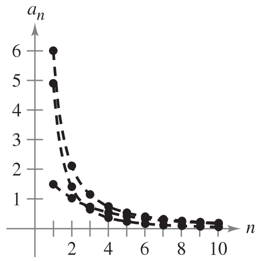
$$\sum_{n=1}^{\infty} \frac{6}{n^{3/2}}, \quad \sum_{n=1}^{\infty} \frac{6}{n^{3/2}+3}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{6}{n\sqrt{n^2+0.5}}$$

(a) Identify the series in each figure.

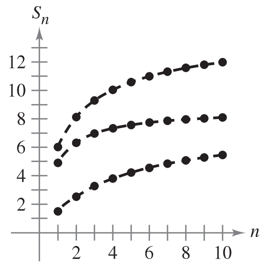
(b) Which series is a p-series? Does it converge or diverge?

(c) For the series that are not p-series, how do the magnitudes of the terms compare with the magnitudes of the terms of the p-series? What conclusion can you draw about the convergence or divergence of the series?

(d) Explain the relationship between the magnitudes of the terms of the series and the magnitudes of the terms of the partial sums.



Graphs of terms



Graphs of partial sums

2. ***Graphical Analysis*** The figures show the graphs of the first 10 terms, and the graphs of the first 10 terms of the sequence of partial sums, of each series.

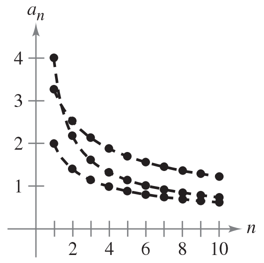
$$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}, \quad \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}-0.5}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{4}{\sqrt{n+0.5}}$$

(a) Identify the series in each figure.

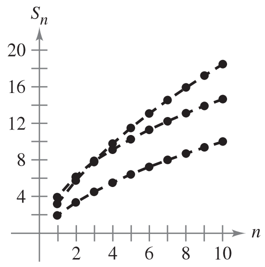
(b) Which series is a p-series? Does it converge or diverge?

(c) For the series that are not p-series, how do the magnitudes of the terms compare with the magnitudes of the terms of the p-series? What conclusion can you draw about the convergence or divergence of the series?

(d) Explain the relationship between the magnitudes of the terms of the series and the magnitudes of the terms of the partial sums.



Graphs of terms



Graphs of partial sums

In Exercises 3–14, use the Direct Comparison Test to determine the convergence or divergence of the series.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

6. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n+1}$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{5^n+3}$

9. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n+1}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

11. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$

13. $\displaystyle\sum_{n=0}^{\infty} e^{-n^2}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n-1}$

In Exercises 15–28, use the Limit Comparison Test to determine the convergence or divergence of the series.

15. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{4^n+1}$

17. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{5^n+1}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{n+3}{n(n^2+4)}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+1}, \quad k > 2$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n+\sqrt{n^2+4}}$

27. $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$

28. $\displaystyle\sum_{n=1}^{\infty} \tan\frac{1}{n}$

In Exercises 29–36, test for convergence or divergence, using each test at least once. Identify which test was used.

(a) *n*th-Term Test	**(b) Geometric Series Test**
(c) *p*-Series Test	**(d) Telescoping Series Test**
(e) Integral Test	**(f) Direct Comparison Test**
(g) Limit Comparison Test	

29. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$

30. $\displaystyle\sum_{n=0}^{\infty} 7\left(-\frac{1}{7}\right)^n$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n+1}$

32. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3-8}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n-2}$

34. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

37. Use the Limit Comparison Test with the harmonic series to show that the series $\sum a_n$ (where $0 < a_n < a_{n-1}$) diverges if $\lim\limits_{n\to\infty} na_n$ is finite and nonzero.

38. Prove that, if $P(n)$ and $Q(n)$ are polynomials of degree j and k, respectively, then the series

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

converges if $j < k - 1$ and diverges if $j \geq k - 1$.

In Exercises 39–42, use the polynomial test given in Exercise 38 to determine whether the series converges or diverges.

39. $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \frac{5}{26} + \cdots$

40. $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \cdots$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$

In Exercises 43 and 44, use the divergence test given in Exercise 37 to show that the series diverges.

43. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5n^4 + 3}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2 + 1}{4n^3 + 2}$

In Exercises 45–48, determine the convergence or divergence of the series.

45. $\frac{1}{200} + \frac{1}{400} + \frac{1}{600} + \frac{1}{800} + \cdots$

46. $\frac{1}{200} + \frac{1}{210} + \frac{1}{220} + \frac{1}{230} + \cdots$

47. $\frac{1}{201} + \frac{1}{204} + \frac{1}{209} + \frac{1}{216} + \cdots$

48. $\frac{1}{201} + \frac{1}{208} + \frac{1}{227} + \frac{1}{264} + \cdots$

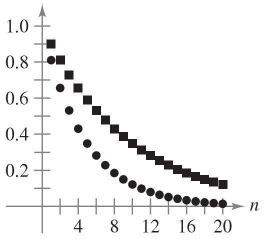
WRITING ABOUT CONCEPTS

49. Review the results of Exercises 45–48. Explain why careful analysis is required to determine the convergence or divergence of a series and why only considering the magnitudes of the terms of a series could be misleading.

50. State the Direct Comparison Test and give an example of its use.

51. State the Limit Comparison Test and give an example of its use.

52. The figure shows the first 20 terms of the convergent series $\sum_{n=1}^{\infty} a_n$ and the first 20 terms of the series $\sum_{n=1}^{\infty} a_n^2$. Identify the two series and explain your reasoning in making the selection.



53. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n - 1)^2}$.

(a) Verify that the series converges.

(b) Use a graphing utility to complete the table.

n	5	10	20	50	100
S_n					

(c) The sum of the series is $\pi^2/8$. Find the sum of the series

$$\sum_{n=3}^{\infty} \frac{1}{(2n - 1)^2}.$$

(d) Use a graphing utility to find the sum of the series

$$\sum_{n=10}^{\infty} \frac{1}{(2n - 1)^2}.$$

CAPSTONE

54. It appears that the terms of the series

$$\frac{1}{1000} + \frac{1}{1001} + \frac{1}{1002} + \frac{1}{1003} + \cdots$$

are less than the corresponding terms of the convergent series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots.$$

If the statement above is correct, the first series converges. Is this correct? Why or why not? Make a statement about how the divergence or convergence of a series is affected by inclusion or exclusion of the first finite number of terms.

True or False? **In Exercises 55–60, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

55. If $0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ diverges.

56. If $0 < a_{n+10} \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

57. If $a_n + b_n \leq c_n$ and $\sum_{n=1}^{\infty} c_n$ converges, then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge. (Assume that the terms of all three series are positive.)

58. If $a_n \leq b_n + c_n$ and $\sum_{n=1}^{\infty} a_n$ diverges, then the series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ both diverge. (Assume that the terms of all three series are positive.)

59. If $0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

60. If $0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

61. Prove that if the nonnegative series

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

converge, then so does the series

$$\sum_{n=1}^{\infty} a_n b_n.$$

62. Use the result of Exercise 61 to prove that if the nonnegative series $\sum_{n=1}^{\infty} a_n$ converges, then so does the series $\sum_{n=1}^{\infty} a_n{}^2$.

63. Find two series that demonstrate the result of Exercise 61.

64. Find two series that demonstrate the result of Exercise 62.

65. Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. Prove that if $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\Sigma\, b_n$ converges, $\Sigma\, a_n$ also converges.

66. Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. Prove that if $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\Sigma\, b_n$ diverges, $\Sigma\, a_n$ also diverges.

67. Use the result of Exercise 65 to show that each series converges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\pi^n}$

68. Use the result of Exercise 66 to show that each series diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

69. Suppose that $\Sigma\, a_n$ is a series with positive terms. Prove that if $\Sigma\, a_n$ converges, then $\Sigma \sin a_n$ also converges.

70. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$ converges.

71. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n\sqrt{n}}$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$.

PUTNAM EXAM CHALLENGE

72. Is the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^{(n+1)/n}}$$

convergent? Prove your statement.

73. Prove that if $\displaystyle\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is

$$\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}.$$

These problems were composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.

SECTION PROJECT

Solera Method

Most wines are produced entirely from grapes grown in a single year. Sherry, however, is a complex mixture of older wines with new wines. This is done with a sequence of barrels (called a solera) stacked on top of each other, as shown in the photo.

© Owen Franken/Corbis

The oldest wine is in the bottom tier of barrels, and the newest is in the top tier. Each year, half of each barrel in the bottom tier is bottled as sherry. The bottom barrels are then refilled with the wine from the barrels above. This process is repeated throughout the solera, with new wine being added to the top barrels. A mathematical model for the amount of n-year-old wine that is removed from a solera (with k tiers) each year is

$$f(n, k) = \binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n+1}, \quad k \le n.$$

(a) Consider a solera that has five tiers, numbered $k = 1, 2, 3, 4$, and 5. In 1990 ($n = 0$), half of each barrel in the top tier (tier 1) was refilled with new wine. How much of this wine was removed from the solera in 1991? In 1992? In 1993? . . . In 2005? During which year(s) was the greatest amount of the 1990 wine removed from the solera?

(b) In part (a), let a_n be the amount of 1990 wine that is removed from the solera in year n. Evaluate

$$\sum_{n=0}^{\infty} a_n.$$

■ **FOR FURTHER INFORMATION** See the article "Finding Vintage Concentrations in a Sherry Solera" by Rhodes Peele and John T. MacQueen in the *UMAP Modules*.

9.5 Alternating Series

- **Use the Alternating Series Test to determine whether an infinite series converges.**
- **Use the Alternating Series Remainder to approximate the sum of an alternating series.**
- **Classify a convergent series as absolutely or conditionally convergent.**
- **Rearrange an infinite series to obtain a different sum.**

Alternating Series

So far, most series you have dealt with have had positive terms. In this section and the following section, you will study series that contain both positive and negative terms. The simplest such series is an **alternating series,** whose terms alternate in sign. For example, the geometric series

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}$$
$$= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdot\cdot\cdot$$

is an *alternating geometric series* with $r = -\frac{1}{2}$. Alternating series occur in two ways: either the odd terms are negative or the even terms are negative.

THEOREM 9.14 ALTERNATING SERIES TEST

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

converge if the following two conditions are met.

1. $\lim\limits_{n\to\infty} a_n = 0$ **2.** $a_{n+1} \le a_n$, for all n

PROOF Consider the alternating series $\Sigma(-1)^{n+1}a_n$. For this series, the partial sum (where $2n$ is even)

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdot\cdot\cdot + (a_{2n-1} - a_{2n})$$

has all nonnegative terms, and therefore $\{S_{2n}\}$ is a nondecreasing sequence. But you can also write

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdot\cdot\cdot - (a_{2n-2} - a_{2n-1}) - a_{2n}$$

which implies that $S_{2n} \le a_1$ for every integer n. So, $\{S_{2n}\}$ is a bounded, nondecreasing sequence that converges to some value L. Because $S_{2n-1} - a_{2n} = S_{2n}$ and $a_{2n} \to 0$, you have

$$\lim_{n\to\infty} S_{2n-1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n}$$
$$= L + \lim_{n\to\infty} a_{2n} = L.$$

Because both S_{2n} and S_{2n-1} converge to the same limit L, it follows that $\{S_n\}$ also converges to L. Consequently, the given alternating series converges. ■

NOTE The second condition in the Alternating Series Test can be modified to require only that $0 < a_{n+1} \le a_n$ for all n greater than some integer N. ■

EXAMPLE 1 Using the Alternating Series Test

NOTE The series in Example 1 is called the *alternating harmonic series.* More is said about this series in Example 7.

Determine the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$.

Solution Note that $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$. So, the first condition of Theorem 9.14 is satisfied. Also note that the second condition of Theorem 9.14 is satisfied because

$$a_{n+1} = \frac{1}{n+1} \le \frac{1}{n} = a_n$$

for all n. So, applying the Alternating Series Test, you can conclude that the series converges.

EXAMPLE 2 Using the Alternating Series Test

Determine the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}$.

Solution To apply the Alternating Series Test, note that, for $n \ge 1$,

$$\begin{aligned} \frac{1}{2} &\le \frac{n}{n+1} \\ \frac{2^{n-1}}{2^n} &\le \frac{n}{n+1} \\ (n+1)2^{n-1} &\le n2^n \\ \frac{n+1}{2^n} &\le \frac{n}{2^{n-1}}. \end{aligned}$$

So, $a_{n+1} = (n+1)/2^n \le n/2^{n-1} = a_n$ for all n. Furthermore, by L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{x}{2^{x-1}} = \lim_{x\to\infty} \frac{1}{2^{x-1}(\ln 2)} = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{n}{2^{n-1}} = 0.$$

Therefore, by the Alternating Series Test, the series converges.

EXAMPLE 3 When the Alternating Series Test Does Not Apply

NOTE In Example 3(a), remember that whenever a series does not pass the first condition of the Alternating Series Test, you can use the nth-Term Test for Divergence to conclude that the series diverges.

a. The alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n} = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \cdots$$

passes the second condition of the Alternating Series Test because $a_{n+1} \le a_n$ for all n. You cannot apply the Alternating Series Test, however, because the series does not pass the first condition. In fact, the series diverges.

b. The alternating series

$$\frac{2}{1} - \frac{1}{1} + \frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \cdots$$

passes the first condition because a_n approaches 0 as $n \to \infty$. You cannot apply the Alternating Series Test, however, because the series does not pass the second condition. To conclude that the series diverges, you can argue that S_{2N} equals the Nth partial sum of the divergent harmonic series. This implies that the sequence of partial sums diverges. So, the series diverges. ■

Alternating Series Remainder

For a convergent alternating series, the partial sum S_N can be a useful approximation for the sum S of the series. The error involved in using $S \approx S_N$ is the remainder $R_N = S - S_N$.

THEOREM 9.15 ALTERNATING SERIES REMAINDER

If a convergent alternating series satisfies the condition $a_{n+1} \le a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \le a_{N+1}.$$

PROOF The series obtained by deleting the first N terms of the given series satisfies the conditions of the Alternating Series Test and has a sum of R_N.

$$\begin{aligned} R_N = S - S_N &= \sum_{n=1}^{\infty} (-1)^{n+1} a_n - \sum_{n=1}^{N} (-1)^{n+1} a_n \\ &= (-1)^N a_{N+1} + (-1)^{N+1} a_{N+2} + (-1)^{N+2} a_{N+3} + \cdots \\ &= (-1)^N (a_{N+1} - a_{N+2} + a_{N+3} - \cdots) \\ |R_N| &= a_{N+1} - a_{N+2} + a_{N+3} - a_{N+4} + a_{N+5} - \cdots \\ &= a_{N+1} - (a_{N+2} - a_{N+3}) - (a_{N+4} - a_{N+5}) - \cdots \le a_{N+1} \end{aligned}$$

Consequently, $|S - S_N| = |R_N| \le a_{N+1}$, which establishes the theorem. ■

EXAMPLE 4 Approximating the Sum of an Alternating Series

Approximate the sum of the following series by its first six terms.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n!}\right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \cdots$$

Solution The series converges by the Alternating Series Test because

$$\frac{1}{(n+1)!} \le \frac{1}{n!} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n!} = 0.$$

The sum of the first six terms is

$$S_6 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \frac{91}{144} \approx 0.63194$$

and, by the Alternating Series Remainder, you have

$$|S - S_6| = |R_6| \le a_7 = \frac{1}{5040} \approx 0.0002.$$

So, the sum S lies between $0.63194 - 0.0002$ and $0.63194 + 0.0002$, and you have

$$0.63174 \le S \le 0.63214.$$ ■

TECHNOLOGY Later, in Section 9.10, you will be able to show that the series in Example 4 converges to

$$\frac{e-1}{e} \approx 0.63212.$$

For now, try using a computer to obtain an approximation of the sum of the series. How many terms do you need to obtain an approximation that is within 0.00001 unit of the actual sum?

Absolute and Conditional Convergence

Occasionally, a series may have both positive and negative terms and not be an alternating series. For instance, the series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdot \cdot \cdot$$

has both positive and negative terms, yet it is not an alternating series. One way to obtain some information about the convergence of this series is to investigate the convergence of the series

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|.$$

By direct comparison, you have $|\sin n| \leq 1$ for all n, so

$$\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}, \quad n \geq 1.$$

Therefore, by the Direct Comparison Test, the series $\sum \left|\frac{\sin n}{n^2}\right|$ converges. The next theorem tells you that the original series also converges.

THEOREM 9.16 ABSOLUTE CONVERGENCE

If the series $\Sigma|a_n|$ converges, then the series $\Sigma\, a_n$ also converges.

PROOF Because $0 \leq a_n + |a_n| \leq 2|a_n|$ for all n, the series

$$\sum_{n=1}^{\infty} (a_n + |a_n|)$$

converges by comparison with the convergent series

$$\sum_{n=1}^{\infty} 2|a_n|.$$

Furthermore, because $a_n = (a_n + |a_n|) - |a_n|$, you can write

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

where both series on the right converge. So, it follows that $\Sigma\, a_n$ converges. ■

The converse of Theorem 9.16 is not true. For instance, the **alternating harmonic series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdot \cdot \cdot$$

converges by the Alternating Series Test. Yet the harmonic series diverges. This type of convergence is called **conditional.**

DEFINITIONS OF ABSOLUTE AND CONDITIONAL CONVERGENCE

1. $\Sigma\, a_n$ is **absolutely convergent** if $\Sigma\, |a_n|$ converges.

2. $\Sigma\, a_n$ is **conditionally convergent** if $\Sigma\, a_n$ converges but $\Sigma\, |a_n|$ diverges.

EXAMPLE 5 Absolute and Conditional Convergence

Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\,n!}{2^n}=\frac{0!}{2^0}-\frac{1!}{2^1}+\frac{2!}{2^2}-\frac{3!}{2^3}+\cdots$

b. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}=-\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}-\cdots$

Solution

a. By the nth-Term Test for Divergence, you can conclude that this series diverges.

b. The given series can be shown to be convergent by the Alternating Series Test. Moreover, because the p-series

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt{n}}\right|=\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots$$

diverges, the given series is *conditionally* convergent.

EXAMPLE 6 Absolute and Conditional Convergence

Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n(n+1)/2}}{3^n}=-\frac{1}{3}-\frac{1}{9}+\frac{1}{27}+\frac{1}{81}-\cdots$

b. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n+1)}=-\frac{1}{\ln 2}+\frac{1}{\ln 3}-\frac{1}{\ln 4}+\frac{1}{\ln 5}-\cdots$

Solution

a. This is *not* an alternating series. However, because

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n(n+1)/2}}{3^n}\right|=\sum_{n=1}^{\infty}\frac{1}{3^n}$$

is a convergent geometric series, you can apply Theorem 9.16 to conclude that the given series is *absolutely* convergent (and therefore convergent).

b. In this case, the Alternating Series Test indicates that the given series converges. However, the series

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\ln(n+1)}\right|=\frac{1}{\ln 2}+\frac{1}{\ln 3}+\frac{1}{\ln 4}+\cdots$$

diverges by direct comparison with the terms of the harmonic series. Therefore, the given series is *conditionally* convergent. ■

Rearrangement of Series

A finite sum such as $(1+3-2+5-4)$ can be rearranged without changing the value of the sum. This is not necessarily true of an infinite series—it depends on whether the series is absolutely convergent (every rearrangement has the same sum) or conditionally convergent.

FOR FURTHER INFORMATION Georg Friedrich Bernhard Riemann (1826–1866) proved that if Σa_n is conditionally convergent and S is any real number, the terms of the series can be rearranged to converge to S. For more on this topic, see the article "Riemann's Rearrangement Theorem" by Stewart Galanor in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

EXAMPLE 7 Rearrangement of a Series

The alternating harmonic series converges to ln 2. That is,

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

(See Exercise 59, Section 9.10.)

Rearrange the series to produce a different sum.

Solution Consider the following rearrangement.

$$\begin{aligned} 1 &- \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \cdots \\ &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \cdots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \cdots \\ &= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots\right) = \frac{1}{2}(\ln 2) \end{aligned}$$

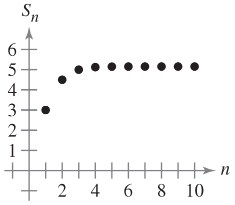
By rearranging the terms, you obtain a sum that is half the original sum. ■

9.5 Exercises

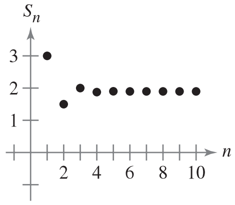
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

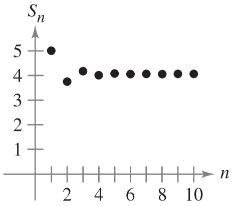
(a)



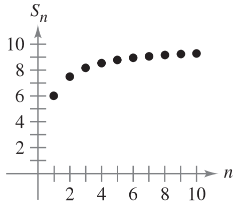
(b)



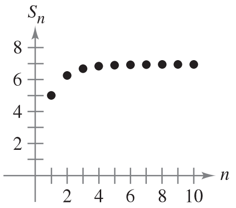
(c)



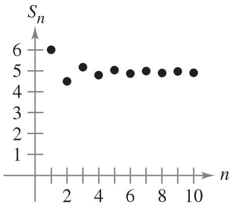
(d)



(e)



(f)



1. $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^2}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}6}{n^2}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n!}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}3}{n!}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n2^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}10}{n2^n}$

Numerical and Graphical Analysis **In Exercises 7–10, explore the Alternating Series Remainder.**

(a) Use a graphing utility to find the indicated partial sum S_n and complete the table.

n	1	2	3	4	5	6	7	8	9	10
S_n										

(b) Use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum.

(c) What pattern exists between the plot of the successive points in part (b) relative to the horizontal line representing the sum of the series? Do the distances between the successive points and the horizontal line increase or decrease?

(d) Discuss the relationship between the answers in part (c) and the Alternating Series Remainder as given in Theorem 9.15.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin 1$

In Exercises 11–36, determine the convergence or divergence of the series.

11. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

12. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{3n+2}$

13. $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{3^n}$

14. $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{e^n}$

15. $\sum_{n=1}^{\infty} \frac{(-1)^{n}(5n-1)}{4n+1}$

16. $\sum_{n=1}^{\infty} \frac{(-1)^{n}n}{\ln(n+1)}$

17. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+2}$

18. $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\ln(n+1)}$

19. $\sum_{n=1}^{\infty} \frac{(-1)^{n}n^2}{n^2+5}$

20. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n^2+5}$

21. $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\sqrt{n}}$

22. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2}{n^2+4}$

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$

24. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln(n+1)}{n+1}$

25. $\sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi}{2}$

26. $\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{(2n-1)\pi}{2}$

27. $\sum_{n=1}^{\infty} \cos n\pi$

28. $\sum_{n=1}^{\infty} \frac{1}{n}\cos n\pi$

29. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!}$

30. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!}$

31. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$

32. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{\sqrt[3]{n}}$

33. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n!}{1\cdot 3\cdot 5\cdots(2n-1)}$

34. $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 4\cdot 7\cdots(3n-2)}$

35. $\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n-e^{-n}} = \sum_{n=1}^{\infty}(-1)^{n+1}\operatorname{csch} n$

36. $\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n+e^{-n}} = \sum_{n=1}^{\infty}(-1)^{n+1}\operatorname{sech} n$

In Exercises 37–40, approximate the sum of the series by using the first six terms. (See Example 4.)

37. $\sum_{n=0}^{\infty} \frac{(-1)^{n}2}{n!}$

38. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}4}{\ln(n+1)}$

39. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3}{n^2}$

40. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{2^n}$

In Exercises 41–46, (a) use Theorem 9.15 to determine the number of terms required to approximate the sum of the convergent series with an error of less than 0.001, and (b) use a graphing utility to approximate the sum of the series with an error of less than 0.001.

41. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} = \frac{1}{e}$

42. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2^n n!} = \frac{1}{\sqrt{e}}$

43. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!} = \sin 1$

44. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n)!} = \cos 1$

45. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$

46. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n} = \ln\frac{5}{4}$

In Exercises 47–50, use Theorem 9.15 to determine the number of terms required to approximate the sum of the series with an error of less than 0.001.

47. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

48. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

49. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3-1}$

50. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$

In Exercises 51–70, determine whether the series converges conditionally or absolutely, or diverges.

51. $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{2^n}$

52. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

53. $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n!}$

54. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n^3}$

55. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+3)^2}$

56. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$

57. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

58. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

59. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2}{(n+1)^2}$

60. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+3)}{n+10}$

61. $\sum_{n=2}^{\infty} \frac{(-1)^{n}}{n\ln n}$

62. $\sum_{n=0}^{\infty} (-1)^{n}e^{-n^2}$

63. $\sum_{n=2}^{\infty} \frac{(-1)^{n}n}{n^3-5}$

64. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4/3}}$

65. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!}$

66. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{\sqrt{n+4}}$

67. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$

68. $\sum_{n=1}^{\infty} (-1)^{n+1}\arctan n$

69. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$

70. $\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi/2]}{n}$

WRITING ABOUT CONCEPTS

71. Define an alternating series.

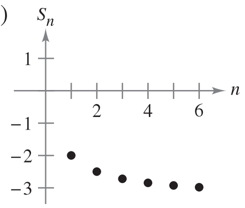
72. State the Alternating Series Test.

73. Give the remainder after *N* terms of a convergent alternating series.

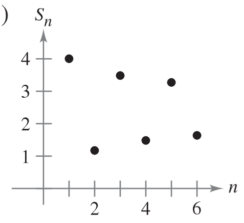
74. In your own words, state the difference between absolute and conditional convergence of an alternating series.

75. The graphs of the sequences of partial sums of two series are shown in the figures. Which graph represents the partial sums of an alternating series? Explain.

(a)



(b)



CAPSTONE

76. Do you agree with the following statements? Why or why not?

(a) If both $\Sigma\, a_n$ and $\Sigma\,(-a_n)$ converge, then $\Sigma\,|a_n|$ converges.

(b) If $\Sigma\, a_n$ diverges, then $\Sigma\,|a_n|$ diverges.

True or False? **In Exercises 77 and 78, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

77. For the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the partial sum S_{100} is an overestimate of the sum of the series.

78. If $\Sigma\, a_n$ and $\Sigma\, b_n$ both converge, then $\Sigma\, a_nb_n$ converges.

In Exercises 79 and 80, find the values of *p* for which the series converges.

79. $\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^p}\right)$

80. $\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n+p}\right)$

81. Prove that if $\Sigma\,|a_n|$ converges, then $\Sigma\, a_n^2$ converges. Is the converse true? If not, give an example that shows it is false.

82. Use the result of Exercise 79 to give an example of an alternating *p*-series that converges, but whose corresponding *p*-series diverges.

83. Give an example of a series that demonstrates the statement you proved in Exercise 81.

84. Find all values of x for which the series $\Sigma\,(x^n/n)$ (a) converges absolutely and (b) converges conditionally.

85. Consider the following series.

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{9}+\frac{1}{8}-\frac{1}{27}+\cdots+\frac{1}{2^n}-\frac{1}{3^n}+\cdots$$

(a) Does the series meet the conditions of Theorem 9.14? Explain why or why not.

(b) Does the series converge? If so, what is the sum?

86. Consider the following series.

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n,\ a_n=\begin{cases}\dfrac{1}{\sqrt{n}}, & \text{if } n \text{ is odd}\\[2mm] \dfrac{1}{n^3}, & \text{if } n \text{ is even}\end{cases}$$

(a) Does the series meet the conditions of Theorem 9.14? Explain why or why not.

(b) Does the series converge? If so, what is the sum?

Review **In Exercises 87–96, test for convergence or divergence and identify the test used.**

87. $\sum_{n=1}^{\infty}\frac{10}{n^{3/2}}$

88. $\sum_{n=1}^{\infty}\frac{3}{n^2+5}$

89. $\sum_{n=1}^{\infty}\frac{3^n}{n^2}$

90. $\sum_{n=1}^{\infty}\frac{1}{2^n+1}$

91. $\sum_{n=0}^{\infty}5\left(\frac{7}{8}\right)^n$

92. $\sum_{n=1}^{\infty}\frac{3n^2}{2n^2+1}$

93. $\sum_{n=1}^{\infty}100e^{-n/2}$

94. $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+4}$

95. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}4}{3n^2-1}$

96. $\sum_{n=2}^{\infty}\frac{\ln n}{n}$

97. The following argument, that $0 = 1$, is *incorrect*. Describe the error.

$$\begin{aligned}0 &= 0+0+0+\cdots\\ &= (1-1)+(1-1)+(1-1)+\cdots\\ &= 1+(-1+1)+(-1+1)+\cdots\\ &= 1+0+0+\cdots\\ &= 1\end{aligned}$$

98. The following argument, $2 = 1$, is *incorrect*. Describe the error. Multiply each side of the alternating harmonic series

$$S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\cdots$$

by 2 to get

$$2S=2-1+\frac{2}{3}-\frac{1}{2}+\frac{2}{5}-\frac{1}{3}+\frac{2}{7}-\frac{1}{4}+\frac{2}{9}-\frac{1}{5}+\cdots$$

Now collect terms with like denominators (as indicated by the arrows) to get

$$2S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\cdots$$

The resulting series is the same one that you started with. So, $2S = S$ and divide each side by S to get $2 = 1$.

■ **FOR FURTHER INFORMATION** For more on this exercise, see the article "Riemann's Rearrangement Theorem" by Stewart Galanor in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

PUTNAM EXAM CHALLENGE

99. Assume as known the (true) fact that the alternating harmonic series

(1) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots$

is convergent, and denote its sum by s. Rearrange the series (1) as follows:

(2) $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots$

Assume as known the (true) fact that the series (2) is also convergent, and denote its sum by S. Denote by s_k, S_k the kth partial sum of the series (1) and (2), respectively. Prove each statement.

(i) $S_{3n}=s_{4n}+\frac{1}{2}s_{2n}$, (ii) $S\neq s$

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9.6 The Ratio and Root Tests

- **Use the Ratio Test to determine whether a series converges or diverges.**
- **Use the Root Test to determine whether a series converges or diverges.**
- **Review the tests for convergence and divergence of an infinite series.**

The Ratio Test

This section begins with a test for absolute convergence—the **Ratio Test.**

THEOREM 9.17 RATIO TEST

Let $\Sigma\, a_n$ be a series with nonzero terms.

1. $\Sigma\, a_n$ converges absolutely if $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1.$

2. $\Sigma\, a_n$ diverges if $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty.$

3. The Ratio Test is inconclusive if $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1.$

PROOF To prove Property 1, assume that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r < 1$$

and choose R such that $0 \le r < R < 1$. By the definition of the limit of a sequence, there exists some $N > 0$ such that $|a_{n+1}/a_n| < R$ for all $n > N$. Therefore, you can write the following inequalities.

$$\begin{aligned}
|a_{N+1}| &< |a_N|R\\
|a_{N+2}| &< |a_{N+1}|R < |a_N|R^2\\
|a_{N+3}| &< |a_{N+2}|R < |a_{N+1}|R^2 < |a_N|R^3\\
&\ \ \vdots
\end{aligned}$$

The geometric series $\Sigma\, |a_N|R^n = |a_N|R + |a_N|R^2 + \cdots + |a_N|R^n + \cdots$ converges, and so, by the Direct Comparison Test, the series

$$\sum_{n=1}^{\infty} |a_{N+n}| = |a_{N+1}| + |a_{N+2}| + \cdots + |a_{N+n}| + \cdots$$

also converges. This in turn implies that the series $\Sigma\, |a_n|$ converges, because discarding a finite number of terms $(n = N - 1)$ does not affect convergence. Consequently, by Theorem 9.16, the series $\Sigma\, a_n$ converges absolutely. The proof of Property 2 is similar and is left as an exercise (see Exercise 99). ■

NOTE The fact that the Ratio Test is inconclusive when $|a_{n+1}/a_n| \to 1$ can be seen by comparing the two series $\Sigma\,(1/n)$ and $\Sigma\,(1/n^2)$. The first series diverges and the second one converges, but in both cases

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1.$$

■

Although the Ratio Test is not a cure for all ills related to testing for convergence, it is particularly useful for series that *converge rapidly*. Series involving factorials or exponentials are frequently of this type.

EXAMPLE 1 Using the Ratio Test

Determine the convergence or divergence of

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

Solution Because $a_n = 2^n/n!$, you can write the following.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left[\frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!}\right]\\ &= \lim_{n\to\infty}\left[\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right]\\ &= \lim_{n\to\infty}\frac{2}{n+1}\\ &= 0 < 1\end{aligned}$$

This series converges because the limit of $|a_{n+1}/a_n|$ is less than 1.

STUDY TIP A step frequently used in applications of the Ratio Test involves simplifying quotients of factorials. In Example 1, for instance, notice that

$$\frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1}.$$

EXAMPLE 2 Using the Ratio Test

Determine whether each series converges or diverges.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ **b.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution

a. This series converges because the limit of $|a_{n+1}/a_n|$ is less than 1.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left[(n+1)^2\left(\frac{2^{n+2}}{3^{n+1}}\right)\left(\frac{3^n}{n^2 2^{n+1}}\right)\right]\\ &= \lim_{n\to\infty}\frac{2(n+1)^2}{3n^2}\\ &= \frac{2}{3} < 1\end{aligned}$$

b. This series diverges because the limit of $|a_{n+1}/a_n|$ is greater than 1.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{(n+1)!}\left(\frac{n!}{n^n}\right)\right]\\ &= \lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{(n+1)}\left(\frac{1}{n^n}\right)\right]\\ &= \lim_{n\to\infty}\frac{(n+1)^n}{n^n}\\ &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\\ &= e > 1\end{aligned}$$

■

EXAMPLE 3 A Failure of the Ratio Test

Determine the convergence or divergence of $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$.

Solution The limit of $|a_{n+1}/a_n|$ is equal to 1.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left[\left(\frac{\sqrt{n+1}}{n+2}\right)\left(\frac{n+1}{\sqrt{n}}\right)\right]\\ &= \lim_{n\to\infty}\left[\sqrt{\frac{n+1}{n}}\left(\frac{n+1}{n+2}\right)\right]\\ &= \sqrt{1}(1)\\ &= 1\end{aligned}$$

NOTE The Ratio Test is also inconclusive for any p-series.

So, the Ratio Test is inconclusive. To determine whether the series converges, you need to try a different test. In this case, you can apply the Alternating Series Test. To show that $a_{n+1} \le a_n$, let

$$f(x) = \frac{\sqrt{x}}{x+1}.$$

Then the derivative is

$$f'(x) = \frac{-x+1}{2\sqrt{x}(x+1)^2}.$$

Because the derivative is negative for $x > 1$, you know that f is a decreasing function. Also, by L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{x}}{x+1} &= \lim_{x\to\infty}\frac{1/\left(2\sqrt{x}\right)}{1}\\ &= \lim_{x\to\infty}\frac{1}{2\sqrt{x}}\\ &= 0.\end{aligned}$$

Therefore, by the Alternating Series Test, the series converges. ■

The series in Example 3 is *conditionally convergent*. This follows from the fact that the series

$$\sum_{n=1}^{\infty} |a_n|$$

diverges $\left(\text{by the Limit Comparison Test with } \Sigma\, 1/\sqrt{n}\right)$, but the series

$$\sum_{n=1}^{\infty} a_n$$

converges.

TECHNOLOGY A computer or programmable calculator can reinforce the conclusion that the series in Example 3 converges *conditionally*. By adding the first 100 terms of the series, you obtain a sum of about -0.2. (The sum of the first 100 terms of the series $\Sigma\,|a_n|$ is about 17.)

The Root Test

The next test for convergence or divergence of series works especially well for series involving nth powers. The proof of this theorem is similar to the proof given for the Ratio Test, and is left as an exercise (see Exercise 100).

THEOREM 9.18 ROOT TEST

Let $\Sigma\, a_n$ be a series.

1. $\Sigma\, a_n$ converges absolutely if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$.
2. $\Sigma\, a_n$ diverges if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.
3. The Root Test is inconclusive if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

NOTE The Root Test is always inconclusive for any p-series. ■

EXAMPLE 4 Using the Root Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}.$$

Solution You can apply the Root Test as follows.

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\frac{e^{2n}}{n^n}}\\ &= \lim_{n\to\infty} \frac{e^{2n/n}}{n^{n/n}}\\ &= \lim_{n\to\infty} \frac{e^2}{n}\\ &= 0 < 1\end{aligned}$$

Because this limit is less than 1, you can conclude that the series converges absolutely (and therefore converges). ■

■ **FOR FURTHER INFORMATION** For more information on the usefulness of the Root Test, see the article "*N*! and the Root Test" by Charles C. Mumma II in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

To see the usefulness of the Root Test for the series in Example 4, try applying the Ratio Test to that series. When you do this, you obtain the following.

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left[\frac{e^{2(n+1)}}{(n+1)^{n+1}} \div \frac{e^{2n}}{n^n}\right]\\ &= \lim_{n\to\infty} e^2 \frac{n^n}{(n+1)^{n+1}}\\ &= \lim_{n\to\infty} e^2 \left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right)\\ &= 0\end{aligned}$$

Note that this limit is not as easily evaluated as the limit obtained by the Root Test in Example 4.

Strategies for Testing Series

You have now studied 10 tests for determining the convergence or divergence of an infinite series. (See the summary in the table on page 646.) Skill in choosing and applying the various tests will come only with practice. Below is a set of guidelines for choosing an appropriate test.

GUIDELINES FOR TESTING A SERIES FOR CONVERGENCE OR DIVERGENCE

1. Does the nth term approach 0? If not, the series diverges.
2. Is the series one of the special types—geometric, p-series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

In some instances, more than one test is applicable. However, your objective should be to learn to choose the most efficient test.

EXAMPLE 5 Applying the Strategies for Testing Series

Determine the convergence or divergence of each series.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$ **b.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$ **c.** $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1}$ **e.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n+1}$ **f.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

g. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

Solution

a. For this series, the limit of the nth term is not 0 $\left(a_n \to \frac{1}{3} \text{ as } n \to \infty\right)$. So, by the nth-Term Test, the series diverges.

b. This series is geometric. Moreover, because the ratio $r = \pi/6$ of the terms is less than 1 in absolute value, you can conclude that the series converges.

c. Because the function $f(x) = xe^{-x^2}$ is easily integrated, you can use the Integral Test to conclude that the series converges.

d. The nth term of this series can be compared to the nth term of the harmonic series. After using the Limit Comparison Test, you can conclude that the series diverges.

e. This is an alternating series whose nth term approaches 0. Because $a_{n+1} \le a_n$, you can use the Alternating Series Test to conclude that the series converges.

f. The nth term of this series involves a factorial, which indicates that the Ratio Test may work well. After applying the Ratio Test, you can conclude that the series diverges.

g. The nth term of this series involves a variable that is raised to the nth power, which indicates that the Root Test may work well. After applying the Root Test, you can conclude that the series converges. ■

SUMMARY OF TESTS FOR SERIES

Test	**Series**	**Condition(s) of Convergence**	**Condition(s) of Divergence**	**Comment**
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \dfrac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \dfrac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\lvert R_N\rvert \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\,dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$ or $= \infty$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \dfrac{a_{n+1}}{a_n}\right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \dfrac{a_{n+1}}{a_n}\right\rvert > 1$ or $= \infty$	Test is inconclusive if $\lim_{n\to\infty} \left\lvert \dfrac{a_{n+1}}{a_n}\right\rvert = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

9.6 Exercises

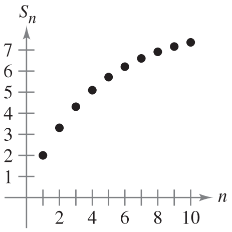
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, verify the formula.

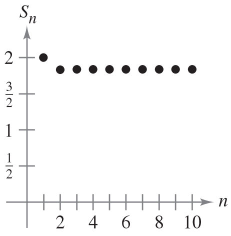
1. $\dfrac{(n+1)!}{(n-2)!} = (n+1)(n)(n-1)$
2. $\dfrac{(2k-2)!}{(2k)!} = \dfrac{1}{(2k)(2k-1)}$
3. $1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2k-1) = \dfrac{(2k)!}{2^k k!}$
4. $\dfrac{1}{1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2k-5)} = \dfrac{2^k k!(2k-3)(2k-1)}{(2k)!}, \quad k \geq 3$

In Exercises 5–10, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

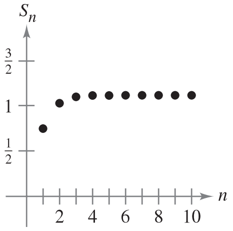
(a)



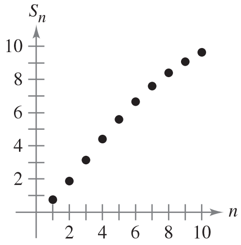
(b)



(c)



(d)



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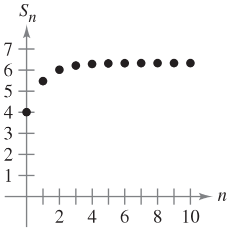
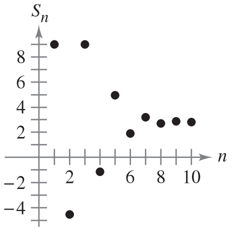


(f)



5. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$
6. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n\left(\frac{1}{n!}\right)$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n!}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4}{(2n)!}$
9. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n$
10. $\displaystyle\sum_{n=0}^{\infty} 4e^{-n}$

Numerical, Graphical, and Analytic Analysis **In Exercises 11 and 12, (a) verify that the series converges. (b) Use a graphing utility to find the indicated partial sum S_n and complete the table. (c) Use a graphing utility to graph the first 10 terms of the sequence of partial sums. (d) Use the table to estimate the sum of the series. (e) Explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.**

n	5	10	15	20	25
S_n					

11. $\displaystyle\sum_{n=1}^{\infty} n^2\left(\frac{5}{8}\right)^n$
12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n!}$

In Exercises 13–34, use the Ratio Test to determine the convergence or divergence of the series.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$
14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$
15. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$
16. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$
17. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$
18. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{10}{9}\right)^n$
19. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}$
20. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{4^n}$
21. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n^2}$
22. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$
23. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$
24. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$
25. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n3^n}$
26. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$
27. $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n!}$
28. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$
29. $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{(n+1)^n}$
30. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$
31. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{2^n+1}$
32. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$
33. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2n+1)}$
34. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n[2 \cdot 4 \cdot 6 \cdot \cdot \cdot (2n)]}{2 \cdot 5 \cdot 8 \cdot \cdot \cdot (3n-1)}$

In Exercises 35–50, use the Root Test to determine the convergence or divergence of the series.

35. $\sum_{n=1}^{\infty} \frac{1}{5^n}$

36. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

37. $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

38. $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

39. $\sum_{n=2}^{\infty} \left(\frac{2n+1}{n-1}\right)^n$

40. $\sum_{n=1}^{\infty} \left(\frac{4n+3}{2n-1}\right)^n$

41. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

42. $\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^{3n}$

43. $\sum_{n=1}^{\infty} \left(2\sqrt[n]{n}+1\right)^n$

44. $\sum_{n=0}^{\infty} e^{-3n}$

45. $\sum_{n=1}^{\infty} \frac{n}{3^n}$

46. $\sum_{n=1}^{\infty} \left(\frac{n}{500}\right)^n$

47. $\sum_{n=1}^{\infty} \left(\frac{1}{n}-\frac{1}{n^2}\right)^n$

48. $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

49. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

50. $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

In Exercises 51–68, determine the convergence or divergence of the series using any appropriate test from this chapter. Identify the test used.

51. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}5}{n}$

52. $\sum_{n=1}^{\infty} \frac{100}{n}$

53. $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$

54. $\sum_{n=1}^{\infty} \left(\frac{2\pi}{3}\right)^n$

55. $\sum_{n=1}^{\infty} \frac{5n}{2n-1}$

56. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

57. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n}$

58. $\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$

59. $\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$

60. $\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$

61. $\sum_{n=1}^{\infty} \frac{\cos n}{3^n}$

62. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

63. $\sum_{n=1}^{\infty} \frac{n7^n}{n!}$

64. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

65. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$

66. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n2^n}$

67. $\sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$

68. $\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{18^n(2n-1)n!}$

In Exercises 69–72, identify the two series that are the same.

69. (a) $\sum_{n=1}^{\infty} \frac{n5^n}{n!}$

(b) $\sum_{n=0}^{\infty} \frac{n5^n}{(n+1)!}$

(c) $\sum_{n=0}^{\infty} \frac{(n+1)5^{n+1}}{(n+1)!}$

70. (a) $\sum_{n=4}^{\infty} n\left(\frac{3}{4}\right)^n$

(b) $\sum_{n=0}^{\infty} (n+1)\left(\frac{3}{4}\right)^n$

(c) $\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^{n-1}$

71. (a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)!}$

72. (a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)2^{n-1}}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)2^n}$

In Exercises 73 and 74, write an equivalent series with the index of summation beginning at $n = 0$.

73. $\sum_{n=1}^{\infty} \frac{n}{7^n}$

74. $\sum_{n=2}^{\infty} \frac{9^n}{(n-2)!}$

In Exercises 75 and 76, (a) determine the number of terms required to approximate the sum of the series with an error less than 0.0001, and (b) use a graphing utility to approximate the sum of the series with an error less than 0.0001.

75. $\sum_{k=1}^{\infty} \frac{(-3)^k}{2^k k!}$

76. $\sum_{k=0}^{\infty} \frac{(-3)^k}{1 \cdot 3 \cdot 5 \cdots (2k+1)}$

In Exercises 77–82, the terms of a series $\sum_{n=1}^{\infty} a_n$ are defined recursively. Determine the convergence or divergence of the series. Explain your reasoning.

77. $a_1 = \frac{1}{2},\ a_{n+1} = \frac{4n-1}{3n+2}a_n$

78. $a_1 = 2,\ a_{n+1} = \frac{2n+1}{5n-4}a_n$

79. $a_1 = 1,\ a_{n+1} = \frac{\sin n + 1}{\sqrt{n}}a_n$

80. $a_1 = \frac{1}{5},\ a_{n+1} = \frac{\cos n + 1}{n}a_n$

81. $a_1 = \frac{1}{3},\ a_{n+1} = \left(1 + \frac{1}{n}\right)a_n$

82. $a_1 = \frac{1}{4},\ a_{n+1} = \sqrt[n]{a_n}$

In Exercises 83–86, use the Ratio Test or the Root Test to determine the convergence or divergence of the series.

83. $1 + \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots$

84. $1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \frac{5}{3^4} + \frac{6}{3^5} + \cdots$

85. $\frac{1}{(\ln 3)^3} + \frac{1}{(\ln 4)^4} + \frac{1}{(\ln 5)^5} + \frac{1}{(\ln 6)^6} + \cdots$

86. $1 + \frac{1 \cdot 3}{1 \cdot 2 \cdot 3} + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$
$+ \frac{1 \cdot 3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \cdots$

In Exercises 87–92, find the values of x for which the series converges.

87. $\sum_{n=0}^{\infty} 2\left(\frac{x}{3}\right)^n$

88. $\sum_{n=0}^{\infty} \left(\frac{x+1}{4}\right)^n$

89. $\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n}$

90. $\sum_{n=0}^{\infty} 2(x-1)^n$

91. $\sum_{n=0}^{\infty} n!\left(\frac{x}{2}\right)^n$

92. $\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}$

WRITING ABOUT CONCEPTS

93. State the Ratio Test.

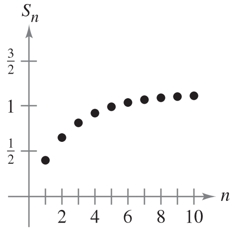
94. State the Root Test.

95. You are told that the terms of a positive series appear to approach zero rapidly as n approaches infinity. In fact, $a_7 \leq 0.0001$. Given no other information, does this imply that the series converges? Support your conclusion with examples.

96. The graph shows the first 10 terms of the sequence of partial sums of the convergent series

$$\sum_{n=1}^{\infty} \left(\frac{2n}{3n+2}\right)^n.$$

Find a series such that the terms of its sequence of partial sums are less than the corresponding terms of the sequence in the figure, but such that the series diverges. Explain your reasoning.



97. Using the Ratio Test, it is determined that an alternating series converges. Does the series converge conditionally or absolutely? Explain.

CAPSTONE

98. What can you conclude about the convergence or divergence of $\Sigma\, a_n$ for each of the following conditions? Explain your reasoning.

(a) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$ (b) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

(c) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{3}{2}$ (d) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 2$

(e) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ (f) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = e$

99. Prove Property 2 of Theorem 9.17.

100. Prove Theorem 9.18. (*Hint for Property 1:* If the limit equals $r < 1$, choose a real number R such that $r < R < 1$. By the definitions of the limit, there exists some $N > 0$ such that $\sqrt[n]{|a_n|} < R$ for $n > N$.)

In Exercises 101–104, verify that the Ratio Test is inconclusive for the p-series.

101. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

102. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

103. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

104. $\sum_{n=1}^{\infty} \frac{1}{n^p}$

105. Show that the Root Test is inconclusive for the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

106. Show that the Ratio Test and the Root Test are both inconclusive for the logarithmic p-series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}.$$

107. Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(xn)!}$$

when (a) $x = 1$, (b) $x = 2$, (c) $x = 3$, and (d) x is a positive integer.

108. Show that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|.$$

109. ***Writing*** Read the article "A Differentiation Test for Absolute Convergence" by Yaser S. Abu-Mostafa in *Mathematics Magazine*. Then write a paragraph that describes the test. Include examples of series that converge and examples of series that diverge.

PUTNAM EXAM CHALLENGE

110. Is the following series convergent or divergent?

$$1 + \frac{1}{2}\cdot\frac{19}{7} + \frac{2!}{3^2}\left(\frac{19}{7}\right)^2 + \frac{3!}{4^3}\left(\frac{19}{7}\right)^3 + \frac{4!}{5^4}\left(\frac{19}{7}\right)^4 + \cdots$$

111. Show that if the series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

converges, then the series

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} + \cdots$$

converges also.

These problems were composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.

9.7 Taylor Polynomials and Approximations

- **Find polynomial approximations of elementary functions and compare them with the elementary functions.**
- **Find Taylor and Maclaurin polynomial approximations of elementary functions.**
- **Use the remainder of a Taylor polynomial.**

Polynomial Approximations of Elementary Functions

The goal of this section is to show how polynomial functions can be used as approximations for other elementary functions. To find a polynomial function P that approximates another function f, begin by choosing a number c in the domain of f at which f and P have the same value. That is,

$$P(c) = f(c).$$

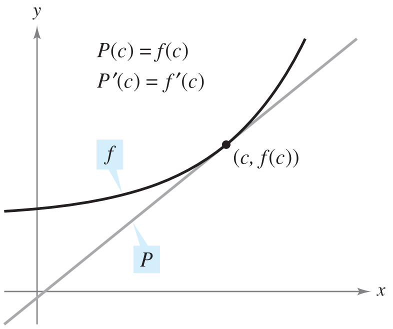
Graphs of f and P pass through $(c, f(c))$.

The approximating polynomial is said to be **expanded about c** or **centered at c.** Geometrically, the requirement that $P(c) = f(c)$ means that the graph of P passes through the point $(c, f(c))$. Of course, there are many polynomials whose graphs pass through the point $(c, f(c))$. Your task is to find a polynomial whose graph resembles the graph of f near this point. One way to do this is to impose the additional requirement that the slope of the polynomial function be the same as the slope of the graph of f at the point $(c, f(c))$.

$$P'(c) = f'(c)$$

Graphs of f and P have the same slope at $(c, f(c))$.

With these two requirements, you can obtain a simple linear approximation of f, as shown in Figure 9.10.



Near $(c, f(c))$, the graph of P can be used to approximate the graph of f.
Figure 9.10

EXAMPLE 1 First-Degree Polynomial Approximation of $f(x) = e^x$

For the function $f(x) = e^x$, find a first-degree polynomial function

$$P_1(x) = a_0 + a_1x$$

whose value and slope agree with the value and slope of f at $x = 0$.

Solution Because $f(x) = e^x$ and $f'(x) = e^x$, the value and the slope of f, at $x = 0$, are given by

$$f(0) = e^0 = 1$$

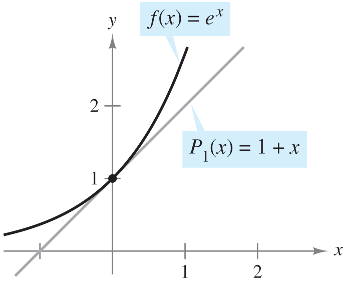
and

$$f'(0) = e^0 = 1.$$

Because $P_1(x) = a_0 + a_1x$, you can use the condition that $P_1(0) = f(0)$ to conclude that $a_0 = 1$. Moreover, because $P_1'(x) = a_1$, you can use the condition that $P_1'(0) = f'(0)$ to conclude that $a_1 = 1$. Therefore,

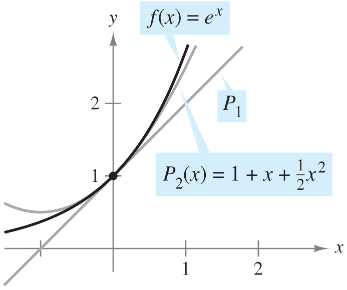
$$P_1(x) = 1 + x.$$

Figure 9.11 shows the graphs of $P_1(x) = 1 + x$ and $f(x) = e^x$. ■



P_1 is the first-degree polynomial approximation of $f(x) = e^x$.
Figure 9.11

NOTE Example 1 isn't the first time you have used a linear function to approximate another function. The same procedure was used as the basis for Newton's Method. ■



P_2 is the second-degree polynomial approximation of $f(x) = e^x$.
Figure 9.12

In Figure 9.12 you can see that, at points near (0, 1), the graph of

$$P_1(x) = 1 + x$$

1st-degree approximation

is reasonably close to the graph of $f(x) = e^x$. However, as you move away from (0, 1), the graphs move farther and farther from each other and the accuracy of the approximation decreases. To improve the approximation, you can impose yet another requirement— that the values of the second derivatives of P and f agree when $x = 0$. The polynomial, P_2, of least degree that satisfies all three requirements $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$ can be shown to be

$$P_2(x) = 1 + x + \frac{1}{2}x^2.$$

2nd-degree approximation

Moreover, in Figure 9.12, you can see that P_2 is a better approximation of f than P_1. If you continue this pattern, requiring that the values of $P_n(x)$ and its first n derivatives match those of $f(x) = e^x$ at $x = 0$, you obtain the following.

$$P_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$
$$\approx e^x$$

nth-degree approximation

EXAMPLE 2 Third-Degree Polynomial Approximation of $f(x) = e^x$

Construct a table comparing the values of the polynomial

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$$

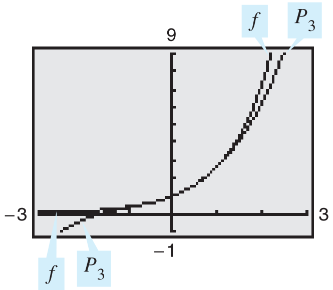
3rd-degree approximation

with $f(x) = e^x$ for several values of x near 0.

Solution Using a calculator or a computer, you can obtain the results shown in the table. Note that for $x = 0$, the two functions have the same value, but that as x moves farther away from 0, the accuracy of the approximating polynomial $P_3(x)$ decreases.

x	-1.0	-0.2	-0.1	0	0.1	0.2	1.0
e^x	0.3679	0.81873	0.904837	1	1.105171	1.22140	2.7183
$P_3(x)$	0.3333	0.81867	0.904833	1	1.105167	1.22133	2.6667

■



P_3 is the third-degree polynomial approximation of $f(x) = e^x$.
Figure 9.13

TECHNOLOGY A graphing utility can be used to compare the graph of the approximating polynomial with the graph of the function f. For instance, in Figure 9.13, the graph of

$$P_3(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3$$

3rd-degree approximation

is compared with the graph of $f(x) = e^x$. If you have access to a graphing utility, try comparing the graphs of

$$P_4(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4$$

4th-degree approximation

$$P_5(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \tfrac{1}{120}x^5$$

5th-degree approximation

$$P_6(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \tfrac{1}{120}x^5 + \tfrac{1}{720}x^6$$

6th-degree approximation

with the graph of f. What do you notice?

Taylor and Maclaurin Polynomials

The Granger Collection

BROOK TAYLOR (1685–1731)

Although Taylor was not the first to seek polynomial approximations of transcendental functions, his account published in 1715 was one of the first comprehensive works on the subject.

The polynomial approximation of $f(x) = e^x$ given in Example 2 is expanded about $c = 0$. For expansions about an arbitrary value of c, it is convenient to write the polynomial in the form

$$P_n(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \cdot \cdot \cdot + a_n(x - c)^n.$$

In this form, repeated differentiation produces

$$\begin{aligned}
P_n'(x) &= a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + \cdot \cdot \cdot + na_n(x - c)^{n-1}\\
P_n''(x) &= 2a_2 + 2(3a_3)(x - c) + \cdot \cdot \cdot + n(n - 1)a_n(x - c)^{n-2}\\
P_n'''(x) &= 2(3a_3) + \cdot \cdot \cdot + n(n - 1)(n - 2)a_n(x - c)^{n-3}\\
&\vdots\\
P_n^{(n)}(x) &= n(n - 1)(n - 2) \cdot \cdot \cdot (2)(1)a_n.
\end{aligned}$$

Letting $x = c$, you then obtain

$$P_n(c) = a_0, \qquad P_n'(c) = a_1, \qquad P_n''(c) = 2a_2, \quad . \; . \; . \; , \qquad P_n^{(n)}(c) = n!a_n$$

and because the values of f and its first n derivatives must agree with the values of P_n and its first n derivatives at $x = c$, it follows that

$$f(c) = a_0, \qquad f'(c) = a_1, \qquad \frac{f''(c)}{2!} = a_2, \quad . \; . \; . \; , \qquad \frac{f^{(n)}(c)}{n!} = a_n.$$

With these coefficients, you can obtain the following definition of **Taylor polynomials,** named after the English mathematician Brook Taylor, and **Maclaurin polynomials,** named after the English mathematician Colin Maclaurin (1698–1746).

NOTE Maclaurin polynomials are special types of Taylor polynomials for which $c = 0$.

DEFINITIONS OF *n*TH TAYLOR POLYNOMIAL AND *n*TH MACLAURIN POLYNOMIAL

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is called the ***n*th Taylor polynomial for *f* at *c*.** If $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdot \cdot \cdot + \frac{f^{(n)}(0)}{n!}x^n$$

is also called the ***n*th Maclaurin polynomial for *f*.**

EXAMPLE 3 A Maclaurin Polynomial for $f(x) = e^x$

Find the nth Maclaurin polynomial for $f(x) = e^x$.

Solution From the discussion on page 651, the nth Maclaurin polynomial for

$$f(x) = e^x$$

is given by

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdot \cdot \cdot + \frac{1}{n!}x^n.$$

■

■ **FOR FURTHER INFORMATION** To see how to use series to obtain other approximations to e, see the article "Novel Series-based Approximations to e" by John Knox and Harlan J. Brothers in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

EXAMPLE 4 Finding Taylor Polynomials for ln x

Find the Taylor polynomials P_0, P_1, P_2, P_3, and P_4 for $f(x) = \ln x$ centered at $c = 1$.

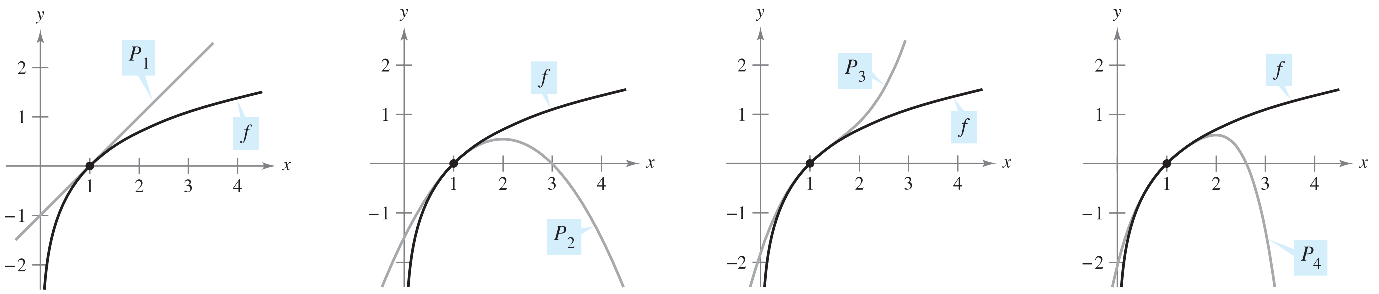
Solution Expanding about $c = 1$ yields the following.

$$\begin{aligned}
f(x) &= \ln x & f(1) &= \ln 1 = 0 \\
f'(x) &= \frac{1}{x} & f'(1) &= \frac{1}{1} = 1 \\
f''(x) &= -\frac{1}{x^2} & f''(1) &= -\frac{1}{1^2} = -1 \\
f'''(x) &= \frac{2!}{x^3} & f'''(1) &= \frac{2!}{1^3} = 2 \\
f^{(4)}(x) &= -\frac{3!}{x^4} & f^{(4)}(1) &= -\frac{3!}{1^4} = -6
\end{aligned}$$

Therefore, the Taylor polynomials are as follows.

$$\begin{aligned}
P_0(x) &= f(1) = 0 \\
P_1(x) &= f(1) + f'(1)(x - 1) = (x - 1) \\
P_2(x) &= f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 \\
&= (x - 1) - \frac{1}{2}(x - 1)^2 \\
P_3(x) &= f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \frac{f'''(1)}{3!}(x - 1)^3 \\
&= (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 \\
P_4(x) &= f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \frac{f'''(1)}{3!}(x - 1)^3 \\
&\quad + \frac{f^{(4)}(1)}{4!}(x - 1)^4 \\
&= (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4
\end{aligned}$$

Figure 9.14 compares the graphs of P_1, P_2, P_3, and P_4 with the graph of $f(x) = \ln x$. Note that near $x = 1$ the graphs are nearly indistinguishable. For instance, $P_4(0.9) \approx -0.105358$ and $\ln(0.9) \approx -0.105361$.



As n increases, the graph of P_n becomes a better and better approximation of the graph of $f(x) = \ln x$ near $x = 1$.
Figure 9.14

■

EXAMPLE 5 Finding Maclaurin Polynomials for cos *x*

Find the Maclaurin polynomials P_0, P_2, P_4, and P_6 for $f(x) = \cos x$. Use $P_6(x)$ to approximate the value of $\cos(0.1)$.

Solution Expanding about $c = 0$ yields the following.

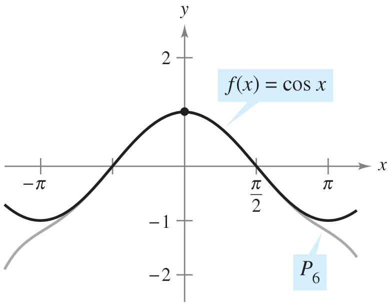
$$\begin{aligned} f(x) &= \cos x & f(0) &= \cos 0 = 1 \\ f'(x) &= -\sin x & f'(0) &= -\sin 0 = 0 \\ f''(x) &= -\cos x & f''(0) &= -\cos 0 = -1 \\ f'''(x) &= \sin x & f'''(0) &= \sin 0 = 0 \end{aligned}$$

Through repeated differentiation, you can see that the pattern 1, 0, -1, 0 continues, and you obtain the following Maclaurin polynomials.

$$P_0(x) = 1, \quad P_2(x) = 1 - \frac{1}{2!}x^2,$$

$$P_4(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4, \quad P_6(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

Using $P_6(x)$, you obtain the approximation $\cos(0.1) \approx 0.995004165$, which coincides with the calculator value to nine decimal places. Figure 9.15 compares the graphs of $f(x) = \cos x$ and P_6. ■



Near (0, 1), the graph of P_6 can be used to approximate the graph of $f(x) = \cos x$.
Figure 9.15

Note in Example 5 that the Maclaurin polynomials for cos *x* have only even powers of *x*. Similarly, the Maclaurin polynomials for sin *x* have only odd powers of *x* (see Exercise 17). This is not generally true of the Taylor polynomials for sin *x* and cos *x* expanded about $c \neq 0$, as you can see in the next example.

EXAMPLE 6 Finding a Taylor Polynomial for sin *x*

Find the third Taylor polynomial for $f(x) = \sin x$, expanded about $c = \pi/6$.

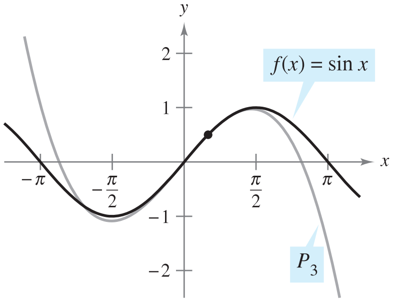
Solution Expanding about $c = \pi/6$ yields the following.

$$\begin{aligned} f(x) &= \sin x & f\left(\frac{\pi}{6}\right) &= \sin\frac{\pi}{6} = \frac{1}{2} \\ f'(x) &= \cos x & f'\left(\frac{\pi}{6}\right) &= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ f''(x) &= -\sin x & f''\left(\frac{\pi}{6}\right) &= -\sin\frac{\pi}{6} = -\frac{1}{2} \\ f'''(x) &= -\cos x & f'''\left(\frac{\pi}{6}\right) &= -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2} \end{aligned}$$

So, the third Taylor polynomial for $f(x) = \sin x$, expanded about $c = \pi/6$, is

$$\begin{aligned} P_3(x) &= f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{f'''\left(\frac{\pi}{6}\right)}{3!}\left(x - \frac{\pi}{6}\right)^3 \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2(2!)}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2(3!)}\left(x - \frac{\pi}{6}\right)^3. \end{aligned}$$

Figure 9.16 compares the graphs of $f(x) = \sin x$ and P_3. ■



Near $(\pi/6, 1/2)$, the graph of P_3 can be used to approximate the graph of $f(x) = \sin x$.
Figure 9.16

Taylor polynomials and Maclaurin polynomials can be used to approximate the value of a function at a specific point. For instance, to approximate the value of $\ln(1.1)$, you can use Taylor polynomials for $f(x) = \ln x$ expanded about $c = 1$, as shown in Example 4, or you can use Maclaurin polynomials, as shown in Example 7.

EXAMPLE 7 Approximation Using Maclaurin Polynomials

Use a fourth Maclaurin polynomial to approximate the value of $\ln(1.1)$.

Solution Because 1.1 is closer to 1 than to 0, you should consider Maclaurin polynomials for the function $g(x) = \ln(1 + x)$.

$$\begin{aligned} g(x) &= \ln(1+x) & g(0) &= \ln(1+0) = 0 \\ g'(x) &= (1+x)^{-1} & g'(0) &= (1+0)^{-1} = 1 \\ g''(x) &= -(1+x)^{-2} & g''(0) &= -(1+0)^{-2} = -1 \\ g'''(x) &= 2(1+x)^{-3} & g'''(0) &= 2(1+0)^{-3} = 2 \\ g^{(4)}(x) &= -6(1+x)^{-4} & g^{(4)}(0) &= -6(1+0)^{-4} = -6 \end{aligned}$$

Note that you obtain the same coefficients as in Example 4. Therefore, the fourth Maclaurin polynomial for $g(x) = \ln(1 + x)$ is

$$\begin{aligned} P_4(x) &= g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \frac{g^{(4)}(0)}{4!}x^4 \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4. \end{aligned}$$

Consequently,

$$\ln(1.1) = \ln(1 + 0.1) \approx P_4(0.1) \approx 0.0953083.$$

Check to see that the fourth Taylor polynomial (from Example 4), evaluated at $x = 1.1$, yields the same result. ■

n	$P_n(0.1)$
1	0.1000000
2	0.0950000
3	0.0953333
4	0.0953083

The table at the left illustrates the accuracy of the Taylor polynomial approximation of the calculator value of $\ln(1.1)$. You can see that as n becomes larger, $P_n(0.1)$ approaches the calculator value of 0.0953102.

On the other hand, the table below illustrates that as you move away from the expansion point $c = 1$, the accuracy of the approximation decreases.

Fourth Taylor Polynomial Approximation of $\ln(1 + x)$

x	0	0.1	0.5	0.75	1.0
$\ln(1 + x)$	0	0.0953102	0.4054651	0.5596158	0.6931472
$P_4(x)$	0	0.0953083	0.4010417	0.5302734	0.5833333

These two tables illustrate two very important points about the accuracy of Taylor (or Maclaurin) polynomials for use in approximations.

1. The approximation is usually better at x-values close to c than at x-values far from c.
2. The approximation is usually better for higher-degree Taylor (or Maclaurin) polynomials than for those of lower degree.

Remainder of a Taylor Polynomial

An approximation technique is of little value without some idea of its accuracy. To measure the accuracy of approximating a function value $f(x)$ by the Taylor polynomial $P_n(x)$, you can use the concept of a **remainder** $R_n(x)$, defined as follows.

$$\underbrace{f(x)}_{\text{Exact value}} = \underbrace{P_n(x)}_{\text{Approximate Value}} + \underbrace{R_n(x)}_{\text{Remainder}}$$

So, $R_n(x) = f(x) - P_n(x)$. The absolute value of $R_n(x)$ is called the **error** associated with the approximation. That is,

$$\text{Error} = |R_n(x)| = |f(x) - P_n(x)|.$$

The next theorem gives a general procedure for estimating the remainder associated with a Taylor polynomial. This important theorem is called **Taylor's Theorem,** and the remainder given in the theorem is called the **Lagrange form of the remainder.** (The proof of the theorem is lengthy, and is given in Appendix A.)

THEOREM 9.19 TAYLOR'S THEOREM

If a function f is differentiable through order $n + 1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}.$$

NOTE One useful consequence of Taylor's Theorem is that

$$|R_n(x)| \le \frac{|x - c|^{n+1}}{(n + 1)!} \max |f^{(n+1)}(z)|$$

where $\max|f^{(n+1)}(z)|$ is the maximum value of $f^{(n+1)}(z)$ between x and c. ■

For $n = 0$, Taylor's Theorem states that if f is differentiable in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(z)(x - c) \qquad \text{or} \qquad f'(z) = \frac{f(x) - f(c)}{x - c}.$$

Do you recognize this special case of Taylor's Theorem? (It is the Mean Value Theorem.)

When applying Taylor's Theorem, you should not expect to be able to find the exact value of z. (If you could do this, an approximation would not be necessary.) Rather, you try to find bounds for $f^{(n+1)}(z)$ from which you are able to tell how large the remainder $R_n(x)$ is.

EXAMPLE 8 Determining the Accuracy of an Approximation

The third Maclaurin polynomial for sin x is given by

$$P_3(x) = x - \frac{x^3}{3!}.$$

Use Taylor's Theorem to approximate sin(0.1) by $P_3(0.1)$ and determine the accuracy of the approximation.

Solution Using Taylor's Theorem, you have

$$\sin x = x - \frac{x^3}{3!} + R_3(x) = x - \frac{x^3}{3!} + \frac{f^{(4)}(z)}{4!}x^4$$

where $0 < z < 0.1$. Therefore,

$$\sin(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} \approx 0.1 - 0.000167 = 0.099833.$$

Because $f^{(4)}(z) = \sin z$, it follows that the error $|R_3(0.1)|$ can be bounded as follows.

$$0 < R_3(0.1) = \frac{\sin z}{4!}(0.1)^4 < \frac{0.0001}{4!} \approx 0.000004$$

This implies that

$$0.099833 < \sin(0.1) = 0.099833 + R_3(x) < 0.099833 + 0.000004$$
$$0.099833 < \sin(0.1) < 0.099837.$$

NOTE Try using a calculator to verify the results obtained in Examples 8 and 9. For Example 8, you obtain

$$\sin(0.1) \approx 0.0998334.$$

For Example 9, you obtain

$$P_3(1.2) \approx 0.1827$$

and

$$\ln(1.2) \approx 0.1823.$$

EXAMPLE 9 Approximating a Value to a Desired Accuracy

Determine the degree of the Taylor polynomial $P_n(x)$ expanded about $c = 1$ that should be used to approximate ln(1.2) so that the error is less than 0.001.

Solution Following the pattern of Example 4, you can see that the $(n + 1)$st derivative of $f(x) = \ln x$ is given by

$$f^{(n+1)}(x) = (-1)^n \frac{n!}{x^{n+1}}.$$

Using Taylor's Theorem, you know that the error $|R_n(1.2)|$ is given by

$$|R_n(1.2)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}(1.2 - 1)^{n+1}\right| = \frac{n!}{z^{n+1}}\left[\frac{1}{(n+1)!}\right](0.2)^{n+1}$$
$$= \frac{(0.2)^{n+1}}{z^{n+1}(n+1)}$$

where $1 < z < 1.2$. In this interval, $(0.2)^{n+1}/[z^{n+1}(n + 1)]$ is less than $(0.2)^{n+1}/(n + 1)$. So, you are seeking a value of n such that

$$\frac{(0.2)^{n+1}}{(n+1)} < 0.001 \quad \Rightarrow \quad 1000 < (n+1)5^{n+1}.$$

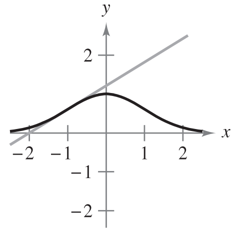
By trial and error, you can determine that the smallest value of n that satisfies this inequality is $n = 3$. So, you would need the third Taylor polynomial to achieve the desired accuracy in approximating ln(1.2). ■

9.7 Exercises

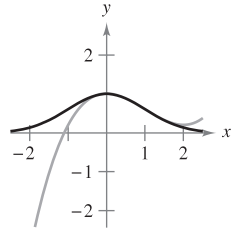
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, match the Taylor polynomial approximation of the function $f(x) = e^{-x^2/2}$ with the corresponding graph. [The graphs are labeled (a), (b), (c), and (d).]

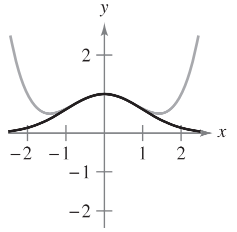
(a)



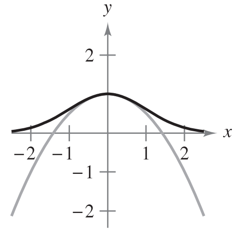
(b)



(c)



(d)



1. $g(x) = -\frac{1}{2}x^2 + 1$

2. $g(x) = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$

3. $g(x) = e^{-1/2}[(x+1) + 1]$

4. $g(x) = e^{-1/2}\left[\frac{1}{3}(x-1)^3 - (x-1) + 1\right]$

In Exercises 5–8, find a first-degree polynomial function P_1 whose value and slope agree with the value and slope of f at $x = c$. Use a graphing utility to graph f and P_1. What is P_1 called?

5. $f(x) = \dfrac{8}{\sqrt{x}}, \quad c = 4$

6. $f(x) = \dfrac{6}{\sqrt[3]{x}}, \quad c = 8$

7. $f(x) = \sec x, \quad c = \dfrac{\pi}{4}$

8. $f(x) = \tan x, \quad c = \dfrac{\pi}{4}$

Graphical and Numerical Analysis **In Exercises 9 and 10, use a graphing utility to graph f and its second-degree polynomial approximation P_2 at $x = c$. Complete the table comparing the values of f and P_2.**

9. $f(x) = \dfrac{4}{\sqrt{x}}, \quad c = 1$

$P_2(x) = 4 - 2(x-1) + \frac{3}{2}(x-1)^2$

x	0	0.8	0.9	1	1.1	1.2	2
$f(x)$							
$P_2(x)$							

10. $f(x) = \sec x, \quad c = \dfrac{\pi}{4}$

$$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$

x	-2.15	0.585	0.685	$\frac{\pi}{4}$	0.885	0.985	1.785
$f(x)$							
$P_2(x)$							

11. ***Conjecture*** Consider the function $f(x) = \cos x$ and its Maclaurin polynomials P_2, P_4, and P_6 (see Example 5).

(a) Use a graphing utility to graph f and the indicated polynomial approximations.

(b) Evaluate and compare the values of $f^{(n)}(0)$ and $P_n^{(n)}(0)$ for $n = 2$, 4, and 6.

(c) Use the results in part (b) to make a conjecture about $f^{(n)}(0)$ and $P_n^{(n)}(0)$.

12. ***Conjecture*** Consider the function $f(x) = x^2e^x$.

(a) Find the Maclaurin polynomials P_2, P_3, and P_4 for f.

(b) Use a graphing utility to graph f, P_2, P_3, and P_4.

(c) Evaluate and compare the values of $f^{(n)}(0)$ and $P_n^{(n)}(0)$ for $n = 2$, 3, and 4.

(d) Use the results in part (c) to make a conjecture about $f^{(n)}(0)$ and $P_n^{(n)}(0)$.

In Exercises 13–24, find the Maclaurin polynomial of degree n for the function.

13. $f(x) = e^{3x}, \quad n = 4$

14. $f(x) = e^{-x}, \quad n = 5$

15. $f(x) = e^{-x/2}, \quad n = 4$

16. $f(x) = e^{x/3}, \quad n = 4$

17. $f(x) = \sin x, \quad n = 5$

18. $f(x) = \sin \pi x, \quad n = 3$

19. $f(x) = xe^x, \quad n = 4$

20. $f(x) = x^2e^{-x}, \quad n = 4$

21. $f(x) = \dfrac{1}{x+1}, \quad n = 5$

22. $f(x) = \dfrac{x}{x+1}, \quad n = 4$

23. $f(x) = \sec x, \quad n = 2$

24. $f(x) = \tan x, \quad n = 3$

In Exercises 25–30, find the nth Taylor polynomial centered at c.

25. $f(x) = \dfrac{2}{x}, \quad n = 3, \quad c = 1$

26. $f(x) = \dfrac{1}{x^2}, \quad n = 4, \quad c = 2$

27. $f(x) = \sqrt{x}, \quad n = 3, \quad c = 4$

28. $f(x) = \sqrt[3]{x}, \quad n = 3, \quad c = 8$

29. $f(x) = \ln x, \quad n = 4, \quad c = 2$

30. $f(x) = x^2 \cos x, \quad n = 2, \quad c = \pi$

CAS **In Exercises 31 and 32, use a computer algebra system to find the indicated Taylor polynomials for the function f. Graph the function and the Taylor polynomials.**

31. $f(x) = \tan \pi x$
(a) $n = 3, \quad c = 0$
(b) $n = 3, \quad c = 1/4$

32. $f(x) = 1/(x^2 + 1)$
(a) $n = 4, \quad c = 0$
(b) $n = 4, \quad c = 1$

33. *Numerical and Graphical Approximations*

(a) Use the Maclaurin polynomials $P_1(x)$, $P_3(x)$, and $P_5(x)$ for $f(x) = \sin x$ to complete the table.

x	0	0.25	0.50	0.75	1.00
$\sin x$	0	0.2474	0.4794	0.6816	0.8415
$P_1(x)$					
$P_3(x)$					
$P_5(x)$					

(b) Use a graphing utility to graph $f(x) = \sin x$ and the Maclaurin polynomials in part (a).

(c) Describe the change in accuracy of a polynomial approximation as the distance from the point where the polynomial is centered increases.

CAPSTONE

34. *Numerical and Graphical Approximations*

(a) Use the Taylor polynomials $P_1(x)$, $P_2(x)$, and $P_4(x)$ for $f(x) = e^x$ centered at $c = 1$ to complete the table.

x	1.00	1.25	1.50	1.75	2.00
e^x	e	3.4903	4.4817	5.7546	7.3891
$P_1(x)$					
$P_2(x)$					
$P_4(x)$					

(b) Use a graphing utility to graph $f(x) = e^x$ and the Taylor polynomials in part (a).

(c) Describe the change in accuracy of polynomial approximations as the degree increases.

Numerical and Graphical Approximations **In Exercises 35 and 36, (a) find the Maclaurin polynomial $P_3(x)$ for $f(x)$, (b) complete the table for $f(x)$ and $P_3(x)$, and (c) sketch the graphs of $f(x)$ and $P_3(x)$ on the same set of coordinate axes.**

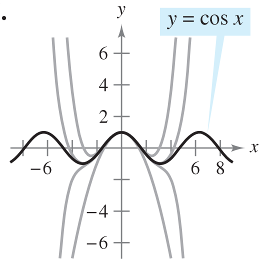
x	-0.75	-0.50	-0.25	0	0.25	0.50	0.75
$f(x)$							
$P_3(x)$							

35. $f(x) = \arcsin x$

36. $f(x) = \arctan x$

In Exercises 37–40, the graph of $y = f(x)$ is shown with four of its Maclaurin polynomials. Identify the Maclaurin polynomials and use a graphing utility to confirm your results.

37.



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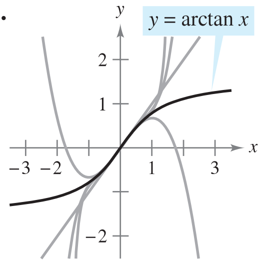

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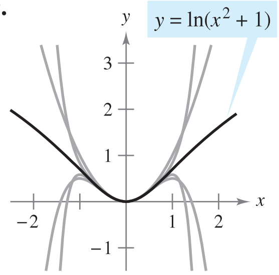

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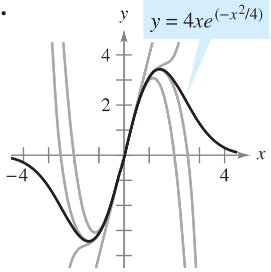


In Exercises 41–44, approximate the function at the given value of x, using the polynomial found in the indicated exercise.

41. $f(x) = e^{3x}, \quad f\left(\frac{1}{2}\right)$, Exercise 13

42. $f(x) = x^2e^{-x}, \quad f\left(\frac{1}{5}\right)$, Exercise 20

43. $f(x) = \ln x, \quad f(2.1)$, Exercise 29

44. $f(x) = x^2 \cos x, \quad f\left(\frac{7\pi}{8}\right)$, Exercise 30

In Exercises 45–48, use Taylor's Theorem to obtain an upper bound for the error of the approximation. Then calculate the exact value of the error.

45. $\cos(0.3) \approx 1 - \dfrac{(0.3)^2}{2!} + \dfrac{(0.3)^4}{4!}$

46. $e \approx 1 + 1 + \dfrac{1^2}{2!} + \dfrac{1^3}{3!} + \dfrac{1^4}{4!} + \dfrac{1^5}{5!}$

47. $\arcsin(0.4) \approx 0.4 + \dfrac{(0.4)^3}{2 \cdot 3}$

48. $\arctan(0.4) \approx 0.4 - \dfrac{(0.4)^3}{3}$

In Exercises 49–52, determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value of x to be less than 0.001.

49. $\sin(0.3)$

50. $\cos(0.1)$

51. $e^{0.6}$

52. $\ln(1.25)$

CAS **In Exercises 53–56, determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value of x to be less than 0.0001. Use a computer algebra system to obtain and evaluate the required derivative.**

53. $f(x) = \ln(x + 1)$, approximate $f(0.5)$.

54. $f(x) = \cos(\pi x^2)$, approximate $f(0.6)$.

55. $f(x) = e^{-\pi x}$, approximate $f(1.3)$.

56. $f(x) = e^{-x}$, approximate $f(1)$.

In Exercises 57–60, determine the values of x for which the function can be replaced by the Taylor polynomial if the error cannot exceed 0.001.

57. $f(x) = e^x \approx 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}, \quad x < 0$

58. $f(x) = \sin x \approx x - \dfrac{x^3}{3!}$

59. $f(x) = \cos x \approx 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$

60. $f(x) = e^{-2x} \approx 1 - 2x + 2x^2 - \dfrac{4}{3}x^3$

WRITING ABOUT CONCEPTS

61. An elementary function is approximated by a polynomial. In your own words, describe what is meant by saying that the polynomial is *expanded about* c or *centered at* c.

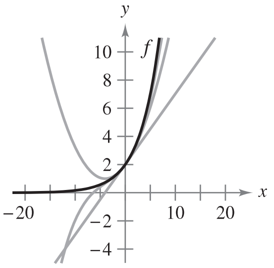
62. When an elementary function f is approximated by a second-degree polynomial P_2 centered at c, what is known about f and P_2 at c? Explain your reasoning.

63. State the definition of an nth-degree Taylor polynomial of f centered at c.

64. Describe the accuracy of the nth-degree Taylor polynomial of f centered at c as the distance between c and x increases.

65. In general, how does the accuracy of a Taylor polynomial change as the degree of the polynomial increases? Explain your reasoning.

66. The graphs show first-, second-, and third-degree polynomial approximations P_1, P_2, and P_3 of a function f. Label the graphs of P_1, P_2, and P_3. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



67. ***Comparing Maclaurin Polynomials***

(a) Compare the Maclaurin polynomials of degree 4 and degree 5, respectively, for the functions $f(x) = e^x$ and $g(x) = xe^x$. What is the relationship between them?

(b) Use the result in part (a) and the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ to find a Maclaurin polynomial of degree 6 for the function $g(x) = x \sin x$.

(c) Use the result in part (a) and the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ to find a Maclaurin polynomial of degree 4 for the function $g(x) = (\sin x)/x$.

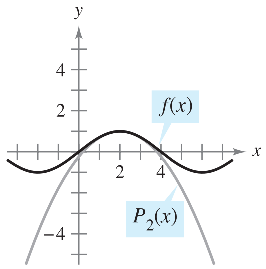
68. ***Differentiating Maclaurin Polynomials***

(a) Differentiate the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ and compare the result with the Maclaurin polynomial of degree 4 for $g(x) = \cos x$.

(b) Differentiate the Maclaurin polynomial of degree 6 for $f(x) = \cos x$ and compare the result with the Maclaurin polynomial of degree 5 for $g(x) = \sin x$.

(c) Differentiate the Maclaurin polynomial of degree 4 for $f(x) = e^x$. Describe the relationship between the two series.

69. ***Graphical Reasoning*** The figure shows the graphs of the function $f(x) = \sin(\pi x/4)$ and the second-degree Taylor polynomial $P_2(x) = 1 - (\pi^2/32)(x - 2)^2$ centered at $x = 2$.



(a) Use the symmetry of the graph of f to write the second-degree Taylor polynomial $Q_2(x)$ for f centered at $x = -2$.

(b) Use a horizontal translation of the result in part (a) to find the second-degree Taylor polynomial $R_2(x)$ for f centered at $x = 6$.

(c) Is it possible to use a horizontal translation of the result in part (a) to write a second-degree Taylor polynomial for f centered at $x = 4$? Explain.

70. Prove that if f is an odd function, then its nth Maclaurin polynomial contains only terms with odd powers of x.

71. Prove that if f is an even function, then its nth Maclaurin polynomial contains only terms with even powers of x.

72. Let $P_n(x)$ be the nth Taylor polynomial for f at c. Prove that $P_n(c) = f(c)$ and $P^{(k)}(c) = f^{(k)}(c)$ for $1 \le k \le n$. (See Exercises 9 and 10.)

73. ***Writing*** The proof in Exercise 72 guarantees that the Taylor polynomial and its derivatives agree with the function and its derivatives at $x = c$. Use the graphs and tables in Exercises 33–36 to discuss what happens to the accuracy of the Taylor polynomial as you move away from $x = c$.

9.8 Power Series

- Understand the definition of a power series.
- Find the radius and interval of convergence of a power series.
- Determine the endpoint convergence of a power series.
- Differentiate and integrate a power series.

Power Series

In Section 9.7, you were introduced to the concept of approximating functions by Taylor polynomials. For instance, the function $f(x) = e^x$ can be *approximated* by its Maclaurin polynomials as follows.

$$e^x \approx 1 + x \qquad \text{1st-degree polynomial}$$

$$e^x \approx 1 + x + \frac{x^2}{2!} \qquad \text{2nd-degree polynomial}$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \qquad \text{3rd-degree polynomial}$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \qquad \text{4th-degree polynomial}$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \qquad \text{5th-degree polynomial}$$

In that section, you saw that the higher the degree of the approximating polynomial, the better the approximation becomes.

In this and the next two sections, you will see that several important types of functions, including

$$f(x) = e^x$$

can be represented *exactly* by an infinite series called a **power series.** For example, the power series representation for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdot \cdot \cdot + \frac{x^n}{n!} + \cdot \cdot \cdot .$$

For each real number x, it can be shown that the infinite series on the right converges to the number e^x. Before doing this, however, some preliminary results dealing with power series will be discussed—beginning with the following definition.

EXPLORATION

Graphical Reasoning Use a graphing utility to approximate the graph of each power series near $x = 0$. (Use the first several terms of each series.) Each series represents a well-known function. What is the function?

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

c. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

d. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

e. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

DEFINITION OF POWER SERIES

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdot \cdot \cdot + a_n x^n + \cdot \cdot \cdot$$

is called a **power series.** More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdot \cdot \cdot + a_n(x-c)^n + \cdot \cdot \cdot$$

is called a **power series centered at *c*,** where c is a constant.

NOTE To simplify the notation for power series, we agree that $(x-c)^0 = 1$, even if $x = c$. ■

EXAMPLE 1 Power Series

a. The following power series is centered at 0.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdot \cdot \cdot$$

b. The following power series is centered at -1.

$$\sum_{n=0}^{\infty} (-1)^n (x+1)^n = 1 - (x+1) + (x+1)^2 - (x+1)^3 + \cdot \cdot \cdot$$

c. The following power series is centered at 1.

$$\sum_{n=1}^{\infty} \frac{1}{n}(x-1)^n = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdot \cdot \cdot$$ ■

Radius and Interval of Convergence

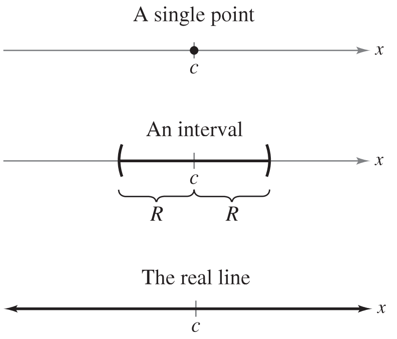
A power series in x can be viewed as a function of x

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

where the *domain of f* is the set of all x for which the power series converges. Determination of the domain of a power series is the primary concern in this section. Of course, every power series converges at its center c because

$$\begin{aligned} f(c) &= \sum_{n=0}^{\infty} a_n(c-c)^n \\ &= a_0(1) + 0 + 0 + \cdot \cdot \cdot + 0 + \cdot \cdot \cdot \\ &= a_0. \end{aligned}$$

So, c always lies in the domain of f. The following important theorem states that the domain of a power series can take three basic forms: a single point, an interval centered at c, or the entire real line, as shown in Figure 9.17. A proof is given in Appendix A.



The domain of a power series has only three basic forms: a single point, an interval centered at c, or the entire real line.
Figure 9.17

THEOREM 9.20 CONVERGENCE OF A POWER SERIES

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$.
3. The series converges absolutely for all x.

The number R is the **radius of convergence** of the power series. If the series converges only at c, the radius of convergence is $R = 0$, and if the series converges for all x, the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

STUDY TIP To determine the radius of convergence of a power series, use the Ratio Test, as demonstrated in Examples 2, 3, and 4.

EXAMPLE 2 Finding the Radius of Convergence

Find the radius of convergence of $\sum_{n=0}^{\infty} n!x^n$.

Solution For $x = 0$, you obtain

$$f(0) = \sum_{n=0}^{\infty} n!0^n = 1 + 0 + 0 + \cdot \cdot \cdot = 1.$$

For any fixed value of x such that $|x| > 0$, let $u_n = n!x^n$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= |x| \lim_{n\to\infty}(n+1) \\ &= \infty.\end{aligned}$$

Therefore, by the Ratio Test, the series diverges for $|x| > 0$ and converges only at its center, 0. So, the radius of convergence is $R = 0$.

EXAMPLE 3 Finding the Radius of Convergence

Find the radius of convergence of

$$\sum_{n=0}^{\infty} 3(x-2)^n.$$

Solution For $x \neq 2$, let $u_n = 3(x-2)^n$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{3(x-2)^{n+1}}{3(x-2)^n}\right| \\ &= \lim_{n\to\infty}|x-2| \\ &= |x-2|.\end{aligned}$$

By the Ratio Test, the series converges if $|x-2| < 1$ and diverges if $|x-2| > 1$. Therefore, the radius of convergence of the series is $R = 1$.

EXAMPLE 4 Finding the Radius of Convergence

Find the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

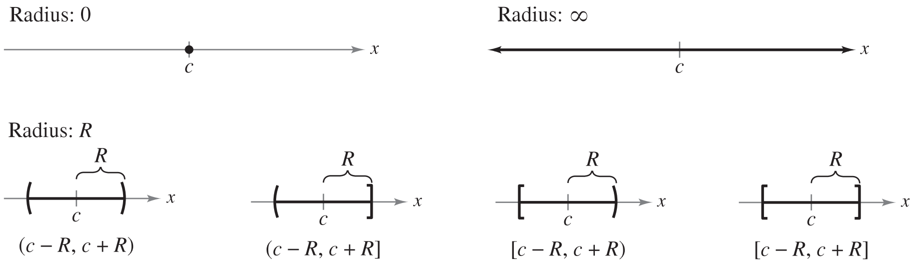
Solution Let $u_n = (-1)^n x^{2n+1}/(2n+1)!$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}}{\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}}\right| \\ &= \lim_{n\to\infty}\frac{x^2}{(2n+3)(2n+2)}.\end{aligned}$$

For any *fixed* value of x, this limit is 0. So, by the Ratio Test, the series converges for all x. Therefore, the radius of convergence is $R = \infty$. ■

Endpoint Convergence

Note that for a power series whose radius of convergence is a finite number R, Theorem 9.20 says nothing about the convergence at the *endpoints* of the interval of convergence. Each endpoint must be tested separately for convergence or divergence. As a result, the interval of convergence of a power series can take any one of the six forms shown in Figure 9.18.



$(c - R, c + R)$ $(c - R, c + R]$ $[c - R, c + R)$ $[c - R, c + R]$

Intervals of convergence
Figure 9.18

EXAMPLE 5 Finding the Interval of Convergence

Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$.

Solution Letting $u_n = x^n/n$ produces

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{\dfrac{x^{n+1}}{(n+1)}}{\dfrac{x^n}{n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{nx}{n+1}\right| \\ &= |x|.\end{aligned}$$

So, by the Ratio Test, the radius of convergence is $R = 1$. Moreover, because the series is centered at 0, it converges in the interval $(-1, 1)$. This interval, however, is not necessarily the *interval of convergence*. To determine this, you must test for convergence at each endpoint. When $x = 1$, you obtain the *divergent* harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdot \cdot \cdot .$$

Diverges when $x = 1$

When $x = -1$, you obtain the *convergent* alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdot \cdot \cdot .$$

Converges when $x = -1$

So, the interval of convergence for the series is $[-1, 1)$, as shown in Figure 9.19.

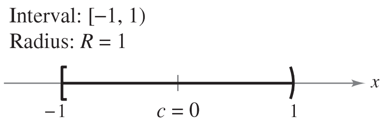


Figure 9.19

EXAMPLE 6 Finding the Interval of Convergence

Find the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(x+1)^n}{2^n}$.

Solution Letting $u_n = (-1)^n(x+1)^n/2^n$ produces

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(x+1)^{n+1}}{2^{n+1}}}{\dfrac{(-1)^n(x+1)^n}{2^n}}\right|\\ &= \lim_{n\to\infty}\left|\frac{2^n(x+1)}{2^{n+1}}\right|\\ &= \left|\frac{x+1}{2}\right|.\end{aligned}$$

By the Ratio Test, the series converges if $|(x+1)/2| < 1$ or $|x+1| < 2$. So, the radius of convergence is $R = 2$. Because the series is centered at $x = -1$, it will converge in the interval $(-3, 1)$. Furthermore, at the endpoints you have

$$\sum_{n=0}^{\infty}\frac{(-1)^n(-2)^n}{2^n} = \sum_{n=0}^{\infty}\frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1 \qquad \text{Diverges when } x = -3$$

and

$$\sum_{n=0}^{\infty}\frac{(-1)^n(2)^n}{2^n} = \sum_{n=0}^{\infty}(-1)^n \qquad \text{Diverges when } x = 1$$

both of which diverge. So, the interval of convergence is $(-3, 1)$, as shown in Figure 9.20.

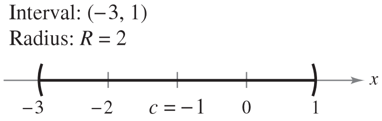


Figure 9.20

EXAMPLE 7 Finding the Interval of Convergence

Find the interval of convergence of

$$\sum_{n=1}^{\infty}\frac{x^n}{n^2}.$$

Solution Letting $u_n = x^n/n^2$ produces

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)^2}{x^n/n^2}\right|\\ &= \lim_{n\to\infty}\left|\frac{n^2x}{(n+1)^2}\right| = |x|.\end{aligned}$$

So, the radius of convergence is $R = 1$. Because the series is centered at $x = 0$, it converges in the interval $(-1, 1)$. When $x = 1$, you obtain the convergent p-series

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots. \qquad \text{Converges when } x = 1$$

When $x = -1$, you obtain the convergent alternating series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots. \qquad \text{Converges when } x = -1$$

Therefore, the interval of convergence for the given series is $[-1, 1]$. ■

The Granger Collection

JAMES GREGORY (1638–1675)

One of the earliest mathematicians to work with power series was a Scotsman, James Gregory. He developed a power series method for interpolating table values—a method that was later used by Brook Taylor in the development of Taylor polynomials and Taylor series.

Differentiation and Integration of Power Series

Power series representation of functions has played an important role in the development of calculus. In fact, much of Newton's work with differentiation and integration was done in the context of power series—especially his work with complicated algebraic functions and transcendental functions. Euler, Lagrange, Leibniz, and the Bernoullis all used power series extensively in calculus.

Once you have defined a function with a power series, it is natural to wonder how you can determine the characteristics of the function. Is it continuous? Differentiable? Theorem 9.21, which is stated without proof, answers these questions.

THEOREM 9.21 PROPERTIES OF FUNCTIONS DEFINED BY POWER SERIES

If the function given by

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n(x-c)^n \\ &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots \end{aligned}$$

has a radius of convergence of $R > 0$, then, on the interval $(c - R, c + R)$, f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} na_n(x-c)^{n-1} \\ &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots \end{aligned}$$

2. $$\begin{aligned} \int f(x)\,dx &= C + \sum_{n=0}^{\infty} a_n\frac{(x-c)^{n+1}}{n+1} \\ &= C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots \end{aligned}$$

The *radius of convergence* of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The *interval of convergence*, however, may differ as a result of the behavior at the endpoints.

Theorem 9.21 states that, in many ways, a function defined by a power series behaves like a polynomial. It is continuous in its interval of convergence, and both its derivative and its antiderivative can be determined by differentiating and integrating each term of the given power series. For instance, the derivative of the power series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \end{aligned}$$

is

$$\begin{aligned} f'(x) &= 1 + (2)\frac{x}{2} + (3)\frac{x^2}{3!} + (4)\frac{x^3}{4!} + \cdots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= f(x). \end{aligned}$$

Notice that $f'(x) = f(x)$. Do you recognize this function?

EXAMPLE 8 Intervals of Convergence for $f(x)$, $f'(x)$, and $\int f(x)\,dx$

Consider the function given by

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots.$$

Find the interval of convergence for each of the following.

a. $\int f(x)\,dx$ **b.** $f(x)$ **c.** $f'(x)$

Solution By Theorem 9.21, you have

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} x^{n-1} \\ &= 1 + x + x^2 + x^3 + \cdots \end{aligned}$$

and

$$\begin{aligned} \int f(x)\,dx &= C + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} \\ &= C + \frac{x^2}{1 \cdot 2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} + \cdots. \end{aligned}$$

By the Ratio Test, you can show that each series has a radius of convergence of $R = 1$. Considering the interval $(-1, 1)$, you have the following.

a. For $\int f(x)\,dx$, the series

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$$ Interval of convergence: $[-1, 1]$

converges for $x = \pm 1$, and its interval of convergence is $[-1, 1]$. See Figure 9.21(a).

b. For $f(x)$, the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$ Interval of convergence: $[-1, 1)$

converges for $x = -1$ and diverges for $x = 1$. So, its interval of convergence is $[-1, 1)$. See Figure 9.21(b).

c. For $f'(x)$, the series

$$\sum_{n=1}^{\infty} x^{n-1}$$ Interval of convergence: $(-1, 1)$

diverges for $x = \pm 1$, and its interval of convergence is $(-1, 1)$. See Figure 9.21(c).

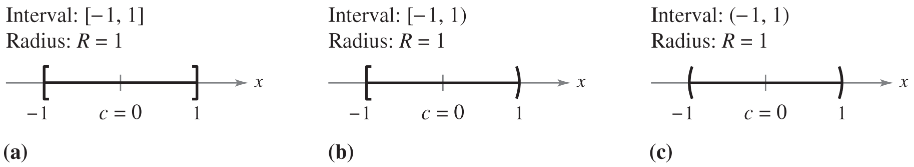


Figure 9.21 ■

From Example 8, it appears that of the three series, the one for the derivative, $f'(x)$, is the least likely to converge at the endpoints. In fact, it can be shown that if the series for $f'(x)$ converges at the endpoints $x = c \pm R$, the series for $f(x)$ will also converge there.

9.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, state where the power series is centered.

1. $\displaystyle\sum_{n=0}^{\infty} nx^n$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 1\cdot 3 \cdots (2n-1)}{2^n n!} x^n$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^3}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi)^{2n}}{(2n)!}$

In Exercises 5–10, find the radius of convergence of the power series.

5. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

6. $\displaystyle\sum_{n=0}^{\infty} (4x)^n$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^n}$

9. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

10. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$

In Exercises 11–34, find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

11. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

12. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{7}\right)^n$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

14. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

16. $\displaystyle\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

17. $\displaystyle\sum_{n=0}^{\infty} (2n)!\left(\frac{x}{3}\right)^n$

18. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{4^n}$

20. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-5)^n}{3^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-4)^n}{n9^n}$

22. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}}$

26. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

28. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

29. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot 3\cdot 4 \cdots (n+1)x^n}{n!}$

32. $\displaystyle\sum_{n=1}^{\infty} \left[\frac{2\cdot 4\cdot 6 \cdots 2n}{3\cdot 5\cdot 7 \cdots (2n+1)}\right] x^{2n+1}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3\cdot 7\cdot 11 \cdots (4n-1)(x-3)^n}{4^n}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x+1)^n}{1\cdot 3\cdot 5 \cdots (2n-1)}$

In Exercises 35 and 36, find the radius of convergence of the power series, where $c > 0$ and k is a positive integer.

35. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$

36. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^k x^n}{(kn)!}$

In Exercises 37–40, find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

37. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{k}\right)^n, \quad k > 0$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-c)^n}{nc^n}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{k(k+1)(k+2)\cdots(k+n-1)x^n}{n!}, \quad k \geq 1$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x-c)^n}{1\cdot 3\cdot 5 \cdots (2n-1)}$

In Exercises 41–44, write an equivalent series with the index of summation beginning at $n = 1$.

41. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

42. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

43. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

44. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

In Exercises 45–48, find the intervals of convergence of (a) $f(x)$, (b) $f'(x)$, (c) $f''(x)$, and (d) $\int f(x)\,dx$. Include a check for convergence at the endpoints of the interval.

45. $\displaystyle f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$

46. $\displaystyle f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$

47. $\displaystyle f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

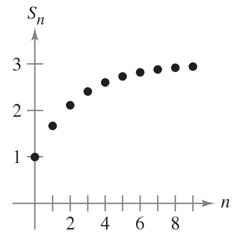
48. $\displaystyle f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n}$

Writing **In Exercises 49–52, match the graph of the first 10 terms of the sequence of partial sums of the series**

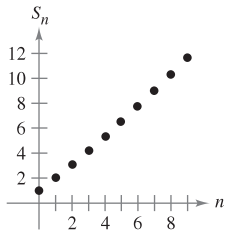
$$g(x) = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

with the indicated value of the function. [The graphs are labeled (a), (b), (c), and (d).] Explain how you made your choice.

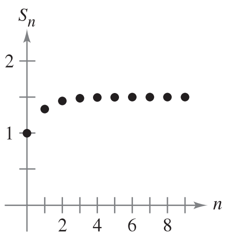
(a)



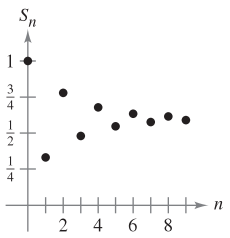
(b)



(c)



(d)



49. $g(1)$

50. $g(2)$

51. $g(3.1)$

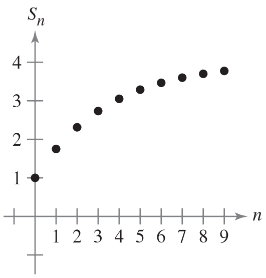
52. $g(-2)$

Writing **In Exercises 53–56, match the graph of the first 10 terms of the sequence of partial sums of the series**

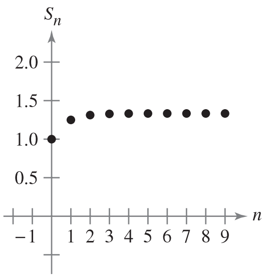
$$g(x) = \sum_{n=0}^{\infty} (2x)^n$$

with the indicated value of the function. [The graphs are labeled (a), (b), (c), and (d).] Explain how you made your choice.

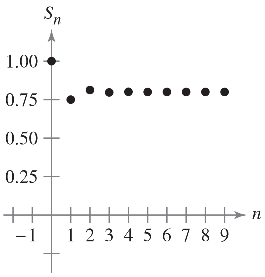
(a)



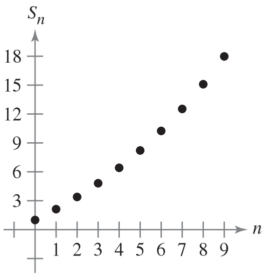
(b)



(c)



(d)



53. $g\left(\frac{1}{8}\right)$

54. $g\left(-\frac{1}{8}\right)$

55. $g\left(\frac{9}{16}\right)$

56. $g\left(\frac{3}{8}\right)$

WRITING ABOUT CONCEPTS

57. Define a power series centered at c.

58. Describe the radius of convergence of a power series. Describe the interval of convergence of a power series.

59. Describe the three basic forms of the domain of a power series.

60. Describe how to differentiate and integrate a power series with a radius of convergence R. Will the series resulting from the operations of differentiation and integration have a different radius of convergence? Explain.

WRITING ABOUT CONCEPTS (continued)

61. Give examples that show that the convergence of a power series at an endpoint of its interval of convergence may be either conditional or absolute. Explain your reasoning.

CAPSTONE

62. Write a power series that has the indicated interval of convergence. Explain your reasoning.

(a) $(-2, 2)$ (b) $(-1, 1]$ (c) $(-1, 0)$ (d) $[-2, 6)$

63. Let $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

(a) Find the intervals of convergence of f and g.

(b) Show that $f'(x) = g(x)$.

(c) Show that $g'(x) = -f(x)$.

(d) Identify the functions f and g.

64. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(a) Find the interval of convergence of f.

(b) Show that $f'(x) = f(x)$.

(c) Show that $f(0) = 1$.

(d) Identify the function f.

In Exercises 65–70, show that the function represented by the power series is a solution of the differential equation.

65. $y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $y'' + y = 0$

66. $y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, $y'' + y = 0$

67. $y = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$, $y'' - y = 0$

68. $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$, $y'' - y = 0$

69. $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$, $y'' - xy' - y = 0$

70. $y = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)}$, $y'' + x^2 y = 0$

71. ***Bessel Function*** The Bessel function of order 0 is

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k!)^2}.$$

(a) Show that the series converges for all x.

(b) Show that the series is a solution of the differential equation $x^2 J_0'' + x J_0' + x^2 J_0 = 0$.

(c) Use a graphing utility to graph the polynomial composed of the first four terms of J_0.

(d) Approximate $\int_0^1 J_0 \, dx$ accurate to two decimal places.

72. ***Bessel Function*** The Bessel function of order 1 is

$$J_1(x) = x\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+1}\, k!(k+1)!}.$$

(a) Show that the series converges for all x.

(b) Show that the series is a solution of the differential equation $x^2 J_1'' + xJ_1' + (x^2 - 1)J_1 = 0$.

(c) Use a graphing utility to graph the polynomial composed of the first four terms of J_1.

(d) Show that $J_0'(x) = -J_1(x)$.

CAS **In Exercises 73–76, the series represents a well-known function. Use a computer algebra system to graph the partial sum S_{10} and identify the function from the graph.**

73. $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

74. $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

75. $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n, \quad -1 < x < 1$

76. $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad -1 \le x \le 1$

77. ***Investigation*** The interval of convergence of the geometric series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n$ is $(-4, 4)$.

(a) Find the sum of the series when $x = \frac{5}{2}$. Use a graphing utility to graph the first six terms of the sequence of partial sums and the horizontal line representing the sum of the series.

(b) Repeat part (a) for $x = -\frac{5}{2}$.

(c) Write a short paragraph comparing the rates of convergence of the partial sums with the sums of the series in parts (a) and (b). How do the plots of the partial sums differ as they converge toward the sum of the series?

(d) Given any positive real number M, there exists a positive integer N such that the partial sum

$$\sum_{n=0}^{N}\left(\frac{5}{4}\right)^n > M.$$

Use a graphing utility to complete the table.

M	10	100	1000	10,000
N				

78. ***Investigation*** The interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty}(3x)^n$ is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

(a) Find the sum of the series when $x = \frac{1}{6}$. Use a graphing utility to graph the first six terms of the sequence of partial sums and the horizontal line representing the sum of the series.

(b) Repeat part (a) for $x = -\frac{1}{6}$.

(c) Write a short paragraph comparing the rates of convergence of the partial sums with the sums of the series in parts (a) and (b). How do the plots of the partial sums differ as they converge toward the sum of the series?

(d) Given any positive real number M, there exists a positive integer N such that the partial sum

$$\sum_{n=0}^{N}\left(3 \cdot \frac{2}{3}\right)^n > M.$$

Use a graphing utility to complete the table.

M	10	100	1000	10,000
N				

True or False? **In Exercises 79–82, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

79. If the power series $\displaystyle\sum_{n=1}^{\infty} a_n x^n$ converges for $x = 2$, then it also converges for $x = -2$.

80. It is possible to find a power series whose interval of convergence is $[0, \infty)$.

81. If the interval of convergence for $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ is $(-1, 1)$, then the interval of convergence for $\displaystyle\sum_{n=0}^{\infty} a_n (x-1)^n$ is $(0, 2)$.

82. If $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 2$, then

$$\int_0^1 f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}.$$

83. Prove that the power series

$$\sum_{n=0}^{\infty} \frac{(n+p)!}{n!(n+q)!}x^n$$

has a radius of convergence of $R = \infty$ if p and q are positive integers.

84. Let $g(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$, where the coefficients are $c_{2n} = 1$ and $c_{2n+1} = 2$ for $n \ge 0$.

(a) Find the interval of convergence of the series.

(b) Find an explicit formula for $g(x)$.

85. Let $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n$, where $c_{n+3} = c_n$ for $n \ge 0$.

(a) Find the interval of convergence of the series.

(b) Find an explicit formula for $f(x)$.

86. Prove that if the power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ has a radius of convergence of R, then $\displaystyle\sum_{n=0}^{\infty} c_n x^{2n}$ has a radius of convergence of $\sqrt{R}$.

87. For $n > 0$, let $R > 0$ and $c_n > 0$. Prove that if the interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} c_n (x - x_0)^n$ is $[x_0 - R, x_0 + R]$, then the series converges conditionally at $x_0 - R$.

9.9 Representation of Functions by Power Series

- **Find a geometric power series that represents a function.**
- **Construct a power series using series operations.**

The Granger Collection

JOSEPH FOURIER (1768–1830)

Some of the early work in representing functions by power series was done by the French mathematician Joseph Fourier. Fourier's work is important in the history of calculus, partly because it forced eighteenth century mathematicians to question the then-prevailing narrow concept of a function. Both Cauchy and Dirichlet were motivated by Fourier's work with series, and in 1837 Dirichlet published the general definition of a function that is used today.

Geometric Power Series

In this section and the next, you will study several techniques for finding a power series that represents a given function.

Consider the function given by $f(x) = 1/(1 - x)$. The form of f closely resembles the sum of a geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1.$$

In other words, if you let $a = 1$ and $r = x$, a power series representation for $1/(1 - x)$, centered at 0, is

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\ &= 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1.\end{aligned}$$

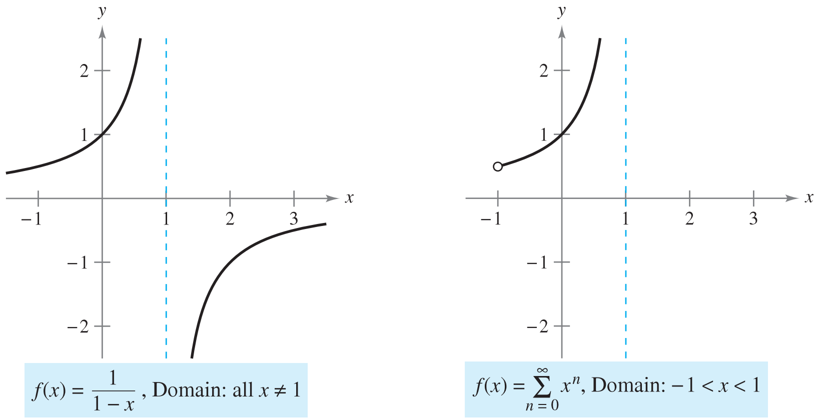
Of course, this series represents $f(x) = 1/(1 - x)$ only on the interval $(-1, 1)$, whereas f is defined for all $x \neq 1$, as shown in Figure 9.22. To represent f in another interval, you must develop a different series. For instance, to obtain the power series centered at -1, you could write

$$\frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{1/2}{1-[(x+1)/2]} = \frac{a}{1-r}$$

which implies that $a = \frac{1}{2}$ and $r = (x + 1)/2$. So, for $|x + 1| < 2$, you have

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{x+1}{2}\right)^n \\ &= \frac{1}{2}\left[1 + \frac{(x+1)}{2} + \frac{(x+1)^2}{4} + \frac{(x+1)^3}{8} + \cdots\right], \quad |x+1| < 2\end{aligned}$$

which converges on the interval $(-3, 1)$.



$f(x) = \dfrac{1}{1-x}$, Domain: all $x \neq 1$

$f(x) = \sum_{n=0}^{\infty} x^n$, Domain: $-1 < x < 1$

Figure 9.22

EXAMPLE 1 Finding a Geometric Power Series Centered at 0

Find a power series for $f(x) = \dfrac{4}{x+2}$, centered at 0.

Solution Writing $f(x)$ in the form $a/(1-r)$ produces

$$\frac{4}{2+x} = \frac{2}{1-(-x/2)} = \frac{a}{1-r}$$

which implies that $a = 2$ and $r = -x/2$. So, the power series for $f(x)$ is

$$\begin{aligned}\frac{4}{x+2} &= \sum_{n=0}^{\infty} ar^n \\ &= \sum_{n=0}^{\infty} 2\left(-\frac{x}{2}\right)^n \\ &= 2\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots\right).\end{aligned}$$

This power series converges when

$$\left|-\frac{x}{2}\right| < 1$$

which implies that the interval of convergence is $(-2, 2)$. ■

Another way to determine a power series for a rational function such as the one in Example 1 is to use long division. For instance, by dividing $2 + x$ into 4, you obtain the result shown at the left.

Long Division

$$\begin{array}{rl} & 2 - x + \tfrac{1}{2}x^2 - \tfrac{1}{4}x^3 + \cdots \\ \cline{2-2} 2 + x \big) & 4 \\ & \underline{4 + 2x} \\ & -2x \\ & \underline{-2x - x^2} \\ & x^2 \\ & \underline{x^2 + \tfrac{1}{2}x^3} \\ & -\tfrac{1}{2}x^3 \\ & \underline{-\tfrac{1}{2}x^3 - \tfrac{1}{4}x^4} \end{array}$$

EXAMPLE 2 Finding a Geometric Power Series Centered at 1

Find a power series for $f(x) = \dfrac{1}{x}$, centered at 1.

Solution Writing $f(x)$ in the form $a/(1-r)$ produces

$$\frac{1}{x} = \frac{1}{1-(-x+1)} = \frac{a}{1-r}$$

which implies that $a = 1$ and $r = 1 - x = -(x-1)$. So, the power series for $f(x)$ is

$$\begin{aligned}\frac{1}{x} &= \sum_{n=0}^{\infty} ar^n \\ &= \sum_{n=0}^{\infty} [-(x-1)]^n \\ &= \sum_{n=0}^{\infty} (-1)^n(x-1)^n \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots.\end{aligned}$$

This power series converges when

$$|x - 1| < 1$$

which implies that the interval of convergence is $(0, 2)$. ■

Operations with Power Series

The versatility of geometric power series will be shown later in this section, following a discussion of power series operations. These operations, used with differentiation and integration, provide a means of developing power series for a variety of elementary functions. (For simplicity, the following properties are stated for a series centered at 0.)

OPERATIONS WITH POWER SERIES

Let $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$.

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$

2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$

3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

The operations described above can change the interval of convergence for the resulting series. For example, in the following addition, the interval of convergence for the sum is the *intersection* of the intervals of convergence of the two original series.

$$\underbrace{\sum_{n=0}^{\infty} x^n}_{} + \underbrace{\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n}_{} = \underbrace{\sum_{n=0}^{\infty} \left(1 + \frac{1}{2^n}\right) x^n}_{}$$

$$(-1, 1) \cap (-2, 2) = (-1, 1)$$

EXAMPLE 3 Adding Two Power Series

Find a power series, centered at 0, for $f(x) = \dfrac{3x - 1}{x^2 - 1}$.

Solution Using partial fractions, you can write $f(x)$ as

$$\frac{3x - 1}{x^2 - 1} = \frac{2}{x + 1} + \frac{1}{x - 1}.$$

By adding the two geometric power series

$$\frac{2}{x + 1} = \frac{2}{1 - (-x)} = \sum_{n=0}^{\infty} 2(-1)^n x^n, \quad |x| < 1$$

and

$$\frac{1}{x - 1} = \frac{-1}{1 - x} = -\sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

you obtain the following power series.

$$\frac{3x - 1}{x^2 - 1} = \sum_{n=0}^{\infty} [2(-1)^n - 1] x^n = 1 - 3x + x^2 - 3x^3 + x^4 - \cdots$$

The interval of convergence for this power series is $(-1, 1)$. ■

EXAMPLE 4 Finding a Power Series by Integration

Find a power series for $f(x) = \ln x$, centered at 1.

Solution From Example 2, you know that

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

Interval of convergence: $(0, 2)$

Integrating this series produces

$$\begin{aligned} \ln x &= \int \frac{1}{x}\,dx + C \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}. \end{aligned}$$

By letting $x = 1$, you can conclude that $C = 0$. Therefore,

$$\begin{aligned} \ln x &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} \\ &= \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdot \cdot \cdot . \end{aligned}$$

Interval of convergence: $(0, 2]$

Note that the series converges at $x = 2$. This is consistent with the observation in the preceding section that integration of a power series may alter the convergence at the endpoints of the interval of convergence. ■

TECHNOLOGY In Section 9.7, the fourth-degree Taylor polynomial for the natural logarithmic function

$$\ln x \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

was used to approximate $\ln(1.1)$.

$$\begin{aligned} \ln(1.1) &\approx (0.1) - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 \\ &\approx 0.0953083 \end{aligned}$$

You now know from Example 4 that this polynomial represents the first four terms of the power series for $\ln x$. Moreover, using the Alternating Series Remainder, you can determine that the error in this approximation is less than

$$\begin{aligned} |R_4| &\le |a_5| \\ &= \frac{1}{5}(0.1)^5 \\ &= 0.000002. \end{aligned}$$

During the seventeenth and eighteenth centuries, mathematical tables for logarithms and values of other transcendental functions were computed in this manner. Such numerical techniques are far from outdated, because it is precisely by such means that many modern calculating devices are programmed to evaluate transcendental functions.

EXAMPLE 5 Finding a Power Series by Integration

Find a power series for $g(x) = \arctan x$, centered at 0.

Solution Because $D_x[\arctan x] = 1/(1 + x^2)$, you can use the series

$$f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n. \qquad \text{Interval of convergence: } (-1, 1)$$

Substituting x^2 for x produces

$$f(x^2) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Finally, by integrating, you obtain

$$\begin{aligned}\arctan x &= \int \frac{1}{1+x^2}\,dx + C\\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} && \text{Let } x = 0, \text{ then } C = 0.\\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots. && \text{Interval of convergence: } (-1, 1)\end{aligned}$$

■

The Granger Collection

SRINIVASA RAMANUJAN (1887–1920)

Series that can be used to approximate π have interested mathematicians for the past 300 years. An amazing series for approximating $1/\pi$ was discovered by the Indian mathematician Srinivasa Ramanujan in 1914 (see Exercise 67). Each successive term of Ramanujan's series adds roughly eight more correct digits to the value of $1/\pi$. For more information about Ramanujan's work, see the article "Ramanujan and Pi" by Jonathan M. Borwein and Peter B. Borwein in *Scientific American*.

It can be shown that the power series developed for arctan x in Example 5 also converges (to arctan x) for $x = \pm 1$. For instance, when $x = 1$, you can write

$$\begin{aligned}\arctan 1 &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\\ &= \frac{\pi}{4}.\end{aligned}$$

However, this series (developed by James Gregory in 1671) does not give us a practical way of approximating π because it converges so slowly that hundreds of terms would have to be used to obtain reasonable accuracy. Example 6 shows how to use *two* different arctangent series to obtain a very good approximation of π using only a few terms. This approximation was developed by John Machin in 1706.

EXAMPLE 6 Approximating π with a Series

Use the trigonometric identity

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$$

to approximate the number π [see Exercise 50(b)].

Solution By using only five terms from each of the series for arctan(1/5) and arctan(1/239), you obtain

$$4\left(4 \arctan \frac{1}{5} - \arctan \frac{1}{239}\right) \approx 3.1415926$$

which agrees with the exact value of π with an error of less than 0.0000001.

■

9.9 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find a geometric power series for the function, centered at 0, (a) by the technique shown in Examples 1 and 2 and (b) by long division.

1. $f(x) = \dfrac{1}{4 - x}$

2. $f(x) = \dfrac{1}{2 + x}$

3. $f(x) = \dfrac{3}{4 + x}$

4. $f(x) = \dfrac{2}{5 - x}$

In Exercises 5–16, find a power series for the function, centered at *c*, and determine the interval of convergence.

5. $f(x) = \dfrac{1}{3 - x}, \quad c = 1$

6. $f(x) = \dfrac{4}{5 - x}, \quad c = -3$

7. $f(x) = \dfrac{1}{1 - 3x}, \quad c = 0$

8. $h(x) = \dfrac{1}{1 - 5x}, \quad c = 0$

9. $g(x) = \dfrac{5}{2x - 3}, \quad c = -3$

10. $f(x) = \dfrac{3}{2x - 1}, \quad c = 2$

11. $f(x) = \dfrac{2}{2x + 3}, \quad c = 0$

12. $f(x) = \dfrac{4}{3x + 2}, \quad c = 3$

13. $g(x) = \dfrac{4x}{x^2 + 2x - 3}, \quad c = 0$

14. $g(x) = \dfrac{3x - 8}{3x^2 + 5x - 2}, \quad c = 0$

15. $f(x) = \dfrac{2}{1 - x^2}, \quad c = 0$

16. $f(x) = \dfrac{5}{5 + x^2}, \quad c = 0$

In Exercises 17–26, use the power series

$$\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

17. $h(x) = \dfrac{-2}{x^2 - 1} = \dfrac{1}{1 + x} + \dfrac{1}{1 - x}$

18. $h(x) = \dfrac{x}{x^2 - 1} = \dfrac{1}{2(1 + x)} - \dfrac{1}{2(1 - x)}$

19. $f(x) = -\dfrac{1}{(x + 1)^2} = \dfrac{d}{dx}\left[\dfrac{1}{x + 1}\right]$

20. $f(x) = \dfrac{2}{(x + 1)^3} = \dfrac{d^2}{dx^2}\left[\dfrac{1}{x + 1}\right]$

21. $f(x) = \ln(x + 1) = \displaystyle\int \frac{1}{x + 1}\,dx$

22. $f(x) = \ln(1 - x^2) = \displaystyle\int \frac{1}{1 + x}\,dx - \int \frac{1}{1 - x}\,dx$

23. $g(x) = \dfrac{1}{x^2 + 1}$

24. $f(x) = \ln(x^2 + 1)$

25. $h(x) = \dfrac{1}{4x^2 + 1}$

26. $f(x) = \arctan 2x$

Graphical and Numerical Analysis **In Exercises 27 and 28, let**

$$S_n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdot \cdot \cdot \pm \frac{x^n}{n}.$$

Use a graphing utility to confirm the inequality graphically. Then complete the table to confirm the inequality numerically.

x	0.0	0.2	0.4	0.6	0.8	1.0
S_n						
$\ln(x + 1)$						
S_{n+1}						

27. $S_2 \le \ln(x + 1) \le S_3$

28. $S_4 \le \ln(x + 1) \le S_5$

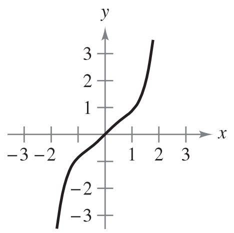
In Exercises 29 and 30, (a) graph several partial sums of the series, (b) find the sum of the series and its radius of convergence, (c) use 50 terms of the series to approximate the sum when $x = 0.5$, and (d) determine what the approximation represents and how good the approximation is.

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x - 1)^n}{n}$

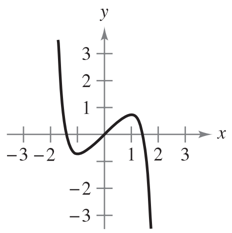
30. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!}$

In Exercises 31–34, match the polynomial approximation of the function $f(x) = \arctan x$ with the correct graph. [The graphs are labeled (a), (b), (c), and (d).]

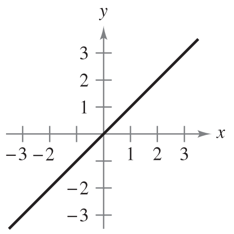
(a)



(b)



(c)



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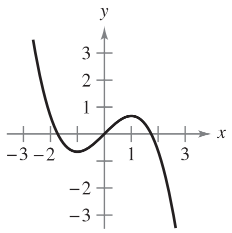


31. $g(x) = x$

32. $g(x) = x - \dfrac{x^3}{3}$

33. $g(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5}$

34. $g(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7}$

In Exercises 35–38, use the series for $f(x) = \arctan x$ to approximate the value, using $R_N \le 0.001$.

35. $\arctan \frac{1}{4}$

36. $\int_0^{3/4} \arctan x^2 \, dx$

37. $\int_0^{1/2} \frac{\arctan x^2}{x} \, dx$

38. $\int_0^{1/2} x^2 \arctan x \, dx$

In Exercises 39–42, use the power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ |x| < 1.$$

Find the series representation of the function and determine its interval of convergence.

39. $f(x) = \dfrac{1}{(1-x)^2}$

40. $f(x) = \dfrac{x}{(1-x)^2}$

41. $f(x) = \dfrac{1+x}{(1-x)^2}$

42. $f(x) = \dfrac{x(1+x)}{(1-x)^2}$

43. ***Probability*** A fair coin is tossed repeatedly. The probability that the first head occurs on the nth toss is $P(n) = \left(\frac{1}{2}\right)^n$. When this game is repeated many times, the average number of tosses required until the first head occurs is

$$E(n) = \sum_{n=1}^{\infty} nP(n).$$

(This value is called the *expected value of n*.) Use the results of Exercises 39–42 to find $E(n)$. Is the answer what you expected? Why or why not?

44. Use the results of Exercises 39–42 to find the sum of each series.

(a) $\frac{1}{3}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ (b) $\frac{1}{10}\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^n$

Writing **In Exercises 45–48, explain how to use the geometric series**

$$g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

to find the series for the function. Do not find the series.

45. $f(x) = \dfrac{1}{1+x}$

46. $f(x) = \dfrac{1}{1-x^2}$

47. $f(x) = \dfrac{5}{1+x}$

48. $f(x) = \ln(1-x)$

49. Prove that $\arctan x + \arctan y = \arctan \dfrac{x+y}{1-xy}$ for $xy \ne 1$ provided the value of the left side of the equation is between $-\pi/2$ and $\pi/2$.

50. Use the result of Exercise 49 to verify each identity.

(a) $\arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4}$

(b) $4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$

[*Hint:* Use Exercise 49 twice to find $4 \arctan \frac{1}{5}$. Then use part (a).]

In Exercises 51 and 52, (a) verify the given equation and (b) use the equation and the series for the arctangent to approximate π to two-decimal-place accuracy.

51. $2 \arctan \frac{1}{2} - \arctan \frac{1}{7} = \frac{\pi}{4}$

52. $\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}$

In Exercises 53–58, find the sum of the convergent series by using a well-known function. Identify the function and explain how you obtained the sum.

53. $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{1}{2^n n}$

54. $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{1}{3^n n}$

55. $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{2^n}{5^n n}$

56. $\sum_{n=0}^{\infty} (-1)^n \dfrac{1}{2n+1}$

57. $\sum_{n=0}^{\infty} (-1)^n \dfrac{1}{2^{2n+1}(2n+1)}$

58. $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{1}{3^{2n-1}(2n-1)}$

WRITING ABOUT CONCEPTS

59. Use the results of Exercises 31–34 to make a geometric argument for why the series approximations of $f(x) = \arctan x$ have only odd powers of x.

60. Use the results of Exercises 31–34 to make a conjecture about the degrees of series approximations of $f(x) = \arctan x$ that have relative extrema.

61. One of the series in Exercises 53–58 converges to its sum at a much lower rate than the other five series. Which is it? Explain why this series converges so slowly. Use a graphing utility to illustrate the rate of convergence.

62. The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is 3. What is the radius of convergence of the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$? Explain.

63. The power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x+1| < 4$. What can you conclude about the series $\sum_{n=0}^{\infty} a_n \dfrac{x^{n+1}}{n+1}$? Explain.

CAPSTONE

64. ***Find the Error*** Describe why the statement is incorrect.

$$\xcancel{\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \left(1 + \frac{1}{5}\right) x^n}$$

In Exercises 65 and 66, find the sum of the series.

65. $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{3^n(2n+1)}$

66. $\sum_{n=0}^{\infty} \dfrac{(-1)^n \pi^{2n+1}}{3^{2n+1}(2n+1)!}$

67. ***Ramanujan and Pi*** Use a graphing utility to show that

$$\frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26{,}390n)}{(n!)396^{4n}} = \frac{1}{\pi}.$$

9.10 Taylor and Maclaurin Series

- **Find a Taylor or Maclaurin series for a function.**
- **Find a binomial series.**
- **Use a basic list of Taylor series to find other Taylor series.**

Taylor Series and Maclaurin Series

Bettmann/Corbis

COLIN MACLAURIN (1698–1746)

The development of power series to represent functions is credited to the combined work of many seventeenth and eighteenth century mathematicians. Gregory, Newton, John and James Bernoulli, Leibniz, Euler, Lagrange, Wallis, and Fourier all contributed to this work. However, the two names that are most commonly associated with power series are Brook Taylor (1685–1731) and Colin Maclaurin.

In Section 9.9, you derived power series for several functions using geometric series with term-by-term differentiation or integration. In this section you will study a *general* procedure for deriving the power series for a function that has derivatives of all orders. The following theorem gives the form that *every* convergent power series must take.

THEOREM 9.22 THE FORM OF A CONVERGENT POWER SERIES

If f is represented by a power series $f(x) = \Sigma\, a_n(x-c)^n$ for all x in an open interval I containing c, then $a_n = f^{(n)}(c)/n!$ and

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdot\cdot\cdot + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdot\cdot\cdot.$$

PROOF Suppose the power series $\Sigma\, a_n(x-c)^n$ has a radius of convergence R. Then, by Theorem 9.21, you know that the nth derivative of f exists for $|x-c| < R$, and by successive differentiation you obtain the following.

$$\begin{aligned}
f^{(0)}(x) &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \cdot\cdot\cdot\\
f^{(1)}(x) &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdot\cdot\cdot\\
f^{(2)}(x) &= 2a_2 + 3!a_3(x-c) + 4\cdot 3a_4(x-c)^2 + \cdot\cdot\cdot\\
f^{(3)}(x) &= 3!a_3 + 4!a_4(x-c) + \cdot\cdot\cdot\\
&\vdots\\
f^{(n)}(x) &= n!a_n + (n+1)!a_{n+1}(x-c) + \cdot\cdot\cdot
\end{aligned}$$

Evaluating each of these derivatives at $x = c$ yields

$$\begin{aligned}
f^{(0)}(c) &= 0!a_0\\
f^{(1)}(c) &= 1!a_1\\
f^{(2)}(c) &= 2!a_2\\
f^{(3)}(c) &= 3!a_3
\end{aligned}$$

and, in general, $f^{(n)}(c) = n!a_n$. By solving for a_n, you find that the coefficients of the power series representation of $f(x)$ are

$$a_n = \frac{f^{(n)}(c)}{n!}.$$ ■

NOTE Be sure you understand Theorem 9.22. The theorem says that *if a power series converges to* $f(x)$, the series must be a Taylor series. The theorem does *not* say that every series formed with the Taylor coefficients $a_n = f^{(n)}(c)/n!$ will converge to $f(x)$.

Notice that the coefficients of the power series in Theorem 9.22 are precisely the coefficients of the Taylor polynomials for $f(x)$ at c as defined in Section 9.7. For this reason, the series is called the **Taylor series** for $f(x)$ at c.

DEFINITION OF TAYLOR AND MACLAURIN SERIES

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdot \cdot \cdot$$

is called the **Taylor series for $f(x)$ at c.** Moreover, if $c = 0$, then the series is the **Maclaurin series for f.**

If you know the pattern for the coefficients of the Taylor polynomials for a function, you can extend the pattern easily to form the corresponding Taylor series. For instance, in Example 4 in Section 9.7, you found the fourth Taylor polynomial for $\ln x$, centered at 1, to be

$$P_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4.$$

From this pattern, you can obtain the Taylor series for $\ln x$ centered at $c = 1$,

$$(x-1) - \frac{1}{2}(x-1)^2 + \cdot \cdot \cdot + \frac{(-1)^{n+1}}{n}(x-1)^n + \cdot \cdot \cdot .$$

EXAMPLE 1 Forming a Power Series

Use the function $f(x) = \sin x$ to form the Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdot \cdot \cdot$$

and determine the interval of convergence.

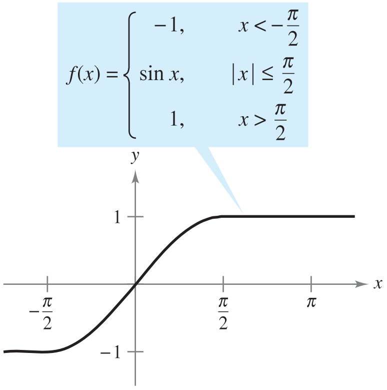
Solution Successive differentiation of $f(x)$ yields

$$\begin{aligned}
f(x) &= \sin x & f(0) &= \sin 0 = 0\\
f'(x) &= \cos x & f'(0) &= \cos 0 = 1\\
f''(x) &= -\sin x & f''(0) &= -\sin 0 = 0\\
f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -\cos 0 = -1\\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= \sin 0 = 0\\
f^{(5)}(x) &= \cos x & f^{(5)}(0) &= \cos 0 = 1
\end{aligned}$$

and so on. The pattern repeats after the third derivative. So, the power series is as follows.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdot \cdot \cdot\\
\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} &= 0 + (1)x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6\\
&\quad + \frac{(-1)}{7!}x^7 + \cdot \cdot \cdot\\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdot \cdot \cdot
\end{aligned}$$

By the Ratio Test, you can conclude that this series converges for all x. ■



$$f(x) = \begin{cases} -1, & x < -\frac{\pi}{2} \\ \sin x, & |x| \le \frac{\pi}{2} \\ 1, & x > \frac{\pi}{2} \end{cases}$$

Figure 9.23

Notice that in Example 1 you cannot conclude that the power series converges to $\sin x$ for all x. You can simply conclude that the power series converges to some function, but you are not sure what function it is. This is a subtle, but important, point in dealing with Taylor or Maclaurin series. To persuade yourself that the series

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdot \cdot \cdot$$

might converge to a function other than f, remember that the derivatives are being evaluated at a single point. It can easily happen that another function will agree with the values of $f^{(n)}(x)$ when $x = c$ and disagree at other x-values. For instance, if you formed the power series (centered at 0) for the function shown in Figure 9.23, you would obtain the same series as in Example 1. You know that the series converges for all x, and yet it obviously cannot converge to both $f(x)$ and $\sin x$ for all x.

Let f have derivatives of all orders in an open interval I centered at c. The Taylor series for f may fail to converge for some x in I. Or, even if it is convergent, it may fail to have $f(x)$ as its sum. Nevertheless, Theorem 9.19 tells us that for each n,

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

Note that in this remainder formula the particular value of z that makes the remainder formula true depends on the values of x and n. If $R_n \to 0$, then the following theorem tells us that the Taylor series for f actually converges to $f(x)$ for all x in I.

THEOREM 9.23 CONVERGENCE OF TAYLOR SERIES

If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

PROOF For a Taylor series, the nth partial sum coincides with the nth Taylor polynomial. That is, $S_n(x) = P_n(x)$. Moreover, because

$$P_n(x) = f(x) - R_n(x)$$

it follows that

$$\begin{aligned} \lim_{n\to\infty} S_n(x) &= \lim_{n\to\infty} P_n(x) \\ &= \lim_{n\to\infty} [f(x) - R_n(x)] \\ &= f(x) - \lim_{n\to\infty} R_n(x). \end{aligned}$$

So, for a given x, the Taylor series (the sequence of partial sums) converges to $f(x)$ if and only if $R_n(x) \to 0$ as $n \to \infty$. ■

NOTE Stated another way, Theorem 9.23 says that a power series formed with Taylor coefficients $a_n = f^{(n)}(c)/n!$ converges to the function from which it was derived at precisely those values for which the remainder approaches 0 as $n \to \infty$. ■

In Example 1, you derived the power series from the sine function and you also concluded that the series converges to some function on the entire real line. In Example 2, you will see that the series actually converges to sin x. The key observation is that although the value of z is not known, it is possible to obtain an upper bound for $|f^{(n+1)}(z)|$.

EXAMPLE 2 A Convergent Maclaurin Series

Show that the Maclaurin series for $f(x) = \sin x$ converges to $\sin x$ for all x.

Solution Using the result in Example 1, you need to show that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdot \cdot \cdot + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdot \cdot \cdot$$

is true for all x. Because

$$f^{(n+1)}(x) = \pm \sin x$$

or

$$f^{(n+1)}(x) = \pm \cos x$$

you know that $|f^{(n+1)}(z)| \leq 1$ for every real number z. Therefore, for any fixed x, you can apply Taylor's Theorem (Theorem 9.19) to conclude that

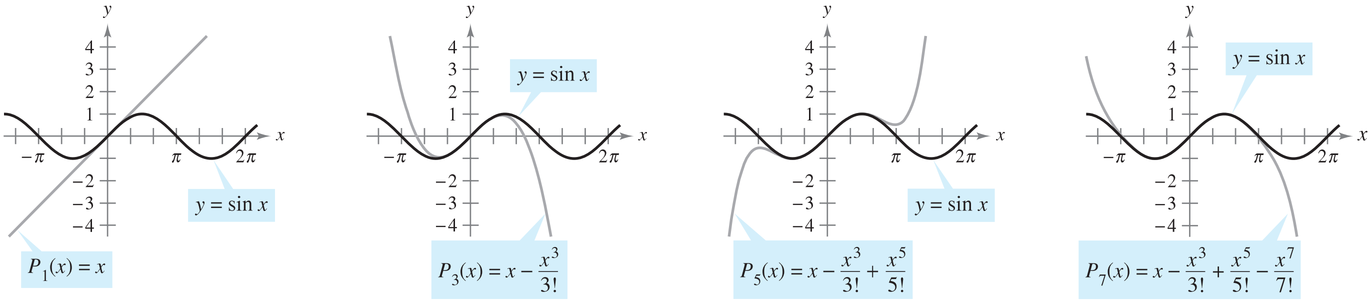
$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

From the discussion in Section 9.1 regarding the relative rates of convergence of exponential and factorial sequences, it follows that for a fixed x

$$\lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

Finally, by the Squeeze Theorem, it follows that for all x, $R_n(x) \to 0$ as $n \to \infty$. So, by Theorem 9.23, the Maclaurin series for sin x converges to sin x for all x.

Figure 9.24 visually illustrates the convergence of the Maclaurin series for sin x by comparing the graphs of the Maclaurin polynomials $P_1(x)$, $P_3(x)$, $P_5(x)$, and $P_7(x)$ with the graph of the sine function. Notice that as the degree of the polynomial increases, its graph more closely resembles that of the sine function.



As n increases, the graph of P_n more closely resembles the sine function.
Figure 9.24

The guidelines for finding a Taylor series for $f(x)$ at c are summarized below.

GUIDELINES FOR FINDING A TAYLOR SERIES

1. Differentiate $f(x)$ several times and evaluate each derivative at c.

$$f(c), f'(c), f''(c), f'''(c), \cdot \cdot \cdot, f^{(n)}(c), \cdot \cdot \cdot$$

Try to recognize a pattern in these numbers.

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$, and determine the interval of convergence for the resulting power series

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdot \cdot \cdot .$$

3. Within this interval of convergence, determine whether the series converges to $f(x)$.

The direct determination of Taylor or Maclaurin coefficients using successive differentiation can be difficult, and the next example illustrates a shortcut for finding the coefficients indirectly—using the coefficients of a known Taylor or Maclaurin series.

EXAMPLE 3 Maclaurin Series for a Composite Function

Find the Maclaurin series for $f(x) = \sin x^2$.

Solution To find the coefficients for this Maclaurin series directly, you must calculate successive derivatives of $f(x) = \sin x^2$. By calculating just the first two,

$$f'(x) = 2x \cos x^2 \quad \text{and} \quad f''(x) = -4x^2 \sin x^2 + 2 \cos x^2$$

you can see that this task would be quite cumbersome. Fortunately, there is an alternative. First consider the Maclaurin series for sin x found in Example 1.

$$\begin{aligned} g(x) &= \sin x \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdot \cdot \cdot \end{aligned}$$

Now, because $\sin x^2 = g(x^2)$, you can substitute x^2 for x in the series for sin x to obtain

$$\begin{aligned} \sin x^2 &= g(x^2) \\ &= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdot \cdot \cdot . \end{aligned}$$

■

Be sure to understand the point illustrated in Example 3. Because direct computation of Taylor or Maclaurin coefficients can be tedious, the most practical way to find a Taylor or Maclaurin series is to develop power series for a *basic list* of elementary functions. From this list, you can determine power series for other functions by the operations of addition, subtraction, multiplication, division, differentiation, integration, and composition with known power series.

Binomial Series

Before presenting the basic list for elementary functions, you will develop one more series—for a function of the form $f(x) = (1 + x)^k$. This produces the **binomial series.**

EXAMPLE 4 Binomial Series

Find the Maclaurin series for $f(x) = (1 + x)^k$ and determine its radius of convergence. Assume that k is not a positive integer.

Solution By successive differentiation, you have

$$\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1\\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k\\
f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1)\\
f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2)\\
&\vdots & &\vdots\\
f^{(n)}(x) &= k \cdot \cdot \cdot (k-n+1)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1) \cdot \cdot \cdot (k-n+1)
\end{aligned}$$

which produces the series

$$1 + kx + \frac{k(k-1)x^2}{2} + \cdot \cdot \cdot + \frac{k(k-1) \cdot \cdot \cdot (k-n+1)x^n}{n!} + \cdot \cdot \cdot .$$

Because $a_{n+1}/a_n \to 1$, you can apply the Ratio Test to conclude that the radius of convergence is $R = 1$. So, the series converges to some function in the interval $(-1, 1)$. ■

Note that Example 4 shows that the Taylor series for $(1 + x)^k$ converges to some function in the interval $(-1, 1)$. However, the example does not show that the series actually converges to $(1 + x)^k$. To do this, you could show that the remainder $R_n(x)$ converges to 0, as illustrated in Example 2.

EXAMPLE 5 Finding a Binomial Series

Find the power series for $f(x) = \sqrt[3]{1 + x}$.

Solution Using the binomial series

$$(1 + x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdot \cdot \cdot$$

let $k = \frac{1}{3}$ and write

$$(1 + x)^{1/3} = 1 + \frac{x}{3} - \frac{2x^2}{3^2 2!} + \frac{2 \cdot 5x^3}{3^3 3!} - \frac{2 \cdot 5 \cdot 8x^4}{3^4 4!} + \cdot \cdot \cdot$$

which converges for $-1 \le x \le 1$. ■

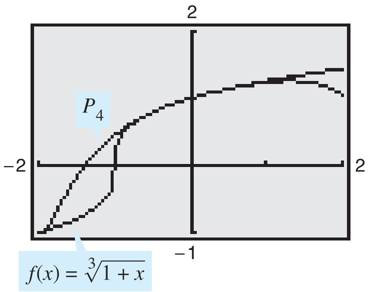


Figure 9.25

TECHNOLOGY Use a graphing utility to confirm the result in Example 5. When you graph the functions

$$f(x) = (1 + x)^{1/3} \quad \text{and} \quad P_4(x) = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243}$$

in the same viewing window, you should obtain the result shown in Figure 9.25.

Deriving Taylor Series from a Basic List

The following list provides the power series for several elementary functions with the corresponding intervals of convergence.

POWER SERIES FOR ELEMENTARY FUNCTIONS

Function	*Interval of Convergence*
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \le 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \le x \le 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \cdots$	$-1 \le x \le 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$	$-1 < x < 1$*

* *The convergence at $x = \pm 1$ depends on the value of k.*

NOTE The binomial series is valid for noninteger values of k. Moreover, if k happens to be a positive integer, the binomial series reduces to a simple binomial expansion. ■

EXAMPLE 6 Deriving a Power Series from a Basic List

Find the power series for $f(x) = \cos\sqrt{x}$.

Solution Using the power series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

you can replace x by $\sqrt{x}$ to obtain the series

$$\cos\sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots.$$

This series converges for all x in the domain of $\cos\sqrt{x}$—that is, for $x \ge 0$. ■

Power series can be multiplied and divided like polynomials. After finding the first few terms of the product (or quotient), you may be able to recognize a pattern.

EXAMPLE 7 Multiplication and Division of Power Series

Find the first three nonzero terms in each Maclaurin series.

a. $e^x \arctan x$ **b.** $\tan x$

Solution

a. Using the Maclaurin series for e^x and $\arctan x$ in the table, you have

$$e^x \arctan x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right).$$

Multiply these expressions and collect like terms as you would in multiplying polynomials.

$$\begin{array}{rrrrrrr}
1 + x & +\,\frac{1}{2}x^2 & +\,\frac{1}{6}x^3 & +\,\frac{1}{24}x^4 & +\cdots & & \\
x & & -\,\frac{1}{3}x^3 & & +\,\frac{1}{5}x^5 & -\cdots & \\ \hline
x & +\,x^2 & +\,\frac{1}{2}x^3 & +\,\frac{1}{6}x^4 & +\,\frac{1}{24}x^5 & +\cdots & \\
 & & -\,\frac{1}{3}x^3 & -\,\frac{1}{3}x^4 & -\,\frac{1}{6}x^5 & -\cdots & \\
 & & & & +\,\frac{1}{5}x^5 & +\cdots & \\ \hline
x & +\,x^2 & +\,\frac{1}{6}x^3 & -\,\frac{1}{6}x^4 & +\,\frac{3}{40}x^5 & +\cdots &
\end{array}$$

So, $e^x \arctan x = x + x^2 + \frac{1}{6}x^3 + \cdots$.

b. Using the Maclaurin series for $\sin x$ and $\cos x$ in the table, you have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}.$$

Divide using long division.

$$\begin{array}{rr}
 & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\ \cline{2-2}
1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \Big) & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots \\
 & x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots \\ \cline{2-2}
 & \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots \\
 & \frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots \\ \cline{2-2}
 & \frac{2}{15}x^5 + \cdots
\end{array}$$

So, $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$. ■

EXAMPLE 8 A Power Series for $\sin^2 x$

Find the power series for $f(x) = \sin^2 x$.

Solution Consider rewriting $\sin^2 x$ as follows.

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{\cos 2x}{2}$$

Now, use the series for $\cos x$.

$$\begin{aligned}
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \\
\cos 2x &= 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \frac{2^8}{8!}x^8 - \cdots \\
-\frac{1}{2}\cos 2x &= -\frac{1}{2} + \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \cdots \\
\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x &= \frac{1}{2} - \frac{1}{2} + \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \cdots \\
&= \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \cdots
\end{aligned}$$

This series converges for $-\infty < x < \infty$. ■

As mentioned in the preceding section, power series can be used to obtain tables of values of transcendental functions. They are also useful for estimating the values of definite integrals for which antiderivatives cannot be found. The next example demonstrates this use.

EXAMPLE 9 Power Series Approximation of a Definite Integral

Use a power series to approximate

$$\int_0^1 e^{-x^2}\,dx$$

with an error of less than 0.01.

Solution Replacing x with $-x^2$ in the series for e^x produces the following.

$$\begin{aligned}
e^{-x^2} &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots \\
\int_0^1 e^{-x^2}\,dx &= \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \cdots\right]_0^1 \\
&= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots
\end{aligned}$$

Summing the first four terms, you have

$$\int_0^1 e^{-x^2}\,dx \approx 0.74$$

which, by the Alternating Series Test, has an error of less than $\frac{1}{216} \approx 0.005$.

■

9.10 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–12, use the definition of Taylor series to find the Taylor series (centered at c) for the function.

1. $f(x) = e^{2x}, \quad c = 0$

2. $f(x) = e^{3x}, \quad c = 0$

3. $f(x) = \cos x, \quad c = \frac{\pi}{4}$

4. $f(x) = \sin x, \quad c = \frac{\pi}{4}$

5. $f(x) = \frac{1}{x}, \quad c = 1$

6. $f(x) = \frac{1}{1-x}, \quad c = 2$

7. $f(x) = \ln x, \quad c = 1$

8. $f(x) = e^{x}, \quad c = 1$

9. $f(x) = \sin 3x, \quad c = 0$

10. $f(x) = \ln(x^2 + 1), \quad c = 0$

11. $f(x) = \sec x, \quad c = 0$ (first three nonzero terms)

12. $f(x) = \tan x, \quad c = 0$ (first three nonzero terms)

In Exercises 13–16, prove that the Maclaurin series for the function converges to the function for all x.

13. $f(x) = \cos x$

14. $f(x) = e^{-2x}$

15. $f(x) = \sinh x$

16. $f(x) = \cosh x$

In Exercises 17–26, use the binomial series to find the Maclaurin series for the function.

17. $f(x) = \frac{1}{(1+x)^2}$

18. $f(x) = \frac{1}{(1+x)^4}$

19. $f(x) = \frac{1}{\sqrt{1-x}}$

20. $f(x) = \frac{1}{\sqrt{1-x^2}}$

21. $f(x) = \frac{1}{\sqrt{4+x^2}}$

22. $f(x) = \frac{1}{(2+x)^3}$

23. $f(x) = \sqrt{1+x}$

24. $f(x) = \sqrt[4]{1+x}$

25. $f(x) = \sqrt{1+x^2}$

26. $f(x) = \sqrt{1+x^3}$

In Exercises 27–40, find the Maclaurin series for the function. (Use the table of power series for elementary functions.)

27. $f(x) = e^{x^2/2}$

28. $g(x) = e^{-3x}$

29. $f(x) = \ln(1+x)$

30. $f(x) = \ln(1+x^2)$

31. $g(x) = \sin 3x$

32. $f(x) = \sin \pi x$

33. $f(x) = \cos 4x$

34. $f(x) = \cos \pi x$

35. $f(x) = \cos x^{3/2}$

36. $g(x) = 2 \sin x^3$

37. $f(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh x$

38. $f(x) = e^x + e^{-x} = 2 \cosh x$

39. $f(x) = \cos^2 x$

40. $f(x) = \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

$\left(\textit{Hint:}\text{ Integrate the series for } \frac{1}{\sqrt{x^2+1}}.\right)$

In Exercises 41–44, find the Maclaurin series for the function. (See Example 7.)

41. $f(x) = x \sin x$

42. $h(x) = x \cos x$

43. $g(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

44. $f(x) = \begin{cases} \dfrac{\arcsin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

In Exercises 45 and 46, use a power series and the fact that $i^2 = -1$ to verify the formula.

45. $g(x) = \frac{1}{2i}(e^{ix} - e^{-ix}) = \sin x$

46. $g(x) = \frac{1}{2}(e^{ix} + e^{-ix}) = \cos x$

In Exercises 47–52, find the first four nonzero terms of the Maclaurin series for the function by multiplying or dividing the appropriate power series. Use the table of power series for elementary functions on page 684. Use a graphing utility to graph the function and its corresponding polynomial approximation.

47. $f(x) = e^x \sin x$

48. $g(x) = e^x \cos x$

49. $h(x) = \cos x \ln(1+x)$

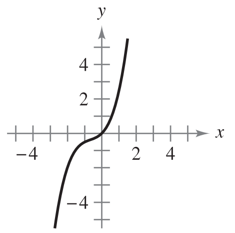
50. $f(x) = e^x \ln(1+x)$

51. $g(x) = \frac{\sin x}{1+x}$

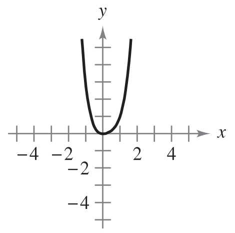
52. $f(x) = \frac{e^x}{1+x}$

In Exercises 53–56, match the polynomial with its graph. [The graphs are labeled (a), (b), (c), and (d).] Factor a common factor from each polynomial and identify the function approximated by the remaining Taylor polynomial.

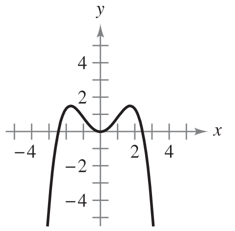
(a)



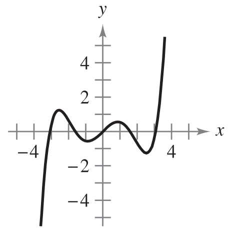
(b)



(c)



(d)



53. $y = x^2 - \frac{x^4}{3!}$

54. $y = x - \frac{x^3}{2!} + \frac{x^5}{4!}$

55. $y = x + x^2 + \frac{x^3}{2!}$

56. $y = x^2 - x^3 + x^4$

In Exercises 57 and 58, find a Maclaurin series for $f(x)$.

57. $f(x) = \int_0^x (e^{-t^2} - 1)\, dt$

58. $f(x) = \int_0^x \sqrt{1 + t^3}\, dt$

In Exercises 59–62, verify the sum. Then use a graphing utility to approximate the sum with an error of less than 0.0001.

59. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2$

60. $\sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(2n+1)!}\right] = \sin 1$

61. $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$

62. $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n!}\right) = \frac{e-1}{e}$

In Exercises 63–66, use the series representation of the function f to find $\lim_{x \to 0} f(x)$ (if it exists).

63. $f(x) = \frac{1 - \cos x}{x}$

64. $f(x) = \frac{\sin x}{x}$

65. $f(x) = \frac{e^x - 1}{x}$

66. $f(x) = \frac{\ln(x + 1)}{x}$

In Exercises 67–74, use a power series to approximate the value of the integral with an error of less than 0.0001. (In Exercises 69 and 71, assume that the integrand is defined as 1 when $x = 0$.)

67. $\int_0^1 e^{-x^3}\, dx$

68. $\int_0^{1/4} x \ln(x + 1)\, dx$

69. $\int_0^1 \frac{\sin x}{x}\, dx$

70. $\int_0^1 \cos x^2\, dx$

71. $\int_0^{1/2} \frac{\arctan x}{x}\, dx$

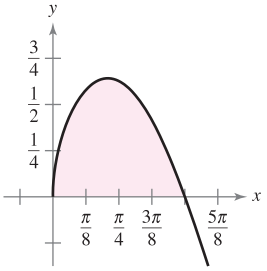
72. $\int_0^{1/2} \arctan x^2\, dx$

73. $\int_{0.1}^{0.3} \sqrt{1 + x^3}\, dx$

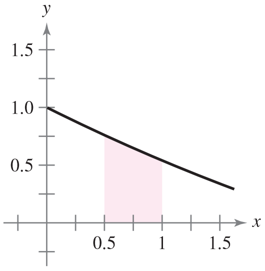
74. $\int_0^{0.2} \sqrt{1 + x^2}\, dx$

Area **In Exercises 75 and 76, use a power series to approximate the area of the region. Use a graphing utility to verify the result.**

75. $\int_0^{\pi/2} \sqrt{x} \cos x\, dx$

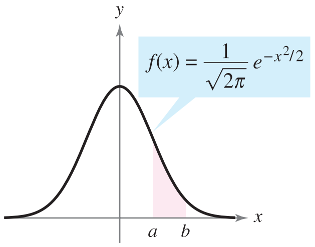


76. $\int_{0.5}^1 \cos \sqrt{x}\, dx$



Probability **In Exercises 77 and 78, approximate the normal probability with an error of less than 0.0001, where the probability is given by**

$$P(a < x < b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx.$$



77. $P(0 < x < 1)$

78. $P(1 < x < 2)$

CAS **In Exercises 79–82, use a computer algebra system to find the fifth-degree Taylor polynomial (centered at c) for the function. Graph the function and the polynomial. Use the graph to determine the largest interval on which the polynomial is a reasonable approximation of the function.**

79. $f(x) = x \cos 2x, \quad c = 0$

80. $f(x) = \sin \frac{x}{2} \ln(1 + x), \quad c = 0$

81. $g(x) = \sqrt{x} \ln x, \quad c = 1$

82. $h(x) = \sqrt[3]{x} \arctan x, \quad c = 1$

WRITING ABOUT CONCEPTS

83. State the guidelines for finding a Taylor series.

84. If f is an even function, what must be true about the coefficients a_n in the Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n?$$

Explain your reasoning.

85. Define the binomial series. What is its radius of convergence?

CAPSTONE

86. Explain how to use the series

$$g(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

to find the series for each function. Do not find the series.

(a) $f(x) = e^{-x}$

(b) $f(x) = e^{3x}$

(c) $f(x) = xe^x$

(d) $f(x) = e^{2x} + e^{-2x}$

87. ***Projectile Motion*** A projectile fired from the ground follows the trajectory given by

$$y = \left(\tan\theta - \frac{g}{kv_0\cos\theta}\right)x - \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_0\cos\theta}\right)$$

where v_0 is the initial speed, θ is the angle of projection, g is the acceleration due to gravity, and k is the drag factor caused by air resistance. Using the power series representation

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 < x < 1$$

verify that the trajectory can be rewritten as

$$y = (\tan\theta)x + \frac{gx^2}{2v_0^2\cos^2\theta} + \frac{kgx^3}{3v_0^3\cos^3\theta} + \frac{k^2gx^4}{4v_0^4\cos^4\theta} + \cdots.$$

88. ***Projectile Motion*** Use the result of Exercise 87 to determine the series for the path of a projectile launched from ground level at an angle of $\theta = 60°$, with an initial speed of $v_0 = 64$ feet per second and a drag factor of $k = \frac{1}{16}$.

89. ***Investigation*** Consider the function f defined by

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

(a) Sketch a graph of the function.

(b) Use the alternative form of the definition of the derivative (Section 2.1) and L'Hôpital's Rule to show that $f'(0) = 0$. [By continuing this process, it can be shown that $f^{(n)}(0) = 0$ for $n > 1$.]

(c) Using the result in part (b), find the Maclaurin series for f. Does the series converge to f?

90. ***Investigation***

(a) Find the power series centered at 0 for the function

$$f(x) = \frac{\ln(x^2 + 1)}{x^2}.$$

(b) Use a graphing utility to graph f and the eighth-degree Taylor polynomial $P_8(x)$ for f.

(c) Complete the table, where

$$F(x) = \int_0^x \frac{\ln(t^2 + 1)}{t^2}\,dt \quad \text{and} \quad G(x) = \int_0^x P_8(t)\,dt.$$

x	0.25	0.50	0.75	1.00	1.50	2.00
$F(x)$						
$G(x)$						

(d) Describe the relationship between the graphs of f and P_8 and the results given in the table in part (c).

91. Prove that $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any real x.

92. Find the Maclaurin series for

$$f(x) = \ln\frac{1 + x}{1 - x}$$

and determine its radius of convergence. Use the first four terms of the series to approximate ln 3.

In Exercises 93–96, evaluate the binomial coefficient using the formula

$$\binom{k}{n} = \frac{k(k-1)(k-2)(k-3)\cdots(k-n+1)}{n!}$$

where k is a real number, n is a positive integer, and

$$\binom{k}{0} = 1.$$

93. $\binom{5}{3}$

94. $\binom{-2}{2}$

95. $\binom{0.5}{4}$

96. $\binom{-1/3}{5}$

97. Write the power series for $(1 + x)^k$ in terms of binomial coefficients.

98. Prove that e is irrational. [*Hint:* Assume that $e = p/q$ is rational (p and q are integers) and consider

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots.]$$

99. Show that the Maclaurin series for the function

$$g(x) = \frac{x}{1 - x - x^2}$$

is

$$\sum_{n=1}^{\infty} F_n x^n$$

where F_n is the nth Fibonacci number with $F_1 = F_2 = 1$ and $F_n = F_{n-2} + F_{n-1}$, for $n \geq 3$.

(*Hint:* Write

$$\frac{x}{1 - x - x^2} = a_0 + a_1x + a_2x^2 + \cdots$$

and multiply each side of this equation by $1 - x - x^2$.)

PUTNAM EXAM CHALLENGE

100. Assume that $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ for all x on an interval of length at least 2. Show that $|f'(x)| \leq 2$ on the interval.

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9 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

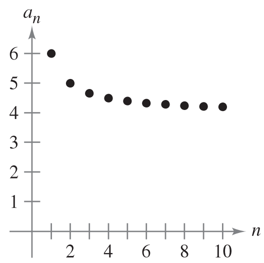
In Exercises 1 and 2, write an expression for the *n*th term of the sequence.

1. $\frac{1}{2}, \frac{1}{3}, \frac{1}{7}, \frac{1}{25}, \frac{1}{121}, \ldots$

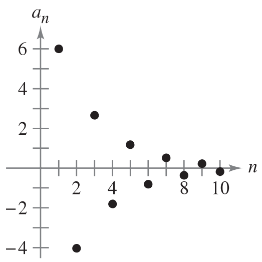
2. $\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \ldots$

In Exercises 3–6, match the sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)



(b)



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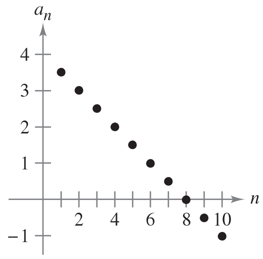

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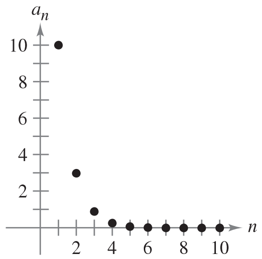


3. $a_n = 4 + \frac{2}{n}$

4. $a_n = 4 - \frac{1}{2}n$

5. $a_n = 10(0.3)^{n-1}$

6. $a_n = 6\left(-\frac{2}{3}\right)^{n-1}$

In Exercises 7 and 8, use a graphing utility to graph the first 10 terms of the sequence. Use the graph to make an inference about the convergence or divergence of the sequence. Verify your inference analytically and, if the sequence converges, find its limit.

7. $a_n = \frac{5n + 2}{n}$

8. $a_n = \sin \frac{n\pi}{2}$

In Exercises 9–18, determine the convergence or divergence of the sequence with the given *n*th term. If the sequence converges, find its limit. (*b* and *c* are positive real numbers.)

9. $a_n = \left(\frac{7}{8}\right)^n + 3$

10. $a_n = 1 + \frac{5}{n + 1}$

11. $a_n = \frac{n^3 + 1}{n^2}$

12. $a_n = \frac{1}{\sqrt{n}}$

13. $a_n = \frac{n}{n^2 + 1}$

14. $a_n = \frac{n}{\ln n}$

15. $a_n = \sqrt{n + 1} - \sqrt{n}$

16. $a_n = \left(1 + \frac{1}{2n}\right)^n$

17. $a_n = \frac{\sin \sqrt{n}}{\sqrt{n}}$

18. $a_n = (b^n + c^n)^{1/n}$

19. ***Compound Interest*** A deposit of \$8000 is made in an account that earns 5% interest compounded quarterly. The balance in the account after n quarters is

$$A_n = 8000\left(1 + \frac{0.05}{4}\right)^n, \quad n = 1, 2, 3, \ldots.$$

(a) Compute the first eight terms of the sequence $\{A_n\}$.

(b) Find the balance in the account after 10 years by computing the 40th term of the sequence.

20. ***Depreciation*** A company buys a machine for \$175,000. During the next 5 years the machine will depreciate at a rate of 30% per year. (That is, at the end of each year, the depreciated value will be 70% of what it was at the beginning of the year.)

(a) Find a formula for the nth term of the sequence that gives the value V of the machine t full years after it was purchased.

(b) Find the depreciated value of the machine at the end of 5 full years.

Numerical, Graphical, and Analytic Analysis **In Exercises 21–24, (a) use a graphing utility to find the indicated partial sum S_n and complete the table, and (b) use a graphing utility to graph the first 10 terms of the sequence of partial sums.**

n	5	10	15	20	25
S_n					

21. $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^{n-1}$

22. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!}$

24. $\sum_{n=1}^{\infty} \frac{1}{n(n + 1)}$

In Exercises 25–28, find the sum of the convergent series.

25. $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

26. $\sum_{n=0}^{\infty} \frac{2^{n+2}}{3^n}$

27. $\sum_{n=1}^{\infty} [(0.6)^n + (0.8)^n]$

28. $\sum_{n=0}^{\infty} \left[\left(\frac{2}{3}\right)^n - \frac{1}{(n + 1)(n + 2)}\right]$

In Exercises 29 and 30, (a) write the repeating decimal as a geometric series and (b) write its sum as the ratio of two integers.

29. $0.\overline{09}$

30. $0.\overline{64}$

In Exercises 31–34, determine the convergence or divergence of the series.

31. $\sum_{n=0}^{\infty} (1.67)^n$

32. $\sum_{n=0}^{\infty} (0.67)^n$

33. $\sum_{n=2}^{\infty} \frac{(-1)^n n}{\ln n}$

34. $\sum_{n=0}^{\infty} \frac{2n + 1}{3n + 2}$

35. ***Distance*** A ball is dropped from a height of 8 meters. Each time it drops h meters, it rebounds $0.7h$ meters. Find the total distance traveled by the ball.

36. ***Salary*** You accept a job that pays a salary of \$42,000 the first year. During the next 39 years, you will receive a 5.5% raise each year. What would be your total compensation over the 40-year period?

37. ***Compound Interest*** A deposit of \$300 is made at the end of each month for 2 years in an account that pays 6% interest, compounded continuously. Determine the balance in the account at the end of 2 years.

38. ***Compound Interest*** A deposit of \$125 is made at the end of each month for 10 years in an account that pays 3.5% interest, compounded monthly. Determine the balance in the account at the end of 10 years.

In Exercises 39–42, determine the convergence or divergence of the series.

39. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^4}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}}$

41. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n}\right)$

42. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{2^n}\right)$

In Exercises 43–48, determine the convergence or divergence of the series.

43. $\displaystyle\sum_{n=1}^{\infty} \frac{6}{5n-1}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+3n}}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2n}}$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)}$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$

In Exercises 49–54, determine the convergence or divergence of the series.

49. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$

50. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{n^2+1}$

51. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2-3}$

52. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$

53. $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^n n}{n-3}$

54. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \ln n^3}{n}$

In Exercises 55–60, determine the convergence or divergence of the series.

55. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n-1}{2n+5}\right)^n$

56. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n}{7n-1}\right)^n$

57. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

58. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^n}$

59. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

60. $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

Numerical, Graphical, and Analytic Analysis **In Exercises 61 and 62, (a) verify that the series converges, (b) use a graphing utility to find the indicated partial sum S_n and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums, and (d) use the table to estimate the sum of the series.**

n	5	10	15	20	25
S_n					

61. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{5}\right)^n$

62. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^3+5}$

63. ***Writing*** Use a graphing utility to complete the table for (a) $p = 2$ and (b) $p = 5$. Write a short paragraph describing and comparing the entries in the tables.

N	5	10	20	30	40
$\displaystyle\sum_{n=1}^{N} \frac{1}{n^p}$					
$\displaystyle\int_N^{\infty} \frac{1}{x^p}\,dx$					

64. ***Writing*** You are told that the terms of a positive series appear to approach zero very slowly as n approaches infinity. (In fact, $a_{75} = 0.7$.) If you are given no other information, can you conclude that the series diverges? Support your answer with an example.

In Exercises 65 and 66, find the third-degree Taylor polynomial centered at c.

65. $f(x) = e^{-3x}, \quad c = 0$

66. $f(x) = \tan x, \quad c = -\dfrac{\pi}{4}$

In Exercises 67–70, use a Taylor polynomial to approximate the function with an error of less than 0.001.

67. $\sin 95^\circ$

68. $\cos(0.75)$

69. $\ln(1.75)$

70. $e^{-0.25}$

71. A Taylor polynomial centered at 0 will be used to approximate the cosine function. Find the degree of the polynomial required to obtain the desired accuracy over each interval.

	Maximum Error	*Interval*
(a)	0.001	$[-0.5, 0.5]$
(b)	0.001	$[-1, 1]$
(c)	0.0001	$[-0.5, 0.5]$
(d)	0.0001	$[-2, 2]$

72. Use a graphing utility to graph the cosine function and the Taylor polynomials in Exercise 71.

In Exercises 73–78, find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

73. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{10}\right)^n$

74. $\displaystyle\sum_{n=0}^{\infty}(2x)^n$

75. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{(n+1)^2}$

76. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n(x-2)^n}{n}$

77. $\displaystyle\sum_{n=0}^{\infty}n!(x-2)^n$

78. $\displaystyle\sum_{n=0}^{\infty}\frac{(x-2)^n}{2^n}$

In Exercises 79 and 80, show that the function represented by the power series is a solution of the differential equation.

79. $\displaystyle y=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{4^n(n!)^2}$

$x^2y''+xy'+x^2y=0$

80. $\displaystyle y=\sum_{n=0}^{\infty}\frac{(-3)^n x^{2n}}{2^n n!}$

$y''+3xy'+3y=0$

In Exercises 81 and 82, find a geometric power series centered at 0 for the function.

81. $g(x)=\dfrac{2}{3-x}$

82. $h(x)=\dfrac{3}{2+x}$

83. Find a power series for the derivative of the function in Exercise 81.

84. Find a power series for the integral of the function in Exercise 82.

In Exercises 85 and 86, find a function represented by the series and give the domain of the function.

85. $1+\dfrac{2}{3}x+\dfrac{4}{9}x^2+\dfrac{8}{27}x^3+\cdots$

86. $8-2(x-3)+\dfrac{1}{2}(x-3)^2-\dfrac{1}{8}(x-3)^3+\cdots$

In Exercises 87–94, find a power series for the function centered at *c*.

87. $f(x)=\sin x,\quad c=\dfrac{3\pi}{4}$

88. $f(x)=\cos x,\quad c=-\dfrac{\pi}{4}$

89. $f(x)=3^x,\quad c=0$

90. $f(x)=\csc x,\quad c=\dfrac{\pi}{2}$

(first three terms)

91. $f(x)=\dfrac{1}{x},\quad c=-1$

92. $f(x)=\sqrt{x},\quad c=4$

93. $g(x)=\sqrt[5]{1+x},\quad c=0$

94. $h(x)=\dfrac{1}{(1+x)^3},\quad c=0$

In Exercises 95–100, find the sum of the convergent series by using a well-known function. Identify the function and explain how you obtained the sum.

95. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{4^n n}$

96. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5^n n}$

97. $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2^n n!}$

98. $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{3^n n!}$

99. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{3^{2n}(2n)!}$

100. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{2n+1}(2n+1)!}$

101. ***Writing*** One of the series in Exercises 45 and 57 converges to its sum at a much lower rate than the other series. Which is it? Explain why this series converges so slowly. Use a graphing utility to illustrate the rate of convergence.

102. Use the binomial series to find the Maclaurin series for

$$f(x)=\frac{1}{\sqrt{1+x^3}}.$$

103. ***Forming Maclaurin Series*** Determine the first four terms of the Maclaurin series for e^{2x}

(a) by using the definition of the Maclaurin series and the formula for the coefficient of the nth term, $a_n=f^{(n)}(0)/n!$.

(b) by replacing x by $2x$ in the series for e^x.

(c) by multiplying the series for e^x by itself, because $e^{2x}=e^x\cdot e^x$.

104. ***Forming Maclaurin Series*** Follow the pattern of Exercise 103 to find the first four terms of the series for $\sin 2x$. (*Hint:* $\sin 2x=2\sin x\cos x$.)

In Exercises 105–108, find the series representation of the function defined by the integral.

105. $\displaystyle\int_0^x\frac{\sin t}{t}\,dt$

106. $\displaystyle\int_0^x\cos\frac{\sqrt{t}}{2}\,dt$

107. $\displaystyle\int_0^x\frac{\ln(t+1)}{t}\,dt$

108. $\displaystyle\int_0^x\frac{e^t-1}{t}\,dt$

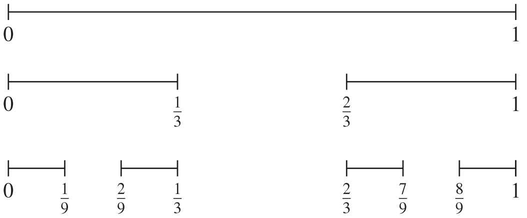
In Exercises 109 and 110, use a power series to find the limit (if it exists). Verify the result by using L'Hôpital's Rule.

109. $\displaystyle\lim_{x\to0^+}\frac{\arctan x}{\sqrt{x}}$

110. $\displaystyle\lim_{x\to0}\frac{\arcsin x}{x}$

P.S. PROBLEM SOLVING

1. The **Cantor set** (Georg Cantor, 1845–1918) is a subset of the unit interval $[0, 1]$. To construct the Cantor set, first remove the middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$ of the interval, leaving two line segments. For the second step, remove the middle third of each of the two remaining segments, leaving four line segments. Continue this procedure indefinitely, as shown in the figure. The Cantor set consists of all numbers in the unit interval $[0, 1]$ that still remain.



(a) Find the total length of all the line segments that are removed.

(b) Write down three numbers that are in the Cantor set.

(c) Let C_n denote the total length of the remaining line segments after n steps. Find $\lim_{n\to\infty} C_n$.

2. (a) Given that $\lim_{x\to\infty} a_{2n} = L$ and $\lim_{x\to\infty} a_{2n+1} = L$, show that $\{a_n\}$ is convergent and $\lim_{x\to\infty} a_n = L$.

(b) Let $a_1 = 1$ and $a_{n+1} = 1 + \dfrac{1}{1 + a_n}$. Write out the first eight terms of $\{a_n\}$. Use part (a) to show that $\lim_{x\to\infty} a_n = \sqrt{2}$. This gives the **continued fraction expansion**

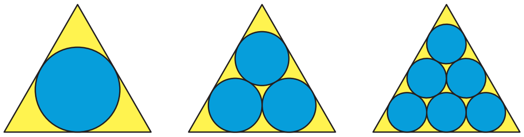
$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}.$$

3. It can be shown that

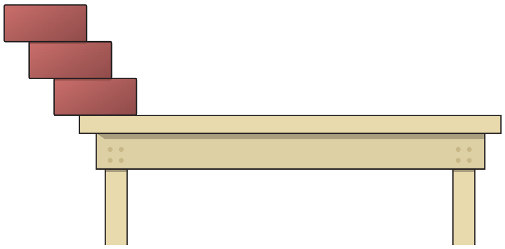
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \text{ [see Example 3(b), Section 9.3].}$$

Use this fact to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

4. Let T be an equilateral triangle with sides of length 1. Let a_n be the number of circles that can be packed tightly in n rows inside the triangle. For example, $a_1 = 1$, $a_2 = 3$, and $a_3 = 6$, as shown in the figure. Let A_n be the combined area of the a_n circles. Find $\lim_{n\to\infty} A_n$.



5. Identical blocks of unit length are stacked on top of each other at the edge of a table. The center of gravity of the top block must lie over the block below it, the center of gravity of the top two blocks must lie over the block below them, and so on (see figure).



(a) If there are three blocks, show that it is possible to stack them so that the left edge of the top block extends $\frac{11}{12}$ unit beyond the edge of the table.

(b) Is it possible to stack the blocks so that the right edge of the top block extends beyond the edge of the table?

(c) How far beyond the table can the blocks be stacked?

6. (a) Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n = 1 + 2x + 3x^2 + x^3 + 2x^4 + 3x^5 + x^6 + \cdots$$

in which the coefficients $a_n = 1, 2, 3, 1, 2, 3, 1, \ldots$ are periodic of period $p = 3$. Find the radius of convergence and the sum of this power series.

(b) Consider a power series

$$\sum_{n=0}^{\infty} a_n x^n$$

in which the coefficients are periodic, $(a_{n+p} = a_p)$ and $a_n > 0$. Find the radius of convergence and the sum of this power series.

7. For what values of the positive constants a and b does the following series converge absolutely? For what values does it converge conditionally?

$$a - \frac{b}{2} + \frac{a}{3} - \frac{b}{4} + \frac{a}{5} - \frac{b}{6} + \frac{a}{7} - \frac{b}{8} + \cdots$$

8. (a) Find a power series for the function

$$f(x) = xe^x$$

centered at 0. Use this representation to find the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n!(n+2)}.$$

(b) Differentiate the power series for $f(x) = xe^x$. Use the result to find the sum of the infinite series

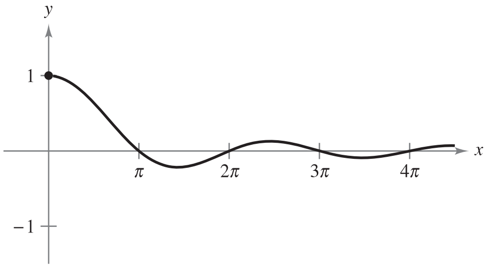
$$\sum_{n=0}^{\infty} \frac{n+1}{n!}.$$

9. Find $f^{(12)}(0)$ if $f(x) = e^{x^2}$. (*Hint:* Do not calculate 12 derivatives.)

10. The graph of the function

$$f(x) = \begin{cases} 1, & x = 0 \\ \dfrac{\sin x}{x}, & x > 0 \end{cases}$$

is shown below. Use the Alternating Series Test to show that the improper integral $\int_1^\infty f(x)\,dx$ converges.



11. (a) Prove that $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^p}\,dx$ converges if and only if $p > 1$.

(b) Determine the convergence or divergence of the series $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n \ln(n^2)}$.

12. (a) Consider the following sequence of numbers defined recursively.

$$a_1 = 3$$
$$a_2 = \sqrt{3}$$
$$a_3 = \sqrt{3 + \sqrt{3}}$$
$$\vdots$$
$$a_{n+1} = \sqrt{3 + a_n}$$

Write the decimal approximations for the first six terms of this sequence. Prove that the sequence converges, and find its limit.

(b) Consider the following sequence defined recursively by $a_1 = \sqrt{a}$ and $a_{n+1} = \sqrt{a + a_n}$, where $a > 2$.

$$\sqrt{a},\ \sqrt{a + \sqrt{a}},\ \sqrt{a + \sqrt{a + \sqrt{a}}}, \ldots$$

Prove that this sequence converges, and find its limit.

13. Let $\{a_n\}$ be a sequence of positive numbers satisfying $\displaystyle\lim_{n\to\infty} (a_n)^{1/n} = L < \frac{1}{r}$, $r > 0$. Prove that the series $\displaystyle\sum_{n=1}^{\infty} a_n r^n$ converges.

14. Consider the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}}$.

(a) Find the first five terms of the sequence of partial sums.

(b) Show that the Ratio Test is inconclusive for this series.

(c) Use the Root Test to test for the convergence or divergence of this series.

15. Derive each identity using the appropriate geometric series.

(a) $\dfrac{1}{0.99} = 1.01010101\ldots$ (b) $\dfrac{1}{0.98} = 1.0204081632\ldots$

16. Consider an idealized population with the characteristic that each member of the population produces one offspring at the end of every time period. Each member has a life span of three time periods and the population begins with 10 newborn members. The following table shows the population during the first five time periods.

	Time Period				
Age Bracket	**1**	**2**	**3**	**4**	**5**
0–1	10	10	20	40	70
1–2		10	10	20	40
2–3			10	10	20
Total	10	20	40	70	130

The sequence for the total population has the property that

$$S_n = S_{n-1} + S_{n-2} + S_{n-3}, \qquad n > 3.$$

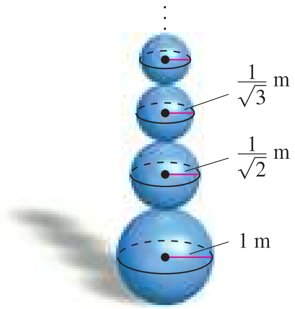
Find the total population during each of the next five time periods.

17. Imagine you are stacking an infinite number of spheres of decreasing radii on top of each other, as shown in the figure. The radii of the spheres are 1 meter, $1/\sqrt{2}$ meter, $1/\sqrt{3}$ meter, etc. The spheres are made of a material that weighs 1 newton per cubic meter.

(a) How high is this infinite stack of spheres?

(b) What is the total surface area of all the spheres in the stack?

(c) Show that the weight of the stack is finite.



18. (a) Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{2n}.$$

(b) Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \left(\sin\frac{1}{2n} - \sin\frac{1}{2n+1}\right).$$

10

Conics, Parametric Equations, and Polar Coordinates

In this chapter, you will analyze and write equations of conics using their properties. You will also learn how to write and graph parametric equations and polar equations, and see how calculus can be used to study these graphs. In addition to the rectangular equations of conics, you will also study polar equations of conics.

In this chapter, you should learn the following.

- How to analyze and write equations of a parabola, an ellipse, and a hyperbola. (10.1)
- How to sketch a curve represented by parametric equations. (10.2)
- How to use a set of parametric equations to find the slope of a tangent line to a curve and the arc length of a curve. (10.3)
- How to sketch the graph of an equation in polar form, find the slope of a tangent line to a polar graph, and identify special polar graphs. (10.4)
- How to find the area of a region bounded by a polar graph and find the arc length of a polar graph. (10.5)
- How to analyze and write a polar equation of a conic. (10.6)

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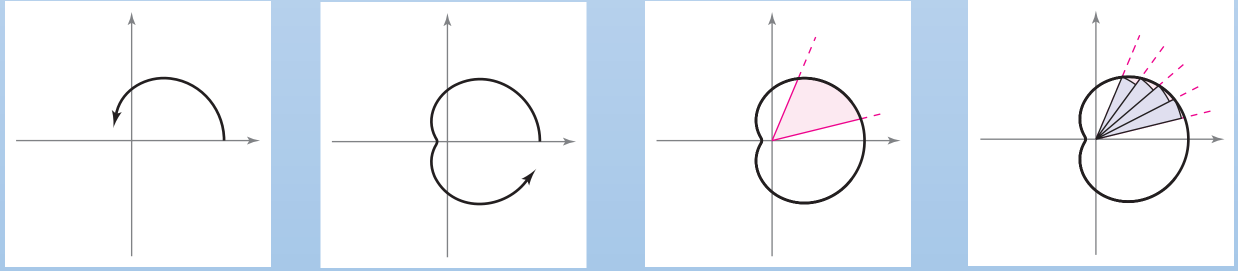
The path of a baseball hit at a particular height at an angle with the horizontal can be modeled using parametric equations. How can a set of parametric equations be used to find the minimum angle at which the ball must leave the bat in order for the hit to be a home run? (See Section 10.2, Exercise 75.)



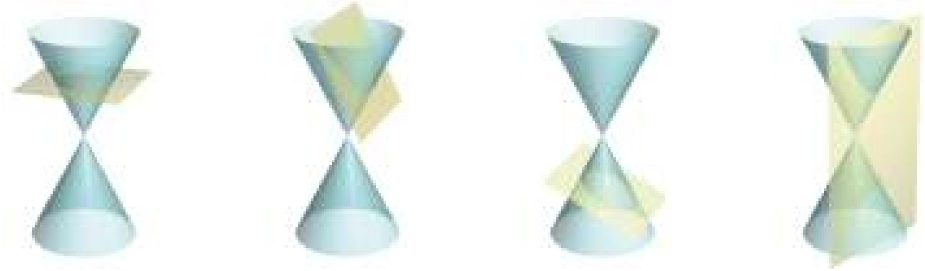
In the polar coordinate system, graphing an equation involves tracing a curve about a fixed point called the pole. Consider a region bounded by a curve and by the rays that contain the endpoints of an interval on the curve. You can use sectors of circles to approximate the area of such a region. In Section 10.5, you will see how the limit process can be used to find this area.

10.1 Conics and Calculus

- **Understand the definition of a conic section.**
- **Analyze and write equations of parabolas using properties of parabolas.**
- **Analyze and write equations of ellipses using properties of ellipses.**
- **Analyze and write equations of hyperbolas using properties of hyperbolas.**

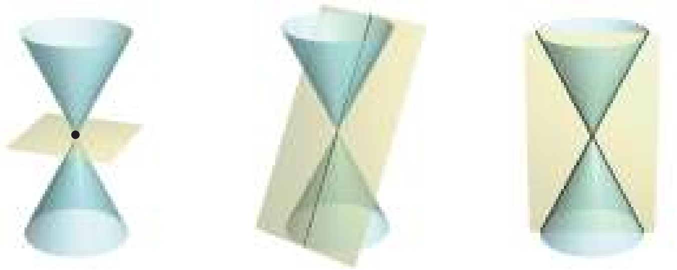
Conic Sections

Each **conic section** (or simply **conic**) can be described as the intersection of a plane and a double-napped cone. Notice in Figure 10.1 that for the four basic conics, the intersecting plane does not pass through the vertex of the cone. When the plane passes through the vertex, the resulting figure is a **degenerate conic,** as shown in Figure 10.2.



Circle Parabola Ellipse Hyperbola

Conic sections
Figure 10.1



Point Line Two intersecting lines

Degenerate conics
Figure 10.2

Bettmann/Corbis

HYPATIA (370–415 A.D.)

The Greeks discovered conic sections sometime between 600 and 300 B.C. By the beginning of the Alexandrian period, enough was known about conics for Apollonius (262–190 B.C.) to produce an eight-volume work on the subject. Later, toward the end of the Alexandrian period, Hypatia wrote a textbook entitled *On the Conics of Apollonius.* Her death marked the end of major mathematical discoveries in Europe for several hundred years.

The early Greeks were largely concerned with the geometric properties of conics. It was not until 1900 years later, in the early seventeenth century, that the broader applicability of conics became apparent. Conics then played a prominent role in the development of calculus.

There are several ways to study conics. You could begin as the Greeks did by defining the conics in terms of the intersections of planes and cones, or you could define them algebraically in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$ General second-degree equation

However, a third approach, in which each of the conics is defined as a **locus** (collection) of points satisfying a certain geometric property, works best. For example, a circle can be defined as the collection of all points (x, y) that are equidistant from a fixed point (h, k). This locus definition easily produces the standard equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2.$$ Standard equation of a circle

For information about rotating second-degree equations in two variables, see Appendix D.

FOR FURTHER INFORMATION To learn more about the mathematical activities of Hypatia, see the article "Hypatia and Her Mathematics" by Michael A. B. Deakin in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

Parabolas

A **parabola** is the set of all points (x, y) that are equidistant from a fixed line called the **directrix** and a fixed point called the **focus** not on the line. The midpoint between the focus and the directrix is the **vertex,** and the line passing through the focus and the vertex is the **axis** of the parabola. Note in Figure 10.3 that a parabola is symmetric with respect to its axis.

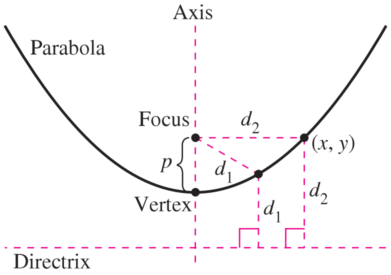


Figure 10.3

THEOREM 10.1 STANDARD EQUATION OF A PARABOLA

The **standard form** of the equation of a parabola with vertex (h, k) and directrix $y = k - p$ is

$$(x - h)^2 = 4p(y - k). \qquad \text{Vertical axis}$$

For directrix $x = h - p$, the equation is

$$(y - k)^2 = 4p(x - h). \qquad \text{Horizontal axis}$$

The focus lies on the axis p units (*directed distance*) from the vertex. The coordinates of the focus are as follows.

$(h, k + p)$ Vertical axis

$(h + p, k)$ Horizontal axis

EXAMPLE 1 Finding the Focus of a Parabola

Find the focus of the parabola given by $y = \dfrac{1}{2} - x - \dfrac{1}{2}x^2$.

Solution To find the focus, convert to standard form by completing the square.

$$\begin{aligned} y &= \tfrac{1}{2} - x - \tfrac{1}{2}x^2 && \text{Write original equation.}\\ y &= \tfrac{1}{2}(1 - 2x - x^2) && \text{Factor out } \tfrac{1}{2}.\\ 2y &= 1 - 2x - x^2 && \text{Multiply each side by 2.}\\ 2y &= 1 - (x^2 + 2x) && \text{Group terms.}\\ 2y &= 2 - (x^2 + 2x + 1) && \text{Add and subtract 1 on right side.}\\ x^2 + 2x + 1 &= -2y + 2 &&\\ (x + 1)^2 &= -2(y - 1) && \text{Write in standard form.} \end{aligned}$$

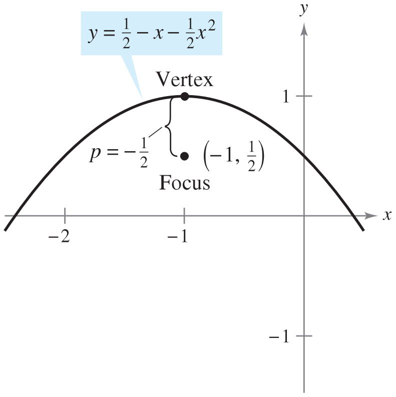
Comparing this equation with $(x - h)^2 = 4p(y - k)$, you can conclude that

$$h = -1, \qquad k = 1, \qquad \text{and} \qquad p = -\tfrac{1}{2}.$$

Because p is negative, the parabola opens downward, as shown in Figure 10.4. So, the focus of the parabola is p units from the vertex, or

$$(h, k + p) = \left(-1, \tfrac{1}{2}\right). \qquad \text{Focus}$$

■

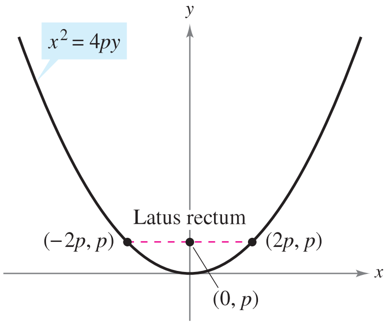


Parabola with a vertical axis, $p < 0$
Figure 10.4

A line segment that passes through the focus of a parabola and has endpoints on the parabola is called a **focal chord.** The specific focal chord perpendicular to the axis of the parabola is the **latus rectum.** The next example shows how to determine the length of the latus rectum and the length of the corresponding intercepted arc.

EXAMPLE 2 Focal Chord Length and Arc Length

Find the length of the latus rectum of the parabola given by $x^2 = 4py$. Then find the length of the parabolic arc intercepted by the latus rectum.



Length of latus rectum: $4p$
Figure 10.5

Solution Because the latus rectum passes through the focus $(0, p)$ and is perpendicular to the y-axis, the coordinates of its endpoints are $(-x, p)$ and (x, p). Substituting p for y in the equation of the parabola produces

$$x^2 = 4p(p) \quad \Rightarrow \quad x = \pm 2p.$$

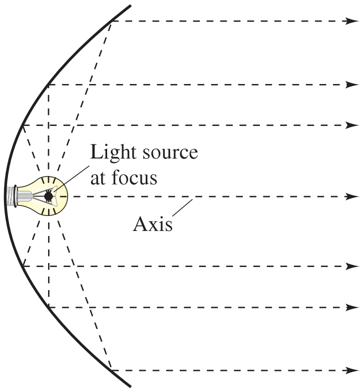
So, the endpoints of the latus rectum are $(-2p, p)$ and $(2p, p)$, and you can conclude that its length is $4p$, as shown in Figure 10.5. In contrast, the length of the intercepted arc is

$$\begin{aligned}
s &= \int_{-2p}^{2p} \sqrt{1 + (y')^2}\,dx && \text{Use arc length formula.}\\
&= 2\int_0^{2p} \sqrt{1 + \left(\frac{x}{2p}\right)^2}\,dx && y = \frac{x^2}{4p} \;\Rightarrow\; y' = \frac{x}{2p}\\
&= \frac{1}{p}\int_0^{2p} \sqrt{4p^2 + x^2}\,dx && \text{Simplify.}\\
&= \frac{1}{2p}\Big[x\sqrt{4p^2 + x^2} + 4p^2 \ln\left|x + \sqrt{4p^2 + x^2}\right|\Big]_0^{2p} && \text{Theorem 8.2}\\
&= \frac{1}{2p}\left[2p\sqrt{8p^2} + 4p^2 \ln\left(2p + \sqrt{8p^2}\right) - 4p^2 \ln(2p)\right]\\
&= 2p\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]\\
&\approx 4.59p.
\end{aligned}$$

■

One widely used property of a parabola is its reflective property. In physics, a surface is called **reflective** if the tangent line at any point on the surface makes equal angles with an incoming ray and the resulting outgoing ray. The angle corresponding to the incoming ray is the **angle of incidence,** and the angle corresponding to the outgoing ray is the **angle of reflection.** One example of a reflective surface is a flat mirror.

Another type of reflective surface is that formed by revolving a parabola about its axis. A special property of parabolic reflectors is that they allow us to direct all incoming rays parallel to the axis through the focus of the parabola—this is the principle behind the design of the parabolic mirrors used in reflecting telescopes. Conversely, all light rays emanating from the focus of a parabolic reflector used in a flashlight are parallel, as shown in Figure 10.6.



Parabolic reflector: light is reflected in parallel rays.
Figure 10.6

THEOREM 10.2 REFLECTIVE PROPERTY OF A PARABOLA

Let P be a point on a parabola. The tangent line to the parabola at the point P makes equal angles with the following two lines.

1. The line passing through P and the focus
2. The line passing through P parallel to the axis of the parabola

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The icon ↻ indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

Bettmann/Corbis

NICOLAUS COPERNICUS (1473–1543)

Copernicus began to study planetary motion when asked to revise the calendar. At that time, the exact length of the year could not be accurately predicted using the theory that Earth was the center of the universe.

Ellipses

More than a thousand years after the close of the Alexandrian period of Greek mathematics, Western civilization finally began a Renaissance of mathematical and scientific discovery. One of the principal figures in this rebirth was the Polish astronomer Nicolaus Copernicus. In his work *On the Revolutions of the Heavenly Spheres*, Copernicus claimed that all of the planets, including Earth, revolved about the sun in circular orbits. Although some of Copernicus's claims were invalid, the controversy set off by his heliocentric theory motivated astronomers to search for a mathematical model to explain the observed movements of the sun and planets. The first to find an accurate model was the German astronomer Johannes Kepler (1571–1630). Kepler discovered that the planets move about the sun in elliptical orbits, with the sun not as the center but as a focal point of the orbit.

The use of ellipses to explain the movements of the planets is only one of many practical and aesthetic uses. As with parabolas, you will begin your study of this second type of conic by defining it as a locus of points. Now, however, *two* focal points are used rather than one.

An **ellipse** is the set of all points (x, y) the sum of whose distances from two distinct fixed points called **foci** is constant. (See Figure 10.7.) The line through the foci intersects the ellipse at two points, called the **vertices.** The chord joining the vertices is the **major axis,** and its midpoint is the **center** of the ellipse. The chord perpendicular to the major axis at the center is the **minor axis** of the ellipse. (See Figure 10.8.)

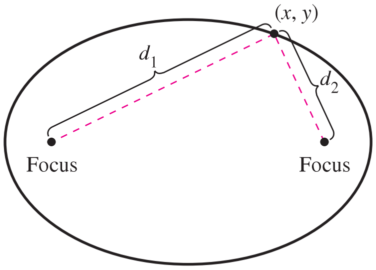


Figure 10.7

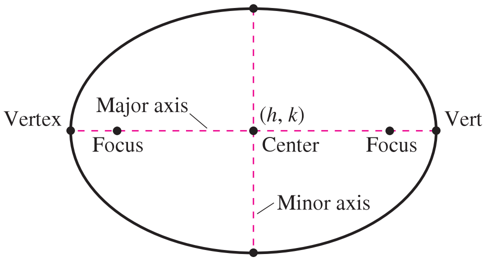


Figure 10.8

■ **FOR FURTHER INFORMATION** To learn about how an ellipse may be "exploded" into a parabola, see the article "Exploding the Ellipse" by Arnold Good in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*. ■

THEOREM 10.3 STANDARD EQUATION OF AN ELLIPSE

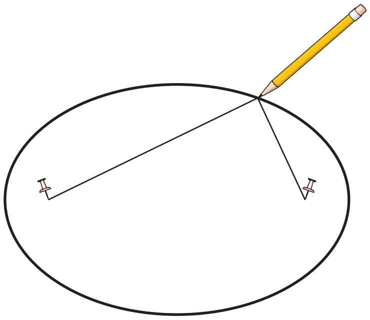
The standard form of the equation of an ellipse with center (h, k) and major and minor axes of lengths $2a$ and $2b$, where $a > b$, is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad \text{Major axis is horizontal.}$$

or

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1. \qquad \text{Major axis is vertical.}$$

The foci lie on the major axis, c units from the center, with $c^2 = a^2 - b^2$.



If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse.
Figure 10.9

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure 10.9.

EXAMPLE 3 Analyzing an Ellipse

Find the center, vertices, and foci of the ellipse given by

$$4x^2 + y^2 - 8x + 4y - 8 = 0.$$

Solution By completing the square, you can write the original equation in standard form.

$$\begin{aligned} 4x^2 + y^2 - 8x + 4y - 8 &= 0 && \text{Write original equation.} \\ 4x^2 - 8x + y^2 + 4y &= 8 \\ 4(x^2 - 2x + 1) + (y^2 + 4y + 4) &= 8 + 4 + 4 \\ 4(x-1)^2 + (y+2)^2 &= 16 \\ \frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} &= 1 && \text{Write in standard form.} \end{aligned}$$

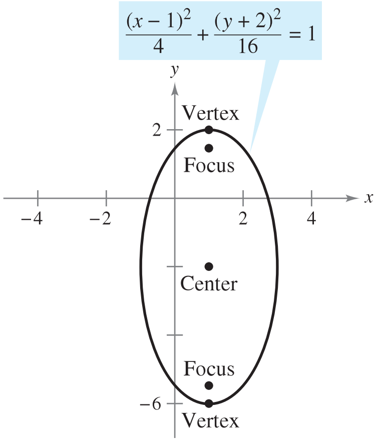
So, the major axis is parallel to the y-axis, where $h = 1$, $k = -2$, $a = 4$, $b = 2$, and $c = \sqrt{16 - 4} = 2\sqrt{3}$. So, you obtain the following.

Center: $(1, -2)$ (h, k)

Vertices: $(1, -6)$ and $(1, 2)$ $(h, k \pm a)$

Foci: $\left(1, -2 - 2\sqrt{3}\right)$ and $\left(1, -2 + 2\sqrt{3}\right)$ $(h, k \pm c)$

The graph of the ellipse is shown in Figure 10.10. ■



Ellipse with a vertical major axis
Figure 10.10

NOTE If the constant term $F = -8$ in the equation in Example 3 had been greater than or equal to 8, you would have obtained one of the following degenerate cases.

1. $F = 8$, single point, $(1, -2)$: $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 0$

2. $F > 8$, no solution points: $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} < 0$ ■

EXAMPLE 4 The Orbit of the Moon

The moon orbits Earth in an elliptical path with the center of Earth at one focus, as shown in Figure 10.11. The major and minor axes of the orbit have lengths of 768,800 kilometers and 767,640 kilometers, respectively. Find the greatest and least distances (the apogee and perigee) from Earth's center to the moon's center.

Solution Begin by solving for a and b.

$$\begin{aligned} 2a &= 768{,}800 && \text{Length of major axis} \\ a &= 384{,}400 && \text{Solve for } a. \\ 2b &= 767{,}640 && \text{Length of minor axis} \\ b &= 383{,}820 && \text{Solve for } b. \end{aligned}$$

Now, using these values, you can solve for c as follows.

$$c = \sqrt{a^2 - b^2} \approx 21{,}108$$

The greatest distance between the center of Earth and the center of the moon is $a + c \approx 405{,}508$ kilometers, and the least distance is $a - c \approx 363{,}292$ kilometers. ■

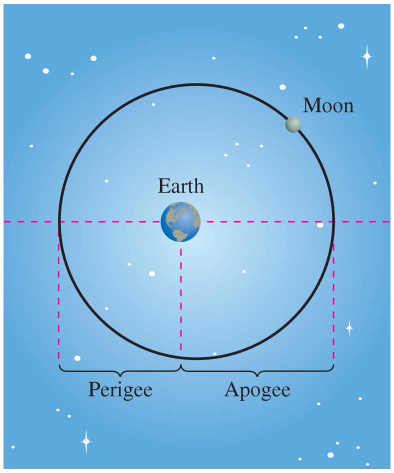


Figure 10.11

■ **FOR FURTHER INFORMATION** For more information on some uses of the reflective properties of conics, see the article "Parabolic Mirrors, Elliptic and Hyperbolic Lenses" by Mohsen Maesumi in *The American Mathematical Monthly*. Also see the article "The Geometry of Microwave Antennas" by William R. Parzynski in *Mathematics Teacher*.

Theorem 10.2 presented a reflective property of parabolas. Ellipses have a similar reflective property. You are asked to prove the following theorem in Exercise 112.

THEOREM 10.4 REFLECTIVE PROPERTY OF AN ELLIPSE

Let P be a point on an ellipse. The tangent line to the ellipse at point P makes equal angles with the lines through P and the foci.

One of the reasons that astronomers had difficulty in detecting that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to the center of the sun, making the orbits nearly circular. To measure the ovalness of an ellipse, you can use the concept of **eccentricity.**

DEFINITION OF ECCENTRICITY OF AN ELLIPSE

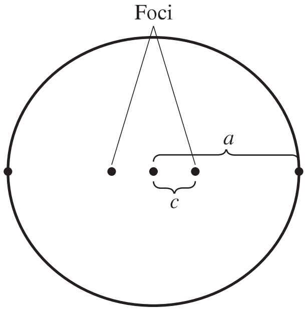
The **eccentricity** e of an ellipse is given by the ratio

$$e = \frac{c}{a}.$$

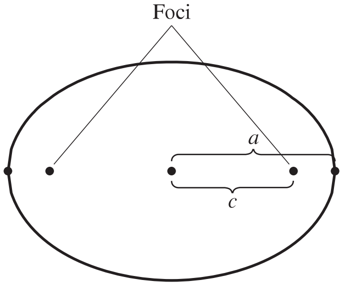
To see how this ratio is used to describe the shape of an ellipse, note that because the foci of an ellipse are located along the major axis between the vertices and the center, it follows that

$$0 < c < a.$$

For an ellipse that is nearly circular, the foci are close to the center and the ratio c/a is small, and for an elongated ellipse, the foci are close to the vertices and the ratio c/a is close to 1, as shown in Figure 10.12. Note that $0 < e < 1$ for every ellipse.



(a) $\frac{c}{a}$ is small.



(b) $\frac{c}{a}$ is close to 1.

Eccentricity is the ratio $\frac{c}{a}$.

Figure 10.12

The orbit of the moon has an eccentricity of $e = 0.0549$, and the eccentricities of the eight planetary orbits are as follows.

Mercury:	$e = 0.2056$	Jupiter:	$e = 0.0484$
Venus:	$e = 0.0068$	Saturn:	$e = 0.0542$
Earth:	$e = 0.0167$	Uranus:	$e = 0.0472$
Mars:	$e = 0.0934$	Neptune:	$e = 0.0086$

You can use integration to show that the area of an ellipse is $A = \pi ab$. For instance, the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is given by

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\int_0^{\pi/2} a^2 \cos^2\theta\,d\theta. \qquad \text{Trigonometric substitution } x = a\sin\theta.$$

However, it is not so simple to find the *circumference* of an ellipse. The next example shows how to use eccentricity to set up an "elliptic integral" for the circumference of an ellipse.

EXAMPLE 5 Finding the Circumference of an Ellipse

Show that the circumference of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ is

$$4a\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta. \qquad e = \frac{c}{a}$$

Solution Because the given ellipse is symmetric with respect to both the x-axis and the y-axis, you know that its circumference C is four times the arc length of $y = (b/a)\sqrt{a^2 - x^2}$ in the first quadrant. The function y is differentiable for all x in the interval $[0, a]$ except at $x = a$. So, the circumference is given by the improper integral

$$C = \lim_{d\to a} 4\int_0^d \sqrt{1 + (y')^2}\, dx = 4\int_0^a \sqrt{1 + (y')^2}\, dx = 4\int_0^a \sqrt{1 + \frac{b^2x^2}{a^2(a^2 - x^2)}}\, dx.$$

Using the trigonometric substitution $x = a \sin \theta$, you obtain

$$\begin{aligned} C &= 4\int_0^{\pi/2} \sqrt{1 + \frac{b^2 \sin^2 \theta}{a^2 \cos^2 \theta}}(a \cos \theta)\, d\theta \\ &= 4\int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}\, d\theta \\ &= 4\int_0^{\pi/2} \sqrt{a^2(1 - \sin^2 \theta) + b^2 \sin^2 \theta}\, d\theta \\ &= 4\int_0^{\pi/2} \sqrt{a^2 - (a^2 - b^2)\sin^2 \theta}\, d\theta. \end{aligned}$$

Because $e^2 = c^2/a^2 = (a^2 - b^2)/a^2$, you can rewrite this integral as

$$C = 4a\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta.$$

■

AREA AND CIRCUMFERENCE OF AN ELLIPSE

In his work with elliptic orbits in the early 1600's, Johannes Kepler successfully developed a formula for the area of an ellipse, $A = \pi ab$. He was less successful in developing a formula for the circumference of an ellipse, however; the best he could do was to give the approximate formula $C = \pi(a + b)$.

A great deal of time has been devoted to the study of elliptic integrals. Such integrals generally do not have elementary antiderivatives. To find the circumference of an ellipse, you must usually resort to an approximation technique.

EXAMPLE 6 Approximating the Value of an Elliptic Integral

Use the elliptic integral in Example 5 to approximate the circumference of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

Solution Because $e^2 = c^2/a^2 = (a^2 - b^2)/a^2 = 9/25$, you have

$$C = (4)(5)\int_0^{\pi/2} \sqrt{1 - \frac{9 \sin^2 \theta}{25}}\, d\theta.$$

Applying Simpson's Rule with $n = 4$ produces

$$\begin{aligned} C &\approx 20\left(\frac{\pi}{6}\right)\left(\frac{1}{4}\right)[1 + 4(0.9733) + 2(0.9055) + 4(0.8323) + 0.8] \\ &\approx 28.36. \end{aligned}$$

So, the ellipse has a circumference of about 28.36 units, as shown in Figure 10.13.

■

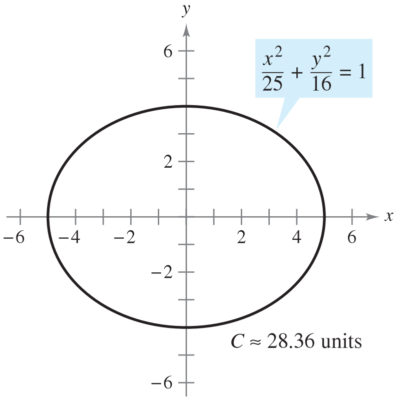


Figure 10.13

Hyperbolas

The definition of a hyperbola is similar to that of an ellipse. For an ellipse, the *sum* of the distances between the foci and a point on the ellipse is fixed, whereas for a hyperbola, the absolute value of the *difference* between these distances is fixed.

A **hyperbola** is the set of all points (x, y) for which the absolute value of the difference between the distances from two distinct fixed points called **foci** is constant. (See Figure 10.14.) The line through the two foci intersects a hyperbola at two points called the **vertices.** The line segment connecting the vertices is the **transverse axis,** and the midpoint of the transverse axis is the **center** of the hyperbola. One distinguishing feature of a hyperbola is that its graph has two separate *branches*.

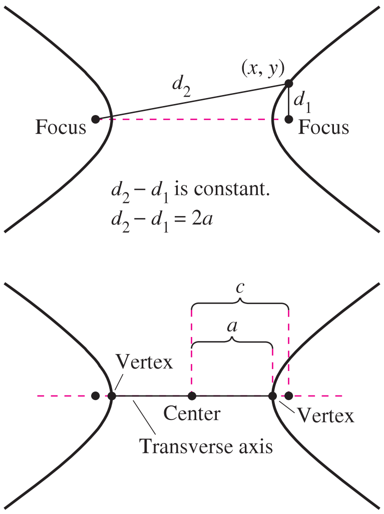


Figure 10.14

THEOREM 10.5 STANDARD EQUATION OF A HYPERBOLA

The standard form of the equation of a hyperbola with center at (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$

or

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \qquad \text{Transverse axis is vertical.}$$

The vertices are a units from the center, and the foci are c units from the center, where, $c^2 = a^2 + b^2$.

NOTE The constants a, b, and c do not have the same relationship for hyperbolas as they do for ellipses. For hyperbolas, $c^2 = a^2 + b^2$, but for ellipses, $c^2 = a^2 - b^2$. ■

An important aid in sketching the graph of a hyperbola is the determination of its **asymptotes,** as shown in Figure 10.15. Each hyperbola has two asymptotes that intersect at the center of the hyperbola. The asymptotes pass through the vertices of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k). The line segment of length $2b$ joining $(h, k + b)$ and $(h, k - b)$ is referred to as the **conjugate axis** of the hyperbola.

THEOREM 10.6 ASYMPTOTES OF A HYPERBOLA

For a *horizontal* transverse axis, the equations of the asymptotes are

$$y = k + \frac{b}{a}(x-h) \qquad \text{and} \qquad y = k - \frac{b}{a}(x-h).$$

For a *vertical* transverse axis, the equations of the asymptotes are

$$y = k + \frac{a}{b}(x-h) \qquad \text{and} \qquad y = k - \frac{a}{b}(x-h).$$

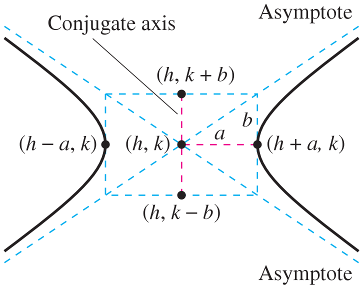


Figure 10.15

In Figure 10.15 you can see that the asymptotes coincide with the diagonals of the rectangle with dimensions $2a$ and $2b$, centered at (h, k). This provides you with a quick means of sketching the asymptotes, which in turn aids in sketching the hyperbola.

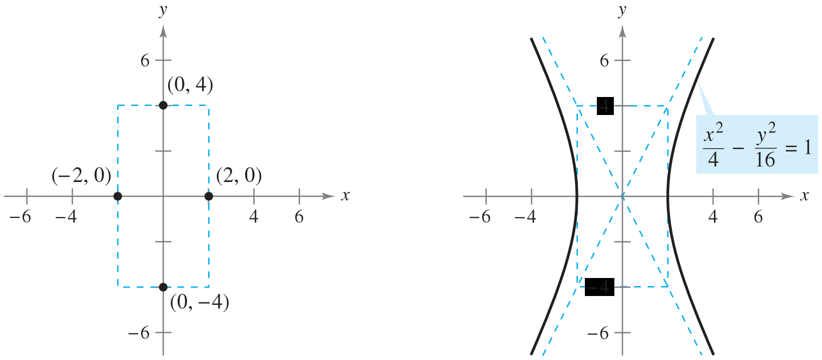
EXAMPLE 7 Using Asymptotes to Sketch a Hyperbola

Sketch the graph of the hyperbola whose equation is $4x^2 - y^2 = 16$.

Solution Begin by rewriting the equation in standard form.

$$\frac{x^2}{4} - \frac{y^2}{16} = 1$$

The transverse axis is horizontal and the vertices occur at $(-2, 0)$ and $(2, 0)$. The ends of the conjugate axis occur at $(0, -4)$ and $(0, 4)$. Using these four points, you can sketch the rectangle shown in Figure 10.16(a). By drawing the asymptotes through the corners of this rectangle, you can complete the sketch as shown in Figure 10.16(b).



(a) **(b)**

Figure 10.16

■

TECHNOLOGY You can use a graphing utility to verify the graph obtained in Example 7 by solving the original equation for y and graphing the following equations.

$$y_1 = \sqrt{4x^2 - 16}$$
$$y_2 = -\sqrt{4x^2 - 16}$$

DEFINITION OF ECCENTRICITY OF A HYPERBOLA

The **eccentricity** e of a hyperbola is given by the ratio

$$e = \frac{c}{a}.$$

As with an ellipse, the **eccentricity** of a hyperbola is $e = c/a$. Because $c > a$ for hyperbolas, it follows that $e > 1$ for hyperbolas. If the eccentricity is large, the branches of the hyperbola are nearly flat. If the eccentricity is close to 1, the branches of the hyperbola are more pointed, as shown in Figure 10.17.

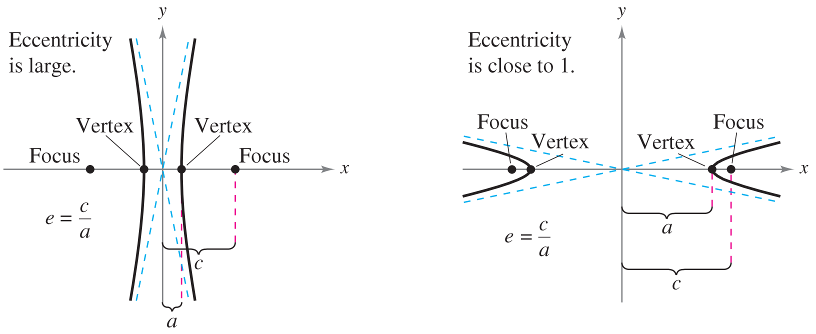
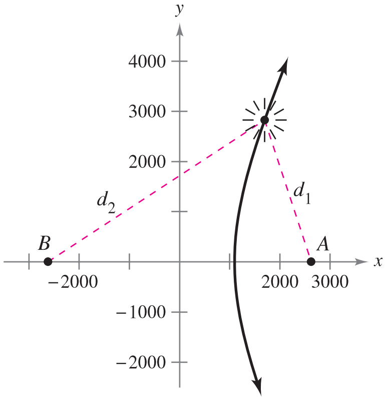


Figure 10.17

The following application was developed during World War II. It shows how the properties of hyperbolas can be used in radar and other detection systems.

EXAMPLE 8 A Hyperbolic Detection System

Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds before microphone B. Where was the explosion?



$2c = 5280$
$d_2 - d_1 = 2a = 2200$
Figure 10.18

Solution Assuming that sound travels at 1100 feet per second, you know that the explosion took place 2200 feet farther from B than from A, as shown in Figure 10.18. The locus of all points that are 2200 feet closer to A than to B is one branch of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$, where

$$c = \frac{1 \text{ mile}}{2} = \frac{5280 \text{ ft}}{2} = 2640 \text{ feet}$$

and

$$a = \frac{2200 \text{ ft}}{2} = 1100 \text{ feet.}$$

Because $c^2 = a^2 + b^2$, it follows that

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= 5{,}759{,}600 \end{aligned}$$

and you can conclude that the explosion occurred somewhere on the right branch of the hyperbola given by

$$\frac{x^2}{1{,}210{,}000} - \frac{y^2}{5{,}759{,}600} = 1.$$

■

Mary Evans Picture Library

CAROLINE HERSCHEL (1750–1848)

The first woman to be credited with detecting a new comet was the English astronomer Caroline Herschel. During her life, Caroline Herschel discovered a total of eight new comets.

In Example 8, you were able to determine only the hyperbola on which the explosion occurred, but not the exact location of the explosion. If, however, you had received the sound at a third position C, then two other hyperbolas would be determined. The exact location of the explosion would be the point at which these three hyperbolas intersect.

Another interesting application of conics involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. The center of the sun is a focus of each orbit, and each orbit has a vertex at the point at which the comet is closest to the sun. Undoubtedly, many comets with parabolic or hyperbolic orbits have not been identified—such comets pass through our solar system only once. Only comets with elliptical orbits such as Halley's comet remain in our solar system.

The type of orbit for a comet can be determined as follows.

1. Ellipse: $v < \sqrt{2GM/p}$
2. Parabola: $v = \sqrt{2GM/p}$
3. Hyperbola: $v > \sqrt{2GM/p}$

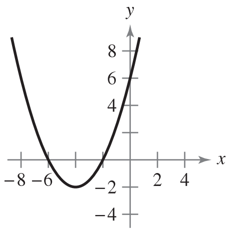
In these three formulas, p is the distance between one vertex and one focus of the comet's orbit (in meters), v is the velocity of the comet at the vertex (in meters per second), $M \approx 1.989 \times 10^{30}$ kilograms is the mass of the sun, and $G \approx 6.67 \times 10^{-8}$ cubic meters per kilogram-second squared is the gravitational constant.

10.1 Exercises

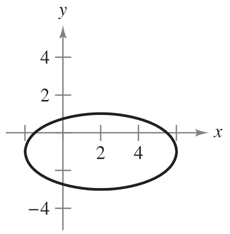
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

(a)



(b)



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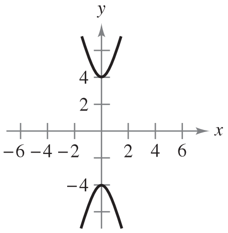

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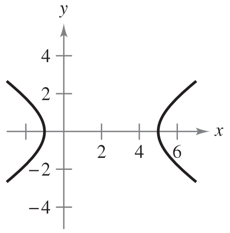

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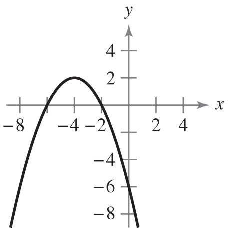

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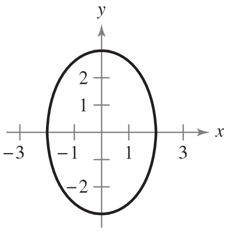
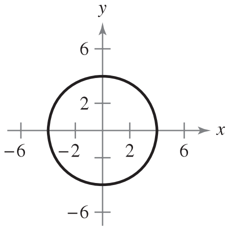
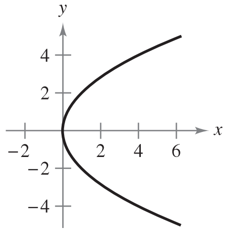


(g)



(h)



1. $y^2 = 4x$
2. $(x + 4)^2 = 2(y + 2)$
3. $(x + 4)^2 = -2(y - 2)$
4. $\dfrac{(x - 2)^2}{16} + \dfrac{(y + 1)^2}{4} = 1$
5. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$
6. $\dfrac{x^2}{16} + \dfrac{y^2}{16} = 1$
7. $\dfrac{y^2}{16} - \dfrac{x^2}{1} = 1$
8. $\dfrac{(x - 2)^2}{9} - \dfrac{y^2}{4} = 1$

In Exercises 9–16, find the vertex, focus, and directrix of the parabola, and sketch its graph.

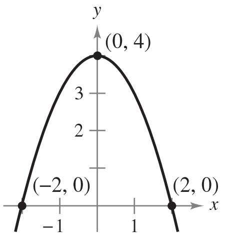
9. $y^2 = -8x$
10. $x^2 + 6y = 0$
11. $(x + 5) + (y - 3)^2 = 0$
12. $(x - 6)^2 + 8(y + 7) = 0$
13. $y^2 - 4y - 4x = 0$
14. $y^2 + 6y + 8x + 25 = 0$
15. $x^2 + 4x + 4y - 4 = 0$
16. $y^2 + 4y + 8x - 12 = 0$

In Exercises 17–20, find the vertex, focus, and directrix of the parabola. Then use a graphing utility to graph the parabola.

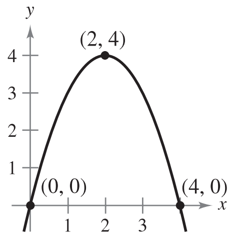
17. $y^2 + x + y = 0$
18. $y = -\frac{1}{6}(x^2 - 8x + 6)$
19. $y^2 - 4x - 4 = 0$
20. $x^2 - 2x + 8y + 9 = 0$

In Exercises 21–28, find an equation of the parabola.

21. Vertex: $(5, 4)$
    Focus: $(3, 4)$
22. Vertex: $(-2, 1)$
    Focus: $(-2, -1)$
23. Vertex: $(0, 5)$
    Directrix: $y = -3$
24. Focus: $(2, 2)$
    Directrix: $x = -2$
25.



26.



27. Axis is parallel to y-axis; graph passes through $(0, 3)$, $(3, 4)$, and $(4, 11)$.
28. Directrix: $y = -2$; endpoints of latus rectum are $(0, 2)$ and $(8, 2)$.

In Exercises 29–34, find the center, foci, vertices, and eccentricity of the ellipse, and sketch its graph.

29. $16x^2 + y^2 = 16$
30. $3x^2 + 7y^2 = 63$
31. $\dfrac{(x - 3)^2}{16} + \dfrac{(y - 1)^2}{25} = 1$
32. $(x + 4)^2 + \dfrac{(y + 6)^2}{1/4} = 1$
33. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$
34. $16x^2 + 25y^2 - 64x + 150y + 279 = 0$

In Exercises 35–38, find the center, foci, and vertices of the ellipse. Use a graphing utility to graph the ellipse.

35. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$
36. $36x^2 + 9y^2 + 48x - 36y + 43 = 0$
37. $x^2 + 2y^2 - 3x + 4y + 0.25 = 0$
38. $2x^2 + y^2 + 4.8x - 6.4y + 3.12 = 0$

In Exercises 39–44, find an equation of the ellipse.

39. Center: $(0, 0)$
    Focus: $(5, 0)$
    Vertex: $(6, 0)$
40. Vertices: $(0, 3)$, $(8, 3)$
    Eccentricity: $\frac{3}{4}$
41. Vertices: $(3, 1)$, $(3, 9)$
    Minor axis length: 6
42. Foci: $(0, \pm 9)$
    Major axis length: 22
43. Center: $(0, 0)$
    Major axis: horizontal
    Points on the ellipse:
    $(3, 1)$, $(4, 0)$
44. Center: $(1, 2)$
    Major axis: vertical
    Points on the ellipse:
    $(1, 6)$, $(3, 2)$

In Exercises 45–52, find the center, foci, and vertices of the hyperbola, and sketch its graph using asymptotes as an aid.

45. $y^2 - \frac{x^2}{9} = 1$

46. $\frac{x^2}{25} - \frac{y^2}{16} = 1$

47. $\frac{(x-1)^2}{4} - \frac{(y+2)^2}{1} = 1$

48. $\frac{(y+3)^2}{225} - \frac{(x-5)^2}{64} = 1$

49. $9x^2 - y^2 - 36x - 6y + 18 = 0$

50. $y^2 - 16x^2 + 64x - 208 = 0$

51. $x^2 - 9y^2 + 2x - 54y - 80 = 0$

52. $9x^2 - 4y^2 + 54x + 8y + 78 = 0$

In Exercises 53–56, find the center, foci, and vertices of the hyperbola. Use a graphing utility to graph the hyperbola and its asymptotes.

53. $9y^2 - x^2 + 2x + 54y + 62 = 0$

54. $9x^2 - y^2 + 54x + 10y + 55 = 0$

55. $3x^2 - 2y^2 - 6x - 12y - 27 = 0$

56. $3y^2 - x^2 + 6x - 12y = 0$

In Exercises 57–64, find an equation of the hyperbola.

57. Vertices: $(\pm 1, 0)$
Asymptotes: $y = \pm 5x$

58. Vertices: $(0, \pm 4)$
Asymptotes: $y = \pm 2x$

59. Vertices: $(2, \pm 3)$
Point on graph: $(0, 5)$

60. Vertices: $(2, \pm 3)$
Foci: $(2, \pm 5)$

61. Center: $(0, 0)$
Vertex: $(0, 2)$
Focus: $(0, 4)$

62. Center: $(0, 0)$
Vertex: $(6, 0)$
Focus: $(10, 0)$

63. Vertices: $(0, 2), (6, 2)$
Asymptotes: $y = \frac{2}{3}x$
$y = 4 - \frac{2}{3}x$

64. Focus: $(20, 0)$
Asymptotes: $y = \pm\frac{3}{4}x$

In Exercises 65 and 66, find equations for (a) the tangent lines and (b) the normal lines to the hyperbola for the given value of *x*.

65. $\frac{x^2}{9} - y^2 = 1, \quad x = 6$

66. $\frac{y^2}{4} - \frac{x^2}{2} = 1, \quad x = 4$

In Exercises 67–76, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

67. $x^2 + 4y^2 - 6x + 16y + 21 = 0$

68. $4x^2 - y^2 - 4x - 3 = 0$

69. $y^2 - 8y - 8x = 0$

70. $25x^2 - 10x - 200y - 119 = 0$

71. $4x^2 + 4y^2 - 16y + 15 = 0$

72. $y^2 - 4y = x + 5$

73. $9x^2 + 9y^2 - 36x + 6y + 34 = 0$

74. $2x(x - y) = y(3 - y - 2x)$

75. $3(x - 1)^2 = 6 + 2(y + 1)^2$

76. $9(x + 3)^2 = 36 - 4(y - 2)^2$

WRITING ABOUT CONCEPTS

77. (a) Give the definition of a parabola.

(b) Give the standard forms of a parabola with vertex at (h, k).

(c) In your own words, state the reflective property of a parabola.

78. (a) Give the definition of an ellipse.

(b) Give the standard forms of an ellipse with center at (h, k).

79. (a) Give the definition of a hyperbola.

(b) Give the standard forms of a hyperbola with center at (h, k).

(c) Write equations for the asymptotes of a hyperbola.

80. Define the eccentricity of an ellipse. In your own words, describe how changes in the eccentricity affect the ellipse.

81. ***Solar Collector*** A solar collector for heating water is constructed with a sheet of stainless steel that is formed into the shape of a parabola (see figure). The water will flow through a pipe that is located at the focus of the parabola. At what distance from the vertex is the pipe?

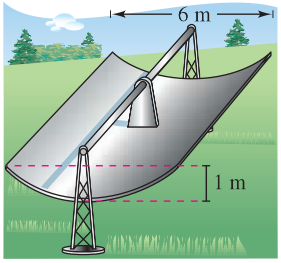


Figure for 81

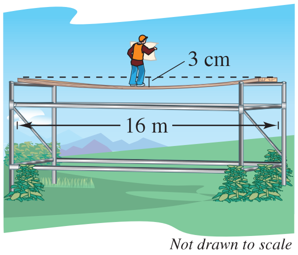


Figure for 82

82. ***Beam Deflection*** A simply supported beam that is 16 meters long has a load concentrated at the center (see figure). The deflection of the beam at its center is 3 centimeters. Assume that the shape of the deflected beam is parabolic.

(a) Find an equation of the parabola. (Assume that the origin is at the center of the beam.)

(b) How far from the center of the beam is the deflection 1 centimeter?

83. Find an equation of the tangent line to the parabola $y = ax^2$ at $x = x_0$. Prove that the x-intercept of this tangent line is $(x_0/2, 0)$.

84. (a) Prove that any two distinct tangent lines to a parabola intersect.

(b) Demonstrate the result of part (a) by finding the point of intersection of the tangent lines to the parabola $x^2 - 4x - 4y = 0$ at the points $(0, 0)$ and $(6, 3)$.

85. (a) Prove that if any two tangent lines to a parabola intersect at right angles, their point of intersection must lie on the directrix.

(b) Demonstrate the result of part (a) by proving that the tangent lines to the parabola $x^2 - 4x - 4y + 8 = 0$ at the points $(-2, 5)$ and $\left(3, \frac{5}{4}\right)$ intersect at right angles, and that the point of intersection lies on the directrix.

86. Find the point on the graph of $x^2 = 8y$ that is closest to the focus of the parabola.

87. ***Radio and Television Reception*** In mountainous areas, reception of radio and television is sometimes poor. Consider an idealized case where a hill is represented by the graph of the parabola $y = x - x^2$, a transmitter is located at the point $(-1, 1)$, and a receiver is located on the other side of the hill at the point $(x_0, 0)$. What is the closest the receiver can be to the hill while still maintaining unobstructed reception?

88. ***Modeling Data*** The table shows the average amounts of time A (in minutes) women spent watching television each day for the years 1999 through 2005. *(Source: Nielsen Media Research)*

Year	1999	2000	2001	2002	2003	2004	2005
A	280	286	291	298	305	307	317

(a) Use the regression capabilities of a graphing utility to find a model of the form $A = at^2 + bt + c$ for the data. Let t represent the year, with $t = 9$ corresponding to 1999.

(b) Use a graphing utility to plot the data and graph the model.

(c) Find dA/dt and sketch its graph for $9 \le t \le 15$. What information about the average amount of time women spent watching television is given by the graph of the derivative?

89. ***Architecture*** A church window is bounded above by a parabola and below by the arc of a circle (see figure). Find the surface area of the window.

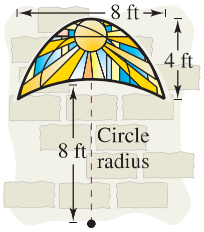


Figure for 89

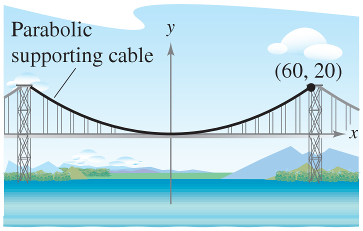


Figure for 91

90. ***Arc Length*** Find the arc length of the parabola $4x - y^2 = 0$ over the interval $0 \le y \le 4$.

91. ***Bridge Design*** A cable of a suspension bridge is suspended (in the shape of a parabola) between two towers that are 120 meters apart and 20 meters above the roadway (see figure). The cables touch the roadway midway between the towers.

(a) Find an equation for the parabolic shape of each cable.

(b) Find the length of the parabolic supporting cable.

92. ***Surface Area*** A satellite signal receiving dish is formed by revolving the parabola given by $x^2 = 20y$ about the y-axis. The radius of the dish is r feet. Verify that the surface area of the dish is given by

$$2\pi\int_0^r x\sqrt{1 + \left(\frac{x}{10}\right)^2}\,dx = \frac{\pi}{15}\left[(100 + r^2)^{3/2} - 1000\right].$$

93. ***Investigation*** Sketch the graphs of $x^2 = 4py$ for $p = \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}$, and 2 on the same coordinate axes. Discuss the change in the graphs as p increases.

94. ***Area*** Find a formula for the area of the shaded region in the figure.

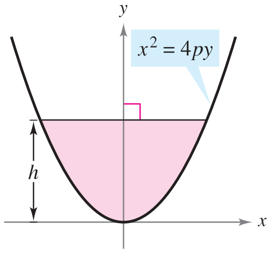


Figure for 94

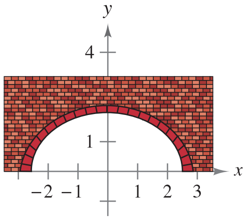


Figure for 96

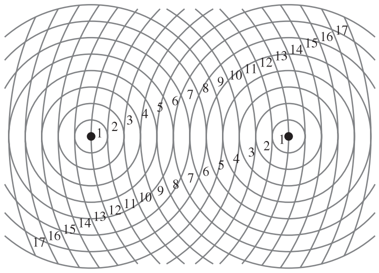
95. ***Writing*** On page 699, it was noted that an ellipse can be drawn using two thumbtacks, a string of fixed length (greater than the distance between the tacks), and a pencil. If the ends of the string are fastened at the tacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse.

(a) What is the length of the string in terms of a?

(b) Explain why the path is an ellipse.

96. ***Construction of a Semielliptical Arch*** A fireplace arch is to be constructed in the shape of a semiellipse. The opening is to have a height of 2 feet at the center and a width of 5 feet along the base (see figure). The contractor draws the outline of the ellipse by the method shown in Exercise 95. Where should the tacks be placed and what should be the length of the piece of string?

97. Sketch the ellipse that consists of all points (x, y) such that the sum of the distances between (x, y) and two fixed points is 16 units, and the foci are located at the centers of the two sets of concentric circles in the figure. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



98. ***Orbit of Earth*** Earth moves in an elliptical orbit with the sun at one of the foci. The length of half of the major axis is 149,598,000 kilometers, and the eccentricity is 0.0167. Find the minimum distance (*perihelion*) and the maximum distance (*aphelion*) of Earth from the sun.

99. ***Satellite Orbit*** The *apogee* (the point in orbit farthest from Earth) and the *perigee* (the point in orbit closest to Earth) of an elliptical orbit of an Earth satellite are given by A and P. Show that the eccentricity of the orbit is

$$e = \frac{A - P}{A + P}.$$

100. ***Explorer 18*** On November 27, 1963, the United States launched the research satellite Explorer 18. Its low and high points above the surface of Earth were 119 miles and 123,000 miles. Find the eccentricity of its elliptical orbit.

101. ***Explorer 55*** On November 20, 1975, the United States launched the research satellite Explorer 55. Its low and high points above the surface of Earth were 96 miles and 1865 miles. Find the eccentricity of its elliptical orbit.

CAPSTONE

102. Consider the equation

$$9x^2 + 4y^2 - 36x - 24y - 36 = 0.$$

(a) Classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

(b) Change the $4y^2$-term in the equation to $-4y^2$. Classify the graph of the new equation.

(c) Change the $9x^2$-term in the original equation to $4x^2$. Classify the graph of the new equation.

(d) Describe one way you could change the original equation so that its graph is a parabola.

103. ***Halley's Comet*** Probably the most famous of all comets, Halley's comet, has an elliptical orbit with the sun at one focus. Its maximum distance from the sun is approximately 35.29 AU (1 astronomical unit $\approx 92.956 \times 10^6$ miles), and its minimum distance is approximately 0.59 AU. Find the eccentricity of the orbit.

104. The equation of an ellipse with its center at the origin can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

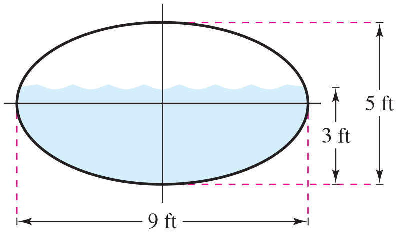
Show that as $e \to 0$, with a remaining fixed, the ellipse approaches a circle.

105. Consider a particle traveling clockwise on the elliptical path

$$\frac{x^2}{100} + \frac{y^2}{25} = 1.$$

The particle leaves the orbit at the point $(-8, 3)$ and travels in a straight line tangent to the ellipse. At what point will the particle cross the y-axis?

106. ***Volume*** The water tank on a fire truck is 16 feet long, and its cross sections are ellipses. Find the volume of water in the partially filled tank as shown in the figure.



In Exercises 107 and 108, determine the points at which *dy/dx* is zero or does not exist to locate the endpoints of the major and minor axes of the ellipse.

107. $16x^2 + 9y^2 + 96x + 36y + 36 = 0$

108. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

Area and Volume **In Exercises 109 and 110, find (a) the area of the region bounded by the ellipse, (b) the volume and surface area of the solid generated by revolving the region about its major axis (prolate spheroid), and (c) the volume and surface area of the solid generated by revolving the region about its minor axis (oblate spheroid).**

109. $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$

110. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

111. ***Arc Length*** Use the integration capabilities of a graphing utility to approximate to two-decimal-place accuracy the elliptical integral representing the circumference of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{49} = 1.$$

112. Prove Theorem 10.4 by showing that the tangent line to an ellipse at a point P makes equal angles with lines through P and the foci (see figure). [*Hint:* (1) Find the slope of the tangent line at P, (2) find the slopes of the lines through P and each focus, and (3) use the formula for the tangent of the angle between two lines.]

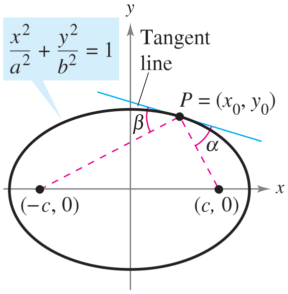


Figure for 112

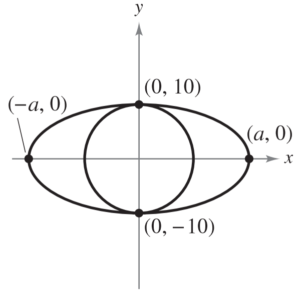


Figure for 113

113. ***Geometry*** The area of the ellipse in the figure is twice the area of the circle. What is the length of the major axis?

114. ***Conjecture***

(a) Show that the equation of an ellipse can be written as

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{a^2(1 - e^2)} = 1.$$

(b) Use a graphing utility to graph the ellipse

$$\frac{(x - 2)^2}{4} + \frac{(y - 3)^2}{4(1 - e^2)} = 1$$

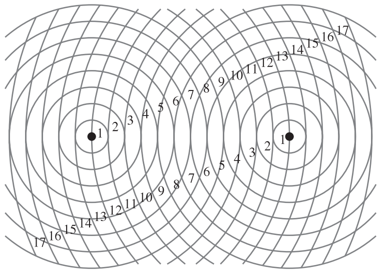
for $e = 0.95$, $e = 0.75$, $e = 0.5$, $e = 0.25$, and $e = 0$.

(c) Use the results of part (b) to make a conjecture about the change in the shape of the ellipse as e approaches 0.

115. Find an equation of the hyperbola such that for any point on the hyperbola, the difference between its distances from the points $(2, 2)$ and $(10, 2)$ is 6.

116. Find an equation of the hyperbola such that for any point on the hyperbola, the difference between its distances from the points $(-3, 0)$ and $(-3, 3)$ is 2.

117. Sketch the hyperbola that consists of all points (x, y) such that the difference of the distances between (x, y) and two fixed points is 10 units, and the foci are located at the centers of the two sets of concentric circles in the figure. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*



118. Consider a hyperbola centered at the origin with a horizontal transverse axis. Use the definition of a hyperbola to derive its standard form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

119. ***Sound Location*** A rifle positioned at point $(-c, 0)$ is fired at a target positioned at point $(c, 0)$. A person hears the sound of the rifle and the sound of the bullet hitting the target at the same time. Prove that the person is positioned on one branch of the hyperbola given by

$$\frac{x^2}{c^2 v_s^2/v_m^2} - \frac{y^2}{c^2(v_m^2 - v_s^2)/v_m^2} = 1$$

where v_m is the muzzle velocity of the rifle and v_s is the speed of sound, which is about 1100 feet per second.

120. ***Navigation*** LORAN (long distance radio navigation) for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci. Assume that two stations, 300 miles apart, are positioned on a rectangular coordinate system at $(-150, 0)$ and $(150, 0)$ and that a ship is traveling on a path with coordinates $(x, 75)$ (see figure). Find the x-coordinate of the position of the ship if the time difference between the pulses from the transmitting stations is 1000 microseconds (0.001 second).

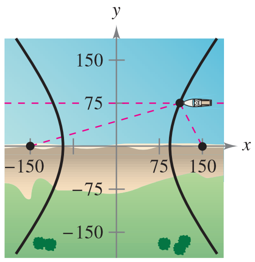


Figure for 120

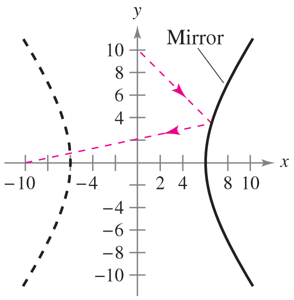


Figure for 121

121. ***Hyperbolic Mirror*** A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at the focus will be reflected to the other focus. The mirror in the figure has the equation $(x^2/36) - (y^2/64) = 1$. At which point on the mirror will light from the point $(0, 10)$ be reflected to the other focus?

122. Show that the equation of the tangent line to $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point (x_0, y_0) is $(x_0/a^2)x - (y_0/b^2)y = 1$.

123. Show that the graphs of the equations intersect at right angles:

$$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1.$$

124. Prove that the graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is one of the following (except in degenerate cases).

	Conic	*Condition*
(a)	Circle	$A = C$
(b)	Parabola	$A = 0$ or $C = 0$ (but not both)
(c)	Ellipse	$AC > 0$
(d)	Hyperbola	$AC < 0$

True or False? **In Exercises 125–130, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

125. It is possible for a parabola to intersect its directrix.

126. The point on a parabola closest to its focus is its vertex.

127. If C is the circumference of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b < a$$

then $2\pi b \le C \le 2\pi a$.

128. If $D \neq 0$ or $E \neq 0$, then the graph of $y^2 - x^2 + Dx + Ey = 0$ is a hyperbola.

129. If the asymptotes of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ intersect at right angles, then $a = b$.

130. Every tangent line to a hyperbola intersects the hyperbola only at the point of tangency.

PUTNAM EXAM CHALLENGE

131. For a point P on an ellipse, let d be the distance from the center of the ellipse to the line tangent to the ellipse at P. Prove that $(PF_1)(PF_2)d^2$ is constant as P varies on the ellipse, where PF_1 and PF_2 are the distances from P to the foci F_1 and F_2 of the ellipse.

132. Find the minimum value of $(u - v)^2 + \left(\sqrt{2 - u^2} - \dfrac{9}{v}\right)^2$ for $0 < u < \sqrt{2}$ and $v > 0$.

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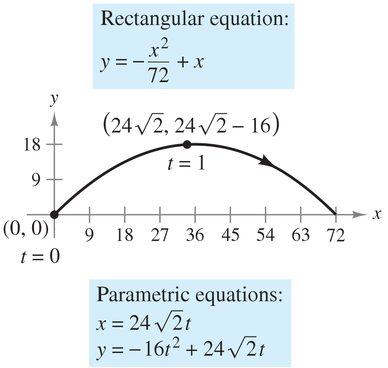
10.2 Plane Curves and Parametric Equations

- **Sketch the graph of a curve given by a set of parametric equations.**
- **Eliminate the parameter in a set of parametric equations.**
- **Find a set of parametric equations to represent a curve.**
- **Understand two classic calculus problems, the tautochrone and brachistochrone problems.**

Plane Curves and Parametric Equations



Curvilinear motion: two variables for position, one variable for time
Figure 10.19

Until now, you have been representing a graph by a single equation involving *two* variables. In this section you will study situations in which *three* variables are used to represent a curve in the plane.

Consider the path followed by an object that is propelled into the air at an angle of 45°. If the initial velocity of the object is 48 feet per second, the object travels the parabolic path given by

$$y = -\frac{x^2}{72} + x \qquad \text{Rectangular equation}$$

as shown in Figure 10.19. However, this equation does not tell the whole story. Although it does tell you *where* the object has been, it doesn't tell you *when* the object was at a given point (x, y). To determine this time, you can introduce a third variable t, called a **parameter.** By writing both x and y as functions of t, you obtain the **parametric equations**

$$x = 24\sqrt{2}\,t \qquad \text{Parametric equation for } x$$

and

$$y = -16t^2 + 24\sqrt{2}\,t. \qquad \text{Parametric equation for } y$$

From this set of equations, you can determine that at time $t = 0$, the object is at the point $(0, 0)$. Similarly, at time $t = 1$, the object is at the point $\left(24\sqrt{2}, 24\sqrt{2} - 16\right)$, and so on. (You will learn a method for determining this particular set of parametric equations—the equations of motion—later, in Section 12.3.)

For this particular motion problem, x and y are continuous functions of t, and the resulting path is called a **plane curve.**

DEFINITION OF A PLANE CURVE

If f and g are continuous functions of t on an interval I, then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are called **parametric equations** and t is called the **parameter.** The set of points (x, y) obtained as t varies over the interval I is called the **graph** of the parametric equations. Taken together, the parametric equations and the graph are called a **plane curve,** denoted by C.

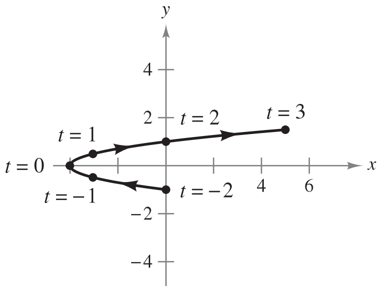
NOTE At times it is important to distinguish between a graph (the set of points) and a curve (the points together with their defining parametric equations). When it is important, we will make the distinction explicit. When it is not important, we will use C to represent the graph or the curve. ■

When sketching (by hand) a curve represented by a set of parametric equations, you can plot points in the xy-plane. Each set of coordinates (x, y) is determined from a value chosen for the parameter t. By plotting the resulting points in order of increasing values of t, the curve is traced out in a specific direction. This is called the **orientation** of the curve.

EXAMPLE 1 Sketching a Curve

Sketch the curve described by the parametric equations

$$x = t^2 - 4 \quad \text{and} \quad y = \frac{t}{2}, \quad -2 \le t \le 3.$$



Parametric equations:
$x = t^2 - 4$ and $y = \frac{t}{2}, -2 \le t \le 3$

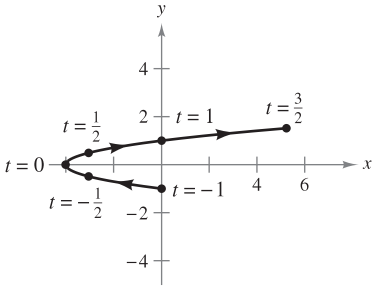
Figure 10.20

Solution For values of t on the given interval, the parametric equations yield the points (x, y) shown in the table.

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

By plotting these points in order of increasing t and using the continuity of f and g, you obtain the curve C shown in Figure 10.20. Note that the arrows on the curve indicate its orientation as t increases from -2 to 3. ■

NOTE From the Vertical Line Test, you can see that the graph shown in Figure 10.20 does not define y as a function of x. This points out one benefit of parametric equations—they can be used to represent graphs that are more general than graphs of functions. ■



Parametric equations:
$x = 4t^2 - 4$ and $y = t, -1 \le t \le \frac{3}{2}$

Figure 10.21

It often happens that two different sets of parametric equations have the same graph. For example, the set of parametric equations

$$x = 4t^2 - 4 \quad \text{and} \quad y = t, \quad -1 \le t \le \frac{3}{2}$$

has the same graph as the set given in Example 1. (See Figure 10.21.) However, comparing the values of t in Figures 10.20 and 10.21, you can see that the second graph is traced out more *rapidly* (considering t as time) than the first graph. So, in applications, different parametric representations can be used to represent various *speeds* at which objects travel along a given path.

TECHNOLOGY Most graphing utilities have a *parametric* graphing mode. If you have access to such a utility, use it to confirm the graphs shown in Figures 10.20 and 10.21. Does the curve given by the parametric equations

$$x = 4t^2 - 8t \quad \text{and} \quad y = 1 - t, \quad -\tfrac{1}{2} \le t \le 2$$

represent the same graph as that shown in Figures 10.20 and 10.21? What do you notice about the *orientation* of this curve?

Eliminating the Parameter

Finding a rectangular equation that represents the graph of a set of parametric equations is called **eliminating the parameter.** For instance, you can eliminate the parameter from the set of parametric equations in Example 1 as follows.

Parametric equations	→	Solve for t in one equation.	→	Substitute into second equation.	→	Rectangular equation
$x = t^2 - 4$ $y = t/2$		$t = 2y$		$x = (2y)^2 - 4$		$x = 4y^2 - 4$

Once you have eliminated the parameter, you can recognize that the equation $x = 4y^2 - 4$ represents a parabola with a horizontal axis and vertex at $(-4, 0)$, as shown in Figure 10.20.

The range of x and y implied by the parametric equations may be altered by the change to rectangular form. In such instances the domain of the rectangular equation must be adjusted so that its graph matches the graph of the parametric equations. Such a situation is demonstrated in the next example.

EXAMPLE 2 Adjusting the Domain After Eliminating the Parameter

Sketch the curve represented by the equations

$$x = \frac{1}{\sqrt{t+1}} \quad \text{and} \quad y = \frac{t}{t+1}, \quad t > -1$$

by eliminating the parameter and adjusting the domain of the resulting rectangular equation.

Solution Begin by solving one of the parametric equations for t. For instance, you can solve the first equation for t as follows.

$$x = \frac{1}{\sqrt{t+1}} \qquad \text{Parametric equation for } x$$

$$x^2 = \frac{1}{t+1} \qquad \text{Square each side.}$$

$$t + 1 = \frac{1}{x^2}$$

$$t = \frac{1}{x^2} - 1 = \frac{1 - x^2}{x^2} \qquad \text{Solve for } t.$$

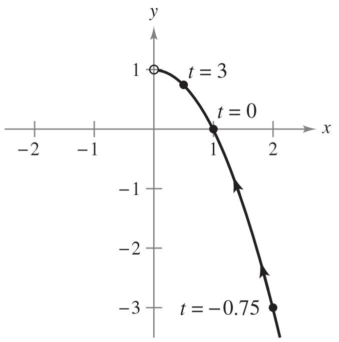
Now, substituting into the parametric equation for y produces

$$y = \frac{t}{t+1} \qquad \text{Parametric equation for } y$$

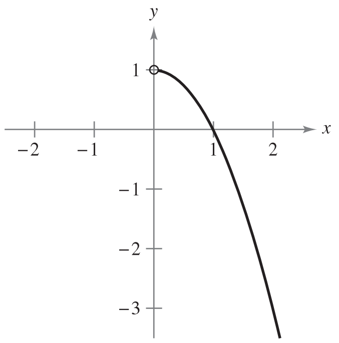
$$y = \frac{(1 - x^2)/x^2}{[(1 - x^2)/x^2] + 1} \qquad \text{Substitute } (1 - x^2)/x^2 \text{ for } t.$$

$$y = 1 - x^2. \qquad \text{Simplify.}$$

The rectangular equation, $y = 1 - x^2$, is defined for all values of x, but from the parametric equation for x you can see that the curve is defined only when $t > -1$. This implies that you should restrict the domain of x to positive values, as shown in Figure 10.22. ■



Parametric equations:
$x = \frac{1}{\sqrt{t+1}}, \; y = \frac{t}{t+1}, \; t > -1$



Rectangular equation:
$y = 1 - x^2, \; x > 0$

Figure 10.22

It is not necessary for the parameter in a set of parametric equations to represent time. The next example uses an *angle* as the parameter.

EXAMPLE 3 Using Trigonometry to Eliminate a Parameter

Sketch the curve represented by

$$x = 3\cos\theta \quad \text{and} \quad y = 4\sin\theta, \quad 0 \le \theta \le 2\pi$$

by eliminating the parameter and finding the corresponding rectangular equation.

Solution Begin by solving for cos θ and sin θ in the given equations.

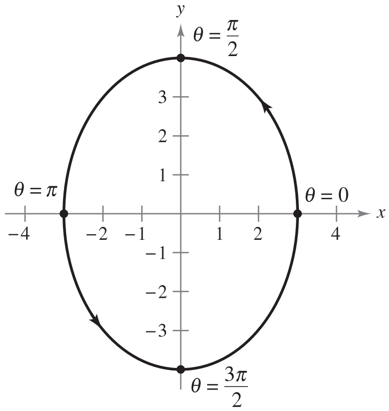
$$\cos\theta = \frac{x}{3} \quad \text{and} \quad \sin\theta = \frac{y}{4} \qquad \text{Solve for } \cos\theta \text{ and } \sin\theta.$$

Next, make use of the identity $\sin^2\theta + \cos^2\theta = 1$ to form an equation involving only x and y.

$$\cos^2\theta + \sin^2\theta = 1 \qquad \text{Trigonometric identity}$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \qquad \text{Substitute.}$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1 \qquad \text{Rectangular equation}$$



Parametric equations:
$x = 3\cos\theta,\ y = 4\sin\theta$
Rectangular equation:
$\frac{x^2}{9} + \frac{y^2}{16} = 1$

Figure 10.23

From this rectangular equation you can see that the graph is an ellipse centered at (0, 0), with vertices at (0, 4) and (0, −4) and minor axis of length $2b = 6$, as shown in Figure 10.23. Note that the ellipse is traced out *counterclockwise* as θ varies from 0 to 2π. ■

Using the technique shown in Example 3, you can conclude that the graph of the parametric equations

$$x = h + a\cos\theta \quad \text{and} \quad y = k + b\sin\theta, \quad 0 \le \theta \le 2\pi$$

is the ellipse (traced counterclockwise) given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

The graph of the parametric equations

$$x = h + a\sin\theta \quad \text{and} \quad y = k + b\cos\theta, \quad 0 \le \theta \le 2\pi$$

is also the ellipse (traced clockwise) given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Use a graphing utility in *parametric* mode to graph several ellipses.

In Examples 2 and 3, it is important to realize that eliminating the parameter is primarily an *aid to curve sketching*. If the parametric equations represent the path of a moving object, the graph alone is not sufficient to describe the object's motion. You still need the parametric equations to tell you the *position*, *direction*, and *speed* at a given time.

Finding Parametric Equations

The first three examples in this section illustrate techniques for sketching the graph represented by a set of parametric equations. You will now investigate the reverse problem. How can you determine a set of parametric equations for a given graph or a given physical description? From the discussion following Example 1, you know that such a representation is not unique. This is demonstrated further in the following example, which finds two different parametric representations for a given graph.

EXAMPLE 4 Finding Parametric Equations for a Given Graph

Find a set of parametric equations that represents the graph of $y = 1 - x^2$, using each of the following parameters.

a. $t = x$ **b.** The slope $m = \dfrac{dy}{dx}$ at the point (x, y)

Solution

a. Letting $x = t$ produces the parametric equations

$$x = t \quad \text{and} \quad y = 1 - x^2 = 1 - t^2.$$

b. To write x and y in terms of the parameter m, you can proceed as follows.

$$m = \frac{dy}{dx} = -2x \qquad \text{Differentiate } y = 1 - x^2.$$

$$x = -\frac{m}{2} \qquad \text{Solve for } x.$$

This produces a parametric equation for x. To obtain a parametric equation for y, substitute $-m/2$ for x in the original equation.

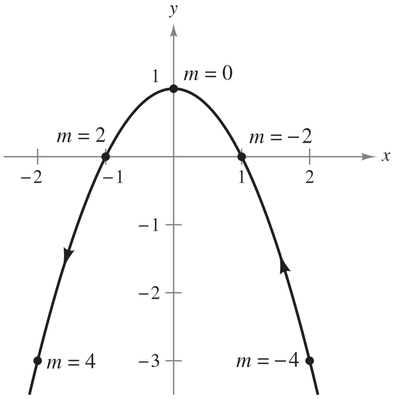
$$y = 1 - x^2 \qquad \text{Write original rectangular equation.}$$

$$y = 1 - \left(-\frac{m}{2}\right)^2 \qquad \text{Substitute } -m/2 \text{ for } x.$$

$$y = 1 - \frac{m^2}{4} \qquad \text{Simplify.}$$

So, the parametric equations are

$$x = -\frac{m}{2} \quad \text{and} \quad y = 1 - \frac{m^2}{4}.$$



Rectangular equation: $y = 1 - x^2$
Parametric equations:
$x = -\dfrac{m}{2}, y = 1 - \dfrac{m^2}{4}$

Figure 10.24

In Figure 10.24, note that the resulting curve has a right-to-left orientation as determined by the direction of increasing values of slope m. For part (a), the curve would have the opposite orientation. ■

TECHNOLOGY To be efficient at using a graphing utility, it is important that you develop skill in representing a graph by a set of parametric equations. The reason for this is that many graphing utilities have only three graphing modes—(1) functions, (2) parametric equations, and (3) polar equations. Most graphing utilities are not programmed to graph a general equation. For instance, suppose you want to graph the hyperbola $x^2 - y^2 = 1$. To graph the hyperbola in *function* mode, you need two equations: $y = \sqrt{x^2 - 1}$ and $y = -\sqrt{x^2 - 1}$. In *parametric* mode, you can represent the graph by $x = \sec t$ and $y = \tan t$.

CYCLOIDS

Galileo first called attention to the cycloid, once recommending that it be used for the arches of bridges. Pascal once spent 8 days attempting to solve many of the problems of cycloids, such as finding the area under one arch, and the volume of the solid of revolution formed by revolving the curve about a line. The cycloid has so many interesting properties and has caused so many quarrels among mathematicians that it has been called "the Helen of geometry" and "the apple of discord."

■ **FOR FURTHER INFORMATION** For more information on cycloids, see the article "The Geometry of Rolling Curves" by John Bloom and Lee Whitt in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

EXAMPLE 5 Parametric Equations for a Cycloid

Determine the curve traced by a point P on the circumference of a circle of radius a rolling along a straight line in a plane. Such a curve is called a **cycloid.**

Solution Let the parameter θ be the measure of the circle's rotation, and let the point $P = (x, y)$ begin at the origin. When $\theta = 0$, P is at the origin. When $\theta = \pi$, P is at a maximum point $(\pi a, 2a)$. When $\theta = 2\pi$, P is back on the x-axis at $(2\pi a, 0)$. From Figure 10.25, you can see that $\angle APC = 180° - \theta$. So,

$$\sin\theta = \sin(180° - \theta) = \sin(\angle APC) = \frac{AC}{a} = \frac{BD}{a}$$

$$\cos\theta = -\cos(180° - \theta) = -\cos(\angle APC) = \frac{AP}{-a}$$

which implies that $AP = -a\cos\theta$ and $BD = a\sin\theta$.

Because the circle rolls along the x-axis, you know that $OD = \widehat{PD} = a\theta$. Furthermore, because $BA = DC = a$, you have

$$x = OD - BD = a\theta - a\sin\theta$$
$$y = BA + AP = a - a\cos\theta.$$

So, the parametric equations are $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$.

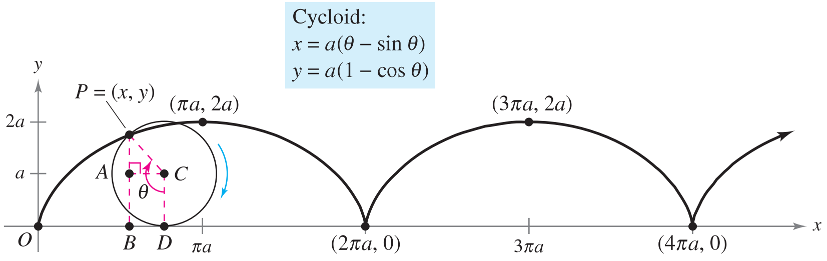


Figure 10.25 ■

TECHNOLOGY Some graphing utilities allow you to simulate the motion of an object that is moving in the plane or in space. If you have access to such a utility, use it to trace out the path of the cycloid shown in Figure 10.25.

The cycloid in Figure 10.25 has sharp corners at the values $x = 2n\pi a$. Notice that the derivatives $x'(\theta)$ and $y'(\theta)$ are both zero at the points for which $\theta = 2n\pi$.

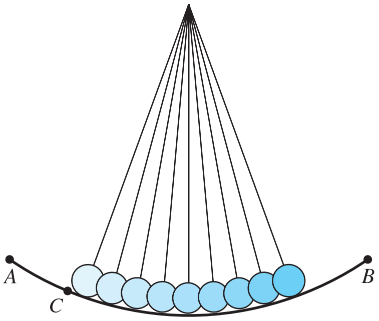
$$\begin{aligned} x(\theta) &= a(\theta - \sin\theta) & y(\theta) &= a(1 - \cos\theta) \\ x'(\theta) &= a - a\cos\theta & y'(\theta) &= a\sin\theta \\ x'(2n\pi) &= 0 & y'(2n\pi) &= 0 \end{aligned}$$

Between these points, the cycloid is called **smooth.**

DEFINITION OF A SMOOTH CURVE

A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called **smooth** if f' and g' are continuous on I and not simultaneously 0, except possibly at the endpoints of I. The curve C is called **piecewise smooth** if it is smooth on each subinterval of some partition of I.

The Tautochrone and Brachistochrone Problems



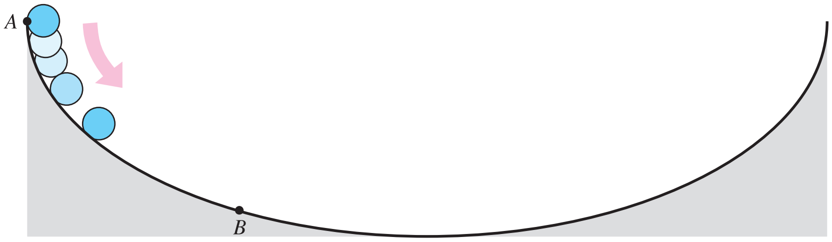
The time required to complete a full swing of the pendulum when starting from point *C* is only approximately the same as when starting from point *A*.
Figure 10.26

The type of curve described in Example 5 is related to one of the most famous pairs of problems in the history of calculus. The first problem (called the **tautochrone problem**) began with Galileo's discovery that the time required to complete a full swing of a given pendulum is *approximately* the same whether it makes a large movement at high speed or a small movement at lower speed (see Figure 10.26). Late in his life, Galileo (1564–1642) realized that he could use this principle to construct a clock. However, he was not able to conquer the mechanics of actual construction. Christian Huygens (1629–1695) was the first to design and construct a working model. In his work with pendulums, Huygens realized that a pendulum does not take exactly the same time to complete swings of varying lengths. (This doesn't affect a pendulum clock, because the length of the circular arc is kept constant by giving the pendulum a slight boost each time it passes its lowest point.) But, in studying the problem, Huygens discovered that a ball rolling back and forth on an inverted cycloid does complete each cycle in exactly the same time.

The Granger Collection

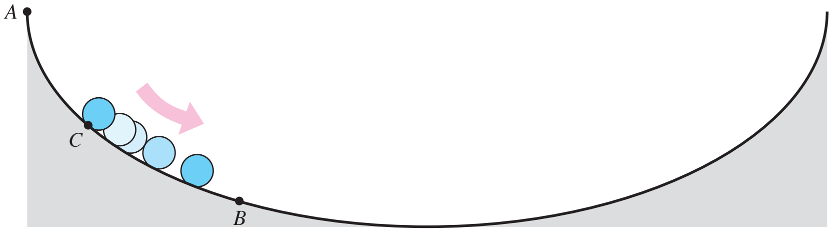
JAMES BERNOULLI (1654–1705)

James Bernoulli, also called Jacques, was the older brother of John. He was one of several accomplished mathematicians of the Swiss Bernoulli family. James's mathematical accomplishments have given him a prominent place in the early development of calculus.



An inverted cycloid is the path down which a ball will roll in the shortest time.
Figure 10.27

The second problem, which was posed by John Bernoulli in 1696, is called the **brachistochrone problem**—in Greek, *brachys* means short and *chronos* means time. The problem was to determine the path down which a particle will slide from point *A* to point *B* in the *shortest time*. Several mathematicians took up the challenge, and the following year the problem was solved by Newton, Leibniz, L'Hôpital, John Bernoulli, and James Bernoulli. As it turns out, the solution is not a straight line from *A* to *B*, but an inverted cycloid passing through the points *A* and *B*, as shown in Figure 10.27. The amazing part of the solution is that a particle starting at rest at *any* other point *C* of the cycloid between *A* and *B* will take exactly the same time to reach *B*, as shown in Figure 10.28.



A ball starting at point *C* takes the same time to reach point *B* as one that starts at point *A*.
Figure 10.28

FOR FURTHER INFORMATION To see a proof of the famous brachistochrone problem, see the article "A New Minimization Proof for the Brachistochrone" by Gary Lawlor in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

10.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. Consider the parametric equations $x = \sqrt{t}$ and $y = 3 - t$.

(a) Construct a table of values for $t = 0, 1, 2, 3$, and 4.

(b) Plot the points (x, y) generated in the table, and sketch a graph of the parametric equations. Indicate the orientation of the graph.

(c) Use a graphing utility to confirm your graph in part (b).

(d) Find the rectangular equation by eliminating the parameter, and sketch its graph. Compare the graph in part (b) with the graph of the rectangular equation.

2. Consider the parametric equations $x = 4\cos^2\theta$ and $y = 2\sin\theta$.

(a) Construct a table of values for $\theta = -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}$, and $\frac{\pi}{2}$.

(b) Plot the points (x, y) generated in the table, and sketch a graph of the parametric equations. Indicate the orientation of the graph.

(c) Use a graphing utility to confirm your graph in part (b).

(d) Find the rectangular equation by eliminating the parameter, and sketch its graph. Compare the graph in part (b) with the graph of the rectangular equation.

(e) If values of θ were selected from the interval $[\pi/2, 3\pi/2]$ for the table in part (a), would the graph in part (b) be different? Explain.

In Exercises 3–20, sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.

3. $x = 2t - 3, \quad y = 3t + 1$

4. $x = 5 - 4t, \quad y = 2 + 5t$

5. $x = t + 1, \quad y = t^2$

6. $x = 2t^2, \quad y = t^4 + 1$

7. $x = t^3, \quad y = \frac{t^2}{2}$

8. $x = t^2 + t, \quad y = t^2 - t$

9. $x = \sqrt{t}, \quad y = t - 5$

10. $x = \sqrt[4]{t}, \quad y = 8 - t$

11. $x = t - 3, \quad y = \frac{t}{t - 3}$

12. $x = 1 + \frac{1}{t}, \quad y = t - 1$

13. $x = 2t, \quad y = |t - 2|$

14. $x = |t - 1|, \quad y = t + 2$

15. $x = e^t, \quad y = e^{3t} + 1$

16. $x = e^{-t}, \quad y = e^{2t} - 1$

17. $x = \sec\theta, \quad y = \cos\theta, \quad 0 \le \theta < \pi/2, \quad \pi/2 < \theta \le \pi$

18. $x = \tan^2\theta, \quad y = \sec^2\theta$

19. $x = 8\cos\theta, \quad y = 8\sin\theta$

20. $x = 3\cos\theta, \quad y = 7\sin\theta$

In Exercises 21–32, use a graphing utility to graph the curve represented by the parametric equations (indicate the orientation of the curve). Eliminate the parameter and write the corresponding rectangular equation.

21. $x = 6\sin 2\theta, y = 4\cos 2\theta$

22. $x = \cos\theta, y = 2\sin 2\theta$

23. $x = 4 + 2\cos\theta$
$y = -1 + \sin\theta$

24. $x = -2 + 3\cos\theta$
$y = -5 + 3\sin\theta$

25. $x = -3 + 4\cos\theta$
$y = 2 + 5\sin\theta$

26. $x = \sec\theta$
$y = \tan\theta$

27. $x = 4\sec\theta, \quad y = 3\tan\theta$

28. $x = \cos^3\theta, \quad y = \sin^3\theta$

29. $x = t^3, \quad y = 3\ln t$

30. $x = \ln 2t, \quad y = t^2$

31. $x = e^{-t}, \quad y = e^{3t}$

32. $x = e^{2t}, \quad y = e^t$

Comparing Plane Curves **In Exercises 33–36, determine any differences between the curves of the parametric equations. Are the graphs the same? Are the orientations the same? Are the curves smooth? Explain.**

33. (a) $x = t$
$y = 2t + 1$
(b) $x = \cos\theta$
$y = 2\cos\theta + 1$
(c) $x = e^{-t}$
$y = 2e^{-t} + 1$
(d) $x = e^t$
$y = 2e^t + 1$

34. (a) $x = 2\cos\theta$
$y = 2\sin\theta$
(b) $x = \sqrt{4t^2 - 1}/|t|$
$y = 1/t$
(c) $x = \sqrt{t}$
$y = \sqrt{4 - t}$
(d) $x = -\sqrt{4 - e^{2t}}$
$y = e^t$

35. (a) $x = \cos\theta$
$y = 2\sin^2\theta$
$0 < \theta < \pi$
(b) $x = \cos(-\theta)$
$y = 2\sin^2(-\theta)$
$0 < \theta < \pi$

36. (a) $x = t + 1, y = t^3$
(b) $x = -t + 1, y = (-t)^3$

37. ***Conjecture***

(a) Use a graphing utility to graph the curves represented by the two sets of parametric equations.

$x = 4\cos t \qquad x = 4\cos(-t)$
$y = 3\sin t \qquad y = 3\sin(-t)$

(b) Describe the change in the graph when the sign of the parameter is changed.

(c) Make a conjecture about the change in the graph of parametric equations when the sign of the parameter is changed.

(d) Test your conjecture with another set of parametric equations.

38. ***Writing*** Review Exercises 33–36 and write a short paragraph describing how the graphs of curves represented by different sets of parametric equations can differ even though eliminating the parameter from each yields the same rectangular equation.

In Exercises 39–42, eliminate the parameter and obtain the standard form of the rectangular equation.

39. Line through (x_1, y_1) and (x_2, y_2):
$x = x_1 + t(x_2 - x_1), \quad y = y_1 + t(y_2 - y_1)$

40. Circle: $x = h + r\cos\theta, \quad y = k + r\sin\theta$

41. Ellipse: $x = h + a\cos\theta, \quad y = k + b\sin\theta$

42. Hyperbola: $x = h + a\sec\theta, \quad y = k + b\tan\theta$

In Exercises 43–50, use the results of Exercises 39–42 to find a set of parametric equations for the line or conic.

43. Line: passes through $(0, 0)$ and $(4, -7)$

44. Line: passes through $(1, 4)$ and $(5, -2)$

45. Circle: center: $(3, 1)$; radius: 2

46. Circle: center: $(-6, 2)$; radius: 4

47. Ellipse: vertices: $(\pm 10, 0)$; foci: $(\pm 8, 0)$

48. Ellipse: vertices: $(4, 7)$, $(4, -3)$; foci: $(4, 5)$, $(4, -1)$

49. Hyperbola: vertices: $(\pm 4, 0)$; foci: $(\pm 5, 0)$

50. Hyperbola: vertices: $(0, \pm 1)$; foci: $(0, \pm 2)$

In Exercises 51–54, find two different sets of parametric equations for the rectangular equation.

51. $y = 6x - 5$

52. $y = 4/(x - 1)$

53. $y = x^3$

54. $y = x^2$

In Exercises 55–58, find a set of parametric equations for the rectangular equation that satisfies the given condition.

55. $y = 2x - 5$, $t = 0$ at the point $(3, 1)$

56. $y = 4x + 1$, $t = -1$ at the point $(-2, -7)$

57. $y = x^2$, $t = 4$ at the point $(4, 16)$

58. $y = 4 - x^2$, $t = 1$ at the point $(1, 3)$

In Exercises 59–66, use a graphing utility to graph the curve represented by the parametric equations. Indicate the direction of the curve. Identify any points at which the curve is not smooth.

59. Cycloid: $x = 2(\theta - \sin\theta)$, $y = 2(1 - \cos\theta)$

60. Cycloid: $x = \theta + \sin\theta$, $y = 1 - \cos\theta$

61. Prolate cycloid: $x = \theta - \frac{3}{2}\sin\theta$, $y = 1 - \frac{3}{2}\cos\theta$

62. Prolate cycloid: $x = 2\theta - 4\sin\theta$, $y = 2 - 4\cos\theta$

63. Hypocycloid: $x = 3\cos^3\theta$, $y = 3\sin^3\theta$

64. Curtate cycloid: $x = 2\theta - \sin\theta$, $y = 2 - \cos\theta$

65. Witch of Agnesi: $x = 2\cot\theta$, $y = 2\sin^2\theta$

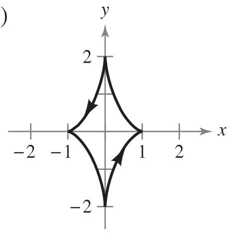
66. Folium of Descartes: $x = 3t/(1 + t^3)$, $y = 3t^2/(1 + t^3)$

WRITING ABOUT CONCEPTS

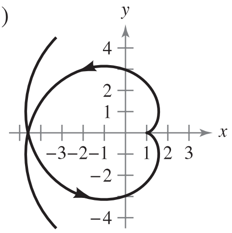
67. Explain the process of sketching a plane curve given by parametric equations. What is meant by the orientation of the curve?

68. Match each set of parametric equations with the correct graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).] Explain your reasoning.

(a)

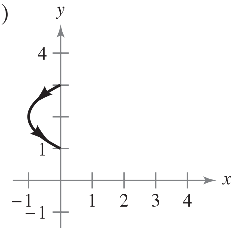


(b)



WRITING ABOUT CONCEPTS (continued)

(c)



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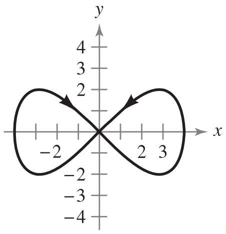

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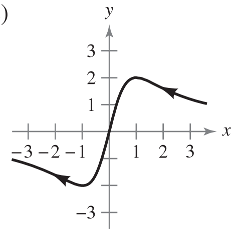
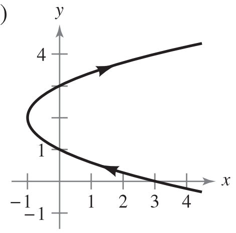


(f)



(i) $x = t^2 - 1$, $y = t + 2$

(ii) $x = \sin^2\theta - 1$, $y = \sin\theta + 2$

(iii) Lissajous curve: $x = 4\cos\theta$, $y = 2\sin 2\theta$

(iv) Evolute of ellipse: $x = \cos^3\theta$, $y = 2\sin^3\theta$

(v) Involute of circle: $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$

(vi) Serpentine curve: $x = \cot\theta$, $y = 4\sin\theta\cos\theta$

69. ***Curtate Cycloid*** A wheel of radius a rolls along a line without slipping. The curve traced by a point P that is b units from the center $(b < a)$ is called a **curtate cycloid** (see figure). Use the angle θ to find a set of parametric equations for this curve.

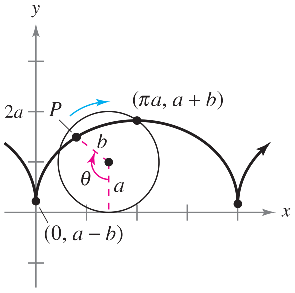


Figure for 69

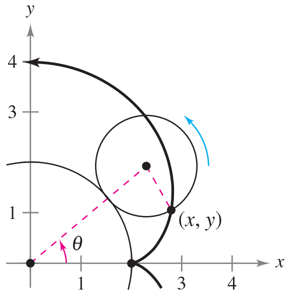


Figure for 70

70. ***Epicycloid*** A circle of radius 1 rolls around the outside of a circle of radius 2 without slipping. The curve traced by a point on the circumference of the smaller circle is called an epicycloid (see figure). Use the angle θ to find a set of parametric equations for this curve.

True or False? **In Exercises 71–73, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

71. The graph of the parametric equations $x = t^2$ and $y = t^2$ is the line $y = x$.

72. If y is a function of t and x is a function of t, then y is a function of x.

73. The curve represented by the parametric equations $x = t$ and $y = \cos t$ can be written as an equation of the form $y = f(x)$.

CAPSTONE

74. Consider the parametric equations $x = 8 \cos t$ and $y = 8 \sin t$.

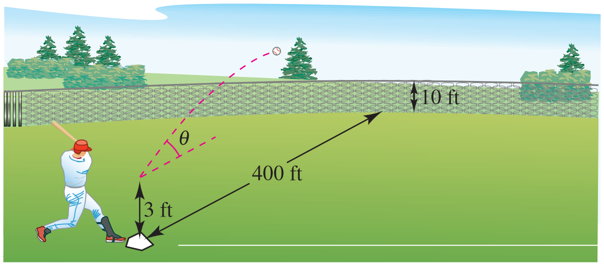
(a) Describe the curve represented by the parametric equations.

(b) How does the curve represented by the parametric equations $x = 8 \cos t + 3$ and $y = 8 \sin t + 6$ compare to the curve described in part (a)?

(c) How does the original curve change when cosine and sine are interchanged?

Projectile Motion **In Exercises 75 and 76, consider a projectile launched at a height h feet above the ground and at an angle θ with the horizontal. If the initial velocity is v_0 feet per second, the path of the projectile is modeled by the parametric equations $x = (v_0 \cos \theta)t$ and $y = h + (v_0 \sin \theta)t - 16t^2$.**

75. The center field fence in a ballpark is 10 feet high and 400 feet from home plate. The ball is hit 3 feet above the ground. It leaves the bat at an angle of θ degrees with the horizontal at a speed of 100 miles per hour (see figure).



(a) Write a set of parametric equations for the path of the ball.

(b) Use a graphing utility to graph the path of the ball when $\theta = 15°$. Is the hit a home run?

(c) Use a graphing utility to graph the path of the ball when $\theta = 23°$. Is the hit a home run?

(d) Find the minimum angle at which the ball must leave the bat in order for the hit to be a home run.

76. A rectangular equation for the path of a projectile is $y = 5 + x - 0.005x^2$.

(a) Eliminate the parameter t from the position function for the motion of a projectile to show that the rectangular equation is

$$y = -\frac{16 \sec^2 \theta}{v_0^2} x^2 + (\tan \theta) x + h.$$

(b) Use the result of part (a) to find h, v_0, and θ. Find the parametric equations of the path.

(c) Use a graphing utility to graph the rectangular equation for the path of the projectile. Confirm your answer in part (b) by sketching the curve represented by the parametric equations.

(d) Use a graphing utility to approximate the maximum height of the projectile and its range.

SECTION PROJECT

Cycloids

In Greek, the word *cycloid* means *wheel*, the word *hypocycloid* means *under the wheel*, and the word *epicycloid* means *upon the wheel*. Match the hypocycloid or epicycloid with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

Hypocycloid, H(A, B)

The path traced by a fixed point on a circle of radius B as it rolls around the inside of a circle of radius A

$$x = (A - B) \cos t + B \cos\left(\frac{A - B}{B}\right)t$$

$$y = (A - B) \sin t - B \sin\left(\frac{A - B}{B}\right)t$$

Epicycloid, E(A, B)

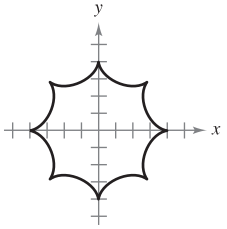
The path traced by a fixed point on a circle of radius B as it rolls around the *outside* of a circle of radius A

$$x = (A + B) \cos t - B \cos\left(\frac{A + B}{B}\right)t$$

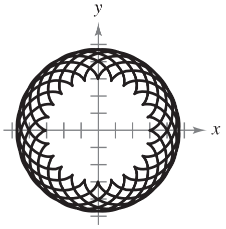
$$y = (A + B) \sin t - B \sin\left(\frac{A + B}{B}\right)t$$

I. $H(8, 3)$
II. $E(8,3)$
III. $H(8, 7)$
IV. $E(24, 3)$
V. $H(24, 7)$
VI. $E(24, 7)$

(a)



(b)



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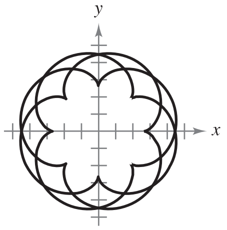

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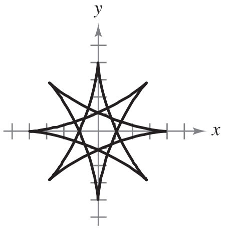

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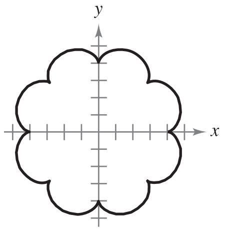

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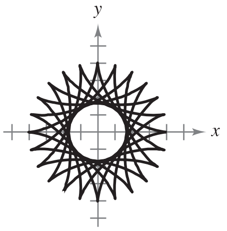


Exercises based on “Mathematical Discovery via Computer Graphics: Hypocycloids and Epicycloids” by Florence S. Gordon and Sheldon P. Gordon, *College Mathematics Journal*, November 1984, p.441. Used by permission of the authors.

10.3 Parametric Equations and Calculus

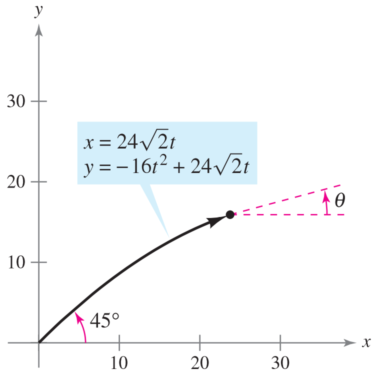
- **Find the slope of a tangent line to a curve given by a set of parametric equations.**
- **Find the arc length of a curve given by a set of parametric equations.**
- **Find the area of a surface of revolution (parametric form).**

Slope and Tangent Lines

Now that you can represent a graph in the plane by a set of parametric equations, it is natural to ask how to use calculus to study plane curves. To begin, let's take another look at the projectile represented by the parametric equations

$$x = 24\sqrt{2}t \qquad \text{and} \qquad y = -16t^2 + 24\sqrt{2}t$$

as shown in Figure 10.29. From the discussion at the beginning of Section 10.2, you know that these equations enable you to locate the position of the projectile at a given time. You also know that the object is initially projected at an angle of 45°. But how can you find the angle θ representing the object's direction at some other time t? The following theorem answers this question by giving a formula for the slope of the tangent line as a function of t.

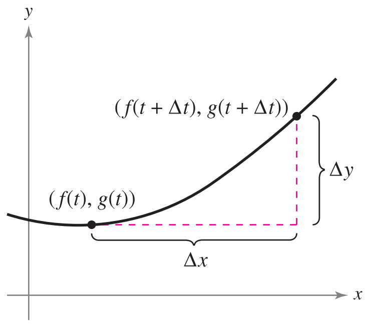


At time t, the angle of elevation of the projectile is θ, the slope of the tangent line at that point.
Figure 10.29

THEOREM 10.7 PARAMETRIC FORM OF THE DERIVATIVE

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{dx}{dt} \neq 0.$$



The slope of the secant line through the points $(f(t), g(t))$ and $(f(t + \Delta t), g(t + \Delta t))$ is $\Delta y/\Delta x$.
Figure 10.30

PROOF In Figure 10.30, consider $\Delta t > 0$ and let

$$\Delta y = g(t + \Delta t) - g(t) \qquad \text{and} \qquad \Delta x = f(t + \Delta t) - f(t).$$

Because $\Delta x \to 0$ as $\Delta t \to 0$, you can write

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\\ &= \lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{f(t + \Delta t) - f(t)}.\end{aligned}$$

Dividing both the numerator and denominator by Δt, you can use the differentiability of f and g to conclude that

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta t \to 0} \frac{[g(t + \Delta t) - g(t)]/\Delta t}{[f(t + \Delta t) - f(t)]/\Delta t}\\ &= \frac{\displaystyle\lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}}{\displaystyle\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}}\\ &= \frac{g'(t)}{f'(t)}\\ &= \frac{dy/dt}{dx/dt}.\end{aligned}$$

■

EXAMPLE 1 Differentiation and Parametric Form

Find dy/dx for the curve given by $x = \sin t$ and $y = \cos t$.

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{\cos t} = -\tan t$$ ■

STUDY TIP The curve traced out in Example 1 is a circle. Use the formula

$$\frac{dy}{dx} = -\tan t$$

to find the slopes at the points (1, 0) and (0, 1).

Because dy/dx is a function of t, you can use Theorem 10.7 repeatedly to find *higher-order* derivatives. For instance,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$$ Second derivative

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{\frac{d}{dt}\left[\frac{d^2y}{dx^2}\right]}{dx/dt}.$$ Third derivative

EXAMPLE 2 Finding Slope and Concavity

For the curve given by

$$x = \sqrt{t} \quad \text{and} \quad y = \frac{1}{4}(t^2 - 4), \quad t \geq 0$$

find the slope and concavity at the point (2, 3).

Solution Because

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(1/2)t}{(1/2)t^{-1/2}} = t^{3/2}$$ Parametric form of first derivative

you can find the second derivative to be

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}[dy/dx]}{dx/dt} = \frac{\frac{d}{dt}[t^{3/2}]}{dx/dt} = \frac{(3/2)t^{1/2}}{(1/2)t^{-1/2}} = 3t.$$ Parametric form of second derivative

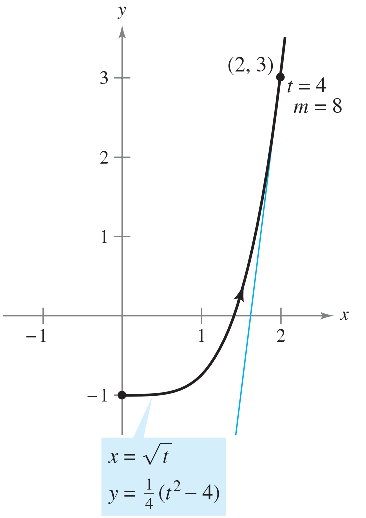
At $(x, y) = (2, 3)$, it follows that $t = 4$, and the slope is

$$\frac{dy}{dx} = (4)^{3/2} = 8.$$

Moreover, when $t = 4$, the second derivative is

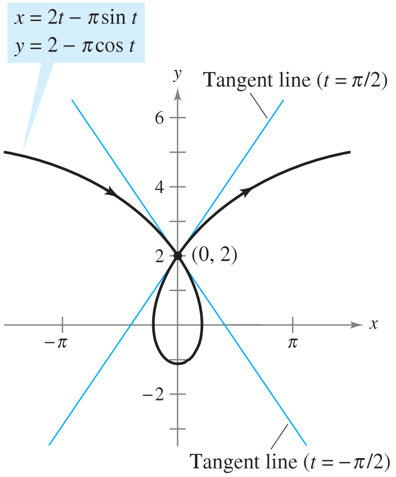
$$\frac{d^2y}{dx^2} = 3(4) = 12 > 0$$

and you can conclude that the graph is concave upward at (2, 3), as shown in Figure 10.31. ■



The graph is concave upward at (2, 3), when $t = 4$.
Figure 10.31

Because the parametric equations $x = f(t)$ and $y = g(t)$ need not define y as a function of x, it is possible for a plane curve to loop around and cross itself. At such points the curve may have more than one tangent line, as shown in the next example.



This prolate cycloid has two tangent lines at the point (0, 2).
Figure 10.32

EXAMPLE 3 A Curve with Two Tangent Lines at a Point

The **prolate cycloid** given by

$$x = 2t - \pi \sin t \qquad \text{and} \qquad y = 2 - \pi \cos t$$

crosses itself at the point (0, 2), as shown in Figure 10.32. Find the equations of both tangent lines at this point.

Solution Because $x = 0$ and $y = 2$ when $t = \pm\pi/2$, and

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\pi \sin t}{2 - \pi \cos t}$$

you have $dy/dx = -\pi/2$ when $t = -\pi/2$ and $dy/dx = \pi/2$ when $t = \pi/2$. So, the two tangent lines at (0, 2) are

$$y - 2 = -\left(\frac{\pi}{2}\right)x \qquad \text{Tangent line when } t = -\frac{\pi}{2}$$

and

$$y - 2 = \left(\frac{\pi}{2}\right)x. \qquad \text{Tangent line when } t = \frac{\pi}{2}$$ ■

If $dy/dt = 0$ and $dx/dt \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a horizontal tangent at $(f(t_0), g(t_0))$. For instance, in Example 3, the given curve has a horizontal tangent at the point $(0, 2 - \pi)$ (when $t = 0$). Similarly, if $dx/dt = 0$ and $dy/dt \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a vertical tangent at $(f(t_0), g(t_0))$.

Arc Length

You have seen how parametric equations can be used to describe the path of a particle moving in the plane. You will now develop a formula for determining the *distance* traveled by the particle along its path.

Recall from Section 7.4 that the formula for the arc length of a curve C given by $y = h(x)$ over the interval $[x_0, x_1]$ is

$$\begin{aligned} s &= \int_{x_0}^{x_1} \sqrt{1 + [h'(x)]^2}\, dx \\ &= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \end{aligned}$$

If C is represented by the parametric equations $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, and if $dx/dt = f'(t) > 0$, you can write

$$\begin{aligned} s = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx &= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, dx \\ &= \int_a^b \sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}}\, \frac{dx}{dt}\, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt. \end{aligned}$$

THEOREM 10.8 ARC LENGTH IN PARAMETRIC FORM

If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \le t \le b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

NOTE When applying the arc length formula to a curve, be sure that the curve is traced out only once on the interval of integration. For instance, the circle given by $x = \cos t$ and $y = \sin t$ is traced out once on the interval $0 \le t \le 2\pi$, but is traced out twice on the interval $0 \le t \le 4\pi$. ■

In the preceding section you saw that if a circle rolls along a line, a point on its circumference will trace a path called a cycloid. If the circle rolls around the circumference of another circle, the path of the point is an **epicycloid.** The next example shows how to find the arc length of an epicycloid.

ARCH OF A CYCLOID

The arc length of an arch of a cycloid was first calculated in 1658 by British architect and mathematician Christopher Wren, famous for rebuilding many buildings and churches in London, including St. Paul's Cathedral.

EXAMPLE 4 Finding Arc Length

A circle of radius 1 rolls around the circumference of a larger circle of radius 4, as shown in Figure 10.33. The epicycloid traced by a point on the circumference of the smaller circle is given by

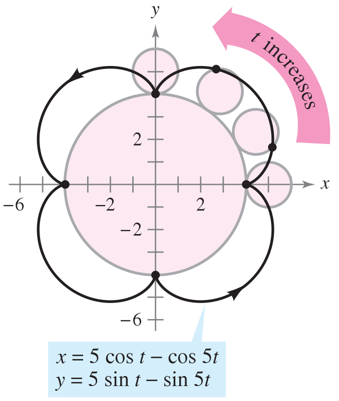
$$x = 5\cos t - \cos 5t \quad \text{and} \quad y = 5\sin t - \sin 5t.$$

Find the distance traveled by the point in one complete trip about the larger circle.

Solution Before applying Theorem 10.8, note in Figure 10.33 that the curve has sharp points when $t = 0$ and $t = \pi/2$. Between these two points, dx/dt and dy/dt are not simultaneously 0. So, the portion of the curve generated from $t = 0$ to $t = \pi/2$ is smooth. To find the total distance traveled by the point, you can find the arc length of that portion lying in the first quadrant and multiply by 4.

$$\begin{aligned}
s &= 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt && \text{Parametric form for arc length}\\
&= 4\int_0^{\pi/2} \sqrt{(-5\sin t + 5\sin 5t)^2 + (5\cos t - 5\cos 5t)^2}\, dt\\
&= 20\int_0^{\pi/2} \sqrt{2 - 2\sin t \sin 5t - 2\cos t \cos 5t}\, dt\\
&= 20\int_0^{\pi/2} \sqrt{2 - 2\cos 4t}\, dt\\
&= 20\int_0^{\pi/2} \sqrt{4\sin^2 2t}\, dt && \text{Trigonometric identity}\\
&= 40\int_0^{\pi/2} \sin 2t\, dt\\
&= -20\Big[\cos 2t\Big]_0^{\pi/2}\\
&= 40
\end{aligned}$$

For the epicycloid shown in Figure 10.33, an arc length of 40 seems about right because the circumference of a circle of radius 6 is $2\pi r = 12\pi \approx 37.7$. ■



$x = 5\cos t - \cos 5t$
$y = 5\sin t - \sin 5t$

An epicycloid is traced by a point on the smaller circle as it rolls around the larger circle.
Figure 10.33

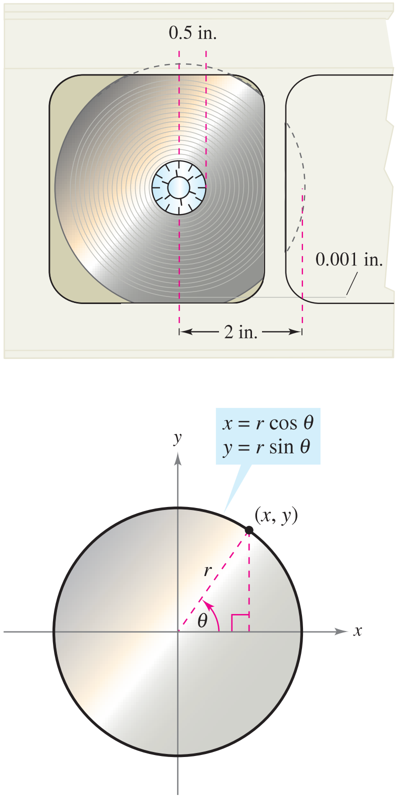


Figure 10.34

EXAMPLE 5 Length of a Recording Tape

A recording tape 0.001 inch thick is wound around a reel whose inner radius is 0.5 inch and whose outer radius is 2 inches, as shown in Figure 10.34. How much tape is required to fill the reel?

Solution To create a model for this problem, assume that as the tape is wound around the reel, its distance r from the center increases linearly at a rate of 0.001 inch per revolution, or

$$r = (0.001)\frac{\theta}{2\pi} = \frac{\theta}{2000\pi}, \qquad 1000\pi \le \theta \le 4000\pi$$

where θ is measured in radians. You can determine that the coordinates of the point (x, y) corresponding to a given radius are

$$x = r\cos\theta$$

and

$$y = r\sin\theta.$$

Substituting for r, you obtain the parametric equations

$$x = \left(\frac{\theta}{2000\pi}\right)\cos\theta \quad \text{and} \quad y = \left(\frac{\theta}{2000\pi}\right)\sin\theta.$$

You can use the arc length formula to determine that the total length of the tape is

$$\begin{aligned} s &= \int_{1000\pi}^{4000\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= \frac{1}{2000\pi}\int_{1000\pi}^{4000\pi} \sqrt{(-\theta\sin\theta + \cos\theta)^2 + (\theta\cos\theta + \sin\theta)^2}\, d\theta \\ &= \frac{1}{2000\pi}\int_{1000\pi}^{4000\pi} \sqrt{\theta^2 + 1}\, d\theta \\ &= \frac{1}{2000\pi}\left(\frac{1}{2}\right)\Big[\theta\sqrt{\theta^2+1} + \ln\left|\theta + \sqrt{\theta^2+1}\right|\Big]_{1000\pi}^{4000\pi} && \text{Integration tables (Appendix B), Formula 26} \\ &\approx 11{,}781 \text{ inches} \\ &\approx 982 \text{ feet.} \end{aligned}$$

■ **FOR FURTHER INFORMATION** For more information on the mathematics of recording tape, see "Tape Counters" by Richard L. Roth in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

NOTE The graph of $r = a\theta$ is called the **spiral of Archimedes.** The graph of $r = \theta/2000\pi$ (in Example 5) is of this form.

The length of the tape in Example 5 can be approximated by adding the circumferences of circular pieces of tape. The smallest circle has a radius of 0.501 and the largest has a radius of 2.

$$\begin{aligned} s &\approx 2\pi(0.501) + 2\pi(0.502) + 2\pi(0.503) + \cdots + 2\pi(2.000) \\ &= \sum_{i=1}^{1500} 2\pi(0.5 + 0.001i) \\ &= 2\pi[1500(0.5) + 0.001(1500)(1501)/2] \\ &\approx 11{,}786 \text{ inches} \end{aligned}$$

Area of a Surface of Revolution

You can use the formula for the area of a surface of revolution in rectangular form to develop a formula for surface area in parametric form.

THEOREM 10.9 AREA OF A SURFACE OF REVOLUTION

If a smooth curve C given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \le t \le b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by the following.

1. $$S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$ Revolution about the x-axis: $g(t) \ge 0$

2. $$S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$ Revolution about the y-axis: $f(t) \ge 0$

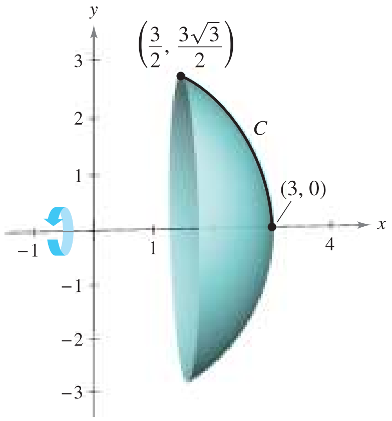
These formulas are easy to remember if you think of the differential of arc length as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Then the formulas are written as follows.

1. $S = 2\pi \int_a^b g(t)\, ds$ **2.** $S = 2\pi \int_a^b f(t)\, ds$

EXAMPLE 6 Finding the Area of a Surface of Revolution



The surface of revolution has a surface area of 9π.
Figure 10.35

Let C be the arc of the circle

$$x^2 + y^2 = 9$$

from $(3, 0)$ to $\left(3/2, 3\sqrt{3}/2\right)$, as shown in Figure 10.35. Find the area of the surface formed by revolving C about the x-axis.

Solution You can represent C parametrically by the equations

$$x = 3\cos t \quad \text{and} \quad y = 3\sin t, \quad 0 \le t \le \pi/3.$$

(Note that you can determine the interval for t by observing that $t = 0$ when $x = 3$ and $t = \pi/3$ when $x = 3/2$.) On this interval, C is smooth and y is nonnegative, and you can apply Theorem 10.9 to obtain a surface area of

$$\begin{aligned}
S &= 2\pi \int_0^{\pi/3} (3\sin t)\sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt && \text{Formula for area of a surface of revolution}\\
&= 6\pi \int_0^{\pi/3} \sin t\sqrt{9(\sin^2 t + \cos^2 t)}\, dt\\
&= 6\pi \int_0^{\pi/3} 3\sin t\, dt && \text{Trigonometric identity}\\
&= -18\pi\Big[\cos t\Big]_0^{\pi/3}\\
&= -18\pi\left(\frac{1}{2} - 1\right)\\
&= 9\pi.
\end{aligned}$$

■

10.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find dy/dx.

1. $x = t^2,\ y = 7 - 6t$

2. $x = \sqrt[3]{t},\ y = 4 - t$

3. $x = \sin^2 \theta,\ y = \cos^2 \theta$

4. $x = 2e^{\theta},\ y = e^{-\theta/2}$

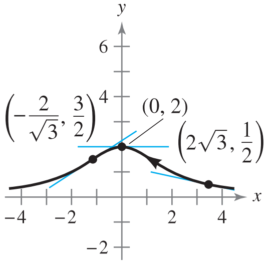
In Exercises 5–14, find dy/dx and d^2y/dx^2, and find the slope and concavity (if possible) at the given value of the parameter.

	Parametric Equations	*Point*
5.	$x = 4t,\ y = 3t - 2$	$t = 3$
6.	$x = \sqrt{t},\ y = 3t - 1$	$t = 1$
7.	$x = t + 1,\ y = t^2 + 3t$	$t = -1$
8.	$x = t^2 + 5t + 4,\ y = 4t$	$t = 0$
9.	$x = 4 \cos \theta,\ y = 4 \sin \theta$	$\theta = \dfrac{\pi}{4}$
10.	$x = \cos \theta,\ y = 3 \sin \theta$	$\theta = 0$
11.	$x = 2 + \sec \theta,\ y = 1 + 2 \tan \theta$	$\theta = \dfrac{\pi}{6}$
12.	$x = \sqrt{t},\ y = \sqrt{t - 1}$	$t = 2$
13.	$x = \cos^3 \theta,\ y = \sin^3 \theta$	$\theta = \dfrac{\pi}{4}$
14.	$x = \theta - \sin \theta,\ y = 1 - \cos \theta$	$\theta = \pi$

In Exercises 15–18, find an equation of the tangent line at each given point on the curve.

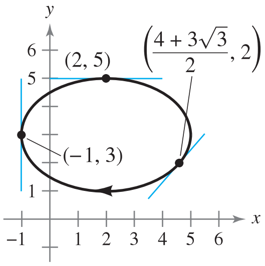
15. $x = 2 \cot \theta$

$y = 2 \sin^2 \theta$



16. $x = 2 - 3 \cos \theta$

$y = 3 + 2 \sin \theta$



17. $x = t^2 - 4$

$y = t^2 - 2t$

$(0, 0), (-3, -1)$, and $(-3, 3)$

18. $x = t^4 + 2$

$y = t^3 + t$

$(2, 0), (3, -2)$, and $(18, 10)$

In Exercises 19–22, (a) use a graphing utility to graph the curve represented by the parametric equations, (b) use a graphing utility to find dx/dt, dy/dt, and dy/dx at the given value of the parameter, (c) find an equation of the tangent line to the curve at the given value of the parameter, and (d) use a graphing utility to graph the curve and the tangent line from part (c).

	Parametric Equations	*Parameter*
19.	$x = 6t,\ y = t^2 + 4$	$t = 1$
20.	$x = t - 2,\ y = \dfrac{1}{t} + 3$	$t = 1$
21.	$x = t^2 - t + 2,\ y = t^3 - 3t$	$t = -1$
22.	$x = 4 \cos \theta,\ y = 3 \sin \theta$	$\theta = \dfrac{3\pi}{4}$

In Exercises 23–26, find the equations of the tangent lines at the point where the curve crosses itself.

23. $x = 2 \sin 2t, \quad y = 3 \sin t$

24. $x = 2 - \pi \cos t, \quad y = 2t - \pi \sin t$

25. $x = t^2 - t, \quad y = t^3 - 3t - 1$

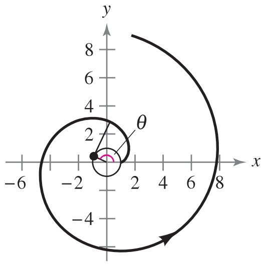
26. $x = t^3 - 6t, \quad y = t^2$

In Exercises 27 and 28, find all points (if any) of horizontal and vertical tangency to the portion of the curve shown.

27. Involute of a circle:

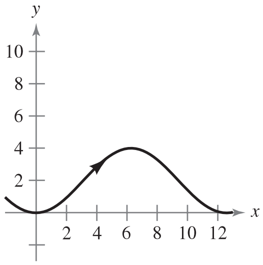
$x = \cos \theta + \theta \sin \theta$

$y = \sin \theta - \theta \cos \theta$



28. $x = 2\theta$

$y = 2(1 - \cos \theta)$



In Exercises 29–38, find all points (if any) of horizontal and vertical tangency to the curve. Use a graphing utility to confirm your results.

29. $x = 4 - t, \quad y = t^2$

30. $x = t + 1, \quad y = t^2 + 3t$

31. $x = t + 4, \quad y = t^3 - 3t$

32. $x = t^2 - t + 2, \quad y = t^3 - 3t$

33. $x = 3 \cos \theta, \quad y = 3 \sin \theta$

34. $x = \cos \theta, \quad y = 2 \sin 2\theta$

35. $x = 5 + 3 \cos \theta, \quad y = -2 + \sin \theta$

36. $x = 4 \cos^2 \theta, \quad y = 2 \sin \theta$

37. $x = \sec \theta, \quad y = \tan \theta$

38. $x = \cos^2 \theta, \quad y = \cos \theta$

In Exercises 39–44, determine the t intervals on which the curve is concave downward or concave upward.

39. $x = 3t^2, \quad y = t^3 - t$

40. $x = 2 + t^2, \quad y = t^2 + t^3$

41. $x = 2t + \ln t, \quad y = 2t - \ln t$

42. $x = t^2, \quad y = \ln t$

43. $x = \sin t, \quad y = \cos t, \quad 0 < t < \pi$

44. $x = 4 \cos t, \quad y = 2 \sin t, \quad 0 < t < 2\pi$

Arc Length **In Exercises 45–48, write an integral that represents the arc length of the curve on the given interval. Do not evaluate the integral.**

	Parametric Equations	*Interval*
45.	$x = 3t - t^2, \quad y = 2t^{3/2}$	$1 \le t \le 3$
46.	$x = \ln t, \quad y = 4t - 3$	$1 \le t \le 5$
47.	$x = e^t + 2, \quad y = 2t + 1$	$-2 \le t \le 2$
48.	$x = t + \sin t, \quad y = t - \cos t$	$0 \le t \le \pi$

Arc Length **In Exercises 49–56, find the arc length of the curve on the given interval.**

	Parametric Equations	*Interval*
49.	$x = 3t + 5, \quad y = 7 - 2t$	$-1 \le t \le 3$
50.	$x = t^2, \quad y = 2t$	$0 \le t \le 2$
51.	$x = 6t^2, \quad y = 2t^3$	$1 \le t \le 4$
52.	$x = t^2 + 1, \quad y = 4t^3 + 3$	$-1 \le t \le 0$
53.	$x = e^{-t} \cos t, \quad y = e^{-t} \sin t$	$0 \le t \le \dfrac{\pi}{2}$
54.	$x = \arcsin t, \quad y = \ln \sqrt{1 - t^2}$	$0 \le t \le \frac{1}{2}$
55.	$x = \sqrt{t}, \quad y = 3t - 1$	$0 \le t \le 1$
56.	$x = t, \quad y = \dfrac{t^5}{10} + \dfrac{1}{6t^3}$	$1 \le t \le 2$

Arc Length **In Exercises 57–60, find the arc length of the curve on the interval $[0, 2\pi]$.**

57. Hypocycloid perimeter: $x = a \cos^3 \theta, y = a \sin^3 \theta$

58. Circle circumference: $x = a \cos \theta, y = a \sin \theta$

59. Cycloid arch: $x = a(\theta - \sin \theta), y = a(1 - \cos \theta)$

60. Involute of a circle: $x = \cos \theta + \theta \sin \theta, y = \sin \theta - \theta \cos \theta$

61. ***Path of a Projectile*** The path of a projectile is modeled by the parametric equations

$$x = (90 \cos 30°)t \quad \text{and} \quad y = (90 \sin 30°)t - 16t^2$$

where x and y are measured in feet.

(a) Use a graphing utility to graph the path of the projectile.

(b) Use a graphing utility to approximate the range of the projectile.

(c) Use the integration capabilities of a graphing utility to approximate the arc length of the path. Compare this result with the range of the projectile.

62. ***Path of a Projectile*** If the projectile in Exercise 61 is launched at an angle θ with the horizontal, its parametric equations are

$$x = (90 \cos \theta)t \quad \text{and} \quad y = (90 \sin \theta)t - 16t^2.$$

Use a graphing utility to find the angle that maximizes the range of the projectile. What angle maximizes the arc length of the trajectory?

63. ***Folium of Descartes*** Consider the parametric equations

$$x = \frac{4t}{1 + t^3} \quad \text{and} \quad y = \frac{4t^2}{1 + t^3}.$$

(a) Use a graphing utility to graph the curve represented by the parametric equations.

(b) Use a graphing utility to find the points of horizontal tangency to the curve.

(c) Use the integration capabilities of a graphing utility to approximate the arc length of the closed loop. (*Hint:* Use symmetry and integrate over the interval $0 \le t \le 1$.)

64. ***Witch of Agnesi*** Consider the parametric equations

$$x = 4 \cot \theta \quad \text{and} \quad y = 4 \sin^2 \theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

(a) Use a graphing utility to graph the curve represented by the parametric equations.

(b) Use a graphing utility to find the points of horizontal tangency to the curve.

(c) Use the integration capabilities of a graphing utility to approximate the arc length over the interval $\pi/4 \le \theta \le \pi/2$.

65. ***Writing***

(a) Use a graphing utility to graph each set of parametric equations.

$x = t - \sin t \qquad x = 2t - \sin(2t)$
$y = 1 - \cos t \qquad y = 1 - \cos(2t)$
$0 \le t \le 2\pi \qquad 0 \le t \le \pi$

(b) Compare the graphs of the two sets of parametric equations in part (a). If the curve represents the motion of a particle and t is time, what can you infer about the average speeds of the particle on the paths represented by the two sets of parametric equations?

(c) Without graphing the curve, determine the time required for a particle to traverse the same path as in parts (a) and (b) if the path is modeled by

$$x = \tfrac{1}{2}t - \sin\left(\tfrac{1}{2}t\right) \quad \text{and} \quad y = 1 - \cos\left(\tfrac{1}{2}t\right).$$

66. ***Writing***

(a) Each set of parametric equations represents the motion of a particle. Use a graphing utility to graph each set.

First Particle	*Second Particle*
$x = 3 \cos t$	$x = 4 \sin t$
$y = 4 \sin t$	$y = 3 \cos t$
$0 \le t \le 2\pi$	$0 \le t \le 2\pi$

(b) Determine the number of points of intersection.

(c) Will the particles ever be at the same place at the same time? If so, identify the point(s).

(d) Explain what happens if the motion of the second particle is represented by

$$x = 2 + 3 \sin t, \quad y = 2 - 4 \cos t, \quad 0 \le t \le 2\pi.$$

Surface Area **In Exercises 67–70, write an integral that represents the area of the surface generated by revolving the curve about the x-axis. Use a graphing utility to approximate the integral.**

	Parametric Equations	*Interval*
67.	$x = 3t, \quad y = t + 2$	$0 \le t \le 4$
68.	$x = \frac{1}{4}t^2, \quad y = t + 3$	$0 \le t \le 3$
69.	$x = \cos^2 \theta, \quad y = \cos \theta$	$0 \le \theta \le \frac{\pi}{2}$
70.	$x = \theta + \sin \theta, \quad y = \theta + \cos \theta$	$0 \le \theta \le \frac{\pi}{2}$

Surface Area **In Exercises 71–76, find the area of the surface generated by revolving the curve about each given axis.**

71. $x = 2t, \quad y = 3t, \quad 0 \le t \le 3,$ (a) x-axis (b) y-axis

72. $x = t, \quad y = 4 - 2t, \quad 0 \le t \le 2,$ (a) x-axis (b) y-axis

73. $x = 5 \cos \theta, \quad y = 5 \sin \theta, \quad 0 \le \theta \le \frac{\pi}{2},$ y-axis

74. $x = \frac{1}{3}t^3, \quad y = t + 1, \quad 1 \le t \le 2,$ y-axis

75. $x = a \cos^3 \theta, \quad y = a \sin^3 \theta, \quad 0 \le \theta \le \pi,$ x-axis

76. $x = a \cos \theta, \quad y = b \sin \theta, \quad 0 \le \theta \le 2\pi,$

(a) x-axis (b) y-axis

WRITING ABOUT CONCEPTS

77. Give the parametric form of the derivative.

In Exercises 78 and 79, mentally determine dy/dx.

78. $x = t, \quad y = 3$

79. $x = t, \quad y = 6t - 5$

80. Give the integral formula for arc length in parametric form.

81. Give the integral formulas for the areas of the surfaces of revolution formed when a smooth curve C is revolved about (a) the x-axis and (b) the y-axis.

CAPSTONE

82. (a) Sketch a graph of a curve defined by the parametric equations $x = g(t)$ and $y = f(t)$ such that $dx/dt > 0$ and $dy/dt < 0$ for all real numbers t.

(b) Sketch a graph of a curve defined by the parametric equations $x = g(t)$ and $y = f(t)$ such that $dx/dt < 0$ and $dy/dt < 0$ for all real numbers t.

83. Use integration by substitution to show that if y is a continuous function of x on the interval $a \le x \le b$, where $x = f(t)$ and $y = g(t)$, then

$$\int_a^b y\,dx = \int_{t_1}^{t_2} g(t) f'(t)\,dt$$

where $f(t_1) = a$, $f(t_2) = b$, and both g and f' are continuous on $[t_1, t_2]$.

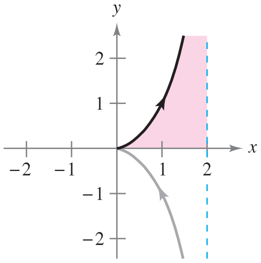
84. ***Surface Area*** A portion of a sphere of radius r is removed by cutting out a circular cone with its vertex at the center of the sphere. The vertex of the cone forms an angle of 2θ. Find the surface area removed from the sphere.

Area **In Exercises 85 and 86, find the area of the region. (Use the result of Exercise 83.)**

85. $x = 2 \sin^2 \theta$

$y = 2 \sin^2 \theta \tan \theta$

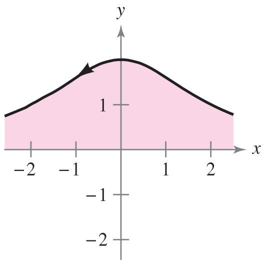
$0 \le \theta < \frac{\pi}{2}$



86. $x = 2 \cot \theta$

$y = 2 \sin^2 \theta$

$0 < \theta < \pi$



CAS ***Areas of Simple Closed Curves*** **In Exercises 87–92, use a computer algebra system and the result of Exercise 83 to match the closed curve with its area. (These exercises were based on "The Surveyor's Area Formula," by Bart Braden, *College Mathematics Journal*, September 1986, pp. 335–337, by permission of the author.)**

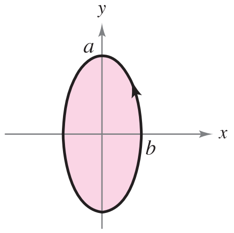
(a) $\frac{8}{3}ab$ **(b)** $\frac{3}{8}\pi a^2$ **(c)** $2\pi a^2$

(d) πab **(e)** $2\pi ab$ **(f)** $6\pi a^2$

87. Ellipse: $(0 \le t \le 2\pi)$

$x = b \cos t$

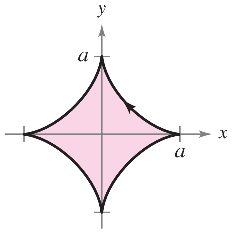
$y = a \sin t$



88. Astroid: $(0 \le t \le 2\pi)$

$x = a \cos^3 t$

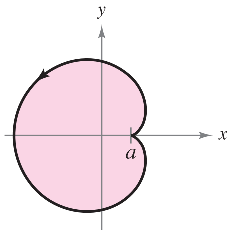
$y = a \sin^3 t$



89. Cardioid: $(0 \le t \le 2\pi)$

$x = 2a \cos t - a \cos 2t$

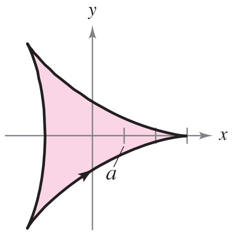
$y = 2a \sin t - a \sin 2t$



90. Deltoid: $(0 \le t \le 2\pi)$

$x = 2a \cos t + a \cos 2t$

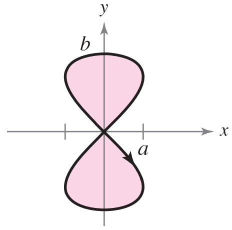
$y = 2a \sin t - a \sin 2t$



91. Hourglass: $(0 \le t \le 2\pi)$

$x = a \sin 2t$

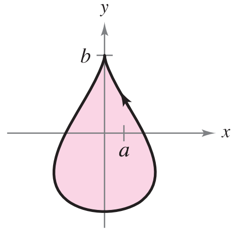
$y = b \sin t$



92. Teardrop: $(0 \le t \le 2\pi)$

$x = 2a \cos t - a \sin 2t$

$y = b \sin t$



Centroid **In Exercises 93 and 94, find the centroid of the region bounded by the graph of the parametric equations and the coordinate axes. (Use the result of Exercise 83.)**

93. $x = \sqrt{t}, \quad y = 4 - t$

94. $x = \sqrt{4 - t}, \quad y = \sqrt{t}$

Volume **In Exercises 95 and 96, find the volume of the solid formed by revolving the region bounded by the graphs of the given equations about the x-axis. (Use the result of Exercise 83.)**

95. $x = 6 \cos \theta, \quad y = 6 \sin \theta$

96. $x = \cos \theta, \quad y = 3 \sin \theta, \quad a > 0$

97. ***Cycloid*** Use the parametric equations

$$x = a(\theta - \sin \theta) \quad \text{and} \quad y = a(1 - \cos \theta),\ a > 0$$

to answer the following.

(a) Find dy/dx and d^2y/dx^2.

(b) Find the equation of the tangent line at the point where $\theta = \pi/6$.

(c) Find all points (if any) of horizontal tangency.

(d) Determine where the curve is concave upward or concave downward.

(e) Find the length of one arc of the curve.

98. Use the parametric equations

$$x = t^2\sqrt{3} \quad \text{and} \quad y = 3t - \frac{1}{3}t^3$$

to answer the following.

(a) Use a graphing utility to graph the curve on the interval $-3 \le t \le 3$.

(b) Find dy/dx and d^2y/dx^2.

(c) Find the equation of the tangent line at the point $\left(\sqrt{3}, \frac{8}{3}\right)$.

(d) Find the length of the curve.

(e) Find the surface area generated by revolving the curve about the x-axis.

99. ***Involute of a Circle*** The involute of a circle is described by the endpoint P of a string that is held taut as it is unwound from a spool that does not turn (see figure). Show that a parametric representation of the involute is

$$x = r(\cos \theta + \theta \sin \theta) \quad \text{and} \quad y = r(\sin \theta - \theta \cos \theta).$$

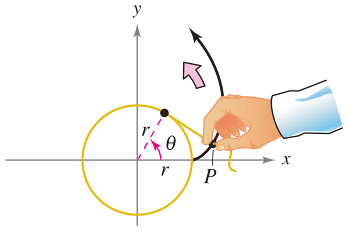


Figure for 99

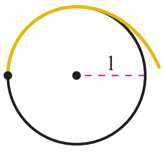


Figure for 100

100. ***Involute of a Circle*** The figure shows a piece of string tied to a circle with a radius of one unit. The string is just long enough to reach the opposite side of the circle. Find the area that is covered when the string is unwound counterclockwise.

101. (a) Use a graphing utility to graph the curve given by

$$x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}, \qquad -20 \le t \le 20.$$

(b) Describe the graph and confirm your result analytically.

(c) Discuss the speed at which the curve is traced as t increases from -20 to 20.

102. ***Tractrix*** A person moves from the origin along the positive y-axis pulling a weight at the end of a 12-meter rope. Initially, the weight is located at the point $(12, 0)$.

(a) In Exercise 96 of Section 8.7, it was shown that the path of the weight is modeled by the rectangular equation

$$y = -12 \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$$

where $0 < x \le 12$. Use a graphing utility to graph the rectangular equation.

(b) Use a graphing utility to graph the parametric equations

$$x = 12 \operatorname{sech} \frac{t}{12} \quad \text{and} \quad y = t - 12 \tanh \frac{t}{12}$$

where $t \ge 0$. How does this graph compare with the graph in part (a)? Which graph (if either) do you think is a better representation of the path?

(c) Use the parametric equations for the tractrix to verify that the distance from the y-intercept of the tangent line to the point of tangency is independent of the location of the point of tangency.

True or False? **In Exercises 103 and 104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

103. If $x = f(t)$ and $y = g(t)$, then $d^2y/dx^2 = g''(t)/f''(t)$.

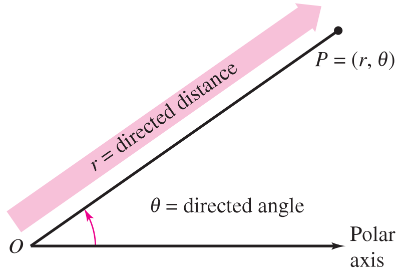
104. The curve given by $x = t^3$, $y = t^2$ has a horizontal tangent at the origin because $dy/dt = 0$ when $t = 0$.

105. ***Recording Tape*** Another method you could use to solve Example 5 is to find the area of the reel with an inner radius of 0.5 inch and an outer radius of 2 inches, and then use the formula for the area of the rectangle where the width is 0.001 inch. Use this method to determine how much tape is required to fill the reel.

10.4 Polar Coordinates and Polar Graphs

- **Understand the polar coordinate system.**
- **Rewrite rectangular coordinates and equations in polar form and vice versa.**
- **Sketch the graph of an equation given in polar form.**
- **Find the slope of a tangent line to a polar graph.**
- **Identify several types of special polar graphs.**

Polar Coordinates



Polar coordinates
Figure 10.36

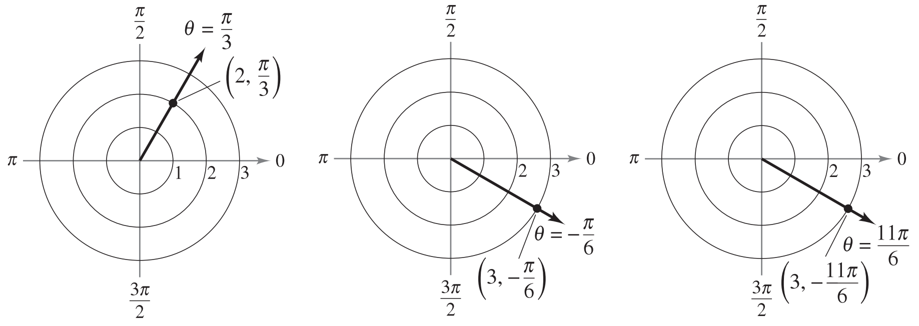
So far, you have been representing graphs as collections of points (x, y) on the rectangular coordinate system. The corresponding equations for these graphs have been in either rectangular or parametric form. In this section you will study a coordinate system called the **polar coordinate system.**

To form the polar coordinate system in the plane, fix a point O, called the **pole** (or **origin**), and construct from O an initial ray called the **polar axis,** as shown in Figure 10.36. Then each point P in the plane can be assigned **polar coordinates** (r, θ), as follows.

$r =$ *directed distance* from O to P

$\theta =$ *directed angle*, counterclockwise from polar axis to segment $\overline{OP}$

Figure 10.37 shows three points on the polar coordinate system. Notice that in this system, it is convenient to locate points with respect to a grid of concentric circles intersected by **radial lines** through the pole.



$\frac{\pi}{2}$
$\theta = \frac{\pi}{3}$
$\left(2, \frac{\pi}{3}\right)$
π
0
1
2
3
$\frac{3\pi}{2}$
$\theta = -\frac{\pi}{6}$
$\left(3, -\frac{\pi}{6}\right)$
$\theta = \frac{11\pi}{6}$
$\left(3, -\frac{11\pi}{6}\right)$

(a) **(b)** **(c)**

Figure 10.37

With rectangular coordinates, each point (x, y) has a unique representation. This is not true with polar coordinates. For instance, the coordinates (r, θ) and $(r, 2\pi + \theta)$ represent the same point [see parts (b) and (c) in Figure 10.37]. Also, because r is a *directed distance*, the coordinates (r, θ) and $(-r, \theta + \pi)$ represent the same point. In general, the point (r, θ) can be written as

$$(r, \theta) = (r, \theta + 2n\pi)$$

or

$$(r, \theta) = (-r, \theta + (2n + 1)\pi)$$

where n is any integer. Moreover, the pole is represented by $(0, \theta)$, where θ is any angle.

POLAR COORDINATES

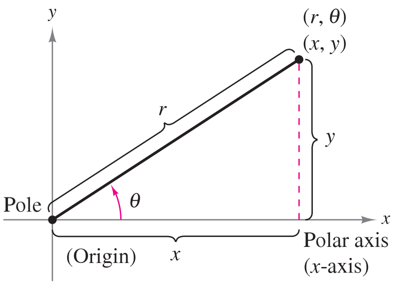
The mathematician credited with first using polar coordinates was James Bernoulli, who introduced them in 1691. However, there is some evidence that it may have been Isaac Newton who first used them.

Coordinate Conversion

To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive x-axis and the pole with the origin, as shown in Figure 10.38. Because (x, y) lies on a circle of radius r, it follows that $r^2 = x^2 + y^2$. Moreover, for $r > 0$, the definitions of the trigonometric functions imply that

$$\tan\theta = \frac{y}{x}, \qquad \cos\theta = \frac{x}{r}, \qquad \text{and} \qquad \sin\theta = \frac{y}{r}.$$

If $r < 0$, you can show that the same relationships hold.



Relating polar and rectangular coordinates
Figure 10.38

THEOREM 10.10 COORDINATE CONVERSION

The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.

1. $x = r\cos\theta$ **2.** $\tan\theta = \dfrac{y}{x}$

$y = r\sin\theta$ $r^2 = x^2 + y^2$

EXAMPLE 1 Polar-to-Rectangular Conversion

a. For the point $(r, \theta) = (2, \pi)$,

$$x = r\cos\theta = 2\cos\pi = -2 \qquad \text{and} \qquad y = r\sin\theta = 2\sin\pi = 0.$$

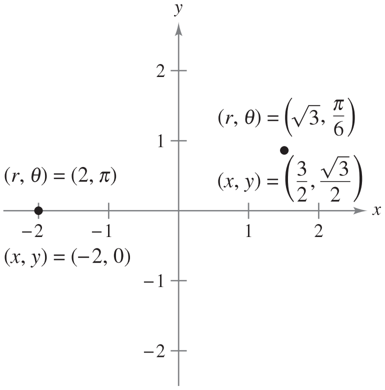
So, the rectangular coordinates are $(x, y) = (-2, 0)$.

b. For the point $(r, \theta) = \left(\sqrt{3}, \pi/6\right)$,

$$x = \sqrt{3}\cos\frac{\pi}{6} = \frac{3}{2} \qquad \text{and} \qquad y = \sqrt{3}\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

So, the rectangular coordinates are $(x, y) = \left(3/2, \sqrt{3}/2\right)$.

See Figure 10.39.



To convert from polar to rectangular coordinates, let $x = r\cos\theta$ and $y = r\sin\theta$.
Figure 10.39

EXAMPLE 2 Rectangular-to-Polar Conversion

a. For the second quadrant point $(x, y) = (-1, 1)$,

$$\tan\theta = \frac{y}{x} = -1 \quad \Rightarrow \quad \theta = \frac{3\pi}{4}.$$

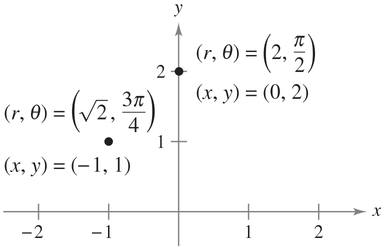
Because θ was chosen to be in the same quadrant as (x, y), you should use a positive value of r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-1)^2 + (1)^2} \\ &= \sqrt{2} \end{aligned}$$

This implies that *one* set of polar coordinates is $(r, \theta) = \left(\sqrt{2}, 3\pi/4\right)$.

b. Because the point $(x, y) = (0, 2)$ lies on the positive y-axis, choose $\theta = \pi/2$ and $r = 2$, and one set of polar coordinates is $(r, \theta) = (2, \pi/2)$.

See Figure 10.40. ■



To convert from rectangular to polar coordinates, let $\tan\theta = y/x$ and $r = \sqrt{x^2 + y^2}$.
Figure 10.40

Polar Graphs

One way to sketch the graph of a polar equation is to convert to rectangular coordinates and then sketch the graph of the rectangular equation.

EXAMPLE 3 Graphing Polar Equations

Describe the graph of each polar equation. Confirm each description by converting to a rectangular equation.

a. $r = 2$ **b.** $\theta = \dfrac{\pi}{3}$ **c.** $r = \sec\theta$

Solution

a. The graph of the polar equation $r = 2$ consists of all points that are two units from the pole. In other words, this graph is a circle centered at the origin with a radius of 2. [See Figure 10.41(a).] You can confirm this by using the relationship $r^2 = x^2 + y^2$ to obtain the rectangular equation

$$x^2 + y^2 = 2^2. \qquad \text{Rectangular equation}$$

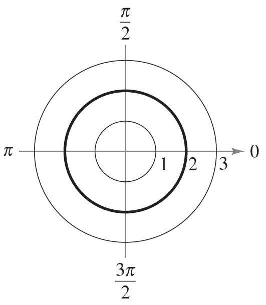
b. The graph of the polar equation $\theta = \pi/3$ consists of all points on the line that makes an angle of $\pi/3$ with the positive x-axis. [See Figure 10.41(b).] You can confirm this by using the relationship $\tan\theta = y/x$ to obtain the rectangular equation

$$y = \sqrt{3}\,x. \qquad \text{Rectangular equation}$$

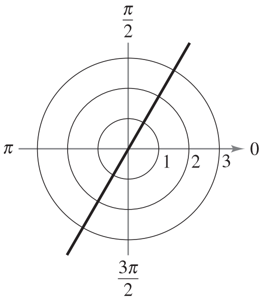
c. The graph of the polar equation $r = \sec\theta$ is not evident by simple inspection, so you can begin by converting to rectangular form using the relationship $r\cos\theta = x$.

$$\begin{aligned} r &= \sec\theta && \text{Polar equation} \\ r\cos\theta &= 1 && \\ x &= 1 && \text{Rectangular equation} \end{aligned}$$

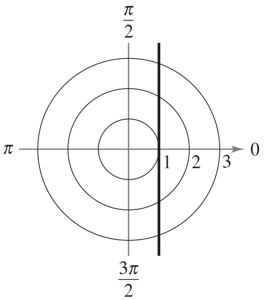
From the rectangular equation, you can see that the graph is a vertical line. [See Figure 10.41(c).] ■



(a) Circle: $r = 2$



(b) Radial line: $\theta = \dfrac{\pi}{3}$



(c) Vertical line: $r = \sec\theta$

Figure 10.41

TECHNOLOGY Sketching the graphs of complicated polar equations *by hand* can be tedious. With technology, however, the task is not difficult. If your graphing utility has a *polar* mode, use it to graph the equations in the exercise set. If your graphing utility doesn't have a *polar* mode, but does have a *parametric* mode, you can graph $r = f(\theta)$ by writing the equation as

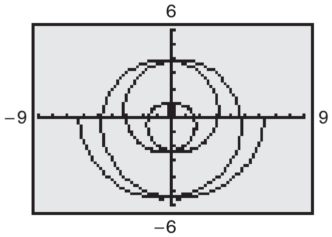
$$x = f(\theta)\cos\theta$$
$$y = f(\theta)\sin\theta.$$

For instance, the graph of $r = \frac{1}{2}\theta$ shown in Figure 10.42 was produced with a graphing calculator in *parametric* mode. This equation was graphed using the parametric equations

$$x = \frac{1}{2}\theta\cos\theta$$

$$y = \frac{1}{2}\theta\sin\theta$$

with the values of θ varying from -4π to 4π. This curve is of the form $r = a\theta$ and is called a **spiral of Archimedes.**



Spiral of Archimedes
Figure 10.42

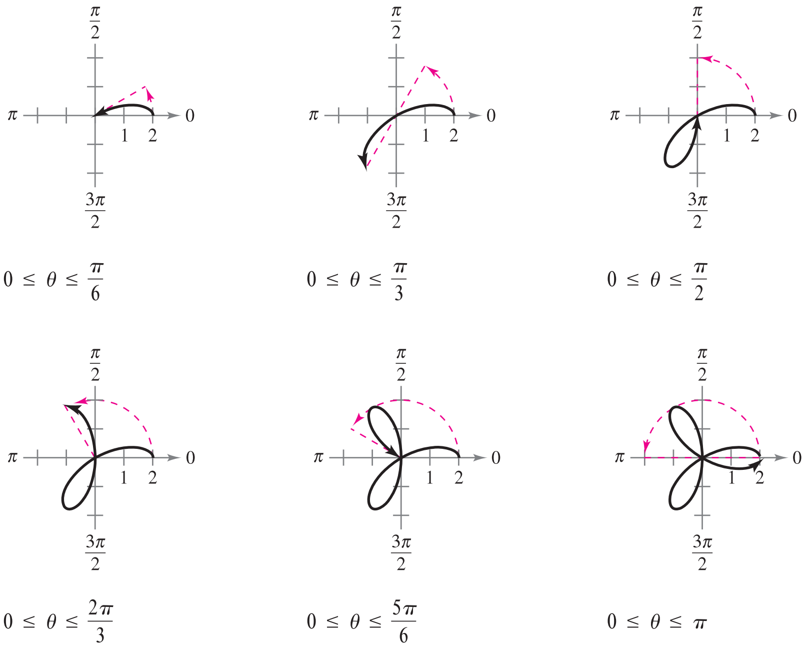
EXAMPLE 4 Sketching a Polar Graph

Sketch the graph of $r = 2 \cos 3\theta$.

Solution Begin by writing the polar equation in parametric form.

$$x = 2 \cos 3\theta \cos \theta \qquad \text{and} \qquad y = 2 \cos 3\theta \sin \theta$$

After some experimentation, you will find that the entire curve, which is called a **rose curve,** can be sketched by letting θ vary from 0 to π, as shown in Figure 10.43. If you try duplicating this graph with a graphing utility, you will find that by letting θ vary from 0 to 2π, you will actually trace the entire curve *twice*.



$0 \le \theta \le \frac{\pi}{6}$ $0 \le \theta \le \frac{\pi}{3}$ $0 \le \theta \le \frac{\pi}{2}$

$0 \le \theta \le \frac{2\pi}{3}$ $0 \le \theta \le \frac{5\pi}{6}$ $0 \le \theta \le \pi$

Figure 10.43

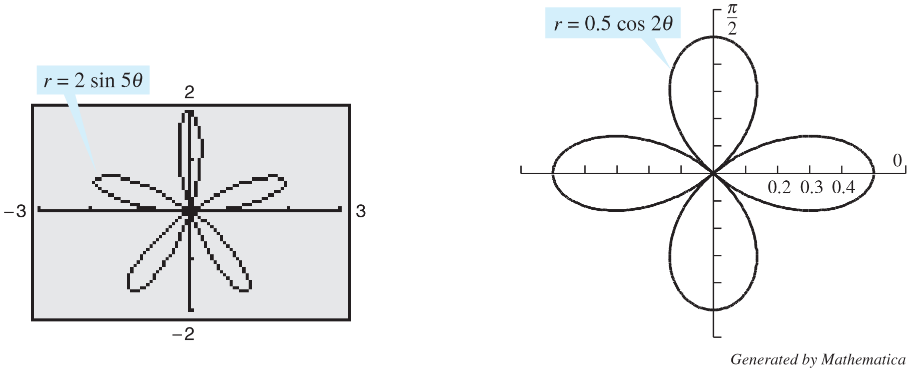
■

NOTE One way to sketch the graph of $r = 2 \cos 3\theta$ by hand is to make a table of values.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
r	2	0	-2	0	2

By extending the table and plotting the points, you will obtain the curve shown in Example 4.

Use a graphing utility to experiment with other rose curves (they are of the form $r = a \cos n\theta$ or $r = a \sin n\theta$). For instance, Figure 10.44 shows the graphs of two other rose curves.



Rose curves
Figure 10.44

Slope and Tangent Lines

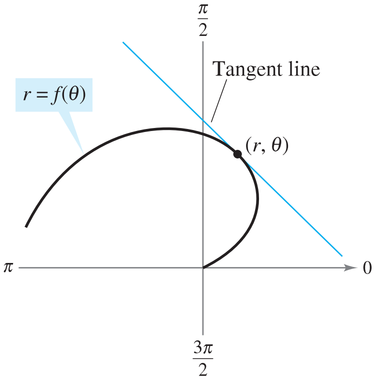
To find the slope of a tangent line to a polar graph, consider a differentiable function given by $r = f(\theta)$. To find the slope in polar form, use the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta \qquad \text{and} \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Using the parametric form of dy/dx given in Theorem 10.7, you have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta}\\ &= \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}\end{aligned}$$

which establishes the following theorem.



Tangent line to polar curve
Figure 10.45

THEOREM 10.11 SLOPE IN POLAR FORM

If f is a differentiable function of θ, then the *slope* of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

provided that $dx/d\theta \neq 0$ at (r, θ). (See Figure 10.45.)

From Theorem 10.11, you can make the following observations.

1. Solutions to $\dfrac{dy}{d\theta} = 0$ yield horizontal tangents, provided that $\dfrac{dx}{d\theta} \neq 0$.
2. Solutions to $\dfrac{dx}{d\theta} = 0$ yield vertical tangents, provided that $\dfrac{dy}{d\theta} \neq 0$.

If $dy/d\theta$ and $dx/d\theta$ are *simultaneously* 0, no conclusion can be drawn about tangent lines.

EXAMPLE 5 Finding Horizontal and Vertical Tangent Lines

Find the horizontal and vertical tangent lines of $r = \sin\theta$, $0 \leq \theta \leq \pi$.

Solution Begin by writing the equation in parametric form.

$$x = r\cos\theta = \sin\theta\cos\theta$$

and

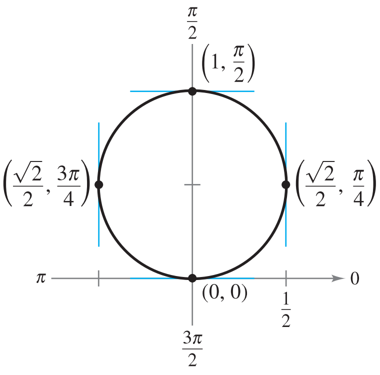
$$y = r\sin\theta = \sin\theta\sin\theta = \sin^2\theta$$

Next, differentiate x and y with respect to θ and set each derivative equal to 0.

$$\frac{dx}{d\theta} = \cos^2\theta - \sin^2\theta = \cos 2\theta = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\frac{dy}{d\theta} = 2\sin\theta\cos\theta = \sin 2\theta = 0 \quad \Rightarrow \quad \theta = 0, \frac{\pi}{2}$$

So, the graph has vertical tangent lines at $\left(\sqrt{2}/2, \pi/4\right)$ and $\left(\sqrt{2}/2, 3\pi/4\right)$, and it has horizontal tangent lines at $(0, 0)$ and $(1, \pi/2)$, as shown in Figure 10.46. ■



$\frac{\pi}{2}$
$\left(1, \frac{\pi}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$
π
0
$(0, 0)$
$\frac{1}{2}$
$\frac{3\pi}{2}$

Horizontal and vertical tangent lines of $r = \sin\theta$
Figure 10.46

EXAMPLE 6 Finding Horizontal and Vertical Tangent Lines

Find the horizontal and vertical tangents to the graph of $r = 2(1 - \cos\theta)$.

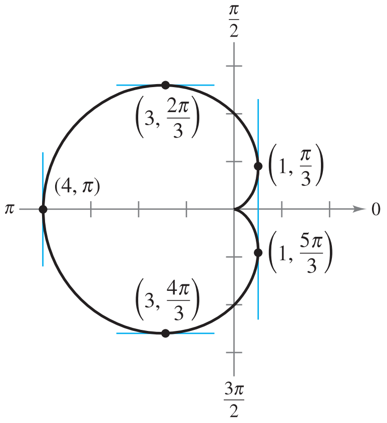
Solution Using $y = r\sin\theta$, differentiate and set $dy/d\theta$ equal to 0.

$$\begin{aligned} y &= r\sin\theta = 2(1-\cos\theta)\sin\theta \\ \frac{dy}{d\theta} &= 2[(1-\cos\theta)(\cos\theta) + \sin\theta(\sin\theta)] \\ &= -2(2\cos\theta+1)(\cos\theta-1) = 0 \end{aligned}$$

So, $\cos\theta = -\frac{1}{2}$ and $\cos\theta = 1$, and you can conclude that $dy/d\theta = 0$ when $\theta = 2\pi/3,\ 4\pi/3$, and 0. Similarly, using $x = r\cos\theta$, you have

$$\begin{aligned} x &= r\cos\theta = 2\cos\theta - 2\cos^2\theta \\ \frac{dx}{d\theta} &= -2\sin\theta + 4\cos\theta\sin\theta = 2\sin\theta(2\cos\theta - 1) = 0. \end{aligned}$$

So, $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$, and you can conclude that $dx/d\theta = 0$ when $\theta = 0$, π, $\pi/3$, and $5\pi/3$. From these results, and from the graph shown in Figure 10.47, you can conclude that the graph has horizontal tangents at $(3, 2\pi/3)$ and $(3, 4\pi/3)$, and has vertical tangents at $(1, \pi/3)$, $(1, 5\pi/3)$, and $(4, \pi)$. This graph is called a **cardioid**. Note that both derivatives $(dy/d\theta$ and $dx/d\theta)$ are 0 when $\theta = 0$. Using this information alone, you don't know whether the graph has a horizontal or vertical tangent line at the pole. From Figure 10.47, however, you can see that the graph has a cusp at the pole. ■



Horizontal and vertical tangent lines of $r = 2(1 - \cos\theta)$
Figure 10.47

Theorem 10.11 has an important consequence. Suppose the graph of $r = f(\theta)$ passes through the pole when $\theta = \alpha$ and $f'(\alpha) \neq 0$. Then the formula for dy/dx simplifies as follows.

$$\frac{dy}{dx} = \frac{f'(\alpha)\sin\alpha + f(\alpha)\cos\alpha}{f'(\alpha)\cos\alpha - f(\alpha)\sin\alpha} = \frac{f'(\alpha)\sin\alpha + 0}{f'(\alpha)\cos\alpha - 0} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha$$

So, the line $\theta = \alpha$ is tangent to the graph at the pole, $(0, \alpha)$.

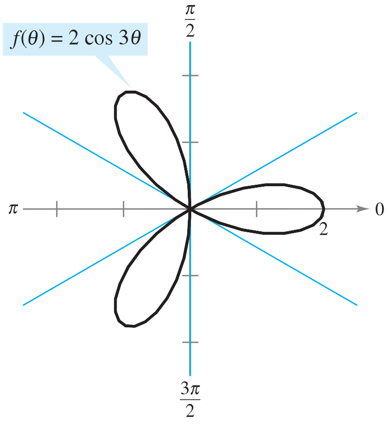
THEOREM 10.12 TANGENT LINES AT THE POLE

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$.

Theorem 10.12 is useful because it states that the zeros of $r = f(\theta)$ can be used to find the tangent lines at the pole. Note that because a polar curve can cross the pole more than once, it can have more than one tangent line at the pole. For example, the rose curve

$$f(\theta) = 2\cos 3\theta$$

has three tangent lines at the pole, as shown in Figure 10.48. For this curve, $f(\theta) = 2\cos 3\theta$ is 0 when θ is $\pi/6$, $\pi/2$, and $5\pi/6$. Moreover, the derivative $f'(\theta) = -6\sin 3\theta$ is not 0 for these values of θ.



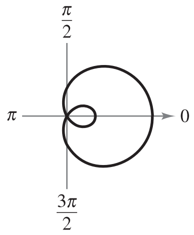
This rose curve has three tangent lines $(\theta = \pi/6,\ \theta = \pi/2$, and $\theta = 5\pi/6)$ at the pole.
Figure 10.48

Special Polar Graphs

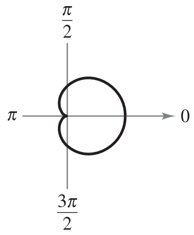
Several important types of graphs have equations that are simpler in polar form than in rectangular form. For example, the polar equation of a circle having a radius of a and centered at the origin is simply $r = a$. Later in the text you will come to appreciate this benefit. For now, several other types of graphs that have simpler equations in polar form are shown below. (Conics are considered in Section 10.6.)

Limaçons

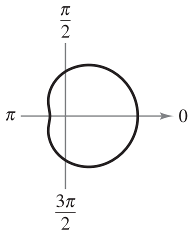
$r = a \pm b \cos \theta$
$r = a \pm b \sin \theta$
$(a > 0, b > 0)$



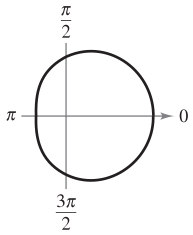
$\frac{a}{b} < 1$
Limaçon with inner loop



$\frac{a}{b} = 1$
Cardioid (heart-shaped)



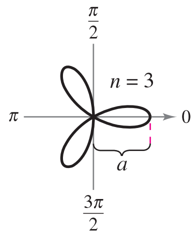
$1 < \frac{a}{b} < 2$
Dimpled limaçon



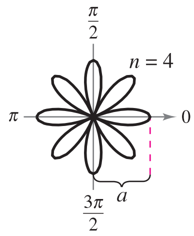
$\frac{a}{b} \geq 2$
Convex limaçon

Rose Curves

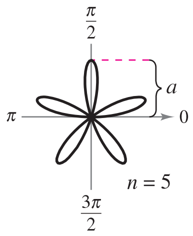
n petals if n is odd
$2n$ petals if n is even
$(n \geq 2)$



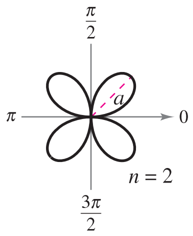
$r = a \cos n\theta$
Rose curve



$r = a \cos n\theta$
Rose curve

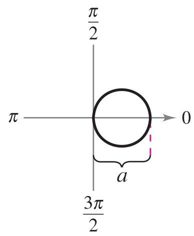


$r = a \sin n\theta$
Rose curve

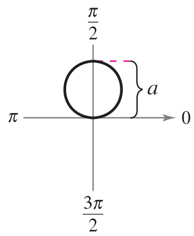


$r = a \sin n\theta$
Rose curve

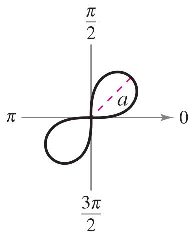
Circles and Lemniscates



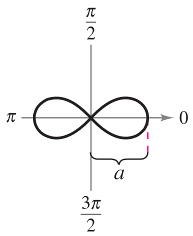
$r = a \cos \theta$
Circle



$r = a \sin \theta$
Circle

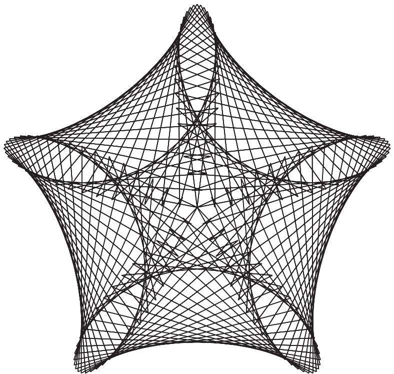


$r^2 = a^2 \sin 2\theta$
Lemniscate



$r^2 = a^2 \cos 2\theta$
Lemniscate

TECHNOLOGY The rose curves described above are of the form $r = a \cos n\theta$ or $r = a \sin n\theta$, where n is a positive integer that is greater than or equal to 2. Use a graphing utility to graph $r = a \cos n\theta$ or $r = a \sin n\theta$ for some noninteger values of n. Are these graphs also rose curves? For example, try sketching the graph of $r = \cos \frac{2}{3}\theta,\ 0 \leq \theta \leq 6\pi$.



Generated by Maple

FOR FURTHER INFORMATION For more information on rose curves and related curves, see the article "A Rose is a Rose . . ." by Peter M. Maurer in *The American Mathematical Monthly*. The computer-generated graph at the left is the result of an algorithm that Maurer calls "The Rose." To view this article, go to the website *www.matharticles.com*.

10.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, plot the point in polar coordinates and find the corresponding rectangular coordinates for the point.

1. $(8, \pi/2)$

2. $(-2, 5\pi/3)$

3. $(-4, -3\pi/4)$

4. $(0, -7\pi/6)$

5. $(\sqrt{2}, 2.36)$

6. $(-3, -1.57)$

In Exercises 7–10, use the *angle* feature of a graphing utility to find the rectangular coordinates for the point given in polar coordinates. Plot the point.

7. $(7, 5\pi/4)$

8. $(-2, 11\pi/6)$

9. $(-4.5, 3.5)$

10. $(9.25, 1.2)$

In Exercises 11–16, the rectangular coordinates of a point are given. Plot the point and find *two* sets of polar coordinates for the point for $0 \le \theta < 2\pi$.

11. $(2, 2)$

12. $(0, -6)$

13. $(-3, 4)$

14. $(4, -2)$

15. $(-1, -\sqrt{3})$

16. $(3, -\sqrt{3})$

In Exercises 17–20, use the *angle* feature of a graphing utility to find one set of polar coordinates for the point given in rectangular coordinates.

17. $(3, -2)$

18. $(3\sqrt{2}, 3\sqrt{2})$

19. $\left(\frac{7}{4}, \frac{5}{2}\right)$

20. $(0, -5)$

21. Plot the point $(4, 3.5)$ if the point is given in (a) rectangular coordinates and (b) polar coordinates.

22. ***Graphical Reasoning***

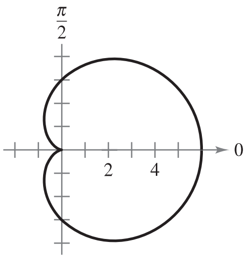
(a) Set the window format of a graphing utility to rectangular coordinates and locate the cursor at any position off the axes. Move the cursor horizontally and vertically. Describe any changes in the displayed coordinates of the points.

(b) Set the window format of a graphing utility to polar coordinates and locate the cursor at any position off the axes. Move the cursor horizontally and vertically. Describe any changes in the displayed coordinates of the points.

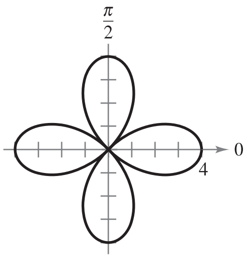
(c) Why are the results in parts (a) and (b) different?

In Exercises 23–26, match the graph with its polar equation. [The graphs are labeled (a), (b), (c), and (d).]

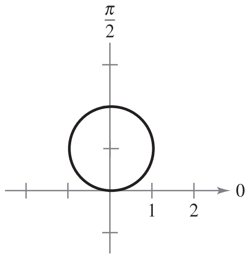
(a)



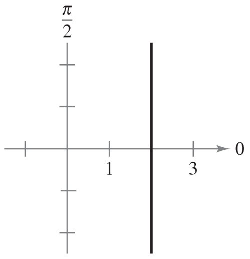
(b)



(c)



(d)



23. $r = 2\sin\theta$

24. $r = 4\cos 2\theta$

25. $r = 3(1 + \cos\theta)$

26. $r = 2\sec\theta$

In Exercises 27–36, convert the rectangular equation to polar form and sketch its graph.

27. $x^2 + y^2 = 9$

28. $x^2 - y^2 = 9$

29. $x^2 + y^2 = a^2$

30. $x^2 + y^2 - 2ax = 0$

31. $y = 8$

32. $x = 12$

33. $3x - y + 2 = 0$

34. $xy = 4$

35. $y^2 = 9x$

36. $(x^2 + y^2)^2 - 9(x^2 - y^2) = 0$

In Exercises 37–46, convert the polar equation to rectangular form and sketch its graph.

37. $r = 4$

38. $r = -5$

39. $r = 3\sin\theta$

40. $r = 5\cos\theta$

41. $r = \theta$

42. $\theta = \dfrac{5\pi}{6}$

43. $r = 3\sec\theta$

44. $r = 2\csc\theta$

45. $r = \sec\theta\tan\theta$

46. $r = \cot\theta\csc\theta$

In Exercises 47–56, use a graphing utility to graph the polar equation. Find an interval for θ over which the graph is traced *only once*.

47. $r = 2 - 5\cos\theta$

48. $r = 3(1 - 4\cos\theta)$

49. $r = 2 + \sin\theta$

50. $r = 4 + 3\cos\theta$

51. $r = \dfrac{2}{1 + \cos\theta}$

52. $r = \dfrac{2}{4 - 3\sin\theta}$

53. $r = 2\cos\left(\dfrac{3\theta}{2}\right)$

54. $r = 3\sin\left(\dfrac{5\theta}{2}\right)$

55. $r^2 = 4\sin 2\theta$

56. $r^2 = \dfrac{1}{\theta}$

57. Convert the equation

$$r = 2(h\cos\theta + k\sin\theta)$$

to rectangular form and verify that it is the equation of a circle. Find the radius and the rectangular coordinates of the center of the circle.

58. ***Distance Formula***

(a) Verify that the Distance Formula for the distance between the two points (r_1, θ_1) and (r_2, θ_2) in polar coordinates is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}.$$

(b) Describe the positions of the points relative to each other for $\theta_1 = \theta_2$. Simplify the Distance Formula for this case. Is the simplification what you expected? Explain.

(c) Simplify the Distance Formula for $\theta_1 - \theta_2 = 90°$. Is the simplification what you expected? Explain.

(d) Choose two points on the polar coordinate system and find the distance between them. Then choose different polar representations of the same two points and apply the Distance Formula again. Discuss the result.

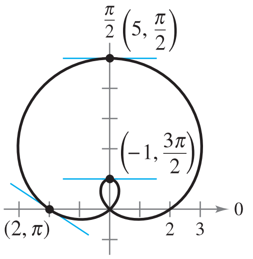
In Exercises 59–62, use the result of Exercise 58 to approximate the distance between the two points in polar coordinates.

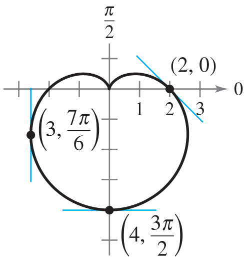
59. $\left(1, \frac{5\pi}{6}\right)$, $\left(4, \frac{\pi}{3}\right)$ **60.** $\left(8, \frac{7\pi}{4}\right)$, $(5, \pi)$

61. $(2, 0.5)$, $(7, 1.2)$ **62.** $(4, 2.5)$, $(12, 1)$

In Exercises 63 and 64, find *dy/dx* and the slopes of the tangent lines shown on the graph of the polar equation.

63. $r = 2 + 3\sin\theta$ **64.** $r = 2(1 - \sin\theta)$





In Exercises 65–68, use a graphing utility to (a) graph the polar equation, (b) draw the tangent line at the given value of *θ*, and (c) find *dy/dx* at the given value of *θ*. (*Hint:* Let the increment between the values of *θ* equal $\pi/24$.)

65. $r = 3(1 - \cos\theta)$, $\theta = \frac{\pi}{2}$ **66.** $r = 3 - 2\cos\theta$, $\theta = 0$

67. $r = 3\sin\theta$, $\theta = \frac{\pi}{3}$ **68.** $r = 4$, $\theta = \frac{\pi}{4}$

In Exercises 69 and 70, find the points of horizontal and vertical tangency (if any) to the polar curve.

69. $r = 1 - \sin\theta$ **70.** $r = a\sin\theta$

In Exercises 71 and 72, find the points of horizontal tangency (if any) to the polar curve.

71. $r = 2\csc\theta + 3$ **72.** $r = a\sin\theta\cos^2\theta$

In Exercises 73–76, use a graphing utility to graph the polar equation and find all points of horizontal tangency.

73. $r = 4\sin\theta\cos^2\theta$ **74.** $r = 3\cos 2\theta\sec\theta$

75. $r = 2\csc\theta + 5$ **76.** $r = 2\cos(3\theta - 2)$

In Exercises 77–84, sketch a graph of the polar equation and find the tangents at the pole.

77. $r = 5\sin\theta$ **78.** $r = 5\cos\theta$

79. $r = 2(1 - \sin\theta)$ **80.** $r = 3(1 - \cos\theta)$

81. $r = 4\cos 3\theta$ **82.** $r = -\sin 5\theta$

83. $r = 3\sin 2\theta$ **84.** $r = 3\cos 2\theta$

In Exercises 85–96, sketch a graph of the polar equation.

85. $r = 8$ **86.** $r = 1$

87. $r = 4(1 + \cos\theta)$ **88.** $r = 1 + \sin\theta$

89. $r = 3 - 2\cos\theta$ **90.** $r = 5 - 4\sin\theta$

91. $r = 3\csc\theta$ **92.** $r = \dfrac{6}{2\sin\theta - 3\cos\theta}$

93. $r = 2\theta$ **94.** $r = \dfrac{1}{\theta}$

95. $r^2 = 4\cos 2\theta$ **96.** $r^2 = 4\sin\theta$

In Exercises 97–100, use a graphing utility to graph the equation and show that the given line is an asymptote of the graph.

	Name of Graph	*Polar Equation*	*Asymptote*
97.	Conchoid	$r = 2 - \sec\theta$	$x = -1$
98.	Conchoid	$r = 2 + \csc\theta$	$y = 1$
99.	Hyperbolic spiral	$r = 2/\theta$	$y = 2$
100.	Strophoid	$r = 2\cos 2\theta\sec\theta$	$x = -2$

WRITING ABOUT CONCEPTS

101. Describe the differences between the rectangular coordinate system and the polar coordinate system.

102. Give the equations for the coordinate conversion from rectangular to polar coordinates and vice versa.

103. How are the slopes of tangent lines determined in polar coordinates? What are tangent lines at the pole and how are they determined?

CAPSTONE

104. Describe the graphs of the following polar equations.

(a) $r = 7$ (b) $r^2 = 7$

(c) $r = \dfrac{7}{\cos\theta}$ (d) $r = \dfrac{7}{\sin\theta}$

(e) $r = 7\cos\theta$ (f) $r = 7\sin\theta$

105. Sketch the graph of $r = 4\sin\theta$ over each interval.

(a) $0 \le \theta \le \frac{\pi}{2}$ (b) $\frac{\pi}{2} \le \theta \le \pi$ (c) $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

106. ***Think About It*** Use a graphing utility to graph the polar equation $r = 6[1 + \cos(\theta - \phi)]$ for (a) $\phi = 0$, (b) $\phi = \pi/4$, and (c) $\phi = \pi/2$. Use the graphs to describe the effect of the angle ϕ. Write the equation as a function of $\sin\theta$ for part (c).

107. Verify that if the curve whose polar equation is $r = f(\theta)$ is rotated about the pole through an angle ϕ, then an equation for the rotated curve is $r = f(\theta - \phi)$.

108. The polar form of an equation of a curve is $r = f(\sin\theta)$. Show that the form becomes

(a) $r = f(-\cos\theta)$ if the curve is rotated counterclockwise $\pi/2$ radians about the pole.

(b) $r = f(-\sin\theta)$ if the curve is rotated counterclockwise π radians about the pole.

(c) $r = f(\cos\theta)$ if the curve is rotated counterclockwise $3\pi/2$ radians about the pole.

In Exercises 109–112, use the results of Exercises 107 and 108.

109. Write an equation for the limaçon $r = 2 - \sin\theta$ after it has been rotated by the given amount. Use a graphing utility to graph the rotated limaçon.

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2}$ (c) π (d) $\dfrac{3\pi}{2}$

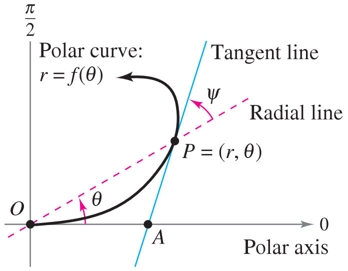
110. Write an equation for the rose curve $r = 2\sin 2\theta$ after it has been rotated by the given amount. Verify the results by using a graphing utility to graph the rotated rose curve.

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{2\pi}{3}$ (d) π

111. Sketch the graph of each equation.

(a) $r = 1 - \sin\theta$ (b) $r = 1 - \sin\left(\theta - \dfrac{\pi}{4}\right)$

112. Prove that the tangent of the angle ψ $(0 \le \psi \le \pi/2)$ between the radial line and the tangent line at the point (r, θ) on the graph of $r = f(\theta)$ (see figure) is given by $\tan\psi = |r/(dr/d\theta)|$.



In Exercises 113–118, use the result of Exercise 112 to find the angle ψ between the radial and tangent lines to the graph for the indicated value of θ. Use a graphing utility to graph the polar equation, the radial line, and the tangent line for the indicated value of θ. Identify the angle ψ.

	Polar Equation	*Value of θ*
113.	$r = 2(1 - \cos\theta)$	$\theta = \pi$
114.	$r = 3(1 - \cos\theta)$	$\theta = 3\pi/4$
115.	$r = 2\cos 3\theta$	$\theta = \pi/4$
116.	$r = 4\sin 2\theta$	$\theta = \pi/6$
117.	$r = \dfrac{6}{1 - \cos\theta}$	$\theta = 2\pi/3$
118.	$r = 5$	$\theta = \pi/6$

True or False? **In Exercises 119–122, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

119. If (r_1, θ_1) and (r_2, θ_2) represent the same point on the polar coordinate system, then $|r_1| = |r_2|$.

120. If (r, θ_1) and (r, θ_2) represent the same point on the polar coordinate system, then $\theta_1 = \theta_2 + 2\pi n$ for some integer n.

121. If $x > 0$, then the point (x, y) on the rectangular coordinate system can be represented by (r, θ) on the polar coordinate system, where $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$.

122. The polar equations $r = \sin 2\theta$, $r = -\sin 2\theta$, and $r = \sin(-2\theta)$ all have the same graph.

SECTION PROJECT

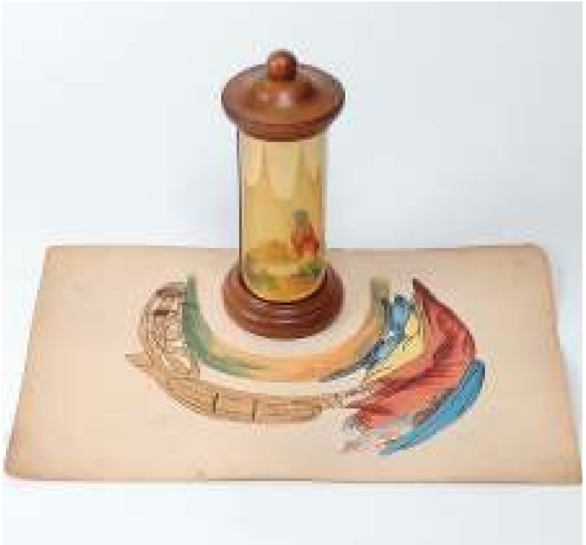
Anamorphic Art

Anamorphic art appears distorted, but when the art is viewed from a particular point or is viewed with a device such as a mirror, it appears to be normal. Use the anamorphic transformations

$$r = y + 16 \qquad \text{and} \qquad \theta = -\frac{\pi}{8}x, \qquad -\frac{3\pi}{4} \le \theta \le \frac{3\pi}{4}$$

to sketch the transformed polar image of the rectangular graph. When the reflection (in a cylindrical mirror centered at the pole) of each polar image is viewed from the polar axis, the viewer will see the original rectangular image.

(a) $y = 3$ (b) $x = 2$ (c) $y = x + 5$ (d) $x^2 + (y - 5)^2 = 5^2$



From the Millington-Barnard Collection of Scientific Apparatus, ca 1855
The University of Mississippi Museum, Oxford, Mississippi

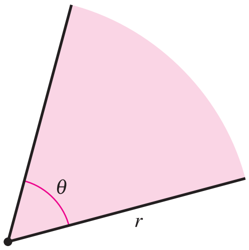
This example of anamorphic art is from the Millington-Barnard Collection at the University of Mississippi. When the reflection of the transformed "polar painting" is viewed in the mirror, the viewer sees the distorted art in its proper proportions.

■ **FOR FURTHER INFORMATION** For more information on anamorphic art, see the article "Anamorphisms" by Philip Hickin in the *Mathematical Gazette*.

10.5 Area and Arc Length in Polar Coordinates

- Find the area of a region bounded by a polar graph.
- Find the points of intersection of two polar graphs.
- Find the arc length of a polar graph.
- Find the area of a surface of revolution (polar form).

Area of a Polar Region



The area of a sector of a circle is $A = \frac{1}{2}\theta r^2$.
Figure 10.49

The development of a formula for the area of a polar region parallels that for the area of a region on the rectangular coordinate system, but uses sectors of a circle instead of rectangles as the basic elements of area. In Figure 10.49, note that the area of a circular sector of radius r is given by $\frac{1}{2}\theta r^2$, provided θ is measured in radians.

Consider the function given by $r = f(\theta)$, where f is continuous and nonnegative on the interval given by $\alpha \le \theta \le \beta$. The region bounded by the graph of f and the radial lines $\theta = \alpha$ and $\theta = \beta$ is shown in Figure 10.50(a). To find the area of this region, partition the interval $[\alpha, \beta]$ into n equal subintervals

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdot \cdot \cdot < \theta_{n-1} < \theta_n = \beta.$$

Then approximate the area of the region by the sum of the areas of the n sectors, as shown in Figure 10.50(b).

$$\text{Radius of } i\text{th sector} = f(\theta_i)$$

$$\text{Central angle of } i\text{th sector} = \frac{\beta - \alpha}{n} = \Delta\theta$$

$$A \approx \sum_{i=1}^{n} \left(\frac{1}{2}\right) \Delta\theta [f(\theta_i)]^2$$

Taking the limit as $n \to \infty$ produces

$$\begin{aligned} A &= \lim_{n\to\infty} \frac{1}{2} \sum_{i=1}^{n} [f(\theta_i)]^2 \, \Delta\theta \\ &= \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta \end{aligned}$$

which leads to the following theorem.

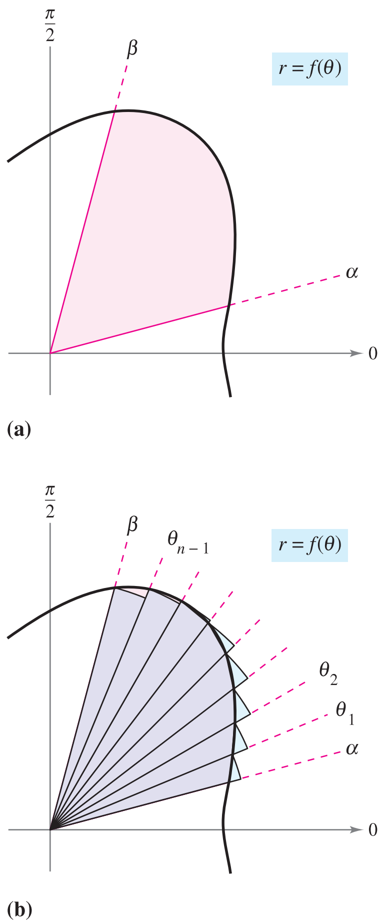


Figure 10.50

THEOREM 10.13 AREA IN POLAR COORDINATES

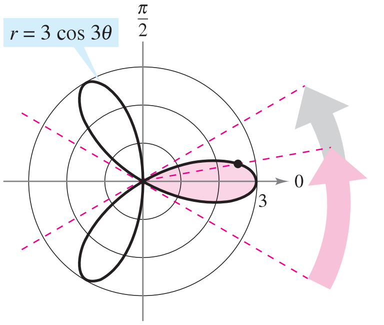
If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \le 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$\begin{aligned} A &= \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta \\ &= \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta. \qquad 0 < \beta - \alpha \le 2\pi \end{aligned}$$

NOTE You can use the same formula to find the area of a region bounded by the graph of a continuous *nonpositive* function. However, the formula is not necessarily valid if f takes on both positive *and* negative values in the interval $[\alpha, \beta]$. ■

EXAMPLE 1 Finding the Area of a Polar Region

Find the area of one petal of the rose curve given by $r = 3\cos 3\theta$.



The area of one petal of the rose curve that lies between the radial lines $\theta = -\pi/6$ and $\theta = \pi/6$ is $3\pi/4$.
Figure 10.51

Solution In Figure 10.51, you can see that the petal on the right is traced as θ increases from $-\pi/6$ to $\pi/6$. So, the area is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} (3\cos 3\theta)^2\,d\theta \qquad \text{Formula for area in polar coordinates}$$

$$= \frac{9}{2}\int_{-\pi/6}^{\pi/6} \frac{1+\cos 6\theta}{2}\,d\theta \qquad \text{Trigonometric identity}$$

$$= \frac{9}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_{-\pi/6}^{\pi/6}$$

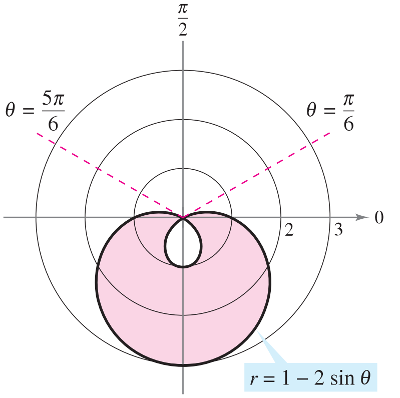
$$= \frac{9}{4}\left(\frac{\pi}{6} + \frac{\pi}{6}\right) = \frac{3\pi}{4}.$$

■

NOTE To find the area of the region lying inside all three petals of the rose curve in Example 1, you could not simply integrate between 0 and 2π. In doing this you would obtain $9\pi/2$, which is twice the area of the three petals. The duplication occurs because the rose curve is traced twice as θ increases from 0 to 2π. ■

EXAMPLE 2 Finding the Area Bounded by a Single Curve

Find the area of the region lying between the inner and outer loops of the limaçon $r = 1 - 2\sin\theta$.



The area between the inner and outer loops is approximately 8.34.
Figure 10.52

Solution In Figure 10.52, note that the inner loop is traced as θ increases from $\pi/6$ to $5\pi/6$. So, the area inside the *inner loop* is

$$A_1 = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 - 2\sin\theta)^2\,d\theta \qquad \text{Formula for area in polar coordinates}$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 - 4\sin\theta + 4\sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[1 - 4\sin\theta + 4\left(\frac{1-\cos 2\theta}{2}\right)\right]d\theta \qquad \text{Trigonometric identity}$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3 - 4\sin\theta - 2\cos 2\theta)\,d\theta \qquad \text{Simplify.}$$

$$= \frac{1}{2}\Big[3\theta + 4\cos\theta - \sin 2\theta\Big]_{\pi/6}^{5\pi/6}$$

$$= \frac{1}{2}\left(2\pi - 3\sqrt{3}\right)$$

$$= \pi - \frac{3\sqrt{3}}{2}.$$

In a similar way, you can integrate from $5\pi/6$ to $13\pi/6$ to find that the area of the region lying inside the *outer loop* is $A_2 = 2\pi + \left(3\sqrt{3}/2\right)$. The area of the region lying between the two loops is the difference of A_2 and A_1.

$$A = A_2 - A_1 = \left(2\pi + \frac{3\sqrt{3}}{2}\right) - \left(\pi - \frac{3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3} \approx 8.34$$

■

Points of Intersection of Polar Graphs

Because a point may be represented in different ways in polar coordinates, care must be taken in determining the points of intersection of two polar graphs. For example, consider the points of intersection of the graphs of

$$r = 1 - 2\cos\theta \quad \text{and} \quad r = 1$$

as shown in Figure 10.53. If, as with rectangular equations, you attempted to find the points of intersection by solving the two equations simultaneously, you would obtain

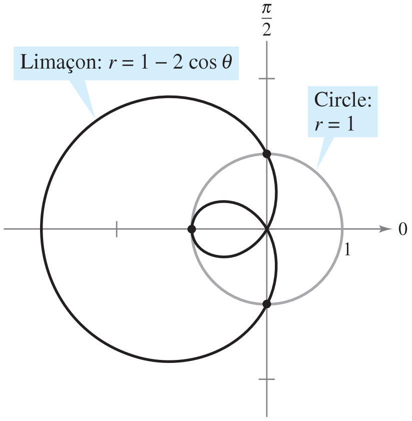
$$r = 1 - 2\cos\theta \qquad \text{First equation}$$
$$1 = 1 - 2\cos\theta \qquad \text{Substitute } r = 1 \text{ from 2nd equation into 1st equation.}$$
$$\cos\theta = 0 \qquad \text{Simplify.}$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}. \qquad \text{Solve for } \theta.$$

The corresponding points of intersection are $(1, \pi/2)$ and $(1, 3\pi/2)$. However, from Figure 10.53 you can see that there is a *third* point of intersection that did not show up when the two polar equations were solved simultaneously. (This is one reason why you should sketch a graph when finding the area of a polar region.) The reason the third point was not found is that it does not occur with the same coordinates in the two graphs. On the graph of $r = 1$, the point occurs with coordinates $(1, \pi)$, but on the graph of $r = 1 - 2\cos\theta$, the point occurs with coordinates $(-1, 0)$.

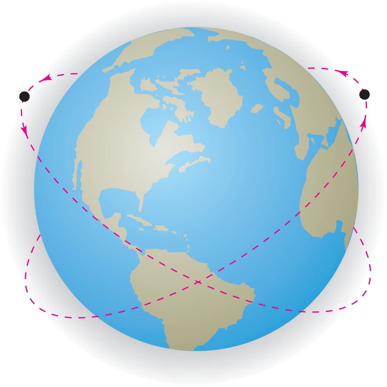
You can compare the problem of finding points of intersection of two polar graphs with that of finding collision points of two satellites in intersecting orbits about Earth, as shown in Figure 10.54. The satellites will not collide as long as they reach the points of intersection at different times (θ-values). Collisions will occur only at the points of intersection that are "simultaneous points"—those reached at the same time (θ-value).

FOR FURTHER INFORMATION For more information on using technology to find points of intersection, see the article "Finding Points of Intersection of Polar-Coordinate Graphs" by Warren W. Esty in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

NOTE Because the pole can be represented by $(0, \theta)$, where θ is *any* angle, you should check separately for the pole when finding points of intersection. ■



Three points of intersection: $(1, \pi/2)$, $(-1, 0)$, $(1, 3\pi/2)$
Figure 10.53



The paths of satellites can cross without causing a collision.
Figure 10.54

EXAMPLE 3 Finding the Area of a Region Between Two Curves

Find the area of the region common to the two regions bounded by the following curves.

$$r = -6\cos\theta \qquad \text{Circle}$$

$$r = 2 - 2\cos\theta \qquad \text{Cardioid}$$

Solution Because both curves are symmetric with respect to the x-axis, you can work with the upper half-plane, as shown in Figure 10.55. The gray shaded region lies between the circle and the radial line $\theta = 2\pi/3$. Because the circle has coordinates $(0, \pi/2)$ at the pole, you can integrate between $\pi/2$ and $2\pi/3$ to obtain the area of this region. The region that is shaded red is bounded by the radial lines $\theta = 2\pi/3$ and $\theta = \pi$ and the cardioid. So, you can find the area of this second region by integrating between $2\pi/3$ and π. The sum of these two integrals gives the area of the common region lying *above* the radial line $\theta = \pi$.

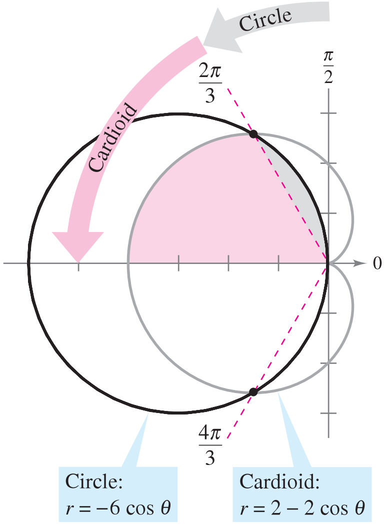


Figure 10.55

$$\frac{A}{2} = \overbrace{\frac{1}{2}\int_{\pi/2}^{2\pi/3} (-6\cos\theta)^2\,d\theta}^{\text{Region between circle and radial line } \theta = 2\pi/3} + \overbrace{\frac{1}{2}\int_{2\pi/3}^{\pi} (2 - 2\cos\theta)^2\,d\theta}^{\text{Region between cardioid and radial lines } \theta = 2\pi/3 \text{ and } \theta = \pi}$$

$$\begin{aligned}
&= 18\int_{\pi/2}^{2\pi/3} \cos^2\theta\,d\theta + \frac{1}{2}\int_{2\pi/3}^{\pi} (4 - 8\cos\theta + 4\cos^2\theta)\,d\theta \\
&= 9\int_{\pi/2}^{2\pi/3} (1 + \cos 2\theta)\,d\theta + \int_{2\pi/3}^{\pi} (3 - 4\cos\theta + \cos 2\theta)\,d\theta \\
&= 9\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/2}^{2\pi/3} + \left[3\theta - 4\sin\theta + \frac{\sin 2\theta}{2}\right]_{2\pi/3}^{\pi} \\
&= 9\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{2}\right) + \left(3\pi - 2\pi + 2\sqrt{3} + \frac{\sqrt{3}}{4}\right) \\
&= \frac{5\pi}{2} \\
&\approx 7.85
\end{aligned}$$

Finally, multiplying by 2, you can conclude that the total area is 5π. ■

NOTE To check the reasonableness of the result obtained in Example 3, note that the area of the circular region is $\pi r^2 = 9\pi$. So, it seems reasonable that the area of the region lying inside the circle and the cardioid is 5π. ■

To see the benefit of polar coordinates for finding the area in Example 3, consider the following integral, which gives the comparable area in rectangular coordinates.

$$\frac{A}{2} = \int_{-4}^{-3/2} \sqrt{2\sqrt{1 - 2x} - x^2 - 2x + 2}\,dx + \int_{-3/2}^{0} \sqrt{-x^2 - 6x}\,dx$$

Use the integration capabilities of a graphing utility to show that you obtain the same area as that found in Example 3.

Arc Length in Polar Form

The formula for the length of a polar arc can be obtained from the arc length formula for a curve described by parametric equations. (See Exercise 89.)

NOTE When applying the arc length formula to a polar curve, be sure that the curve is traced out only once on the interval of integration. For instance, the rose curve given by $r = \cos 3\theta$ is traced out once on the interval $0 \le \theta \le \pi$, but is traced out twice on the interval $0 \le \theta \le 2\pi$.

THEOREM 10.14 ARC LENGTH OF A POLAR CURVE

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

EXAMPLE 4 Finding the Length of a Polar Curve

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid

$$r = f(\theta) = 2 - 2\cos\theta$$

as shown in Figure 10.56.

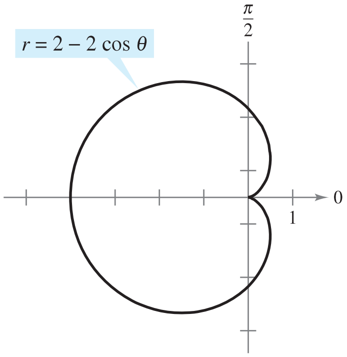


Figure 10.56

Solution Because $f'(\theta) = 2\sin\theta$, you can find the arc length as follows.

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \qquad \text{Formula for arc length of a polar curve}$$

$$= \int_{0}^{2\pi} \sqrt{(2 - 2\cos\theta)^2 + (2\sin\theta)^2}\, d\theta$$

$$= 2\sqrt{2}\int_{0}^{2\pi} \sqrt{1 - \cos\theta}\, d\theta \qquad \text{Simplify.}$$

$$= 2\sqrt{2}\int_{0}^{2\pi} \sqrt{2\sin^2\frac{\theta}{2}}\, d\theta \qquad \text{Trigonometric identity}$$

$$= 4\int_{0}^{2\pi} \sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \ge 0 \text{ for } 0 \le \theta \le 2\pi$$

$$= 8\left[-\cos\frac{\theta}{2}\right]_{0}^{2\pi}$$

$$= 8(1 + 1)$$

$$= 16$$

In the fifth step of the solution, it is legitimate to write

$$\sqrt{2\sin^2(\theta/2)} = \sqrt{2}\sin(\theta/2)$$

rather than

$$\sqrt{2\sin^2(\theta/2)} = \sqrt{2}\,|\sin(\theta/2)|$$

because $\sin(\theta/2) \ge 0$ for $0 \le \theta \le 2\pi$. ■

NOTE Using Figure 10.56, you can determine the reasonableness of this answer by comparing it with the circumference of a circle. For example, a circle of radius $\frac{5}{2}$ has a circumference of $5\pi \approx 15.7$. ■

Area of a Surface of Revolution

The polar coordinate versions of the formulas for the area of a surface of revolution can be obtained from the parametric versions given in Theorem 10.9, using the equations $x = r\cos\theta$ and $y = r\sin\theta$.

NOTE When using Theorem 10.15, check to see that the graph of $r = f(\theta)$ is traced only once on the interval $\alpha \le \theta \le \beta$. For example, the circle given by $r = \cos\theta$ is traced only once on the interval $0 \le \theta \le \pi$.

THEOREM 10.15 AREA OF A SURFACE OF REVOLUTION

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line is as follows.

1. $S = 2\pi \int_\alpha^\beta f(\theta)\sin\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$ About the polar axis

2. $S = 2\pi \int_\alpha^\beta f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$ About the line $\theta = \frac{\pi}{2}$

EXAMPLE 5 Finding the Area of a Surface of Revolution

Find the area of the surface formed by revolving the circle $r = f(\theta) = \cos\theta$ about the line $\theta = \pi/2$, as shown in Figure 10.57.

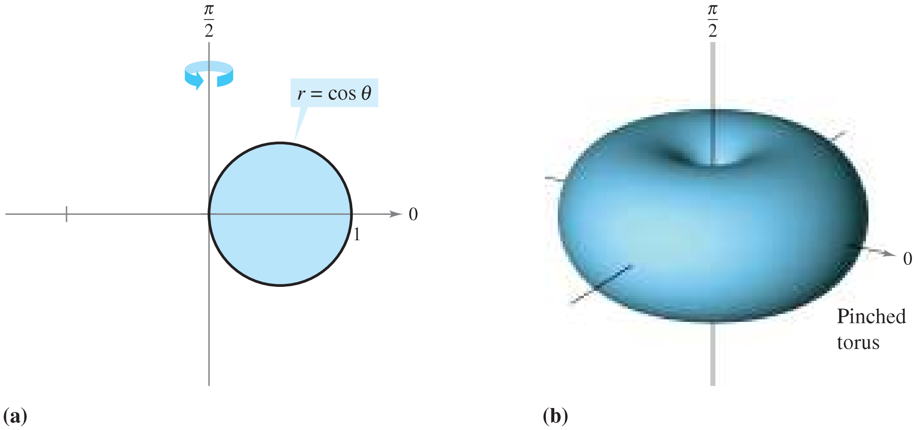


Figure 10.57

Solution You can use the second formula given in Theorem 10.15 with $f'(\theta) = -\sin\theta$. Because the circle is traced once as θ increases from 0 to π, you have

$$S = 2\pi \int_\alpha^\beta f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$$ Formula for area of a surface of revolution

$$= 2\pi \int_0^\pi \cos\theta(\cos\theta)\sqrt{\cos^2\theta + \sin^2\theta}\,d\theta$$

$$= 2\pi \int_0^\pi \cos^2\theta\,d\theta$$ Trigonometric identity

$$= \pi \int_0^\pi (1 + \cos 2\theta)\,d\theta$$ Trigonometric identity

$$= \pi\left[\theta + \frac{\sin 2\theta}{2}\right]_0^\pi = \pi^2.$$

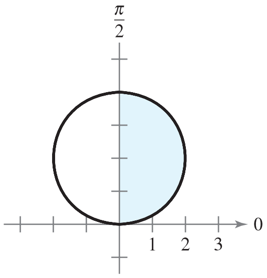
■

10.5 Exercises

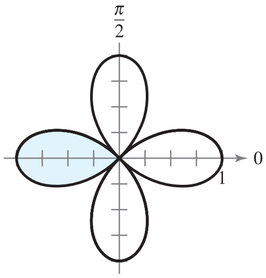
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, write an integral that represents the area of the shaded region of the figure. Do not evaluate the integral.

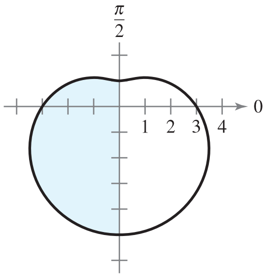
1. $r = 4 \sin \theta$



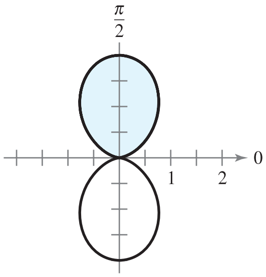
2. $r = \cos 2\theta$



3. $r = 3 - 2 \sin \theta$



4. $r = 1 - \cos 2\theta$



In Exercises 5–16, find the area of the region.

5. Interior of $r = 6 \sin \theta$

6. Interior of $r = 3 \cos \theta$

7. One petal of $r = 2 \cos 3\theta$

8. One petal of $r = 4 \sin 3\theta$

9. One petal of $r = \sin 2\theta$

10. One petal of $r = \cos 5\theta$

11. Interior of $r = 1 - \sin \theta$

12. Interior of $r = 1 - \sin \theta$ (above the polar axis)

13. Interior of $r = 5 + 2 \sin \theta$

14. Interior of $r = 4 - 4 \cos \theta$

15. Interior of $r^2 = 4 \cos 2\theta$

16. Interior of $r^2 = 6 \sin 2\theta$

In Exercises 17–24, use a graphing utility to graph the polar equation and find the area of the given region.

17. Inner loop of $r = 1 + 2 \cos \theta$

18. Inner loop of $r = 2 - 4 \cos \theta$

19. Inner loop of $r = 1 + 2 \sin \theta$

20. Inner loop of $r = 4 - 6 \sin \theta$

21. Between the loops of $r = 1 + 2 \cos \theta$

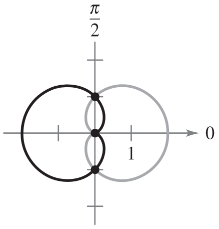
22. Between the loops of $r = 2(1 + 2 \sin \theta)$

23. Between the loops of $r = 3 - 6 \sin \theta$

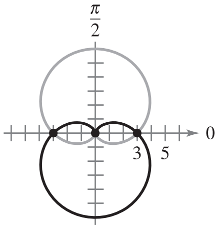
24. Between the loops of $r = \frac{1}{2} + \cos \theta$

In Exercises 25–34, find the points of intersection of the graphs of the equations.

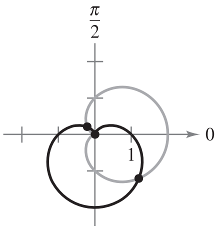
25. $r = 1 + \cos \theta$
$r = 1 - \cos \theta$



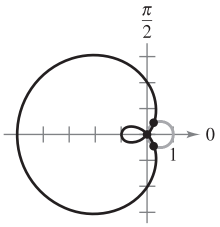
26. $r = 3(1 + \sin \theta)$
$r = 3(1 - \sin \theta)$



27. $r = 1 + \cos \theta$
$r = 1 - \sin \theta$



28. $r = 2 - 3 \cos \theta$
$r = \cos \theta$



29. $r = 4 - 5 \sin \theta$
$r = 3 \sin \theta$

30. $r = 1 + \cos \theta$
$r = 3 \cos \theta$

31. $r = \dfrac{\theta}{2}$
$r = 2$

32. $\theta = \dfrac{\pi}{4}$
$r = 2$

33. $r = 2 \sin 2\theta$
$r = 1$

34. $r = 3 + \sin \theta$
$r = 2 \csc \theta$

In Exercises 35 and 36, use a graphing utility to graph the polar equations and approximate the points of intersection of the graphs. Confirm your results analytically.

35. $r = 2 + 3 \cos \theta$
$r = \dfrac{\sec \theta}{2}$

36. $r = 3(1 - \cos \theta)$
$r = \dfrac{6}{1 - \cos \theta}$

Writing **In Exercises 37 and 38, use a graphing utility to graph the polar equations and find the points of intersection of the graphs. Watch the graphs as they are traced in the viewing window. Explain why the pole is not a point of intersection obtained by solving the equations simultaneously.**

37. $r = \cos \theta$
$r = 2 - 3 \sin \theta$

38. $r = 4 \sin \theta$
$r = 2(1 + \sin \theta)$

In Exercises 39–46, use a graphing utility to graph the polar equations and find the area of the given region.

39. Common interior of $r = 4 \sin 2\theta$ and $r = 2$

40. Common interior of $r = 2(1 + \cos \theta)$ and $r = 2(1 - \cos \theta)$

41. Common interior of $r = 3 - 2 \sin \theta$ and $r = -3 + 2 \sin \theta$

42. Common interior of $r = 5 - 3 \sin \theta$ and $r = 5 - 3 \cos \theta$

43. Common interior of $r = 4 \sin \theta$ and $r = 2$

44. Common interior of $r = 2 \cos \theta$ and $r = 2 \sin \theta$

45. Inside $r = 2 \cos \theta$ and outside $r = 1$

46. Inside $r = 3 \sin \theta$ and outside $r = 1 + \sin \theta$

In Exercises 47–50, find the area of the region.

47. Inside $r = a(1 + \cos \theta)$ and outside $r = a \cos \theta$

48. Inside $r = 2a \cos \theta$ and outside $r = a$

49. Common interior of $r = a(1 + \cos \theta)$ and $r = a \sin \theta$

50. Common interior of $r = a \cos \theta$ and $r = a \sin \theta$, where $a > 0$

51. ***Antenna Radiation*** The radiation from a transmitting antenna is not uniform in all directions. The intensity from a particular antenna is modeled by $r = a \cos^2 \theta$.

(a) Convert the polar equation to rectangular form.

(b) Use a graphing utility to graph the model for $a = 4$ and $a = 6$.

(c) Find the area of the geographical region between the two curves in part (b).

52. ***Area*** The area inside one or more of the three interlocking circles $r = 2a \cos \theta$, $r = 2a \sin \theta$, and $r = a$ is divided into seven regions. Find the area of each region.

53. ***Conjecture*** Find the area of the region enclosed by

$$r = a \cos(n\theta)$$

for $n = 1, 2, 3, \ldots$. Use the results to make a conjecture about the area enclosed by the function if n is even and if n is odd.

54. ***Area*** Sketch the strophoid

$$r = \sec \theta - 2 \cos \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Convert this equation to rectangular coordinates. Find the area enclosed by the loop.

In Exercises 55–60, find the length of the curve over the given interval.

	Polar Equation	*Interval*
55.	$r = 8$	$0 \le \theta \le 2\pi$
56.	$r = a$	$0 \le \theta \le 2\pi$
57.	$r = 4 \sin \theta$	$0 \le \theta \le \pi$
58.	$r = 2a \cos \theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
59.	$r = 1 + \sin \theta$	$0 \le \theta \le 2\pi$
60.	$r = 8(1 + \cos \theta)$	$0 \le \theta \le 2\pi$

In Exercises 61–66, use a graphing utility to graph the polar equation over the given interval. Use the integration capabilities of the graphing utility to approximate the length of the curve accurate to two decimal places.

61. $r = 2\theta, \quad 0 \le \theta \le \frac{\pi}{2}$

62. $r = \sec \theta, \quad 0 \le \theta \le \frac{\pi}{3}$

63. $r = \frac{1}{\theta}, \quad \pi \le \theta \le 2\pi$

64. $r = e^{\theta}, \quad 0 \le \theta \le \pi$

65. $r = \sin(3 \cos \theta), \quad 0 \le \theta \le \pi$

66 $r = 2 \sin(2 \cos \theta), \quad 0 \le \theta \le \pi$

In Exercises 67–70, find the area of the surface formed by revolving the curve about the given line.

	Polar Equation	*Interval*	*Axis of Revolution*
67.	$r = 6 \cos \theta$	$0 \le \theta \le \frac{\pi}{2}$	Polar axis
68.	$r = a \cos \theta$	$0 \le \theta \le \frac{\pi}{2}$	$\theta = \frac{\pi}{2}$
69.	$r = e^{a\theta}$	$0 \le \theta \le \frac{\pi}{2}$	$\theta = \frac{\pi}{2}$
70.	$r = a(1 + \cos \theta)$	$0 \le \theta \le \pi$	Polar axis

In Exercises 71 and 72, use the integration capabilities of a graphing utility to approximate to two decimal places the area of the surface formed by revolving the curve about the polar axis.

71. $r = 4 \cos 2\theta, \quad 0 \le \theta \le \frac{\pi}{4}$

72. $r = \theta, \quad 0 \le \theta \le \pi$

WRITING ABOUT CONCEPTS

73. Explain why finding points of intersection of polar graphs may require further analysis beyond solving two equations simultaneously.

74. Which integral yields the arc length of $r = 3(1 - \cos 2\theta)$? State why the other integrals are incorrect.

(a) $3\int_{0}^{2\pi} \sqrt{(1 - \cos 2\theta)^2 + 4 \sin^2 2\theta}\, d\theta$

(b) $12\int_{0}^{\pi/4} \sqrt{(1 - \cos 2\theta)^2 + 4 \sin^2 2\theta}\, d\theta$

(c) $3\int_{0}^{\pi} \sqrt{(1 - \cos 2\theta)^2 + 4 \sin^2 2\theta}\, d\theta$

(d) $6\int_{0}^{\pi/2} \sqrt{(1 - \cos 2\theta)^2 + 4 \sin^2 2\theta}\, d\theta$

75. Give the integral formulas for the area of the surface of revolution formed when the graph of $r = f(\theta)$ is revolved about (a) the x-axis and (b) the y-axis.

CAPSTONE

76. For each polar equation, sketch its graph, determine the interval that traces the graph only once, and find the area of the region bounded by the graph using a geometric formula and integration.

(a) $r = 10 \cos \theta$ (b) $r = 5 \sin \theta$

77. ***Surface Area of a Torus*** Find the surface area of the torus generated by revolving the circle given by $r = 2$ about the line $r = 5 \sec \theta$.

78. ***Surface Area of a Torus*** Find the surface area of the torus generated by revolving the circle given by $r = a$ about the line $r = b \sec \theta$, where $0 < a < b$.

79. ***Approximating Area*** Consider the circle $r = 8 \cos \theta$.

(a) Find the area of the circle.

(b) Complete the table giving the areas A of the sectors of the circle between $\theta = 0$ and the values of θ in the table.

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A							

(c) Use the table in part (b) to approximate the values of θ for which the sector of the circle composes $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ of the total area of the circle.

(d) Use a graphing utility to approximate, to two decimal places, the angles θ for which the sector of the circle composes $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ of the total area of the circle.

(e) Do the results of part (d) depend on the radius of the circle? Explain.

80. ***Approximate Area*** Consider the circle $r = 3 \sin \theta$.

(a) Find the area of the circle.

(b) Complete the table giving the areas A of the sectors of the circle between $\theta = 0$ and the values of θ in the table.

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A							

(c) Use the table in part (b) to approximate the values of θ for which the sector of the circle composes $\frac{1}{8}$, $\frac{1}{4}$, and $\frac{1}{2}$ of the total area of the circle.

(d) Use a graphing utility to approximate, to two decimal places, the angles θ for which the sector of the circle composes $\frac{1}{8}$, $\frac{1}{4}$, and $\frac{1}{2}$ of the total area of the circle.

81. What conic section does the following polar equation represent?

$$r = a \sin \theta + b \cos \theta$$

82. ***Area*** Find the area of the circle given by $r = \sin \theta + \cos \theta$. Check your result by converting the polar equation to rectangular form, then using the formula for the area of a circle.

83. ***Spiral of Archimedes*** The curve represented by the equation $r = a\theta$, where a is a constant, is called the spiral of Archimedes.

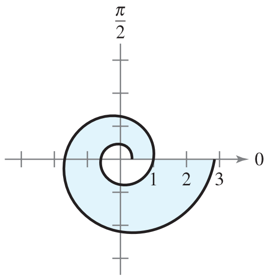
(a) Use a graphing utility to graph $r = \theta$, where $\theta \geq 0$. What happens to the graph of $r = a\theta$ as a increases? What happens if $\theta \leq 0$?

(b) Determine the points on the spiral $r = a\theta$ $(a > 0, \theta \geq 0)$, where the curve crosses the polar axis.

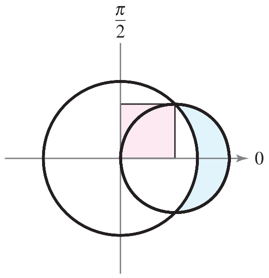
(c) Find the length of $r = \theta$ over the interval $0 \leq \theta \leq 2\pi$.

(d) Find the area under the curve $r = \theta$ for $0 \leq \theta \leq 2\pi$.

84. ***Logarithmic Spiral*** The curve represented by the equation $r = ae^{b\theta}$, where a and b are constants, is called a **logarithmic spiral.** The figure shows the graph of $r = e^{\theta/6}$, $-2\pi \leq \theta \leq 2\pi$. Find the area of the shaded region.



85. The larger circle in the figure is the graph of $r = 1$. Find the polar equation of the smaller circle such that the shaded regions are equal.



86. ***Folium of Descartes*** A curve called the **folium of Descartes** can be represented by the parametric equations

$$x = \frac{3t}{1 + t^3} \quad \text{and} \quad y = \frac{3t^2}{1 + t^3}.$$

(a) Convert the parametric equations to polar form.

(b) Sketch the graph of the polar equation from part (a).

(c) Use a graphing utility to approximate the area enclosed by the loop of the curve.

True or False? **In Exercises 87 and 88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

87. If $f(\theta) > 0$ for all θ and $g(\theta) < 0$ for all θ, then the graphs of $r = f(\theta)$ and $r = g(\theta)$ do not intersect.

88. If $f(\theta) = g(\theta)$ for $\theta = 0$, $\pi/2$, and $3\pi/2$, then the graphs of $r = f(\theta)$ and $r = g(\theta)$ have at least four points of intersection.

89. Use the formula for the arc length of a curve in parametric form to derive the formula for the arc length of a polar curve.

10.6 Polar Equations of Conics and Kepler's Laws

- **Analyze and write polar equations of conics.**
- **Understand and use Kepler's Laws of planetary motion.**

Polar Equations of Conics

In this chapter you have seen that the rectangular equations of ellipses and hyperbolas take simple forms when the origin lies at their *centers*. As it happens, there are many important applications of conics in which it is more convenient to use one of the foci as the reference point (the origin) for the coordinate system. For example, the sun lies at a focus of Earth's orbit. Similarly, the light source of a parabolic reflector lies at its focus. In this section you will see that polar equations of conics take simple forms if one of the foci lies at the pole.

The following theorem uses the concept of *eccentricity*, as defined in Section 10.1, to classify the three basic types of conics. A proof of this theorem is given in Appendix A.

EXPLORATION

Graphing Conics Set a graphing utility to *polar* mode and enter polar equations of the form

$$r = \frac{a}{1 \pm b\cos\theta}$$

or

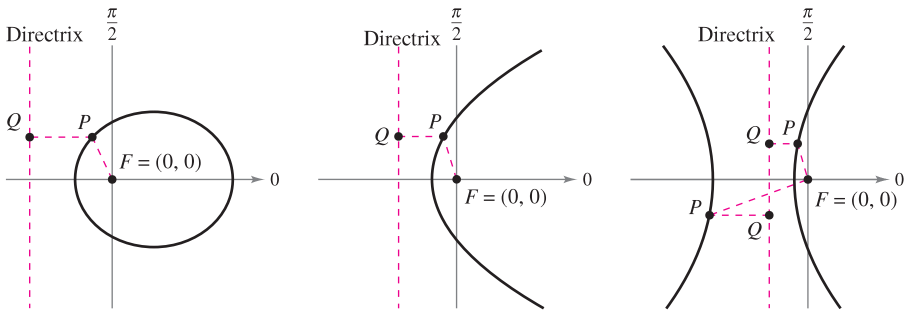
$$r = \frac{a}{1 \pm b\sin\theta}.$$

As long as $a \neq 0$, the graph should be a conic. What values of a and b produce parabolas? What values produce ellipses? What values produce hyperbolas?

THEOREM 10.16 CLASSIFICATION OF CONICS BY ECCENTRICITY

Let F be a fixed point (*focus*) and let D be a fixed line (*directrix*) in the plane. Let P be another point in the plane and let e (*eccentricity*) be the ratio of the distance between P and F to the distance between P and D. The collection of all points P with a given eccentricity is a conic.

1. The conic is an ellipse if $0 < e < 1$.
2. The conic is a parabola if $e = 1$.
3. The conic is a hyperbola if $e > 1$.



Ellipse: $0 < e < 1$

$\frac{PF}{PQ} < 1$

Parabola: $e = 1$

$PF = PQ$

Hyperbola: $e > 1$

$\frac{PF}{PQ} = \frac{P'F}{P'Q'} > 1$

Figure 10.58

In Figure 10.58, note that for each type of conic the pole corresponds to the fixed point (focus) given in the definition. The benefit of this location can be seen in the proof of the following theorem.

THEOREM 10.17 POLAR EQUATIONS OF CONICS

The graph of a polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

is a conic, where $e > 0$ is the eccentricity and $|d|$ is the distance between the focus at the pole and its corresponding directrix.

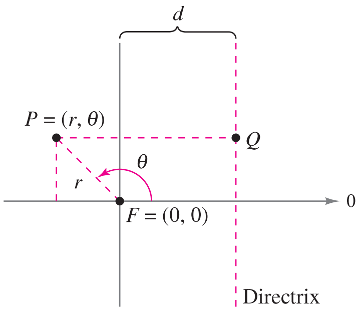


Figure 10.59

PROOF The following is a proof for $r = ed/(1 + e\cos\theta)$ with $d > 0$. In Figure 10.59, consider a vertical directrix d units to the right of the focus $F = (0, 0)$. If $P = (r, \theta)$ is a point on the graph of $r = ed/(1 + e\cos\theta)$, the distance between P and the directrix can be shown to be

$$PQ = |d - x| = |d - r\cos\theta| = \left|\frac{r(1 + e\cos\theta)}{e} - r\cos\theta\right| = \left|\frac{r}{e}\right|.$$

Because the distance between P and the pole is simply $PF = |r|$, the ratio of PF to PQ is $PF/PQ = |r|/|r/e| = |e| = e$ and, by Theorem 10.16, the graph of the equation must be a conic. The proofs of the other cases are similar. ■

The four types of equations indicated in Theorem 10.17 can be classified as follows, where $d > 0$.

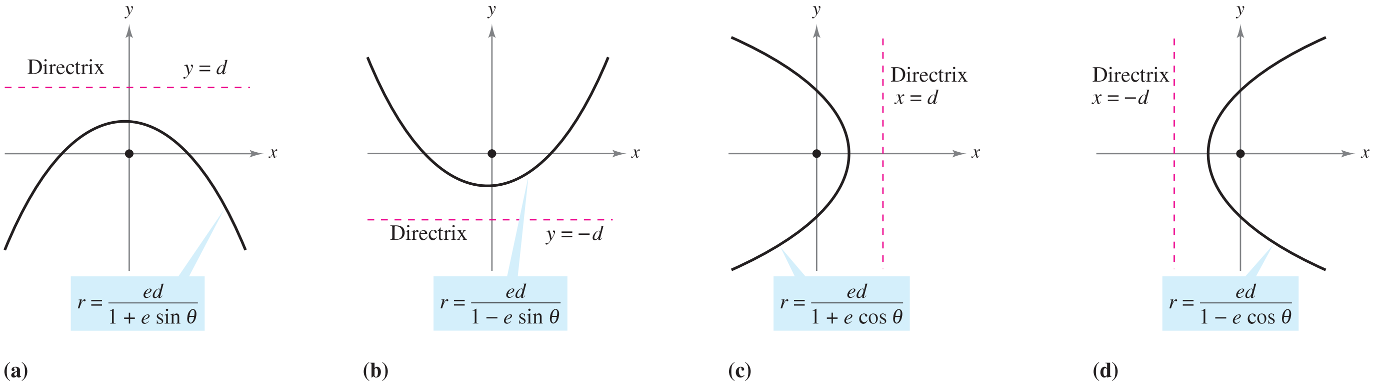
a. Horizontal directrix above the pole: $r = \dfrac{ed}{1 + e\sin\theta}$

b. Horizontal directrix below the pole: $r = \dfrac{ed}{1 - e\sin\theta}$

c. Vertical directrix to the right of the pole: $r = \dfrac{ed}{1 + e\cos\theta}$

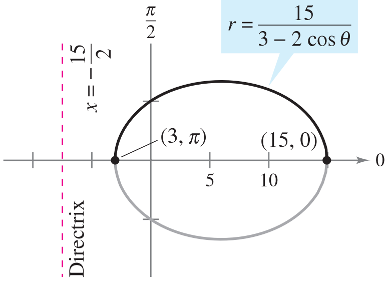
d. Vertical directrix to the left of the pole: $r = \dfrac{ed}{1 - e\cos\theta}$

Figure 10.60 illustrates these four possibilities for a parabola.



The four types of polar equations for a parabola
Figure 10.60

EXAMPLE 1 Determining a Conic from Its Equation



The graph of the conic is an ellipse with $e = \frac{2}{3}$.
Figure 10.61

Sketch the graph of the conic given by $r = \dfrac{15}{3 - 2\cos\theta}$.

Solution To determine the type of conic, rewrite the equation as

$$r = \frac{15}{3 - 2\cos\theta}$$

Write original equation.

$$= \frac{5}{1 - (2/3)\cos\theta}.$$

Divide numerator and denominator by 3.

So, the graph is an ellipse with $e = \frac{2}{3}$. You can sketch the upper half of the ellipse by plotting points from $\theta = 0$ to $\theta = \pi$, as shown in Figure 10.61. Then, using symmetry with respect to the polar axis, you can sketch the lower half. ■

For the ellipse in Figure 10.61, the major axis is horizontal and the vertices lie at $(15, 0)$ and $(3, \pi)$. So, the length of the *major* axis is $2a = 18$. To find the length of the minor axis, you can use the equations $e = c/a$ and $b^2 = a^2 - c^2$ to conclude that

$$b^2 = a^2 - c^2 = a^2 - (ea)^2 = a^2(1 - e^2).$$

Ellipse

Because $e = \frac{2}{3}$, you have

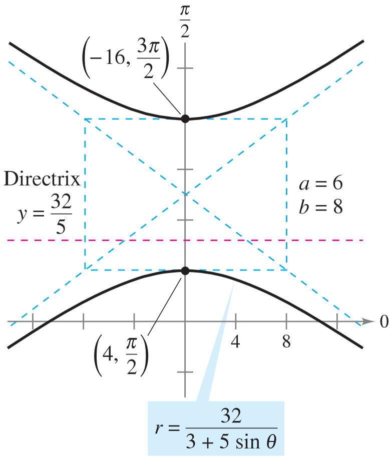
$$b^2 = 9^2\left[1 - \left(\tfrac{2}{3}\right)^2\right] = 45$$

which implies that $b = \sqrt{45} = 3\sqrt{5}$. So, the length of the minor axis is $2b = 6\sqrt{5}$. A similar analysis for hyperbolas yields

$$b^2 = c^2 - a^2 = (ea)^2 - a^2 = a^2(e^2 - 1).$$

Hyperbola

EXAMPLE 2 Sketching a Conic from Its Polar Equation



The graph of the conic is a hyperbola with $e = \frac{5}{3}$.
Figure 10.62

Sketch the graph of the polar equation $r = \dfrac{32}{3 + 5\sin\theta}$.

Solution Dividing the numerator and denominator by 3 produces

$$r = \frac{32/3}{1 + (5/3)\sin\theta}.$$

Because $e = \frac{5}{3} > 1$, the graph is a hyperbola. Because $d = \frac{32}{5}$, the directrix is the line $y = \frac{32}{5}$. The transverse axis of the hyperbola lies on the line $\theta = \pi/2$, and the vertices occur at

$$(r, \theta) = \left(4, \frac{\pi}{2}\right) \quad \text{and} \quad (r, \theta) = \left(-16, \frac{3\pi}{2}\right).$$

Because the length of the transverse axis is 12, you can see that $a = 6$. To find b, write

$$b^2 = a^2(e^2 - 1) = 6^2\left[\left(\frac{5}{3}\right)^2 - 1\right] = 64.$$

Therefore, $b = 8$. Finally, you can use a and b to determine the asymptotes of the hyperbola and obtain the sketch shown in Figure 10.62. ■

Kepler's Laws

Kepler's Laws, named after the German astronomer Johannes Kepler, can be used to describe the orbits of the planets about the sun.

1. Each planet moves in an elliptical orbit with the sun as a focus.
2. A ray from the sun to the planet sweeps out equal areas of the ellipse in equal times.
3. The square of the period is proportional to the cube of the mean distance between the planet and the sun.*

Although Kepler derived these laws empirically, they were later validated by Newton. In fact, Newton was able to show that each law can be deduced from a set of universal laws of motion and gravitation that govern the movement of all heavenly bodies, including comets and satellites. This is shown in the next example, involving the comet named after the English mathematician and physicist Edmund Halley (1656–1742).

Mary Evans Picture Library

JOHANNES KEPLER (1571–1630)

Kepler formulated his three laws from the extensive data recorded by Danish astronomer Tycho Brahe, and from direct observation of the orbit of Mars.

EXAMPLE 3 Halley's Comet

Halley's comet has an elliptical orbit with the sun at one focus and has an eccentricity of $e \approx 0.967$. The length of the major axis of the orbit is approximately 35.88 astronomical units (AU). (An astronomical unit is defined as the mean distance between Earth and the sun, 93 million miles.) Find a polar equation for the orbit. How close does Halley's comet come to the sun?

Solution Using a vertical axis, you can choose an equation of the form

$$r = \frac{ed}{(1 + e \sin \theta)}.$$

Because the vertices of the ellipse occur when $\theta = \pi/2$ and $\theta = 3\pi/2$, you can determine the length of the major axis to be the sum of the r-values of the vertices, as shown in Figure 10.63. That is,

$$\begin{aligned} 2a &= \frac{0.967d}{1 + 0.967} + \frac{0.967d}{1 - 0.967} \\ 35.88 &\approx 29.79d. \qquad 2a \approx 35.88 \end{aligned}$$

So, $d \approx 1.204$ and $ed \approx (0.967)(1.204) \approx 1.164$. Using this value in the equation produces

$$r = \frac{1.164}{1 + 0.967 \sin \theta}$$

where r is measured in astronomical units. To find the closest point to the sun (the focus), you can write $c = ea \approx (0.967)(17.94) \approx 17.35$. Because c is the distance between the focus and the center, the closest point is

$$\begin{aligned} a - c &\approx 17.94 - 17.35 \\ &\approx 0.59 \text{ AU} \\ &\approx 55{,}000{,}000 \text{ miles.} \end{aligned}$$

■

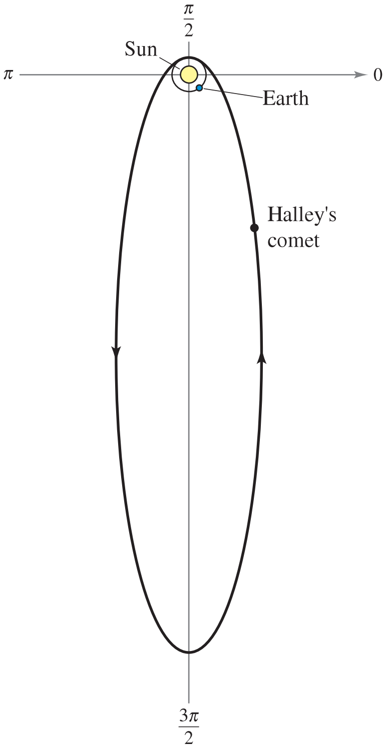


Figure 10.63

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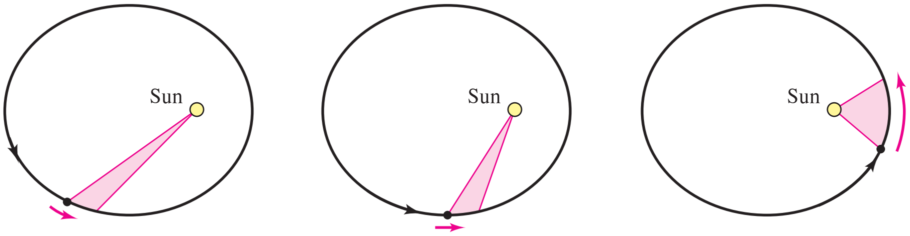
* If Earth is used as a reference with a period of 1 year and a distance of 1 astronomical unit, the proportionality constant is 1. For example, because Mars has a mean distance to the sun of $D = 1.524$ AU, its period P is given by $D^3 = P^2$. So, the period for Mars is $P = 1.88$.

Kepler's Second Law states that as a planet moves about the sun, a ray from the sun to the planet sweeps out equal areas in equal times. This law can also be applied to comets or asteroids with elliptical orbits. For example, Figure 10.64 shows the orbit of the asteroid Apollo about the sun. Applying Kepler's Second Law to this asteroid, you know that the closer it is to the sun, the greater its velocity, because a short ray must be moving quickly to sweep out as much area as a long ray.



A ray from the sun to the asteroid Apollo sweeps out equal areas in equal times.
Figure 10.64

EXAMPLE 4 The Asteroid Apollo

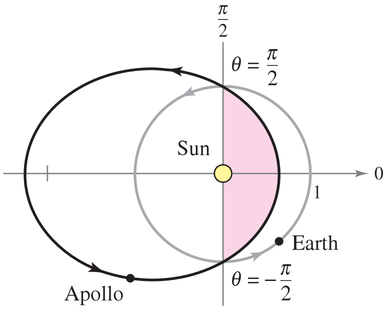


Figure 10.65

The asteroid Apollo has a period of 661 Earth days, and its orbit is approximated by the ellipse

$$r = \frac{1}{1 + (5/9)\cos\theta} = \frac{9}{9 + 5\cos\theta}$$

where r is measured in astronomical units. How long does it take Apollo to move from the position given by $\theta = -\pi/2$ to $\theta = \pi/2$, as shown in Figure 10.65?

Solution Begin by finding the area swept out as θ increases from $-\pi/2$ to $\pi/2$.

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta \qquad \text{Formula for area of a polar graph}$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(\frac{9}{9 + 5\cos\theta}\right)^2 d\theta$$

Using the substitution $u = \tan(\theta/2)$, as discussed in Section 8.6, you obtain

$$A = \frac{81}{112}\left[\frac{-5\sin\theta}{9 + 5\cos\theta} + \frac{18}{\sqrt{56}}\arctan\frac{\sqrt{56}\tan(\theta/2)}{14}\right]_{-\pi/2}^{\pi/2} \approx 0.90429.$$

Because the major axis of the ellipse has length $2a = 81/28$ and the eccentricity is $e = 5/9$, you can determine that $b = a\sqrt{1 - e^2} = 9/\sqrt{56}$. So, the area of the ellipse is

$$\text{Area of ellipse} = \pi ab = \pi\left(\frac{81}{56}\right)\left(\frac{9}{\sqrt{56}}\right) \approx 5.46507.$$

Because the time required to complete the orbit is 661 days, you can apply Kepler's Second Law to conclude that the time t required to move from the position $\theta = -\pi/2$ to $\theta = \pi/2$ is given by

$$\frac{t}{661} = \frac{\text{area of elliptical segment}}{\text{area of ellipse}} \approx \frac{0.90429}{5.46507}$$

which implies that $t \approx 109$ days. ■

10.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Graphical Reasoning **In Exercises 1–4, use a graphing utility to graph the polar equation when (a) $e = 1$, (b) $e = 0.5$, and (c) $e = 1.5$. Identify the conic.**

1. $r = \dfrac{2e}{1 + e\cos\theta}$ **2.** $r = \dfrac{2e}{1 - e\cos\theta}$

3. $r = \dfrac{2e}{1 - e\sin\theta}$ **4.** $r = \dfrac{2e}{1 + e\sin\theta}$

5. ***Writing*** Consider the polar equation

$$r = \frac{4}{1 + e\sin\theta}.$$

(a) Use a graphing utility to graph the equation for $e = 0.1$, $e = 0.25$, $e = 0.5$, $e = 0.75$, and $e = 0.9$. Identify the conic and discuss the change in its shape as $e \to 1^-$ and $e \to 0^+$.

(b) Use a graphing utility to graph the equation for $e = 1$. Identify the conic.

(c) Use a graphing utility to graph the equation for $e = 1.1$, $e = 1.5$, and $e = 2$. Identify the conic and discuss the change in its shape as $e \to 1^+$ and $e \to \infty$.

6. Consider the polar equation

$$r = \frac{4}{1 - 0.4\cos\theta}.$$

(a) Identify the conic without graphing the equation.

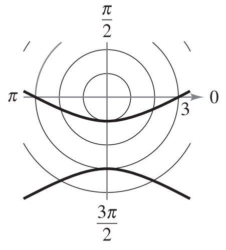
(b) Without graphing the following polar equations, describe how each differs from the polar equation above.

$$r = \frac{4}{1 + 0.4\cos\theta}, \quad r = \frac{4}{1 - 0.4\sin\theta}$$

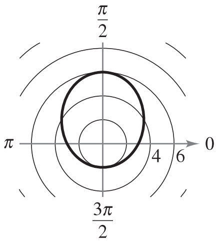
(c) Verify the results of part (b) graphically.

In Exercises 7–12, match the polar equation with the correct graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

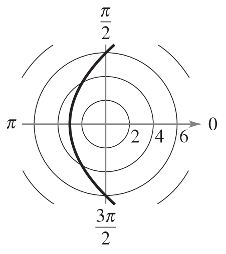
(a)



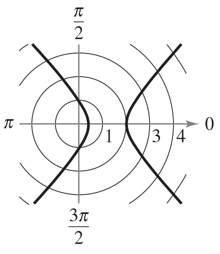
(b)



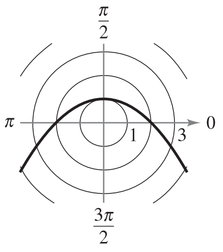
(c)



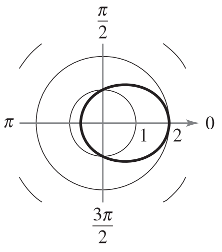
(d)



(e)



(f)



7. $r = \dfrac{6}{1 - \cos\theta}$ **8.** $r = \dfrac{2}{2 - \cos\theta}$

9. $r = \dfrac{3}{1 - 2\sin\theta}$ **10.** $r = \dfrac{2}{1 + \sin\theta}$

11. $r = \dfrac{6}{2 - \sin\theta}$ **12.** $r = \dfrac{2}{2 + 3\cos\theta}$

In Exercises 13–26, find the eccentricity and the distance from the pole to the directrix of the conic. Then sketch and identify the graph. Use a graphing utility to confirm your results.

13. $r = \dfrac{1}{1 - \cos\theta}$ **14.** $r = \dfrac{1}{1 + \sin\theta}$

15. $r = \dfrac{-4}{1 - \sin\theta}$ **16.** $r = \dfrac{4}{1 + \cos\theta}$

17. $r = \dfrac{6}{2 + \cos\theta}$

18. $r = \dfrac{10}{5 + 4\sin\theta}$

19. $r(2 + \sin\theta) = 4$

20. $r(3 - 2\cos\theta) = 6$

21. $r = \dfrac{5}{-1 + 2\cos\theta}$

22. $r = \dfrac{-6}{3 + 7\sin\theta}$

23. $r = \dfrac{3}{2 + 6\sin\theta}$

24. $r = \dfrac{8}{1 + 4\cos\theta}$

25. $r = \dfrac{300}{-12 + 6\sin\theta}$

26. $r = \dfrac{180}{15 - 3.75\cos\theta}$

In Exercises 27–30, use a graphing utility to graph the polar equation. Identify the graph and find its eccentricity.

27. $r = \dfrac{3}{-4 + 2\sin\theta}$ **28.** $r = \dfrac{-15}{2 + 8\sin\theta}$

29. $r = \dfrac{-10}{1 - \cos\theta}$ **30.** $r = \dfrac{6}{6 + 7\cos\theta}$

In Exercises 31–34, use a graphing utility to graph the conic. Describe how the graph differs from the graph in the indicated exercise.

31. $r = \dfrac{-4}{1 - \sin(\theta - \pi/4)}$ (See Exercise 15.)

32. $r = \dfrac{4}{1 + \cos(\theta - \pi/3)}$ (See Exercise 16.)

33. $r = \dfrac{6}{2 + \cos(\theta + \pi/6)}$ (See Exercise 17.)

34. $r = \dfrac{-6}{3 + 7\sin(\theta + 2\pi/3)}$ (See Exercise 22.)

35. Write the equation for the ellipse rotated $\pi/6$ radian clockwise from the ellipse

$$r = \frac{8}{8 + 5\cos\theta}.$$

36. Write the equation for the parabola rotated $\pi/4$ radian counterclockwise from the parabola

$$r = \frac{9}{1 + \sin\theta}.$$

In Exercises 37–48, find a polar equation for the conic with its focus at the pole. (For convenience, the equation for the directrix is given in rectangular form.)

	Conic	*Eccentricity*	*Directrix*
37.	Parabola	$e = 1$	$x = -3$
38.	Parabola	$e = 1$	$y = 4$
39.	Ellipse	$e = \frac{1}{2}$	$y = 1$
40.	Ellipse	$e = \frac{3}{4}$	$y = -2$
41.	Hyperbola	$e = 2$	$x = 1$
42.	Hyperbola	$e = \frac{3}{2}$	$x = -1$

	Conic	*Vertex or Vertices*
43.	Parabola	$\left(1, -\frac{\pi}{2}\right)$
44.	Parabola	$(5, \pi)$
45.	Ellipse	$(2, 0),\ (8, \pi)$
46.	Ellipse	$\left(2, \frac{\pi}{2}\right),\ \left(4, \frac{3\pi}{2}\right)$
47.	Hyperbola	$\left(1, \frac{3\pi}{2}\right),\ \left(9, \frac{3\pi}{2}\right)$
48.	Hyperbola	$(2, 0),\ (10, 0)$

49. Find a polar equation for the ellipse with focus $(0, 0)$, eccentricity $\frac{1}{2}$, and a directrix at $r = 4\sec\theta$.

50. Find a polar equation for the hyperbola with focus $(0, 0)$, eccentricity 2, and a directrix at $r = -8\csc\theta$.

WRITING ABOUT CONCEPTS

51. Classify the conics by their eccentricities.

52. Identify each conic.

(a) $r = \dfrac{5}{1 - 2\cos\theta}$ (b) $r = \dfrac{5}{10 - \sin\theta}$

(c) $r = \dfrac{5}{3 - 3\cos\theta}$ (d) $r = \dfrac{5}{1 - 3\sin(\theta - \pi/4)}$

53. Describe what happens to the distance between the directrix and the center of an ellipse if the foci remain fixed and e approaches 0.

CAPSTONE

54. Explain how the graph of each conic differs from the graph of $r = \dfrac{4}{1 + \sin\theta}$.

(a) $r = \dfrac{4}{1 - \cos\theta}$ (b) $r = \dfrac{4}{1 - \sin\theta}$

(c) $r = \dfrac{4}{1 + \cos\theta}$ (d) $r = \dfrac{4}{1 - \sin(\theta - \pi/4)}$

55. Show that the polar equation for $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$r^2 = \frac{b^2}{1 - e^2\cos^2\theta}.$$ Ellipse

56. Show that the polar equation for $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$$r^2 = \frac{-b^2}{1 - e^2\cos^2\theta}.$$ Hyperbola

In Exercises 57–60, use the results of Exercises 55 and 56 to write the polar form of the equation of the conic.

57. Ellipse: focus at $(4, 0)$; vertices at $(5, 0)$, $(5, \pi)$

58. Hyperbola: focus at $(5, 0)$; vertices at $(4, 0)$, $(4, \pi)$

59. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

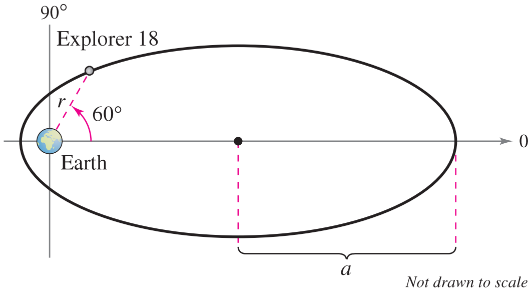
60. $\dfrac{x^2}{4} + y^2 = 1$

In Exercises 61–64, use the integration capabilities of a graphing utility to approximate to two decimal places the area of the region bounded by the graph of the polar equation.

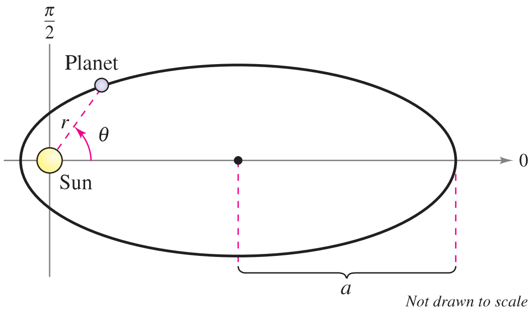
61. $r = \dfrac{3}{2 - \cos\theta}$ **62.** $r = \dfrac{9}{4 + \cos\theta}$

63. $r = \dfrac{2}{3 - 2\sin\theta}$ **64.** $r = \dfrac{3}{6 + 5\sin\theta}$

65. ***Explorer 18*** On November 27, 1963, the United States launched Explorer 18. Its low and high points above the surface of Earth were approximately 119 miles and 123,000 miles (see figure). The center of Earth is the focus of the orbit. Find the polar equation for the orbit and find the distance between the surface of Earth and the satellite when $\theta = 60°$. (Assume that the radius of Earth is 4000 miles.)



66. ***Planetary Motion*** The planets travel in elliptical orbits with the sun as a focus, as shown in the figure.



(a) Show that the polar equation of the orbit is given by

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta}$$

where e is the eccentricity.

(b) Show that the minimum distance (*perihelion*) from the sun to the planet is $r = a(1 - e)$ and the maximum distance (*aphelion*) is $r = a(1 + e)$.

In Exercises 67–70, use Exercise 66 to find the polar equation of the elliptical orbit of the planet, and the perihelion and aphelion distances.

67. Earth $a = 1.496 \times 10^8$ kilometers
$e = 0.0167$

68. Saturn $a = 1.427 \times 10^9$ kilometers
$e = 0.0542$

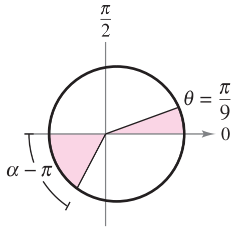
69. Neptune $a = 4.498 \times 10^9$ kilometers
$e = 0.0086$

70. Mercury $a = 5.791 \times 10^7$ kilometers
$e = 0.2056$

CAS **71.** ***Planetary Motion*** In Exercise 69, the polar equation for the elliptical orbit of Neptune was found. Use the equation and a computer algebra system to perform each of the following.

(a) Approximate the area swept out by a ray from the sun to the planet as θ increases from 0 to $\pi/9$. Use this result to determine the number of years required for the planet to move through this arc if the period of one revolution around the sun is 165 years.

(b) By trial and error, approximate the angle α such that the area swept out by a ray from the sun to the planet as θ increases from π to α equals the area found in part (a) (see figure). Does the ray sweep through a larger or smaller angle than in part (a) to generate the same area? Why is this the case?



(c) Approximate the distances the planet traveled in parts (a) and (b). Use these distances to approximate the average number of kilometers per year the planet traveled in the two cases.

72. ***Comet Hale-Bopp*** The comet Hale-Bopp has an elliptical orbit with the sun at one focus and has an eccentricity of $e \approx 0.995$. The length of the major axis of the orbit is approximately 500 astronomical units.

(a) Find the length of its minor axis.

(b) Find a polar equation for the orbit.

(c) Find the perihelion and aphelion distances.

In Exercises 73 and 74, let r_0 represent the distance from the focus to the nearest vertex, and let r_1 represent the distance from the focus to the farthest vertex.

73. Show that the eccentricity of an ellipse can be written as

$$e = \frac{r_1 - r_0}{r_1 + r_0}. \text{ Then show that } \frac{r_1}{r_0} = \frac{1 + e}{1 - e}.$$

74. Show that the eccentricity of a hyperbola can be written as

$$e = \frac{r_1 + r_0}{r_1 - r_0}. \text{ Then show that } \frac{r_1}{r_0} = \frac{e + 1}{e - 1}.$$

In Exercises 75 and 76, show that the graphs of the given equations intersect at right angles.

75. $r = \dfrac{ed}{1 + \sin\theta}$ and $r = \dfrac{ed}{1 - \sin\theta}$

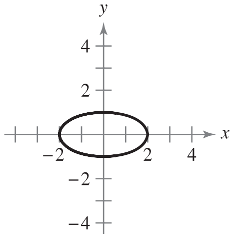
76. $r = \dfrac{c}{1 + \cos\theta}$ and $r = \dfrac{d}{1 - \cos\theta}$

10 REVIEW EXERCISES

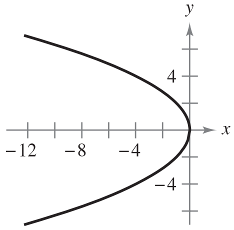
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, match the equation with the correct graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

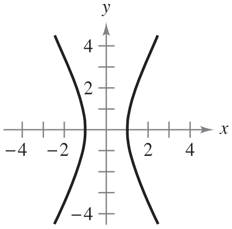
(a)



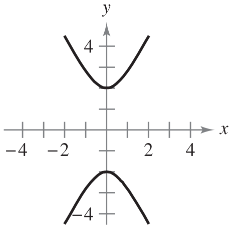
(b)



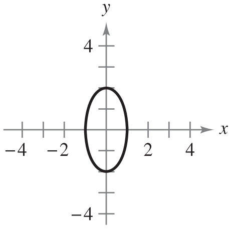
(c)



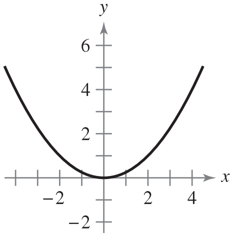
(d)



(e)



(f)



1. $4x^2 + y^2 = 4$

2. $4x^2 - y^2 = 4$

3. $y^2 = -4x$

4. $y^2 - 4x^2 = 4$

5. $x^2 + 4y^2 = 4$

6. $x^2 = 4y$

In Exercises 7–12, analyze the equation and sketch its graph. Use a graphing utility to confirm your results.

7. $16x^2 + 16y^2 - 16x + 24y - 3 = 0$

8. $y^2 - 12y - 8x + 20 = 0$

9. $3x^2 - 2y^2 + 24x + 12y + 24 = 0$

10. $5x^2 + y^2 - 20x + 19 = 0$

11. $3x^2 + 2y^2 - 12x + 12y + 29 = 0$

12. $12x^2 - 12y^2 - 12x + 24y - 45 = 0$

In Exercises 13 and 14, find an equation of the parabola.

13. Vertex: $(0, 2)$; directrix: $x = -3$

14. Vertex: $(2, 6)$; focus: $(2, 4)$

In Exercises 15 and 16, find an equation of the ellipse.

15. Vertices: $(-5, 0), (7, 0)$; foci: $(-3, 0), (5, 0)$

16. Center: $(0, 0)$; solution points: $(1, 2)$, $(2, 0)$

In Exercises 17 and 18, find an equation of the hyperbola.

17. Vertices: $(\pm 7, 0)$; foci: $(\pm 9, 0)$

18. Foci: $(0, \pm 8)$; asymptotes: $y = \pm 4x$

In Exercises 19 and 20, use a graphing utility to approximate the perimeter of the ellipse.

19. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

20. $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$

21. A line is tangent to the parabola $y = x^2 - 2x + 2$ and perpendicular to the line $y = x - 2$. Find the equation of the line.

22. A line is tangent to the parabola $3x^2 + y = x - 6$ and perpendicular to the line $2x + y = 5$. Find the equation of the line.

23. ***Satellite Antenna*** A cross section of a large parabolic antenna is modeled by the graph of

$$y = \frac{x^2}{200}, \quad -100 \le x \le 100.$$

The receiving and transmitting equipment is positioned at the focus.

(a) Find the coordinates of the focus.

(b) Find the surface area of the antenna.

24. ***Fire Truck*** Consider a fire truck with a water tank 16 feet long whose vertical cross sections are ellipses modeled by the equation

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

(a) Find the volume of the tank.

(b) Find the force on the end of the tank when it is full of water. (The density of water is 62.4 pounds per cubic foot.)

(c) Find the depth of the water in the tank if it is $\frac{3}{4}$ full (by volume) and the truck is on level ground.

(d) Approximate the tank's surface area.

In Exercises 25–32, sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.

25. $x = 1 + 8t, \ y = 3 - 4t$

26. $x = t - 6, \ y = t^2$

27. $x = e^t - 1, \ y = e^{3t}$

28. $x = e^{4t}, \ y = t + 4$

29. $x = 6 \cos \theta, \ y = 6 \sin \theta$

30. $x = 2 + 5 \cos t, \ y = 3 + 2 \sin t$

31. $x = 2 + \sec \theta, \ y = 3 + \tan \theta$

32. $x = 5 \sin^3 \theta, \ y = 5 \cos^3 \theta$

In Exercises 33–36, find a parametric representation of the line or conic.

33. Line: passes through $(-2, 6)$ and $(3, 2)$

34. Circle: center at $(-4, -5)$; radius 3

35. Ellipse: center at $(-3, 4)$; horizontal major axis of length 8 and minor axis of length 6

36. Hyperbola: vertices at $(0, \pm 4)$; foci at $(0, \pm 5)$

37. ***Rotary Engine*** The rotary engine was developed by Felix Wankel in the 1950s. It features a rotor, which is a modified equilateral triangle. The rotor moves in a chamber that, in two dimensions, is an epitrochoid. Use a graphing utility to graph the chamber modeled by the parametric equations

$$x = \cos 3\theta + 5\cos\theta$$

and

$$y = \sin 3\theta + 5\sin\theta.$$

38. ***Serpentine Curve*** Consider the parametric equations $x = 2\cot\theta$ and $y = 4\sin\theta\cos\theta$, $0 < \theta < \pi$.

(a) Use a graphing utility to graph the curve.

(b) Eliminate the parameter to show that the rectangular equation of the serpentine curve is $(4 + x^2)y = 8x$.

In Exercises 39–48, (a) find dy/dx and all points of horizontal tangency, (b) eliminate the parameter where possible, and (c) sketch the curve represented by the parametric equations.

39. $x = 2 + 5t, \quad y = 1 - 4t$

40. $x = t - 6, \quad y = t^2$

41. $x = \dfrac{1}{t}, \quad y = 2t + 3$

42. $x = \dfrac{1}{t}, \quad y = t^2$

43. $x = \dfrac{1}{2t + 1}$, $\quad y = \dfrac{1}{t^2 - 2t}$

44. $x = 2t - 1$, $\quad y = \dfrac{1}{t^2 - 2t}$

45. $x = 5 + \cos\theta$, $\quad y = 3 + 4\sin\theta$

46. $x = 10\cos\theta$, $\quad y = 10\sin\theta$

47. $x = \cos^3\theta$, $\quad y = 4\sin^3\theta$

48. $x = e^t$, $\quad y = e^{-t}$

In Exercises 49–52, find all points (if any) of horizontal and vertical tangency to the curve. Use a graphing utility to confirm your results.

49. $x = 5 - t, \quad y = 2t^2$

50. $x = t + 2, \quad y = t^3 - 2t$

51. $x = 2 + 2\sin\theta, \quad y = 1 + \cos\theta$

52. $x = 2 - 2\cos\theta, \quad y = 2\sin 2\theta$

In Exercises 53 and 54, (a) use a graphing utility to graph the curve represented by the parametric equations, (b) use a graphing utility to find $dx/d\theta$, $dy/d\theta$, and dy/dx for $\theta = \pi/6$, and (c) use a graphing utility to graph the tangent line to the curve when $\theta = \pi/6$.

53. $x = \cot\theta$, $\quad y = \sin 2\theta$

54. $x = 2\theta - \sin\theta$, $\quad y = 2 - \cos\theta$

Arc Length **In Exercises 55 and 56, find the arc length of the curve on the given interval.**

55. $x = r(\cos\theta + \theta\sin\theta)$, $\quad y = r(\sin\theta - \theta\cos\theta)$, $\quad 0 \le \theta \le \pi$

56. $x = 6\cos\theta$, $\quad y = 6\sin\theta$, $\quad 0 \le \theta \le \pi$

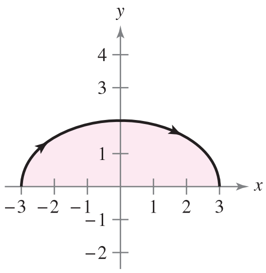
Surface Area **In Exercises 57 and 58, find the area of the surface generated by revolving the curve about (a) the x-axis and (b) the y-axis.**

57. $x = t, \quad y = 3t, \quad 0 \le t \le 2$

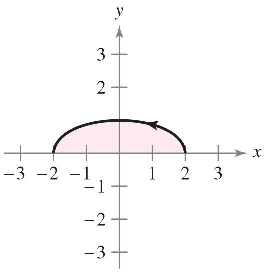
58. $x = 2\cos\theta, \quad y = 2\sin\theta, \quad 0 \le \theta \le \dfrac{\pi}{2}$

Area **In Exercises 59 and 60, find the area of the region.**

59. $x = 3\sin\theta$, $\quad y = 2\cos\theta$, $\quad -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$



60. $x = 2\cos\theta$, $\quad y = \sin\theta$, $\quad 0 \le \theta \le \pi$



In Exercises 61–64, plot the point in polar coordinates and find the corresponding rectangular coordinates of the point.

61. $\left(5, \dfrac{3\pi}{2}\right)$

62. $\left(-6, \dfrac{7\pi}{6}\right)$

63. $\left(\sqrt{3}, 1.56\right)$

64. $(-2, -2.45)$

In Exercises 65–68, the rectangular coordinates of a point are given. Plot the point and find *two* sets of polar coordinates of the point for $0 \le \theta < 2\pi$.

65. $(4, -4)$

66. $(0, -7)$

67. $(-1, 3)$

68. $\left(-\sqrt{3}, -\sqrt{3}\right)$

In Exercises 69–76, convert the polar equation to rectangular form.

69. $r = 3\cos\theta$

70. $r = 10$

71. $r = -2(1 + \cos\theta)$

72. $r = \dfrac{1}{2 - \cos\theta}$

73. $r^2 = \cos 2\theta$

74. $r = 4\sec\left(\theta - \dfrac{\pi}{3}\right)$

75. $r = 4\cos 2\theta \sec\theta$

76. $\theta = \dfrac{3\pi}{4}$

In Exercises 77–80, convert the rectangular equation to polar form.

77. $(x^2 + y^2)^2 = ax^2y$

78. $x^2 + y^2 - 4x = 0$

79. $x^2 + y^2 = a^2\left(\arctan\dfrac{y}{x}\right)^2$

80. $(x^2 + y^2)\left(\arctan\dfrac{y}{x}\right)^2 = a^2$

In Exercises 81–92, sketch a graph of the polar equation.

81. $r = 6$

82. $\theta = \dfrac{\pi}{10}$

83. $r = -\sec\theta$

84. $r = 5\csc\theta$

85. $r = -2(1 + \cos\theta)$

86. $r = 3 - 4\cos\theta$

87. $r = 4 - 3\cos\theta$

88. $r = 4\theta$

89. $r = -3\cos 2\theta$

90. $r = \cos 5\theta$

91. $r^2 = 4\sin^2 2\theta$

92. $r^2 = \cos 2\theta$

In Exercises 93–96, use a graphing utility to graph the polar equation.

93. $r = \dfrac{3}{\cos(\theta - \pi/4)}$

94. $r = 2\sin\theta\cos^2\theta$

95. $r = 4\cos 2\theta\sec\theta$

96. $r = 4(\sec\theta - \cos\theta)$

In Exercises 97 and 98, (a) find the tangents at the pole, (b) find all points of vertical and horizontal tangency, and (c) use a graphing utility to graph the polar equation and draw a tangent line to the graph for $\theta = \pi/6$.

97. $r = 1 - 2\cos\theta$

98. $r^2 = 4\sin 2\theta$

In Exercises 99 and 100, show that the graphs of the polar equations are orthogonal at the points of intersection. Use a graphing utility to confirm your results graphically.

99. $r = 1 + \cos\theta$
$r = 1 - \cos\theta$

100. $r = a\sin\theta$
$r = a\cos\theta$

In Exercises 101–106, find the area of the region.

101. One petal of $r = 3\cos 5\theta$

102. One petal of $r = 2\sin 6\theta$

103. Interior of $r = 2 + \cos\theta$

104. Interior of $r = 5(1 - \sin\theta)$

105. Interior of $r^2 = 4\sin 2\theta$

106. Common interior of $r = 4\cos\theta$ and $r = 2$

107. Find the points of intersection of the graphs of $r = 1 - \cos\theta$ and $r = 1 + \sin\theta$.

108. Find the points of intersection of the graphs of $r = 1 + \sin\theta$ and $r = 3\sin\theta$.

In Exercises 109–112, use a graphing utility to graph the polar equation. Set up an integral for finding the area of the given region and use the integration capabilities of a graphing utility to approximate the integral accurate to two decimal places.

109. Interior of $r = \sin\theta\cos^2\theta$

110. Interior of $r = 4\sin 3\theta$

111. Common interior of $r = 3$ and $r^2 = 18\sin 2\theta$

112. Region bounded by the polar axis and $r = e^{\theta}$ for $0 \le \theta \le \pi$

In Exercises 113 and 114, find the length of the curve over the given interval.

	Polar Equation	*Interval*
113.	$r = a(1 - \cos\theta)$	$0 \le \theta \le \pi$
114.	$r = a\cos 2\theta$	$-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

In Exercises 115 and 116, write an integral that represents the area of the surface formed by revolving the curve about the given line. Use a graphing utility to approximate the integral.

	Polar Equation	*Interval*	*Axis of Revolution*
115.	$r = 1 + 4\cos\theta$	$0 \le \theta \le \dfrac{\pi}{2}$	Polar axis
116.	$r = 2\sin\theta$	$0 \le \theta \le \dfrac{\pi}{2}$	$\theta = \dfrac{\pi}{2}$

In Exercises 117–122, sketch and identify the graph. Use a graphing utility to confirm your results.

117. $r = \dfrac{6}{1 - \sin\theta}$

118. $r = \dfrac{2}{1 + \cos\theta}$

119. $r = \dfrac{6}{3 + 2\cos\theta}$

120. $r = \dfrac{4}{5 - 3\sin\theta}$

121. $r = \dfrac{4}{2 - 3\sin\theta}$

122. $r = \dfrac{8}{2 - 5\cos\theta}$

In Exercises 123–128, find a polar equation for the line or conic with its focus at the pole.

123. Circle
Center: $(5, \pi/2)$
Solution point: $(0, 0)$

124. Line
Solution point: $(0, 0)$
Slope: $\sqrt{3}$

125. Parabola
Vertex: $(2, \pi)$

126. Parabola
Vertex: $(2, \pi/2)$

127. Ellipse
Vertices: $(5, 0)$, $(1, \pi)$

128. Hyperbola
Vertices: $(1, 0)$, $(7, 0)$

P.S. PROBLEM SOLVING

1. Consider the parabola $x^2 = 4y$ and the focal chord $y = \frac{3}{4}x + 1$.

(a) Sketch the graph of the parabola and the focal chord.

(b) Show that the tangent lines to the parabola at the endpoints of the focal chord intersect at right angles.

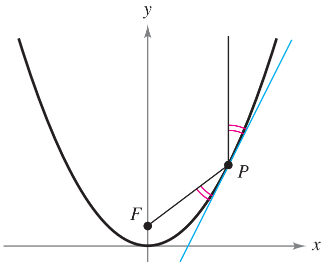
(c) Show that the tangent lines to the parabola at the endpoints of the focal chord intersect on the directrix of the parabola.

2. Consider the parabola $x^2 = 4py$ and one of its focal chords.

(a) Show that the tangent lines to the parabola at the endpoints of the focal chord intersect at right angles.

(b) Show that the tangent lines to the parabola at the endpoints of the focal chord intersect on the directrix of the parabola.

3. Prove Theorem 10.2, Reflective Property of a Parabola, as shown in the figure.



4. Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

with foci F_1 and F_2, as shown in the figure. Let T be the tangent line at a point M on the hyperbola. Show that incoming rays of light aimed at one focus are reflected by a hyperbolic mirror toward the other focus.

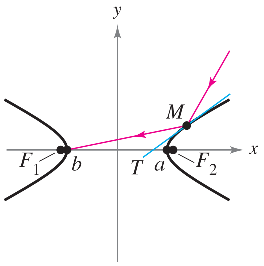


Figure for 4

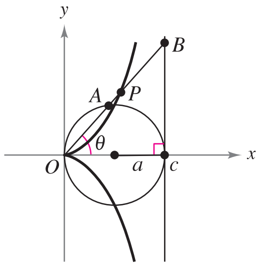


Figure for 5

5. Consider a circle of radius a tangent to the y-axis and the line $x = 2a$, as shown in the figure. Let A be the point where the segment OB intersects the circle. The **cissoid of Diocles** consists of all points P such that $OP = AB$.

(a) Find a polar equation of the cissoid.

(b) Find a set of parametric equations for the cissoid that does not contain trigonometric functions.

(c) Find a rectangular equation of the cissoid.

6. Consider the region bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$, with eccentricity $e = c/a$.

(a) Show that the area of the region is πab.

(b) Show that the solid (oblate spheroid) generated by revolving the region about the minor axis of the ellipse has a volume of $V = 4\pi^2 b/3$ and a surface area of

$$S = 2\pi a^2 + \pi\left(\frac{b^2}{e}\right)\ln\left(\frac{1+e}{1-e}\right).$$

(c) Show that the solid (prolate spheroid) generated by revolving the region about the major axis of the ellipse has a volume of $V = 4\pi ab^2/3$ and a surface area of

$$S = 2\pi b^2 + 2\pi\left(\frac{ab}{e}\right)\arcsin e.$$

7. The curve given by the parametric equations

$$x(t) = \frac{1-t^2}{1+t^2} \quad \text{and} \quad y(t) = \frac{t(1-t^2)}{1+t^2}$$

is called a **strophoid.**

(a) Find a rectangular equation of the strophoid.

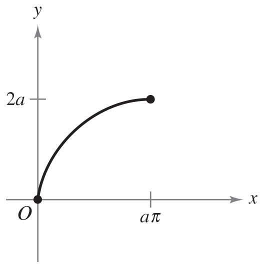
(b) Find a polar equation of the strophoid.

(c) Sketch a graph of the strophoid.

(d) Find the equations of the two tangent lines at the origin.

(e) Find the points on the graph at which the tangent lines are horizontal.

8. Find a rectangular equation of the portion of the cycloid given by the parametric equations $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, $0 \le \theta \le \pi$, as shown in the figure.



9. Consider the **cornu spiral** given by

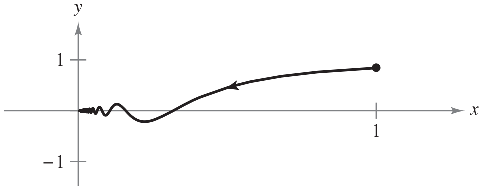
$$x(t) = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du \quad \text{and} \quad y(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du.$$

(a) Use a graphing utility to graph the spiral over the interval $-\pi \le t \le \pi$.

(b) Show that the cornu spiral is symmetric with respect to the origin.

(c) Find the length of the cornu spiral from $t = 0$ to $t = a$. What is the length of the spiral from $t = -\pi$ to $t = \pi$?

10. A particle is moving along the path described by the parametric equations $x = 1/t$ and $y = \sin t/t$, for $1 \le t < \infty$, as shown in the figure. Find the length of this path.



11. Let a and b be positive constants. Find the area of the region in the first quadrant bounded by the graph of the polar equation

$$r = \frac{ab}{(a \sin \theta + b \cos \theta)}, \quad 0 \le \theta \le \frac{\pi}{2}.$$

12. Consider the right triangle shown in the figure.

(a) Show that the area of the triangle is $A(\alpha) = \dfrac{1}{2}\displaystyle\int_0^{\alpha} \sec^2 \theta \, d\theta$.

(b) Show that $\tan \alpha = \displaystyle\int_0^{\alpha} \sec^2 \theta \, d\theta$.

(c) Use part (b) to derive the formula for the derivative of the tangent function.

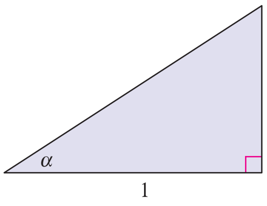


Figure for 12

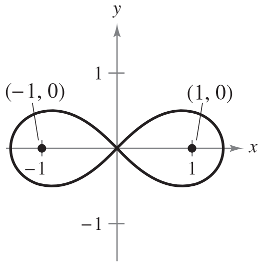
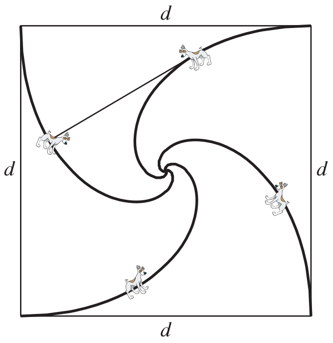


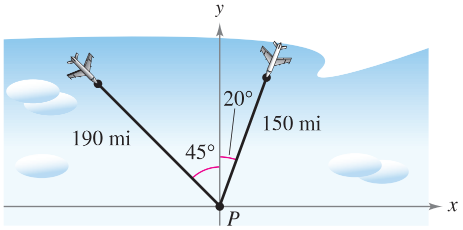
Figure for 13

13. Determine the polar equation of the set of all points (r, θ), the product of whose distances from the points $(1, 0)$ and $(-1, 0)$ is equal to 1, as shown in the figure.

14. Four dogs are located at the corners of a square with sides of length d. The dogs all move counterclockwise at the same speed directly toward the next dog, as shown in the figure. Find the polar equation of a dog's path as it spirals toward the center of the square.



15. An air traffic controller spots two planes at the same altitude flying toward each other (see figure). Their flight paths are 20° and 315°. One plane is 150 miles from point P with a speed of 375 miles per hour. The other is 190 miles from point P with a speed of 450 miles per hour.



(a) Find parametric equations for the path of each plane where t is the time in hours, with $t = 0$ corresponding to the time at which the air traffic controller spots the planes.

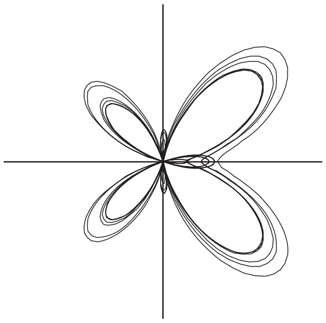
(b) Use the result of part (a) to write the distance between the planes as a function of t.

(c) Use a graphing utility to graph the function in part (b). When will the distance between the planes be minimum? If the planes must keep a separation of at least 3 miles, is the requirement met?

16. Use a graphing utility to graph the curve shown below. The curve is given by

$$r = e^{\cos \theta} - 2 \cos 4\theta + \sin^5 \frac{\theta}{12}.$$

Over what interval must θ vary to produce the curve?



■ **FOR FURTHER INFORMATION** For more information on this curve, see the article "A Study in Step Size" by Temple H. Fay in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

17. Use a graphing utility to graph the polar equation $r = \cos 5\theta + n \cos \theta$ for $0 \le \theta < \pi$ and for the integers $n = -5$ to $n = 5$. What values of n produce the "heart" portion of the curve? What values of n produce the "bell" portion? (This curve, created by Michael W. Chamberlin, appeared in *The College Mathematics Journal*)

11 Vectors and the Geometry of Space

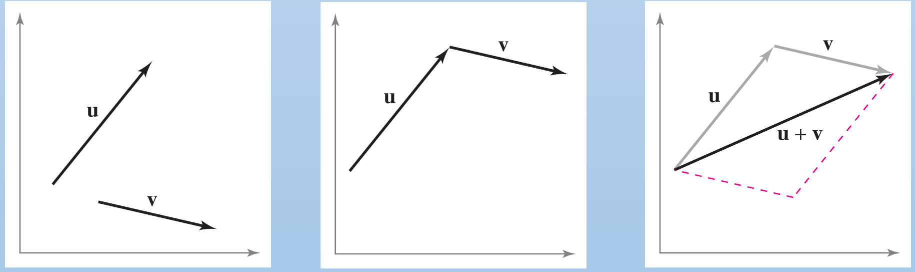
This chapter introduces vectors and the three-dimensional coordinate system. Vectors are used to represent lines and planes, and are also used to represent quantities such as force and velocity. The three-dimensional coordinate system is used to represent surfaces such as ellipsoids and elliptical cones. Much of the material in the remaining chapters relies on an understanding of this system.

In this chapter, you should learn the following.

- How to write vectors, perform basic vector operations, and represent vectors graphically. (11.1)
- How to plot points in a three-dimensional coordinate system and analyze vectors in space. (11.2)
- How to find the dot product of two vectors (in the plane or in space). (11.3)
- How to find the cross product of two vectors (in space). (11.4)
- How to find equations of lines and planes in space, and how to sketch their graphs. (11.5)
- How to recognize and write equations of cylindrical and quadric surfaces and of surfaces of revolution. (11.6)
- How to use cylindrical and spherical coordinates to represent surfaces in space. (11.7)

Mark Hunt/Hunt Stock

Two tugboats are pushing an ocean liner, as shown above. Each boat is exerting a force of 400 pounds. What is the resultant force on the ocean liner? (See Section 11.1, Example 7.)



Vectors indicate quantities that involve both magnitude and direction. In Chapter 11, you will study operations of vectors in the plane and in space. You will also learn how to represent vector operations geometrically. For example, the graphs shown above represent vector addition in the plane.

11.1 Vectors in the Plane

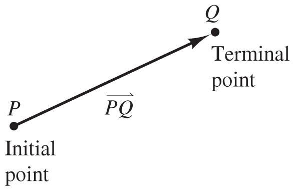
- **Write the component form of a vector.**
- **Perform vector operations and interpret the results geometrically.**
- **Write a vector as a linear combination of standard unit vectors.**
- **Use vectors to solve problems involving force or velocity.**

Component Form of a Vector

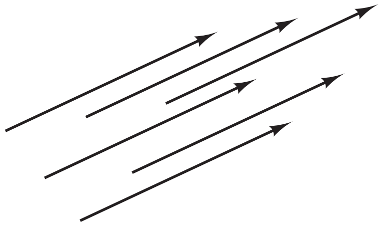
Many quantities in geometry and physics, such as area, volume, temperature, mass, and time, can be characterized by a single real number scaled to appropriate units of measure. These are called **scalar quantities,** and the real number associated with each is called a **scalar.**

Other quantities, such as force, velocity, and acceleration, involve both magnitude and direction and cannot be characterized completely by a single real number. A **directed line segment** is used to represent such a quantity, as shown in Figure 11.1. The directed line segment $\overrightarrow{PQ}$ has **initial point** P and **terminal point** Q, and its **length** (or **magnitude**) is denoted by $\|\overrightarrow{PQ}\|$. Directed line segments that have the same length and direction are **equivalent,** as shown in Figure 11.2. The set of all directed line segments that are equivalent to a given directed line segment $\overrightarrow{PQ}$ is a **vector in the plane** and is denoted by $\mathbf{v} = \overrightarrow{PQ}$. In typeset material, vectors are usually denoted by lowercase, boldface letters such as **u**, **v**, and **w**. When written by hand, however, vectors are often denoted by letters with arrows above them, such as $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Be sure you understand that a vector represents a *set* of directed line segments (each having the same length and direction). In practice, however, it is common not to distinguish between a vector and one of its representatives.



A directed line segment
Figure 11.1



Equivalent directed line segments
Figure 11.2

EXAMPLE 1 Vector Representation by Directed Line Segments

Let **v** be represented by the directed line segment from (0, 0) to (3, 2), and let **u** be represented by the directed line segment from (1, 2) to (4, 4). Show that **v** and **u** are equivalent.

Solution Let $P(0, 0)$ and $Q(3, 2)$ be the initial and terminal points of **v**, and let $R(1, 2)$ and $S(4, 4)$ be the initial and terminal points of **u**, as shown in Figure 11.3. You can use the Distance Formula to show that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the *same length.*

$$\|\overrightarrow{PQ}\| = \sqrt{(3-0)^2 + (2-0)^2} = \sqrt{13} \qquad \text{Length of } \overrightarrow{PQ}$$

$$\|\overrightarrow{RS}\| = \sqrt{(4-1)^2 + (4-2)^2} = \sqrt{13} \qquad \text{Length of } \overrightarrow{RS}$$

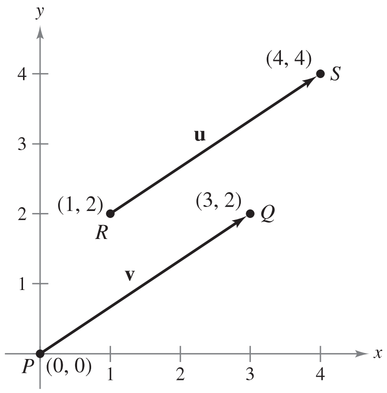
Both line segments have the *same direction*, because they both are directed toward the upper right on lines having the same slope.

$$\text{Slope of } \overrightarrow{PQ} = \frac{2-0}{3-0} = \frac{2}{3}$$

and

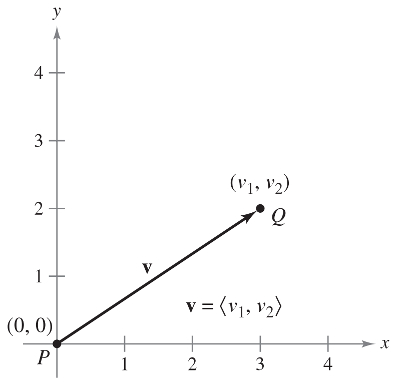
$$\text{Slope of } \overrightarrow{RS} = \frac{4-2}{4-1} = \frac{2}{3}$$

Because $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same length and direction, you can conclude that the two vectors are equivalent. That is, **v** and **u** are equivalent. ■



The vectors **u** and **v** are equivalent.
Figure 11.3

The directed line segment whose initial point is the origin is often the most convenient representative of a set of equivalent directed line segments such as those shown in Figure 11.3. This representation of **v** is said to be in **standard position.** A directed line segment whose initial point is the origin can be uniquely represented by the coordinates of its terminal point $Q(v_1, v_2)$, as shown in Figure 11.4.



A vector in standard position
Figure 11.4

DEFINITION OF COMPONENT FORM OF A VECTOR IN THE PLANE

If **v** is a vector in the plane whose initial point is the origin and whose terminal point is (v_1, v_2), then the **component form of v** is given by

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

The coordinates v_1 and v_2 are called the **components of v.** If both the initial point and the terminal point lie at the origin, then **v** is called the **zero vector** and is denoted by $\mathbf{0} = \langle 0, 0 \rangle$.

This definition implies that two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.

The following procedures can be used to convert directed line segments to component form or vice versa.

1. If $P(p_1, p_2)$ and $Q(q_1, q_2)$ are the initial and terminal points of a directed line segment, the component form of the vector **v** represented by $\overrightarrow{PQ}$ is $\langle v_1, v_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle$. Moreover, from the Distance Formula you can see that the **length** (or **magnitude**) **of v** is

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}\\ &= \sqrt{v_1^2 + v_2^2}.\end{aligned}$$

Length of a vector

2. If $\mathbf{v} = \langle v_1, v_2 \rangle$, **v** can be represented by the directed line segment, in standard position, from $P(0, 0)$ to $Q(v_1, v_2)$.

The length of **v** is also called the **norm of v.** If $\|\mathbf{v}\| = 1$, **v** is a **unit vector.** Moreover, $\|\mathbf{v}\| = 0$ if and only if **v** is the zero vector **0**.

EXAMPLE 2 Finding the Component Form and Length of a Vector

Find the component form and length of the vector **v** that has initial point $(3, -7)$ and terminal point $(-2, 5)$.

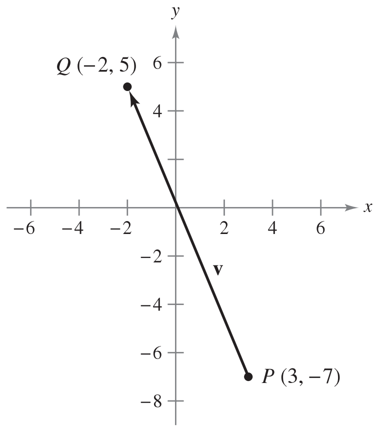
Solution Let $P(3, -7) = (p_1, p_2)$ and $Q(-2, 5) = (q_1, q_2)$. Then the components of $\mathbf{v} = \langle v_1, v_2 \rangle$ are

$$\begin{aligned}v_1 &= q_1 - p_1 = -2 - 3 = -5\\ v_2 &= q_2 - p_2 = 5 - (-7) = 12.\end{aligned}$$

So, as shown in Figure 11.5, $\mathbf{v} = \langle -5, 12 \rangle$, and the length of **v** is

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{(-5)^2 + 12^2}\\ &= \sqrt{169}\\ &= 13.\end{aligned}$$

■



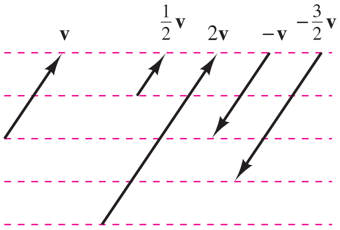
Component form of **v**: $\mathbf{v} = \langle -5, 12 \rangle$
Figure 11.5

Vector Operations

DEFINITIONS OF VECTOR ADDITION AND SCALAR MULTIPLICATION

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let c be a scalar.

1. The **vector sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$.
2. The **scalar multiple** of c and $\mathbf{u}$ is the vector $c\mathbf{u} = \langle cu_1, cu_2 \rangle$.
3. The **negative** of $\mathbf{v}$ is the vector
$$-\mathbf{v} = (-1)\mathbf{v} = \langle -v_1, -v_2 \rangle.$$
4. The **difference** of $\mathbf{u}$ and $\mathbf{v}$ is
$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle u_1 - v_1, u_2 - v_2 \rangle.$$



The scalar multiplication of v
Figure 11.6

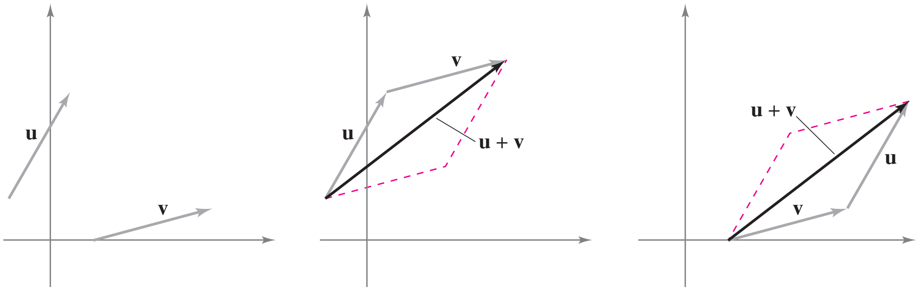
Geometrically, the scalar multiple of a vector $\mathbf{v}$ and a scalar c is the vector that is $|c|$ times as long as $\mathbf{v}$, as shown in Figure 11.6. If c is positive, $c\mathbf{v}$ has the same direction as $\mathbf{v}$. If c is negative, $c\mathbf{v}$ has the opposite direction.

The sum of two vectors can be represented geometrically by positioning the vectors (without changing their magnitudes or directions) so that the initial point of one coincides with the terminal point of the other, as shown in Figure 11.7. The vector $\mathbf{u} + \mathbf{v}$, called the **resultant vector,** is the diagonal of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as its adjacent sides.

The Granger Collection

WILLIAM ROWAN HAMILTON (1805–1865)

Some of the earliest work with vectors was done by the Irish mathematician William Rowan Hamilton. Hamilton spent many years developing a system of vector-like quantities called *quaternions*. Although Hamilton was convinced of the benefits of quaternions, the operations he defined did not produce good models for physical phenomena. It wasn't until the latter half of the nineteenth century that the Scottish physicist James Maxwell (1831–1879) restructured Hamilton's quaternions in a form useful for representing physical quantities such as force, velocity, and acceleration.



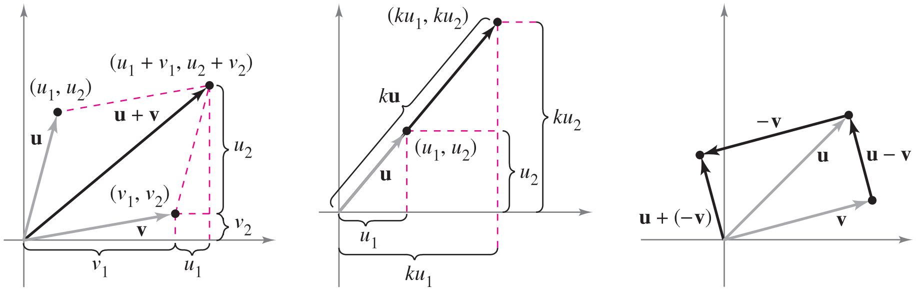
To find **u** + **v**,

(1) move the initial point of **v** to the terminal point of **u,** or

(2) move the initial point of **u** to the terminal point of **v**.

Figure 11.7

Figure 11.8 shows the equivalence of the geometric and algebraic definitions of vector addition and scalar multiplication, and presents (at far right) a geometric interpretation of $\mathbf{u} - \mathbf{v}$.



Vector addition

Scalar multiplication

Vector subtraction

Figure 11.8

EXAMPLE 3 Vector Operations

Given $\mathbf{v} = \langle -2, 5\rangle$ and $\mathbf{w} = \langle 3, 4\rangle$, find each of the vectors.

a. $\frac{1}{2}\mathbf{v}$ **b.** $\mathbf{w} - \mathbf{v}$ **c.** $\mathbf{v} + 2\mathbf{w}$

Solution

a. $\frac{1}{2}\mathbf{v} = \left\langle \frac{1}{2}(-2), \frac{1}{2}(5)\right\rangle = \left\langle -1, \frac{5}{2}\right\rangle$

b. $\mathbf{w} - \mathbf{v} = \langle w_1 - v_1, w_2 - v_2\rangle = \langle 3 - (-2), 4 - 5\rangle = \langle 5, -1\rangle$

c. Using $2\mathbf{w} = \langle 6, 8\rangle$, you have

$$\begin{aligned}\mathbf{v} + 2\mathbf{w} &= \langle -2, 5\rangle + \langle 6, 8\rangle \\ &= \langle -2 + 6, 5 + 8\rangle \\ &= \langle 4, 13\rangle.\end{aligned}$$

■

Vector addition and scalar multiplication share many properties of ordinary arithmetic, as shown in the following theorem.

THEOREM 11.1 PROPERTIES OF VECTOR OPERATIONS

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ Commutative Property
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ Associative Property
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ Additive Identity Property
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ Additive Inverse Property
5. $c(d\mathbf{u}) = (cd)\mathbf{u}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ Distributive Property
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ Distributive Property
8. $1(\mathbf{u}) = \mathbf{u}, 0(\mathbf{u}) = \mathbf{0}$

PROOF The proof of the *Associative Property* of vector addition uses the Associative Property of addition of real numbers.

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= [\langle u_1, u_2\rangle + \langle v_1, v_2\rangle] + \langle w_1, w_2\rangle \\ &= \langle u_1 + v_1, u_2 + v_2\rangle + \langle w_1, w_2\rangle \\ &= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2\rangle \\ &= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)\rangle \\ &= \langle u_1, u_2\rangle + \langle v_1 + w_1, v_2 + w_2\rangle = \mathbf{u} + (\mathbf{v} + \mathbf{w})\end{aligned}$$

Similarly, the proof of the *Distributive Property* of vectors depends on the Distributive Property of real numbers.

$$\begin{aligned}(c + d)\mathbf{u} &= (c + d)\langle u_1, u_2\rangle \\ &= \langle (c + d)u_1, (c + d)u_2\rangle \\ &= \langle cu_1 + du_1, cu_2 + du_2\rangle \\ &= \langle cu_1, cu_2\rangle + \langle du_1, du_2\rangle = c\mathbf{u} + d\mathbf{u}\end{aligned}$$

The other properties can be proved in a similar manner. ■

The Granger Collection

EMMY NOETHER (1882–1935)

One person who contributed to our knowledge of axiomatic systems was the German mathematician Emmy Noether. Noether is generally recognized as the leading woman mathematician in recent history.

Any set of vectors (with an accompanying set of scalars) that satisfies the eight properties given in Theorem 11.1 is a **vector space.*** The eight properties are the *vector space axioms*. So, this theorem states that the set of vectors in the plane (with the set of real numbers) forms a vector space.

THEOREM 11.2 LENGTH OF A SCALAR MULTIPLE

Let $\mathbf{v}$ be a vector and let c be a scalar. Then

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|. \qquad |c| \text{ is the absolute value of } c.$$

PROOF Because $c\mathbf{v} = \langle cv_1, cv_2\rangle$, it follows that

$$\begin{aligned}\|c\mathbf{v}\| = \|\langle cv_1, cv_2\rangle\| &= \sqrt{(cv_1)^2 + (cv_2)^2}\\ &= \sqrt{c^2v_1^2 + c^2v_2^2}\\ &= \sqrt{c^2(v_1^2 + v_2^2)}\\ &= |c|\sqrt{v_1^2 + v_2^2}\\ &= |c|\,\|\mathbf{v}\|.\end{aligned}$$

■

FOR FURTHER INFORMATION For more information on Emmy Noether, see the article "Emmy Noether, Greatest Woman Mathematician" by Clark Kimberling in *The Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given vector. The following theorem gives a procedure for doing this.

THEOREM 11.3 UNIT VECTOR IN THE DIRECTION OF v

If $\mathbf{v}$ is a nonzero vector in the plane, then the vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

has length 1 and the same direction as $\mathbf{v}$.

PROOF Because $1/\|\mathbf{v}\|$ is positive and $\mathbf{u} = (1/\|\mathbf{v}\|)\mathbf{v}$, you can conclude that $\mathbf{u}$ has the same direction as $\mathbf{v}$. To see that $\|\mathbf{u}\| = 1$, note that

$$\begin{aligned}\|\mathbf{u}\| &= \left\|\left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}\right\|\\ &= \left|\frac{1}{\|\mathbf{v}\|}\right|\,\|\mathbf{v}\|\\ &= \frac{1}{\|\mathbf{v}\|}\,\|\mathbf{v}\|\\ &= 1.\end{aligned}$$

So, $\mathbf{u}$ has length 1 and the same direction as $\mathbf{v}$. ■

In Theorem 11.3, $\mathbf{u}$ is called a **unit vector in the direction of v.** The process of multiplying $\mathbf{v}$ by $1/\|\mathbf{v}\|$ to get a unit vector is called **normalization of v.**

---

* *For more information about vector spaces, see* Elementary Linear Algebra, *Sixth Edition, by Larson and Falvo (Boston: Houghton Mifflin Harcourt Publishing Company, 2009).*

EXAMPLE 4 Finding a Unit Vector

Find a unit vector in the direction of $\mathbf{v} = \langle -2, 5\rangle$ and verify that it has length 1.

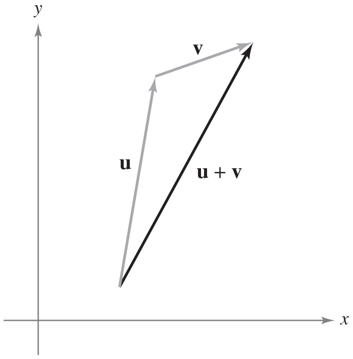
Solution From Theorem 11.3, the unit vector in the direction of $\mathbf{v}$ is

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -2, 5\rangle}{\sqrt{(-2)^2 + (5)^2}} = \frac{1}{\sqrt{29}}\langle -2, 5\rangle = \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle.$$

This vector has length 1, because

$$\sqrt{\left(\frac{-2}{\sqrt{29}}\right)^2 + \left(\frac{5}{\sqrt{29}}\right)^2} = \sqrt{\frac{4}{29} + \frac{25}{29}} = \sqrt{\frac{29}{29}} = 1.$$

■



Triangle inequality
Figure 11.9

Generally, the length of the sum of two vectors is not equal to the sum of their lengths. To see this, consider the vectors **u** and **v** as shown in Figure 11.9. By considering **u** and **v** as two sides of a triangle, you can see that the length of the third side is $\|\mathbf{u} + \mathbf{v}\|$, and you have

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

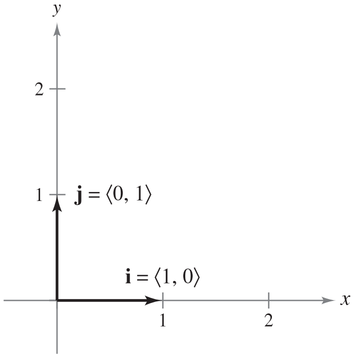
Equality occurs only if the vectors **u** and **v** have the *same direction*. This result is called the **triangle inequality** for vectors. (You are asked to prove this in Exercise 91, Section 11.3.)

Standard Unit Vectors

The unit vectors $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are called the **standard unit vectors** in the plane and are denoted by

$$\mathbf{i} = \langle 1, 0\rangle \quad \text{and} \quad \mathbf{j} = \langle 0, 1\rangle$$

Standard unit vectors



Standard unit vectors **i** and **j**
Figure 11.10

as shown in Figure 11.10. These vectors can be used to represent any vector uniquely, as follows.

$$\mathbf{v} = \langle v_1, v_2\rangle = \langle v_1, 0\rangle + \langle 0, v_2\rangle = v_1\langle 1, 0\rangle + v_2\langle 0, 1\rangle = v_1\mathbf{i} + v_2\mathbf{j}$$

The vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ is called a **linear combination** of **i** and **j**. The scalars v_1 and v_2 are called the **horizontal** and **vertical components of v.**

EXAMPLE 5 Writing a Linear Combination of Unit Vectors

Let **u** be the vector with initial point $(2, -5)$ and terminal point $(-1, 3)$, and let $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$. Write each vector as a linear combination of **i** and **j**.

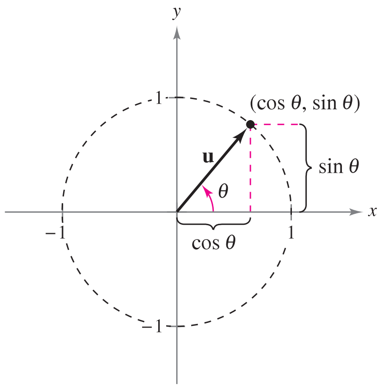
a. **u** **b.** $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v}$

Solution

a. $\mathbf{u} = \langle q_1 - p_1, q_2 - p_2\rangle$
$= \langle -1 - 2, 3 - (-5)\rangle$
$= \langle -3, 8\rangle = -3\mathbf{i} + 8\mathbf{j}$

b. $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v} = 2(-3\mathbf{i} + 8\mathbf{j}) - 3(2\mathbf{i} - \mathbf{j})$
$= -6\mathbf{i} + 16\mathbf{j} - 6\mathbf{i} + 3\mathbf{j}$
$= -12\mathbf{i} + 19\mathbf{j}$

■



The angle θ from the positive x-axis to the vector **u**
Figure 11.11

If **u** is a unit vector and θ is the angle (measured counterclockwise) from the positive x-axis to **u**, then the terminal point of **u** lies on the unit circle, and you have

$$\mathbf{u} = \langle \cos\theta, \sin\theta \rangle = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$

Unit vector

as shown in Figure 11.11. Moreover, it follows that any other nonzero vector **v** making an angle θ with the positive x-axis has the same direction as **u**, and you can write

$$\mathbf{v} = \|\mathbf{v}\|\langle \cos\theta, \sin\theta \rangle = \|\mathbf{v}\| \cos\theta\mathbf{i} + \|\mathbf{v}\| \sin\theta\mathbf{j}.$$

EXAMPLE 6 Writing a Vector of Given Magnitude and Direction

The vector **v** has a magnitude of 3 and makes an angle of $30° = \pi/6$ with the positive x-axis. Write **v** as a linear combination of the unit vectors **i** and **j**.

Solution Because the angle between **v** and the positive x-axis is $\theta = \pi/6$, you can write the following.

$$\begin{aligned}\mathbf{v} &= \|\mathbf{v}\| \cos\theta\mathbf{i} + \|\mathbf{v}\| \sin\theta\mathbf{j}\\ &= 3\cos\frac{\pi}{6}\mathbf{i} + 3\sin\frac{\pi}{6}\mathbf{j}\\ &= \frac{3\sqrt{3}}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}\end{aligned}$$

■

Applications of Vectors

Vectors have many applications in physics and engineering. One example is force. A vector can be used to represent force, because force has both magnitude and direction. If two or more forces are acting on an object, then the **resultant force** on the object is the vector sum of the vector forces.

EXAMPLE 7 Finding the Resultant Force

Two tugboats are pushing an ocean liner, as shown in Figure 11.12. Each boat is exerting a force of 400 pounds. What is the resultant force on the ocean liner?

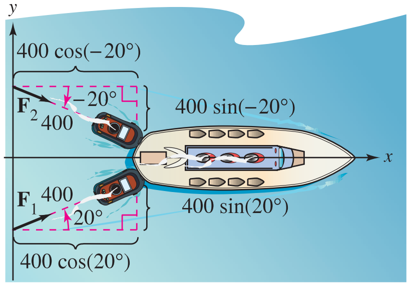
Solution Using Figure 11.12, you can represent the forces exerted by the first and second tugboats as

$$\begin{aligned}\mathbf{F}_1 &= 400\langle \cos 20°, \sin 20° \rangle\\ &= 400\cos(20°)\mathbf{i} + 400\sin(20°)\mathbf{j}\\ \mathbf{F}_2 &= 400\langle \cos(-20°), \sin(-20°) \rangle\\ &= 400\cos(20°)\mathbf{i} - 400\sin(20°)\mathbf{j}.\end{aligned}$$

The resultant force on the ocean liner is

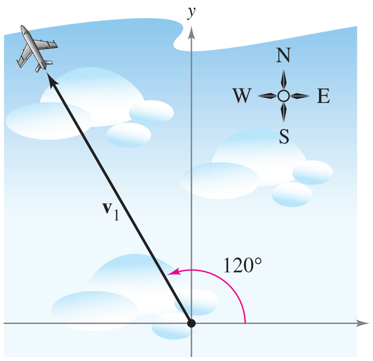
$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2\\ &= [400\cos(20°)\mathbf{i} + 400\sin(20°)\mathbf{j}] + [400\cos(20°)\mathbf{i} - 400\sin(20°)\mathbf{j}]\\ &= 800\cos(20°)\mathbf{i} \approx 752\mathbf{i}.\end{aligned}$$

So, the resultant force on the ocean liner is approximately 752 pounds in the direction of the positive x-axis. ■

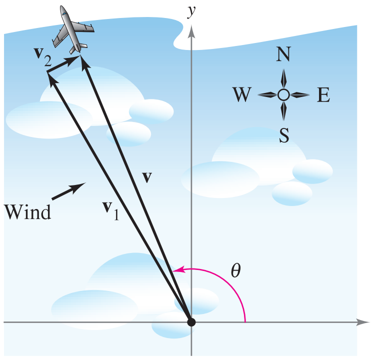


The resultant force on the ocean liner that is exerted by the two tugboats
Figure 11.12

In surveying and navigation, a **bearing** is a direction that measures the acute angle that a path or line of sight makes with a fixed north-south line. In air navigation, bearings are measured in degrees clockwise from north.



(a) Direction without wind



(b) Direction with wind

Figure 11.13

EXAMPLE 8 Finding a Velocity

An airplane is traveling at a fixed altitude with a negligible wind factor. The airplane is traveling at a speed of 500 miles per hour with a bearing of 330°, as shown in Figure 11.13(a). As the airplane reaches a certain point, it encounters wind with a velocity of 70 miles per hour in the direction N 45° E (45° east of north), as shown in Figure 11.13(b). What are the resultant speed and direction of the airplane?

Solution Using Figure 11.13(a), represent the velocity of the airplane (alone) as

$$\mathbf{v}_1 = 500\cos(120°)\mathbf{i} + 500\sin(120°)\mathbf{j}.$$

The velocity of the wind is represented by the vector

$$\mathbf{v}_2 = 70\cos(45°)\mathbf{i} + 70\sin(45°)\mathbf{j}.$$

The resultant velocity of the airplane (in the wind) is

$$\begin{aligned}\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 &= 500\cos(120°)\mathbf{i} + 500\sin(120°)\mathbf{j} + 70\cos(45°)\mathbf{i} + 70\sin(45°)\mathbf{j}\\ &\approx -200.5\mathbf{i} + 482.5\mathbf{j}.\end{aligned}$$

To find the resultant speed and direction, write $\mathbf{v} = \|\mathbf{v}\|(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$. Because $\|\mathbf{v}\| \approx \sqrt{(-200.5)^2 + (482.5)^2} \approx 522.5$, you can write

$$\mathbf{v} \approx 522.5\left(\frac{-200.5}{522.5}\mathbf{i} + \frac{482.5}{522.5}\mathbf{j}\right) \approx 522.5\left[\cos(112.6°)\mathbf{i} + \sin(112.6°)\mathbf{j}\right].$$

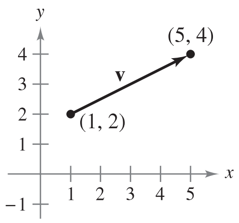
The new speed of the airplane, as altered by the wind, is approximately 522.5 miles per hour in a path that makes an angle of 112.6° with the positive *x*-axis. ■

11.1 Exercises

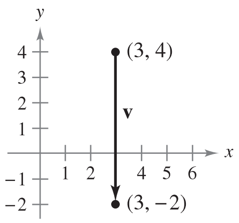
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, (a) find the component form of the vector v and (b) sketch the vector with its initial point at the origin.

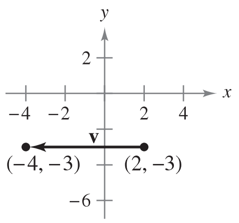
1.



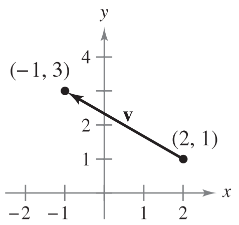
2.



3.



4.



In Exercises 5–8, find the vectors u and v whose initial and terminal points are given. Show that u and v are equivalent.

5. **u**: (3, 2), (5, 6)
v: (1, 4), (3, 8)

6. **u**: (−4, 0), (1, 8)
v: (2, −1), (7, 7)

7. **u**: (0, 3), (6, −2)
v: (3, 10), (9, 5)

8. **u**: (−4, −1), (11, −4)
v: (10, 13), (25, 10)

In Exercises 9–16, the initial and terminal points of a vector v are given. (a) Sketch the given directed line segment, (b) write the vector in component form, (c) write the vector as the linear combination of the standard unit vectors i and j, and (d) sketch the vector with its initial point at the origin.

	Initial Point	*Terminal Point*		*Initial Point*	*Terminal Point*
9.	(2, 0)	(5, 5)	**10.**	(4, −6)	(3, 6)
11.	(8, 3)	(6, −1)	**12.**	(0, −4)	(−5, −1)

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

	Initial Point	*Terminal Point*		*Initial Point*	*Terminal Point*
13.	$(6, 2)$	$(6, 6)$	**14.**	$(7, -1)$	$(-3, -1)$
15.	$\left(\frac{3}{2}, \frac{4}{3}\right)$	$\left(\frac{1}{2}, 3\right)$	**16.**	$(0.12, 0.60)$	$(0.84, 1.25)$

In Exercises 17 and 18, sketch each scalar multiple of v.

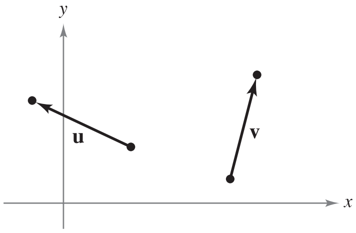
17. $\mathbf{v} = \langle 3, 5\rangle$

(a) $2\mathbf{v}$ (b) $-3\mathbf{v}$ (c) $\frac{7}{2}\mathbf{v}$ (d) $\frac{2}{3}\mathbf{v}$

18. $\mathbf{v} = \langle -2, 3\rangle$

(a) $4\mathbf{v}$ (b) $-\frac{1}{2}\mathbf{v}$ (c) $0\mathbf{v}$ (d) $-6\mathbf{v}$

In Exercises 19–22, use the figure to sketch a graph of the vector. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



19. $-\mathbf{u}$

20. $2\mathbf{u}$

21. $\mathbf{u} - \mathbf{v}$

22. $\mathbf{u} + 2\mathbf{v}$

In Exercises 23 and 24, find (a) $\frac{2}{3}\mathbf{u}$, (b) $\mathbf{v} - \mathbf{u}$, and (c) $2\mathbf{u} + 5\mathbf{v}$.

23. $\mathbf{u} = \langle 4, 9\rangle$
$\mathbf{v} = \langle 2, -5\rangle$

24. $\mathbf{u} = \langle -3, -8\rangle$
$\mathbf{v} = \langle 8, 25\rangle$

In Exercises 25–28, find the vector v where $\mathbf{u} = \langle 2, -1\rangle$ and $\mathbf{w} = \langle 1, 2\rangle$. Illustrate the vector operations geometrically.

25. $\mathbf{v} = \frac{3}{2}\mathbf{u}$

26. $\mathbf{v} = \mathbf{u} + \mathbf{w}$

27. $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$

28. $\mathbf{v} = 5\mathbf{u} - 3\mathbf{w}$

In Exercises 29 and 30, the vector v and its initial point are given. Find the terminal point.

29. $\mathbf{v} = \langle -1, 3\rangle$; Initial point: $(4, 2)$

30. $\mathbf{v} = \langle 4, -9\rangle$; Initial point: $(5, 3)$

In Exercises 31–36, find the magnitude of v.

31. $\mathbf{v} = 7\mathbf{i}$

32. $\mathbf{v} = -3\mathbf{i}$

33. $\mathbf{v} = \langle 4, 3\rangle$

34. $\mathbf{v} = \langle 12, -5\rangle$

35. $\mathbf{v} = 6\mathbf{i} - 5\mathbf{j}$

36. $\mathbf{v} = -10\mathbf{i} + 3\mathbf{j}$

In Exercises 37–40, find the unit vector in the direction of v and verify that it has length 1.

37. $\mathbf{v} = \langle 3, 12\rangle$

38. $\mathbf{v} = \langle -5, 15\rangle$

39. $\mathbf{v} = \left\langle \frac{3}{2}, \frac{5}{2}\right\rangle$

40. $\mathbf{v} = \langle -6.2, 3.4\rangle$

In Exercises 41–44, find the following.

(a) $\|\mathbf{u}\|$ **(b)** $\|\mathbf{v}\|$ **(c)** $\|\mathbf{u} + \mathbf{v}\|$

(d) $\left\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\right\|$ **(e)** $\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\|$ **(f)** $\left\|\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|}\right\|$

41. $\mathbf{u} = \langle 1, -1\rangle$
$\mathbf{v} = \langle -1, 2\rangle$

42. $\mathbf{u} = \langle 0, 1\rangle$
$\mathbf{v} = \langle 3, -3\rangle$

43. $\mathbf{u} = \left\langle 1, \frac{1}{2}\right\rangle$
$\mathbf{v} = \langle 2, 3\rangle$

44. $\mathbf{u} = \langle 2, -4\rangle$
$\mathbf{v} = \langle 5, 5\rangle$

In Exercises 45 and 46, sketch a graph of u, v, and u + v. Then demonstrate the triangle inequality using the vectors u and v.

45. $\mathbf{u} = \langle 2, 1\rangle, \quad \mathbf{v} = \langle 5, 4\rangle$

46. $\mathbf{u} = \langle -3, 2\rangle, \quad \mathbf{v} = \langle 1, -2\rangle$

In Exercises 47–50, find the vector v with the given magnitude and the same direction as u.

	Magnitude	*Direction*
47.	$\|\mathbf{v}\| = 6$	$\mathbf{u} = \langle 0, 3\rangle$
48.	$\|\mathbf{v}\| = 4$	$\mathbf{u} = \langle 1, 1\rangle$
49.	$\|\mathbf{v}\| = 5$	$\mathbf{u} = \langle -1, 2\rangle$
50.	$\|\mathbf{v}\| = 2$	$\mathbf{u} = \left\langle \sqrt{3}, 3\right\rangle$

In Exercises 51–54, find the component form of v given its magnitude and the angle it makes with the positive *x*-axis.

51. $\|\mathbf{v}\| = 3, \quad \theta = 0°$

52. $\|\mathbf{v}\| = 5, \quad \theta = 120°$

53. $\|\mathbf{v}\| = 2, \quad \theta = 150°$

54. $\|\mathbf{v}\| = 4, \quad \theta = 3.5°$

In Exercises 55–58, find the component form of u + v given the lengths of u and v and the angles that u and v make with the positive *x*-axis.

55. $\|\mathbf{u}\| = 1, \quad \theta_{\mathbf{u}} = 0°$
$\|\mathbf{v}\| = 3, \quad \theta_{\mathbf{v}} = 45°$

56. $\|\mathbf{u}\| = 4, \quad \theta_{\mathbf{u}} = 0°$
$\|\mathbf{v}\| = 2, \quad \theta_{\mathbf{v}} = 60°$

57. $\|\mathbf{u}\| = 2, \quad \theta_{\mathbf{u}} = 4$
$\|\mathbf{v}\| = 1, \quad \theta_{\mathbf{v}} = 2$

58. $\|\mathbf{u}\| = 5, \quad \theta_{\mathbf{u}} = -0.5$
$\|\mathbf{v}\| = 5, \quad \theta_{\mathbf{v}} = 0.5$

WRITING ABOUT CONCEPTS

59. In your own words, state the difference between a scalar and a vector. Give examples of each.

60. Give geometric descriptions of the operations of addition of vectors and multiplication of a vector by a scalar.

61. Identify the quantity as a scalar or as a vector. Explain your reasoning.

(a) The muzzle velocity of a gun

(b) The price of a company's stock

62. Identify the quantity as a scalar or as a vector. Explain your reasoning.

(a) The air temperature in a room

(b) The weight of a car

In Exercises 63–68, find a and b such that $\mathbf{v} = a\mathbf{u} + b\mathbf{w}$, where $\mathbf{u} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 1, -1 \rangle$.

63. $\mathbf{v} = \langle 2, 1 \rangle$

64. $\mathbf{v} = \langle 0, 3 \rangle$

65. $\mathbf{v} = \langle 3, 0 \rangle$

66. $\mathbf{v} = \langle 3, 3 \rangle$

67. $\mathbf{v} = \langle 1, 1 \rangle$

68. $\mathbf{v} = \langle -1, 7 \rangle$

In Exercises 69–74, find a unit vector (a) parallel to and (b) perpendicular to the graph of f at the given point. Then sketch the graph of f and sketch the vectors at the given point.

	Function	*Point*
69.	$f(x) = x^2$	$(3, 9)$
70.	$f(x) = -x^2 + 5$	$(1, 4)$
71.	$f(x) = x^3$	$(1, 1)$
72.	$f(x) = x^3$	$(-2, -8)$
73.	$f(x) = \sqrt{25 - x^2}$	$(3, 4)$
74.	$f(x) = \tan x$	$\left(\frac{\pi}{4}, 1\right)$

In Exercises 75 and 76, find the component form of v given the magnitudes of u and u + v and the angles that u and u + v make with the positive x-axis.

75. $\|\mathbf{u}\| = 1, \theta = 45^\circ$
$\|\mathbf{u} + \mathbf{v}\| = \sqrt{2}, \theta = 90^\circ$

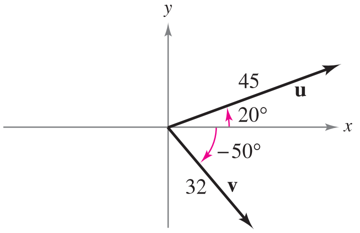
76. $\|\mathbf{u}\| = 4, \theta = 30^\circ$
$\|\mathbf{u} + \mathbf{v}\| = 6, \theta = 120^\circ$

77. ***Programming*** You are given the magnitudes of **u** and **v** and the angles that **u** and **v** make with the positive x-axis. Write a program for a graphing utility in which the output is the following.

(a) $\mathbf{u} + \mathbf{v}$ (b) $\|\mathbf{u} + \mathbf{v}\|$

(c) The angle that $\mathbf{u} + \mathbf{v}$ makes with the positive x-axis

(d) Use the program to find the magnitude and direction of the resultant of the vectors shown.



CAPSTONE

78. The initial and terminal points of vector **v** are $(3, -4)$ and $(9, 1)$, respectively.

(a) Write **v** in component form.

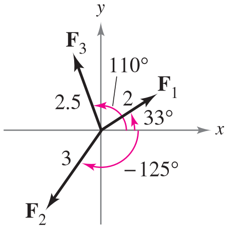
(b) Write **v** as the linear combination of the standard unit vectors **i** and **j**.

(c) Sketch **v** with its initial point at the origin.

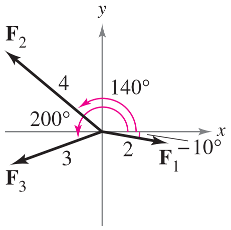
(d) Find the magnitude of **v**.

In Exercises 79 and 80, use a graphing utility to find the magnitude and direction of the resultant of the vectors.

79.



80.



81. ***Resultant Force*** Forces with magnitudes of 500 pounds and 200 pounds act on a machine part at angles of 30° and $-45°$, respectively, with the x-axis (see figure). Find the direction and magnitude of the resultant force.

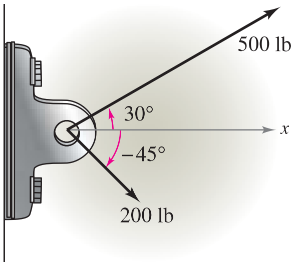


Figure for 81

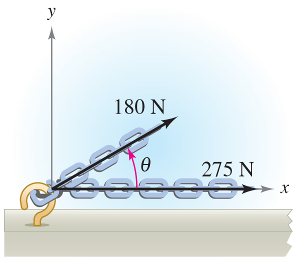


Figure for 82

82. ***Numerical and Graphical Analysis*** Forces with magnitudes of 180 newtons and 275 newtons act on a hook (see figure). The angle between the two forces is θ degrees.

(a) If $\theta = 30°$, find the direction and magnitude of the resultant force.

(b) Write the magnitude M and direction α of the resultant force as functions of θ, where $0° \leq \theta \leq 180°$.

(c) Use a graphing utility to complete the table.

θ	0°	30°	60°	90°	120°	150°	180°
M							
α							

(d) Use a graphing utility to graph the two functions M and α.

(e) Explain why one of the functions decreases for increasing values of θ whereas the other does not.

83. ***Resultant Force*** Three forces with magnitudes of 75 pounds, 100 pounds, and 125 pounds act on an object at angles of 30°, 45°, and 120°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant force.

84. ***Resultant Force*** Three forces with magnitudes of 400 newtons, 280 newtons, and 350 newtons act on an object at angles of $-30°$, 45°, and 135°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant force.

85. ***Think About It*** Consider two forces of equal magnitude acting on a point.

(a) If the magnitude of the resultant is the sum of the magnitudes of the two forces, make a conjecture about the angle between the forces.

(b) If the resultant of the forces is **0**, make a conjecture about the angle between the forces.

(c) Can the magnitude of the resultant be greater than the sum of the magnitudes of the two forces? Explain.

86. ***Graphical Reasoning*** Consider two forces $\mathbf{F}_1 = \langle 20, 0\rangle$ and $\mathbf{F}_2 = 10\langle \cos\theta, \sin\theta\rangle$.

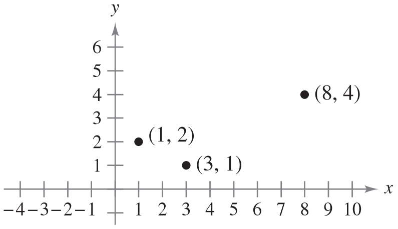
(a) Find $\|\mathbf{F}_1 + \mathbf{F}_2\|$.

(b) Determine the magnitude of the resultant as a function of θ. Use a graphing utility to graph the function for $0 \le \theta < 2\pi$.

(c) Use the graph in part (b) to determine the range of the function. What is its maximum and for what value of θ does it occur? What is its minimum and for what value of θ does it occur?

(d) Explain why the magnitude of the resultant is never 0.

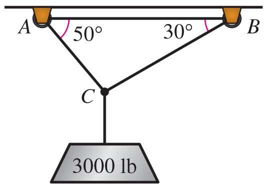
87. Three vertices of a parallelogram are $(1, 2)$, $(3, 1)$, and $(8, 4)$. Find the three possible fourth vertices (see figure).



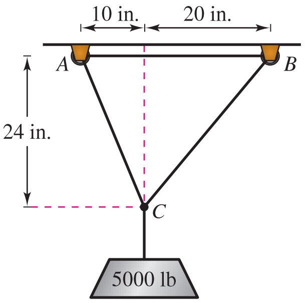
88. Use vectors to find the points of trisection of the line segment with endpoints $(1, 2)$ and $(7, 5)$.

Cable Tension **In Exercises 89 and 90, use the figure to determine the tension in each cable supporting the given load.**

89.



90.



91. ***Projectile Motion*** A gun with a muzzle velocity of 1200 feet per second is fired at an angle of 6° above the horizontal. Find the vertical and horizontal components of the velocity.

92. ***Shared Load*** To carry a 100-pound cylindrical weight, two workers lift on the ends of short ropes tied to an eyelet on the top center of the cylinder. One rope makes a 20° angle away from the vertical and the other makes a 30° angle (see figure).

(a) Find each rope's tension if the resultant force is vertical.

(b) Find the vertical component of each worker's force.

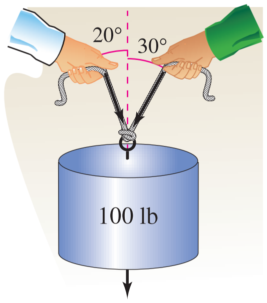


Figure for 92

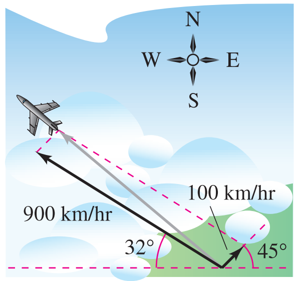


Figure for 93

93. ***Navigation*** A plane is flying with a bearing of 302°. Its speed with respect to the air is 900 kilometers per hour. The wind at the plane's altitude is from the southwest at 100 kilometers per hour (see figure). What is the true direction of the plane, and what is its speed with respect to the ground?

94. ***Navigation*** A plane flies at a constant groundspeed of 400 miles per hour due east and encounters a 50-mile-per-hour wind from the northwest. Find the airspeed and compass direction that will allow the plane to maintain its groundspeed and eastward direction.

True or False? **In Exercises 95–100, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

95. If **u** and **v** have the same magnitude and direction, then **u** and **v** are equivalent.

96. If **u** is a unit vector in the direction of **v**, then $\mathbf{v} = \|\mathbf{v}\|\,\mathbf{u}$.

97. If $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then $a^2 + b^2 = 1$.

98. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \mathbf{0}$, then $a = -b$.

99. If $a = b$, then $\|a\mathbf{i} + b\mathbf{j}\| = \sqrt{2}a$.

100. If **u** and **v** have the same magnitude but opposite directions, then $\mathbf{u} + \mathbf{v} = \mathbf{0}$.

101. Prove that $\mathbf{u} = (\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j}$ and $\mathbf{v} = (\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ are unit vectors for any angle θ.

102. ***Geometry*** Using vectors, prove that the line segment joining the midpoints of two sides of a triangle is parallel to, and one-half the length of, the third side.

103. ***Geometry*** Using vectors, prove that the diagonals of a parallelogram bisect each other.

104. Prove that the vector $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$ bisects the angle between **u** and **v**.

105. Consider the vector $\mathbf{u} = \langle x, y\rangle$. Describe the set of all points (x, y) such that $\|\mathbf{u}\| = 5$.

PUTNAM EXAM CHALLENGE

106. A coast artillery gun can fire at any angle of elevation between 0° and 90° in a fixed vertical plane. If air resistance is neglected and the muzzle velocity is constant $(= v_0)$, determine the set H of points in the plane and above the horizontal which can be hit.

This problem was composed by the Committee on the Putnam Prize Competition.
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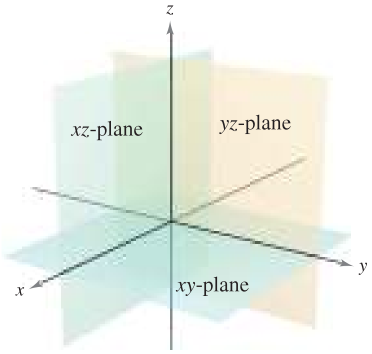
11.2 Space Coordinates and Vectors in Space

- **Understand the three-dimensional rectangular coordinate system.**
- **Analyze vectors in space.**
- **Use three-dimensional vectors to solve real-life problems.**

Coordinates in Space



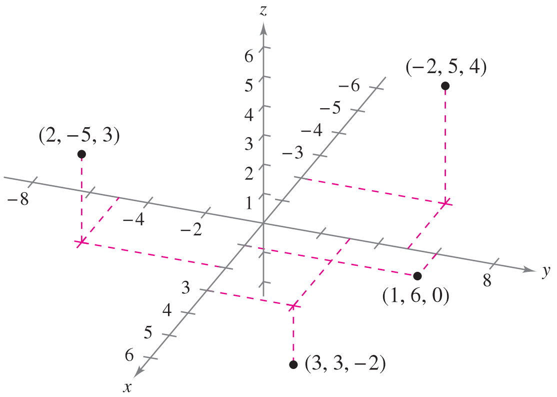
The three-dimensional coordinate system
Figure 11.14

Up to this point in the text, you have been primarily concerned with the two-dimensional coordinate system. Much of the remaining part of your study of calculus will involve the three-dimensional coordinate system.

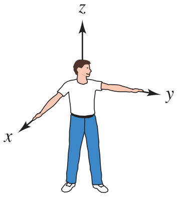
Before extending the concept of a vector to three dimensions, you must be able to identify points in the **three-dimensional coordinate system.** You can construct this system by passing a z-axis perpendicular to both the x- and y-axes at the origin. Figure 11.14 shows the positive portion of each coordinate axis. Taken as pairs, the axes determine three **coordinate planes:** the ***xy*-plane,** the ***xz*-plane,** and the ***yz*-plane.** These three coordinate planes separate three-space into eight **octants.** The first octant is the one for which all three coordinates are positive. In this three-dimensional system, a point P in space is determined by an ordered triple (x, y, z) where x, y, and z are as follows.

$x =$ directed distance from yz-plane to P
$y =$ directed distance from xz-plane to P
$z =$ directed distance from xy-plane to P

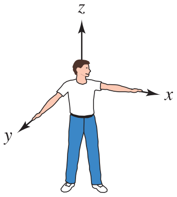
Several points are shown in Figure 11.15.



Points in the three-dimensional coordinate system are represented by ordered triples.
Figure 11.15



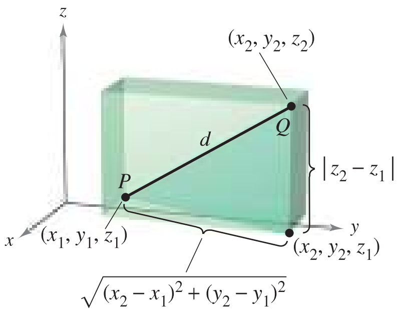
Right-handed system
Figure 11.16



Left-handed system

A three-dimensional coordinate system can have either a **left-handed** or a **right-handed** orientation. To determine the orientation of a system, imagine that you are standing at the origin, with your arms pointing in the direction of the positive x- and y-axes, and with the z-axis pointing up, as shown in Figure 11.16. The system is right-handed or left-handed depending on which hand points along the x-axis. In this text, you will work exclusively with the right-handed system.

NOTE The three-dimensional rotatable graphs that are available in the premium eBook for this text will help you visualize points or objects in a three-dimensional coordinate system. ■



The distance between two points in space
Figure 11.17

Many of the formulas established for the two-dimensional coordinate system can be extended to three dimensions. For example, to find the distance between two points in space, you can use the Pythagorean Theorem twice, as shown in Figure 11.17. By doing this, you will obtain the formula for the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Distance Formula

EXAMPLE 1 Finding the Distance Between Two Points in Space

The distance between the points $(2, -1, 3)$ and $(1, 0, -2)$ is

$$\begin{aligned} d &= \sqrt{(1 - 2)^2 + (0 + 1)^2 + (-2 - 3)^2} \\ &= \sqrt{1 + 1 + 25} \\ &= \sqrt{27} \\ &= 3\sqrt{3}. \end{aligned}$$

Distance Formula

■

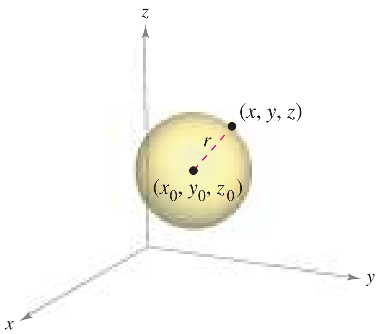


Figure 11.18

A **sphere** with center at (x_0, y_0, z_0) and radius r is defined to be the set of all points (x, y, z) such that the distance between (x, y, z) and (x_0, y_0, z_0) is r. You can use the Distance Formula to find the **standard equation of a sphere** of radius r, centered at (x_0, y_0, z_0). If (x, y, z) is an arbitrary point on the sphere, the equation of the sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Equation of sphere

as shown in Figure 11.18. Moreover, the midpoint of the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Midpoint Formula

EXAMPLE 2 Finding the Equation of a Sphere

Find the standard equation of the sphere that has the points $(5, -2, 3)$ and $(0, 4, -3)$ as endpoints of a diameter.

Solution Using the Midpoint Formula, the center of the sphere is

$$\left(\frac{5 + 0}{2}, \frac{-2 + 4}{2}, \frac{3 - 3}{2}\right) = \left(\frac{5}{2}, 1, 0\right).$$

Midpoint Formula

By the Distance Formula, the radius is

$$r = \sqrt{\left(0 - \frac{5}{2}\right)^2 + (4 - 1)^2 + (-3 - 0)^2} = \sqrt{\frac{97}{4}} = \frac{\sqrt{97}}{2}.$$

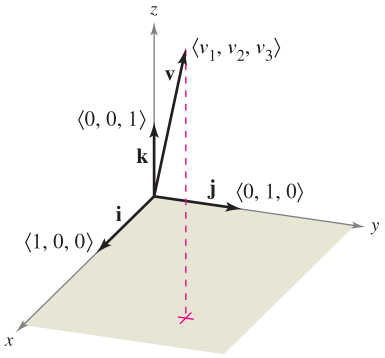
Therefore, the standard equation of the sphere is

$$\left(x - \frac{5}{2}\right)^2 + (y - 1)^2 + z^2 = \frac{97}{4}.$$

Equation of sphere

■

Vectors in Space



The standard unit vectors in space
Figure 11.19

In space, vectors are denoted by ordered triples $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. The **zero vector** is denoted by $\mathbf{0} = \langle 0, 0, 0\rangle$. Using the unit vectors $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$ in the direction of the positive z-axis, the **standard unit vector notation** for $\mathbf{v}$ is

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

as shown in Figure 11.19. If $\mathbf{v}$ is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, as shown in Figure 11.20, the component form of $\mathbf{v}$ is given by subtracting the coordinates of the initial point from the coordinates of the terminal point, as follows.

$$\mathbf{v} = \langle v_1, v_2, v_3\rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3\rangle$$

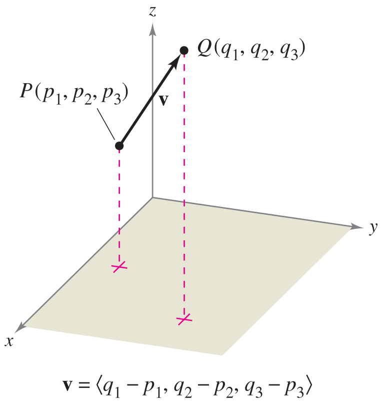

z
$Q(q_1, q_2, q_3)$
$P(p_1, p_2, p_3)$
v
y
x
$\mathbf{v} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3\rangle$

Figure 11.20

VECTORS IN SPACE

Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ be vectors in space and let c be a scalar.

1. *Equality of Vectors:* $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$.
2. *Component Form:* If $\mathbf{v}$ is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, then
$$\mathbf{v} = \langle v_1, v_2, v_3\rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3\rangle.$$
3. *Length:* $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
4. *Unit Vector in the Direction of* $\mathbf{v}$: $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\dfrac{1}{\|\mathbf{v}\|}\right)\langle v_1, v_2, v_3\rangle, \quad \mathbf{v} \neq \mathbf{0}$
5. *Vector Addition:* $\mathbf{v} + \mathbf{u} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3\rangle$
6. *Scalar Multiplication:* $c\mathbf{v} = \langle cv_1, cv_2, cv_3\rangle$

NOTE The properties of vector addition and scalar multiplication given in Theorem 11.1 are also valid for vectors in space. ■

EXAMPLE 3 Finding the Component Form of a Vector in Space

Find the component form and magnitude of the vector $\mathbf{v}$ having initial point $(-2, 3, 1)$ and terminal point $(0, -4, 4)$. Then find a unit vector in the direction of $\mathbf{v}$.

Solution The component form of $\mathbf{v}$ is

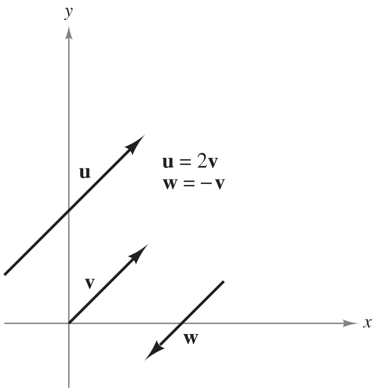
$$\begin{aligned}\mathbf{v} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3\rangle &= \langle 0 - (-2), -4 - 3, 4 - 1\rangle \\ &= \langle 2, -7, 3\rangle\end{aligned}$$

which implies that its magnitude is

$$\|\mathbf{v}\| = \sqrt{2^2 + (-7)^2 + 3^2} = \sqrt{62}.$$

The unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{62}}\langle 2, -7, 3\rangle = \left\langle \frac{2}{\sqrt{62}}, \frac{-7}{\sqrt{62}}, \frac{3}{\sqrt{62}}\right\rangle.$$ ■



Parallel vectors
Figure 11.21

Recall from the definition of scalar multiplication that positive scalar multiples of a nonzero vector $\mathbf{v}$ have the same direction as $\mathbf{v}$, whereas negative multiples have the direction opposite of $\mathbf{v}$. In general, two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if there is some scalar c such that $\mathbf{u} = c\mathbf{v}$.

DEFINITION OF PARALLEL VECTORS

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if there is some scalar c such that $\mathbf{u} = c\mathbf{v}$.

For example, in Figure 11.21, the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are parallel because $\mathbf{u} = 2\mathbf{v}$ and $\mathbf{w} = -\mathbf{v}$.

EXAMPLE 4 Parallel Vectors

Vector $\mathbf{w}$ has initial point $(2, -1, 3)$ and terminal point $(-4, 7, 5)$. Which of the following vectors is parallel to $\mathbf{w}$?

a. $\mathbf{u} = \langle 3, -4, -1 \rangle$

b. $\mathbf{v} = \langle 12, -16, 4 \rangle$

Solution Begin by writing $\mathbf{w}$ in component form.

$$\mathbf{w} = \langle -4 - 2, 7 - (-1), 5 - 3 \rangle = \langle -6, 8, 2 \rangle$$

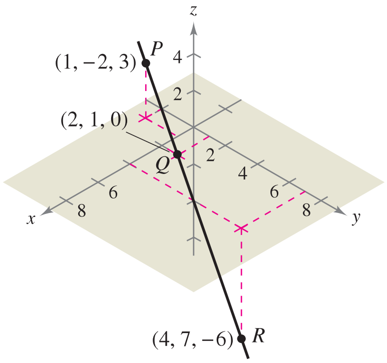
a. Because $\mathbf{u} = \langle 3, -4, -1 \rangle = -\frac{1}{2}\langle -6, 8, 2 \rangle = -\frac{1}{2}\mathbf{w}$, you can conclude that $\mathbf{u}$ is parallel to $\mathbf{w}$.

b. In this case, you want to find a scalar c such that

$$\langle 12, -16, 4 \rangle = c\langle -6, 8, 2 \rangle.$$

$$\begin{aligned} 12 &= -6c \rightarrow c = -2 \\ -16 &= 8c \rightarrow c = -2 \\ 4 &= 2c \rightarrow c = 2 \end{aligned}$$

Because there is no c for which the equation has a solution, the vectors are not parallel.



The points P, Q, and R lie on the same line.
Figure 11.22

EXAMPLE 5 Using Vectors to Determine Collinear Points

Determine whether the points $P(1, -2, 3)$, $Q(2, 1, 0)$, and $R(4, 7, -6)$ are collinear.

Solution The component forms of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\overrightarrow{PQ} = \langle 2 - 1, 1 - (-2), 0 - 3 \rangle = \langle 1, 3, -3 \rangle$$

and

$$\overrightarrow{PR} = \langle 4 - 1, 7 - (-2), -6 - 3 \rangle = \langle 3, 9, -9 \rangle.$$

These two vectors have a common initial point. So, P, Q, and R lie on the same line if and only if $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel—which they are because $\overrightarrow{PR} = 3\,\overrightarrow{PQ}$, as shown in Figure 11.22. ■

EXAMPLE 6 Standard Unit Vector Notation

a. Write the vector $\mathbf{v} = 4\mathbf{i} - 5\mathbf{k}$ in component form.

b. Find the terminal point of the vector $\mathbf{v} = 7\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, given that the initial point is $P(-2, 3, 5)$.

Solution

a. Because $\mathbf{j}$ is missing, its component is 0 and

$$\mathbf{v} = 4\mathbf{i} - 5\mathbf{k} = \langle 4, 0, -5\rangle.$$

b. You need to find $Q(q_1, q_2, q_3)$ such that $\mathbf{v} = \overrightarrow{PQ} = 7\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. This implies that $q_1 - (-2) = 7$, $q_2 - 3 = -1$, and $q_3 - 5 = 3$. The solution of these three equations is $q_1 = 5$, $q_2 = 2$, and $q_3 = 8$. Therefore, Q is $(5, 2, 8)$. ■

Application

EXAMPLE 7 Measuring Force

A television camera weighing 120 pounds is supported by a tripod, as shown in Figure 11.23. Represent the force exerted on each leg of the tripod as a vector.

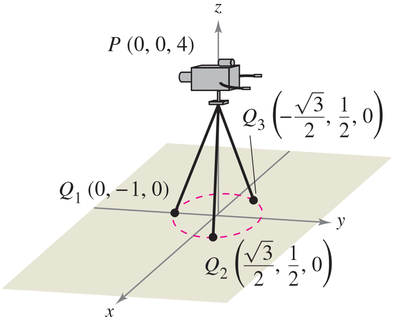

z
$P\ (0, 0, 4)$
$Q_3\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$
$Q_1\ (0, -1, 0)$
y
$Q_2\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$
x

Figure 11.23

Solution Let the vectors $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ represent the forces exerted on the three legs. From Figure 11.23, you can determine the directions of $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ to be as follows.

$$\overrightarrow{PQ}_1 = \langle 0 - 0, -1 - 0, 0 - 4\rangle = \langle 0, -1, -4\rangle$$

$$\overrightarrow{PQ}_2 = \left\langle \frac{\sqrt{3}}{2} - 0, \frac{1}{2} - 0, 0 - 4\right\rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}, -4\right\rangle$$

$$\overrightarrow{PQ}_3 = \left\langle -\frac{\sqrt{3}}{2} - 0, \frac{1}{2} - 0, 0 - 4\right\rangle = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, -4\right\rangle$$

Because each leg has the same length, and the total force is distributed equally among the three legs, you know that $\|\mathbf{F}_1\| = \|\mathbf{F}_2\| = \|\mathbf{F}_3\|$. So, there exists a constant c such that

$$\mathbf{F}_1 = c\langle 0, -1, -4\rangle, \quad \mathbf{F}_2 = c\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}, -4\right\rangle, \quad \text{and} \quad \mathbf{F}_3 = c\left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, -4\right\rangle.$$

Let the total force exerted by the object be given by $\mathbf{F} = \langle 0, 0, -120\rangle$. Then, using the fact that

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

you can conclude that $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ all have a vertical component of -40. This implies that $c(-4) = -40$ and $c = 10$. Therefore, the forces exerted on the legs can be represented by

$$\mathbf{F}_1 = \langle 0, -10, -40\rangle$$
$$\mathbf{F}_2 = \langle 5\sqrt{3}, 5, -40\rangle$$
$$\mathbf{F}_3 = \langle -5\sqrt{3}, 5, -40\rangle.$$

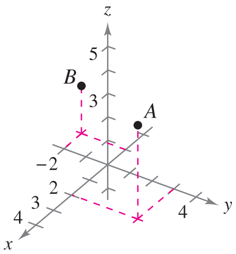
■

11.2 Exercises

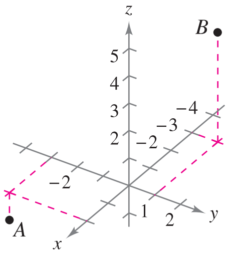
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, approximate the coordinates of the points.

1.



2.



In Exercises 3–6, plot the points on the same three-dimensional coordinate system.

3. (a) $(2, 1, 3)$ (b) $(-1, 2, 1)$

4. (a) $(3, -2, 5)$ (b) $\left(\frac{3}{2}, 4, -2\right)$

5. (a) $(5, -2, 2)$ (b) $(5, -2, -2)$

6. (a) $(0, 4, -5)$ (b) $(4, 0, 5)$

In Exercises 7–10, find the coordinates of the point.

7. The point is located three units behind the yz-plane, four units to the right of the xz-plane, and five units above the xy-plane.

8. The point is located seven units in front of the yz-plane, two units to the left of the xz-plane, and one unit below the xy-plane.

9. The point is located on the x-axis, 12 units in front of the yz-plane.

10. The point is located in the yz-plane, three units to the right of the xz-plane, and two units above the xy-plane.

11. ***Think About It*** What is the z-coordinate of any point in the xy-plane?

12. ***Think About It*** What is the x-coordinate of any point in the yz-plane?

In Exercises 13–24, determine the location of a point (x, y, z) that satisfies the condition(s).

13. $z = 6$

14. $y = 2$

15. $x = -3$

16. $z = -\frac{5}{2}$

17. $y < 0$

18. $x > 0$

19. $|y| \le 3$

20. $|x| > 4$

21. $xy > 0, \quad z = -3$

22. $xy < 0, \quad z = 4$

23. $xyz < 0$

24. $xyz > 0$

In Exercises 25–28, find the distance between the points.

25. $(0, 0, 0), \quad (-4, 2, 7)$

26. $(-2, 3, 2), \quad (2, -5, -2)$

27. $(1, -2, 4), \quad (6, -2, -2)$

28. $(2, 2, 3), \quad (4, -5, 6)$

In Exercises 29–32, find the lengths of the sides of the triangle with the indicated vertices, and determine whether the triangle is a right triangle, an isosceles triangle, or neither.

29. $(0, 0, 4), (2, 6, 7), (6, 4, -8)$

30. $(3, 4, 1), (0, 6, 2), (3, 5, 6)$

31. $(-1, 0, -2), (-1, 5, 2), (-3, -1, 1)$

32. $(4, -1, -1), (2, 0, -4), (3, 5, -1)$

33. ***Think About It*** The triangle in Exercise 29 is translated five units upward along the z-axis. Determine the coordinates of the translated triangle.

34. ***Think About It*** The triangle in Exercise 30 is translated three units to the right along the y-axis. Determine the coordinates of the translated triangle.

In Exercises 35 and 36, find the coordinates of the midpoint of the line segment joining the points.

35. $(5, -9, 7), (-2, 3, 3)$

36. $(4, 0, -6), (8, 8, 20)$

In Exercises 37–40, find the standard equation of the sphere.

37. Center: $(0, 2, 5)$
Radius: 2

38. Center: $(4, -1, 1)$
Radius: 5

39. Endpoints of a diameter: $(2, 0, 0), (0, 6, 0)$

40. Center: $(-3, 2, 4)$, tangent to the yz-plane

In Exercises 41–44, complete the square to write the equation of the sphere in standard form. Find the center and radius.

41. $x^2 + y^2 + z^2 - 2x + 6y + 8z + 1 = 0$

42. $x^2 + y^2 + z^2 + 9x - 2y + 10z + 19 = 0$

43. $9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$

44. $4x^2 + 4y^2 + 4z^2 - 24x - 4y + 8z - 23 = 0$

In Exercises 45–48, describe the solid satisfying the condition.

45. $x^2 + y^2 + z^2 \le 36$

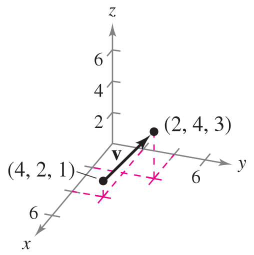
46. $x^2 + y^2 + z^2 > 4$

47. $x^2 + y^2 + z^2 < 4x - 6y + 8z - 13$

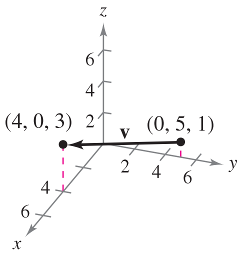
48. $x^2 + y^2 + z^2 > -4x + 6y - 8z - 13$

In Exercises 49–52, (a) find the component form of the vector v, (b) write the vector using standard unit vector notation, and (c) sketch the vector with its initial point at the origin.

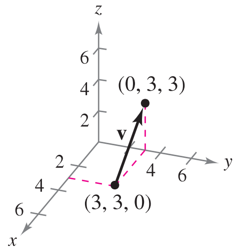
49.



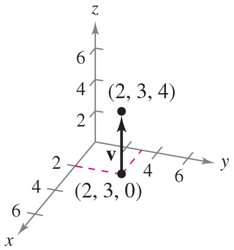
50.



51.



52.



In Exercises 53–56, find the component form and magnitude of the vector v with the given initial and terminal points. Then find a unit vector in the direction of v.

	Initial Point	*Terminal Point*
53.	$(3, 2, 0)$	$(4, 1, 6)$
54.	$(4, -5, 2)$	$(-1, 7, -3)$
55.	$(-4, 3, 1)$	$(-5, 3, 0)$
56.	$(1, -2, 4)$	$(2, 4, -2)$

In Exercises 57 and 58, the initial and terminal points of a vector v are given. (a) Sketch the directed line segment, (b) find the component form of the vector, (c) write the vector using standard unit vector notation, and (d) sketch the vector with its initial point at the origin.

57. Initial point: $(-1, 2, 3)$
Terminal point: $(3, 3, 4)$

58. Initial point: $(2, -1, -2)$
Terminal point: $(-4, 3, 7)$

In Exercises 59 and 60, the vector v and its initial point are given. Find the terminal point.

59. $\mathbf{v} = \langle 3, -5, 6\rangle$
Initial point: $(0, 6, 2)$

60. $\mathbf{v} = \left\langle 1, -\frac{2}{3}, \frac{1}{2}\right\rangle$
Initial point: $\left(0, 2, \frac{5}{2}\right)$

In Exercises 61 and 62, find each scalar multiple of v and sketch its graph.

61. $\mathbf{v} = \langle 1, 2, 2\rangle$
(a) $2\mathbf{v}$ (b) $-\mathbf{v}$
(c) $\frac{3}{2}\mathbf{v}$ (d) $0\mathbf{v}$

62. $\mathbf{v} = \langle 2, -2, 1\rangle$
(a) $-\mathbf{v}$ (b) $2\mathbf{v}$
(c) $\frac{1}{2}\mathbf{v}$ (d) $\frac{5}{2}\mathbf{v}$

In Exercises 63–68, find the vector z, given that $\mathbf{u} = \langle 1, 2, 3\rangle$, $\mathbf{v} = \langle 2, 2, -1\rangle$, and $\mathbf{w} = \langle 4, 0, -4\rangle$.

63. $\mathbf{z} = \mathbf{u} - \mathbf{v}$

64. $\mathbf{z} = \mathbf{u} - \mathbf{v} + 2\mathbf{w}$

65. $\mathbf{z} = 2\mathbf{u} + 4\mathbf{v} - \mathbf{w}$

66. $\mathbf{z} = 5\mathbf{u} - 3\mathbf{v} - \frac{1}{2}\mathbf{w}$

67. $2\mathbf{z} - 3\mathbf{u} = \mathbf{w}$

68. $2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = \mathbf{0}$

In Exercises 69–72, determine which of the vectors is (are) parallel to z. Use a graphing utility to confirm your results.

69. $\mathbf{z} = \langle 3, 2, -5\rangle$
(a) $\langle -6, -4, 10\rangle$
(b) $\left\langle 2, \frac{4}{3}, -\frac{10}{3}\right\rangle$
(c) $\langle 6, 4, 10\rangle$
(d) $\langle 1, -4, 2\rangle$

70. $\mathbf{z} = \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{3}{4}\mathbf{k}$
(a) $6\mathbf{i} - 4\mathbf{j} + 9\mathbf{k}$
(b) $-\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{3}{2}\mathbf{k}$
(c) $12\mathbf{i} + 9\mathbf{k}$
(d) $\frac{3}{4}\mathbf{i} - \mathbf{j} + \frac{9}{8}\mathbf{k}$

71. $\mathbf{z}$ has initial point $(1, -1, 3)$ and terminal point $(-2, 3, 5)$.
(a) $-6\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$ (b) $4\mathbf{j} + 2\mathbf{k}$

72. $\mathbf{z}$ has initial point $(5, 4, 1)$ and terminal point $(-2, -4, 4)$.
(a) $\langle 7, 6, 2\rangle$ (b) $\langle 14, 16, -6\rangle$

In Exercises 73–76, use vectors to determine whether the points are collinear.

73. $(0, -2, -5), (3, 4, 4), (2, 2, 1)$

74. $(4, -2, 7), (-2, 0, 3), (7, -3, 9)$

75. $(1, 2, 4), (2, 5, 0), (0, 1, 5)$

76. $(0, 0, 0), (1, 3, -2), (2, -6, 4)$

In Exercises 77 and 78, use vectors to show that the points form the vertices of a parallelogram.

77. $(2, 9, 1), (3, 11, 4), (0, 10, 2), (1, 12, 5)$

78. $(1, 1, 3), (9, -1, -2), (11, 2, -9), (3, 4, -4)$

In Exercises 79–84, find the magnitude of v.

79. $\mathbf{v} = \langle 0, 0, 0\rangle$

80. $\mathbf{v} = \langle 1, 0, 3\rangle$

81. $\mathbf{v} = 3\mathbf{j} - 5\mathbf{k}$

82. $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$

83. $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$

84. $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$

In Exercises 85–88, find a unit vector (a) in the direction of v and (b) in the direction opposite of v.

85. $\mathbf{v} = \langle 2, -1, 2\rangle$

86. $\mathbf{v} = \langle 6, 0, 8\rangle$

87. $\mathbf{v} = \langle 3, 2, -5\rangle$

88. $\mathbf{v} = \langle 8, 0, 0\rangle$

89. ***Programming*** You are given the component forms of the vectors $\mathbf{u}$ and $\mathbf{v}$. Write a program for a graphing utility in which the output is (a) the component form of $\mathbf{u} + \mathbf{v}$, (b) $\|\mathbf{u} + \mathbf{v}\|$, (c) $\|\mathbf{u}\|$, and (d) $\|\mathbf{v}\|$. (e) Run the program for the vectors $\mathbf{u} = \langle -1, 3, 4\rangle$ and $\mathbf{v} = \langle 5, 4.5, -6\rangle$.

CAPSTONE

90. Consider the two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, and let s and t be real numbers. Describe the geometric figure generated by the terminal points of the three vectors $t\mathbf{v}$, $\mathbf{u} + t\mathbf{v}$, and $s\mathbf{u} + t\mathbf{v}$.

In Exercises 91 and 92, determine the values of *c* that satisfy the equation. Let $\mathbf{u} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

91. $\|c\mathbf{v}\| = 7$

92. $\|c\mathbf{u}\| = 4$

In Exercises 93–96, find the vector v with the given magnitude and direction u.

	Magnitude	*Direction*
93.	10	$\mathbf{u} = \langle 0, 3, 3\rangle$
94.	3	$\mathbf{u} = \langle 1, 1, 1\rangle$
95.	$\frac{3}{2}$	$\mathbf{u} = \langle 2, -2, 1\rangle$
96.	7	$\mathbf{u} = \langle -4, 6, 2\rangle$

In Exercises 97 and 98, sketch the vector v and write its component form.

97. **v** lies in the yz-plane, has magnitude 2, and makes an angle of 30° with the positive y-axis.

98. **v** lies in the xz-plane, has magnitude 5, and makes an angle of 45° with the positive z-axis.

In Exercises 99 and 100, use vectors to find the point that lies two-thirds of the way from P to Q.

99. $P(4, 3, 0)$, $Q(1, -3, 3)$ **100.** $P(1, 2, 5)$, $Q(6, 8, 2)$

101. Let $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$, and $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$.

(a) Sketch **u** and **v**.

(b) If $\mathbf{w} = \mathbf{0}$, show that a and b must both be zero.

(c) Find a and b such that $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(d) Show that no choice of a and b yields $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

102. ***Writing*** The initial and terminal points of the vector **v** are (x_1, y_1, z_1) and (x, y, z). Describe the set of all points (x, y, z) such that $\|\mathbf{v}\| = 4$.

WRITING ABOUT CONCEPTS

103. A point in the three-dimensional coordinate system has coordinates (x_0, y_0, z_0). Describe what each coordinate measures.

104. Give the formula for the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

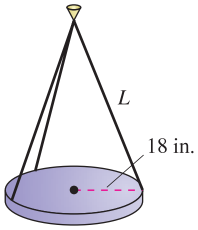
105. Give the standard equation of a sphere of radius r, centered at (x_0, y_0, z_0).

106. State the definition of parallel vectors.

107. Let A, B, and C be vertices of a triangle. Find $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

108. Let $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle 1, 1, 1 \rangle$. Describe the set of all points (x, y, z) such that $\|\mathbf{r} - \mathbf{r}_0\| = 2$.

109. ***Numerical, Graphical, and Analytic Analysis*** The lights in an auditorium are 24-pound discs of radius 18 inches. Each disc is supported by three equally spaced cables that are L inches long (see figure).



(a) Write the tension T in each cable as a function of L. Determine the domain of the function.

(b) Use a graphing utility and the function in part (a) to complete the table.

L	20	25	30	35	40	45	50
T							

(c) Use a graphing utility to graph the function in part (a). Determine the asymptotes of the graph.

(d) Confirm the asymptotes of the graph in part (c) analytically.

(e) Determine the minimum length of each cable if a cable is designed to carry a maximum load of 10 pounds.

110. ***Think About It*** Suppose the length of each cable in Exercise 109 has a fixed length $L = a$, and the radius of each disc is r_0 inches. Make a conjecture about the limit $\lim_{r_0 \to a^-} T$ and give a reason for your answer.

111. ***Diagonal of a Cube*** Find the component form of the unit vector **v** in the direction of the diagonal of the cube shown in the figure.

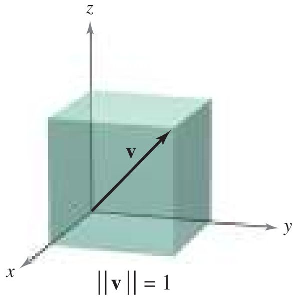


Figure for 111

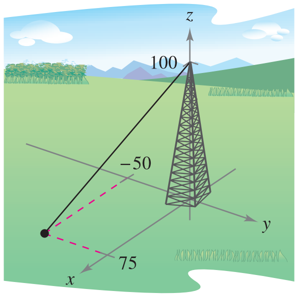


Figure for 112

112. ***Tower Guy Wire*** The guy wire supporting a 100-foot tower has a tension of 550 pounds. Using the distances shown in the figure, write the component form of the vector **F** representing the tension in the wire.

113. ***Load Supports*** Find the tension in each of the supporting cables in the figure if the weight of the crate is 500 newtons.

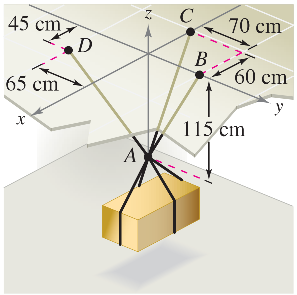


Figure for 113

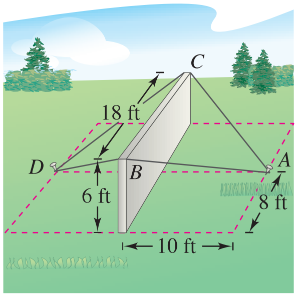


Figure for 114

114. ***Construction*** A precast concrete wall is temporarily kept in its vertical position by ropes (see figure). Find the total force exerted on the pin at position A. The tensions in AB and AC are 420 pounds and 650 pounds.

115. Write an equation whose graph consists of the set of points $P(x, y, z)$ that are twice as far from $A(0, -1, 1)$ as from $B(1, 2, 0)$.

11.3 The Dot Product of Two Vectors

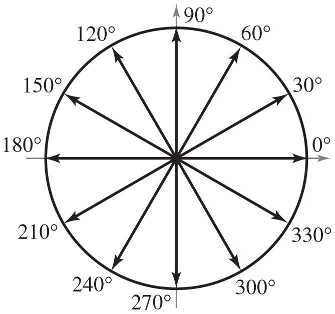
- **Use properties of the dot product of two vectors.**
- **Find the angle between two vectors using the dot product.**
- **Find the direction cosines of a vector in space.**
- **Find the projection of a vector onto another vector.**
- **Use vectors to find the work done by a constant force.**

The Dot Product

So far you have studied two operations with vectors—vector addition and multiplication by a scalar—each of which yields another vector. In this section you will study a third vector operation, called the **dot product.** This product yields a scalar, rather than a vector.

EXPLORATION

Interpreting a Dot Product
Several vectors are shown below on the unit circle. Find the dot products of several pairs of vectors. Then find the angle between each pair that you used. Make a conjecture about the relationship between the dot product of two vectors and the angle between the vectors.



DEFINITION OF DOT PRODUCT

The **dot product** of $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ is

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2.$$

The **dot product** of $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ is

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

NOTE Because the dot product of two vectors yields a scalar, it is also called the **scalar product** (or **inner product**) of the two vectors. ■

THEOREM 11.4 PROPERTIES OF THE DOT PRODUCT

Let **u**, **v**, and **w** be vectors in the plane or in space and let c be a scalar.

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$	Commutative Property
2. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$	Distributive Property
3. $c(\mathbf{u}\cdot\mathbf{v}) = c\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot c\mathbf{v}$	
4. $\mathbf{0}\cdot\mathbf{v} = 0$	
5. $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$	

PROOF To prove the first property, let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= u_1v_1 + u_2v_2 + u_3v_3\\ &= v_1u_1 + v_2u_2 + v_3u_3\\ &= \mathbf{v}\cdot\mathbf{u}.\end{aligned}$$

For the fifth property, let $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then

$$\begin{aligned}\mathbf{v}\cdot\mathbf{v} &= v_1^2 + v_2^2 + v_3^2\\ &= \left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2\\ &= \|\mathbf{v}\|^2.\end{aligned}$$

Proofs of the other properties are left to you. ■

EXAMPLE 1 Finding Dot Products

Given $\mathbf{u} = \langle 2, -2\rangle$, $\mathbf{v} = \langle 5, 8\rangle$, and $\mathbf{w} = \langle -4, 3\rangle$, find each of the following.

a. $\mathbf{u} \cdot \mathbf{v}$ **b.** $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

c. $\mathbf{u} \cdot (2\mathbf{v})$ **d.** $\|\mathbf{w}\|^2$

Solution

a. $\mathbf{u} \cdot \mathbf{v} = \langle 2, -2\rangle \cdot \langle 5, 8\rangle = 2(5) + (-2)(8) = -6$

b. $(\mathbf{u} \cdot \mathbf{v})\mathbf{w} = -6\langle -4, 3\rangle = \langle 24, -18\rangle$

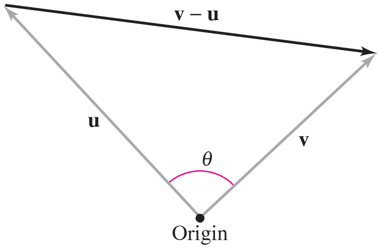
c. $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-6) = -12$ Theorem 11.4

d.
$$\begin{aligned}\|\mathbf{w}\|^2 &= \mathbf{w} \cdot \mathbf{w} && \text{Theorem 11.4}\\ &= \langle -4, 3\rangle \cdot \langle -4, 3\rangle && \text{Substitute } \langle -4, 3\rangle \text{ for } \mathbf{w}.\\ &= (-4)(-4) + (3)(3) && \text{Definition of dot product}\\ &= 25 && \text{Simplify.}\end{aligned}$$

Notice that the result of part (b) is a *vector* quantity, whereas the results of the other three parts are *scalar* quantities. ■

Angle Between Two Vectors

The **angle between two nonzero vectors** is the angle θ, $0 \le \theta \le \pi$, between their respective standard position vectors, as shown in Figure 11.24. The next theorem shows how to find this angle using the dot product. (Note that the angle between the zero vector and another vector is not defined here.)



The angle between two vectors
Figure 11.24

THEOREM 11.5 ANGLE BETWEEN TWO VECTORS

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

PROOF Consider the triangle determined by vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{v} - \mathbf{u}$, as shown in Figure 11.24. By the Law of Cosines, you can write

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta.$$

Using the properties of the dot product, the left side can be rewritten as

$$\begin{aligned}\|\mathbf{v} - \mathbf{u}\|^2 &= (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u})\\ &= (\mathbf{v} - \mathbf{u}) \cdot \mathbf{v} - (\mathbf{v} - \mathbf{u}) \cdot \mathbf{u}\\ &= \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u}\\ &= \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2\end{aligned}$$

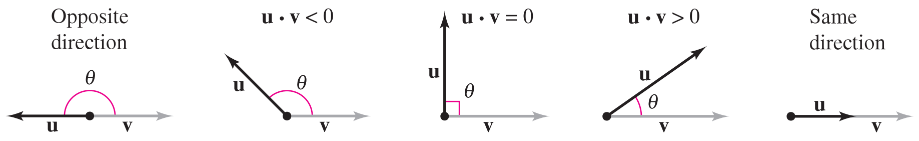
and substitution back into the Law of Cosines yields

$$\begin{aligned}\|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta\\ -2\mathbf{u} \cdot \mathbf{v} &= -2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta\\ \cos\theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.\end{aligned}$$
■

If the angle between two vectors is known, rewriting Theorem 11.5 in the form

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta$$ Alternative form of dot product

produces an alternative way to calculate the dot product. From this form, you can see that because $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are always positive, $\mathbf{u} \cdot \mathbf{v}$ and $\cos\theta$ will always have the same sign. Figure 11.25 shows the possible orientations of two vectors.



$\theta = \pi$, $\cos\theta = -1$ | $\pi/2 < \theta < \pi$, $-1 < \cos\theta < 0$ | $\theta = \pi/2$, $\cos\theta = 0$ | $0 < \theta < \pi/2$, $0 < \cos\theta < 1$ | $\theta = 0$, $\cos\theta = 1$

Figure 11.25

From Theorem 11.5, you can see that two nonzero vectors meet at a right angle if and only if their dot product is zero. Two such vectors are said to be **orthogonal.**

DEFINITION OF ORTHOGONAL VECTORS

The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

NOTE The terms "perpendicular," "orthogonal," and "normal" all mean essentially the same thing—meeting at right angles. However, it is common to say that two vectors are *orthogonal*, two lines or planes are *perpendicular*, and a vector is *normal* to a given line or plane. ■

From this definition, it follows that the zero vector is orthogonal to every vector $\mathbf{u}$, because $\mathbf{0} \cdot \mathbf{u} = 0$. Moreover, for $0 \le \theta \le \pi$, you know that $\cos\theta = 0$ if and only if $\theta = \pi/2$. So, you can use Theorem 11.5 to conclude that two *nonzero* vectors are orthogonal if and only if the angle between them is $\pi/2$.

EXAMPLE 2 Finding the Angle Between Two Vectors

For $\mathbf{u} = \langle 3, -1, 2\rangle$, $\mathbf{v} = \langle -4, 0, 2\rangle$, $\mathbf{w} = \langle 1, -1, -2\rangle$, and $\mathbf{z} = \langle 2, 0, -1\rangle$, find the angle between each pair of vectors.

a. $\mathbf{u}$ and $\mathbf{v}$ **b.** $\mathbf{u}$ and $\mathbf{w}$ **c.** $\mathbf{v}$ and $\mathbf{z}$

Solution

a. $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{-12 + 0 + 4}{\sqrt{14}\sqrt{20}} = \dfrac{-8}{2\sqrt{14}\sqrt{5}} = \dfrac{-4}{\sqrt{70}}$

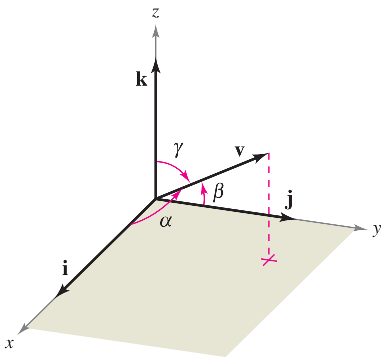
Because $\mathbf{u} \cdot \mathbf{v} < 0$, $\theta = \arccos \dfrac{-4}{\sqrt{70}} \approx 2.069$ radians.

b. $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{u}\|\,\|\mathbf{w}\|} = \dfrac{3 + 1 - 4}{\sqrt{14}\sqrt{6}} = \dfrac{0}{\sqrt{84}} = 0$

Because $\mathbf{u} \cdot \mathbf{w} = 0$, $\mathbf{u}$ and $\mathbf{w}$ are *orthogonal*. So, $\theta = \pi/2$.

c. $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{z}}{\|\mathbf{v}\|\,\|\mathbf{z}\|} = \dfrac{-8 + 0 - 2}{\sqrt{20}\sqrt{5}} = \dfrac{-10}{\sqrt{100}} = -1$

Consequently, $\theta = \pi$. Note that $\mathbf{v}$ and $\mathbf{z}$ are parallel, with $\mathbf{v} = -2\mathbf{z}$. ■



Direction angles
Figure 11.26

Direction Cosines

For a vector in the plane, you have seen that it is convenient to measure direction in terms of the angle, measured counterclockwise, *from* the positive x-axis *to* the vector. In space it is more convenient to measure direction in terms of the angles *between* the nonzero vector $\mathbf{v}$ and the three unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, as shown in Figure 11.26. The angles α, β, and γ are the **direction angles of v,** and $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the **direction cosines of v.** Because

$$\mathbf{v}\cdot\mathbf{i} = \|\mathbf{v}\|\,\|\mathbf{i}\|\cos\alpha = \|\mathbf{v}\|\cos\alpha$$

and

$$\mathbf{v}\cdot\mathbf{i} = \langle v_1, v_2, v_3\rangle \cdot \langle 1, 0, 0\rangle = v_1$$

it follows that $\cos\alpha = v_1/\|\mathbf{v}\|$. By similar reasoning with the unit vectors $\mathbf{j}$ and $\mathbf{k}$, you have

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}$$

α is the angle between **v** and **i**.

$$\cos\beta = \frac{v_2}{\|\mathbf{v}\|}$$

β is the angle between **v** and **j**.

$$\cos\gamma = \frac{v_3}{\|\mathbf{v}\|}.$$

γ is the angle between **v** and **k**.

Consequently, any nonzero vector $\mathbf{v}$ in space has the normalized form

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{v_1}{\|\mathbf{v}\|}\mathbf{i} + \frac{v_2}{\|\mathbf{v}\|}\mathbf{j} + \frac{v_3}{\|\mathbf{v}\|}\mathbf{k} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

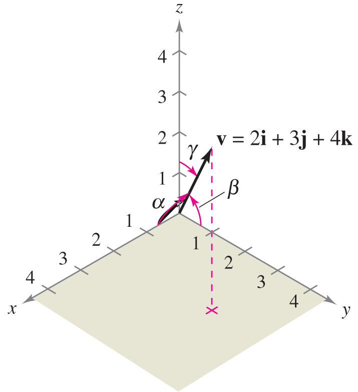
and because $\mathbf{v}/\|\mathbf{v}\|$ is a unit vector, it follows that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

EXAMPLE 3 Finding Direction Angles

Find the direction cosines and angles for the vector $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, and show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

α = angle between **v** and **i**
β = angle between **v** and **j**
γ = angle between **v** and **k**



The direction angles of **v**
Figure 11.27

Solution Because $\|\mathbf{v}\| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$, you can write the following.

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|} = \frac{2}{\sqrt{29}} \;\Rightarrow\; \alpha \approx 68.2^\circ$$

Angle between **v** and **i**

$$\cos\beta = \frac{v_2}{\|\mathbf{v}\|} = \frac{3}{\sqrt{29}} \;\Rightarrow\; \beta \approx 56.1^\circ$$

Angle between **v** and **j**

$$\cos\gamma = \frac{v_3}{\|\mathbf{v}\|} = \frac{4}{\sqrt{29}} \;\Rightarrow\; \gamma \approx 42.0^\circ$$

Angle between **v** and **k**

Furthermore, the sum of the squares of the direction cosines is

$$\begin{aligned}\cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \frac{4}{29} + \frac{9}{29} + \frac{16}{29}\\ &= \frac{29}{29}\\ &= 1.\end{aligned}$$

See Figure 11.27. ■

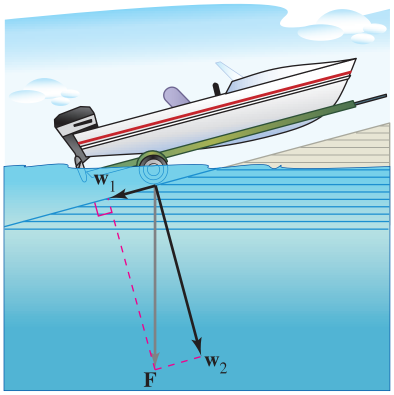
Projections and Vector Components

You have already seen applications in which two vectors are added to produce a resultant vector. Many applications in physics and engineering pose the reverse problem—decomposing a given vector into the sum of two **vector components.** The following physical example enables you to see the usefulness of this procedure.

Consider a boat on an inclined ramp, as shown in Figure 11.28. The force $\mathbf{F}$ due to gravity pulls the boat *down* the ramp and *against* the ramp. These two forces, $\mathbf{w}_1$ and $\mathbf{w}_2$, are orthogonal—they are called the vector components of $\mathbf{F}$.

$$\mathbf{F} = \mathbf{w}_1 + \mathbf{w}_2 \qquad \text{Vector components of } \mathbf{F}$$

The forces $\mathbf{w}_1$ and $\mathbf{w}_2$ help you analyze the effect of gravity on the boat. For example, $\mathbf{w}_1$ indicates the force necessary to keep the boat from rolling down the ramp, whereas $\mathbf{w}_2$ indicates the force that the tires must withstand.

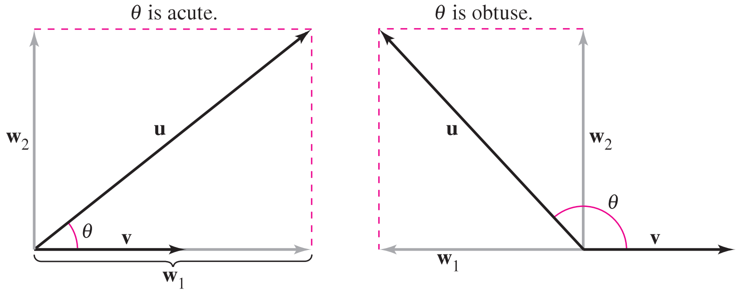


The force due to gravity pulls the boat against the ramp and down the ramp.
Figure 11.28

DEFINITIONS OF PROJECTION AND VECTOR COMPONENTS

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors. Moreover, let $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$, and $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$, as shown in Figure 11.29.

1. $\mathbf{w}_1$ is called the **projection of u onto v** or the **vector component of u along v,** and is denoted by $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$.
2. $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ is called the **vector component of u orthogonal to v.**



$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} =$ projection of $\mathbf{u}$ onto $\mathbf{v} =$ vector component of $\mathbf{u}$ along $\mathbf{v}$
$\mathbf{w}_2 =$ vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$
Figure 11.29

EXAMPLE 4 Finding a Vector Component of u Orthogonal to v

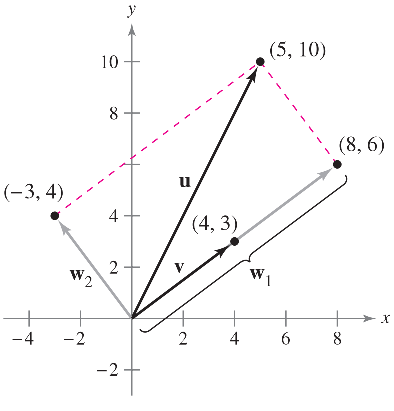
Find the vector component of $\mathbf{u} = \langle 5, 10\rangle$ that is orthogonal to $\mathbf{v} = \langle 4, 3\rangle$, given that $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 8, 6\rangle$ and

$$\mathbf{u} = \langle 5, 10\rangle = \mathbf{w}_1 + \mathbf{w}_2.$$

Solution Because $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$, it follows that $\mathbf{w}_2$ is the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$. So, you have

$$\begin{aligned}\mathbf{w}_2 &= \mathbf{u} - \mathbf{w}_1\\ &= \langle 5, 10\rangle - \langle 8, 6\rangle\\ &= \langle -3, 4\rangle.\end{aligned}$$

Check to see that $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$, as shown in Figure 11.30. ■



$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$
Figure 11.30

From Example 4, you can see that it is easy to find the vector component $\mathbf{w}_2$ once you have found the projection, $\mathbf{w}_1$, of $\mathbf{u}$ onto $\mathbf{v}$. To find this projection, use the dot product given in the theorem below, which you will prove in Exercise 92.

NOTE Note the distinction between the terms "component" and "vector component." For example, using the standard unit vectors with $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, u_1 is the *component* of $\mathbf{u}$ in the direction of $\mathbf{i}$ and $u_1\mathbf{i}$ is the *vector component* in the direction of $\mathbf{i}$.

THEOREM 11.6 PROJECTION USING THE DOT PRODUCT

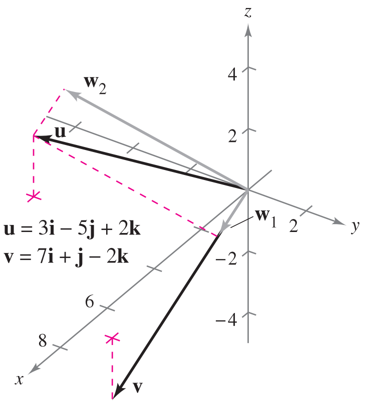
If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then the projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

The projection of $\mathbf{u}$ onto $\mathbf{v}$ can be written as a scalar multiple of a unit vector in the direction of $\mathbf{v}$. That is,

$$\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} = (k)\frac{\mathbf{v}}{\|\mathbf{v}\|} \quad \Rightarrow \quad k = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|} = \|\mathbf{u}\|\cos\theta.$$

The scalar k is called the **component of u in the direction of v.**



z
4
2
−2
−4
$\mathbf{w}_2$
$\mathbf{u}$
$\mathbf{w}_1$
2
y
6
8
x
$\mathbf{v}$
$\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$
$\mathbf{v} = 7\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$

Figure 11.31

EXAMPLE 5 Decomposing a Vector into Vector Components

Find the projection of $\mathbf{u}$ onto $\mathbf{v}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ for the vectors $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 7\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ shown in Figure 11.31.

Solution The projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\mathbf{w}_1 = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{12}{54}\right)(7\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \frac{14}{9}\mathbf{i} + \frac{2}{9}\mathbf{j} - \frac{4}{9}\mathbf{k}.$$

The vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ is the vector

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}) - \left(\frac{14}{9}\mathbf{i} + \frac{2}{9}\mathbf{j} - \frac{4}{9}\mathbf{k}\right) = \frac{13}{9}\mathbf{i} - \frac{47}{9}\mathbf{j} + \frac{22}{9}\mathbf{k}.$$

EXAMPLE 6 Finding a Force

A 600-pound boat sits on a ramp inclined at 30°, as shown in Figure 11.32. What force is required to keep the boat from rolling down the ramp?

Solution Because the force due to gravity is vertical and downward, you can represent the gravitational force by the vector $\mathbf{F} = -600\mathbf{j}$. To find the force required to keep the boat from rolling down the ramp, project $\mathbf{F}$ onto a unit vector $\mathbf{v}$ in the direction of the ramp, as follows.

$$\mathbf{v} = \cos 30°\mathbf{i} + \sin 30°\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} \qquad \text{Unit vector along ramp}$$

Therefore, the projection of $\mathbf{F}$ onto $\mathbf{v}$ is given by

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{F} = \left(\frac{\mathbf{F}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = (\mathbf{F}\cdot\mathbf{v})\mathbf{v} = (-600)\left(\frac{1}{2}\right)\mathbf{v} = -300\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right).$$

The magnitude of this force is 300, and therefore a force of 300 pounds is required to keep the boat from rolling down the ramp. ■

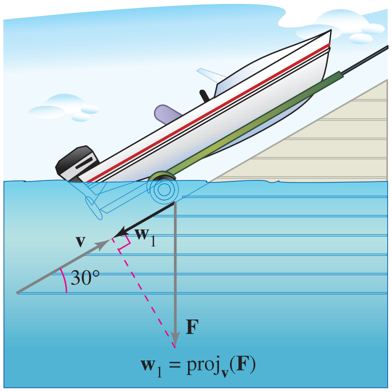
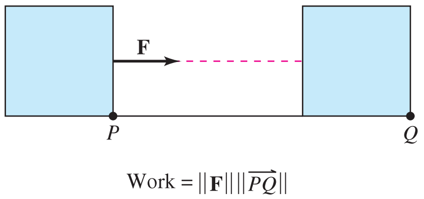
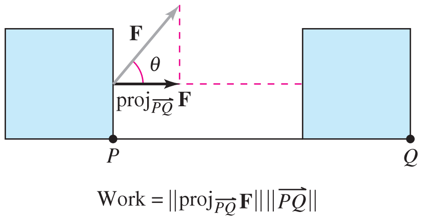


Figure 11.32



(a) Force acts along the line of motion.


F
θ
$\text{proj}_{\overrightarrow{PQ}}\mathbf{F}$
P
Q
Work = $\|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\|\overrightarrow{PQ}\|$

(b) Force acts at angle θ with the line of motion.

Figure 11.33

Work

The work W done by the constant force $\mathbf{F}$ acting along the line of motion of an object is given by

$$W = (\text{magnitude of force})(\text{distance}) = \|\mathbf{F}\|\,\|\overrightarrow{PQ}\|$$

as shown in Figure 11.33(a). If the constant force $\mathbf{F}$ is not directed along the line of motion, you can see from Figure 11.33(b) that the work W done by the force is

$$W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\| = (\cos\theta)\|\mathbf{F}\|\,\|\overrightarrow{PQ}\| = \mathbf{F}\cdot\overrightarrow{PQ}.$$

This notion of work is summarized in the following definition.

DEFINITION OF WORK

The work W done by a constant force $\mathbf{F}$ as its point of application moves along the vector $\overrightarrow{PQ}$ is given by either of the following.

1. $W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\|$ Projection form
2. $W = \mathbf{F}\cdot\overrightarrow{PQ}$ Dot product form

EXAMPLE 7 Finding Work

To close a sliding door, a person pulls on a rope with a constant force of 50 pounds at a constant angle of 60°, as shown in Figure 11.34. Find the work done in moving the door 12 feet to its closed position.

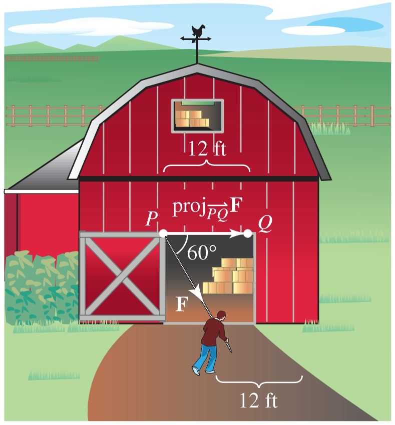


Figure 11.34

Solution Using a projection, you can calculate the work as follows.

$$\begin{aligned} W &= \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\| && \text{Projection form for work} \\ &= \cos(60^\circ)\,\|\mathbf{F}\|\,\|\overrightarrow{PQ}\| \\ &= \frac{1}{2}(50)(12) \\ &= 300 \text{ foot-pounds} \end{aligned}$$

11.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, find (a) $\mathbf{u}\cdot\mathbf{v}$, (b) $\mathbf{u}\cdot\mathbf{u}$, (c) $\|\mathbf{u}\|^2$, (d) $(\mathbf{u}\cdot\mathbf{v})\mathbf{v}$, and (e) $\mathbf{u}\cdot(2\mathbf{v})$.

1. $\mathbf{u} = \langle 3, 4\rangle$, $\mathbf{v} = \langle -1, 5\rangle$
2. $\mathbf{u} = \langle 4, 10\rangle$, $\mathbf{v} = \langle -2, 3\rangle$
3. $\mathbf{u} = \langle 6, -4\rangle$, $\mathbf{v} = \langle -3, 2\rangle$
4. $\mathbf{u} = \langle -4, 8\rangle$, $\mathbf{v} = \langle 7, 5\rangle$
5. $\mathbf{u} = \langle 2, -3, 4\rangle$, $\mathbf{v} = \langle 0, 6, 5\rangle$
6. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{i}$
7. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$
   $\mathbf{v} = \mathbf{i} - \mathbf{k}$
8. $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
   $\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

In Exercises 9 and 10, find $\mathbf{u}\cdot\mathbf{v}$.

9. $\|\mathbf{u}\| = 8$, $\|\mathbf{v}\| = 5$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/3$.
10. $\|\mathbf{u}\| = 40$, $\|\mathbf{v}\| = 25$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $5\pi/6$.

In Exercises 11–18, find the angle θ between the vectors.

11. $\mathbf{u} = \langle 1, 1\rangle$, $\mathbf{v} = \langle 2, -2\rangle$
12. $\mathbf{u} = \langle 3, 1\rangle$, $\mathbf{v} = \langle 2, -1\rangle$
13. $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j}$
14. $\mathbf{u} = \cos\left(\dfrac{\pi}{6}\right)\mathbf{i} + \sin\left(\dfrac{\pi}{6}\right)\mathbf{j}$
    $\mathbf{v} = \cos\left(\dfrac{3\pi}{4}\right)\mathbf{i} + \sin\left(\dfrac{3\pi}{4}\right)\mathbf{j}$
15. $\mathbf{u} = \langle 1, 1, 1\rangle$
    $\mathbf{v} = \langle 2, 1, -1\rangle$
16. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
    $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$
17. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$
    $\mathbf{v} = -2\mathbf{j} + 3\mathbf{k}$
18. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
    $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

In Exercises 19–26, determine whether u and v are orthogonal, parallel, or neither.

19. $\mathbf{u} = \langle 4, 0\rangle$, $\mathbf{v} = \langle 1, 1\rangle$
20. $\mathbf{u} = \langle 2, 18\rangle$, $\mathbf{v} = \left\langle \frac{3}{2}, -\frac{1}{6}\right\rangle$

21. $\mathbf{u} = \langle 4, 3 \rangle$
$\mathbf{v} = \left\langle \frac{1}{2}, -\frac{2}{3} \right\rangle$

22. $\mathbf{u} = -\frac{1}{3}(\mathbf{i} - 2\mathbf{j})$
$\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$

23. $\mathbf{u} = \mathbf{j} + 6\mathbf{k}$
$\mathbf{v} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

24. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

25. $\mathbf{u} = \langle 2, -3, 1 \rangle$
$\mathbf{v} = \langle -1, -1, -1 \rangle$

26. $\mathbf{u} = \langle \cos\theta, \sin\theta, -1 \rangle$
$\mathbf{v} = \langle \sin\theta, -\cos\theta, 0 \rangle$

In Exercises 27–30, the vertices of a triangle are given. Determine whether the triangle is an acute triangle, an obtuse triangle, or a right triangle. Explain your reasoning.

27. $(1, 2, 0), (0, 0, 0), (-2, 1, 0)$

28. $(-3, 0, 0), (0, 0, 0), (1, 2, 3)$

29. $(2, 0, 1), (0, 1, 2), (-0.5, 1.5, 0)$

30. $(2, -7, 3), (-1, 5, 8), (4, 6, -1)$

In Exercises 31–34, find the direction cosines of u and demonstrate that the sum of the squares of the direction cosines is 1.

31. $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

32. $\mathbf{u} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

33. $\mathbf{u} = \langle 0, 6, -4 \rangle$

34. $\mathbf{u} = \langle a, b, c \rangle$

In Exercises 35–38, find the direction angles of the vector.

35. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

36. $\mathbf{u} = -4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

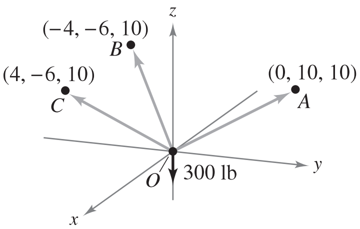
37. $\mathbf{u} = \langle -1, 5, 2 \rangle$

38. $\mathbf{u} = \langle -2, 6, 1 \rangle$

In Exercises 39 and 40, use a graphing utility to find the magnitude and direction angles of the resultant of forces $\mathbf{F}_1$ and $\mathbf{F}_2$ with initial points at the origin. The magnitude and terminal point of each vector are given.

	Vector	*Magnitude*	*Terminal Point*
39.	$\mathbf{F}_1$	50 lb	$(10, 5, 3)$
	$\mathbf{F}_2$	80 lb	$(12, 7, -5)$
40.	$\mathbf{F}_1$	300 N	$(-20, -10, 5)$
	$\mathbf{F}_2$	100 N	$(5, 15, 0)$

41. ***Load-Supporting Cables*** A load is supported by three cables, as shown in the figure. Find the direction angles of the load-supporting cable *OA*.



42. ***Load-Supporting Cables*** The tension in the cable *OA* in Exercise 41 is 200 newtons. Determine the weight of the load.

In Exercises 43–50, (a) find the projection of u onto v, and (b) find the vector component of u orthogonal to v.

43. $\mathbf{u} = \langle 6, 7 \rangle, \quad \mathbf{v} = \langle 1, 4 \rangle$

44. $\mathbf{u} = \langle 9, 7 \rangle, \quad \mathbf{v} = \langle 1, 3 \rangle$

45. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}, \quad \mathbf{v} = 5\mathbf{i} + \mathbf{j}$

46. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}, \quad \mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$

47. $\mathbf{u} = \langle 0, 3, 3 \rangle, \quad \mathbf{v} = \langle -1, 1, 1 \rangle$

48. $\mathbf{u} = \langle 8, 2, 0 \rangle, \quad \mathbf{v} = \langle 2, 1, -1 \rangle$

49. $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad \mathbf{v} = 3\mathbf{j} + 4\mathbf{k}$

50. $\mathbf{u} = \mathbf{i} + 4\mathbf{k}, \quad \mathbf{v} = 3\mathbf{i} + 2\mathbf{k}$

WRITING ABOUT CONCEPTS

51. Define the dot product of vectors **u** and **v**.

52. State the definition of orthogonal vectors. If vectors are neither parallel nor orthogonal, how do you find the angle between them? Explain.

53. Determine which of the following are defined for nonzero vectors **u**, **v**, and **w**. Explain your reasoning.

(a) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$ (b) $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

(c) $\mathbf{u} \cdot \mathbf{v} + \mathbf{w}$ (d) $\|\mathbf{u}\| \cdot (\mathbf{v} + \mathbf{w})$

54. Describe direction cosines and direction angles of a vector **v**.

55. Give a geometric description of the projection of **u** onto **v**.

56. What can be said about the vectors **u** and **v** if (a) the projection of **u** onto **v** equals **u** and (b) the projection of **u** onto **v** equals **0**?

57. If the projection of **u** onto **v** has the same magnitude as the projection of **v** onto **u**, can you conclude that $\|\mathbf{u}\| = \|\mathbf{v}\|$? Explain.

CAPSTONE

58. What is known about θ, the angle between two nonzero vectors **u** and **v**, if

(a) $\mathbf{u} \cdot \mathbf{v} = 0$? (b) $\mathbf{u} \cdot \mathbf{v} > 0$? (c) $\mathbf{u} \cdot \mathbf{v} < 0$?

59. ***Revenue*** The vector $\mathbf{u} = \langle 3240, 1450, 2235 \rangle$ gives the numbers of hamburgers, chicken sandwiches, and cheeseburgers, respectively, sold at a fast-food restaurant in one week. The vector $\mathbf{v} = \langle 1.35, 2.65, 1.85 \rangle$ gives the prices (in dollars) per unit for the three food items. Find the dot product $\mathbf{u} \cdot \mathbf{v}$, and explain what information it gives.

60. ***Revenue*** Repeat Exercise 59 after increasing prices by 4%. Identify the vector operation used to increase prices by 4%.

61. ***Programming*** Given vectors **u** and **v** in component form, write a program for a graphing utility in which the output is (a) $\|\mathbf{u}\|$, (b) $\|\mathbf{v}\|$, and (c) the angle between **u** and **v**.

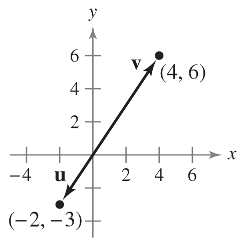
62. ***Programming*** Use the program you wrote in Exercise 61 to find the angle between the vectors $\mathbf{u} = \langle 8, -4, 2 \rangle$ and $\mathbf{v} = \langle 2, 5, 2 \rangle$.

63. ***Programming*** Given vectors **u** and **v** in component form, write a program for a graphing utility in which the output is the component form of the projection of **u** onto **v**.

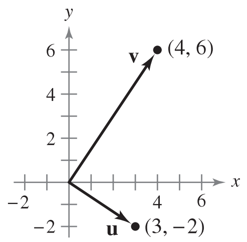
64. ***Programming*** Use the program you wrote in Exercise 63 to find the projection of **u** onto **v** for $\mathbf{u} = \langle 5, 6, 2 \rangle$ and $\mathbf{v} = \langle -1, 3, 4 \rangle$.

Think About It **In Exercises 65 and 66, use the figure to determine mentally the projection of u onto v. (The coordinates of the terminal points of the vectors in standard position are given.) Verify your results analytically.**

65.



66.



In Exercises 67–70, find two vectors in opposite directions that are orthogonal to the vector u. (The answers are not unique.)

67. $\mathbf{u} = -\frac{1}{4}\mathbf{i} + \frac{3}{2}\mathbf{j}$

68. $\mathbf{u} = 9\mathbf{i} - 4\mathbf{j}$

69. $\mathbf{u} = \langle 3, 1, -2 \rangle$

70. $\mathbf{u} = \langle 4, -3, 6 \rangle$

71. ***Braking Load*** A 48,000-pound truck is parked on a 10° slope (see figure). Assume the only force to overcome is that due to gravity. Find (a) the force required to keep the truck from rolling down the hill and (b) the force perpendicular to the hill.

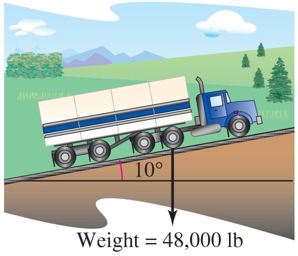


Figure for 71

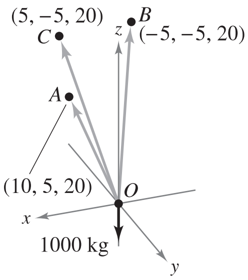


Figure for 72

72. ***Load-Supporting Cables*** Find the magnitude of the projection of the load-supporting cable OA onto the positive z-axis as shown in the figure.

73. ***Work*** An object is pulled 10 feet across a floor, using a force of 85 pounds. The direction of the force is 60° above the horizontal (see figure). Find the work done.

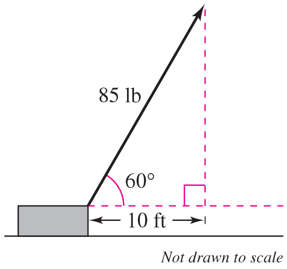


Figure for 73

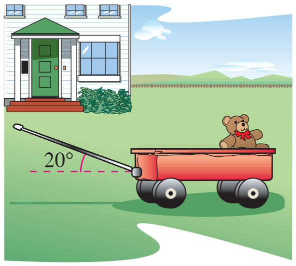


Figure for 74

74. ***Work*** A toy wagon is pulled by exerting a force of 25 pounds on a handle that makes a 20° angle with the horizontal (see figure in left column). Find the work done in pulling the wagon 50 feet.

75. ***Work*** A car is towed using a force of 1600 newtons. The chain used to pull the car makes a 25° angle with the horizontal. Find the work done in towing the car 2 kilometers.

76. ***Work*** A sled is pulled by exerting a force of 100 newtons on a rope that makes a 25° angle with the horizontal. Find the work done in pulling the sled 40 meters.

True or False? **In Exercises 77 and 78, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

77. If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{v} = \mathbf{w}$.

78. If **u** and **v** are orthogonal to **w**, then $\mathbf{u} + \mathbf{v}$ is orthogonal to **w**.

79. Find the angle between a cube's diagonal and one of its edges.

80. Find the angle between the diagonal of a cube and the diagonal of one of its sides.

In Exercises 81–84, (a) find all points of intersection of the graphs of the two equations, (b) find the unit tangent vectors to each curve at their points of intersection, and (c) find the angles $(0° \leq \theta \leq 90°)$ between the curves at their points of intersection.

81. $y = x^2, \quad y = x^{1/3}$

82. $y = x^3, \quad y = x^{1/3}$

83. $y = 1 - x^2, \quad y = x^2 - 1$

84. $(y + 1)^2 = x, \quad y = x^3 - 1$

85. Use vectors to prove that the diagonals of a rhombus are perpendicular.

86. Use vectors to prove that a parallelogram is a rectangle if and only if its diagonals are equal in length.

87. ***Bond Angle*** Consider a regular tetrahedron with vertices $(0, 0, 0)$, $(k, k, 0)$, $(k, 0, k)$, and $(0, k, k)$, where k is a positive real number.

(a) Sketch the graph of the tetrahedron.

(b) Find the length of each edge.

(c) Find the angle between any two edges.

(d) Find the angle between the line segments from the centroid $(k/2, k/2, k/2)$ to two vertices. This is the bond angle for a molecule such as CH_4 or $PbCl_4$, where the structure of the molecule is a tetrahedron.

88. Consider the vectors $\mathbf{u} = \langle \cos \alpha, \sin \alpha, 0 \rangle$ and $\mathbf{v} = \langle \cos \beta, \sin \beta, 0 \rangle$, where $\alpha > \beta$. Find the dot product of the vectors and use the result to prove the identity

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

89. Prove that $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}$.

90. Prove the **Cauchy-Schwarz Inequality** $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$.

91. Prove the triangle inequality $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

92. Prove Theorem 11.6.

11.4 The Cross Product of Two Vectors in Space

- **Find the cross product of two vectors in space.**
- **Use the triple scalar product of three vectors in space.**

The Cross Product

Many applications in physics, engineering, and geometry involve finding a vector in space that is orthogonal to two given vectors. In this section you will study a product that will yield such a vector. It is called the **cross product,** and it is most conveniently defined and calculated using the standard unit vector form. Because the cross product yields a vector, it is also called the **vector product.**

DEFINITION OF CROSS PRODUCT OF TWO VECTORS IN SPACE

Let

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \quad \text{and} \quad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

be vectors in space. The **cross product** of **u** and **v** is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

NOTE Be sure you see that this definition applies only to three-dimensional vectors. The cross product is not defined for two-dimensional vectors. ■

A convenient way to calculate $\mathbf{u} \times \mathbf{v}$ is to use the following *determinant form* with cofactor expansion. (This 3×3 determinant form is used simply to help remember the formula for the cross product—it is technically not a determinant because the entries of the corresponding matrix are not all real numbers.)

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{matrix} \\ \longleftarrow \text{Put “}\mathbf{u}\text{” in Row 2.} \\ \longleftarrow \text{Put “}\mathbf{v}\text{” in Row 3.} \end{matrix}$$

$$= \begin{vmatrix} \textcircled{\mathbf{i}} & \cancel{\mathbf{j}} & \cancel{\mathbf{k}} \\ \cancel{u_1} & u_2 & u_3 \\ \cancel{v_1} & v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \cancel{\mathbf{i}} & \textcircled{\mathbf{j}} & \cancel{\mathbf{k}} \\ u_1 & \cancel{u_2} & u_3 \\ v_1 & \cancel{v_2} & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \cancel{\mathbf{i}} & \cancel{\mathbf{j}} & \textcircled{\mathbf{k}} \\ u_1 & u_2 & \cancel{u_3} \\ v_1 & v_2 & \cancel{v_3} \end{vmatrix}\mathbf{k}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Note the minus sign in front of the **j**-component. Each of the three 2×2 determinants can be evaluated by using the following diagonal pattern.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Here are a couple of examples.

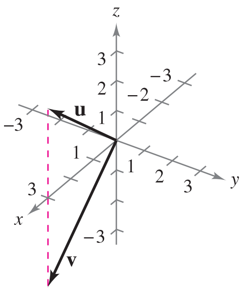
$$\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} = (2)(-1) - (4)(3) = -2 - 12 = -14$$

$$\begin{vmatrix} 4 & 0 \\ -6 & 3 \end{vmatrix} = (4)(3) - (0)(-6) = 12$$

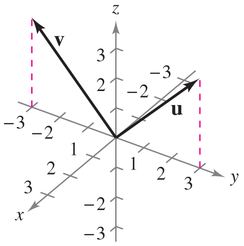
EXPLORATION

Geometric Property of the Cross Product Three pairs of vectors are shown below. Use the definition to find the cross product of each pair. Sketch all three vectors in a three-dimensional system. Describe any relationships among the three vectors. Use your description to write a conjecture about **u**, **v**, and $\mathbf{u} \times \mathbf{v}$.

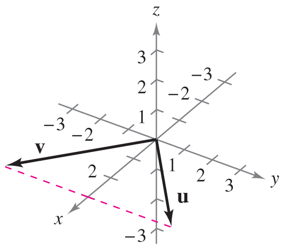
a. $\mathbf{u} = \langle 3, 0, 3\rangle,\ \mathbf{v} = \langle 3, 0, -3\rangle$



b. $\mathbf{u} = \langle 0, 3, 3\rangle,\ \mathbf{v} = \langle 0, -3, 3\rangle$



c. $\mathbf{u} = \langle 3, 3, 0\rangle,\ \mathbf{v} = \langle 3, -3, 0\rangle$



NOTATION FOR DOT AND CROSS PRODUCTS

The notation for the dot product and cross product of vectors was first introduced by the American physicist Josiah Willard Gibbs (1839–1903). In the early 1880s, Gibbs built a system to represent physical quantities called "vector analysis." The system was a departure from Hamilton's theory of quaternions.

EXAMPLE 1 Finding the Cross Product

Given $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, find each of the following.

a. $\mathbf{u} \times \mathbf{v}$ **b.** $\mathbf{v} \times \mathbf{u}$ **c.** $\mathbf{v} \times \mathbf{v}$

Solution

$$\textbf{a. } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}\mathbf{k}$$

$$= (4 - 1)\mathbf{i} - (-2 - 3)\mathbf{j} + (1 + 6)\mathbf{k}$$

$$= 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

$$\textbf{b. } \mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{k}$$

$$= (1 - 4)\mathbf{i} - (3 + 2)\mathbf{j} + (-6 - 1)\mathbf{k}$$

$$= -3\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$$

Note that this result is the negative of that in part (a).

$$\textbf{c. } \mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = \mathbf{0}$$

■

The results obtained in Example 1 suggest some interesting *algebraic* properties of the cross product. For instance, $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$, and $\mathbf{v} \times \mathbf{v} = \mathbf{0}$. These properties, and several others, are summarized in the following theorem.

THEOREM 11.7 ALGEBRAIC PROPERTIES OF THE CROSS PRODUCT

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space, and let c be a scalar.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

PROOF To prove Property 1, let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then,

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

and

$$\mathbf{v} \times \mathbf{u} = (v_2u_3 - v_3u_2)\mathbf{i} - (v_1u_3 - v_3u_1)\mathbf{j} + (v_1u_2 - v_2u_1)\mathbf{k}$$

which implies that $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$. Proofs of Properties 2, 3, 5, and 6 are left as exercises (see Exercises 59–62). ■

Note that Property 1 of Theorem 11.7 indicates that the cross product is *not commutative*. In particular, this property indicates that the vectors $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ have equal lengths but opposite directions. The following theorem lists some other *geometric* properties of the cross product of two vectors.

NOTE It follows from Properties 1 and 2 in Theorem 11.8 that if $\mathbf{n}$ is a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u} \times \mathbf{v} = \pm(\|\mathbf{u}\|\ \|\mathbf{v}\| \sin\theta)\mathbf{n}.$$

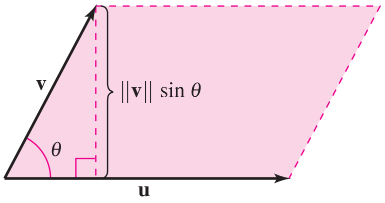
THEOREM 11.8 GEOMETRIC PROPERTIES OF THE CROSS PRODUCT

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\ \|\mathbf{v}\| \sin\theta$
3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.
4. $\|\mathbf{u} \times \mathbf{v}\| =$ area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

PROOF To prove Property 2, note because $\cos\theta = (\mathbf{u} \cdot \mathbf{v})/(\|\mathbf{u}\|\,\|\mathbf{v}\|)$, it follows that

$$\begin{aligned}
\|\mathbf{u}\|\ \|\mathbf{v}\| \sin\theta &= \|\mathbf{u}\|\ \|\mathbf{v}\|\sqrt{1-\cos^2\theta}\\
&= \|\mathbf{u}\|\ \|\mathbf{v}\|\sqrt{1-\frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}}\\
&= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2}\\
&= \sqrt{(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2}\\
&= \sqrt{(u_2v_3-u_3v_2)^2+(u_1v_3-u_3v_1)^2+(u_1v_2-u_2v_1)^2}\\
&= \|\mathbf{u}\times\mathbf{v}\|.
\end{aligned}$$



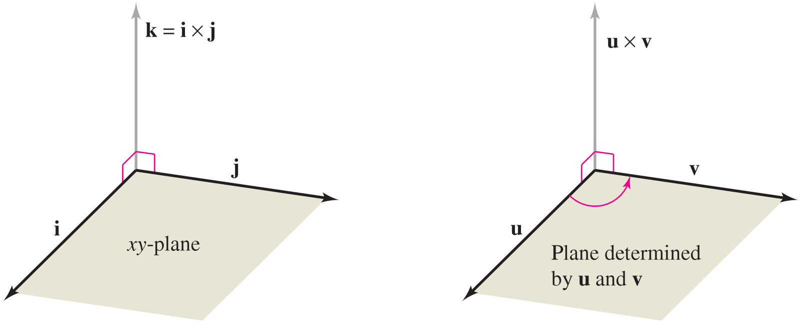
The vectors **u** and **v** form adjacent sides of a parallelogram.
Figure 11.35

To prove Property 4, refer to Figure 11.35, which is a parallelogram having $\mathbf{v}$ and $\mathbf{u}$ as adjacent sides. Because the height of the parallelogram is $\|\mathbf{v}\| \sin\theta$, the area is

$$\begin{aligned}
\text{Area} &= (\text{base})(\text{height})\\
&= \|\mathbf{u}\|\ \|\mathbf{v}\| \sin\theta\\
&= \|\mathbf{u}\times\mathbf{v}\|.
\end{aligned}$$

Proofs of Properties 1 and 3 are left as exercises (see Exercises 63 and 64). ■

Both $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ are perpendicular to the plane determined by $\mathbf{u}$ and $\mathbf{v}$. One way to remember the orientations of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is to compare them with the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k} = \mathbf{i} \times \mathbf{j}$, as shown in Figure 11.36. The three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a *right-handed system*, whereas the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{v} \times \mathbf{u}$ form a *left-handed system*.



Right-handed systems
Figure 11.36

EXAMPLE 2 Using the Cross Product

Find a unit vector that is orthogonal to both

$$\mathbf{u} = \mathbf{i} - 4\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{v} = 2\mathbf{i} + 3\mathbf{j}.$$

Solution The cross product $\mathbf{u} \times \mathbf{v}$, as shown in Figure 11.37, is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 1 \\ 2 & 3 & 0 \end{vmatrix} \qquad \text{Cross product} \\ &= -3\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}\end{aligned}$$

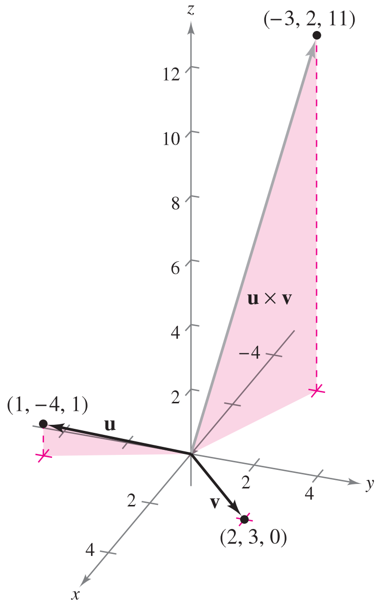
Because

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 11^2} = \sqrt{134}$$

a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = -\frac{3}{\sqrt{134}}\mathbf{i} + \frac{2}{\sqrt{134}}\mathbf{j} + \frac{11}{\sqrt{134}}\mathbf{k}.$$

■



The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
Figure 11.37

NOTE In Example 2, note that you could have used the cross product $\mathbf{v} \times \mathbf{u}$ to form a unit vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. With that choice, you would have obtained the negative of the unit vector found in the example. ■

EXAMPLE 3 Geometric Application of the Cross Product

Show that the quadrilateral with vertices at the following points is a parallelogram, and find its area.

$$A = (5, 2, 0) \qquad B = (2, 6, 1)$$
$$C = (2, 4, 7) \qquad D = (5, 0, 6)$$

Solution From Figure 11.38 you can see that the sides of the quadrilateral correspond to the following four vectors.

$$\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k} \qquad \overrightarrow{CD} = 3\mathbf{i} - 4\mathbf{j} - \mathbf{k} = -\overrightarrow{AB}$$
$$\overrightarrow{AD} = 0\mathbf{i} - 2\mathbf{j} + 6\mathbf{k} \qquad \overrightarrow{CB} = 0\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} = -\overrightarrow{AD}$$

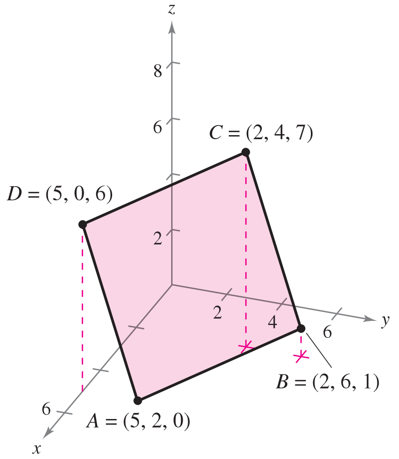
So, $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and $\overrightarrow{AD}$ is parallel to $\overrightarrow{CB}$, and you can conclude that the quadrilateral is a parallelogram with $\overrightarrow{AB}$ and $\overrightarrow{AD}$ as adjacent sides. Moreover, because

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AD} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix} \qquad \text{Cross product} \\ &= 26\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}\end{aligned}$$

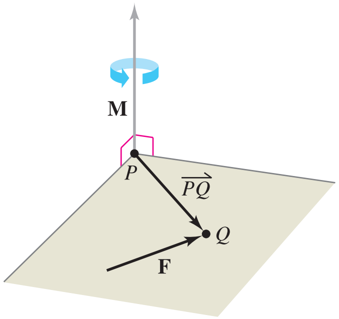
the area of the parallelogram is

$$\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \sqrt{1036} \approx 32.19.$$

Is the parallelogram a rectangle? You can determine whether it is by finding the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$. ■



The area of the parallelogram is approximately 32.19.
Figure 11.38



The moment of **F** about *P*
Figure 11.39

In physics, the cross product can be used to measure **torque**—the **moment M of a force F about a point *P*,** as shown in Figure 11.39. If the point of application of the force is Q, the moment of **F** about P is given by

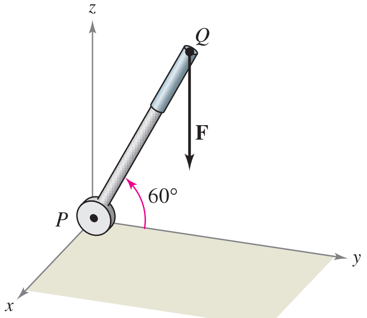
$$\mathbf{M} = \overrightarrow{PQ} \times \mathbf{F}.$$

Moment of **F** about *P*

The magnitude of the moment **M** measures the tendency of the vector $\overrightarrow{PQ}$ to rotate counterclockwise (using the right-hand rule) about an axis directed along the vector **M**.

EXAMPLE 4 An Application of the Cross Product

A vertical force of 50 pounds is applied to the end of a one-foot lever that is attached to an axle at point P, as shown in Figure 11.40. Find the moment of this force about the point P when $\theta = 60°$.



A vertical force of 50 pounds is applied at point Q.
Figure 11.40

Solution If you represent the 50-pound force as $\mathbf{F} = -50\mathbf{k}$ and the lever as

$$\overrightarrow{PQ} = \cos(60°)\mathbf{j} + \sin(60°)\mathbf{k} = \frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$$

the moment of **F** about P is given by

$$\mathbf{M} = \overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -50 \end{vmatrix} = -25\mathbf{i}.$$

Moment of **F** about *P*

The magnitude of this moment is 25 foot-pounds. ■

NOTE In Example 4, note that the moment (the tendency of the lever to rotate about its axle) is dependent on the angle θ. When $\theta = \pi/2$, the moment is 0. The moment is greatest when $\theta = 0$. ■

The Triple Scalar Product

For vectors **u**, **v**, and **w** in space, the dot product of **u** and $\mathbf{v} \times \mathbf{w}$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

is called the **triple scalar product,** as defined in Theorem 11.9. The proof of this theorem is left as an exercise (see Exercise 67).

■ **FOR FURTHER INFORMATION** To see how the cross product is used to model the torque of the robot arm of a space shuttle, see the article "The Long Arm of Calculus" by Ethan Berkove and Rich Marchand in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

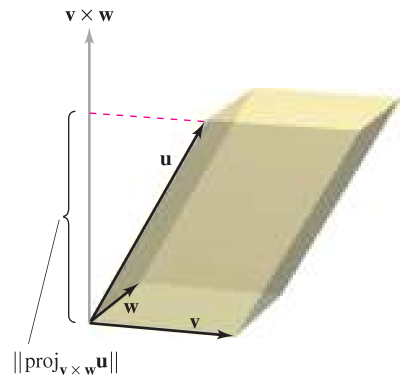
THEOREM 11.9 THE TRIPLE SCALAR PRODUCT

For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, the triple scalar product is given by

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

NOTE The value of a determinant is multiplied by -1 if two rows are interchanged. After two such interchanges, the value of the determinant will be unchanged. So, the following triple scalar products are equivalent.

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$ ■


$\mathbf{v} \times \mathbf{w}$
$\mathbf{u}$
$\mathbf{w}$
$\mathbf{v}$
$\|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\|$

Area of base $= \|\mathbf{v} \times \mathbf{w}\|$
Volume of parallelepiped $= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$
Figure 11.41

If the vectors **u**, **v**, and **w** do not lie in the same plane, the triple scalar product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ can be used to determine the volume of the parallelepiped (a polyhedron, all of whose faces are parallelograms) with **u**, **v**, and **w** as adjacent edges, as shown in Figure 11.41. This is established in the following theorem.

THEOREM 11.10 GEOMETRIC PROPERTY OF THE TRIPLE SCALAR PRODUCT

The volume V of a parallelepiped with vectors **u**, **v**, and **w** as adjacent edges is given by

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

PROOF In Figure 11.41, note that

$$\|\mathbf{v} \times \mathbf{w}\| = \text{area of base}$$

and

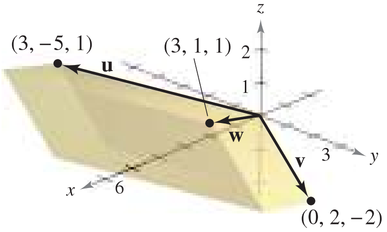
$$\|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\| = \text{height of parallelepiped.}$$

Therefore, the volume is

$$\begin{aligned} V = (\text{height})(\text{area of base}) &= \|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\|\,\|\mathbf{v} \times \mathbf{w}\| \\ &= \left|\frac{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}{\|\mathbf{v} \times \mathbf{w}\|}\right| \|\mathbf{v} \times \mathbf{w}\| \\ &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|. \end{aligned}$$

■

EXAMPLE 5 Volume by the Triple Scalar Product



The parallelepiped has a volume of 36.
Figure 11.42

Find the volume of the parallelepiped shown in Figure 11.42 having $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ as adjacent edges.

Solution By Theorem 11.10, you have

$$\begin{aligned} V &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \qquad \text{Triple scalar product} \\ &= \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix} \\ &= 3\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - (-5)\begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + (1)\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\ &= 3(4) + 5(6) + 1(-6) \\ &= 36. \end{aligned}$$

■

A natural consequence of Theorem 11.10 is that the volume of the parallelepiped is 0 if and only if the three vectors are coplanar. That is, if the vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ have the same initial point, they lie in the same plane if and only if

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0.$$

11.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the cross product of the unit vectors and sketch your result.

1. $\mathbf{j} \times \mathbf{i}$ **2.** $\mathbf{i} \times \mathbf{j}$

3. $\mathbf{j} \times \mathbf{k}$ **4.** $\mathbf{k} \times \mathbf{j}$

5. $\mathbf{i} \times \mathbf{k}$ **6.** $\mathbf{k} \times \mathbf{i}$

In Exercises 7–10, find (a) $\mathbf{u} \times \mathbf{v}$, (b) $\mathbf{v} \times \mathbf{u}$, and (c) $\mathbf{v} \times \mathbf{v}$.

7. $\mathbf{u} = -2\mathbf{i} + 4\mathbf{j}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$

8. $\mathbf{u} = 3\mathbf{i} + 5\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

9. $\mathbf{u} = \langle 7, 3, 2\rangle$, $\mathbf{v} = \langle 1, -1, 5\rangle$

10. $\mathbf{u} = \langle 3, -2, -2\rangle$, $\mathbf{v} = \langle 1, 5, 1\rangle$

In Exercises 11–16, find $\mathbf{u} \times \mathbf{v}$ and show that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

11. $\mathbf{u} = \langle 12, -3, 0\rangle$, $\mathbf{v} = \langle -2, 5, 0\rangle$

12. $\mathbf{u} = \langle -1, 1, 2\rangle$, $\mathbf{v} = \langle 0, 1, 0\rangle$

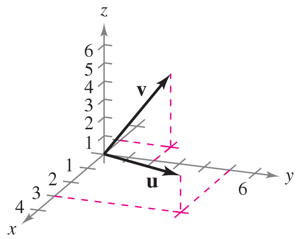
13. $\mathbf{u} = \langle 2, -3, 1\rangle$, $\mathbf{v} = \langle 1, -2, 1\rangle$

14. $\mathbf{u} = \langle -10, 0, 6\rangle$, $\mathbf{v} = \langle 5, -3, 0\rangle$

15. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

16. $\mathbf{u} = \mathbf{i} + 6\mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

Think About It **In Exercises 17–20, use the vectors $\mathbf{u}$ and $\mathbf{v}$ shown in the figure to sketch a vector in the direction of the indicated cross product in a right-handed system.**



17. $\mathbf{u} \times \mathbf{v}$ **18.** $\mathbf{v} \times \mathbf{u}$

19. $(-\mathbf{v}) \times \mathbf{u}$ **20.** $\mathbf{u} \times (\mathbf{u} \times \mathbf{v})$

CAS **In Exercises 21–24, use a computer algebra system to find $\mathbf{u} \times \mathbf{v}$ and a unit vector orthogonal to $\mathbf{u}$ and $\mathbf{v}$.**

21. $\mathbf{u} = \langle 4, -3.5, 7\rangle$, $\mathbf{v} = \langle 2.5, 9, 3\rangle$

22. $\mathbf{u} = \langle -8, -6, 4\rangle$, $\mathbf{v} = \langle 10, -12, -2\rangle$

23. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{v} = 0.4\mathbf{i} - 0.8\mathbf{j} + 0.2\mathbf{k}$

24. $\mathbf{u} = 0.7\mathbf{k}$, $\mathbf{v} = 1.5\mathbf{i} + 6.2\mathbf{k}$

25. ***Programming*** Given the vectors $\mathbf{u}$ and $\mathbf{v}$ in component form, write a program for a graphing utility in which the output is $\mathbf{u} \times \mathbf{v}$ and $\|\mathbf{u} \times \mathbf{v}\|$.

26. ***Programming*** Use the program you wrote in Exercise 25 to find $\mathbf{u} \times \mathbf{v}$ and $\|\mathbf{u} \times \mathbf{v}\|$ for $\mathbf{u} = \langle -2, 6, 10\rangle$ and $\mathbf{v} = \langle 3, 8, 5\rangle$.

Area **In Exercises 27–30, find the area of the parallelogram that has the given vectors as adjacent sides. Use a computer algebra system or a graphing utility to verify your result.**

27. $\mathbf{u} = \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$

28. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$

29. $\mathbf{u} = \langle 3, 2, -1\rangle$, $\mathbf{v} = \langle 1, 2, 3\rangle$

30. $\mathbf{u} = \langle 2, -1, 0\rangle$, $\mathbf{v} = \langle -1, 2, 0\rangle$

Area **In Exercises 31 and 32, verify that the points are the vertices of a parallelogram, and find its area.**

31. $A(0, 3, 2)$, $B(1, 5, 5)$, $C(6, 9, 5)$, $D(5, 7, 2)$

32. $A(2, -3, 1)$, $B(6, 5, -1)$, $C(7, 2, 2)$, $D(3, -6, 4)$

Area **In Exercises 33–36, find the area of the triangle with the given vertices. $\left(\textit{Hint: } \frac{1}{2}\|\mathbf{u} \times \mathbf{v}\| \text{ is the area of the triangle having } \mathbf{u} \text{ and } \mathbf{v} \text{ as adjacent sides.}\right)$**

33. $A(0, 0, 0)$, $B(1, 0, 3)$, $C(-3, 2, 0)$

34. $A(2, -3, 4)$, $B(0, 1, 2)$, $C(-1, 2, 0)$

35. $A(2, -7, 3)$, $B(-1, 5, 8)$, $C(4, 6, -1)$

36. $A(1, 2, 0)$, $B(-2, 1, 0)$, $C(0, 0, 0)$

37. ***Torque*** A child applies the brakes on a bicycle by applying a downward force of 20 pounds on the pedal when the crank makes a 40° angle with the horizontal (see figure). The crank is 6 inches in length. Find the torque at P.

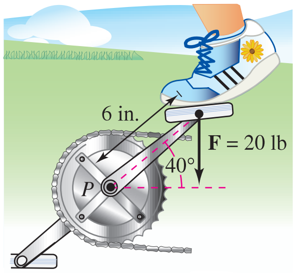


Figure for 37

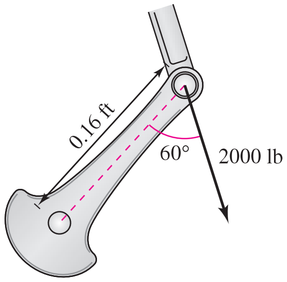


Figure for 38

38. ***Torque*** Both the magnitude and the direction of the force on a crankshaft change as the crankshaft rotates. Find the torque on the crankshaft using the position and data shown in the figure.

39. ***Optimization*** A force of 56 pounds acts on the pipe wrench shown in the figure on the next page.

(a) Find the magnitude of the moment about O by evaluating $\|\overrightarrow{OA} \times \mathbf{F}\|$. Use a graphing utility to graph the resulting function of θ.

(b) Use the result of part (a) to determine the magnitude of the moment when $\theta = 45°$.

(c) Use the result of part (a) to determine the angle θ when the magnitude of the moment is maximum. Is the answer what you expected? Why or why not?

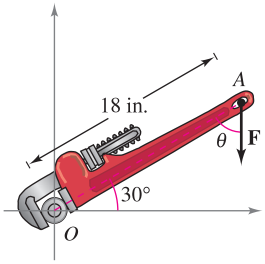


Figure for 39

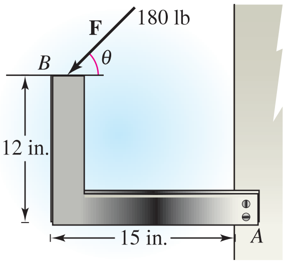


Figure for 40

40. ***Optimization*** A force of 180 pounds acts on the bracket shown in the figure.

(a) Determine the vector $\overrightarrow{AB}$ and the vector **F** representing the force. (**F** will be in terms of θ.)

(b) Find the magnitude of the moment about A by evaluating $\|\overrightarrow{AB} \times \mathbf{F}\|$.

(c) Use the result of part (b) to determine the magnitude of the moment when $\theta = 30°$.

(d) Use the result of part (b) to determine the angle θ when the magnitude of the moment is maximum. At that angle, what is the relationship between the vectors **F** and $\overrightarrow{AB}$? Is it what you expected? Why or why not?

(e) Use a graphing utility to graph the function for the magnitude of the moment about A for $0° \leq \theta \leq 180°$. Find the zero of the function in the given domain. Interpret the meaning of the zero in the context of the problem.

In Exercises 41–44, find $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

41. $\mathbf{u} = \mathbf{i}$
$\mathbf{v} = \mathbf{j}$
$\mathbf{w} = \mathbf{k}$

42. $\mathbf{u} = \langle 1, 1, 1 \rangle$
$\mathbf{v} = \langle 2, 1, 0 \rangle$
$\mathbf{w} = \langle 0, 0, 1 \rangle$

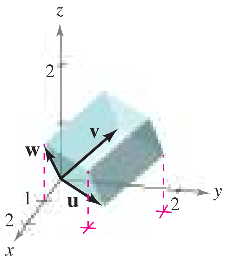
43. $\mathbf{u} = \langle 2, 0, 1 \rangle$
$\mathbf{v} = \langle 0, 3, 0 \rangle$
$\mathbf{w} = \langle 0, 0, 1 \rangle$

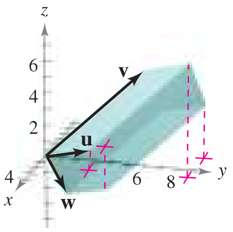
44. $\mathbf{u} = \langle 2, 0, 0 \rangle$
$\mathbf{v} = \langle 1, 1, 1 \rangle$
$\mathbf{w} = \langle 0, 2, 2 \rangle$

Volume **In Exercises 45 and 46, use the triple scalar product to find the volume of the parallelepiped having adjacent edges u, v, and w.**

45. $\mathbf{u} = \mathbf{i} + \mathbf{j}$
$\mathbf{v} = \mathbf{j} + \mathbf{k}$
$\mathbf{w} = \mathbf{i} + \mathbf{k}$

46. $\mathbf{u} = \langle 1, 3, 1 \rangle$
$\mathbf{v} = \langle 0, 6, 6 \rangle$
$\mathbf{w} = \langle -4, 0, -4 \rangle$





Volume **In Exercises 47 and 48, find the volume of the parallelepiped with the given vertices.**

47. $(0, 0, 0), (3, 0, 0), (0, 5, 1), (2, 0, 5)$
$(3, 5, 1), (5, 0, 5), (2, 5, 6), (5, 5, 6)$

48. $(0, 0, 0), (0, 4, 0), (-3, 0, 0), (-1, 1, 5)$
$(-3, 4, 0), (-1, 5, 5), (-4, 1, 5), (-4, 5, 5)$

49. If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = 0$, what can you conclude about **u** and **v**?

50. Identify the dot products that are equal. Explain your reasoning. (Assume **u**, **v**, and **w** are nonzero vectors.)

(a) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
(b) $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}$
(c) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
(d) $(\mathbf{u} \times -\mathbf{w}) \cdot \mathbf{v}$
(e) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$
(f) $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})$
(g) $(-\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
(h) $(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$

WRITING ABOUT CONCEPTS

51. Define the cross product of vectors **u** and **v**.

52. State the geometric properties of the cross product.

53. If the magnitudes of two vectors are doubled, how will the magnitude of the cross product of the vectors change? Explain.

CAPSTONE

54. The vertices of a triangle in space are (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3). Explain how to find a vector perpendicular to the triangle.

True or False? **In Exercises 55–58, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

55. It is possible to find the cross product of two vectors in a two-dimensional coordinate system.

56. If **u** and **v** are vectors in space that are nonzero and nonparallel, then $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.

57. If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

58. If $\mathbf{u} \neq \mathbf{0}$, $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

In Exercises 59–66, prove the property of the cross product.

59. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

60. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$

61. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

62. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

63. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v**.

64. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if **u** and **v** are scalar multiples of each other.

65. Prove that $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|$ if **u** and **v** are orthogonal.

66. Prove that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

67. Prove Theorem 11.9.

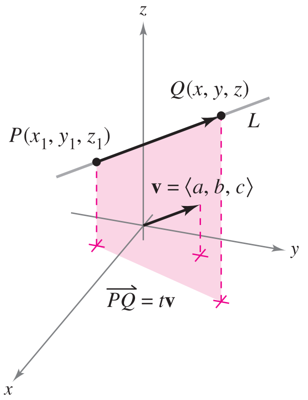
11.5 Lines and Planes in Space

- **Write a set of parametric equations for a line in space.**
- **Write a linear equation to represent a plane in space.**
- **Sketch the plane given by a linear equation.**
- **Find the distances between points, planes, and lines in space.**

Lines in Space

In the plane, *slope* is used to determine an equation of a line. In space, it is more convenient to use *vectors* to determine the equation of a line.

In Figure 11.43, consider the line L through the point $P(x_1, y_1, z_1)$ and parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$. The vector $\mathbf{v}$ is a **direction vector** for the line L, and a, b, and c are **direction numbers.** One way of describing the line L is to say that it consists of all points $Q(x, y, z)$ for which the vector $\overrightarrow{PQ}$ is parallel to $\mathbf{v}$. This means that $\overrightarrow{PQ}$ is a scalar multiple of $\mathbf{v}$, and you can write $\overrightarrow{PQ} = t\mathbf{v}$, where t is a scalar (a real number).



Line L and its direction vector $\mathbf{v}$
Figure 11.43

$$\overrightarrow{PQ} = \langle x - x_1, y - y_1, z - z_1\rangle = \langle at, bt, ct\rangle = t\mathbf{v}$$

By equating corresponding components, you can obtain **parametric equations** of a line in space.

THEOREM 11.11 PARAMETRIC EQUATIONS OF A LINE IN SPACE

A line L parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$ and passing through the point $P(x_1, y_1, z_1)$ is represented by the **parametric equations**

$$x = x_1 + at, \quad y = y_1 + bt, \quad \text{and} \quad z = z_1 + ct.$$

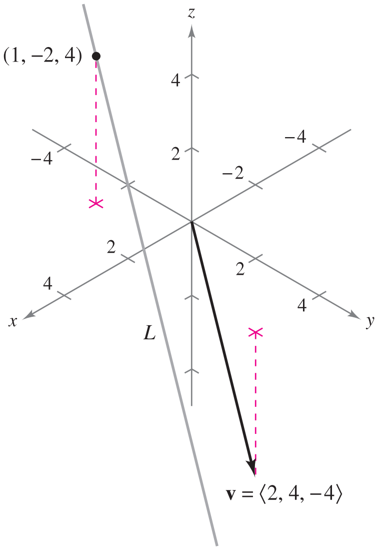
If the direction numbers a, b, and c are all nonzero, you can eliminate the parameter t to obtain **symmetric equations** of the line.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Symmetric equations

EXAMPLE 1 Finding Parametric and Symmetric Equations

Find parametric and symmetric equations of the line L that passes through the point $(1, -2, 4)$ and is parallel to $\mathbf{v} = \langle 2, 4, -4\rangle$.



The vector $\mathbf{v}$ is parallel to the line L.
Figure 11.44

Solution To find a set of parametric equations of the line, use the coordinates $x_1 = 1$, $y_1 = -2$, and $z_1 = 4$ and direction numbers $a = 2$, $b = 4$, and $c = -4$ (see Figure 11.44).

$$x = 1 + 2t, \quad y = -2 + 4t, \quad z = 4 - 4t$$

Parametric equations

Because a, b, and c are all nonzero, a set of symmetric equations is

$$\frac{x - 1}{2} = \frac{y + 2}{4} = \frac{z - 4}{-4}.$$

Symmetric equations ■

Neither parametric equations nor symmetric equations of a given line are unique. For instance, in Example 1, by letting $t = 1$ in the parametric equations you would obtain the point $(3, 2, 0)$. Using this point with the direction numbers $a = 2$, $b = 4$, and $c = -4$ would produce a different set of parametric equations

$$x = 3 + 2t, \quad y = 2 + 4t, \quad \text{and} \quad z = -4t.$$

EXAMPLE 2 Parametric Equations of a Line Through Two Points

Find a set of parametric equations of the line that passes through the points $(-2, 1, 0)$ and $(1, 3, 5)$.

Solution Begin by using the points $P(-2, 1, 0)$ and $Q(1, 3, 5)$ to find a direction vector for the line passing through P and Q, given by

$$\mathbf{v} = \overrightarrow{PQ} = \langle 1 - (-2), 3 - 1, 5 - 0 \rangle = \langle 3, 2, 5 \rangle = \langle a, b, c \rangle.$$

Using the direction numbers $a = 3$, $b = 2$, and $c = 5$ with the point $P(-2, 1, 0)$, you can obtain the parametric equations

$$x = -2 + 3t, \quad y = 1 + 2t, \quad \text{and} \quad z = 5t.$$ ■

NOTE As t varies over all real numbers, the parametric equations in Example 2 determine the points (x, y, z) on the line. In particular, note that $t = 0$ and $t = 1$ give the original points $(-2, 1, 0)$ and $(1, 3, 5)$. ■

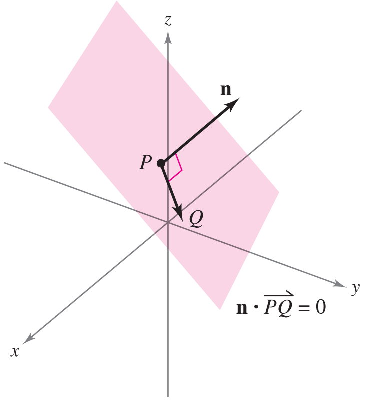
Planes in Space

You have seen how an equation of a line in space can be obtained from a point on the line and a vector *parallel* to it. You will now see that an equation of a plane in space can be obtained from a point in the plane and a vector *normal* (perpendicular) to the plane.

Consider the plane containing the point $P(x_1, y_1, z_1)$ having a nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$, as shown in Figure 11.45. This plane consists of all points $Q(x, y, z)$ for which vector $\overrightarrow{PQ}$ is orthogonal to $\mathbf{n}$. Using the dot product, you can write the following.

$$\begin{aligned} \mathbf{n} \cdot \overrightarrow{PQ} &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle &= 0 \\ a(x - x_1) + b(y - y_1) + c(z - z_1) &= 0 \end{aligned}$$

The third equation of the plane is said to be in **standard form.**



The normal vector **n** is orthogonal to each vector $\overrightarrow{PQ}$ in the plane.
Figure 11.45

THEOREM 11.12 STANDARD EQUATION OF A PLANE IN SPACE

The plane containing the point (x_1, y_1, z_1) and having normal vector $\mathbf{n} = \langle a, b, c \rangle$ can be represented by the **standard form** of the equation of a plane

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

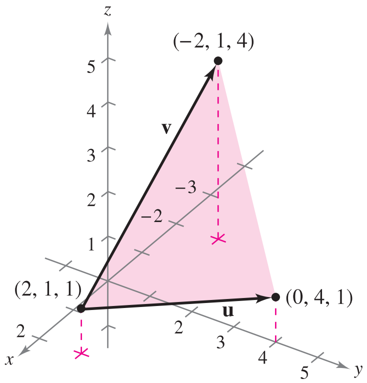
By regrouping terms, you obtain the **general form** of the equation of a plane in space.

$$ax + by + cz + d = 0$$

General form of equation of plane

Given the general form of the equation of a plane, it is easy to find a normal vector to the plane. Simply use the coefficients of x, y, and z and write $\mathbf{n} = \langle a, b, c\rangle$.

EXAMPLE 3 Finding an Equation of a Plane in Three-Space



A plane determined by **u** and **v**
Figure 11.46

Find the general equation of the plane containing the points $(2, 1, 1)$, $(0, 4, 1)$, and $(-2, 1, 4)$.

Solution To apply Theorem 11.12 you need a point in the plane and a vector that is normal to the plane. There are three choices for the point, but no normal vector is given. To obtain a normal vector, use the cross product of vectors **u** and **v** extending from the point $(2, 1, 1)$ to the points $(0, 4, 1)$ and $(-2, 1, 4)$, as shown in Figure 11.46. The component forms of **u** and **v** are

$$\begin{aligned}\mathbf{u} &= \langle 0-2, 4-1, 1-1\rangle = \langle -2, 3, 0\rangle\\ \mathbf{v} &= \langle -2-2, 1-1, 4-1\rangle = \langle -4, 0, 3\rangle\end{aligned}$$

and it follows that

$$\begin{aligned}\mathbf{n} &= \mathbf{u}\times\mathbf{v}\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2 & 3 & 0\\ -4 & 0 & 3\end{vmatrix}\\ &= 9\mathbf{i} + 6\mathbf{j} + 12\mathbf{k}\\ &= \langle a, b, c\rangle\end{aligned}$$

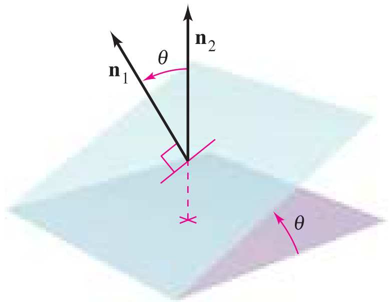
is normal to the given plane. Using the direction numbers for **n** and the point $(x_1, y_1, z_1) = (2, 1, 1)$, you can determine an equation of the plane to be

$$\begin{aligned}a(x-x_1) + b(y-y_1) + c(z-z_1) &= 0\\ 9(x-2) + 6(y-1) + 12(z-1) &= 0 && \text{Standard form}\\ 9x + 6y + 12z - 36 &= 0 && \text{General form}\\ 3x + 2y + 4z - 12 &= 0. && \text{Simplified general form}\end{aligned}$$

■

NOTE In Example 3, check to see that each of the three original points satisfies the equation $3x + 2y + 4z - 12 = 0$. ■

Two distinct planes in three-space either are parallel or intersect in a line. If they intersect, you can determine the angle $(0 \le \theta \le \pi/2)$ between them from the angle between their normal vectors, as shown in Figure 11.47. Specifically, if vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal to two intersecting planes, the angle θ between the normal vectors is equal to the angle between the two planes and is given by



The angle θ between two planes
Figure 11.47

$$\boxed{\cos\theta = \frac{|\mathbf{n}_1\cdot\mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}.} \qquad \text{Angle between two planes}$$

Consequently, two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are

1. *perpendicular* if $\mathbf{n}_1\cdot\mathbf{n}_2 = 0$.
2. *parallel* if $\mathbf{n}_1$ is a scalar multiple of $\mathbf{n}_2$.

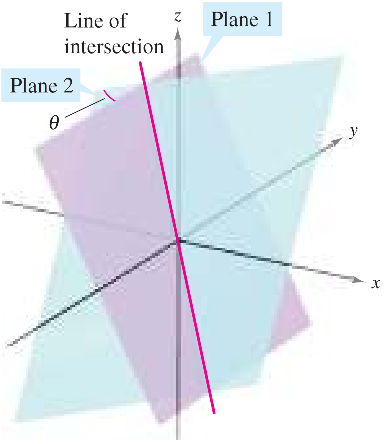


Figure 11.48

EXAMPLE 4 Finding the Line of Intersection of Two Planes

Find the angle between the two planes given by

$$x - 2y + z = 0 \qquad \text{Equation of plane 1}$$
$$2x + 3y - 2z = 0 \qquad \text{Equation of plane 2}$$

and find parametric equations of their line of intersection (see Figure 11.48).

Solution Normal vectors for the planes are $\mathbf{n}_1 = \langle 1, -2, 1\rangle$ and $\mathbf{n}_2 = \langle 2, 3, -2\rangle$. Consequently, the angle between the two planes is determined as follows.

$$\begin{aligned}\cos\theta &= \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} && \text{Cosine of angle between } \mathbf{n}_1 \text{ and } \mathbf{n}_2\\ &= \frac{|-6|}{\sqrt{6}\,\sqrt{17}}\\ &= \frac{6}{\sqrt{102}}\\ &\approx 0.59409\end{aligned}$$

This implies that the angle between the two planes is $\theta \approx 53.55°$. You can find the line of intersection of the two planes by simultaneously solving the two linear equations representing the planes. One way to do this is to multiply the first equation by -2 and add the result to the second equation.

$$\begin{array}{rcr} x - 2y + z = 0 & \Rightarrow & -2x + 4y - 2z = 0\\ 2x + 3y - 2z = 0 & & \underline{2x + 3y - 2z = 0}\\ & & 7y - 4z = 0 \quad \Rightarrow \quad y = \dfrac{4z}{7}\end{array}$$

Substituting $y = 4z/7$ back into one of the original equations, you can determine that $x = z/7$. Finally, by letting $t = z/7$, you obtain the parametric equations

$$x = t, \quad y = 4t, \quad \text{and} \quad z = 7t \qquad \text{Line of intersection}$$

which indicate that 1, 4, and 7 are direction numbers for the line of intersection. ■

Note that the direction numbers in Example 4 can be obtained from the cross product of the two normal vectors as follows.

$$\begin{aligned}\mathbf{n}_1 \times \mathbf{n}_2 &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 1\\ 2 & 3 & -2\end{vmatrix}\\ &= \begin{vmatrix}-2 & 1\\ 3 & -2\end{vmatrix}\mathbf{i} - \begin{vmatrix}1 & 1\\ 2 & -2\end{vmatrix}\mathbf{j} + \begin{vmatrix}1 & -2\\ 2 & 3\end{vmatrix}\mathbf{k}\\ &= \mathbf{i} + 4\mathbf{j} + 7\mathbf{k}\end{aligned}$$

This means that the line of intersection of the two planes is parallel to the cross product of their normal vectors.

NOTE The three-dimensional rotatable graphs that are available in the premium eBook for this text can help you visualize surfaces such as those shown in Figure 11.48. If you have access to these graphs, you should use them to help your spatial intuition when studying this section and other sections in the text that deal with vectors, curves, or surfaces in space. ■

Sketching Planes in Space

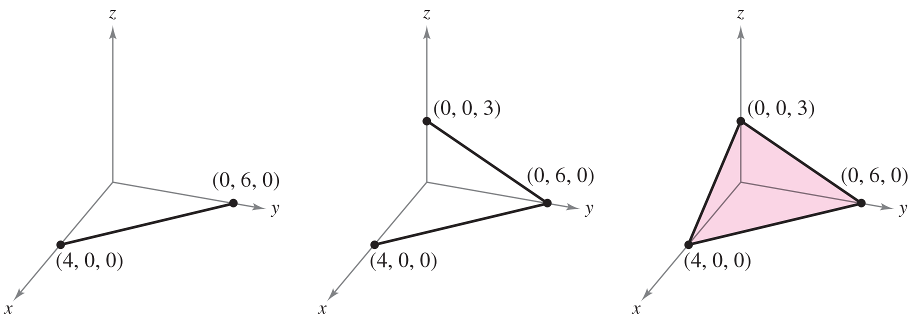
If a plane in space intersects one of the coordinate planes, the line of intersection is called the **trace** of the given plane in the coordinate plane. To sketch a plane in space, it is helpful to find its points of intersection with the coordinate axes and its traces in the coordinate planes. For example, consider the plane given by

$$3x + 2y + 4z = 12. \qquad \text{Equation of plane}$$

You can find the *xy*-trace by letting $z = 0$ and sketching the line

$$3x + 2y = 12 \qquad xy\text{-trace}$$

in the *xy*-plane. This line intersects the *x*-axis at $(4, 0, 0)$ and the *y*-axis at $(0, 6, 0)$. In Figure 11.49, this process is continued by finding the *yz*-trace and the *xz*-trace, and then shading the triangular region lying in the first octant.

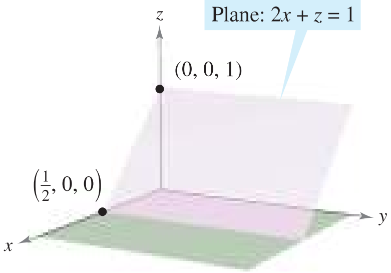


xy-trace $(z = 0)$: $3x + 2y = 12$

yz-trace $(x = 0)$: $2y + 4z = 12$

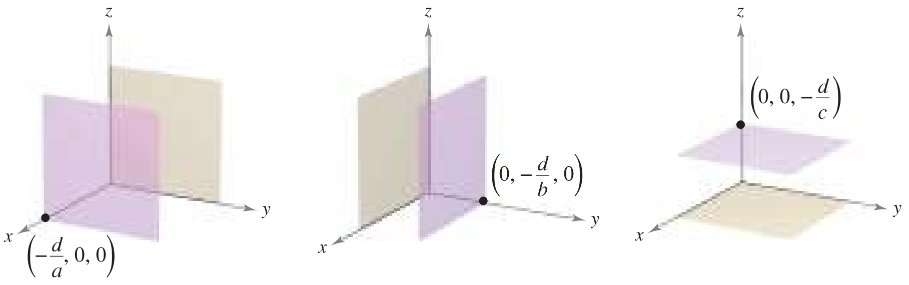
xz-trace $(y = 0)$: $3x + 4z = 12$

Traces of the plane $3x + 2y + 4z = 12$
Figure 11.49



Plane $2x + z = 1$ is parallel to the *y*-axis.
Figure 11.50

If an equation of a plane has a missing variable, such as $2x + z = 1$, the plane must be *parallel to the axis* represented by the missing variable, as shown in Figure 11.50. If two variables are missing from an equation of a plane, it is *parallel to the coordinate plane* represented by the missing variables, as shown in Figure 11.51.



Plane $ax + d = 0$ is parallel to the *yz*-plane

Plane $by + d = 0$ is parallel to the *xz*-plane

Plane $cz + d = 0$ is parallel to the *xy*-plane

Figure 11.51

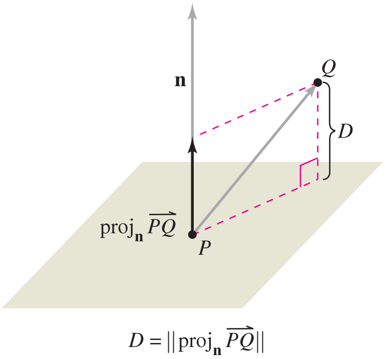
Distances Between Points, Planes, and Lines

This section is concluded with the following discussion of two basic types of problems involving distance in space.

1. Finding the distance between a point and a plane
2. Finding the distance between a point and a line

The solutions of these problems illustrate the versatility and usefulness of vectors in coordinate geometry: the first problem uses the *dot product* of two vectors, and the second problem uses the *cross product*.

The distance D between a point Q and a plane is the length of the shortest line segment connecting Q to the plane, as shown in Figure 11.52. If P is *any* point in the plane, you can find this distance by projecting the vector $\overrightarrow{PQ}$ onto the normal vector $\mathbf{n}$. The length of this projection is the desired distance.



n
Q
D
$\text{proj}_{\mathbf{n}} \overrightarrow{PQ}$
P
$D = \|\text{proj}_{\mathbf{n}} \overrightarrow{PQ}\|$

The distance between a point and a plane
Figure 11.52

THEOREM 11.13 DISTANCE BETWEEN A POINT AND A PLANE

The distance between a plane and a point Q (not in the plane) is

$$D = \|\text{proj}_{\mathbf{n}} \overrightarrow{PQ}\| = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

where P is a point in the plane and $\mathbf{n}$ is normal to the plane.

To find a point in the plane given by $ax + by + cz + d = 0$ $(a \neq 0)$, let $y = 0$ and $z = 0$. Then, from the equation $ax + d = 0$, you can conclude that the point $(-d/a, 0, 0)$ lies in the plane.

EXAMPLE 5 Finding the Distance Between a Point and a Plane

Find the distance between the point $Q(1, 5, -4)$ and the plane given by

$$3x - y + 2z = 6.$$

Solution You know that $\mathbf{n} = \langle 3, -1, 2\rangle$ is normal to the given plane. To find a point in the plane, let $y = 0$ and $z = 0$, and obtain the point $P(2, 0, 0)$. The vector from P to Q is given by

$$\begin{aligned}\overrightarrow{PQ} &= \langle 1 - 2, 5 - 0, -4 - 0\rangle \\ &= \langle -1, 5, -4\rangle.\end{aligned}$$

Using the Distance Formula given in Theorem 11.13 produces

$$\begin{aligned}D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} &= \frac{|\langle -1, 5, -4\rangle \cdot \langle 3, -1, 2\rangle|}{\sqrt{9 + 1 + 4}} \\ &= \frac{|-3 - 5 - 8|}{\sqrt{14}} \\ &= \frac{16}{\sqrt{14}}.\end{aligned}$$

Distance between a point and a plane

■

NOTE The choice of the point P in Example 5 is arbitrary. Try choosing a different point in the plane to verify that you obtain the same distance. ■

From Theorem 11.13, you can determine that the distance between the point $Q(x_0, y_0, z_0)$ and the plane given by $ax + by + cz + d = 0$ is

$$D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}}$$

or

$$\boxed{D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

Distance between a point and a plane

where $P(x_1, y_1, z_1)$ is a point in the plane and $d = -(ax_1 + by_1 + cz_1)$.

EXAMPLE 6 Finding the Distance Between Two Parallel Planes

Find the distance between the two parallel planes given by

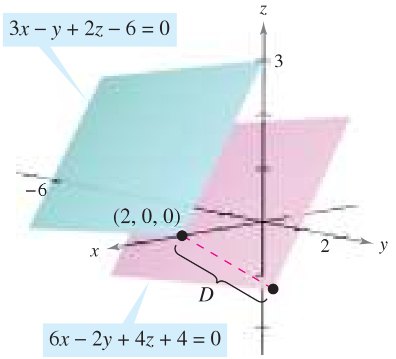
$$3x - y + 2z - 6 = 0 \quad \text{and} \quad 6x - 2y + 4z + 4 = 0.$$

Solution The two planes are shown in Figure 11.53. To find the distance between the planes, choose a point in the first plane, say $(x_0, y_0, z_0) = (2, 0, 0)$. Then, from the second plane, you can determine that $a = 6$, $b = -2$, $c = 4$, and $d = 4$, and conclude that the distance is

$$\begin{aligned} D &= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|6(2) + (-2)(0) + (4)(0) + 4|}{\sqrt{6^2 + (-2)^2 + 4^2}} \\ &= \frac{16}{\sqrt{56}} = \frac{8}{\sqrt{14}} \approx 2.14. \end{aligned}$$

Distance between a point and a plane

■



The distance between the parallel planes is approximately 2.14.
Figure 11.53

The formula for the distance between a point and a line in space resembles that for the distance between a point and a plane—except that you replace the dot product with the length of the cross product and the normal vector **n** with a direction vector for the line.

THEOREM 11.14 DISTANCE BETWEEN A POINT AND A LINE IN SPACE

The distance between a point Q and a line in space is given by

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|}$$

where **u** is a direction vector for the line and P is a point on the line.

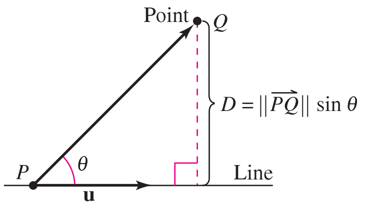
PROOF In Figure 11.54, let D be the distance between the point Q and the given line. Then $D = \|\overrightarrow{PQ}\| \sin\theta$, where θ is the angle between **u** and $\overrightarrow{PQ}$. By Property 2 of Theorem 11.8, you have

$$\|\mathbf{u}\|\,\|\overrightarrow{PQ}\| \sin\theta = \|\mathbf{u} \times \overrightarrow{PQ}\| = \|\overrightarrow{PQ} \times \mathbf{u}\|.$$

Consequently,

$$D = \|\overrightarrow{PQ}\| \sin\theta = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|}.$$

■



The distance between a point and a line
Figure 11.54

EXAMPLE 7 Finding the Distance Between a Point and a Line

Find the distance between the point $Q(3, -1, 4)$ and the line given by

$$x = -2 + 3t, \quad y = -2t, \quad \text{and} \quad z = 1 + 4t.$$

Solution Using the direction numbers 3, -2, and 4, you know that a direction vector for the line is

$$\mathbf{u} = \langle 3, -2, 4 \rangle. \qquad \text{Direction vector for line}$$

To find a point on the line, let $t = 0$ and obtain

$$P = (-2, 0, 1). \qquad \text{Point on the line}$$

So,

$$\overrightarrow{PQ} = \langle 3 - (-2), -1 - 0, 4 - 1 \rangle = \langle 5, -1, 3 \rangle$$

and you can form the cross product

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 3 \\ 3 & -2 & 4 \end{vmatrix} = 2\mathbf{i} - 11\mathbf{j} - 7\mathbf{k} = \langle 2, -11, -7 \rangle.$$

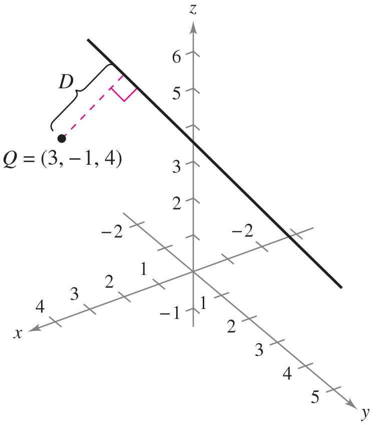
Finally, using Theorem 11.14, you can find the distance to be

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|}$$

$$= \frac{\sqrt{174}}{\sqrt{29}}$$

$$= \sqrt{6} \approx 2.45. \qquad \text{See Figure 11.55.}$$

■



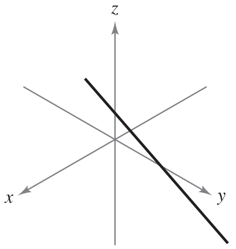
The distance between the point Q and the line is $\sqrt{6} \approx 2.45$.
Figure 11.55

11.5 Exercises

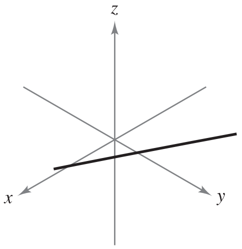
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, the figure shows the graph of a line given by the parametric equations. (a) Draw an arrow on the line to indicate its orientation. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*. (b) Find the coordinates of two points, P and Q, on the line. Determine the vector $\overrightarrow{PQ}$. What is the relationship between the components of the vector and the coefficients of t in the parametric equations? Why is this true? (c) Determine the coordinates of any points of intersection with the coordinate planes. If the line does not intersect a coordinate plane, explain why.

1. $x = 1 + 3t$
$y = 2 - t$
$z = 2 + 5t$



2. $x = 2 - 3t$
$y = 2$
$z = 1 - t$



In Exercises 3 and 4, determine whether each point lies on the line.

3. $x = -2 + t, y = 3t, z = 4 + t$
(a) $(0, 6, 6)$ (b) $(2, 3, 5)$

4. $\dfrac{x - 3}{2} = \dfrac{y - 7}{8} = z + 2$
(a) $(7, 23, 0)$ (b) $(1, -1, -3)$

In Exercises 5–10, find sets of (a) parametric equations and (b) symmetric equations of the line through the point parallel to the given vector or line (if possible). (For each line, write the direction numbers as integers.)

	Point	*Parallel to*
5.	$(0, 0, 0)$	$\mathbf{v} = \langle 3, 1, 5 \rangle$
6.	$(0, 0, 0)$	$\mathbf{v} = \left\langle -2, \frac{5}{2}, 1 \right\rangle$
7.	$(-2, 0, 3)$	$\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$
8.	$(-3, 0, 2)$	$\mathbf{v} = 6\mathbf{j} + 3\mathbf{k}$
9.	$(1, 0, 1)$	$x = 3 + 3t, y = 5 - 2t, z = -7 + t$
10.	$(-3, 5, 4)$	$\dfrac{x - 1}{3} = \dfrac{y + 1}{-2} = z - 3$

In Exercises 11–14, find sets of (a) parametric equations and (b) symmetric equations of the line through the two points (if possible). (For each line, write the direction numbers as integers.)

11. $(5, -3, -2), \left(-\frac{2}{3}, \frac{2}{3}, 1\right)$

12. $(0, 4, 3), (-1, 2, 5)$

13. $(7, -2, 6), (-3, 0, 6)$

14. $(0, 0, 25), (10, 10, 0)$

In Exercises 15–22, find a set of parametric equations of the line.

15. The line passes through the point $(2, 3, 4)$ and is parallel to the xz-plane and the yz-plane.

16. The line passes through the point $(-4, 5, 2)$ and is parallel to the xy-plane and the yz-plane.

17. The line passes through the point $(2, 3, 4)$ and is perpendicular to the plane given by $3x + 2y - z = 6$.

18. The line passes through the point $(-4, 5, 2)$ and is perpendicular to the plane given by $-x + 2y + z = 5$.

19. The line passes through the point $(5, -3, -4)$ and is parallel to $\mathbf{v} = \langle 2, -1, 3 \rangle$.

20. The line passes through the point $(-1, 4, -3)$ and is parallel to $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$.

21. The line passes through the point $(2, 1, 2)$ and is parallel to the line $x = -t,\ y = 1 + t,\ z = -2 + t$.

22. The line passes through the point $(-6, 0, 8)$ and is parallel to the line $x = 5 - 2t,\ y = -4 + 2t,\ z = 0$.

In Exercises 23–26, find the coordinates of a point *P* on the line and a vector v parallel to the line.

23. $x = 3 - t, \quad y = -1 + 2t, \quad z = -2$

24. $x = 4t, \quad y = 5 - t, \quad z = 4 + 3t$

25. $\dfrac{x - 7}{4} = \dfrac{y + 6}{2} = z + 2$

26. $\dfrac{x + 3}{5} = \dfrac{y}{8} = \dfrac{z - 3}{6}$

In Exercises 27–30, determine if any of the lines are parallel or identical.

27. L_1: $x = 6 - 3t, \quad y = -2 + 2t, \quad z = 5 + 4t$

L_2: $x = 6t, \quad y = 2 - 4t, \quad z = 13 - 8t$

L_3: $x = 10 - 6t, \quad y = 3 + 4t, \quad z = 7 + 8t$

L_4: $x = -4 + 6t, \quad y = 3 + 4t, \quad z = 5 - 6t$

28. L_1: $x = 3 + 2t, \quad y = -6t, \quad z = 1 - 2t$

L_2: $x = 1 + 2t, \quad y = -1 - t, \quad z = 3t$

L_3: $x = -1 + 2t, \quad y = 3 - 10t, \quad z = 1 - 4t$

L_4: $x = 5 + 2t, \quad y = 1 - t, \quad z = 8 + 3t$

29. L_1: $\dfrac{x - 8}{4} = \dfrac{y + 5}{-2} = \dfrac{z + 9}{3}$

L_2: $\dfrac{x + 7}{2} = \dfrac{y - 4}{1} = \dfrac{z + 6}{5}$

L_3: $\dfrac{x + 4}{-8} = \dfrac{y - 1}{4} = \dfrac{z + 18}{-6}$

L_4: $\dfrac{x - 2}{-2} = \dfrac{y + 3}{1} = \dfrac{z - 4}{1.5}$

30. L_1: $\dfrac{x - 3}{2} = \dfrac{y - 2}{1} = \dfrac{z + 2}{2}$

L_2: $\dfrac{x - 1}{4} = \dfrac{y - 1}{2} = \dfrac{z + 3}{4}$

L_3: $\dfrac{x + 2}{1} = \dfrac{y - 1}{0.5} = \dfrac{z - 3}{1}$

L_4: $\dfrac{x - 3}{2} = \dfrac{y + 1}{4} = \dfrac{z - 2}{-1}$

In Exercises 31–34, determine whether the lines intersect, and if so, find the point of intersection and the cosine of the angle of intersection.

31. $x = 4t + 2, \quad y = 3, \quad z = -t + 1$

$x = 2s + 2, \quad y = 2s + 3, \quad z = s + 1$

32. $x = -3t + 1, y = 4t + 1, z = 2t + 4$

$x = 3s + 1, y = 2s + 4, z = -s + 1$

33. $\dfrac{x}{3} = \dfrac{y - 2}{-1} = z + 1, \quad \dfrac{x - 1}{4} = y + 2 = \dfrac{z + 3}{-3}$

34. $\dfrac{x - 2}{-3} = \dfrac{y - 2}{6} = z - 3, \quad \dfrac{x - 3}{2} = y + 5 = \dfrac{z + 2}{4}$

CAS **In Exercises 35 and 36, use a computer algebra system to graph the pair of intersecting lines and find the point of intersection.**

35. $x = 2t + 3, y = 5t - 2, z = -t + 1$

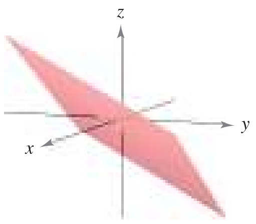
$x = -2s + 7, y = s + 8, z = 2s - 1$

36. $x = 2t - 1, y = -4t + 10, z = t$

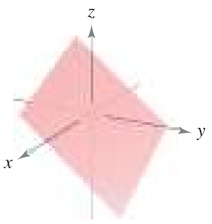
$x = -5s - 12, y = 3s + 11, z = -2s - 4$

Cross Product **In Exercises 37 and 38, (a) find the coordinates of three points *P*, *Q*, and *R* in the plane, and determine the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. (b) Find $\overrightarrow{PQ} \times \overrightarrow{PR}$. What is the relationship between the components of the cross product and the coefficients of the equation of the plane? Why is this true?**

37. $4x - 3y - 6z = 6$



38. $2x + 3y + 4z = 4$



In Exercises 39 and 40, determine whether the plane passes through each point.

39. $x + 2y - 4z - 1 = 0$

(a) $(-7, 2, -1)$ (b) $(5, 2, 2)$

40. $2x + y + 3z - 6 = 0$

(a) $(3, 6, -2)$ (b) $(-1, 5, -1)$

In Exercises 41–46, find an equation of the plane passing through the point perpendicular to the given vector or line.

	Point	*Perpendicular to*
41.	$(1, 3, -7)$	$\mathbf{n} = \mathbf{j}$
42.	$(0, -1, 4)$	$\mathbf{n} = \mathbf{k}$
43.	$(3, 2, 2)$	$\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
44.	$(0, 0, 0)$	$\mathbf{n} = -3\mathbf{i} + 2\mathbf{k}$
45.	$(-1, 4, 0)$	$x = -1 + 2t, y = 5 - t, z = 3 - 2t$
46.	$(3, 2, 2)$	$\dfrac{x-1}{4} = y + 2 = \dfrac{z+3}{-3}$

In Exercises 47–58, find an equation of the plane.

47. The plane passes through $(0, 0, 0)$, $(2, 0, 3)$, and $(-3, -1, 5)$.

48. The plane passes through $(3, -1, 2)$, $(2, 1, 5)$, and $(1, -2, -2)$.

49. The plane passes through $(1, 2, 3)$, $(3, 2, 1)$, and $(-1, -2, 2)$.

50. The plane passes through the point $(1, 2, 3)$ and is parallel to the yz-plane.

51. The plane passes through the point $(1, 2, 3)$ and is parallel to the xy-plane.

52. The plane contains the y-axis and makes an angle of $\pi/6$ with the positive x-axis.

53. The plane contains the lines given by

$$\frac{x-1}{-2} = y - 4 = z \quad \text{and} \quad \frac{x-2}{-3} = \frac{y-1}{4} = \frac{z-2}{-1}.$$

54. The plane passes through the point $(2, 2, 1)$ and contains the line given by

$$\frac{x}{2} = \frac{y-4}{-1} = z.$$

55. The plane passes through the points $(2, 2, 1)$ and $(-1, 1, -1)$ and is perpendicular to the plane $2x - 3y + z = 3$.

56. The plane passes through the points $(3, 2, 1)$ and $(3, 1, -5)$ and is perpendicular to the plane $6x + 7y + 2z = 10$.

57. The plane passes through the points $(1, -2, -1)$ and $(2, 5, 6)$ and is parallel to the x-axis.

58. The plane passes through the points $(4, 2, 1)$ and $(-3, 5, 7)$ and is parallel to the z-axis.

In Exercises 59 and 60, sketch a graph of the line and find the points (if any) where the line intersects the *xy*-, *xz*-, and *yz*-planes.

59. $x = 1 - 2t, \quad y = -2 + 3t, \quad z = -4 + t$

60. $\dfrac{x-2}{3} = y + 1 = \dfrac{z-3}{2}$

In Exercises 61–64, find an equation of the plane that contains all the points that are equidistant from the given points.

61. $(2, 2, 0), \quad (0, 2, 2)$

62. $(1, 0, 2), \quad (2, 0, 1)$

63. $(-3, 1, 2), \quad (6, -2, 4)$

64. $(-5, 1, -3), \quad (2, -1, 6)$

In Exercises 65–70, determine whether the planes are parallel, orthogonal, or neither. If they are neither parallel nor orthogonal, find the angle of intersection.

65. $5x - 3y + z = 4$
$x + 4y + 7z = 1$

66. $3x + y - 4z = 3$
$-9x - 3y + 12z = 4$

67. $x - 3y + 6z = 4$
$5x + y - z = 4$

68. $3x + 2y - z = 7$
$x - 4y + 2z = 0$

69. $x - 5y - z = 1$
$5x - 25y - 5z = -3$

70. $2x - z = 1$
$4x + y + 8z = 10$

In Exercises 71–78, sketch a graph of the plane and label any intercepts.

71 $4x + 2y + 6z = 12$

72. $3x + 6y + 2z = 6$

73. $2x - y + 3z = 4$

74. $2x - y + z = 4$

75. $x + z = 6$

76. $2x + y = 8$

77. $x = 5$

78. $z = 8$

CAS **In Exercises 79–82, use a computer algebra system to graph the plane.**

79. $2x + y - z = 6$

80. $x - 3z = 3$

81. $-5x + 4y - 6z = -8$

82. $2.1x - 4.7y - z = -3$

In Exercises 83–86, determine if any of the planes are parallel or identical.

83. P_1: $15x - 6y + 24z = 17$
P_2: $-5x + 2y - 8z = 6$
P_3: $6x - 4y + 4z = 9$
P_4: $3x - 2y - 2z = 4$

84. P_1: $2x - y + 3z = 8$
P_2: $3x - 5y - 2z = 6$
P_3: $8x - 4y + 12z = 5$
P_4: $-4x - 2y + 6z = 11$

85. P_1: $3x - 2y + 5z = 10$
P_2: $-6x + 4y - 10z = 5$
P_3: $-3x + 2y + 5z = 8$
P_4: $75x - 50y + 125z = 250$

86. P_1: $-60x + 90y + 30z = 27$
P_2: $6x - 9y - 3z = 2$
P_3: $-20x + 30y + 10z = 9$
P_4: $12x - 18y + 6z = 5$

In Exercises 87–90, describe the family of planes represented by the equation, where *c* is any real number.

87. $x + y + z = c$

88. $x + y = c$

89. $cy + z = 0$

90. $x + cz = 0$

In Exercises 91 and 92, (a) find the angle between the two planes, and (b) find a set of parametric equations for the line of intersection of the planes.

91. $3x + 2y - z = 7$
$x - 4y + 2z = 0$

92. $6x - 3y + z = 5$
$-x + y + 5z = 5$

In Exercises 93–96, find the point(s) of intersection (if any) of the plane and the line. Also determine whether the line lies in the plane.

93. $2x - 2y + z = 12, \quad x - \frac{1}{2} = \frac{y + (3/2)}{-1} = \frac{z + 1}{2}$

94. $2x + 3y = -5, \quad \frac{x - 1}{4} = \frac{y}{2} = \frac{z - 3}{6}$

95. $2x + 3y = 10, \quad \frac{x - 1}{3} = \frac{y + 1}{-2} = z - 3$

96. $5x + 3y = 17, \quad \frac{x - 4}{2} = \frac{y + 1}{-3} = \frac{z + 2}{5}$

In Exercises 97–100, find the distance between the point and the plane.

97. $(0, 0, 0)$
$2x + 3y + z = 12$

98. $(0, 0, 0)$
$5x + y - z = 9$

99. $(2, 8, 4)$
$2x + y + z = 5$

100. $(1, 3, -1)$
$3x - 4y + 5z = 6$

In Exercises 101–104, verify that the two planes are parallel, and find the distance between the planes.

101. $x - 3y + 4z = 10$
$x - 3y + 4z = 6$

102. $4x - 4y + 9z = 7$
$4x - 4y + 9z = 18$

103. $-3x + 6y + 7z = 1$
$6x - 12y - 14z = 25$

104. $2x - 4z = 4$
$2x - 4z = 10$

In Exercises 105–108, find the distance between the point and the line given by the set of parametric equations.

105. $(1, 5, -2); \quad x = 4t - 2, \quad y = 3, \quad z = -t + 1$

106. $(1, -2, 4); \quad x = 2t, \quad y = t - 3, \quad z = 2t + 2$

107. $(-2, 1, 3); \quad x = 1 - t, \quad y = 2 + t, \quad z = -2t$

108. $(4, -1, 5); \quad x = 3, \quad y = 1 + 3t, \quad z = 1 + t$

In Exercises 109 and 110, verify that the lines are parallel, and find the distance between them.

109. L_1: $x = 2 - t, \quad y = 3 + 2t, \quad z = 4 + t$
L_2: $x = 3t, \quad y = 1 - 6t, \quad z = 4 - 3t$

110. L_1: $x = 3 + 6t, \quad y = -2 + 9t, \quad z = 1 - 12t$
L_2: $x = -1 + 4t, \quad y = 3 + 6t, \quad z = -8t$

WRITING ABOUT CONCEPTS

111. Give the parametric equations and the symmetric equations of a line in space. Describe what is required to find these equations.

112. Give the standard equation of a plane in space. Describe what is required to find this equation.

113. Describe a method of finding the line of intersection of two planes.

114. Describe each surface given by the equations $x = a$, $y = b$, and $z = c$.

WRITING ABOUT CONCEPTS (continued)

115. Describe a method for determining when two planes

$a_1x + b_1y + c_1z + d_1 = 0$ and

$a_2x + b_2y + c_2z + d_2 = 0$

are (a) parallel and (b) perpendicular. Explain your reasoning.

116. Let L_1 and L_2 be nonparallel lines that do not intersect. Is it possible to find a nonzero vector $\mathbf{v}$ such that $\mathbf{v}$ is perpendicular to both L_1 and L_2? Explain your reasoning.

117. Find an equation of the plane with x-intercept $(a, 0, 0)$, y-intercept $(0, b, 0)$, and z-intercept $(0, 0, c)$. (Assume a, b, and c are nonzero.)

CAPSTONE

118. Match the equation or set of equations with the description it represents.

(a) Set of parametric equations of a line

(b) Set of symmetric equations of a line

(c) Standard equation of a plane in space

(d) General form of an equation of a plane in space

(i) $(x - 6)/2 = (y + 1)/-3 = z/1$

(ii) $2x - 7y + 5z + 10 = 0$

(iii) $x = 4 + 7t, y = 3 + t, z = 3 - 3t$

(iv) $2(x - 1) + (y + 3) - 4(z - 5) = 0$

119. Describe and find an equation for the surface generated by all points (x, y, z) that are four units from the point $(3, -2, 5)$.

120. Describe and find an equation for the surface generated by all points (x, y, z) that are four units from the plane $4x - 3y + z = 10$.

121. ***Modeling Data*** Per capita consumptions (in gallons) of different types of milk in the United States from 1999 through 2005 are shown in the table. Consumptions of flavored milk, plain reduced-fat milk, and plain light and skim milks are represented by the variables x, y, and z, respectively. *(Source: U.S. Department of Agriculture)*

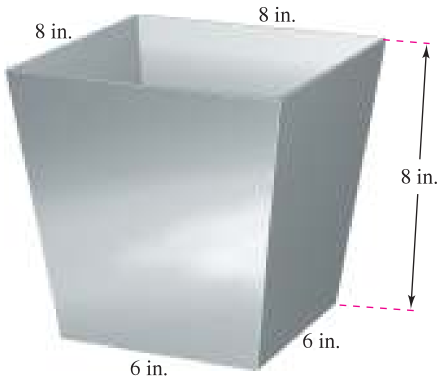
Year	1999	2000	2001	2002	2003	2004	2005
x	1.4	1.4	1.4	1.6	1.6	1.7	1.7
y	7.3	7.1	7.0	7.0	6.9	6.9	6.9
z	6.2	6.1	5.9	5.8	5.6	5.5	5.6

A model for the data is given by $0.92x - 1.03y + z = 0.02$.

(a) Complete a fourth row in the table using the model to approximate z for the given values of x and y. Compare the approximations with the actual values of z.

(b) According to this model, any increases in consumption of two types of milk will have what effect on the consumption of the third type?

122. ***Mechanical Design*** The figure shows a chute at the top of a grain elevator of a combine that funnels the grain into a bin. Find the angle between two adjacent sides.



123. ***Distance*** Two insects are crawling along different lines in three-space. At time t (in minutes), the first insect is at the point (x, y, z) on the line $x = 6 + t$, $y = 8 - t$, $z = 3 + t$. Also, at time t, the second insect is at the point (x, y, z) on the line $x = 1 + t$, $y = 2 + t$, $z = 2t$.

Assume that distances are given in inches.

(a) Find the distance between the two insects at time $t = 0$.

(b) Use a graphing utility to graph the distance between the insects from $t = 0$ to $t = 10$.

(c) Using the graph from part (b), what can you conclude about the distance between the insects?

(d) How close to each other do the insects get?

124. Find the standard equation of the sphere with center $(-3, 2, 4)$ that is tangent to the plane given by $2x + 4y - 3z = 8$.

125. Find the point of intersection of the plane $3x - y + 4z = 7$ and the line through $(5, 4, -3)$ that is perpendicular to this plane.

126. Show that the plane $2x - y - 3z = 4$ is parallel to the line $x = -2 + 2t$, $y = -1 + 4t$, $z = 4$, and find the distance between them.

127. Find the point of intersection of the line through $(1, -3, 1)$ and $(3, -4, 2)$, and the plane given by $x - y + z = 2$.

128. Find a set of parametric equations for the line passing through the point $(1, 0, 2)$ that is parallel to the plane given by $x + y + z = 5$, and perpendicular to the line $x = t$, $y = 1 + t$, $z = 1 + t$.

True or False? **In Exercises 129–134, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

129. If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ is any vector in the plane given by $a_2x + b_2y + c_2z + d_2 = 0$, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

130. Every two lines in space are either intersecting or parallel.

131. Two planes in space are either intersecting or parallel.

132. If two lines L_1 and L_2 are parallel to a plane P, then L_1 and L_2 are parallel.

133. Two planes perpendicular to a third plane in space are parallel.

134. A plane and a line in space are either intersecting or parallel.

SECTION PROJECT

Distances in Space

You have learned two distance formulas in this section—the distance between a point and a plane, and the distance between a point and a line. In this project you will study a third distance problem—the distance between two skew lines. Two lines in space are *skew* if they are neither parallel nor intersecting (see figure).

(a) Consider the following two lines in space.

L_1: $x = 4 + 5t$, $y = 5 + 5t$, $z = 1 - 4t$

L_2: $x = 4 + s$, $y = -6 + 8s$, $z = 7 - 3s$

(i) Show that these lines are not parallel.

(ii) Show that these lines do not intersect, and therefore are skew lines.

(iii) Show that the two lines lie in parallel planes.

(iv) Find the distance between the parallel planes from part (iii). This is the distance between the original skew lines.

(b) Use the procedure in part (a) to find the distance between the lines.

L_1: $x = 2t$, $y = 4t$, $z = 6t$

L_2: $x = 1 - s$, $y = 4 + s$, $z = -1 + s$

(c) Use the procedure in part (a) to find the distance between the lines.

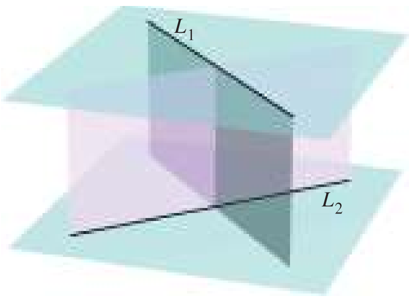
L_1: $x = 3t$, $y = 2 - t$, $z = -1 + t$

L_2: $x = 1 + 4s$, $y = -2 + s$, $z = -3 - 3s$

(d) Develop a formula for finding the distance between the skew lines.

L_1: $x = x_1 + a_1t$, $y = y_1 + b_1t$, $z = z_1 + c_1t$

L_2: $x = x_2 + a_2s$, $y = y_2 + b_2s$, $z = z_2 + c_2s$



11.6 Surfaces in Space

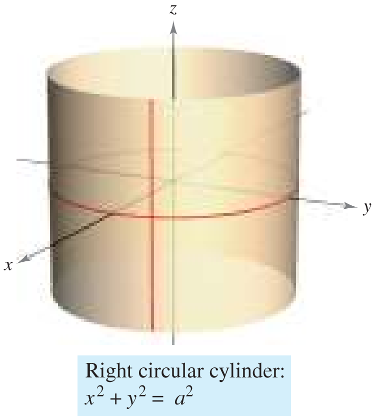
- **Recognize and write equations of cylindrical surfaces.**
- **Recognize and write equations of quadric surfaces.**
- **Recognize and write equations of surfaces of revolution.**

Cylindrical Surfaces

The first five sections of this chapter contained the vector portion of the preliminary work necessary to study vector calculus and the calculus of space. In this and the next section, you will study surfaces in space and alternative coordinate systems for space. You have already studied two special types of surfaces.

1. Spheres: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$ Section 11.2

2. Planes: $ax + by + cz + d = 0$ Section 11.5

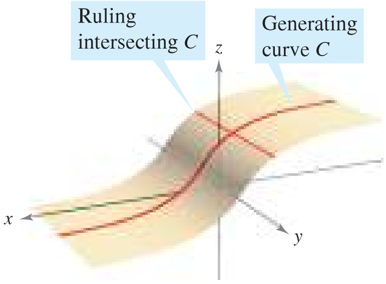


Rulings are parallel to z-axis.
Figure 11.56

A third type of surface in space is called a **cylindrical surface,** or simply a **cylinder.** To define a cylinder, consider the familiar right circular cylinder shown in Figure 11.56. You can imagine that this cylinder is generated by a vertical line moving around the circle $x^2 + y^2 = a^2$ in the xy-plane. This circle is called a **generating curve** for the cylinder, as indicated in the following definition.

DEFINITION OF A CYLINDER

Let C be a curve in a plane and let L be a line not in a parallel plane. The set of all lines parallel to L and intersecting C is called a **cylinder.** C is called the **generating curve** (or **directrix**) of the cylinder, and the parallel lines are called **rulings.**



Cylinder: Rulings intersect C and are parallel to the given line.
Figure 11.57

NOTE Without loss of generality, you can assume that C lies in one of the three coordinate planes. Moreover, this text restricts the discussion to *right* cylinders—cylinders whose rulings are perpendicular to the coordinate plane containing C, as shown in Figure 11.57. ■

For the right circular cylinder shown in Figure 11.56, the equation of the generating curve is

$$x^2 + y^2 = a^2.$$ Equation of generating curve in xy-plane

To find an equation of the cylinder, note that you can generate any one of the rulings by fixing the values of x and y and then allowing z to take on all real values. In this sense, the value of z is arbitrary and is, therefore, not included in the equation. In other words, the equation of this cylinder is simply the equation of its generating curve.

$$x^2 + y^2 = a^2$$ Equation of cylinder in space

EQUATIONS OF CYLINDERS

The equation of a cylinder whose rulings are parallel to one of the coordinate axes contains only the variables corresponding to the other two axes.

EXAMPLE 1 Sketching a Cylinder

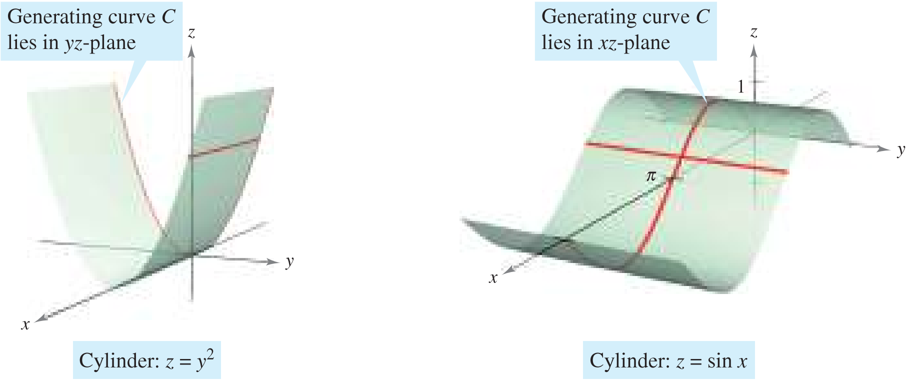
Sketch the surface represented by each equation.

a. $z = y^2$ **b.** $z = \sin x, \quad 0 \le x \le 2\pi$

Solution

a. The graph is a cylinder whose generating curve, $z = y^2$, is a parabola in the yz-plane. The rulings of the cylinder are parallel to the x-axis, as shown in Figure 11.58(a).

b. The graph is a cylinder generated by the sine curve in the xz-plane. The rulings are parallel to the y-axis, as shown in Figure 11.58(b).



Generating curve C lies in yz-plane

Cylinder: $z = y^2$

Generating curve C lies in xz-plane

Cylinder: $z = \sin x$

(a) Rulings are parallel to x-axis. **(b)** Rulings are parallel to y-axis.

Figure 11.58

Quadric Surfaces

The fourth basic type of surface in space is a **quadric surface.** Quadric surfaces are the three-dimensional analogs of conic sections.

STUDY TIP In the table on pages 814 and 815, only one of several orientations of each quadric surface is shown. If the surface is oriented along a different axis, its standard equation will change accordingly, as illustrated in Examples 2 and 3. The fact that the two types of paraboloids have one variable raised to the first power can be helpful in classifying quadric surfaces. The other four types of basic quadric surfaces have equations that are of *second degree* in all three variables.

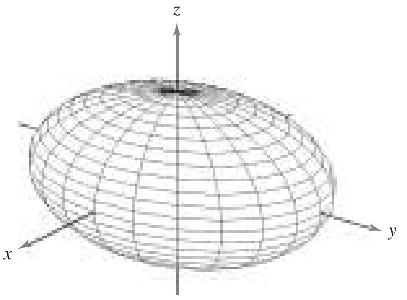
QUADRIC SURFACE

The equation of a **quadric surface** in space is a second-degree equation in three variables. The **general form** of the equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

There are six basic types of quadric surfaces: **ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, elliptic paraboloid, and hyperbolic paraboloid.**

The intersection of a surface with a plane is called the **trace of the surface** in the plane. To visualize a surface in space, it is helpful to determine its traces in some well-chosen planes. The traces of quadric surfaces are conics. These traces, together with the **standard form** of the equation of each quadric surface, are shown in the table on pages 814 and 815.

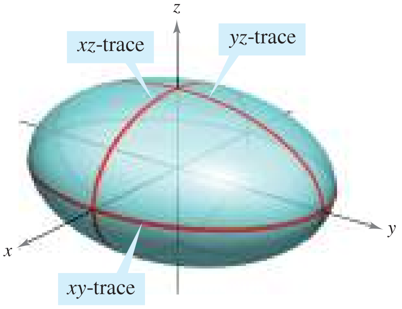


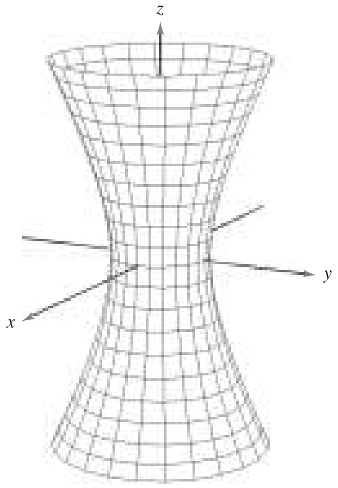
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Ellipse	Parallel to xz-plane
Ellipse	Parallel to yz-plane

The surface is a sphere if $a = b = c \neq 0$.


z
xz-trace
yz-trace
y
x
xy-trace

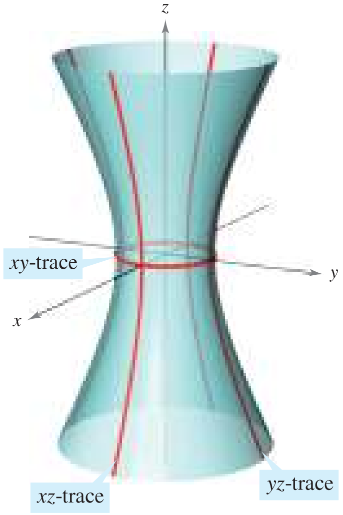


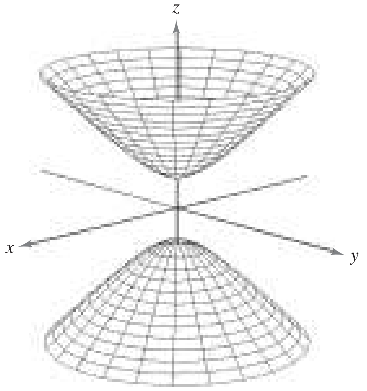
Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.


z
xy-trace
y
x
xz-trace
yz-trace

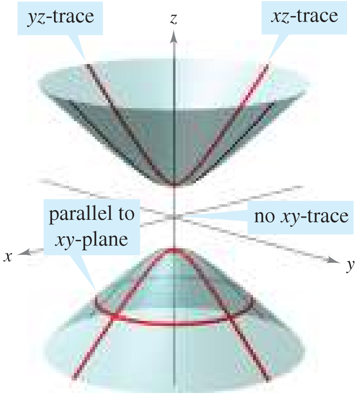


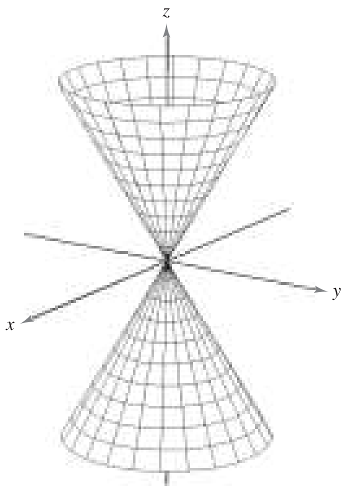
Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.


yz-trace
z
xz-trace
parallel to xy-plane
no xy-trace
x
y

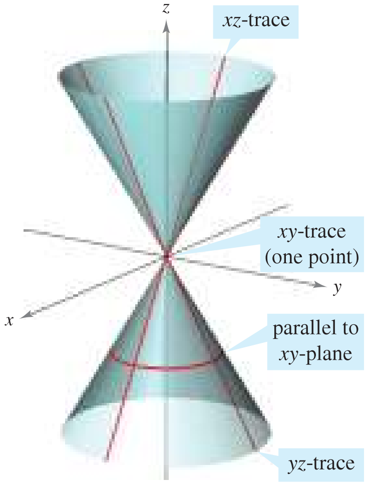


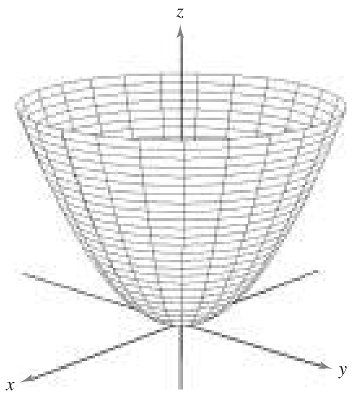
Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.


z
xz-trace
xy-trace
(one point)
y
x
parallel to
xy-plane
yz-trace

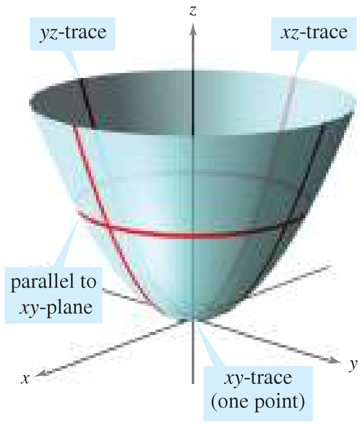


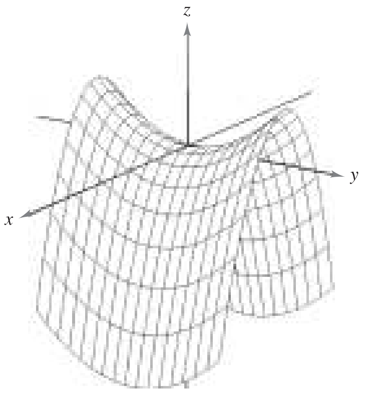
Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.


yz-trace
z
xz-trace
parallel to
xy-plane
x
y
xy-trace
(one point)

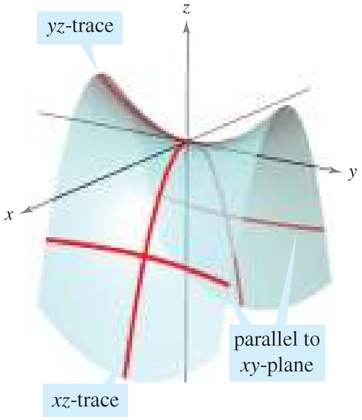


Hyperbolic Paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

Trace	*Plane*
Hyperbola	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.


yz-trace
z
y
x
parallel to
xy-plane
xz-trace

To classify a quadric surface, begin by writing the surface in standard form. Then, determine several traces taken in the coordinate planes *or* taken in planes that are parallel to the coordinate planes.

EXAMPLE 2 Sketching a Quadric Surface

Classify and sketch the surface given by $4x^2 - 3y^2 + 12z^2 + 12 = 0$.

Solution Begin by writing the equation in standard form.

$$4x^2 - 3y^2 + 12z^2 + 12 = 0$$ Write original equation.

$$\frac{x^2}{-3} + \frac{y^2}{4} - z^2 - 1 = 0$$ Divide by -12.

$$\frac{y^2}{4} - \frac{x^2}{3} - \frac{z^2}{1} = 1$$ Standard form

From the table on pages 814 and 815, you can conclude that the surface is a hyperboloid of two sheets with the y-axis as its axis. To sketch the graph of this surface, it helps to find the traces in the coordinate planes.

xy-trace $(z = 0)$: $\frac{y^2}{4} - \frac{x^2}{3} = 1$ Hyperbola

xz-trace $(y = 0)$: $\frac{x^2}{3} + \frac{z^2}{1} = -1$ No trace

yz-trace $(x = 0)$: $\frac{y^2}{4} - \frac{z^2}{1} = 1$ Hyperbola

The graph is shown in Figure 11.59.

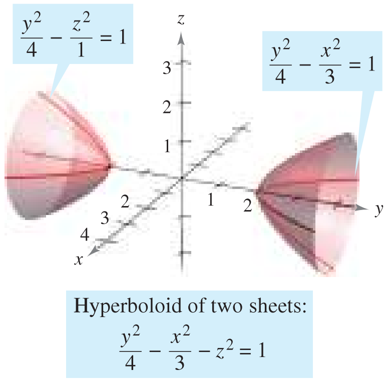

$\frac{y^2}{4} - \frac{z^2}{1} = 1$
z
3
2
1
$\frac{y^2}{4} - \frac{x^2}{3} = 1$
2
3
4
x
1
2
y
Hyperboloid of two sheets:
$\frac{y^2}{4} - \frac{x^2}{3} - z^2 = 1$

Figure 11.59

EXAMPLE 3 Sketching a Quadric Surface

Classify and sketch the surface given by $x - y^2 - 4z^2 = 0$.

Solution Because x is raised only to the first power, the surface is a paraboloid. The axis of the paraboloid is the x-axis. In the standard form, the equation is

$$x = y^2 + 4z^2.$$ Standard form

Some convenient traces are as follows.

xy-trace $(z = 0)$: $x = y^2$ Parabola

xz-trace $(y = 0)$: $x = 4z^2$ Parabola

parallel to yz-plane $(x = 4)$: $\frac{y^2}{4} + \frac{z^2}{1} = 1$ Ellipse

The surface is an *elliptic* paraboloid, as shown in Figure 11.60. ■

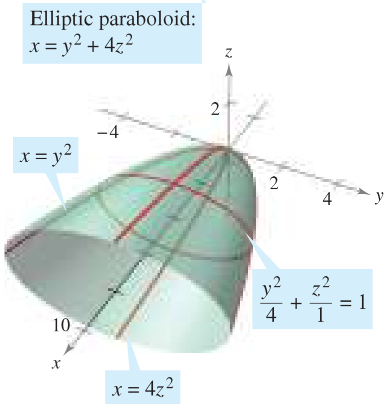


Figure 11.60

Some second-degree equations in x, y, and z do not represent any of the basic types of quadric surfaces. Here are two examples.

$$x^2 + y^2 + z^2 = 0$$ Single point

$$x^2 + y^2 = 1$$ Right circular cylinder

For a quadric surface not centered at the origin, you can form the standard equation by completing the square, as demonstrated in Example 4.

EXAMPLE 4 A Quadric Surface Not Centered at the Origin

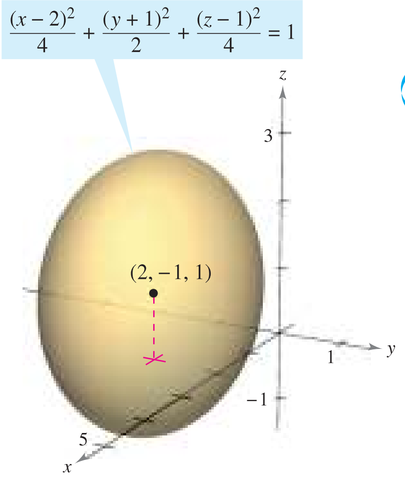
Classify and sketch the surface given by

$$x^2 + 2y^2 + z^2 - 4x + 4y - 2z + 3 = 0.$$

Solution Completing the square for each variable produces the following.

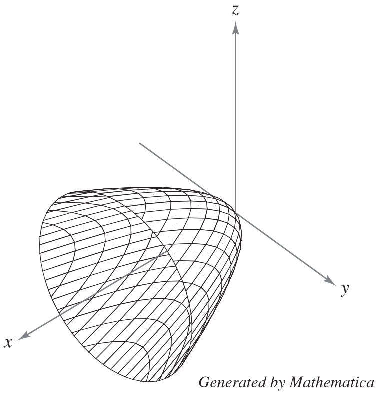
$$(x^2 - 4x + \quad) + 2(y^2 + 2y + \quad) + (z^2 - 2z + \quad) = -3$$
$$(x^2 - 4x + 4) + 2(y^2 + 2y + 1) + (z^2 - 2z + 1) = -3 + 4 + 2 + 1$$
$$(x - 2)^2 + 2(y + 1)^2 + (z - 1)^2 = 4$$
$$\frac{(x - 2)^2}{4} + \frac{(y + 1)^2}{2} + \frac{(z - 1)^2}{4} = 1$$

From this equation, you can see that the quadric surface is an ellipsoid that is centered at $(2, -1, 1)$. Its graph is shown in Figure 11.61. ■


$\frac{(x-2)^2}{4} + \frac{(y+1)^2}{2} + \frac{(z-1)^2}{4} = 1$
z
3
(2, −1, 1)
1
y
−1
5
x

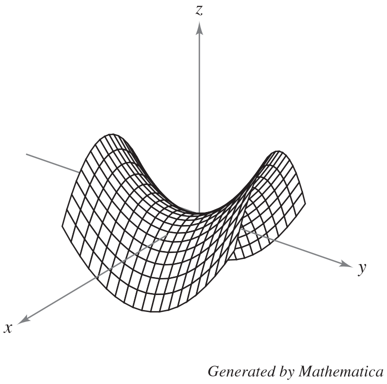
An ellipsoid centered at $(2, -1, 1)$
Figure 11.61

TECHNOLOGY A computer algebra system can help you visualize a surface in space.* Most of these computer algebra systems create three-dimensional illusions by sketching several traces of the surface and then applying a "hidden-line" routine that blocks out portions of the surface that lie behind other portions of the surface. Two examples of figures that were generated by *Mathematica* are shown below.



Elliptic paraboloid

$$x = \frac{y^2}{2} + \frac{z^2}{2}$$



Hyperbolic paraboloid

$$z = \frac{y^2}{16} - \frac{x^2}{16}$$

Using a graphing utility to graph a surface in space requires practice. For one thing, you must know enough about the surface to be able to specify a *viewing window* that gives a representative view of the surface. Also, you can often improve the view of a surface by rotating the axes. For instance, note that the elliptic paraboloid in the figure is seen from a line of sight that is "higher" than the line of sight used to view the hyperbolic paraboloid.

**Some 3-D graphing utilities require surfaces to be entered with parametric equations. For a discussion of this technique, see Section 15.5.*

Surfaces of Revolution

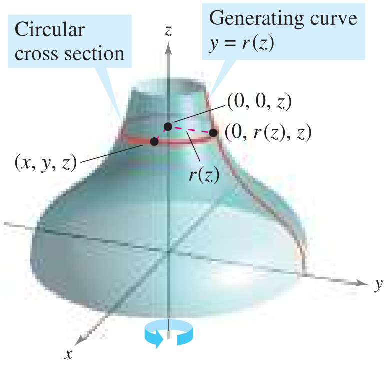


Figure 11.62

The fifth special type of surface you will study is called a **surface of revolution.** In Section 7.4, you studied a method for finding the *area* of such a surface. You will now look at a procedure for finding its *equation*. Consider the graph of the **radius function**

$$y = r(z) \qquad \text{Generating curve}$$

in the yz-plane. If this graph is revolved about the z-axis, it forms a surface of revolution, as shown in Figure 11.62. The trace of the surface in the plane $z = z_0$ is a circle whose radius is $r(z_0)$ and whose equation is

$$x^2 + y^2 = [r(z_0)]^2. \qquad \text{Circular trace in plane: } z = z_0$$

Replacing z_0 with z produces an equation that is valid for all values of z. In a similar manner, you can obtain equations for surfaces of revolution for the other two axes, and the results are summarized as follows.

SURFACE OF REVOLUTION

If the graph of a radius function r is revolved about one of the coordinate axes, the equation of the resulting surface of revolution has one of the following forms.

1. Revolved about the x-axis: $y^2 + z^2 = [r(x)]^2$
2. Revolved about the y-axis: $x^2 + z^2 = [r(y)]^2$
3. Revolved about the z-axis: $x^2 + y^2 = [r(z)]^2$

EXAMPLE 5 Finding an Equation for a Surface of Revolution

a. An equation for the surface of revolution formed by revolving the graph of

$$y = \frac{1}{z} \qquad \text{Radius function}$$

about the z-axis is

$$x^2 + y^2 = [r(z)]^2 \qquad \text{Revolved about the } z\text{-axis}$$

$$x^2 + y^2 = \left(\frac{1}{z}\right)^2. \qquad \text{Substitute } 1/z \text{ for } r(z).$$

b. To find an equation for the surface formed by revolving the graph of $9x^2 = y^3$ about the y-axis, solve for x in terms of y to obtain

$$x = \tfrac{1}{3}y^{3/2} = r(y). \qquad \text{Radius function}$$

So, the equation for this surface is

$$x^2 + z^2 = [r(y)]^2 \qquad \text{Revolved about the } y\text{-axis}$$

$$x^2 + z^2 = \left(\tfrac{1}{3}y^{3/2}\right)^2 \qquad \text{Substitute } \tfrac{1}{3}y^{3/2} \text{ for } r(y).$$

$$x^2 + z^2 = \tfrac{1}{9}y^3. \qquad \text{Equation of surface}$$

The graph is shown in Figure 11.63. ■

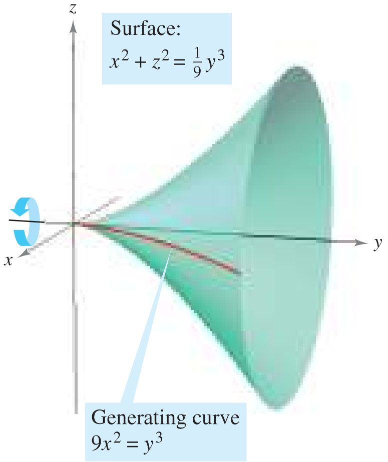

z
Surface: $x^2 + z^2 = \frac{1}{9}y^3$
y
x
Generating curve $9x^2 = y^3$

Figure 11.63

The generating curve for a surface of revolution is not unique. For instance, the surface

$$x^2 + z^2 = e^{-2y}$$

can be formed by revolving either the graph of $x = e^{-y}$ about the y-axis or the graph of $z = e^{-y}$ about the y-axis, as shown in Figure 11.64.

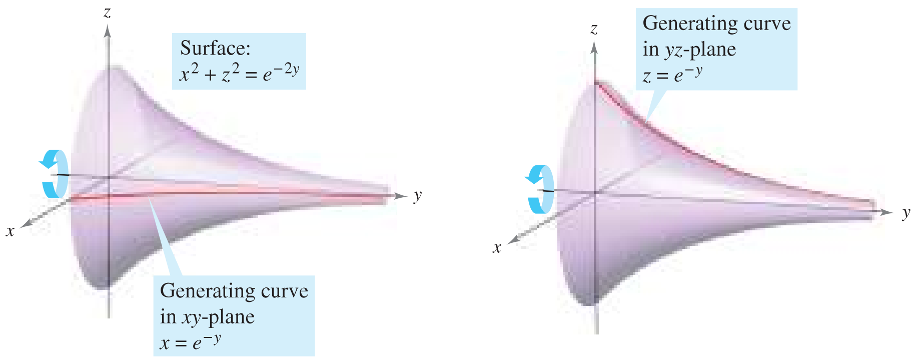

z
Surface:
$x^2 + z^2 = e^{-2y}$
y
x
Generating curve
in xy-plane
$x = e^{-y}$
z
Generating curve
in yz-plane
$z = e^{-y}$
y
x

Figure 11.64

EXAMPLE 6 Finding a Generating Curve for a Surface of Revolution

Find a generating curve and the axis of revolution for the surface given by

$$x^2 + 3y^2 + z^2 = 9.$$

Solution You now know that the equation has one of the following forms.

$$x^2 + y^2 = [r(z)]^2 \quad \text{Revolved about } z\text{-axis}$$
$$y^2 + z^2 = [r(x)]^2 \quad \text{Revolved about } x\text{-axis}$$
$$x^2 + z^2 = [r(y)]^2 \quad \text{Revolved about } y\text{-axis}$$

Because the coefficients of x^2 and z^2 are equal, you should choose the third form and write

$$x^2 + z^2 = 9 - 3y^2.$$

The y-axis is the axis of revolution. You can choose a generating curve from either of the following traces.

$$x^2 = 9 - 3y^2 \quad \text{Trace in } xy\text{-plane}$$
$$z^2 = 9 - 3y^2 \quad \text{Trace in } yz\text{-plane}$$

For example, using the first trace, the generating curve is the semiellipse given by

$$x = \sqrt{9 - 3y^2}. \quad \text{Generating curve}$$

The graph of this surface is shown in Figure 11.65. ■

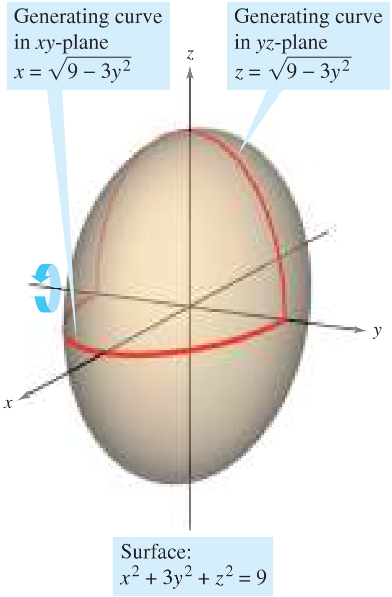

Generating curve
in xy-plane
$x = \sqrt{9 - 3y^2}$
z
Generating curve
in yz-plane
$z = \sqrt{9 - 3y^2}$
y
x
Surface:
$x^2 + 3y^2 + z^2 = 9$

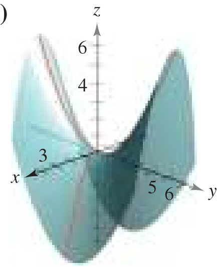
Figure 11.65

11.6 Exercises

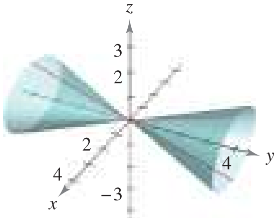
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

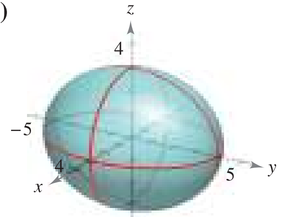
(a)



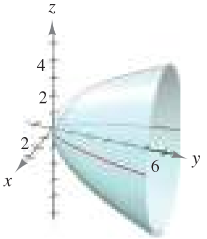
(b)



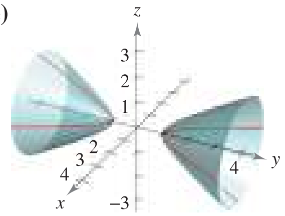
(c)



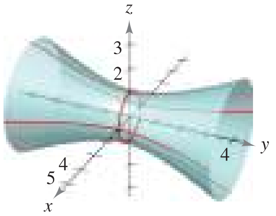
(d)



(e)



(f)



1. $\dfrac{x^2}{9} + \dfrac{y^2}{16} + \dfrac{z^2}{9} = 1$

2. $15x^2 - 4y^2 + 15z^2 = -4$

3. $4x^2 - y^2 + 4z^2 = 4$

4. $y^2 = 4x^2 + 9z^2$

5. $4x^2 - 4y + z^2 = 0$

6. $4x^2 - y^2 + 4z = 0$

In Exercises 7–16, describe and sketch the surface.

7. $y = 5$

8. $z = 2$

9. $y^2 + z^2 = 9$

10. $x^2 + z^2 = 25$

11. $x^2 - y = 0$

12. $y^2 + z = 6$

13. $4x^2 + y^2 = 4$

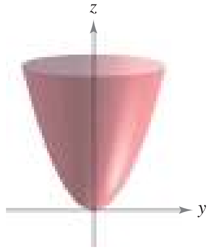
14. $y^2 - z^2 = 16$

15. $z - \sin y = 0$

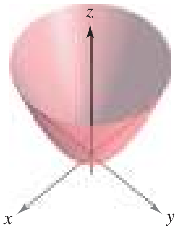
16. $z - e^y = 0$

17. ***Think About It*** The four figures are graphs of the quadric surface $z = x^2 + y^2$. Match each of the four graphs with the point in space from which the paraboloid is viewed. The four points are $(0, 0, 20)$, $(0, 20, 0)$, $(20, 0, 0)$, and $(10, 10, 20)$.

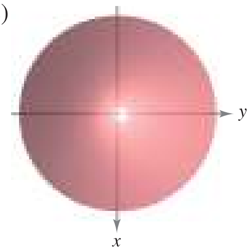
(a)



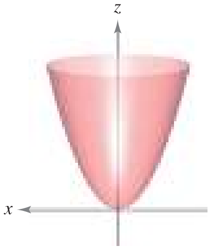
(b)



(c)



(d)



Figures for 17

CAS **18.** Use a computer algebra system to graph a view of the cylinder $y^2 + z^2 = 4$ from each point.

(a) $(10, 0, 0)$

(b) $(0, 10, 0)$

(c) $(10, 10, 10)$

In Exercises 19–32, identify and sketch the quadric surface. Use a computer algebra system to confirm your sketch.

19. $x^2 + \dfrac{y^2}{4} + z^2 = 1$

20. $\dfrac{x^2}{16} + \dfrac{y^2}{25} + \dfrac{z^2}{25} = 1$

21. $16x^2 - y^2 + 16z^2 = 4$

22. $-8x^2 + 18y^2 + 18z^2 = 2$

23. $4x^2 - y^2 - z^2 = 1$

24. $z^2 - x^2 - \dfrac{y^2}{4} = 1$

25. $x^2 - y + z^2 = 0$

26. $z = x^2 + 4y^2$

27. $x^2 - y^2 + z = 0$

28. $3z = -y^2 + x^2$

29. $z^2 = x^2 + \dfrac{y^2}{9}$

30. $x^2 = 2y^2 + 2z^2$

31. $16x^2 + 9y^2 + 16z^2 - 32x - 36y + 36 = 0$

32. $9x^2 + y^2 - 9z^2 - 54x - 4y - 54z + 4 = 0$

CAS **In Exercises 33–42, use a computer algebra system to graph the surface. (*Hint:* It may be necessary to solve for z and acquire two equations to graph the surface.)**

33. $z = 2\cos x$

34. $z = x^2 + 0.5y^2$

35. $z^2 = x^2 + 7.5y^2$

36. $3.25y = x^2 + z^2$

37. $x^2 + y^2 = \left(\dfrac{2}{z}\right)^2$

38. $x^2 + y^2 = e^{-z}$

39. $z = 10 - \sqrt{|xy|}$

40. $z = \dfrac{-x}{8 + x^2 + y^2}$

41. $6x^2 - 4y^2 + 6z^2 = -36$

42. $9x^2 + 4y^2 - 8z^2 = 72$

In Exercises 43–46, sketch the region bounded by the graphs of the equations.

43. $z = 2\sqrt{x^2 + y^2}, \; z = 2$

44. $z = \sqrt{4 - x^2}, \; y = \sqrt{4 - x^2}, \; x = 0, \; y = 0, \; z = 0$

45. $x^2 + y^2 = 1, \; x + z = 2, \; z = 0$

46. $z = \sqrt{4 - x^2 - y^2}, \; y = 2z, \; z = 0$

In Exercises 47–52, find an equation for the surface of revolution generated by revolving the curve in the indicated coordinate plane about the given axis.

	Equation of Curve	*Coordinate Plane*	*Axis of Revolution*
47.	$z^2 = 4y$	yz-plane	y-axis
48.	$z = 3y$	yz-plane	y-axis
49.	$z = 2y$	yz-plane	z-axis
50.	$2z = \sqrt{4 - x^2}$	xz-plane	x-axis
51.	$xy = 2$	xy-plane	x-axis
52.	$z = \ln y$	yz-plane	z-axis

In Exercises 53 and 54, find an equation of a generating curve given the equation of its surface of revolution.

53. $x^2 + y^2 - 2z = 0$

54. $x^2 + z^2 = \cos^2 y$

WRITING ABOUT CONCEPTS

55. State the definition of a cylinder.

56. What is meant by the trace of a surface? How do you find a trace?

57. Identify the six quadric surfaces and give the standard form of each.

CAPSTONE

58. What does the equation $z = x^2$ represent in the xz-plane? What does it represent in three-space?

In Exercises 59 and 60, use the shell method to find the volume of the solid below the surface of revolution and above the *xy*-plane.

59. The curve $z = 4x - x^2$ in the xz-plane is revolved about the z-axis.

60. The curve $z = \sin y\ (0 \le y \le \pi)$ in the yz-plane is revolved about the z-axis.

In Exercises 61 and 62, analyze the trace when the surface

$$z = \tfrac{1}{2}x^2 + \tfrac{1}{4}y^2$$

is intersected by the indicated planes.

61. Find the lengths of the major and minor axes and the coordinates of the foci of the ellipse generated when the surface is intersected by the planes given by

(a) $z = 2$ and (b) $z = 8$.

62. Find the coordinates of the focus of the parabola formed when the surface is intersected by the planes given by

(a) $y = 4$ and (b) $x = 2$.

In Exercises 63 and 64, find an equation of the surface satisfying the conditions, and identify the surface.

63. The set of all points equidistant from the point $(0, 2, 0)$ and the plane $y = -2$

64. The set of all points equidistant from the point $(0, 0, 4)$ and the xy-plane

65. ***Geography*** Because of the forces caused by its rotation, Earth is an oblate ellipsoid rather than a sphere. The equatorial radius is 3963 miles and the polar radius is 3950 miles. Find an equation of the ellipsoid. (Assume that the center of Earth is at the origin and that the trace formed by the plane $z = 0$ corresponds to the equator.)

66. ***Machine Design*** The top of a rubber bushing designed to absorb vibrations in an automobile is the surface of revolution generated by revolving the curve $z = \frac{1}{2}y^2 + 1\ (0 \le y \le 2)$ in the yz-plane about the z-axis.

(a) Find an equation for the surface of revolution.

(b) All measurements are in centimeters and the bushing is set on the xy-plane. Use the shell method to find its volume.

(c) The bushing has a hole of diameter 1 centimeter through its center and parallel to the axis of revolution. Find the volume of the rubber bushing.

67. Determine the intersection of the hyperbolic paraboloid $z = y^2/b^2 - x^2/a^2$ with the plane $bx + ay - z = 0$. (Assume $a, b > 0$.)

68. Explain why the curve of intersection of the surfaces $x^2 + 3y^2 - 2z^2 + 2y = 4$ and $2x^2 + 6y^2 - 4z^2 - 3x = 2$ lies in a plane.

True or False? **In Exercises 69–72, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

69. A sphere is an ellipsoid.

70. The generating curve for a surface of revolution is unique.

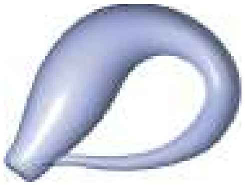
71. All traces of an ellipsoid are ellipses.

72. All traces of a hyperboloid of one sheet are hyperboloids.

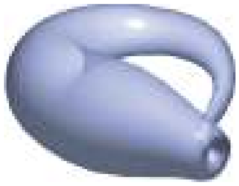
73. ***Think About It*** Three types of classic "topological" surfaces are shown below. The sphere and torus have both an "inside" and an "outside." Does the Klein bottle have both an inside and an outside? Explain.

Sphere

Torus



Klein bottle



Klein bottle

11.7 Cylindrical and Spherical Coordinates

- **Use cylindrical coordinates to represent surfaces in space.**
- **Use spherical coordinates to represent surfaces in space.**

Cylindrical Coordinates

You have already seen that some two-dimensional graphs are easier to represent in polar coordinates than in rectangular coordinates. A similar situation exists for surfaces in space. In this section, you will study two alternative space-coordinate systems. The first, the **cylindrical coordinate system,** is an extension of polar coordinates in the plane to three-dimensional space.

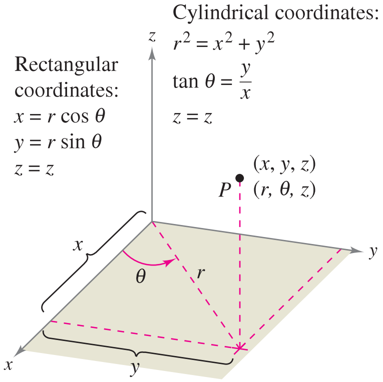


Figure 11.66

THE CYLINDRICAL COORDINATE SYSTEM

In a **cylindrical coordinate system,** a point P in space is represented by an ordered triple (r, θ, z).

1. (r, θ) is a polar representation of the projection of P in the xy-plane.
2. z is the directed distance from (r, θ) to P.

To convert from rectangular to cylindrical coordinates (or vice versa), use the following conversion guidelines for polar coordinates, as illustrated in Figure 11.66.

Cylindrical to rectangular:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

Rectangular to cylindrical:

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}, \qquad z = z$$

The point $(0, 0, 0)$ is called the **pole.** Moreover, because the representation of a point in the polar coordinate system is not unique, it follows that the representation in the cylindrical coordinate system is also not unique.

EXAMPLE 1 Converting from Cylindrical to Rectangular Coordinates

Convert the point $(r, \theta, z) = \left(4, \frac{5\pi}{6}, 3\right)$ to rectangular coordinates.

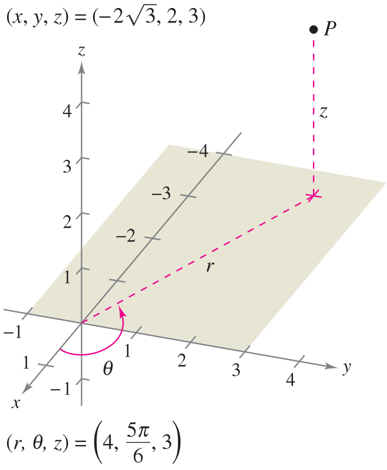
Solution Using the cylindrical-to-rectangular conversion equations produces

$$x = 4\cos\frac{5\pi}{6} = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

$$y = 4\sin\frac{5\pi}{6} = 4\left(\frac{1}{2}\right) = 2$$

$$z = 3.$$

So, in rectangular coordinates, the point is $(x, y, z) = \left(-2\sqrt{3}, 2, 3\right)$, as shown in Figure 11.67. ■



$(x, y, z) = (-2\sqrt{3}, 2, 3)$
P
z
4
3
2
1
−4
−3
−2
z
r
−1
1
−1
θ
1
2
3
4
y
x
$(r, \theta, z) = \left(4, \frac{5\pi}{6}, 3\right)$

Figure 11.67

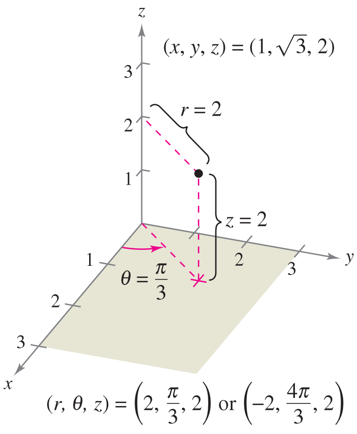

z
$(x, y, z) = (1, \sqrt{3}, 2)$
3
2
1
$r = 2$
$z = 2$
1
2
3
2
3
y
x
$\theta = \frac{\pi}{3}$
$(r, \theta, z) = \left(2, \frac{\pi}{3}, 2\right)$ or $\left(-2, \frac{4\pi}{3}, 2\right)$

Figure 11.68

EXAMPLE 2 Converting from Rectangular to Cylindrical Coordinates

Convert the point $(x, y, z) = (1, \sqrt{3}, 2)$ to cylindrical coordinates.

Solution Use the rectangular-to-cylindrical conversion equations.

$$r = \pm\sqrt{1 + 3} = \pm 2$$

$$\tan\theta = \sqrt{3} \Longrightarrow \theta = \arctan\left(\sqrt{3}\right) + n\pi = \frac{\pi}{3} + n\pi$$

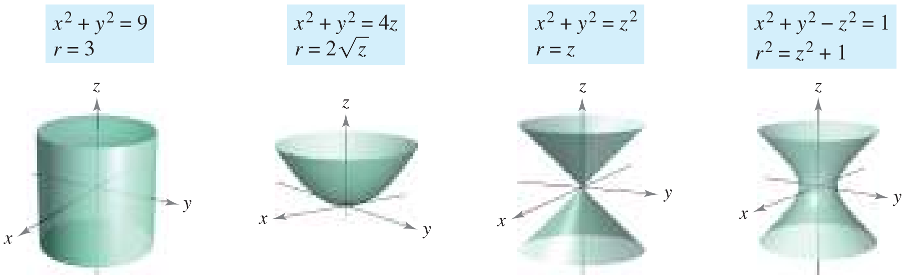
$$z = 2$$

You have two choices for r and infinitely many choices for θ. As shown in Figure 11.68, two convenient representations of the point are

$\left(2, \frac{\pi}{3}, 2\right)$ $r > 0$ and θ in Quadrant I

$\left(-2, \frac{4\pi}{3}, 2\right)$. $r < 0$ and θ in Quadrant III ■

Cylindrical coordinates are especially convenient for representing cylindrical surfaces and surfaces of revolution with the z-axis as the axis of symmetry, as shown in Figure 11.69.


$x^2 + y^2 = 9$
$r = 3$
$x^2 + y^2 = 4z$
$r = 2\sqrt{z}$
$x^2 + y^2 = z^2$
$r = z$
$x^2 + y^2 - z^2 = 1$
$r^2 = z^2 + 1$
z
x
y

Cylinder Paraboloid Cone Hyperboloid

Figure 11.69

Vertical planes containing the z-axis and horizontal planes also have simple cylindrical coordinate equations, as shown in Figure 11.70.

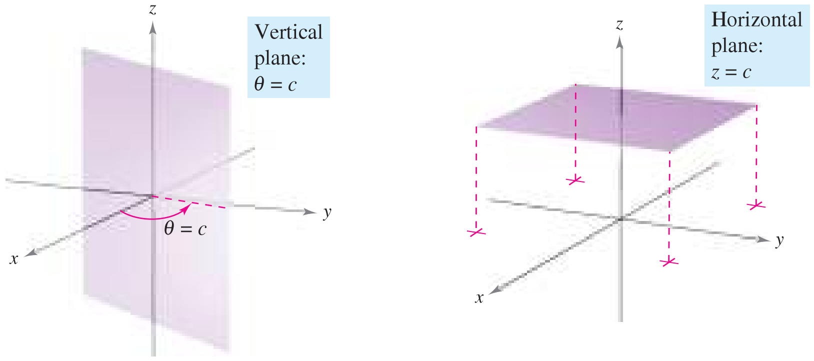

z
Vertical plane: $\theta = c$
y
x
$\theta = c$
z
Horizontal plane: $z = c$
y
x

Figure 11.70

EXAMPLE 3 Rectangular-to-Cylindrical Conversion

Find an equation in cylindrical coordinates for the surface represented by each rectangular equation.

a. $x^2 + y^2 = 4z^2$

b. $y^2 = x$

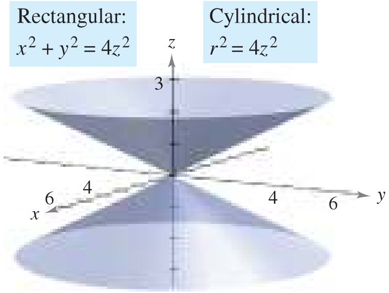


Figure 11.71

Solution

a. From the preceding section, you know that the graph $x^2 + y^2 = 4z^2$ is an elliptic cone with its axis along the z-axis, as shown in Figure 11.71. If you replace $x^2 + y^2$ with r^2, the equation in cylindrical coordinates is

$$x^2 + y^2 = 4z^2 \qquad \text{Rectangular equation}$$
$$r^2 = 4z^2. \qquad \text{Cylindrical equation}$$

b. The graph of the surface $y^2 = x$ is a parabolic cylinder with rulings parallel to the z-axis, as shown in Figure 11.72. By replacing y^2 with $r^2 \sin^2 \theta$ and x with $r \cos \theta$, you obtain the following equation in cylindrical coordinates.

$$y^2 = x \qquad \text{Rectangular equation}$$
$$r^2 \sin^2 \theta = r \cos \theta \qquad \text{Substitute } r \sin \theta \text{ for } y \text{ and } r \cos \theta \text{ for } x.$$
$$r(r \sin^2 \theta - \cos \theta) = 0 \qquad \text{Collect terms and factor.}$$
$$r \sin^2 \theta - \cos \theta = 0 \qquad \text{Divide each side by } r.$$
$$r = \frac{\cos \theta}{\sin^2 \theta} \qquad \text{Solve for } r.$$
$$r = \csc \theta \cot \theta \qquad \text{Cylindrical equation}$$

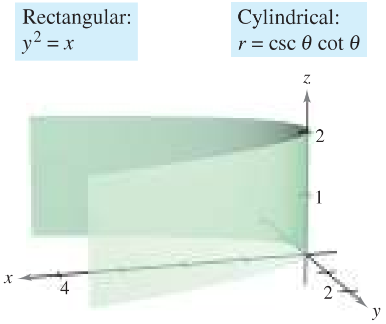


Figure 11.72

Note that this equation includes a point for which $r = 0$, so nothing was lost by dividing each side by the factor r. ■

Converting from cylindrical coordinates to rectangular coordinates is less straightforward than converting from rectangular coordinates to cylindrical coordinates, as demonstrated in Example 4.

EXAMPLE 4 Cylindrical-to-Rectangular Conversion

Find an equation in rectangular coordinates for the surface represented by the cylindrical equation

$$r^2 \cos 2\theta + z^2 + 1 = 0.$$

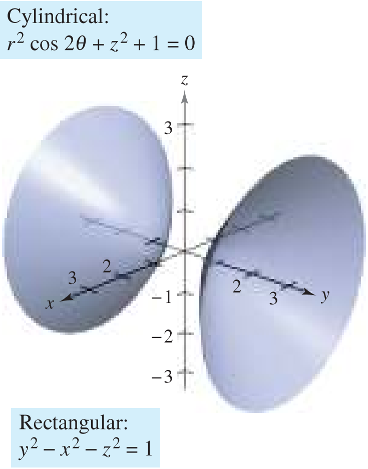


Figure 11.73

Solution

$$r^2 \cos 2\theta + z^2 + 1 = 0 \qquad \text{Cylindrical equation}$$
$$r^2(\cos^2 \theta - \sin^2 \theta) + z^2 + 1 = 0 \qquad \text{Trigonometric identity}$$
$$r^2 \cos^2 \theta - r^2 \sin^2 \theta + z^2 = -1$$
$$x^2 - y^2 + z^2 = -1 \qquad \text{Replace } r \cos \theta \text{ with } x \text{ and } r \sin \theta \text{ with } y.$$
$$y^2 - x^2 - z^2 = 1 \qquad \text{Rectangular equation}$$

This is a hyperboloid of two sheets whose axis lies along the y-axis, as shown in Figure 11.73. ■

Spherical Coordinates

In the **spherical coordinate system,** each point is represented by an ordered triple: the first coordinate is a distance, and the second and third coordinates are angles. This system is similar to the latitude-longitude system used to identify points on the surface of Earth. For example, the point on the surface of Earth whose latitude is 40° North (of the equator) and whose longitude is 80° West (of the prime meridian) is shown in Figure 11.74. Assuming that the Earth is spherical and has a radius of 4000 miles, you would label this point as

$$(4000, -80°, 50°).$$

Radius

80° clockwise from prime meridian

50° down from North Pole

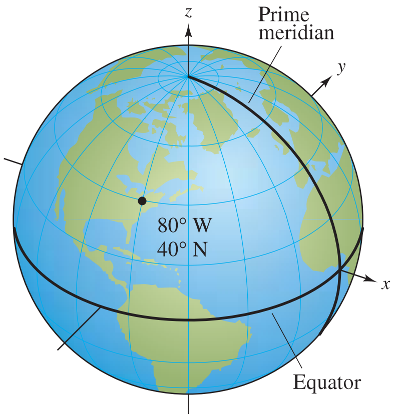


Figure 11.74

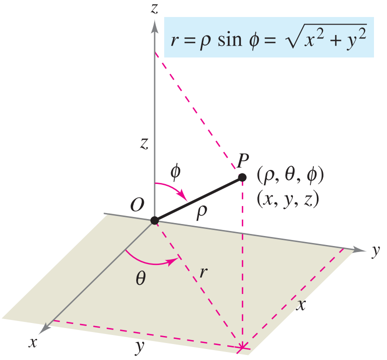
THE SPHERICAL COORDINATE SYSTEM

In a **spherical coordinate system,** a point P in space is represented by an ordered triple (ρ, θ, ϕ).

1. ρ is the distance between P and the origin, $\rho \geq 0$.
2. θ is the same angle used in cylindrical coordinates for $r \geq 0$.
3. ϕ is the angle *between* the positive z-axis and the line segment $\overrightarrow{OP}$, $0 \leq \phi \leq \pi$.

Note that the first and third coordinates, ρ and ϕ, are nonnegative. ρ is the lowercase Greek letter *rho*, and ϕ is the lowercase Greek letter *phi*.

The relationship between rectangular and spherical coordinates is illustrated in Figure 11.75. To convert from one system to the other, use the following.



Spherical coordinates
Figure 11.75

Spherical to rectangular:

$$x = \rho \sin \phi \cos \theta, \qquad y = \rho \sin \phi \sin \theta, \qquad z = \rho \cos \phi$$

Rectangular to spherical:

$$\rho^2 = x^2 + y^2 + z^2, \qquad \tan \theta = \frac{y}{x}, \qquad \phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

To change coordinates between the cylindrical and spherical systems, use the following.

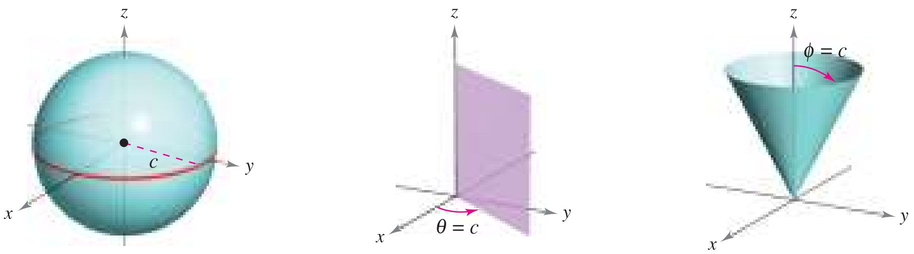
Spherical to cylindrical $(r \geq 0)$***:***

$$r^2 = \rho^2 \sin^2 \phi, \qquad \theta = \theta, \qquad z = \rho \cos \phi$$

Cylindrical to spherical $(r \geq 0)$***:***

$$\rho = \sqrt{r^2 + z^2}, \qquad \theta = \theta, \qquad \phi = \arccos\left(\frac{z}{\sqrt{r^2 + z^2}}\right)$$

The spherical coordinate system is useful primarily for surfaces in space that have a *point* or *center* of symmetry. For example, Figure 11.76 shows three surfaces with simple spherical equations.



Sphere: $\rho = c$

Vertical half-plane: $\theta = c$

Half-cone: $\phi = c$ $\left(0 < c < \frac{\pi}{2}\right)$

Figure 11.76

EXAMPLE 5 Rectangular-to-Spherical Conversion

Find an equation in spherical coordinates for the surface represented by each rectangular equation.

a. Cone: $x^2 + y^2 = z^2$

b. Sphere: $x^2 + y^2 + z^2 - 4z = 0$

Solution

a. Making the appropriate replacements for x, y, and z in the given equation yields the following.

$$\begin{aligned} x^2 + y^2 &= z^2 \\ \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta &= \rho^2 \cos^2 \phi \\ \rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) &= \rho^2 \cos^2 \phi \\ \rho^2 \sin^2 \phi &= \rho^2 \cos^2 \phi \\ \frac{\sin^2 \phi}{\cos^2 \phi} &= 1 && \rho \ge 0 \\ \tan^2 \phi &= 1 && \phi = \pi/4 \text{ or } \phi = 3\pi/4 \end{aligned}$$

The equation $\phi = \pi/4$ represents the *upper* half-cone, and the equation $\phi = 3\pi/4$ represents the *lower* half-cone.

b. Because $\rho^2 = x^2 + y^2 + z^2$ and $z = \rho \cos \phi$, the given equation has the following spherical form.

$$\rho^2 - 4\rho \cos \phi = 0 \quad \Rightarrow \quad \rho(\rho - 4 \cos \phi) = 0$$

Temporarily discarding the possibility that $\rho = 0$, you have the spherical equation

$$\rho - 4 \cos \phi = 0 \qquad \text{or} \qquad \rho = 4 \cos \phi.$$

Note that the solution set for this equation includes a point for which $\rho = 0$, so nothing is lost by discarding the factor ρ. The sphere represented by the equation $\rho = 4 \cos \phi$ is shown in Figure 11.77. ■

Rectangular: $x^2 + y^2 + z^2 - 4z = 0$

Spherical: $\rho = 4 \cos \phi$

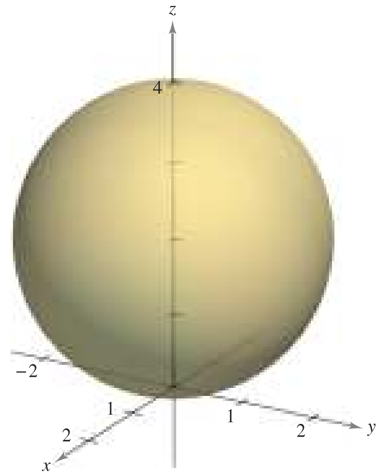


Figure 11.77

11.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, convert the point from cylindrical coordinates to rectangular coordinates.

1. $(-7, 0, 5)$

2. $(2, -\pi, -4)$

3. $(3, \pi/4, 1)$

4. $(6, -\pi/4, 2)$

5. $(4, 7\pi/6, 3)$

6. $(-0.5, 4\pi/3, 8)$

In Exercises 7–12, convert the point from rectangular coordinates to cylindrical coordinates.

7. $(0, 5, 1)$

8. $\left(2\sqrt{2}, -2\sqrt{2}, 4\right)$

9. $(2, -2, -4)$

10. $(3, -3, 7)$

11. $\left(1, \sqrt{3}, 4\right)$

12. $\left(2\sqrt{3}, -2, 6\right)$

In Exercises 13–20, find an equation in cylindrical coordinates for the equation given in rectangular coordinates.

13. $z = 4$

14. $x = 9$

15. $x^2 + y^2 + z^2 = 17$

16. $z = x^2 + y^2 - 11$

17. $y = x^2$

18. $x^2 + y^2 = 8x$

19. $y^2 = 10 - z^2$

20. $x^2 + y^2 + z^2 - 3z = 0$

In Exercises 21–28, find an equation in rectangular coordinates for the equation given in cylindrical coordinates, and sketch its graph.

21. $r = 3$

22. $z = 2$

23. $\theta = \pi/6$

24. $r = \frac{1}{2}z$

25. $r^2 + z^2 = 5$

26. $z = r^2 \cos^2 \theta$

27. $r = 2 \sin \theta$

28. $r = 2 \cos \theta$

In Exercises 29–34, convert the point from rectangular coordinates to spherical coordinates.

29. $(4, 0, 0)$

30. $(-4, 0, 0)$

31. $\left(-2, 2\sqrt{3}, 4\right)$

32. $\left(2, 2, 4\sqrt{2}\right)$

33. $\left(\sqrt{3}, 1, 2\sqrt{3}\right)$

34. $(-1, 2, 1)$

In Exercises 35–40, convert the point from spherical coordinates to rectangular coordinates.

35. $(4, \pi/6, \pi/4)$

36. $(12, 3\pi/4, \pi/9)$

37. $(12, -\pi/4, 0)$

38. $(9, \pi/4, \pi)$

39. $(5, \pi/4, 3\pi/4)$

40. $(6, \pi, \pi/2)$

In Exercises 41–48, find an equation in spherical coordinates for the equation given in rectangular coordinates.

41. $y = 2$

42. $z = 6$

43. $x^2 + y^2 + z^2 = 49$

44. $x^2 + y^2 - 3z^2 = 0$

45. $x^2 + y^2 = 16$

46. $x = 13$

47. $x^2 + y^2 = 2z^2$

48. $x^2 + y^2 + z^2 - 9z = 0$

In Exercises 49–56, find an equation in rectangular coordinates for the equation given in spherical coordinates, and sketch its graph.

49. $\rho = 5$

50. $\theta = \dfrac{3\pi}{4}$

51. $\phi = \dfrac{\pi}{6}$

52. $\phi = \dfrac{\pi}{2}$

53. $\rho = 4 \cos \phi$

54. $\rho = 2 \sec \phi$

55. $\rho = \csc \phi$

56. $\rho = 4 \csc \phi \sec \theta$

In Exercises 57–64, convert the point from cylindrical coordinates to spherical coordinates.

57. $(4, \pi/4, 0)$

58. $(3, -\pi/4, 0)$

59. $(4, \pi/2, 4)$

60. $(2, 2\pi/3, -2)$

61. $(4, -\pi/6, 6)$

62. $(-4, \pi/3, 4)$

63. $(12, \pi, 5)$

64. $(4, \pi/2, 3)$

In Exercises 65–72, convert the point from spherical coordinates to cylindrical coordinates.

65. $(10, \pi/6, \pi/2)$

66. $(4, \pi/18, \pi/2)$

67. $(36, \pi, \pi/2)$

68. $(18, \pi/3, \pi/3)$

69. $(6, -\pi/6, \pi/3)$

70. $(5, -5\pi/6, \pi)$

71. $(8, 7\pi/6, \pi/6)$

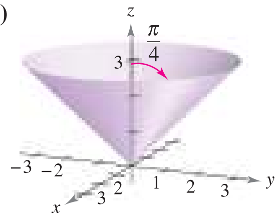
72. $(7, \pi/4, 3\pi/4)$

In Exercises 73–88, use a computer algebra system or graphing utility to convert the point from one system to another among the rectangular, cylindrical, and spherical coordinate systems.

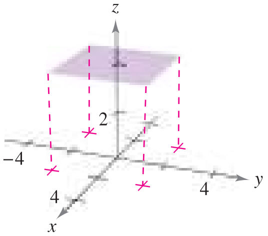
	Rectangular	*Cylindrical*	*Spherical*
73.	$(4, 6, 3)$		
74.	$(6, -2, -3)$		
75.		$(5, \pi/9, 8)$	
76.		$(10, -0.75, 6)$	
77.			$(20, 2\pi/3, \pi/4)$
78.			$(7.5, 0.25, 1)$
79.	$(3, -2, 2)$		
80.	$\left(3\sqrt{2}, 3\sqrt{2}, -3\right)$		
81.	$(5/2, 4/3, -3/2)$		
82.	$(0, -5, 4)$		
83.		$(5, 3\pi/4, -5)$	
84.		$(-2, 11\pi/6, 3)$	
85.		$(-3.5, 2.5, 6)$	
86.		$(8.25, 1.3, -4)$	
87.			$(3, 3\pi/4, \pi/3)$
88.			$(8, -\pi/6, \pi)$

In Exercises 89–94, match the equation (written in terms of cylindrical or spherical coordinates) with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

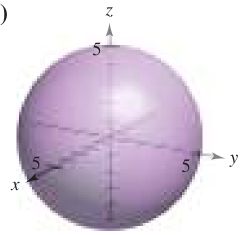
(a)



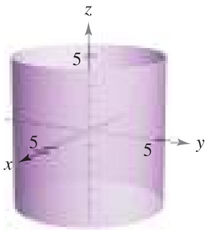
(b)



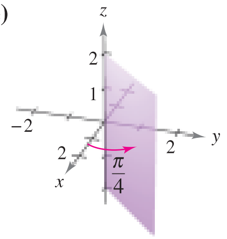
(c)



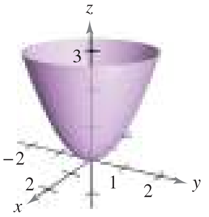
(d)



(e)



(f)



89. $r = 5$

90. $\theta = \frac{\pi}{4}$

91. $\rho = 5$

92. $\phi = \frac{\pi}{4}$

93. $r^2 = z$

94. $\rho = 4 \sec \phi$

WRITING ABOUT CONCEPTS

95. Give the equations for the coordinate conversion from rectangular to cylindrical coordinates and vice versa.

96. Explain why in spherical coordinates the graph of $\theta = c$ is a half-plane and not an entire plane.

97. Give the equations for the coordinate conversion from rectangular to spherical coordinates and vice versa.

CAPSTONE

98. (a) For constants a, b, and c, describe the graphs of the equations $r = a$, $\theta = b$, and $z = c$ in cylindrical coordinates.

(b) For constants a, b, and c, describe the graphs of the equations $\rho = a$, $\theta = b$, and $\phi = c$ in spherical coordinates.

In Exercises 99–106, convert the rectangular equation to an equation in (a) cylindrical coordinates and (b) spherical coordinates.

99. $x^2 + y^2 + z^2 = 25$

100. $4(x^2 + y^2) = z^2$

101. $x^2 + y^2 + z^2 - 2z = 0$

102. $x^2 + y^2 = z$

103. $x^2 + y^2 = 4y$

104. $x^2 + y^2 = 36$

105. $x^2 - y^2 = 9$

106. $y = 4$

In Exercises 107–110, sketch the solid that has the given description in cylindrical coordinates.

107. $0 \le \theta \le \pi/2,\ 0 \le r \le 2,\ 0 \le z \le 4$

108. $-\pi/2 \le \theta \le \pi/2,\ 0 \le r \le 3,\ 0 \le z \le r\cos\theta$

109. $0 \le \theta \le 2\pi,\ 0 \le r \le a,\ r \le z \le a$

110. $0 \le \theta \le 2\pi,\ 2 \le r \le 4,\ z^2 \le -r^2 + 6r - 8$

In Exercises 111–114, sketch the solid that has the given description in spherical coordinates.

111. $0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/6,\ 0 \le \rho \le a \sec \phi$

112. $0 \le \theta \le 2\pi,\ \pi/4 \le \phi \le \pi/2,\ 0 \le \rho \le 1$

113. $0 \le \theta \le \pi/2,\ 0 \le \phi \le \pi/2,\ 0 \le \rho \le 2$

114. $0 \le \theta \le \pi,\ 0 \le \phi \le \pi/2,\ 1 \le \rho \le 3$

Think About It **In Exercises 115–120, find inequalities that describe the solid, and state the coordinate system used. Position the solid on the coordinate system such that the inequalities are as simple as possible.**

115. A cube with each edge 10 centimeters long

116. A cylindrical shell 8 meters long with an inside diameter of 0.75 meter and an outside diameter of 1.25 meters

117. A spherical shell with inside and outside radii of 4 inches and 6 inches, respectively

118. The solid that remains after a hole 1 inch in diameter is drilled through the center of a sphere 6 inches in diameter

119. The solid inside both $x^2 + y^2 + z^2 = 9$ and $\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

120. The solid between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$, and inside the cone $z^2 = x^2 + y^2$

True or False? **In Exercises 121–124, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

121. In cylindrical coordinates, the equation $r = z$ is a cylinder.

122. The equations $\rho = 2$ and $x^2 + y^2 + z^2 = 4$ represent the same surface.

123. The cylindrical coordinates of a point (x, y, z) are unique.

124. The spherical coordinates of a point (x, y, z) are unique.

125. Identify the curve of intersection of the surfaces (in cylindrical coordinates) $z = \sin\theta$ and $r = 1$.

126. Identify the curve of intersection of the surfaces (in spherical coordinates) $\rho = 2 \sec \phi$ and $\rho = 4$.

11 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, let $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PR}$, and (a) write u and v in component form, (b) write u as the linear combination of the standard unit vectors i and j, (c) find the magnitude of v, and (d) find 2u + v.

1. $P = (1, 2)$, $Q = (4, 1)$, $R = (5, 4)$

2. $P = (-2, -1)$, $Q = (5, -1)$, $R = (2, 4)$

In Exercises 3 and 4, find the component form of v given its magnitude and the angle it makes with the positive *x*-axis.

3. $\|\mathbf{v}\| = 8$, $\theta = 60°$

4. $\|\mathbf{v}\| = \frac{1}{2}$, $\theta = 225°$

5. Find the coordinates of the point in the xy-plane four units to the right of the xz-plane and five units behind the yz-plane.

6. Find the coordinates of the point located on the y-axis and seven units to the left of the xz-plane.

In Exercises 7 and 8, determine the location of a point (x, y, z) that satisfies the condition.

7. $yz > 0$

8. $xy < 0$

In Exercises 9 and 10, find the standard equation of the sphere.

9. Center: $(3, -2, 6)$; Diameter: 15

10. Endpoints of a diameter: $(0, 0, 4)$, $(4, 6, 0)$

In Exercises 11 and 12, complete the square to write the equation of the sphere in standard form. Find the center and radius.

11. $x^2 + y^2 + z^2 - 4x - 6y + 4 = 0$

12. $x^2 + y^2 + z^2 - 10x + 6y - 4z + 34 = 0$

In Exercises 13 and 14, the initial and terminal points of a vector are given. (a) Sketch the directed line segment, (b) find the component form of the vector, (c) write the vector using standard unit vector notation, and (d) sketch the vector with its initial point at the origin.

13. Initial point: $(2, -1, 3)$
Terminal point: $(4, 4, -7)$

14. Initial point: $(6, 2, 0)$
Terminal point: $(3, -3, 8)$

In Exercises 15 and 16, use vectors to determine whether the points are collinear.

15. $(3, 4, -1)$, $(-1, 6, 9)$, $(5, 3, -6)$

16. $(5, -4, 7)$, $(8, -5, 5)$, $(11, 6, 3)$

17. Find a unit vector in the direction of $\mathbf{u} = \langle 2, 3, 5 \rangle$.

18. Find the vector $\mathbf{v}$ of magnitude 8 in the direction $\langle 6, -3, 2 \rangle$.

In Exercises 19 and 20, let $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PR}$, and find (a) the component forms of u and v, (b) u · v, and (c) v · v.

19. $P = (5, 0, 0)$, $Q = (4, 4, 0)$, $R = (2, 0, 6)$

20. $P = (2, -1, 3)$, $Q = (0, 5, 1)$, $R = (5, 5, 0)$

In Exercises 21 and 22, determine whether u and v are orthogonal, parallel, or neither.

21. $\mathbf{u} = \langle 7, -2, 3 \rangle$
$\mathbf{v} = \langle -1, 4, 5 \rangle$

22. $\mathbf{u} = \langle -4, 3, -6 \rangle$
$\mathbf{v} = \langle 16, -12, 24 \rangle$

In Exercises 23–26, find the angle θ between the vectors.

23. $\mathbf{u} = 5[\cos(3\pi/4)\mathbf{i} + \sin(3\pi/4)\mathbf{j}]$
$\mathbf{v} = 2[\cos(2\pi/3)\mathbf{i} + \sin(2\pi/3)\mathbf{j}]$

24. $\mathbf{u} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + 5\mathbf{j}$

25. $\mathbf{u} = \langle 10, -5, 15 \rangle$, $\mathbf{v} = \langle -2, 1, -3 \rangle$

26. $\mathbf{u} = \langle 1, 0, -3 \rangle$, $\mathbf{v} = \langle 2, -2, 1 \rangle$

27. Find two vectors in opposite directions that are orthogonal to the vector $\mathbf{u} = \langle 5, 6, -3 \rangle$.

28. ***Work*** An object is pulled 8 feet across a floor using a force of 75 pounds. The direction of the force is 30° above the horizontal. Find the work done.

In Exercises 29–38, let $\mathbf{u} = \langle 3, -2, 1 \rangle$, $\mathbf{v} = \langle 2, -4, -3 \rangle$, and $\mathbf{w} = \langle -1, 2, 2 \rangle$.

29. Show that $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.

30. Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

31. Determine the projection of $\mathbf{w}$ onto $\mathbf{u}$.

32. Find the work done in moving an object along the vector $\mathbf{u}$ if the applied force is $\mathbf{w}$.

33. Determine a unit vector perpendicular to the plane containing $\mathbf{v}$ and $\mathbf{w}$.

34. Show that $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

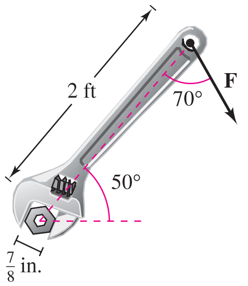
35. Find the volume of the solid whose edges are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

36. Show that $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$.

37. Find the area of the parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$.

38. Find the area of the triangle with adjacent sides $\mathbf{v}$ and $\mathbf{w}$.

39. ***Torque*** The specifications for a tractor state that the torque on a bolt with head size $\frac{7}{8}$ inch cannot exceed 200 foot-pounds. Determine the maximum force $\|\mathbf{F}\|$ that can be applied to the wrench in the figure.



40. ***Volume*** Use the triple scalar product to find the volume of the parallelepiped having adjacent edges $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 2\mathbf{j} + \mathbf{k}$, and $\mathbf{w} = -\mathbf{j} + 2\mathbf{k}$.

In Exercises 41 and 42, find sets of (a) parametric equations and (b) symmetric equations of the line through the two points. (For each line, write the direction numbers as integers.)

41. $(3, 0, 2)$, $(9, 11, 6)$

42. $(-1, 4, 3)$, $(8, 10, 5)$

In Exercises 43–46, (a) find a set of parametric equations for the line, (b) find a set of symmetric equations for the line, and (c) sketch a graph of the line.

43. The line passes through the point $(1, 2, 3)$ and is perpendicular to the xz-plane.

44. The line passes through the point $(1, 2, 3)$ and is parallel to the line given by $x = y = z$.

45. The intersection of the planes $3x - 3y - 7z = -4$ and $x - y + 2z = 3$.

46. The line passes through the point $(0, 1, 4)$ and is perpendicular to $\mathbf{u} = \langle 2, -5, 1\rangle$ and $\mathbf{v} = \langle -3, 1, 4\rangle$.

In Exercises 47–50, find an equation of the plane and sketch its graph.

47. The plane passes through

$(-3, -4, 2)$, $(-3, 4, 1)$, and $(1, 1, -2)$.

48. The plane passes through the point $(-2, 3, 1)$ and is perpendicular to $\mathbf{n} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$.

49. The plane contains the lines given by

$$\frac{x-1}{-2} = y = z + 1$$

and

$$\frac{x+1}{-2} = y - 1 = z - 2.$$

50. The plane passes through the points $(5, 1, 3)$ and $(2, -2, 1)$ and is perpendicular to the plane $2x + y - z = 4$.

51. Find the distance between the point $(1, 0, 2)$ and the plane $2x - 3y + 6z = 6$.

52. Find the distance between the point $(3, -2, 4)$ and the plane $2x - 5y + z = 10$.

53. Find the distance between the planes $5x - 3y + z = 2$ and $5x - 3y + z = -3$.

54. Find the distance between the point $(-5, 1, 3)$ and the line given by $x = 1 + t$, $y = 3 - 2t$, and $z = 5 - t$.

In Exercises 55–64, describe and sketch the surface.

55. $x + 2y + 3z = 6$

56. $y = z^2$

57. $y = \frac{1}{2}z$

58. $y = \cos z$

59. $\dfrac{x^2}{16} + \dfrac{y^2}{9} + z^2 = 1$

60. $16x^2 + 16y^2 - 9z^2 = 0$

61. $\dfrac{x^2}{16} - \dfrac{y^2}{9} + z^2 = -1$

62. $\dfrac{x^2}{25} + \dfrac{y^2}{4} - \dfrac{z^2}{100} = 1$

63. $x^2 + z^2 = 4$

64. $y^2 + z^2 = 16$

65. Find an equation of a generating curve of the surface of revolution $y^2 + z^2 - 4x = 0$.

66. Find an equation of a generating curve of the surface of revolution $x^2 + 2y^2 + z^2 = 3y$.

67. Find an equation for the surface of revolution generated by revolving the curve $z^2 = 2y$ in the yz-plane about the y-axis.

68. Find an equation for the surface of revolution generated by revolving the curve $2x + 3z = 1$ in the xz-plane about the x-axis.

In Exercises 69 and 70, convert the point from rectangular coordinates to (a) cylindrical coordinates and (b) spherical coordinates.

69. $\left(-2\sqrt{2}, 2\sqrt{2}, 2\right)$

70. $\left(\dfrac{\sqrt{3}}{4}, \dfrac{3}{4}, \dfrac{3\sqrt{3}}{2}\right)$

In Exercises 71 and 72, convert the point from cylindrical coordinates to spherical coordinates.

71. $\left(100, -\dfrac{\pi}{6}, 50\right)$

72. $\left(81, -\dfrac{5\pi}{6}, 27\sqrt{3}\right)$

In Exercises 73 and 74, convert the point from spherical coordinates to cylindrical coordinates.

73. $\left(25, -\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right)$

74. $\left(12, -\dfrac{\pi}{2}, \dfrac{2\pi}{3}\right)$

In Exercises 75 and 76, convert the rectangular equation to an equation in (a) cylindrical coordinates and (b) spherical coordinates.

75. $x^2 - y^2 = 2z$

76. $x^2 + y^2 + z^2 = 16$

In Exercises 77 and 78, find an equation in rectangular coordinates for the equation given in cylindrical coordinates, and sketch its graph.

77. $r = 5\cos\theta$

78. $z = 4$

In Exercises 79 and 80, find an equation in rectangular coordinates for the equation given in spherical coordinates, and sketch its graph.

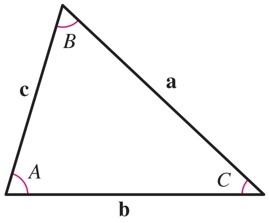
79. $\theta = \dfrac{\pi}{4}$

80. $\rho = 3\cos\phi$

P.S. PROBLEM SOLVING

1. Using vectors, prove the Law of Sines: If **a**, **b**, and **c** are the three sides of the triangle shown in the figure, then

$$\frac{\sin A}{\|\mathbf{a}\|} = \frac{\sin B}{\|\mathbf{b}\|} = \frac{\sin C}{\|\mathbf{c}\|}.$$



2. Consider the function $f(x) = \int_0^x \sqrt{t^4 + 1}\, dt$.

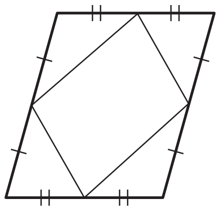
(a) Use a graphing utility to graph the function on the interval $-2 \le x \le 2$.

(b) Find a unit vector parallel to the graph of f at the point $(0, 0)$.

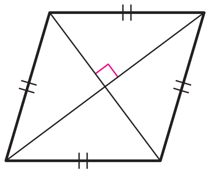
(c) Find a unit vector perpendicular to the graph of f at the point $(0, 0)$.

(d) Find the parametric equations of the tangent line to the graph of f at the point $(0, 0)$.

3. Using vectors, prove that the line segments joining the midpoints of the sides of a parallelogram form a parallelogram (see figure).



4. Using vectors, prove that the diagonals of a rhombus are perpendicular (see figure).



5. (a) Find the shortest distance between the point $Q(2, 0, 0)$ and the line determined by the points $P_1(0, 0, 1)$ and $P_2(0, 1, 2)$.

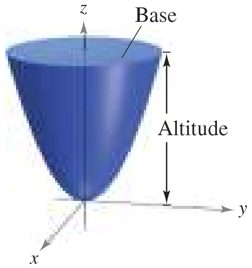
(b) Find the shortest distance between the point $Q(2, 0, 0)$ and the line segment joining the points $P_1(0, 0, 1)$ and $P_2(0, 1, 2)$.

6. Let P_0 be a point in the plane with normal vector **n**. Describe the set of points P in the plane for which $(\mathbf{n} + \overrightarrow{PP_0})$ is orthogonal to $(\mathbf{n} - \overrightarrow{PP_0})$.

7. (a) Find the volume of the solid bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 1$.

(b) Find the volume of the solid bounded below by the elliptic paraboloid $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ and above by the plane $z = k$, where $k > 0$.

(c) Show that the volume of the solid in part (b) is equal to one-half the product of the area of the base times the altitude, as shown in the figure.



8. (a) Use the disk method to find the volume of the sphere $x^2 + y^2 + z^2 = r^2$.

(b) Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

9. Sketch the graph of each equation given in spherical coordinates.

(a) $\rho = 2 \sin \phi$

(b) $\rho = 2 \cos \phi$

10. Sketch the graph of each equation given in cylindrical coordinates.

(a) $r = 2 \cos \theta$

(b) $z = r^2 \cos 2\theta$

11. Prove the following property of the cross product.

$$(\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{z}) = (\mathbf{u} \times \mathbf{v} \cdot \mathbf{z})\mathbf{w} - (\mathbf{u} \times \mathbf{v} \cdot \mathbf{w})\mathbf{z}$$

12. Consider the line given by the parametric equations

$$x = -t + 3, \quad y = \tfrac{1}{2}t + 1, \quad z = 2t - 1$$

and the point $(4, 3, s)$ for any real number s.

(a) Write the distance between the point and the line as a function of s.

(b) Use a graphing utility to graph the function in part (a). Use the graph to find the value of s such that the distance between the point and the line is minimum.

(c) Use the *zoom* feature of a graphing utility to zoom out several times on the graph in part (b). Does it appear that the graph has slant asymptotes? Explain. If it appears to have slant asymptotes, find them.

13. A tetherball weighing 1 pound is pulled outward from the pole by a horizontal force **u** until the rope makes an angle of θ degrees with the pole (see figure).

(a) Determine the resulting tension in the rope and the magnitude of **u** when $\theta = 30°$.

(b) Write the tension T in the rope and the magnitude of **u** as functions of θ. Determine the domains of the functions.

(c) Use a graphing utility to complete the table.

θ	0°	10°	20°	30°	40°	50°	60°
T							
$\lVert \mathbf{u} \rVert$							

(d) Use a graphing utility to graph the two functions for $0° \le \theta \le 60°$.

(e) Compare T and $\lVert \mathbf{u} \rVert$ as θ increases.

(f) Find (if possible) $\displaystyle\lim_{\theta \to \pi/2^-} T$ and $\displaystyle\lim_{\theta \to \pi/2^-} \lVert \mathbf{u} \rVert$. Are the results what you expected? Explain.

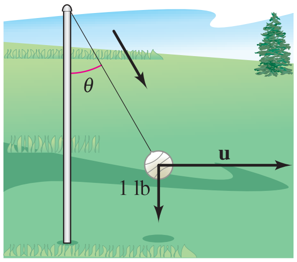


Figure for 13

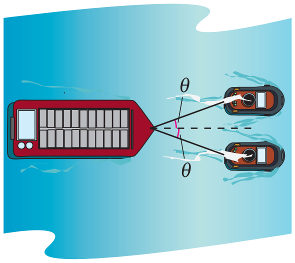


Figure for 14

14. A loaded barge is being towed by two tugboats, and the magnitude of the resultant is 6000 pounds directed along the axis of the barge (see figure). Each towline makes an angle of θ degrees with the axis of the barge.

(a) Find the tension in the towlines if $\theta = 20°$.

(b) Write the tension T of each line as a function of θ. Determine the domain of the function.

(c) Use a graphing utility to complete the table.

θ	10°	20°	30°	40°	50°	60°
T						

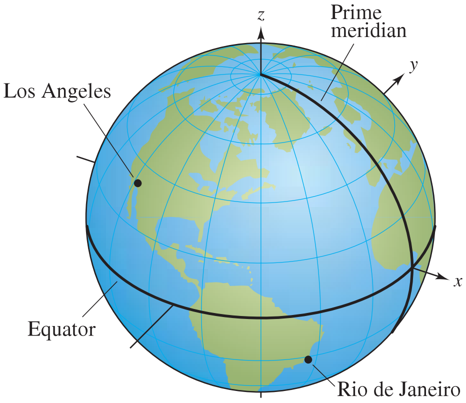
(d) Use a graphing utility to graph the tension function.

(e) Explain why the tension increases as θ increases.

15. Consider the vectors $\mathbf{u} = \langle \cos \alpha, \sin \alpha, 0 \rangle$ and $\mathbf{v} = \langle \cos \beta, \sin \beta, 0 \rangle$, where $\alpha > \beta$. Find the cross product of the vectors and use the result to prove the identity

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta.$$

16. Los Angeles is located at 34.05° North latitude and 118.24° West longitude, and Rio de Janeiro, Brazil is located at 22.90° South latitude and 43.23° West longitude (see figure). Assume that Earth is spherical and has a radius of 4000 miles.



(a) Find the spherical coordinates for the location of each city.

(b) Find the rectangular coordinates for the location of each city.

(c) Find the angle (in radians) between the vectors from the center of Earth to the two cities.

(d) Find the great-circle distance s between the cities. (*Hint:* $s = r\theta$)

(e) Repeat parts (a)–(d) for the cities of Boston, located at 42.36° North latitude and 71.06° West longitude, and Honolulu, located at 21.31° North latitude and 157.86° West longitude.

17. Consider the plane that passes through the points P, R, and S. Show that the distance from a point Q to this plane is

$$\text{Distance} = \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\lVert \mathbf{u} \times \mathbf{v} \rVert}$$

where $\mathbf{u} = \overrightarrow{PR}$, $\mathbf{v} = \overrightarrow{PS}$, and $\mathbf{w} = \overrightarrow{PQ}$.

18. Show that the distance between the parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$\text{Distance} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

19. Show that the curve of intersection of the plane $z = 2y$ and the cylinder $x^2 + y^2 = 1$ is an ellipse.

20. Read the article "Tooth Tables: Solution of a Dental Problem by Vector Algebra" by Gary Hosler Meisters in *Mathematics Magazine*. (To view this article, go to the website *www.matharticles.com*.) Then write a paragraph explaining how vectors and vector algebra can be used in the construction of dental inlays.

12

Vector-Valued Functions

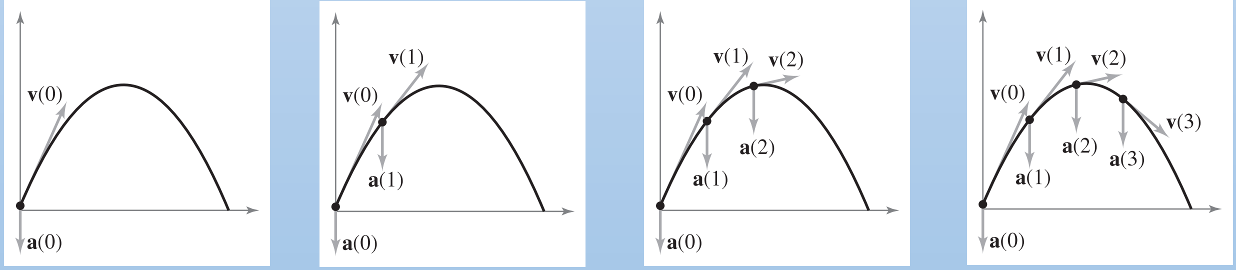
This chapter introduces the concept of vector-valued functions. Vector-valued functions can be used to study curves in the plane and in space. These functions can also be used to study the motion of an object along a curve.

In this chapter, you should learn the following.

- How to analyze and sketch a space curve represented by a vector-valued function. How to apply the concepts of limits and continuity to vector-valued functions. (**12.1**)
- How to differentiate and integrate vector-valued functions. (**12.2**)
- How to describe the velocity and acceleration associated with a vector-valued function and how to use a vector-valued function to analyze projectile motion. (**12.3**)
- How to find tangent vectors and normal vectors. (**12.4**)
- How to find the arc length and curvature of a curve. (**12.5**)

Jerry Driendl/Getty Images

A Ferris wheel is constructed using the basic principles of a bicycle wheel. You can use a vector-valued function to analyze the motion of a Ferris wheel, including its position and velocity. (See P.S. Problem Solving, Exercise 14.)



A *vector-valued function* maps real numbers to vectors. You can use a vector-valued function to represent the motion of a particle along a curve. In Section 12.3, you will use the first and second derivatives of a position vector to find a particle's velocity and acceleration.

12.1 Vector-Valued Functions

- **Analyze and sketch a space curve given by a vector-valued function.**
- **Extend the concepts of limits and continuity to vector-valued functions.**

Space Curves and Vector-Valued Functions

In Section 10.2, a *plane curve* was defined as the set of ordered pairs $(f(t), g(t))$ together with their defining parametric equations

$$x = f(t) \qquad \text{and} \qquad y = g(t)$$

where f and g are continuous functions of t on an interval I. This definition can be extended naturally to three-dimensional space as follows. A **space curve** C is the set of all ordered triples $(f(t), g(t), h(t))$ together with their defining parametric equations

$$x = f(t), \qquad y = g(t), \qquad \text{and} \qquad z = h(t)$$

where f, g, and h are continuous functions of t on an interval I.

Before looking at examples of space curves, a new type of function, called a **vector-valued function,** is introduced. This type of function maps real numbers to vectors.

DEFINITION OF VECTOR-VALUED FUNCTION

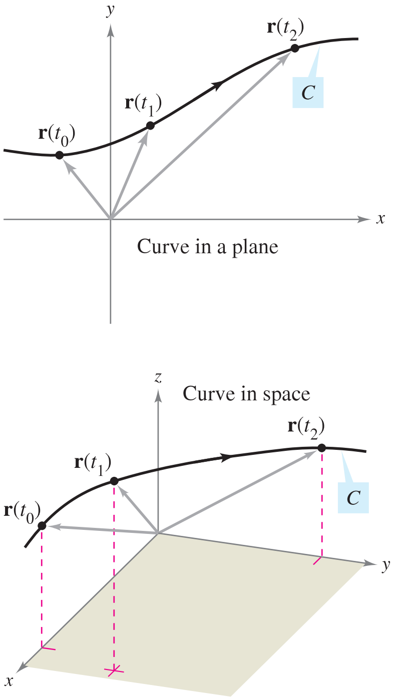
A function of the form

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \qquad \text{Plane}$$

or

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \qquad \text{Space}$$

is a **vector-valued function,** where the **component functions** f, g, and h are real-valued functions of the parameter t. Vector-valued functions are sometimes denoted as $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ or $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$.



Curve C is traced out by the terminal point of position vector $\mathbf{r}(t)$.
Figure 12.1

Technically, a curve in the plane or in space consists of a collection of points and the defining parametric equations. Two different curves can have the same graph. For instance, each of the curves given by

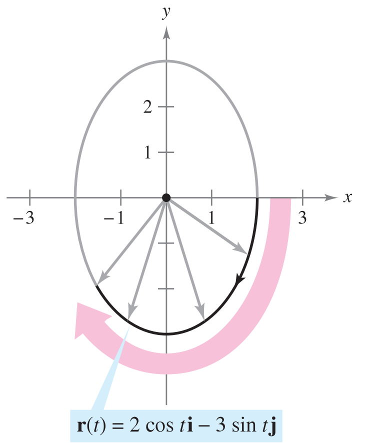
$$\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} \qquad \text{and} \qquad \mathbf{r}(t) = \sin t^2\,\mathbf{i} + \cos t^2\,\mathbf{j}$$

has the unit circle as its graph, but these equations do not represent the same curve—because the circle is traced out in different ways on the graphs.

Be sure you see the distinction between the vector-valued function $\mathbf{r}$ and the real-valued functions f, g, and h. All are functions of the real variable t, but $\mathbf{r}(t)$ is a vector, whereas $f(t)$, $g(t)$, and $h(t)$ are real numbers (for each specific value of t).

Vector-valued functions serve dual roles in the representation of curves. By letting the parameter t represent time, you can use a vector-valued function to represent *motion* along a curve. Or, in the more general case, you can use a vector-valued function to *trace the graph* of a curve. In either case, the terminal point of the position vector $\mathbf{r}(t)$ coincides with the point (x, y) or (x, y, z) on the curve given by the parametric equations, as shown in Figure 12.1. The arrowhead on the curve indicates the curve's *orientation* by pointing in the direction of increasing values of t.

Unless stated otherwise, the **domain** of a vector-valued function $\mathbf{r}$ is considered to be the intersection of the domains of the component functions f, g, and h. For instance, the domain of $\mathbf{r}(t) = \ln t\,\mathbf{i} + \sqrt{1-t}\,\mathbf{j} + t\mathbf{k}$ is the interval $(0, 1]$.



The ellipse is traced clockwise as t increases from 0 to 2π.
Figure 12.2

EXAMPLE 1 Sketching a Plane Curve

Sketch the plane curve represented by the vector-valued function

$$\mathbf{r}(t) = 2\cos t\mathbf{i} - 3\sin t\mathbf{j}, \quad 0 \le t \le 2\pi.$$

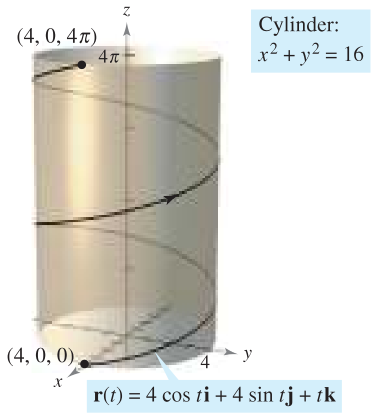
Vector-valued function

Solution From the position vector $\mathbf{r}(t)$, you can write the parametric equations $x = 2\cos t$ and $y = -3\sin t$. Solving for $\cos t$ and $\sin t$ and using the identity $\cos^2 t + \sin^2 t = 1$ produces the rectangular equation

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1.$$

Rectangular equation

The graph of this rectangular equation is the ellipse shown in Figure 12.2. The curve has a *clockwise* orientation. That is, as t increases from 0 to 2π, the position vector $\mathbf{r}(t)$ moves clockwise, and its terminal point traces the ellipse.


z
(4, 0, 4π)
4π
Cylinder:
$x^2 + y^2 = 16$
(4, 0, 0)
4
y
x
$\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + t\mathbf{k}$

As t increases from 0 to 4π, two spirals on the helix are traced out.
Figure 12.3

EXAMPLE 2 Sketching a Space Curve

Sketch the space curve represented by the vector-valued function

$$\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 4\pi.$$

Vector-valued function

Solution From the first two parametric equations $x = 4\cos t$ and $y = 4\sin t$, you can obtain

$$x^2 + y^2 = 16.$$

Rectangular equation

This means that the curve lies on a right circular cylinder of radius 4, centered about the z-axis. To locate the curve on this cylinder, you can use the third parametric equation $z = t$. In Figure 12.3, note that as t increases from 0 to 4π, the point (x, y, z) spirals up the cylinder to produce a **helix.** A real-life example of a helix is shown in the drawing at the lower left. ■

In 1953 Francis Crick and James D. Watson discovered the double helix structure of DNA.

In Examples 1 and 2, you were given a vector-valued function and were asked to sketch the corresponding curve. The next two examples address the reverse problem—finding a vector-valued function to represent a given graph. Of course, if the graph is described parametrically, representation by a vector-valued function is straightforward. For instance, to represent the line in space given by

$$x = 2 + t, \quad y = 3t, \quad \text{and} \quad z = 4 - t$$

you can simply use the vector-valued function given by

$$\mathbf{r}(t) = (2 + t)\mathbf{i} + 3t\mathbf{j} + (4 - t)\mathbf{k}.$$

If a set of parametric equations for the graph is not given, the problem of representing the graph by a vector-valued function boils down to finding a set of parametric equations.

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

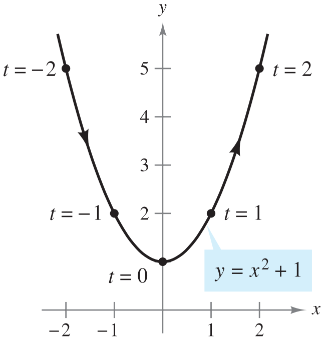
EXAMPLE 3 Representing a Graph by a Vector-Valued Function

Represent the parabola given by $y = x^2 + 1$ by a vector-valued function.

Solution Although there are many ways to choose the parameter t, a natural choice is to let $x = t$. Then $y = t^2 + 1$ and you have

$$\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j}. \qquad \text{Vector-valued function}$$

Note in Figure 12.4 the orientation produced by this particular choice of parameter. Had you chosen $x = -t$ as the parameter, the curve would have been oriented in the opposite direction.



There are many ways to parametrize this graph. One way is to let $x = t$.
Figure 12.4

EXAMPLE 4 Representing a Graph by a Vector-Valued Function

Sketch the space curve C represented by the intersection of the semiellipsoid

$$\frac{x^2}{12} + \frac{y^2}{24} + \frac{z^2}{4} = 1, \quad z \geq 0$$

and the parabolic cylinder $y = x^2$. Then, find a vector-valued function to represent the graph.

Solution The intersection of the two surfaces is shown in Figure 12.5. As in Example 3, a natural choice of parameter is $x = t$. For this choice, you can use the given equation $y = x^2$ to obtain $y = t^2$. Then, it follows that

$$\frac{z^2}{4} = 1 - \frac{x^2}{12} - \frac{y^2}{24} = 1 - \frac{t^2}{12} - \frac{t^4}{24} = \frac{24 - 2t^2 - t^4}{24} = \frac{(6 + t^2)(4 - t^2)}{24}.$$

NOTE Curves in space can be specified in various ways. For instance, the curve in Example 4 is described as the intersection of two surfaces in space.

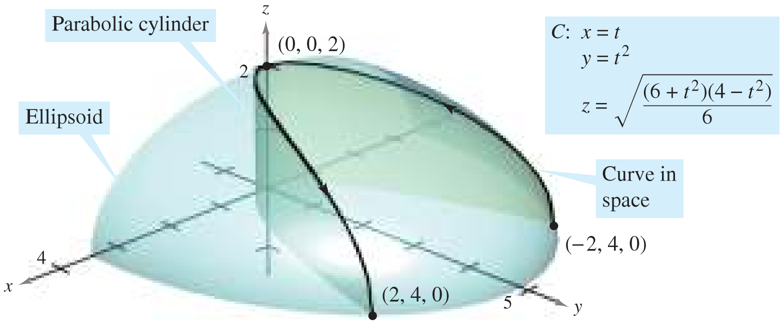
Because the curve lies above the xy-plane, you should choose the positive square root for z and obtain the following parametric equations.

$$x = t, \quad y = t^2, \qquad \text{and} \qquad z = \sqrt{\frac{(6 + t^2)(4 - t^2)}{6}}$$

The resulting vector-valued function is

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \sqrt{\frac{(6 + t^2)(4 - t^2)}{6}}\,\mathbf{k}, \quad -2 \leq t \leq 2. \qquad \text{Vector-valued function}$$

(Note that the $\mathbf{k}$-component of $\mathbf{r}(t)$ implies $-2 \leq t \leq 2$.) From the points $(-2, 4, 0)$ and $(2, 4, 0)$ shown in Figure 12.5, you can see that the curve is traced as t increases from -2 to 2.



The curve C is the intersection of the semiellipsoid and the parabolic cylinder.
Figure 12.5

■

Limits and Continuity

Many techniques and definitions used in the calculus of real-valued functions can be applied to vector-valued functions. For instance, you can add and subtract vector-valued functions, multiply a vector-valued function by a scalar, take the limit of a vector-valued function, differentiate a vector-valued function, and so on. The basic approach is to capitalize on the linearity of vector operations by extending the definitions on a component-by-component basis. For example, to add or subtract two vector-valued functions (in the plane), you can write

$$\begin{aligned}\mathbf{r}_1(t) + \mathbf{r}_2(t) &= [f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] + [f_2(t)\mathbf{i} + g_2(t)\mathbf{j}] && \text{Sum}\\ &= [f_1(t) + f_2(t)]\mathbf{i} + [g_1(t) + g_2(t)]\mathbf{j}\\ \mathbf{r}_1(t) - \mathbf{r}_2(t) &= [f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] - [f_2(t)\mathbf{i} + g_2(t)\mathbf{j}] && \text{Difference}\\ &= [f_1(t) - f_2(t)]\mathbf{i} + [g_1(t) - g_2(t)]\mathbf{j}.\end{aligned}$$

Similarly, to multiply and divide a vector-valued function by a scalar, you can write

$$\begin{aligned}c\mathbf{r}(t) &= c[f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] && \text{Scalar multiplication}\\ &= cf_1(t)\mathbf{i} + cg_1(t)\mathbf{j}\\ \frac{\mathbf{r}(t)}{c} &= \frac{[f_1(t)\mathbf{i} + g_1(t)\mathbf{j}]}{c}, \quad c \neq 0 && \text{Scalar division}\\ &= \frac{f_1(t)}{c}\mathbf{i} + \frac{g_1(t)}{c}\mathbf{j}.\end{aligned}$$

This component-by-component extension of operations with real-valued functions to vector-valued functions is further illustrated in the following definition of the limit of a vector-valued function.

DEFINITION OF THE LIMIT OF A VECTOR-VALUED FUNCTION

1. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then

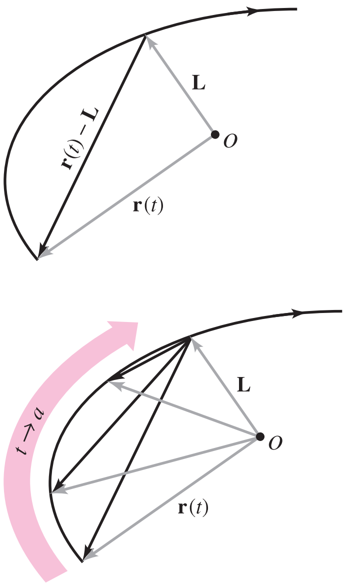
$$\lim_{t\to a}\mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} \qquad \text{Plane}$$

provided f and g have limits as $t \to a$.

2. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t\to a}\mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} + \left[\lim_{t\to a} h(t)\right]\mathbf{k} \qquad \text{Space}$$

provided f, g, and h have limits as $t \to a$.



As t approaches a, $\mathbf{r}(t)$ approaches the limit $\mathbf{L}$. For the limit $\mathbf{L}$ to exist, it is not necessary that $\mathbf{r}(a)$ be defined or that $\mathbf{r}(a)$ be equal to $\mathbf{L}$.
Figure 12.6

If $\mathbf{r}(t)$ approaches the vector $\mathbf{L}$ as $t \to a$, the length of the vector $\mathbf{r}(t) - \mathbf{L}$ approaches 0. That is,

$$\|\mathbf{r}(t) - \mathbf{L}\| \to 0 \qquad \text{as} \qquad t \to a.$$

This is illustrated graphically in Figure 12.6. With this definition of the limit of a vector-valued function, you can develop vector versions of most of the limit theorems given in Chapter 1. For example, the limit of the sum of two vector-valued functions is the sum of their individual limits. Also, you can use the orientation of the curve $\mathbf{r}(t)$ to define one-sided limits of vector-valued functions. The next definition extends the notion of continuity to vector-valued functions.

DEFINITION OF CONTINUITY OF A VECTOR-VALUED FUNCTION

A vector-valued function $\mathbf{r}$ is **continuous at the point** given by $t = a$ if the limit of $\mathbf{r}(t)$ exists as $t \to a$ and

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a).$$

A vector-valued function $\mathbf{r}$ is **continuous on an interval** I if it is continuous at every point in the interval.

From this definition, it follows that a vector-valued function is continuous at $t = a$ if and only if each of its component functions is continuous at $t = a$.

EXAMPLE 5 Continuity of Vector-Valued Functions

Discuss the continuity of the vector-valued function given by

$$\mathbf{r}(t) = t\mathbf{i} + a\mathbf{j} + (a^2 - t^2)\mathbf{k} \qquad a \text{ is a constant.}$$

at $t = 0$.

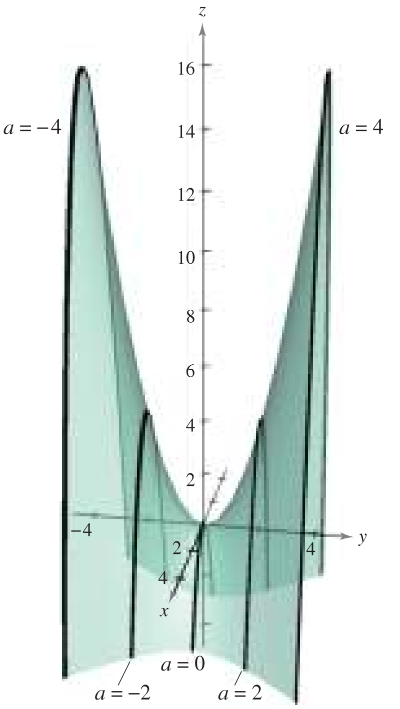
Solution As t approaches 0, the limit is

$$\begin{aligned}\lim_{t \to 0} \mathbf{r}(t) &= \left[\lim_{t \to 0} t\right]\mathbf{i} + \left[\lim_{t \to 0} a\right]\mathbf{j} + \left[\lim_{t \to 0} (a^2 - t^2)\right]\mathbf{k}\\ &= 0\mathbf{i} + a\mathbf{j} + a^2\mathbf{k}\\ &= a\mathbf{j} + a^2\mathbf{k}.\end{aligned}$$

Because

$$\begin{aligned}\mathbf{r}(0) &= (0)\mathbf{i} + (a)\mathbf{j} + (a^2)\mathbf{k}\\ &= a\mathbf{j} + a^2\mathbf{k}\end{aligned}$$

you can conclude that $\mathbf{r}$ is continuous at $t = 0$. By similar reasoning, you can conclude that the vector-valued function $\mathbf{r}$ is continuous at all real-number values of t. ■



For each value of a, the curve represented by the vector-valued function $\mathbf{r}(t) = t\mathbf{i} + a\mathbf{j} + (a^2 - t^2)\mathbf{k}$ is a parabola.
Figure 12.7

For each value of a, the curve represented by the vector-valued function in Example 5,

$$\mathbf{r}(t) = t\mathbf{i} + a\mathbf{j} + (a^2 - t^2)\mathbf{k} \qquad a \text{ is a constant.}$$

is a parabola. You can think of each parabola as the intersection of the vertical plane $y = a$ and the hyperbolic paraboloid

$$y^2 - x^2 = z$$

as shown in Figure 12.7.

TECHNOLOGY Almost any type of three-dimensional sketch is difficult to do by hand, but sketching curves in space is especially difficult. The problem is in trying to create the illusion of three dimensions. Graphing utilities use a variety of techniques to add “three-dimensionality” to graphs of space curves: one way is to show the curve on a surface, as in Figure 12.7.

12.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, find the domain of the vector-valued function.

1. $\mathbf{r}(t) = \dfrac{1}{t+1}\mathbf{i} + \dfrac{t}{2}\mathbf{j} - 3t\mathbf{k}$

2. $\mathbf{r}(t) = \sqrt{4-t^2}\,\mathbf{i} + t^2\mathbf{j} - 6t\mathbf{k}$

3. $\mathbf{r}(t) = \ln t\,\mathbf{i} - e^t\,\mathbf{j} - t\,\mathbf{k}$

4. $\mathbf{r}(t) = \sin t\,\mathbf{i} + 4\cos t\,\mathbf{j} + t\,\mathbf{k}$

5. $\mathbf{r}(t) = \mathbf{F}(t) + \mathbf{G}(t)$ where
$\mathbf{F}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sqrt{t}\,\mathbf{k}, \quad \mathbf{G}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$

6. $\mathbf{r}(t) = \mathbf{F}(t) - \mathbf{G}(t)$ where
$\mathbf{F}(t) = \ln t\,\mathbf{i} + 5t\,\mathbf{j} - 3t^2\mathbf{k}, \quad \mathbf{G}(t) = \mathbf{i} + 4t\,\mathbf{j} - 3t^2\mathbf{k}$

7. $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t)$ where
$\mathbf{F}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}, \quad \mathbf{G}(t) = \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$

8. $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t)$ where
$\mathbf{F}(t) = t^3\mathbf{i} - t\,\mathbf{j} + t\,\mathbf{k}, \quad \mathbf{G}(t) = \sqrt[3]{t}\,\mathbf{i} + \dfrac{1}{t+1}\mathbf{j} + (t+2)\mathbf{k}$

In Exercises 9–12, evaluate (if possible) the vector-valued function at each given value of *t*.

9. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - (t-1)\mathbf{j}$
(a) $\mathbf{r}(1)$ (b) $\mathbf{r}(0)$ (c) $\mathbf{r}(s+1)$
(d) $\mathbf{r}(2+\Delta t) - \mathbf{r}(2)$

10. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$
(a) $\mathbf{r}(0)$ (b) $\mathbf{r}(\pi/4)$ (c) $\mathbf{r}(\theta - \pi)$
(d) $\mathbf{r}(\pi/6 + \Delta t) - \mathbf{r}(\pi/6)$

11. $\mathbf{r}(t) = \ln t\,\mathbf{i} + \dfrac{1}{t}\mathbf{j} + 3t\mathbf{k}$
(a) $\mathbf{r}(2)$ (b) $\mathbf{r}(-3)$ (c) $\mathbf{r}(t-4)$
(d) $\mathbf{r}(1+\Delta t) - \mathbf{r}(1)$

12. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + t^{3/2}\mathbf{j} + e^{-t/4}\mathbf{k}$
(a) $\mathbf{r}(0)$ (b) $\mathbf{r}(4)$ (c) $\mathbf{r}(c+2)$
(d) $\mathbf{r}(9+\Delta t) - \mathbf{r}(9)$

In Exercises 13 and 14, find $\|\mathbf{r}(t)\|$.

13. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + 3t\,\mathbf{j} - 4t\,\mathbf{k}$

14. $\mathbf{r}(t) = \sin 3t\,\mathbf{i} + \cos 3t\,\mathbf{j} + t\,\mathbf{k}$

In Exercises 15–18, represent the line segment from *P* to *Q* by a vector-valued function and by a set of parametric equations.

15. $P(0, 0, 0)$, $Q(3, 1, 2)$ **16.** $P(0, 2, -1)$, $Q(4, 7, 2)$

17. $P(-2, 5, -3)$, $Q(-1, 4, 9)$

18. $P(1, -6, 8)$, $Q(-3, -2, 5)$

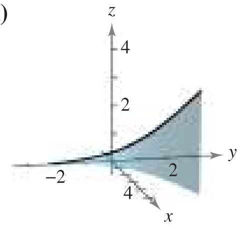
Think About It **In Exercises 19 and 20, find $\mathbf{r}(t) \cdot \mathbf{u}(t)$. Is the result a vector-valued function? Explain.**

19. $\mathbf{r}(t) = (3t-1)\mathbf{i} + \frac{1}{4}t^3\mathbf{j} + 4\mathbf{k}, \quad \mathbf{u}(t) = t^2\mathbf{i} - 8\mathbf{j} + t^3\mathbf{k}$

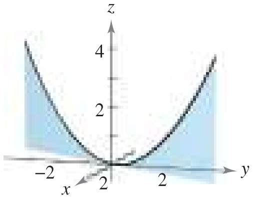
20. $\mathbf{r}(t) = \langle 3\cos t, 2\sin t, t-2\rangle, \quad \mathbf{u}(t) = \langle 4\sin t, -6\cos t, t^2\rangle$

In Exercises 21–24, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]

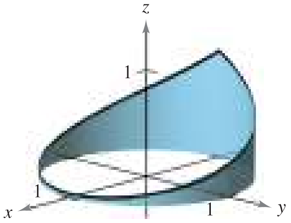
(a)



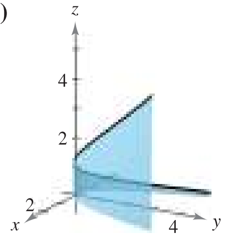
(b)



(c)



(d)



21. $\mathbf{r}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\mathbf{k}, \quad -2 \le t \le 2$

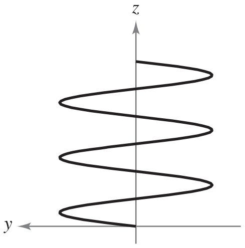
22. $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k}, \quad -1 \le t \le 1$

23. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\,\mathbf{k}, \quad -2 \le t \le 2$

24. $\mathbf{r}(t) = t\,\mathbf{i} + \ln t\,\mathbf{j} + \dfrac{2t}{3}\mathbf{k}, \quad 0.1 \le t \le 5$

25. ***Think About It*** The four figures below are graphs of the vector-valued function $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + (t/4)\mathbf{k}$. Match each of the four graphs with the point in space from which the helix is viewed. The four points are $(0, 0, 20)$, $(20, 0, 0)$, $(-20, 0, 0)$, and $(10, 20, 10)$.

(a)



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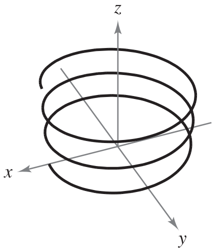
(b)



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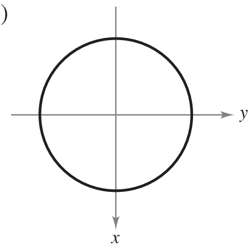
(c)



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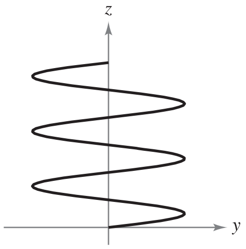
(d)



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26. Sketch the three graphs of the vector-valued function $\mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + 2\mathbf{k}$ as viewed from each point.
(a) $(0, 0, 20)$ (b) $(10, 0, 0)$ (c) $(5, 5, 5)$

In Exercises 27–42, sketch the curve represented by the vector-valued function and give the orientation of the curve.

27. $\mathbf{r}(t) = \frac{t}{4}\mathbf{i} + (t - 1)\mathbf{j}$

28. $\mathbf{r}(t) = (5 - t)\mathbf{i} + \sqrt{t}\,\mathbf{j}$

29. $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$

30. $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$

31. $\mathbf{r}(\theta) = \cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j}$

32. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$

33. $\mathbf{r}(\theta) = 3\sec\theta\,\mathbf{i} + 2\tan\theta\,\mathbf{j}$

34. $\mathbf{r}(t) = 2\cos^3 t\,\mathbf{i} + 2\sin^3 t\,\mathbf{j}$

35. $\mathbf{r}(t) = (-t + 1)\mathbf{i} + (4t + 2)\mathbf{j} + (2t + 3)\mathbf{k}$

36. $\mathbf{r}(t) = t\mathbf{i} + (2t - 5)\mathbf{j} + 3t\mathbf{k}$

37. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\mathbf{k}$

38. $\mathbf{r}(t) = t\mathbf{i} + 3\cos t\mathbf{j} + 3\sin t\mathbf{k}$

39. $\mathbf{r}(t) = 2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + e^{-t}\,\mathbf{k}$

40. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + \frac{3}{2}t\mathbf{k}$

41. $\mathbf{r}(t) = \left\langle t, t^2, \frac{2}{3}t^3\right\rangle$

42. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$

CAS **In Exercises 43–46, use a computer algebra system to graph the vector-valued function and identify the common curve.**

43. $\mathbf{r}(t) = -\frac{1}{2}t^2\mathbf{i} + t\mathbf{j} - \frac{\sqrt{3}}{2}t^2\mathbf{k}$

44. $\mathbf{r}(t) = t\mathbf{i} - \frac{\sqrt{3}}{2}t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

45. $\mathbf{r}(t) = \sin t\,\mathbf{i} + \left(\frac{\sqrt{3}}{2}\cos t - \frac{1}{2}t\right)\mathbf{j} + \left(\frac{1}{2}\cos t + \frac{\sqrt{3}}{2}\right)\mathbf{k}$

46. $\mathbf{r}(t) = -\sqrt{2}\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \sqrt{2}\sin t\,\mathbf{k}$

CAS ***Think About It*** **In Exercises 47 and 48, use a computer algebra system to graph the vector-valued function r(*t*). For each u(*t*), make a conjecture about the transformation (if any) of the graph of r(*t*). Use a computer algebra system to verify your conjecture.**

47. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \frac{1}{2}t\mathbf{k}$

(a) $\mathbf{u}(t) = 2(\cos t - 1)\mathbf{i} + 2\sin t\,\mathbf{j} + \frac{1}{2}t\mathbf{k}$

(b) $\mathbf{u}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 2t\mathbf{k}$

(c) $\mathbf{u}(t) = 2\cos(-t)\mathbf{i} + 2\sin(-t)\mathbf{j} + \frac{1}{2}(-t)\mathbf{k}$

(d) $\mathbf{u}(t) = \frac{1}{2}t\mathbf{i} + 2\sin t\,\mathbf{j} + 2\cos t\,\mathbf{k}$

(e) $\mathbf{u}(t) = 6\cos t\,\mathbf{i} + 6\sin t\,\mathbf{j} + \frac{1}{2}t\mathbf{k}$

48. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$

(a) $\mathbf{u}(t) = t\mathbf{i} + (t^2 - 2)\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$

(b) $\mathbf{u}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$

(c) $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \left(\frac{1}{2}t^3 + 4\right)\mathbf{k}$

(d) $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{8}t^3\mathbf{k}$

(e) $\mathbf{u}(t) = (-t)\mathbf{i} + (-t)^2\mathbf{j} + \frac{1}{2}(-t)^3\mathbf{k}$

In Exercises 49–56, represent the plane curve by a vector-valued function. (There are many correct answers.)

49. $y = x + 5$

50. $2x - 3y + 5 = 0$

51. $y = (x - 2)^2$

52. $y = 4 - x^2$

53. $x^2 + y^2 = 25$

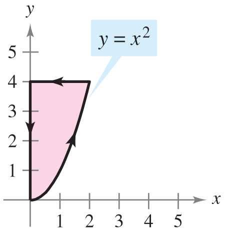
54. $(x - 2)^2 + y^2 = 4$

55. $\frac{x^2}{16} - \frac{y^2}{4} = 1$

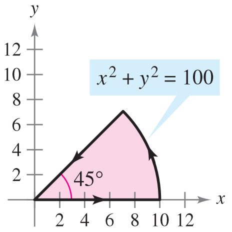
56. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

In Exercises 57 and 58, find vector-valued functions forming the boundaries of the region in the figure. State the interval for the parameter of each function.

57.



58.



In Exercises 59–66, sketch the space curve represented by the intersection of the surfaces. Then represent the curve by a vector-valued function using the given parameter.

	Surfaces	*Parameter*
59.	$z = x^2 + y^2,\quad x + y = 0$	$x = t$
60.	$z = x^2 + y^2,\quad z = 4$	$x = 2\cos t$
61.	$x^2 + y^2 = 4,\quad z = x^2$	$x = 2\sin t$
62.	$4x^2 + 4y^2 + z^2 = 16,\quad x = z^2$	$z = t$
63.	$x^2 + y^2 + z^2 = 4,\quad x + z = 2$	$x = 1 + \sin t$
64.	$x^2 + y^2 + z^2 = 10,\quad x + y = 4$	$x = 2 + \sin t$
65.	$x^2 + z^2 = 4,\quad y^2 + z^2 = 4$	$x = t$ (first octant)
66.	$x^2 + y^2 + z^2 = 16,\quad xy = 4$	$x = t$ (first octant)

67. Show that the vector-valued function

$$\mathbf{r}(t) = t\mathbf{i} + 2t\cos t\mathbf{j} + 2t\sin t\mathbf{k}$$

lies on the cone $4x^2 = y^2 + z^2$. Sketch the curve.

68. Show that the vector-valued function

$$\mathbf{r}(t) = e^{-t}\cos t\,\mathbf{i} + e^{-t}\sin t\mathbf{j} + e^{-t}\mathbf{k}$$

lies on the cone $z^2 = x^2 + y^2$. Sketch the curve.

In Exercises 69–74, find the limit (if it exists).

69. $\lim_{t\to\pi} (t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k})$

70. $\lim_{t\to 2} \left(3t\mathbf{i} + \frac{2}{t^2 - 1}\mathbf{j} + \frac{1}{t}\mathbf{k}\right)$

71. $\lim_{t\to 0} \left(t^2\mathbf{i} + 3t\mathbf{j} + \frac{1 - \cos t}{t}\mathbf{k}\right)$

72. $\lim_{t\to 1} \left(\sqrt{t}\,\mathbf{i} + \frac{\ln t}{t^2 - 1}\mathbf{j} + \frac{1}{t - 1}\mathbf{k}\right)$

73. $\lim_{t\to 0} \left(e^t\mathbf{i} + \frac{\sin t}{t}\mathbf{j} + e^{-t}\mathbf{k}\right)$

74. $\lim_{t\to\infty} \left(e^{-t}\,\mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{t}{t^2 + 1}\mathbf{k}\right)$

In Exercises 75–80, determine the interval(s) on which the vector-valued function is continuous.

75. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j}$

76. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \sqrt{t-1}\,\mathbf{j}$

77. $\mathbf{r}(t) = t\mathbf{i} + \arcsin t\mathbf{j} + (t-1)\mathbf{k}$

78. $\mathbf{r}(t) = 2e^{-t}\mathbf{i} + e^{-t}\mathbf{j} + \ln(t-1)\mathbf{k}$

79. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t\rangle$

80. $\mathbf{r}(t) = \left\langle 8, \sqrt{t}, \sqrt[3]{t}\right\rangle$

WRITING ABOUT CONCEPTS

81. Consider the vector-valued function

$$\mathbf{r}(t) = t^2\mathbf{i} + (t-3)\mathbf{j} + t\mathbf{k}.$$

Write a vector-valued function $\mathbf{s}(t)$ that is the specified transformation of $\mathbf{r}$.

(a) A vertical translation three units upward

(b) A horizontal translation two units in the direction of the negative x-axis

(c) A horizontal translation five units in the direction of the positive y-axis

82. State the definition of continuity of a vector-valued function. Give an example of a vector-valued function that is defined but not continuous at $t = 2$.

CAS **83.** The outer edge of a playground slide is in the shape of a helix of radius 1.5 meters. The slide has a height of 2 meters and makes one complete revolution from top to bottom. Find a vector-valued function for the helix. Use a computer algebra system to graph your function. (There are many correct answers.)

CAPSTONE

84. Which of the following vector-valued functions represent the same graph?

(a) $\mathbf{r}(t) = (-3\cos t + 1)\mathbf{i} + (5\sin t + 2)\mathbf{j} + 4\mathbf{k}$

(b) $\mathbf{r}(t) = 4\mathbf{i} + (-3\cos t + 1)\mathbf{j} + (5\sin t + 2)\mathbf{k}$

(c) $\mathbf{r}(t) = (3\cos t - 1)\mathbf{i} + (-5\sin t - 2)\mathbf{j} + 4\mathbf{k}$

(d) $\mathbf{r}(t) = (-3\cos 2t + 1)\mathbf{i} + (5\sin 2t + 2)\mathbf{j} + 4\mathbf{k}$

85. Let $\mathbf{r}(t)$ and $\mathbf{u}(t)$ be vector-valued functions whose limits exist as $t \to c$. Prove that

$$\lim_{t\to c}\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] = \lim_{t\to c}\mathbf{r}(t) \times \lim_{t\to c}\mathbf{u}(t).$$

86. Let $\mathbf{r}(t)$ and $\mathbf{u}(t)$ be vector-valued functions whose limits exist as $t \to c$. Prove that

$$\lim_{t\to c}\left[\mathbf{r}(t) \cdot \mathbf{u}(t)\right] = \lim_{t\to c}\mathbf{r}(t) \cdot \lim_{t\to c}\mathbf{u}(t).$$

87. Prove that if $\mathbf{r}$ is a vector-valued function that is continuous at c, then $\|\mathbf{r}\|$ is continuous at c.

88. Verify that the converse of Exercise 87 is not true by finding a vector-valued function $\mathbf{r}$ such that $\|\mathbf{r}\|$ is continuous at c but $\mathbf{r}$ is not continuous at c.

In Exercises 89 and 90, two particles travel along the space curves $\mathbf{r}(t)$ and $\mathbf{u}(t)$. A collision will occur at the point of intersection P if both particles are at P at the same time. Do the particles collide? Do their paths intersect?

89. $\mathbf{r}(t) = t^2\mathbf{i} + (9t - 20)\mathbf{j} + t^2\mathbf{k}$

$\mathbf{u}(t) = (3t + 4)\mathbf{i} + t^2\mathbf{j} + (5t - 4)\mathbf{k}$

90. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

$\mathbf{u}(t) = (-2t + 3)\mathbf{i} + 8t\mathbf{j} + (12t + 2)\mathbf{k}$

Think About It **In Exercises 91 and 92, two particles travel along the space curves $\mathbf{r}(t)$ and $\mathbf{u}(t)$.**

91. If $\mathbf{r}(t)$ and $\mathbf{u}(t)$ intersect, will the particles collide?

92. If the particles collide, do their paths $\mathbf{r}(t)$ and $\mathbf{u}(t)$ intersect?

True or False? **In Exercises 93–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

93. If f, g, and h are first-degree polynomial functions, then the curve given by $x = f(t)$, $y = g(t)$, and $z = h(t)$ is a line.

94. If the curve given by $x = f(t)$, $y = g(t)$, and $z = h(t)$ is a line, then f, g, and h are first-degree polynomial functions of t.

95. Two particles travel along the space curves $\mathbf{r}(t)$ and $\mathbf{u}(t)$. The intersection of their paths depends only on the curves traced out by $\mathbf{r}(t)$ and $\mathbf{u}(t)$, while collision depends on the parameterizations.

96. The vector-valued function $\mathbf{r}(t) = t^2\mathbf{i} + t\sin t\,\mathbf{j} + t\cos t\,\mathbf{k}$ lies on the paraboloid $x = y^2 + z^2$.

SECTION PROJECT

Witch of Agnesi

In Section 3.5, you studied a famous curve called the **Witch of Agnesi.** In this project you will take a closer look at this function.

Consider a circle of radius a centered on the y-axis at $(0, a)$. Let A be a point on the horizontal line $y = 2a$, let O be the origin, and let B be the point where the segment OA intersects the circle. A point P is on the Witch of Agnesi if P lies on the horizontal line through B and on the vertical line through A.

(a) Show that the point A is traced out by the vector-valued function

$$\mathbf{r}_A(\theta) = 2a\cot\theta\mathbf{i} + 2a\mathbf{j}, \quad 0 < \theta < \pi$$

where θ is the angle that OA makes with the positive x-axis.

(b) Show that the point B is traced out by the vector-valued function

$$\mathbf{r}_B(\theta) = a\sin 2\theta\mathbf{i} + a(1 - \cos 2\theta)\mathbf{j}, \quad 0 < \theta < \pi.$$

(c) Combine the results of parts (a) and (b) to find the vector-valued function $\mathbf{r}(\theta)$ for the Witch of Agnesi. Use a graphing utility to graph this curve for $a = 1$.

(d) Describe the limits $\displaystyle\lim_{\theta\to 0^+}\mathbf{r}(\theta)$ and $\displaystyle\lim_{\theta\to\pi^-}\mathbf{r}(\theta)$.

(e) Eliminate the parameter θ and determine the rectangular equation of the Witch of Agnesi. Use a graphing utility to graph this function for $a = 1$ and compare your graph with that obtained in part (c).

12.2 Differentiation and Integration of Vector-Valued Functions

- **Differentiate a vector-valued function.**
- **Integrate a vector-valued function.**

Differentiation of Vector-Valued Functions

In Sections 12.3–12.5, you will study several important applications involving the calculus of vector-valued functions. In preparation for that study, this section is devoted to the mechanics of differentiation and integration of vector-valued functions.

The definition of the derivative of a vector-valued function parallels the definition given for real-valued functions.

DEFINITION OF THE DERIVATIVE OF A VECTOR-VALUED FUNCTION

The **derivative of a vector-valued function r** is defined by

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all t for which the limit exists. If $\mathbf{r}'(t)$ exists, then $\mathbf{r}$ is **differentiable at *t*.** If $\mathbf{r}'(t)$ exists for all t in an open interval I, then $\mathbf{r}$ is **differentiable on the interval *I*.** Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

NOTE In addition to $\mathbf{r}'(t)$, other notations for the derivative of a vector-valued function are

$$D_t[\mathbf{r}(t)], \quad \frac{d}{dt}[\mathbf{r}(t)], \quad \text{and} \quad \frac{d\mathbf{r}}{dt}.$$

■

Differentiation of vector-valued functions can be done on a *component-by-component basis*. To see why this is true, consider the function given by

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}.$$

Applying the definition of the derivative produces the following.

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{f(t + \Delta t)\mathbf{i} + g(t + \Delta t)\mathbf{j} - f(t)\mathbf{i} - g(t)\mathbf{j}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left\{ \left[\frac{f(t + \Delta t) - f(t)}{\Delta t} \right] \mathbf{i} + \left[\frac{g(t + \Delta t) - g(t)}{\Delta t} \right] \mathbf{j} \right\} \\
&= \left\{ \lim_{\Delta t \to 0} \left[\frac{f(t + \Delta t) - f(t)}{\Delta t} \right] \right\} \mathbf{i} + \left\{ \lim_{\Delta t \to 0} \left[\frac{g(t + \Delta t) - g(t)}{\Delta t} \right] \right\} \mathbf{j} \\
&= f'(t)\mathbf{i} + g'(t)\mathbf{j}
\end{aligned}$$

This important result is listed in the theorem on the next page. Note that the derivative of the vector-valued function $\mathbf{r}$ is itself a vector-valued function. You can see from Figure 12.8 that $\mathbf{r}'(t)$ is a vector tangent to the curve given by $\mathbf{r}(t)$ and pointing in the direction of increasing t-values.

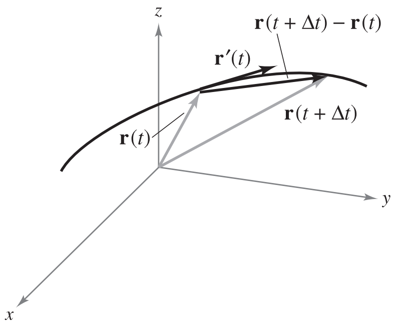


Figure 12.8

THEOREM 12.1 DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are differentiable functions of t, then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}.$$ Plane

2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions of t, then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$ Space

EXAMPLE 1 Differentiation of Vector-Valued Functions

For the vector-valued function given by $\mathbf{r}(t) = t\mathbf{i} + (t^2 + 2)\mathbf{j}$, find $\mathbf{r}'(t)$. Then sketch the plane curve represented by $\mathbf{r}(t)$, and the graphs of $\mathbf{r}(1)$ and $\mathbf{r}'(1)$.

Solution Differentiate on a component-by-component basis to obtain

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}.$$ Derivative

From the position vector $\mathbf{r}(t)$, you can write the parametric equations $x = t$ and $y = t^2 + 2$. The corresponding rectangular equation is $y = x^2 + 2$. When $t = 1$, $\mathbf{r}(1) = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{r}'(1) = \mathbf{i} + 2\mathbf{j}$. In Figure 12.9, $\mathbf{r}(1)$ is drawn starting at the origin, and $\mathbf{r}'(1)$ is drawn starting at the terminal point of $\mathbf{r}(1)$. ■

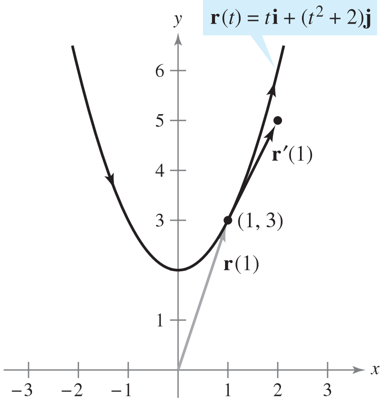


Figure 12.9

Higher-order derivatives of vector-valued functions are obtained by successive differentiation of each component function.

EXAMPLE 2 Higher-Order Differentiation

For the vector-valued function given by $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$, find each of the following.

a. $\mathbf{r}'(t)$ **b.** $\mathbf{r}''(t)$

c. $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$ **d.** $\mathbf{r}'(t) \times \mathbf{r}''(t)$

Solution

a. $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}$ First derivative

b. $\mathbf{r}''(t) = -\cos t\mathbf{i} - \sin t\mathbf{j} + 0\mathbf{k}$

$= -\cos t\mathbf{i} - \sin t\mathbf{j}$ Second derivative

c. $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = \sin t \cos t - \sin t \cos t = 0$ Dot product

d. $$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 2 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$ Cross product

$$= \begin{vmatrix} \cos t & 2 \\ -\sin t & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -\sin t & 2 \\ -\cos t & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix}\mathbf{k}$$

$$= 2\sin t\mathbf{i} - 2\cos t\mathbf{j} + \mathbf{k}$$

Note that the dot product in part (c) is a *real-valued* function, not a vector-valued function. ■

The parametrization of the curve represented by the vector-valued function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is **smooth on an open interval** I if f', g', and h' are continuous on I and $\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t in the interval I.

EXAMPLE 3 Finding Intervals on Which a Curve Is Smooth

Find the intervals on which the epicycloid C given by

$$\mathbf{r}(t) = (5\cos t - \cos 5t)\mathbf{i} + (5\sin t - \sin 5t)\mathbf{j}, \quad 0 \le t \le 2\pi$$

is smooth.

Solution The derivative of $\mathbf{r}$ is

$$\mathbf{r}'(t) = (-5\sin t + 5\sin 5t)\mathbf{i} + (5\cos t - 5\cos 5t)\mathbf{j}.$$

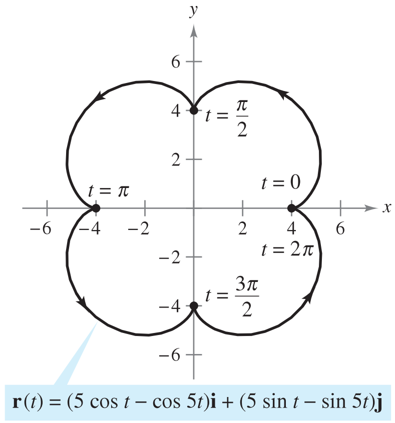
In the interval $[0, 2\pi]$, the only values of t for which

$$\mathbf{r}'(t) = 0\mathbf{i} + 0\mathbf{j}$$

are $t = 0, \pi/2, \pi, 3\pi/2$, and 2π. Therefore, you can conclude that C is smooth in the intervals

$$\left(0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right), \left(\pi, \frac{3\pi}{2}\right), \quad \text{and} \quad \left(\frac{3\pi}{2}, 2\pi\right)$$

as shown in Figure 12.10. ■



The epicycloid is not smooth at the points where it intersects the axes.
Figure 12.10

NOTE In Figure 12.10, note that the curve is not smooth at points at which the curve makes abrupt changes in direction. Such points are called **cusps** or **nodes.** ■

Most of the differentiation rules in Chapter 2 have counterparts for vector-valued functions, and several are listed in the following theorem. Note that the theorem contains three versions of "product rules." Property 3 gives the derivative of the product of a real-valued function w and a vector-valued function $\mathbf{r}$, Property 4 gives the derivative of the dot product of two vector-valued functions, and Property 5 gives the derivative of the cross product of two vector-valued functions (in space). Note that Property 5 applies only to three-dimensional vector-valued functions, because the cross product is not defined for two-dimensional vectors.

THEOREM 12.2 PROPERTIES OF THE DERIVATIVE

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of t, let w be a differentiable real-valued function of t, and let c be a scalar.

1. $D_t[c\mathbf{r}(t)] = c\mathbf{r}'(t)$
2. $D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
3. $D_t[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$
4. $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
5. $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
6. $D_t[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$
7. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

PROOF To prove Property 4, let

$$\mathbf{r}(t) = f_1(t)\mathbf{i} + g_1(t)\mathbf{j} \quad \text{and} \quad \mathbf{u}(t) = f_2(t)\mathbf{i} + g_2(t)\mathbf{j}$$

where f_1, f_2, g_1, and g_2 are differentiable functions of t. Then,

$$\mathbf{r}(t) \cdot \mathbf{u}(t) = f_1(t)f_2(t) + g_1(t)g_2(t)$$

and it follows that

$$\begin{aligned} D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] &= f_1(t)f_2'(t) + f_1'(t)f_2(t) + g_1(t)g_2'(t) + g_1'(t)g_2(t) \\ &= [f_1(t)f_2'(t) + g_1(t)g_2'(t)] + [f_1'(t)f_2(t) + g_1'(t)g_2(t)] \\ &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t). \end{aligned}$$

Proofs of the other properties are left as exercises (see Exercises 77–81 and Exercise 84). ■

EXPLORATION

Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$. Sketch the graph of $\mathbf{r}(t)$. Explain why the graph is a circle of radius 1 centered at the origin. Calculate $\mathbf{r}(\pi/4)$ and $\mathbf{r}'(\pi/4)$. Position the vector $\mathbf{r}'(\pi/4)$ so that its initial point is at the terminal point of $\mathbf{r}(\pi/4)$. What do you observe? Show that $\mathbf{r}(t) \cdot \mathbf{r}(t)$ is constant and that $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ for all t. How does this example relate to Property 7 of Theorem 12.2?

EXAMPLE 4 Using Properties of the Derivative

For the vector-valued functions given by

$$\mathbf{r}(t) = \frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k} \quad \text{and} \quad \mathbf{u}(t) = t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$$

find

a. $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$ and **b.** $D_t[\mathbf{u}(t) \times \mathbf{u}'(t)]$.

Solution

a. Because $\mathbf{r}'(t) = -\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{k}$ and $\mathbf{u}'(t) = 2t\mathbf{i} - 2\mathbf{j}$, you have

$$\begin{aligned} D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t) \\ &= \left(\frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k}\right) \cdot (2t\mathbf{i} - 2\mathbf{j}) \\ &\quad + \left(-\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{k}\right) \cdot (t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}) \\ &= 2 + 2 + (-1) + \frac{1}{t} \\ &= 3 + \frac{1}{t}. \end{aligned}$$

b. Because $\mathbf{u}'(t) = 2t\mathbf{i} - 2\mathbf{j}$ and $\mathbf{u}''(t) = 2\mathbf{i}$, you have

$$\begin{aligned} D_t[\mathbf{u}(t) \times \mathbf{u}'(t)] &= [\mathbf{u}(t) \times \mathbf{u}''(t)] + [\mathbf{u}'(t) \times \mathbf{u}'(t)] \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & -2t & 1 \\ 2 & 0 & 0 \end{vmatrix} + \mathbf{0} \\ &= \begin{vmatrix} -2t & 1 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} t^2 & 1 \\ 2 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} t^2 & -2t \\ 2 & 0 \end{vmatrix}\mathbf{k} \\ &= 0\mathbf{i} - (-2)\mathbf{j} + 4t\mathbf{k} \\ &= 2\mathbf{j} + 4t\mathbf{k}. \end{aligned}$$

■

NOTE Try reworking parts (a) and (b) in Example 4 by first forming the dot and cross products and then differentiating to see that you obtain the same results. ■

Integration of Vector-Valued Functions

The following definition is a rational consequence of the definition of the derivative of a vector-valued function.

DEFINITION OF INTEGRATION OF VECTOR-VALUED FUNCTIONS

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are continuous on $[a, b]$, then the **indefinite integral (antiderivative)** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} \qquad \text{Plane}$$

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j}.$$

2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are continuous on $[a, b]$, then the **indefinite integral (antiderivative)** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} + \left[\int h(t)\,dt\right]\mathbf{k} \qquad \text{Space}$$

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}.$$

The antiderivative of a vector-valued function is a family of vector-valued functions all differing by a constant vector $\mathbf{C}$. For instance, if $\mathbf{r}(t)$ is a three-dimensional vector-valued function, then for the indefinite integral $\int \mathbf{r}(t)\,dt$, you obtain three constants of integration

$$\int f(t)\,dt = F(t) + C_1, \qquad \int g(t)\,dt = G(t) + C_2, \qquad \int h(t)\,dt = H(t) + C_3$$

where $F'(t) = f(t)$, $G'(t) = g(t)$, and $H'(t) = h(t)$. These three *scalar* constants produce one *vector* constant of integration,

$$\begin{aligned}\int \mathbf{r}(t)\,dt &= [F(t) + C_1]\mathbf{i} + [G(t) + C_2]\mathbf{j} + [H(t) + C_3]\mathbf{k}\\ &= [F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k}] + [C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}]\\ &= \mathbf{R}(t) + \mathbf{C}\end{aligned}$$

where $\mathbf{R}'(t) = \mathbf{r}(t)$.

EXAMPLE 5 Integrating a Vector-Valued Function

Find the indefinite integral

$$\int (t\,\mathbf{i} + 3\mathbf{j})\,dt.$$

Solution Integrating on a component-by-component basis produces

$$\int (t\,\mathbf{i} + 3\mathbf{j})\,dt = \frac{t^2}{2}\mathbf{i} + 3t\mathbf{j} + \mathbf{C}.$$ ■

Example 6 shows how to evaluate the definite integral of a vector-valued function.

EXAMPLE 6 Definite Integral of a Vector-Valued Function

Evaluate the integral

$$\int_0^1 \mathbf{r}(t)\,dt = \int_0^1 \left(\sqrt[3]{t}\,\mathbf{i} + \frac{1}{t+1}\mathbf{j} + e^{-t}\mathbf{k}\right) dt.$$

Solution

$$\begin{aligned}\int_0^1 \mathbf{r}(t)\,dt &= \left(\int_0^1 t^{1/3}\,dt\right)\mathbf{i} + \left(\int_0^1 \frac{1}{t+1}\,dt\right)\mathbf{j} + \left(\int_0^1 e^{-t}\,dt\right)\mathbf{k}\\ &= \left[\left(\frac{3}{4}\right)t^{4/3}\right]_0^1 \mathbf{i} + \Big[\ln|t+1|\Big]_0^1 \mathbf{j} + \Big[-e^{-t}\Big]_0^1 \mathbf{k}\\ &= \frac{3}{4}\mathbf{i} + (\ln 2)\mathbf{j} + \left(1 - \frac{1}{e}\right)\mathbf{k}\end{aligned}$$

■

As with real-valued functions, you can narrow the family of antiderivatives of a vector-valued function **r**′ down to a single antiderivative by imposing an initial condition on the vector-valued function **r**. This is demonstrated in the next example.

EXAMPLE 7 The Antiderivative of a Vector-Valued Function

Find the antiderivative of

$$\mathbf{r}'(t) = \cos 2t\mathbf{i} - 2\sin t\mathbf{j} + \frac{1}{1+t^2}\mathbf{k}$$

that satisfies the initial condition $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Solution

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{r}'(t)\,dt\\ &= \left(\int \cos 2t\,dt\right)\mathbf{i} + \left(\int -2\sin t\,dt\right)\mathbf{j} + \left(\int \frac{1}{1+t^2}\,dt\right)\mathbf{k}\\ &= \left(\frac{1}{2}\sin 2t + C_1\right)\mathbf{i} + (2\cos t + C_2)\mathbf{j} + (\arctan t + C_3)\mathbf{k}\end{aligned}$$

Letting $t = 0$ and using the fact that $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, you have

$$\begin{aligned}\mathbf{r}(0) &= (0 + C_1)\mathbf{i} + (2 + C_2)\mathbf{j} + (0 + C_3)\mathbf{k}\\ &= 3\mathbf{i} + (-2)\mathbf{j} + \mathbf{k}.\end{aligned}$$

Equating corresponding components produces

$$C_1 = 3, \qquad 2 + C_2 = -2, \qquad \text{and} \qquad C_3 = 1.$$

So, the antiderivative that satisfies the given initial condition is

$$\mathbf{r}(t) = \left(\frac{1}{2}\sin 2t + 3\right)\mathbf{i} + (2\cos t - 4)\mathbf{j} + (\arctan t + 1)\mathbf{k}.$$

■

12.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, sketch the plane curve represented by the vector-valued function, and sketch the vectors $\mathbf{r}(t_0)$ and $\mathbf{r}'(t_0)$ for the given value of t_0. Position the vectors such that the initial point of $\mathbf{r}(t_0)$ is at the origin and the initial point of $\mathbf{r}'(t_0)$ is at the terminal point of $\mathbf{r}(t_0)$. What is the relationship between $\mathbf{r}'(t_0)$ and the curve?

1. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}, \quad t_0 = 2$

2. $\mathbf{r}(t) = t\mathbf{i} + (t^2 - 1)\mathbf{j}, \quad t_0 = 1$

3. $\mathbf{r}(t) = t^2\mathbf{i} + \dfrac{1}{t}\mathbf{j}, \quad t_0 = 2$

4. $\mathbf{r}(t) = (1 + t)\mathbf{i} + t^3\mathbf{j}, \quad t_0 = 1$

5. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad t_0 = \dfrac{\pi}{2}$

6. $\mathbf{r}(t) = 3\sin \mathrm{ti} + 4\cos t\mathbf{j}, \quad t_0 = \dfrac{\pi}{2}$

7. $\mathbf{r}(t) = \langle e^t, e^{2t}\rangle, \quad t_0 = 0$

8. $\mathbf{r}(t) = \langle e^{-t}, e^t\rangle, \quad t_0 = 0$

In Exercises 9 and 10, (a) sketch the space curve represented by the vector-valued function, and (b) sketch the vectors $\mathbf{r}(t_0)$ and $\mathbf{r}'(t_0)$ for the given value of t_0.

9. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, \quad t_0 = \dfrac{3\pi}{2}$

10. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k}, \quad t_0 = 2$

In Exercises 11–22, find $\mathbf{r}'(t)$.

11. $\mathbf{r}(t) = t^3\mathbf{i} - 3t\mathbf{j}$

12. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (1 - t^3)\mathbf{j}$

13. $\mathbf{r}(t) = \langle 2\cos t, 5\sin t\rangle$

14. $\mathbf{r}(t) = \langle t\cos t, -2\sin t\rangle$

15. $\mathbf{r}(t) = 6t\mathbf{i} - 7t^2\mathbf{j} + t^3\mathbf{k}$

16. $\mathbf{r}(t) = \dfrac{1}{t}\mathbf{i} + 16t\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

17. $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j} + \mathbf{k}$

18. $\mathbf{r}(t) = 4\sqrt{t}\,\mathbf{i} + t^2\sqrt{t}\,\mathbf{j} + \ln t^2\mathbf{k}$

19. $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j} + 5te^t\mathbf{k}$

20. $\mathbf{r}(t) = \langle t^3, \cos 3t, \sin 3t\rangle$

21. $\mathbf{r}(t) = \langle t\sin t, t\cos t, t\rangle$

22. $\mathbf{r}(t) = \langle \arcsin t, \arccos t, 0\rangle$

In Exercises 23–30, find (a) $\mathbf{r}'(t)$, (b) $\mathbf{r}''(t)$, and (c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

23. $\mathbf{r}(t) = t^3\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$

24. $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$

25. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$

26. $\mathbf{r}(t) = 8\cos t\mathbf{i} + 3\sin t\mathbf{j}$

27. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - t\mathbf{j} + \frac{1}{6}t^3\mathbf{k}$

28. $\mathbf{r}(t) = t\mathbf{i} + (2t + 3)\mathbf{j} + (3t - 5)\mathbf{k}$

29. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$

30. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t\rangle$

In Exercises 31 and 32, a vector-valued function and its graph are given. The graph also shows the unit vectors $\mathbf{r}'(t_0)/\|\mathbf{r}'(t_0)\|$ and $\mathbf{r}''(t_0)/\|\mathbf{r}''(t_0)\|$. Find these two unit vectors and identify them on the graph.

31. $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k}, \quad t_0 = -\frac{1}{4}$

32. $\mathbf{r}(t) = \frac{3}{2}t\mathbf{i} + t^2\mathbf{j} + e^{-t}\mathbf{k}, \quad t_0 = \frac{1}{4}$

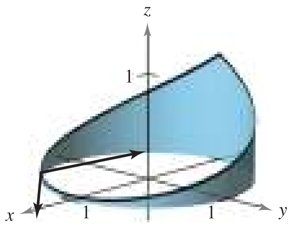


Figure for 31

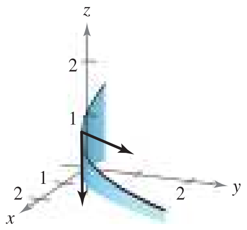


Figure for 32

In Exercises 33–42, find the open interval(s) on which the curve given by the vector-valued function is smooth.

33. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

34. $\mathbf{r}(t) = \dfrac{1}{t - 1}\mathbf{i} + 3t\mathbf{j}$

35. $\mathbf{r}(\theta) = 2\cos^3\theta\mathbf{i} + 3\sin^3\theta\mathbf{j}$

36. $\mathbf{r}(\theta) = (\theta + \sin\theta)\mathbf{i} + (1 - \cos\theta)\mathbf{j}$

37. $\mathbf{r}(\theta) = (\theta - 2\sin\theta)\mathbf{i} + (1 - 2\cos\theta)\mathbf{j}$

38. $\mathbf{r}(t) = \dfrac{2t}{8 + t^3}\mathbf{i} + \dfrac{2t^2}{8 + t^3}\mathbf{j}$

39. $\mathbf{r}(t) = (t - 1)\mathbf{i} + \dfrac{1}{t}\mathbf{j} - t^2\mathbf{k}$

40. $\mathbf{r}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + 3t\mathbf{k}$

41. $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$

42. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (t^2 - 1)\mathbf{j} + \frac{1}{4}t\mathbf{k}$

In Exercises 43 and 44, use the properties of the derivative to find the following.

(a) $\mathbf{r}'(t)$ **(b)** $\mathbf{r}''(t)$ **(c)** $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$

(d) $D_t[3\mathbf{r}(t) - \mathbf{u}(t)]$ **(e)** $D_t[\mathbf{r}(t) \times \mathbf{u}(t)]$ **(f)** $D_t[\|\mathbf{r}(t)\|], \quad t > 0$

43. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}, \quad \mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

44. $\mathbf{r}(t) = t\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k},$

$\mathbf{u}(t) = \dfrac{1}{t}\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

In Exercises 45 and 46, find (a) $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$ and (b) $D_t[\mathbf{r}(t) \times \mathbf{u}(t)]$ in two different ways.

(i) Find the product first, then differentiate.

(ii) Apply the properties of Theorem 12.2.

45. $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}, \quad \mathbf{u}(t) = t^4\mathbf{k}$

46. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \quad \mathbf{u}(t) = \mathbf{j} + t\mathbf{k}$

In Exercises 47 and 48, find the angle θ between $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ as a function of t. Use a graphing utility to graph $\theta(t)$. Use the graph to find any extrema of the function. Find any values of t at which the vectors are orthogonal.

47. $\mathbf{r}(t) = 3\sin t\mathbf{i} + 4\cos t\mathbf{j}$

48. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$

In Exercises 49–52, use the definition of the derivative to find $\mathbf{r}'(t)$.

49. $\mathbf{r}(t) = (3t + 2)\mathbf{i} + (1 - t^2)\mathbf{j}$

50. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \dfrac{3}{t}\mathbf{j} - 2t\mathbf{k}$

51. $\mathbf{r}(t) = \langle t^2, 0, 2t\rangle$

52. $\mathbf{r}(t) = \langle 0, \sin t, 4t\rangle$

In Exercises 53–60, find the indefinite integral.

53. $\displaystyle\int (2t\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt$

54. $\displaystyle\int \left(4t^3\,\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\,\mathbf{k}\right) dt$

55. $\displaystyle\int \left(\frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\,\mathbf{k}\right) dt$

56. $\displaystyle\int \left(\ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right) dt$

57. $\displaystyle\int \left[(2t - 1)\mathbf{i} + 4t^3\mathbf{j} + 3\sqrt{t}\,\mathbf{k}\right] dt$

58. $\displaystyle\int (e^t\,\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k})\,dt$

59. $\displaystyle\int \left(\sec^2 t\mathbf{i} + \frac{1}{1 + t^2}\mathbf{j}\right) dt$

60. $\displaystyle\int (e^{-t}\sin t\mathbf{i} + e^{-t}\cos t\mathbf{j})\,dt$

In Exercises 61–66, evaluate the definite integral.

61. $\displaystyle\int_0^1 (8t\mathbf{i} + t\mathbf{j} - \mathbf{k})\,dt$

62. $\displaystyle\int_{-1}^1 \left(t\mathbf{i} + t^3\mathbf{j} + \sqrt[3]{t}\,\mathbf{k}\right) dt$

63. $\displaystyle\int_0^{\pi/2} [(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + \mathbf{k}]\,dt$

64. $\displaystyle\int_0^{\pi/4} [(\sec t\tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2\sin t\cos t)\mathbf{k}]\,dt$

65. $\displaystyle\int_0^2 (t\mathbf{i} + e^t\,\mathbf{j} - te^t\mathbf{k})\,dt$

66. $\displaystyle\int_0^3 \|t\mathbf{i} + t^2\mathbf{j}\|\,dt$

In Exercises 67–72, find $\mathbf{r}(t)$ for the given conditions.

67. $\mathbf{r}'(t) = 4e^{2t}\mathbf{i} + 3e^t\mathbf{j}, \quad \mathbf{r}(0) = 2\mathbf{i}$

68. $\mathbf{r}'(t) = 3t^2\mathbf{j} + 6\sqrt{t}\,\mathbf{k}, \quad \mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$

69. $\mathbf{r}''(t) = -32\mathbf{j}, \quad \mathbf{r}'(0) = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}, \quad \mathbf{r}(0) = \mathbf{0}$

70. $\mathbf{r}''(t) = -4\cos t\mathbf{j} - 3\sin t\mathbf{k}, \quad \mathbf{r}'(0) = 3\mathbf{k}, \quad \mathbf{r}(0) = 4\mathbf{j}$

71. $\mathbf{r}'(t) = te^{-t^2}\mathbf{i} - e^{-t}\,\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(0) = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k}$

72. $\mathbf{r}'(t) = \dfrac{1}{1 + t^2}\mathbf{i} + \dfrac{1}{t^2}\mathbf{j} + \dfrac{1}{t}\mathbf{k}, \quad \mathbf{r}(1) = 2\mathbf{i}$

WRITING ABOUT CONCEPTS

73. State the definition of the derivative of a vector-valued function. Describe how to find the derivative of a vector-valued function and give its geometric interpretation.

74. How do you find the integral of a vector-valued function?

75. The three components of the derivative of the vector-valued function $\mathbf{u}$ are positive at $t = t_0$. Describe the behavior of $\mathbf{u}$ at $t = t_0$.

76. The z-component of the derivative of the vector-valued function $\mathbf{u}$ is 0 for t in the domain of the function. What does this information imply about the graph of $\mathbf{u}$?

In Exercises 77–84, prove the property. In each case, assume $\mathbf{r}$, $\mathbf{u}$, and $\mathbf{v}$ are differentiable vector-valued functions of t in space, w is a differentiable real-valued function of t, and c is a scalar.

77. $D_t[c\mathbf{r}(t)] = c\mathbf{r}'(t)$

78. $D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$

79. $D_t[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$

80. $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

81. $D_t[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$

82. $D_t[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$

83. $D_t\{\mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)]\} = \mathbf{r}'(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] +$
$\mathbf{r}(t) \cdot [\mathbf{u}'(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}'(t)]$

84. If $\mathbf{r}(t) \cdot \mathbf{r}(t)$ is a constant, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

85. ***Particle Motion*** A particle moves in the xy-plane along the curve represented by the vector-valued function $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$.

(a) Use a graphing utility to graph $\mathbf{r}$. Describe the curve.

(b) Find the minimum and maximum values of $\|\mathbf{r}'\|$ and $\|\mathbf{r}''\|$.

86. ***Particle Motion*** A particle moves in the yz-plane along the curve represented by the vector-valued function $\mathbf{r}(t) = (2\cos t)\mathbf{j} + (3\sin t)\mathbf{k}$.

(a) Describe the curve.

(b) Find the minimum and maximum values of $\|\mathbf{r}'\|$ and $\|\mathbf{r}''\|$.

87. Consider the vector-valued function

$$\mathbf{r}(t) = (e^t\sin t)\mathbf{i} + (e^t\cos t)\mathbf{j}.$$

Show that $\mathbf{r}(t)$ and $\mathbf{r}''(t)$ are always perpendicular to each other.

CAPSTONE

88. ***Investigation*** Consider the vector-valued function $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$.

(a) Sketch the graph of $\mathbf{r}(t)$. Use a graphing utility to verify your graph.

(b) Sketch the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.25)$, and $\mathbf{r}(1.25) - \mathbf{r}(1)$ on the graph in part (a).

(c) Compare the vector $\mathbf{r}'(1)$ with the vector

$$\frac{\mathbf{r}(1.25) - \mathbf{r}(1)}{1.25 - 1}.$$

True or False? **In Exercises 89–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

89. If a particle moves along a sphere centered at the origin, then its derivative vector is always tangent to the sphere.

90. The definite integral of a vector-valued function is a real number.

91. $\dfrac{d}{dt}[\|\mathbf{r}(t)\|] = \|\mathbf{r}'(t)\|$

92. If $\mathbf{r}$ and $\mathbf{u}$ are differentiable vector-valued functions of t, then $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}'(t) \cdot \mathbf{u}'(t)$.

12.3 Velocity and Acceleration

- Describe the velocity and acceleration associated with a vector-valued function.
- Use a vector-valued function to analyze projectile motion.

Velocity and Acceleration

You are now ready to combine your study of parametric equations, curves, vectors, and vector-valued functions to form a model for motion along a curve. You will begin by looking at the motion of an object in the plane. (The motion of an object in space can be developed similarly.)

As an object moves along a curve in the plane, the coordinates x and y of its center of mass are each functions of time t. Rather than using the letters f and g to represent these two functions, it is convenient to write $x = x(t)$ and $y = y(t)$. So, the position vector $\mathbf{r}(t)$ takes the form

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}. \qquad \text{Position vector}$$

The beauty of this vector model for representing motion is that you can use the first and second derivatives of the vector-valued function $\mathbf{r}$ to find the object's velocity and acceleration. (Recall from the preceding chapter that velocity and acceleration are both vector quantities having magnitude and direction.) To find the velocity and acceleration vectors at a given time t, consider a point $Q(x(t + \Delta t), y(t + \Delta t))$ that is approaching the point $P(x(t), y(t))$ along the curve C given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, as shown in Figure 12.11. As $\Delta t \to 0$, the direction of the vector $\overrightarrow{PQ}$ (denoted by $\Delta\mathbf{r}$) approaches the *direction of motion* at time t.

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$$

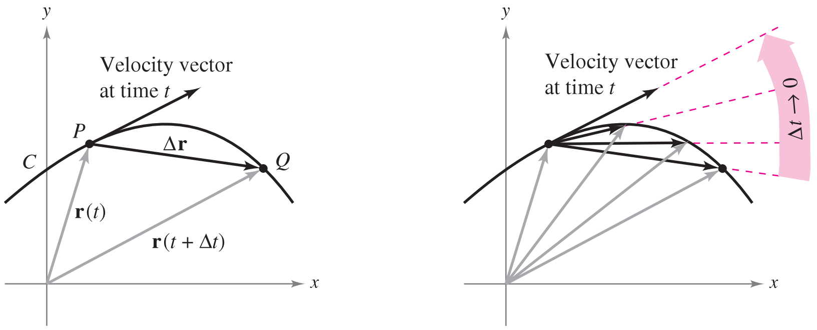
$$\frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

If this limit exists, it is defined as the **velocity vector** or **tangent vector** to the curve at point P. Note that this is the same limit used to define $\mathbf{r}'(t)$. So, the direction of $\mathbf{r}'(t)$ gives the direction of motion at time t. Moreover, the magnitude of the vector $\mathbf{r}'(t)$

$$\|\mathbf{r}'(t)\| = \|x'(t)\mathbf{i} + y'(t)\mathbf{j}\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

gives the **speed** of the object at time t. Similarly, you can use $\mathbf{r}''(t)$ to find acceleration, as indicated in the definitions at the top of the next page.



As $\Delta t \to 0$, $\dfrac{\Delta\mathbf{r}}{\Delta t}$ approaches the velocity vector.

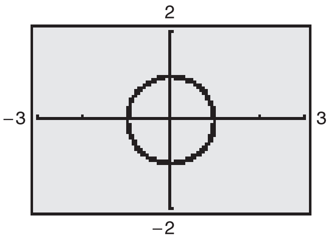
Figure 12.11

EXPLORATION

Exploring Velocity Consider the circle given by

$$\mathbf{r}(t) = (\cos \omega t)\mathbf{i} + (\sin \omega t)\mathbf{j}.$$

Use a graphing utility in *parametric* mode to graph this circle for several values of ω. How does ω affect the velocity of the terminal point as it traces out the curve? For a given value of ω, does the speed appear constant? Does the acceleration appear constant? Explain your reasoning.



DEFINITIONS OF VELOCITY AND ACCELERATION

If x and y are twice-differentiable functions of t, and $\mathbf{r}$ is a vector-valued function given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then the velocity vector, acceleration vector, and speed at time t are as follows.

$$\begin{aligned}\textbf{Velocity} &= \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}\\ \textbf{Acceleration} &= \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j}\\ \textbf{Speed} &= \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}\end{aligned}$$

For motion along a space curve, the definitions are similar. That is, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, you have

$$\begin{aligned}\textbf{Velocity} &= \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}\\ \textbf{Acceleration} &= \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}\\ \textbf{Speed} &= \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}.\end{aligned}$$

EXAMPLE 1 Finding Velocity and Acceleration Along a Plane Curve

NOTE In Example 1, note that the velocity and acceleration vectors are orthogonal at any point in time. This is characteristic of motion at a constant speed. (See Exercise 57.)

Find the velocity vector, speed, and acceleration vector of a particle that moves along the plane curve C described by

$$\mathbf{r}(t) = 2\sin\frac{t}{2}\mathbf{i} + 2\cos\frac{t}{2}\mathbf{j}.\qquad\text{Position vector}$$

Solution

The velocity vector is

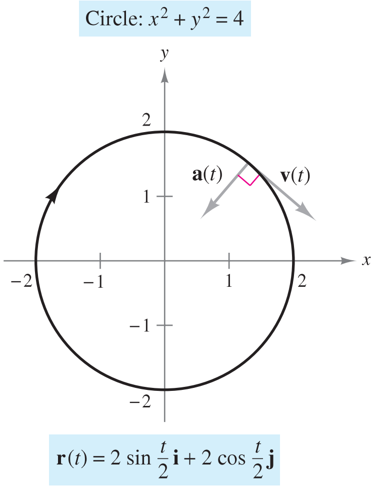
$$\mathbf{v}(t) = \mathbf{r}'(t) = \cos\frac{t}{2}\mathbf{i} - \sin\frac{t}{2}\mathbf{j}.\qquad\text{Velocity vector}$$

The speed (at any time) is

$$\|\mathbf{r}'(t)\| = \sqrt{\cos^2\frac{t}{2} + \sin^2\frac{t}{2}} = 1.\qquad\text{Speed}$$

The acceleration vector is

$$\mathbf{a}(t) = \mathbf{r}''(t) = -\frac{1}{2}\sin\frac{t}{2}\mathbf{i} - \frac{1}{2}\cos\frac{t}{2}\mathbf{j}.\qquad\text{Acceleration vector}$$ ■


Circle: $x^2 + y^2 = 4$
y
x
2
1
−1
−2
−2
−1
1
2
a(t)
v(t)
$\mathbf{r}(t) = 2\sin\frac{t}{2}\mathbf{i} + 2\cos\frac{t}{2}\mathbf{j}$

The particle moves around the circle at a constant speed.
Figure 12.12

The parametric equations for the curve in Example 1 are

$$x = 2\sin\frac{t}{2}\quad\text{and}\quad y = 2\cos\frac{t}{2}.$$

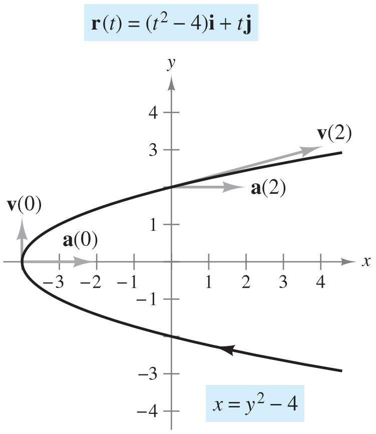
By eliminating the parameter t, you obtain the rectangular equation

$$x^2 + y^2 = 4.\qquad\text{Rectangular equation}$$

So, the curve is a circle of radius 2 centered at the origin, as shown in Figure 12.12. Because the velocity vector

$$\mathbf{v}(t) = \cos\frac{t}{2}\mathbf{i} - \sin\frac{t}{2}\mathbf{j}$$

has a constant magnitude but a changing direction as t increases, the particle moves around the circle at a constant speed.



At each point on the curve, the acceleration vector points to the right.
Figure 12.13

EXAMPLE 2 Sketching Velocity and Acceleration Vectors in the Plane

Sketch the path of an object moving along the plane curve given by

$$\mathbf{r}(t) = (t^2 - 4)\mathbf{i} + t\mathbf{j} \qquad \text{Position vector}$$

and find the velocity and acceleration vectors when $t = 0$ and $t = 2$.

Solution Using the parametric equations $x = t^2 - 4$ and $y = t$, you can determine that the curve is a parabola given by $x = y^2 - 4$, as shown in Figure 12.13. The velocity vector (at any time) is

$$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j} \qquad \text{Velocity vector}$$

and the acceleration vector (at any time) is

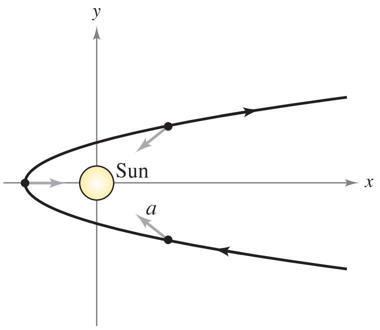
$$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i}. \qquad \text{Acceleration vector}$$

When $t = 0$, the velocity and acceleration vectors are given by

$$\mathbf{v}(0) = 2(0)\mathbf{i} + \mathbf{j} = \mathbf{j} \quad \text{and} \quad \mathbf{a}(0) = 2\mathbf{i}.$$

When $t = 2$, the velocity and acceleration vectors are given by

$$\mathbf{v}(2) = 2(2)\mathbf{i} + \mathbf{j} = 4\mathbf{i} + \mathbf{j} \quad \text{and} \quad \mathbf{a}(2) = 2\mathbf{i}.$$ ■



At each point in the comet's orbit, the acceleration vector points toward the sun.
Figure 12.14

For the object moving along the path shown in Figure 12.13, note that the acceleration vector is constant (it has a magnitude of 2 and points to the right). This implies that the speed of the object is decreasing as the object moves toward the vertex of the parabola, and the speed is increasing as the object moves away from the vertex of the parabola.

This type of motion is *not* characteristic of comets that travel on parabolic paths through our solar system. For such comets, the acceleration vector always points to the origin (the sun), which implies that the comet's speed increases as it approaches the vertex of the path and decreases as it moves away from the vertex. (See Figure 12.14.)

EXAMPLE 3 Sketching Velocity and Acceleration Vectors in Space

Sketch the path of an object moving along the space curve C given by

$$\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j} + 3t\mathbf{k}, \quad t \ge 0 \qquad \text{Position vector}$$

and find the velocity and acceleration vectors when $t = 1$.

Solution Using the parametric equations $x = t$ and $y = t^3$, you can determine that the path of the object lies on the cubic cylinder given by $y = x^3$. Moreover, because $z = 3t$, the object starts at $(0, 0, 0)$ and moves upward as t increases, as shown in Figure 12.15. Because $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j} + 3t\mathbf{k}$, you have

$$\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j} + 3\mathbf{k} \qquad \text{Velocity vector}$$

and

$$\mathbf{a}(t) = \mathbf{r}''(t) = 6t\mathbf{j}. \qquad \text{Acceleration vector}$$

When $t = 1$, the velocity and acceleration vectors are given by

$$\mathbf{v}(1) = \mathbf{r}'(1) = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \mathbf{a}(1) = \mathbf{r}''(1) = 6\mathbf{j}.$$ ■

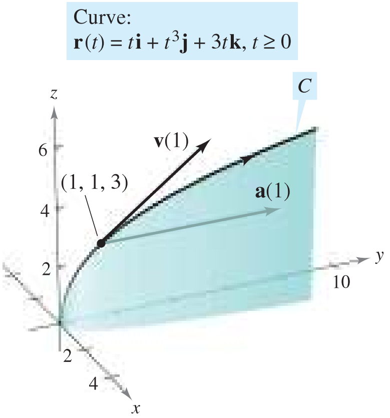


Figure 12.15

So far in this section, you have concentrated on finding the velocity and acceleration by differentiating the position function. Many practical applications involve the reverse problem—finding the position function for a given velocity or acceleration. This is demonstrated in the next example.

EXAMPLE 4 Finding a Position Function by Integration

An object starts from rest at the point $P(1, 2, 0)$ and moves with an acceleration of

$$\mathbf{a}(t) = \mathbf{j} + 2\mathbf{k} \qquad \text{Acceleration vector}$$

where $\|\mathbf{a}(t)\|$ is measured in feet per second per second. Find the location of the object after $t = 2$ seconds.

Solution From the description of the object's motion, you can deduce the following *initial conditions*. Because the object starts from rest, you have

$$\mathbf{v}(0) = \mathbf{0}.$$

Moreover, because the object starts at the point $(x, y, z) = (1, 2, 0)$, you have

$$\begin{aligned}\mathbf{r}(0) &= x(0)\mathbf{i} + y(0)\mathbf{j} + z(0)\mathbf{k}\\ &= 1\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}\\ &= \mathbf{i} + 2\mathbf{j}.\end{aligned}$$

To find the position function, you should integrate twice, each time using one of the initial conditions to solve for the constant of integration. The velocity vector is

$$\begin{aligned}\mathbf{v}(t) = \int \mathbf{a}(t)\,dt &= \int (\mathbf{j} + 2\mathbf{k})\,dt\\ &= t\mathbf{j} + 2t\mathbf{k} + \mathbf{C}\end{aligned}$$

where $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$. Letting $t = 0$ and applying the initial condition $\mathbf{v}(0) = \mathbf{0}$, you obtain

$$\mathbf{v}(0) = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k} = \mathbf{0} \quad \Rightarrow \quad C_1 = C_2 = C_3 = 0.$$

So, the *velocity* at any time t is

$$\mathbf{v}(t) = t\mathbf{j} + 2t\mathbf{k}. \qquad \text{Velocity vector}$$

Integrating once more produces

$$\begin{aligned}\mathbf{r}(t) = \int \mathbf{v}(t)\,dt &= \int (t\mathbf{j} + 2t\mathbf{k})\,dt\\ &= \frac{t^2}{2}\mathbf{j} + t^2\mathbf{k} + \mathbf{C}\end{aligned}$$

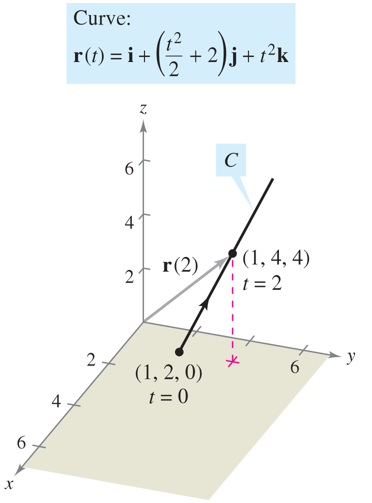
where $\mathbf{C} = C_4\mathbf{i} + C_5\mathbf{j} + C_6\mathbf{k}$. Letting $t = 0$ and applying the initial condition $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$, you have

$$\mathbf{r}(0) = C_4\mathbf{i} + C_5\mathbf{j} + C_6\mathbf{k} = \mathbf{i} + 2\mathbf{j} \quad \Rightarrow \quad C_4 = 1, C_5 = 2, C_6 = 0.$$

So, the *position* vector is

$$\mathbf{r}(t) = \mathbf{i} + \left(\frac{t^2}{2} + 2\right)\mathbf{j} + t^2\mathbf{k}. \qquad \text{Position vector}$$

The location of the object after $t = 2$ seconds is given by $\mathbf{r}(2) = \mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$, as shown in Figure 12.16. ■


Curve:
$\mathbf{r}(t) = \mathbf{i} + \left(\frac{t^2}{2} + 2\right)\mathbf{j} + t^2\mathbf{k}$
z
6
4
2
C
r(2)
(1, 4, 4)
t = 2
2
4
6
x
(1, 2, 0)
t = 0
6
y

The object takes 2 seconds to move from point (1, 2, 0) to point (1, 4, 4) along C.
Figure 12.16

Projectile Motion

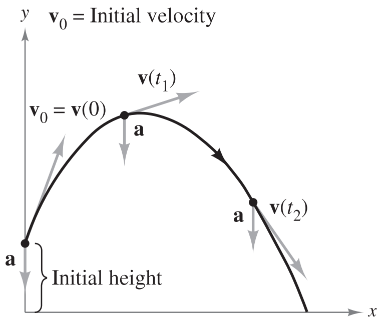

y
$\mathbf{v}_0$ = Initial velocity
$\mathbf{v}(t_1)$
$\mathbf{v}_0 = \mathbf{v}(0)$
a
a
$\mathbf{v}(t_2)$
a
Initial height
x

Figure 12.17

You now have the machinery to derive the parametric equations for the path of a projectile. Assume that gravity is the only force acting on the projectile after it is launched. So, the motion occurs in a vertical plane, which can be represented by the xy-coordinate system with the origin as a point on Earth's surface, as shown in Figure 12.17. For a projectile of mass m, the force due to gravity is

$$\mathbf{F} = -mg\mathbf{j} \qquad \text{Force due to gravity}$$

where the acceleration due to gravity is $g = 32$ feet per second per second, or 9.81 meters per second per second. By **Newton's Second Law of Motion,** this same force produces an acceleration $\mathbf{a} = \mathbf{a}(t)$, and satisfies the equation $\mathbf{F} = m\mathbf{a}$. Consequently, the acceleration of the projectile is given by $m\mathbf{a} = -mg\mathbf{j}$, which implies that

$$\mathbf{a} = -g\mathbf{j}. \qquad \text{Acceleration of projectile}$$

EXAMPLE 5 Derivation of the Position Function for a Projectile

A projectile of mass m is launched from an initial position $\mathbf{r}_0$ with an initial velocity $\mathbf{v}_0$. Find its position vector as a function of time.

Solution Begin with the acceleration $\mathbf{a}(t) = -g\mathbf{j}$ and integrate twice.

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int -g\mathbf{j}\,dt = -gt\mathbf{j} + \mathbf{C}_1$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (-gt\mathbf{j} + \mathbf{C}_1)\,dt = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{C}_1 t + \mathbf{C}_2$$

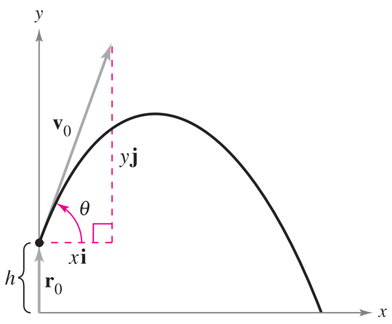
You can use the facts that $\mathbf{v}(0) = \mathbf{v}_0$ and $\mathbf{r}(0) = \mathbf{r}_0$ to solve for the constant vectors $\mathbf{C}_1$ and $\mathbf{C}_2$. Doing this produces $\mathbf{C}_1 = \mathbf{v}_0$ and $\mathbf{C}_2 = \mathbf{r}_0$. Therefore, the position vector is

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{r}_0. \qquad \text{Position vector}$$

■

$\|\mathbf{v}_0\| = v_0$ = initial speed

$\|\mathbf{r}_0\| = h$ = initial height



$x = \|\mathbf{v}_0\| \cos\theta$

$y = \|\mathbf{v}_0\| \sin\theta$

Figure 12.18

In many projectile problems, the constant vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ are not given explicitly. Often you are given the initial height h, the initial speed v_0, and the angle θ at which the projectile is launched, as shown in Figure 12.18. From the given height, you can deduce that $\mathbf{r}_0 = h\mathbf{j}$. Because the speed gives the magnitude of the initial velocity, it follows that $v_0 = \|\mathbf{v}_0\|$ and you can write

$$\begin{aligned}\mathbf{v}_0 &= x\mathbf{i} + y\mathbf{j}\\ &= (\|\mathbf{v}_0\|\cos\theta)\mathbf{i} + (\|\mathbf{v}_0\|\sin\theta)\mathbf{j}\\ &= v_0\cos\theta\mathbf{i} + v_0\sin\theta\mathbf{j}.\end{aligned}$$

So, the position vector can be written in the form

$$\boxed{\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{r}_0} \qquad \text{Position vector}$$

$$= -\frac{1}{2}gt^2\mathbf{j} + tv_0\cos\theta\mathbf{i} + tv_0\sin\theta\mathbf{j} + h\mathbf{j}$$

$$= (v_0\cos\theta)t\mathbf{i} + \left[h + (v_0\sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}.$$

THEOREM 12.3 POSITION FUNCTION FOR A PROJECTILE

Neglecting air resistance, the path of a projectile launched from an initial height h with initial speed v_0 and angle of elevation θ is described by the vector function

$$\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[h + (v_0 \sin \theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$$

where g is the acceleration due to gravity.

EXAMPLE 6 Describing the Path of a Baseball

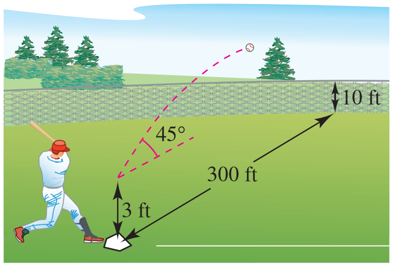


Figure 12.19

A baseball is hit 3 feet above ground level at 100 feet per second and at an angle of 45° with respect to the ground, as shown in Figure 12.19. Find the maximum height reached by the baseball. Will it clear a 10-foot-high fence located 300 feet from home plate?

Solution You are given $h = 3$, $v_0 = 100$, and $\theta = 45°$. So, using $g = 32$ feet per second per second produces

$$\begin{aligned}\mathbf{r}(t) &= \left(100 \cos \frac{\pi}{4}\right)t\mathbf{i} + \left[3 + \left(100 \sin \frac{\pi}{4}\right)t - 16t^2\right]\mathbf{j}\\ &= \left(50\sqrt{2}\,t\right)\mathbf{i} + \left(3 + 50\sqrt{2}\,t - 16t^2\right)\mathbf{j}\\ \mathbf{v}(t) &= \mathbf{r}'(t) = 50\sqrt{2}\,\mathbf{i} + \left(50\sqrt{2} - 32t\right)\mathbf{j}.\end{aligned}$$

The maximum height occurs when

$$y'(t) = 50\sqrt{2} - 32t = 0$$

which implies that

$$\begin{aligned}t &= \frac{25\sqrt{2}}{16}\\ &\approx 2.21 \text{ seconds.}\end{aligned}$$

So, the maximum height reached by the ball is

$$\begin{aligned}y &= 3 + 50\sqrt{2}\left(\frac{25\sqrt{2}}{16}\right) - 16\left(\frac{25\sqrt{2}}{16}\right)^2\\ &= \frac{649}{8}\\ &\approx 81 \text{ feet.}\end{aligned}$$

Maximum height when $t \approx 2.21$ seconds

The ball is 300 feet from where it was hit when

$$300 = x(t) = 50\sqrt{2}\,t.$$

Solving this equation for t produces $t = 3\sqrt{2} \approx 4.24$ seconds. At this time, the height of the ball is

$$\begin{aligned}y &= 3 + 50\sqrt{2}\left(3\sqrt{2}\right) - 16\left(3\sqrt{2}\right)^2\\ &= 303 - 288\\ &= 15 \text{ feet.}\end{aligned}$$

Height when $t \approx 4.24$ seconds

Therefore, the ball clears the 10-foot fence for a home run. ■

12.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, the position vector r describes the path of an object moving in the xy-plane. Sketch a graph of the path and sketch the velocity and acceleration vectors at the given point.

	Position Function	*Point*
1.	$\mathbf{r}(t) = 3t\mathbf{i} + (t-1)\mathbf{j}$	$(3, 0)$
2.	$\mathbf{r}(t) = (6-t)\mathbf{i} + t\mathbf{j}$	$(3, 3)$
3.	$\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$	$(4, 2)$
4.	$\mathbf{r}(t) = t\mathbf{i} + (-t^2+4)\mathbf{j}$	$(1, 3)$
5.	$\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$	$(1, 1)$
6.	$\mathbf{r}(t) = \left(\frac{1}{4}t^3 + 1\right)\mathbf{i} + t\mathbf{j}$	$(3, 2)$
7.	$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$	$\left(\sqrt{2}, \sqrt{2}\right)$
8.	$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$	$(3, 0)$
9.	$\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$	$(\pi, 2)$
10.	$\mathbf{r}(t) = \langle e^{-t}, e^{t}\rangle$	$(1, 1)$

In Exercises 11–20, the position vector r describes the path of an object moving in space. Find the velocity, speed, and acceleration of the object.

11. $\mathbf{r}(t) = t\mathbf{i} + 5t\mathbf{j} + 3t\mathbf{k}$

12. $\mathbf{r}(t) = 4t\mathbf{i} + 4t\mathbf{j} + 2t\mathbf{k}$

13. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

14. $\mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + \dfrac{1}{4}t^2\mathbf{k}$

15. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{9 - t^2}\,\mathbf{k}$

16. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t^{3/2}\mathbf{k}$

17. $\mathbf{r}(t) = \langle 4t, 3\cos t, 3\sin t\rangle$

18. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t^2\rangle$

19. $\mathbf{r}(t) = \langle e^t\cos t, e^t\sin t, e^t\rangle$

20. $\mathbf{r}(t) = \left\langle \ln t, \dfrac{1}{t}, t^4\right\rangle$

Linear Approximation **In Exercises 21 and 22, the graph of the vector-valued function r(t) and a tangent vector to the graph at $t = t_0$ are given.**

(a) Find a set of parametric equations for the tangent line to the graph at $t = t_0$.

(b) Use the equations for the line to approximate $\mathbf{r}(t_0 + 0.1)$.

21. $\mathbf{r}(t) = \left\langle t, -t^2, \frac{1}{4}t^3\right\rangle,\quad t_0 = 1$

22. $\mathbf{r}(t) = \left\langle t, \sqrt{25 - t^2}, \sqrt{25 - t^2}\right\rangle,\quad t_0 = 3$

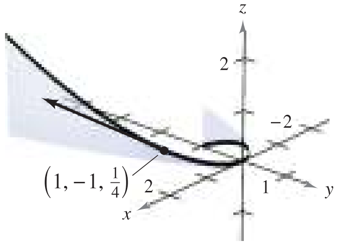


Figure for 21

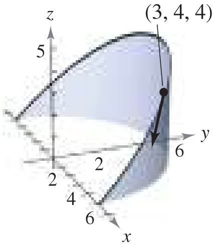


Figure for 22

In Exercises 23–28, use the given acceleration function to find the velocity and position vectors. Then find the position at time $t = 2$.

23. $\mathbf{a}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{v}(0) = \mathbf{0},\quad \mathbf{r}(0) = \mathbf{0}$

24. $\mathbf{a}(t) = 2\mathbf{i} + 3\mathbf{k}$

$\mathbf{v}(0) = 4\mathbf{j},\quad \mathbf{r}(0) = \mathbf{0}$

25. $\mathbf{a}(t) = t\mathbf{j} + t\mathbf{k}$

$\mathbf{v}(1) = 5\mathbf{j},\quad \mathbf{r}(1) = \mathbf{0}$

26. $\mathbf{a}(t) = -32\mathbf{k}$

$\mathbf{v}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k},\quad \mathbf{r}(0) = 5\mathbf{j} + 2\mathbf{k}$

27. $\mathbf{a}(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$

$\mathbf{v}(0) = \mathbf{j} + \mathbf{k},\quad \mathbf{r}(0) = \mathbf{i}$

28. $\mathbf{a}(t) = e^t\mathbf{i} - 8\mathbf{k}$

$\mathbf{v}(0) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k},\quad \mathbf{r}(0) = \mathbf{0}$

Projectile Motion **In Exercises 29–44, use the model for projectile motion, assuming there is no air resistance.**

29. Find the vector-valued function for the path of a projectile launched at a height of 10 feet above the ground with an initial velocity of 88 feet per second and at an angle of 30° above the horizontal. Use a graphing utility to graph the path of the projectile.

30. Determine the maximum height and range of a projectile fired at a height of 3 feet above the ground with an initial velocity of 900 feet per second and at an angle of 45° above the horizontal.

31. A baseball, hit 3 feet above the ground, leaves the bat at an angle of 45° and is caught by an outfielder 3 feet above the ground and 300 feet from home plate. What is the initial speed of the ball, and how high does it rise?

32. A baseball player at second base throws a ball 90 feet to the player at first base. The ball is released at a point 5 feet above the ground with an initial velocity of 50 miles per hour and at an angle of 15° above the horizontal. At what height does the player at first base catch the ball?

33. Eliminate the parameter t from the position function for the motion of a projectile to show that the rectangular equation is

$$y = -\frac{16\sec^2\theta}{v_0^2}x^2 + (\tan\theta)x + h.$$

34. The path of a ball is given by the rectangular equation

$$y = x - 0.005x^2.$$

Use the result of Exercise 33 to find the position function. Then find the speed and direction of the ball at the point at which it has traveled 60 feet horizontally.

35. ***Modeling Data*** After the path of a ball thrown by a baseball player is videotaped, it is analyzed on a television set with a grid covering the screen. The tape is paused three times and the positions of the ball are measured. The coordinates are approximately $(0, 6.0)$, $(15, 10.6)$, and $(30, 13.4)$. (The x-coordinate measures the horizontal distance from the player in feet and the y-coordinate measures the height in feet.)

(a) Use a graphing utility to find a quadratic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Determine the maximum height of the ball.

(d) Find the initial velocity of the ball and the angle at which it was thrown.

36. A baseball is hit from a height of 2.5 feet above the ground with an initial velocity of 140 feet per second and at an angle of 22° above the horizontal. Use a graphing utility to graph the path of the ball and determine whether it will clear a 10-foot-high fence located 375 feet from home plate.

37. The Rogers Centre in Toronto, Ontario has a center field fence that is 10 feet high and 400 feet from home plate. A ball is hit 3 feet above the ground and leaves the bat at a speed of 100 miles per hour.

(a) The ball leaves the bat at an angle of $\theta = \theta_0$ with the horizontal. Write the vector-valued function for the path of the ball.

(b) Use a graphing utility to graph the vector-valued function for $\theta_0 = 10°$, $\theta_0 = 15°$, $\theta_0 = 20°$, and $\theta_0 = 25°$. Use the graphs to approximate the minimum angle required for the hit to be a home run.

(c) Determine analytically the minimum angle required for the hit to be a home run.

38. The quarterback of a football team releases a pass at a height of 7 feet above the playing field, and the football is caught by a receiver 30 yards directly downfield at a height of 4 feet. The pass is released at an angle of 35° with the horizontal.

(a) Find the speed of the football when it is released.

(b) Find the maximum height of the football.

(c) Find the time the receiver has to reach the proper position after the quarterback releases the football.

39. A bale ejector consists of two variable-speed belts at the end of a baler. Its purpose is to toss bales into a trailing wagon. In loading the back of a wagon, a bale must be thrown to a position 8 feet above and 16 feet behind the ejector.

(a) Find the minimum initial speed of the bale and the corresponding angle at which it must be ejected from the baler.

(b) The ejector has a fixed angle of 45°. Find the initial speed required.

40. A bomber is flying at an altitude of 30,000 feet at a speed of 540 miles per hour (see figure). When should the bomb be released for it to hit the target? (Give your answer in terms of the angle of depression from the plane to the target.) What is the speed of the bomb at the time of impact?

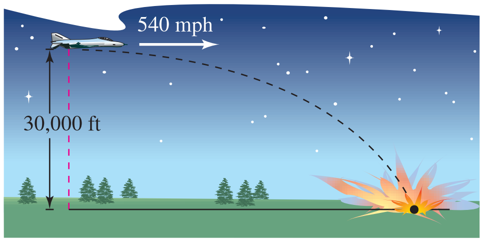


Figure for 40

41. A shot fired from a gun with a muzzle velocity of 1200 feet per second is to hit a target 3000 feet away. Determine the minimum angle of elevation of the gun.

42. A projectile is fired from ground level at an angle of 12° with the horizontal. The projectile is to have a range of 200 feet. Find the minimum initial velocity necessary.

43. Use a graphing utility to graph the paths of a projectile for the given values of θ and v_0. For each case, use the graph to approximate the maximum height and range of the projectile. (Assume that the projectile is launched from ground level.)

(a) $\theta = 10°$, $v_0 = 66$ ft/sec (b) $\theta = 10°$, $v_0 = 146$ ft/sec

(c) $\theta = 45°$, $v_0 = 66$ ft/sec (d) $\theta = 45°$, $v_0 = 146$ ft/sec

(e) $\theta = 60°$, $v_0 = 66$ ft/sec (f) $\theta = 60°$, $v_0 = 146$ ft/sec

44. Find the angles at which an object must be thrown to obtain (a) the maximum range and (b) the maximum height.

Projectile Motion **In Exercises 45 and 46, use the model for projectile motion, assuming there is no air resistance. $[a(t) = -9.8$ meters per second per second$]$**

45. Determine the maximum height and range of a projectile fired at a height of 1.5 meters above the ground with an initial velocity of 100 meters per second and at an angle of 30° above the horizontal.

46. A projectile is fired from ground level at an angle of 8° with the horizontal. The projectile is to have a range of 50 meters. Find the minimum initial velocity necessary.

Cycloidal Motion **In Exercises 47 and 48, consider the motion of a point (or particle) on the circumference of a rolling circle. As the circle rolls, it generates the cycloid $\mathbf{r}(t) = b(\omega t - \sin \omega t)\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$, where ω is the constant angular velocity of the circle and b is the radius of the circle.**

47. Find the velocity and acceleration vectors of the particle. Use the results to determine the times at which the speed of the particle will be (a) zero and (b) maximized.

48. Find the maximum speed of a point on the circumference of an automobile tire of radius 1 foot when the automobile is traveling at 60 miles per hour. Compare this speed with the speed of the automobile.

Circular Motion **In Exercises 49–52, consider a particle moving on a circular path of radius b described by $\mathbf{r}(t) = b \cos \omega t\,\mathbf{i} + b \sin \omega t\,\mathbf{j}$, where $\omega = du/dt$ is the constant angular velocity.**

49. Find the velocity vector and show that it is orthogonal to $\mathbf{r}(t)$.

50. (a) Show that the speed of the particle is $b\omega$.

(b) Use a graphing utility in *parametric* mode to graph the circle for $b = 6$. Try different values of ω. Does the graphing utility draw the circle faster for greater values of ω?

51. Find the acceleration vector and show that its direction is always toward the center of the circle.

52. Show that the magnitude of the acceleration vector is $b\omega^2$.

Circular Motion **In Exercises 53 and 54, use the results of Exercises 49–52.**

53. A stone weighing 1 pound is attached to a two-foot string and is whirled horizontally (see figure). The string will break under a force of 10 pounds. Find the maximum speed the stone can attain without breaking the string. $\left(\text{Use } \mathbf{F} = m\mathbf{a}, \text{ where } m = \frac{1}{32}.\right)$

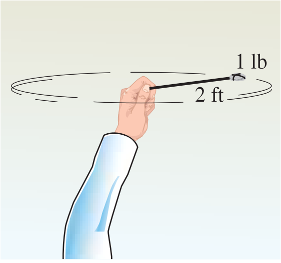


Figure for 53

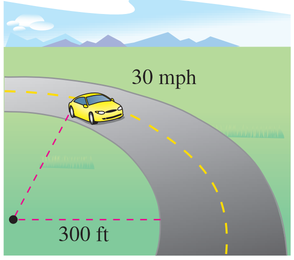


Figure for 54

54. A 3400-pound automobile is negotiating a circular interchange of radius 300 feet at 30 miles per hour (see figure). Assuming the roadway is level, find the force between the tires and the road such that the car stays on the circular path and does not skid. (Use $\mathbf{F} = m\mathbf{a}$, where $m = 3400/32$.) Find the angle at which the roadway should be banked so that no lateral frictional force is exerted on the tires of the automobile.

55. ***Shot-Put Throw*** The path of a shot thrown at an angle θ is

$$\mathbf{r}(t) = (v_0 \cos \theta)t\,\mathbf{i} + \left[h + (v_0 \sin \theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$$

where v_0 is the initial speed, h is the initial height, t is the time in seconds, and g is the acceleration due to gravity. Verify that the shot will remain in the air for a total of

$$t = \frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g} \text{ seconds}$$

and will travel a horizontal distance of

$$\frac{v_0^2 \cos \theta}{g}\left(\sin \theta + \sqrt{\sin^2 \theta + \frac{2gh}{v_0^2}}\right) \text{ feet.}$$

56. ***Shot-Put Throw*** A shot is thrown from a height of $h = 6$ feet with an initial speed of $v_0 = 45$ feet per second and at an angle of $\theta = 42.5°$ with the horizontal. Find the total time of travel and the total horizontal distance traveled.

57. Prove that if an object is traveling at a constant speed, its velocity and acceleration vectors are orthogonal.

58. Prove that an object moving in a straight line at a constant speed has an acceleration of 0.

59. ***Investigation*** A particle moves on an elliptical path given by the vector-valued function $\mathbf{r}(t) = 6 \cos t\,\mathbf{i} + 3 \sin t\,\mathbf{j}$.

(a) Find $\mathbf{v}(t)$, $\|\mathbf{v}(t)\|$, and $\mathbf{a}(t)$.

(b) Use a graphing utility to complete the table.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
Speed					

(c) Graph the elliptical path and the velocity and acceleration vectors at the values of t given in the table in part (b).

(d) Use the results of parts (b) and (c) to describe the geometric relationship between the velocity and acceleration vectors when the speed of the particle is increasing, and when it is decreasing.

CAPSTONE

60. Consider a particle moving on an elliptical path described by $\mathbf{r}(t) = a \cos \omega t\,\mathbf{i} + b \sin \omega t\,\mathbf{j}$, where $\omega = d\theta/dt$ is the constant angular velocity.

(a) Find the velocity vector. What is the speed of the particle?

(b) Find the acceleration vector and show that its direction is always toward the center of the ellipse.

WRITING ABOUT CONCEPTS

61. In your own words, explain the difference between the velocity of an object and its speed.

62. What is known about the speed of an object if the angle between the velocity and acceleration vectors is (a) acute and (b) obtuse?

63. Consider a particle that is moving on the path $\mathbf{r}_1(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$.

(a) Discuss any changes in the position, velocity, or acceleration of the particle if its position is given by the vector-valued function $\mathbf{r}_2(t) = \mathbf{r}_1(2t)$.

(b) Generalize the results for the position function $\mathbf{r}_3(t) = \mathbf{r}_1(\omega t)$.

64. When $t = 0$, an object is at the point $(0, 1)$ and has a velocity vector $\mathbf{v}(0) = -\mathbf{i}$. It moves with an acceleration of $\mathbf{a}(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j}$. Show that the path of the object is a circle.

True or False? **In Exercises 65–68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

65. The acceleration of an object is the derivative of the speed.

66. The velocity of an object is the derivative of the position.

67. The velocity vector points in the direction of motion.

68. If a particle moves along a straight line, then the velocity and acceleration vectors are orthogonal.

12.4 Tangent Vectors and Normal Vectors

- **Find a unit tangent vector at a point on a space curve.**
- **Find the tangential and normal components of acceleration.**

Tangent Vectors and Normal Vectors

In the preceding section, you learned that the velocity vector points in the direction of motion. This observation leads to the following definition, which applies to any smooth curve—not just to those for which the parameter represents time.

DEFINITION OF UNIT TANGENT VECTOR

Let C be a smooth curve represented by $\mathbf{r}$ on an open interval I. The **unit tangent vector** $\mathbf{T}(t)$ at t is defined as

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \quad \mathbf{r}'(t) \neq \mathbf{0}.$$

Recall that a curve is *smooth* on an interval if $\mathbf{r}'$ is continuous and nonzero on the interval. So, "smoothness" is sufficient to guarantee that a curve has a unit tangent vector.

EXAMPLE 1 Finding the Unit Tangent Vector

Find the unit tangent vector to the curve given by

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$$

when $t = 1$.

Solution The derivative of $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}. \qquad \text{Derivative of } \mathbf{r}(t)$$

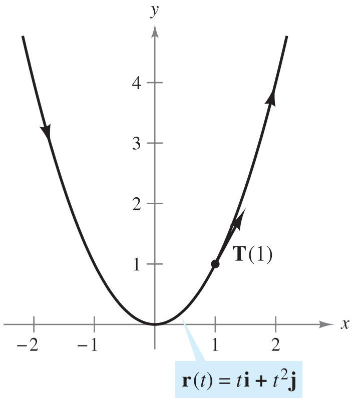
So, the unit tangent vector is

$$\begin{aligned} \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} && \text{Definition of } \mathbf{T}(t) \\ &= \frac{1}{\sqrt{1+4t^2}}(\mathbf{i} + 2t\mathbf{j}). && \text{Substitute for } \mathbf{r}'(t). \end{aligned}$$

When $t = 1$, the unit tangent vector is

$$\mathbf{T}(1) = \frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$$

as shown in Figure 12.20. ■



The direction of the unit tangent vector depends on the orientation of the curve.
Figure 12.20

NOTE In Example 1, note that the direction of the unit tangent vector depends on the orientation of the curve. For instance, if the parabola in Figure 12.20 were given by

$$\mathbf{r}(t) = -(t-2)\mathbf{i} + (t-2)^2\mathbf{j}$$

$\mathbf{T}(1)$ would still represent the unit tangent vector at the point $(1, 1)$, but it would point in the opposite direction. Try verifying this. ■

The **tangent line to a curve** at a point is the line that passes through the point and is parallel to the unit tangent vector. In Example 2, the unit tangent vector is used to find the tangent line at a point on a helix.

EXAMPLE 2 Finding the Tangent Line at a Point on a Curve

Find $\mathbf{T}(t)$ and then find a set of parametric equations for the tangent line to the helix given by

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\mathbf{k}$$

at the point $\left(\sqrt{2}, \sqrt{2}, \dfrac{\pi}{4}\right)$.

Solution The derivative of $\mathbf{r}(t)$ is $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \mathbf{k}$, which implies that $\|\mathbf{r}'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$. Therefore, the unit tangent vector is

$$\begin{aligned}\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\\ &= \frac{1}{\sqrt{5}}(-2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \mathbf{k}). \qquad \text{Unit tangent vector}\end{aligned}$$

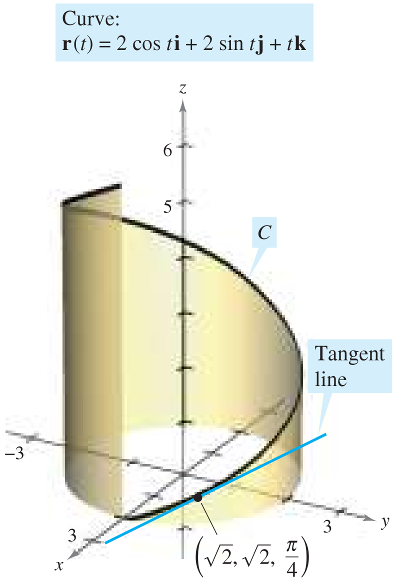
At the point $(\sqrt{2}, \sqrt{2}, \pi/4)$, $t = \pi/4$ and the unit tangent vector is

$$\begin{aligned}\mathbf{T}\left(\frac{\pi}{4}\right) &= \frac{1}{\sqrt{5}}\left(-2\frac{\sqrt{2}}{2}\mathbf{i} + 2\frac{\sqrt{2}}{2}\mathbf{j} + \mathbf{k}\right)\\ &= \frac{1}{\sqrt{5}}(-\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j} + \mathbf{k}).\end{aligned}$$

Using the direction numbers $a = -\sqrt{2}$, $b = \sqrt{2}$, and $c = 1$, and the point $(x_1, y_1, z_1) = (\sqrt{2}, \sqrt{2}, \pi/4)$, you can obtain the following parametric equations (given with parameter s).

$$\begin{aligned}x &= x_1 + as = \sqrt{2} - \sqrt{2}s\\ y &= y_1 + bs = \sqrt{2} + \sqrt{2}s\\ z &= z_1 + cs = \frac{\pi}{4} + s\end{aligned}$$

This tangent line is shown in Figure 12.21. ■


Curve:
$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\mathbf{k}$
z
6
5
C
Tangent line
−3
3
3
y
x
$\left(\sqrt{2}, \sqrt{2}, \frac{\pi}{4}\right)$

The tangent line to a curve at a point is determined by the unit tangent vector at the point.
Figure 12.21

In Example 2, there are infinitely many vectors that are orthogonal to the tangent vector $\mathbf{T}(t)$. One of these is the vector $\mathbf{T}'(t)$. This follows from Property 7 of Theorem 12.2. That is,

$$\mathbf{T}(t) \cdot \mathbf{T}(t) = \|\mathbf{T}(t)\|^2 = 1 \quad \Rightarrow \quad \mathbf{T}(t) \cdot \mathbf{T}'(t) = 0.$$

By normalizing the vector $\mathbf{T}'(t)$, you obtain a special vector called the **principal unit normal vector,** as indicated in the following definition.

DEFINITION OF PRINCIPAL UNIT NORMAL VECTOR

Let C be a smooth curve represented by $\mathbf{r}$ on an open interval I. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the **principal unit normal vector** at t is defined as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

EXAMPLE 3 Finding the Principal Unit Normal Vector

Find $\mathbf{N}(t)$ and $\mathbf{N}(1)$ for the curve represented by

$$\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j}.$$

Solution By differentiating, you obtain

$$\mathbf{r}'(t) = 3\mathbf{i} + 4t\mathbf{j} \qquad \text{and} \qquad \|\mathbf{r}'(t)\| = \sqrt{9 + 16t^2}$$

which implies that the unit tangent vector is

$$\begin{aligned}\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\\ &= \frac{1}{\sqrt{9 + 16t^2}}(3\mathbf{i} + 4t\mathbf{j}). \qquad \text{Unit tangent vector}\end{aligned}$$

Using Theorem 12.2, differentiate $\mathbf{T}(t)$ with respect to t to obtain

$$\begin{aligned}\mathbf{T}'(t) &= \frac{1}{\sqrt{9 + 16t^2}}(4\mathbf{j}) - \frac{16t}{(9 + 16t^2)^{3/2}}(3\mathbf{i} + 4t\mathbf{j})\\ &= \frac{12}{(9 + 16t^2)^{3/2}}(-4t\mathbf{i} + 3\mathbf{j})\end{aligned}$$

$$\|\mathbf{T}'(t)\| = 12\sqrt{\frac{9 + 16t^2}{(9 + 16t^2)^3}} = \frac{12}{9 + 16t^2}.$$

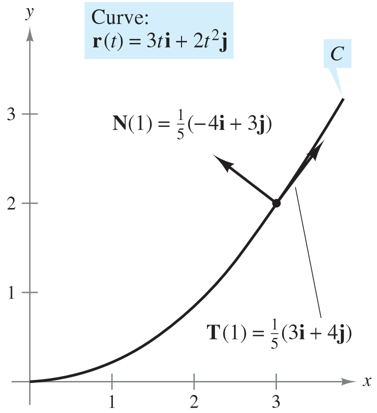
Therefore, the principal unit normal vector is

$$\begin{aligned}\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}\\ &= \frac{1}{\sqrt{9 + 16t^2}}(-4t\mathbf{i} + 3\mathbf{j}). \qquad \text{Principal unit normal vector}\end{aligned}$$

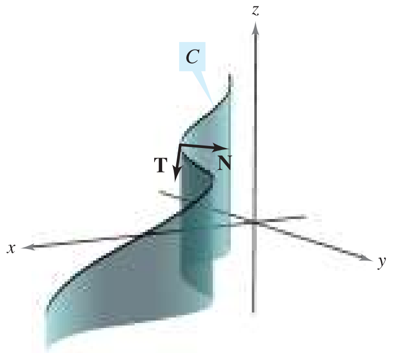
When $t = 1$, the principal unit normal vector is

$$\mathbf{N}(1) = \frac{1}{5}(-4\mathbf{i} + 3\mathbf{j})$$

as shown in Figure 12.22. ■



The principal unit normal vector points toward the concave side of the curve.
Figure 12.22



At any point on a curve, a unit normal vector is orthogonal to the unit tangent vector. The *principal* unit normal vector points in the direction in which the curve is turning.
Figure 12.23

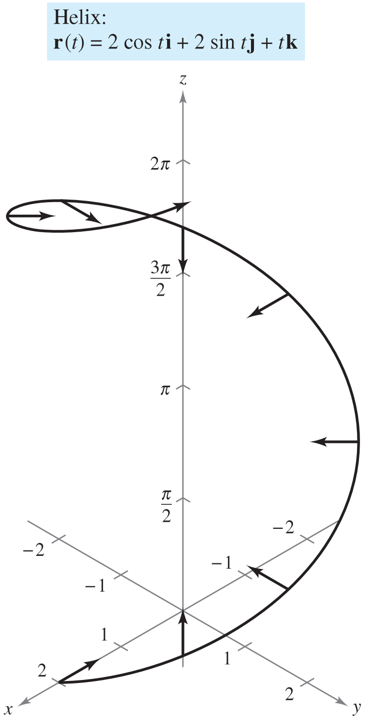
The principal unit normal vector can be difficult to evaluate algebraically. For plane curves, you can simplify the algebra by finding

$$\mathbf{T}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad \text{Unit tangent vector}$$

and observing that $\mathbf{N}(t)$ must be either

$$\mathbf{N}_1(t) = y(t)\mathbf{i} - x(t)\mathbf{j} \qquad \text{or} \qquad \mathbf{N}_2(t) = -y(t)\mathbf{i} + x(t)\mathbf{j}.$$

Because $\sqrt{[x(t)]^2 + [y(t)]^2} = 1$, it follows that both $\mathbf{N}_1(t)$ and $\mathbf{N}_2(t)$ are unit normal vectors. The *principal* unit normal vector $\mathbf{N}$ is the one that points toward the concave side of the curve, as shown in Figure 12.22 (see Exercise 94). This also holds for curves in space. That is, for an object moving along a curve C in space, the vector $\mathbf{T}(t)$ points in the direction the object is moving, whereas the vector $\mathbf{N}(t)$ is orthogonal to $\mathbf{T}(t)$ and points in the direction in which the object is turning, as shown in Figure 12.23.


Helix:
$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$
z
2π
$\frac{3\pi}{2}$
π
$\frac{\pi}{2}$
−2
−1
1
2
x
−2
−1
1
2
y

$\mathbf{N}(t)$ is horizontal and points toward the z-axis.
Figure 12.24

EXAMPLE 4 Finding the Principal Unit Normal Vector

Find the principal unit normal vector for the helix given by

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}.$$

Solution From Example 2, you know that the unit tangent vector is

$$\mathbf{T}(t) = \frac{1}{\sqrt{5}}(-2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}). \qquad \text{Unit tangent vector}$$

So, $\mathbf{T}'(t)$ is given by

$$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}(-2\cos t\mathbf{i} - 2\sin t\mathbf{j}).$$

Because $\|\mathbf{T}'(t)\| = 2/\sqrt{5}$, it follows that the principal unit normal vector is

$$\begin{aligned}\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}\\ &= \frac{1}{2}(-2\cos t\mathbf{i} - 2\sin t\mathbf{j})\\ &= -\cos t\mathbf{i} - \sin t\mathbf{j}. \qquad \text{Principal unit normal vector}\end{aligned}$$

Note that this vector is horizontal and points toward the z-axis, as shown in Figure 12.24. ■

Tangential and Normal Components of Acceleration

Let's return to the problem of describing the motion of an object along a curve. In the preceding section, you saw that for an object traveling at a *constant speed*, the velocity and acceleration vectors are perpendicular. This seems reasonable, because the speed would not be constant if any acceleration were acting in the direction of motion. You can verify this observation by noting that

$$\mathbf{r}''(t) \cdot \mathbf{r}'(t) = 0$$

if $\|\mathbf{r}'(t)\|$ is a constant. (See Property 7 of Theorem 12.2.)

However, for an object traveling at a *variable speed*, the velocity and acceleration vectors are not necessarily perpendicular. For instance, you saw that the acceleration vector for a projectile always points down, regardless of the direction of motion.

In general, part of the acceleration (the tangential component) acts in the line of motion, and part (the normal component) acts perpendicular to the line of motion. In order to determine these two components, you can use the unit vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$, which serve in much the same way as do $\mathbf{i}$ and $\mathbf{j}$ in representing vectors in the plane. The following theorem states that the acceleration vector lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

THEOREM 12.4 ACCELERATION VECTOR

If $\mathbf{r}(t)$ is the position vector for a smooth curve C and $\mathbf{N}(t)$ exists, then the acceleration vector $\mathbf{a}(t)$ lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

PROOF To simplify the notation, write $\mathbf{T}$ for $\mathbf{T}(t)$, $\mathbf{T}'$ for $\mathbf{T}'(t)$, and so on. Because $\mathbf{T} = \mathbf{r}'/\|\mathbf{r}'\| = \mathbf{v}/\|\mathbf{v}\|$, it follows that

$$\mathbf{v} = \|\mathbf{v}\|\mathbf{T}.$$

By differentiating, you obtain

$$\begin{aligned} \mathbf{a} = \mathbf{v}' &= D_t[\|\mathbf{v}\|]\mathbf{T} + \|\mathbf{v}\|\mathbf{T}' && \text{Product Rule} \\ &= D_t[\|\mathbf{v}\|]\mathbf{T} + \|\mathbf{v}\|\mathbf{T}'\left(\frac{\|\mathbf{T}'\|}{\|\mathbf{T}'\|}\right) \\ &= D_t[\|\mathbf{v}\|]\mathbf{T} + \|\mathbf{v}\|\,\|\mathbf{T}'\|\,\mathbf{N}. && \mathbf{N} = \mathbf{T}'/\|\mathbf{T}'\| \end{aligned}$$

Because $\mathbf{a}$ is written as a linear combination of $\mathbf{T}$ and $\mathbf{N}$, it follows that $\mathbf{a}$ lies in the plane determined by $\mathbf{T}$ and $\mathbf{N}$. ■

The coefficients of $\mathbf{T}$ and $\mathbf{N}$ in the proof of Theorem 12.4 are called the **tangential and normal components of acceleration** and are denoted by $a_{\mathbf{T}} = D_t[\|\mathbf{v}\|]$ and $a_{\mathbf{N}} = \|\mathbf{v}\|\,\|\mathbf{T}'\|$. So, you can write

$$\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t).$$

The following theorem gives some convenient formulas for $a_{\mathbf{N}}$ and $a_{\mathbf{T}}$.

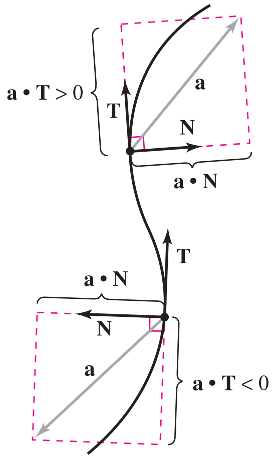
THEOREM 12.5 TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

If $\mathbf{r}(t)$ is the position vector for a smooth curve C [for which $\mathbf{N}(t)$ exists], then the tangential and normal components of acceleration are as follows.

$$a_{\mathbf{T}} = D_t[\|\mathbf{v}\|] = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}$$

$$a_{\mathbf{N}} = \|\mathbf{v}\|\,\|\mathbf{T}'\| = \mathbf{a} \cdot \mathbf{N} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}$$

Note that $a_{\mathbf{N}} \geq 0$. The normal component of acceleration is also called the **centripetal component of acceleration.**



The tangential and normal components of acceleration are obtained by projecting **a** onto **T** and **N**.
Figure 12.25

PROOF Note that a lies in the plane of $\mathbf{T}$ and $\mathbf{N}$. So, you can use Figure 12.25 to conclude that, for any time t, the components of the projection of the acceleration vector onto $\mathbf{T}$ and onto $\mathbf{N}$ are given by $a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T}$ and $a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N}$, respectively. Moreover, because $\mathbf{a} = \mathbf{v}'$ and $\mathbf{T} = \mathbf{v}/\|\mathbf{v}\|$, you have

$$\begin{aligned} a_{\mathbf{T}} &= \mathbf{a} \cdot \mathbf{T} \\ &= \mathbf{T} \cdot \mathbf{a} \\ &= \frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \mathbf{a} \\ &= \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}. \end{aligned}$$

In Exercises 96 and 97, you are asked to prove the other parts of the theorem. ■

NOTE The formulas from Theorem 12.5, together with several other formulas from this chapter, are summarized on page 877. ■

EXAMPLE 5 Tangential and Normal Components of Acceleration

Find the tangential and normal components of acceleration for the position vector given by $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{j} + t^2\mathbf{k}$.

Solution Begin by finding the velocity, speed, and acceleration.

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = 3\mathbf{i} - \mathbf{j} + 2t\mathbf{k}\\ \|\mathbf{v}(t)\| &= \sqrt{9 + 1 + 4t^2} = \sqrt{10 + 4t^2}\\ \mathbf{a}(t) &= \mathbf{r}''(t) = 2\mathbf{k}\end{aligned}$$

By Theorem 12.5, the tangential component of acceleration is

$$a_{\mathbf{T}} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{4t}{\sqrt{10 + 4t^2}}$$

Tangential component of acceleration

and because

$$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = -2\mathbf{i} - 6\mathbf{j}$$

the normal component of acceleration is

$$a_{\mathbf{N}} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \frac{\sqrt{4 + 36}}{\sqrt{10 + 4t^2}} = \frac{2\sqrt{10}}{\sqrt{10 + 4t^2}}.$$

Normal component of acceleration

NOTE In Example 5, you could have used the alternative formula for $a_{\mathbf{N}}$ as follows.

$$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = \sqrt{(2)^2 - \frac{16t^2}{10 + 4t^2}} = \frac{2\sqrt{10}}{\sqrt{10 + 4t^2}}$$

EXAMPLE 6 Finding $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ for a Circular Helix

Find the tangential and normal components of acceleration for the helix given by $\mathbf{r}(t) = b\cos t\mathbf{i} + b\sin t\mathbf{j} + ct\mathbf{k}$, $b > 0$.

Solution

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = -b\sin t\mathbf{i} + b\cos t\mathbf{j} + c\mathbf{k}\\ \|\mathbf{v}(t)\| &= \sqrt{b^2\sin^2 t + b^2\cos^2 t + c^2} = \sqrt{b^2 + c^2}\\ \mathbf{a}(t) &= \mathbf{r}''(t) = -b\cos t\mathbf{i} - b\sin t\mathbf{j}\end{aligned}$$

By Theorem 12.5, the tangential component of acceleration is

$$a_{\mathbf{T}} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{b^2\sin t\cos t - b^2\sin t\cos t + 0}{\sqrt{b^2 + c^2}} = 0.$$

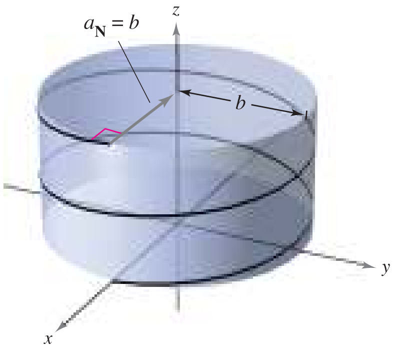
Tangential component of acceleration

Moreover, because $\|\mathbf{a}\| = \sqrt{b^2\cos^2 t + b^2\sin^2 t} = b$, you can use the alternative formula for the normal component of acceleration to obtain

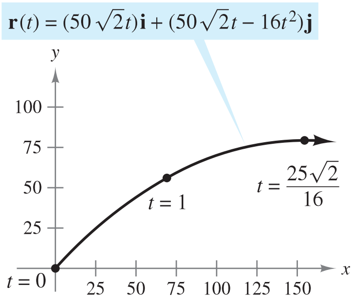
$$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = \sqrt{b^2 - 0^2} = b.$$

Normal component of acceleration

Note that the normal component of acceleration is equal to the magnitude of the acceleration. In other words, because the speed is constant, the acceleration is perpendicular to the velocity. See Figure 12.26.



The normal component of acceleration is equal to the radius of the cylinder around which the helix is spiraling.
Figure 12.26



The path of a projectile
Figure 12.27

EXAMPLE 7 Projectile Motion

The position vector for the projectile shown in Figure 12.27 is given by

$$\mathbf{r}(t) = \left(50\sqrt{2}\,t\right)\mathbf{i} + \left(50\sqrt{2}\,t - 16t^2\right)\mathbf{j}. \qquad \text{Position vector}$$

Find the tangential components of acceleration when $t = 0$, 1, and $25\sqrt{2}/16$.

Solution

$$\mathbf{v}(t) = 50\sqrt{2}\,\mathbf{i} + \left(50\sqrt{2} - 32t\right)\mathbf{j} \qquad \text{Velocity vector}$$

$$\|\mathbf{v}(t)\| = 2\sqrt{50^2 - 16(50)\sqrt{2}\,t + 16^2t^2} \qquad \text{Speed}$$

$$\mathbf{a}(t) = -32\mathbf{j} \qquad \text{Acceleration vector}$$

The tangential component of acceleration is

$$a_{\mathbf{T}}(t) = \frac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{\|\mathbf{v}(t)\|} = \frac{-32(50\sqrt{2} - 32t)}{2\sqrt{50^2 - 16(50)\sqrt{2}\,t + 16^2t^2}}. \qquad \text{Tangential component of acceleration}$$

At the specified times, you have

$$a_{\mathbf{T}}(0) = \frac{-32\left(50\sqrt{2}\right)}{100} = -16\sqrt{2} \approx -22.6$$

$$a_{\mathbf{T}}(1) = \frac{-32\left(50\sqrt{2} - 32\right)}{2\sqrt{50^2 - 16(50)\sqrt{2} + 16^2}} \approx -15.4$$

$$a_{\mathbf{T}}\left(\frac{25\sqrt{2}}{16}\right) = \frac{-32\left(50\sqrt{2} - 50\sqrt{2}\right)}{50\sqrt{2}} = 0.$$

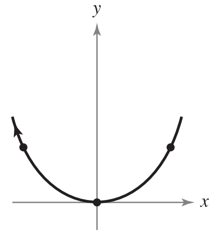
You can see from Figure 12.27 that, at the maximum height, when $t = 25\sqrt{2}/16$, the tangential component is 0. This is reasonable because the direction of motion is horizontal at the point and the tangential component of the acceleration is equal to the horizontal component of the acceleration. ■

12.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, sketch the unit tangent and normal vectors at the given points. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

1.



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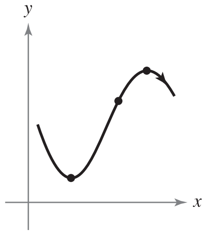

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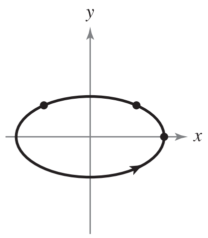

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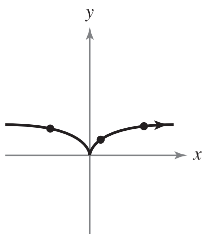


In Exercises 5–10, find the unit tangent vector to the curve at the specified value of the parameter.

5. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}, \quad t = 1$

6. $\mathbf{r}(t) = t^3\mathbf{i} + 2t^2\mathbf{j}, \quad t = 1$

7. $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j}, \quad t = \dfrac{\pi}{4}$

8. $\mathbf{r}(t) = 6\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad t = \dfrac{\pi}{3}$

9. $\mathbf{r}(t) = 3t\mathbf{i} - \ln t\,\mathbf{j}, \quad t = e$

10. $\mathbf{r}(t) = e^t\cos t\,\mathbf{i} + e^t\mathbf{j}, \quad t = 0$

In Exercises 11–16, find the unit tangent vector T(t) and find a set of parametric equations for the line tangent to the space curve at point P.

11. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, \quad P(0, 0, 0)$

12. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{4}{3}\mathbf{k}, \quad P\left(1, 1, \frac{4}{3}\right)$

13. $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + t\mathbf{k}, \quad P(3, 0, 0)$

14. $\mathbf{r}(t) = \left\langle t, t, \sqrt{4 - t^2}\right\rangle, \quad P\left(1, 1, \sqrt{3}\right)$

15. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4\rangle, \quad P\left(\sqrt{2}, \sqrt{2}, 4\right)$

16. $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, 4\sin^2 t\rangle, \quad P\left(1, \sqrt{3}, 1\right)$

CAS **In Exercises 17 and 18, use a computer algebra system to graph the space curve. Then find $\mathbf{T}(t)$ and find a set of parametric equations for the line tangent to the space curve at point P. Graph the tangent line.**

17. $\mathbf{r}(t) = \langle t, t^2, 2t^3/3 \rangle, \quad P(3, 9, 18)$

18. $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + \frac{1}{2}t\,\mathbf{k}, \quad P(0, 4, \pi/4)$

Linear Approximation **In Exercises 19 and 20, find a set of parametric equations for the tangent line to the graph at $t = t_0$ and use the equations for the line to approximate $\mathbf{r}(t_0 + 0.1)$.**

19. $\mathbf{r}(t) = \left\langle t, \ln t, \sqrt{t} \right\rangle, \quad t_0 = 1$

20. $\mathbf{r}(t) = \langle e^{-t}, 2\cos t, 2\sin t \rangle, \quad t_0 = 0$

In Exercises 21 and 22, verify that the space curves intersect at the given values of the parameters. Find the angle between the tangent vectors to the curves at the point of intersection.

21. $\mathbf{r}(t) = \left\langle t - 2, t^2, \frac{1}{2}t \right\rangle, \quad t = 4$

$\mathbf{u}(s) = \left\langle \frac{1}{4}s, 2s, \sqrt[3]{s} \right\rangle, \quad s = 8$

22. $\mathbf{r}(t) = \langle t, \cos t, \sin t \rangle, \quad t = 0$

$\mathbf{u}(s) = \left\langle -\frac{1}{2}\sin^2 s - \sin s, 1 - \frac{1}{2}\sin^2 s - \sin s, \frac{1}{2}\sin s\cos s + \frac{1}{2}s \right\rangle, \quad s = 0$

In Exercises 23–30, find the principal unit normal vector to the curve at the specified value of the parameter.

23. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j}, \quad t = 2$

24. $\mathbf{r}(t) = t\mathbf{i} + \frac{6}{t}\mathbf{j}, \quad t = 3$

25. $\mathbf{r}(t) = \ln t\,\mathbf{i} + (t + 1)\mathbf{j}, \quad t = 2$

26. $\mathbf{r}(t) = \pi\cos t\,\mathbf{i} + \pi\sin t\,\mathbf{j}, \quad t = \frac{\pi}{6}$

27. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\,\mathbf{k}, \quad t = 1$

28. $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}, \quad t = 0$

29. $\mathbf{r}(t) = 6\cos t\,\mathbf{i} + 6\sin t\,\mathbf{j} + \mathbf{k}, \quad t = \frac{3\pi}{4}$

30. $\mathbf{r}(t) = \cos 3t\,\mathbf{i} + 2\sin 3t\,\mathbf{j} + \mathbf{k}, \quad t = \pi$

In Exercises 31–34, find $\mathbf{v}(t)$, $\mathbf{a}(t)$, $\mathbf{T}(t)$, and $\mathbf{N}(t)$ (if it exists) for an object moving along the path given by the vector-valued function $\mathbf{r}(t)$. Use the results to determine the form of the path. Is the speed of the object constant or changing?

31. $\mathbf{r}(t) = 4t\mathbf{i}$

32. $\mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}$

33. $\mathbf{r}(t) = 4t^2\mathbf{i}$

34. $\mathbf{r}(t) = t^2\mathbf{j} + \mathbf{k}$

In Exercises 35–44, find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$ at the given time t for the plane curve $\mathbf{r}(t)$.

35. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}, \quad t = 1$

36. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}, \quad t = 1$

37. $\mathbf{r}(t) = (t - t^3)\mathbf{i} + 2t^2\mathbf{j}, \quad t = 1$

38. $\mathbf{r}(t) = (t^3 - 4t)\mathbf{i} + (t^2 - 1)\mathbf{j}, \quad t = 0$

39. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-2t}\mathbf{j}, \quad t = 0$

40. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k}, \quad t = 0$

41. $\mathbf{r}(t) = e^t\cos t\,\mathbf{i} + e^t\sin t\,\mathbf{j}, \quad t = \frac{\pi}{2}$

42. $\mathbf{r}(t) = a\cos\omega t\,\mathbf{i} + b\sin\omega t\,\mathbf{j}, \quad t = 0$

43. $\mathbf{r}(t) = \langle \cos\omega t + \omega t\sin\omega t, \sin\omega t - \omega t\cos\omega t \rangle, \quad t = t_0$

44. $\mathbf{r}(t) = \langle \omega t - \sin\omega t, 1 - \cos\omega t \rangle, \quad t = t_0$

Circular Motion **In Exercises 45–48, consider an object moving according to the position function**

$$\mathbf{r}(t) = a\cos\omega t\,\mathbf{i} + a\sin\omega t\,\mathbf{j}.$$

45. Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$.

46. Determine the directions of $\mathbf{T}$ and $\mathbf{N}$ relative to the position function $\mathbf{r}$.

47. Determine the speed of the object at any time t and explain its value relative to the value of $a_{\mathbf{T}}$.

48. If the angular velocity ω is halved, by what factor is $a_{\mathbf{N}}$ changed?

In Exercises 49–54, sketch the graph of the plane curve given by the vector-valued function, and, at the point on the curve determined by $\mathbf{r}(t_0)$, sketch the vectors $\mathbf{T}$ and $\mathbf{N}$. Note that $\mathbf{N}$ points toward the concave side of the curve.

Function	*Time*
49. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$	$t_0 = 2$
50. $\mathbf{r}(t) = t^3\mathbf{i} + t\mathbf{j}$	$t_0 = 1$
51. $\mathbf{r}(t) = 4t\mathbf{i} + 4t^2\mathbf{j}$	$t_0 = \frac{1}{4}$
52. $\mathbf{r}(t) = (2t + 1)\mathbf{i} - t^2\mathbf{j}$	$t_0 = 2$
53. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$	$t_0 = \frac{\pi}{4}$
54. $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$	$t_0 = \pi$

In Exercises 55–62, find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$ at the given time t for the space curve $\mathbf{r}(t)$. [*Hint:* Find $\mathbf{a}(t)$, $\mathbf{T}(t)$, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$. Solve for $\mathbf{N}$ in the equation $\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$.]

Function	*Time*
55. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} - 3t\mathbf{k}$	$t = 1$
56. $\mathbf{r}(t) = 4t\mathbf{i} - 4t\mathbf{j} + 2t\mathbf{k}$	$t = 2$
57. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$	$t = \frac{\pi}{3}$
58. $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{j} + t^2\mathbf{k}$	$t = -1$
59. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$	$t = 1$
60. $\mathbf{r}(t) = (2t - 1)\mathbf{i} + t^2\mathbf{j} - 4t\mathbf{k}$	$t = 2$
61. $\mathbf{r}(t) = e^t\sin t\,\mathbf{i} + e^t\cos t\,\mathbf{j} + e^t\mathbf{k}$	$t = 0$
62. $\mathbf{r}(t) = e^t\mathbf{i} + 2t\mathbf{j} + e^{-t}\mathbf{k}$	$t = 0$

CAS **In Exercises 63–66, use a computer algebra system to graph the space curve. Then find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$ at the given time t. Sketch $\mathbf{T}(t)$ and $\mathbf{N}(t)$ on the space curve.**

	Function	*Time*
63.	$\mathbf{r}(t) = 4t\mathbf{i} + 3\cos t\mathbf{j} + 3\sin t\mathbf{k}$	$t = \frac{\pi}{2}$
64.	$\mathbf{r}(t) = (2 + \cos t)\mathbf{i} + (1 - \sin t)\mathbf{j} + \frac{t}{3}\mathbf{k}$	$t = \pi$
65.	$\mathbf{r}(t) = t\mathbf{i} + 3t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$	$t = 2$
66.	$\mathbf{r}(t) = t^2\mathbf{i} + \mathbf{j} + 2t\mathbf{k}$	$t = 1$

WRITING ABOUT CONCEPTS

67. Define the unit tangent vector, the principal unit normal vector, and the tangential and normal components of acceleration.

68. How is the unit tangent vector related to the orientation of a curve? Explain.

69. (a) Describe the motion of a particle if the normal component of acceleration is 0.

(b) Describe the motion of a particle if the tangential component of acceleration is 0.

CAPSTONE

70. An object moves along the path given by

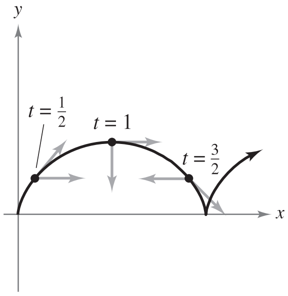
$$\mathbf{r}(t) = 3t\mathbf{i} + 4t\mathbf{j}.$$

Find $v(t)$, $a(t)$, $\mathbf{T}(t)$, and $\mathbf{N}(t)$ (if it exists). What is the form of the path? Is the speed of the object constant or changing?

71. ***Cycloidal Motion*** The figure shows the path of a particle modeled by the vector-valued function

$$\mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t \rangle.$$

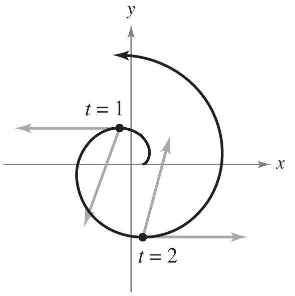
The figure also shows the vectors $\mathbf{v}(t)/\|\mathbf{v}(t)\|$ and $\mathbf{a}(t)/\|\mathbf{a}(t)\|$ at the indicated values of t.



(a) Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ at $t = \frac{1}{2}$, $t = 1$, and $t = \frac{3}{2}$.

(b) Determine whether the speed of the particle is increasing or decreasing at each of the indicated values of t. Give reasons for your answers.

72. ***Motion Along an Involute of a Circle*** The figure shows a particle moving along a path modeled by $\mathbf{r}(t) = \langle \cos \pi t + \pi t \sin \pi t, \sin \pi t - \pi t \cos \pi t \rangle$. The figure also shows the vectors $\mathbf{v}(t)$ and $\mathbf{a}(t)$ for $t = 1$ and $t = 2$.



(a) Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ at $t = 1$ and $t = 2$.

(b) Determine whether the speed of the particle is increasing or decreasing at each of the indicated values of t. Give reasons for your answers.

In Exercises 73–78, find the vectors T and N, and the unit binormal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, for the vector-valued function $\mathbf{r}(t)$ at the given value of t.

73. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + \frac{t}{2}\mathbf{k}$

$t_0 = \frac{\pi}{2}$

74. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^3}{3}\mathbf{k}$

$t_0 = 1$

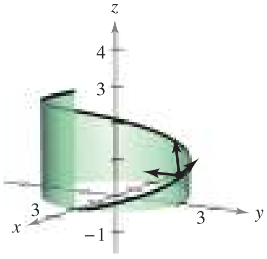


Figure for 73

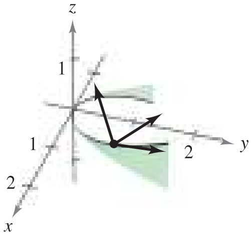


Figure for 74

75. $\mathbf{r}(t) = \mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}$, $t_0 = \frac{\pi}{4}$

76. $\mathbf{r}(t) = 2e^t\mathbf{i} + e^t\cos t\mathbf{j} + e^t\sin t\mathbf{k}$, $t_0 = 0$

77. $\mathbf{r}(t) = 4\sin t\mathbf{i} + 4\cos t\mathbf{j} + 2t\mathbf{k}$, $t_0 = \frac{\pi}{3}$

78. $\mathbf{r}(t) = 3\cos 2t\mathbf{i} + 3\sin 2t\mathbf{j} + t\mathbf{k}$, $t_0 = \frac{\pi}{4}$

79. ***Projectile Motion*** Find the tangential and normal components of acceleration for a projectile fired at an angle θ with the horizontal at an initial speed of v_0. What are the components when the projectile is at its maximum height?

80. ***Projectile Motion*** Use your results from Exercise 79 to find the tangential and normal components of acceleration for a projectile fired at an angle of 45° with the horizontal at an initial speed of 150 feet per second. What are the components when the projectile is at its maximum height?

81. ***Projectile Motion*** A projectile is launched with an initial velocity of 120 feet per second at a height of 5 feet and at an angle of 30° with the horizontal.

(a) Determine the vector-valued function for the path of the projectile.

(b) Use a graphing utility to graph the path and approximate the maximum height and range of the projectile.

(c) Find $\mathbf{v}(t)$, $\|\mathbf{v}(t)\|$, and $\mathbf{a}(t)$.

(d) Use a graphing utility to complete the table.

t	0.5	1.0	1.5	2.0	2.5	3.0
Speed						

(e) Use a graphing utility to graph the scalar functions $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$. How is the speed of the projectile changing when $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ have opposite signs?

82. ***Projectile Motion*** A projectile is launched with an initial velocity of 220 feet per second at a height of 4 feet and at an angle of 45° with the horizontal.

(a) Determine the vector-valued function for the path of the projectile.

(b) Use a graphing utility to graph the path and approximate the maximum height and range of the projectile.

(c) Find $\mathbf{v}(t)$, $\|\mathbf{v}(t)\|$, and $\mathbf{a}(t)$.

(d) Use a graphing utility to complete the table.

t	0.5	1.0	1.5	2.0	2.5	3.0
Speed						

83. ***Air Traffic Control*** Because of a storm, ground controllers instruct the pilot of a plane flying at an altitude of 4 miles to make a 90° turn and climb to an altitude of 4.2 miles. The model for the path of the plane during this maneuver is

$$\mathbf{r}(t) = \langle 10 \cos 10\pi t, 10 \sin 10\pi t, 4 + 4t\rangle, \quad 0 \le t \le \tfrac{1}{20}$$

where t is the time in hours and $\mathbf{r}$ is the distance in miles.

(a) Determine the speed of the plane.

CAS (b) Use a computer algebra system to calculate $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$. Why is one of these equal to 0?

84. ***Projectile Motion*** A plane flying at an altitude of 36,000 feet at a speed of 600 miles per hour releases a bomb. Find the tangential and normal components of acceleration acting on the bomb.

85. ***Centripetal Acceleration*** An object is spinning at a constant speed on the end of a string, according to the position function given in Exercises 45–48.

(a) If the angular velocity ω is doubled, how is the centripetal component of acceleration changed?

(b) If the angular velocity is unchanged but the length of the string is halved, how is the centripetal component of acceleration changed?

86. ***Centripetal Force*** An object of mass m moves at a constant speed v in a circular path of radius r. The force required to produce the centripetal component of acceleration is called the *centripetal force* and is given by $F = mv^2/r$. Newton's Law of Universal Gravitation is given by $F = GMm/d^2$, where d is the distance between the centers of the two bodies of masses M and m, and G is a gravitational constant. Use this law to show that the speed required for circular motion is $v = \sqrt{GM/r}$.

Orbital Speed **In Exercises 87–90, use the result of Exercise 86 to find the speed necessary for the given circular orbit around Earth. Let $GM = 9.56 \times 10^4$ cubic miles per second per second, and assume the radius of Earth is 4000 miles.**

87. The orbit of a space shuttle 115 miles above the surface of Earth

88. The orbit of a space shuttle 245 miles above the surface of Earth

89. The orbit of a heat capacity mapping satellite 385 miles above the surface of Earth

90. The orbit of a communications satellite r miles above the surface of Earth that is in geosynchronous orbit [The satellite completes one orbit per sidereal day (approximately 23 hours, 56 minutes), and therefore appears to remain stationary above a point on Earth.]

True or False? **In Exercises 91 and 92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. If a car's speedometer is constant, then the car cannot be accelerating.

92. If $a_{\mathbf{N}} = 0$ for a moving object, then the object is moving in a straight line.

93. A particle moves along a path modeled by $\mathbf{r}(t) = \cosh(bt)\mathbf{i} + \sinh(bt)\mathbf{j}$, where b is a positive constant.

(a) Show that the path of the particle is a hyperbola.

(b) Show that $\mathbf{a}(t) = b^2\,\mathbf{r}(t)$.

94. Prove that the principal unit normal vector $\mathbf{N}$ points toward the concave side of a plane curve.

95. Prove that the vector $\mathbf{T}'(t)$ is $\mathbf{0}$ for an object moving in a straight line.

96. Prove that $a_{\mathbf{N}} = \dfrac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$.

97. Prove that $a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}$.

PUTNAM EXAM CHALLENGE

98. A particle of unit mass moves on a straight line under the action of a force which is a function $f(v)$ of the velocity v of the particle, but the form of this function is not known. A motion is observed, and the distance x covered in time t is found to be connected with t by the formula $x = at + bt^2 + ct^3$, where a, b, and c have numerical values determined by observation of the motion. Find the function $f(v)$ for the range of v covered by the experiment.

This problem was composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America.

12.5 Arc Length and Curvature

- **Find the arc length of a space curve.**
- **Use the arc length parameter to describe a plane curve or space curve.**
- **Find the curvature of a curve at a point on the curve.**
- **Use a vector-valued function to find frictional force.**

Arc Length

In Section 10.3, you saw that the arc length of a smooth *plane* curve C given by the parametric equations $x = x(t)$ and $y = y(t)$, $a \le t \le b$, is

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

In vector form, where C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, you can rewrite this equation for arc length as

$$s = \int_a^b \|\mathbf{r}'(t)\|\, dt.$$

The formula for the arc length of a plane curve has a natural extension to a smooth curve in *space*, as stated in the following theorem.

THEOREM 12.6 ARC LENGTH OF A SPACE CURVE

If C is a smooth curve given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, on an interval $[a, b]$, then the arc length of C on the interval is

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt.$$

EXPLORATION

Arc Length Formula The formula for the arc length of a space curve is given in terms of the parametric equations used to represent the curve. Does this mean that the arc length of the curve depends on the parameter being used? Would you want this to be true? Explain your reasoning.

Here is a different parametric representation of the curve in Example 1.

$$\mathbf{r}(t) = t^2\mathbf{i} + \frac{4}{3}t^3\mathbf{j} + \frac{1}{2}t^4\mathbf{k}$$

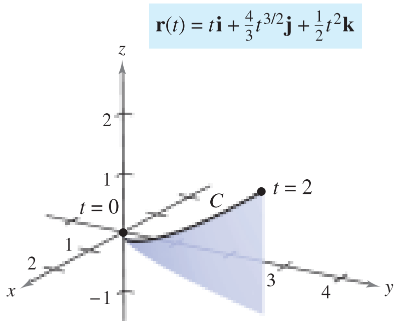
Find the arc length from $t = 0$ to $t = \sqrt{2}$ and compare the result with that found in Example 1.

EXAMPLE 1 Finding the Arc Length of a Curve in Space

Find the arc length of the curve given by

$$\mathbf{r}(t) = t\mathbf{i} + \frac{4}{3}t^{3/2}\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$$

from $t = 0$ to $t = 2$, as shown in Figure 12.28.


$\mathbf{r}(t) = t\mathbf{i} + \frac{4}{3}t^{3/2}\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$
z
2
1
t = 0
C
t = 2
1
2
x
−1
3
4
y

As t increases from 0 to 2, the vector $\mathbf{r}(t)$ traces out a curve.
Figure 12.28

Solution Using $x(t) = t$, $y(t) = \frac{4}{3}t^{3/2}$, and $z(t) = \frac{1}{2}t^2$, you obtain $x'(t) = 1$, $y'(t) = 2t^{1/2}$, and $z'(t) = t$. So, the arc length from $t = 0$ to $t = 2$ is given by

$$s = \int_0^2 \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt \qquad \text{Formula for arc length}$$

$$= \int_0^2 \sqrt{1 + 4t + t^2}\, dt$$

$$= \int_0^2 \sqrt{(t+2)^2 - 3}\, dt \qquad \text{Integration tables (Appendix B), Formula 26}$$

$$= \left[\frac{t+2}{2}\sqrt{(t+2)^2 - 3} - \frac{3}{2}\ln\left|(t+2) + \sqrt{(t+2)^2 - 3}\right|\right]_0^2$$

$$= 2\sqrt{13} - \frac{3}{2}\ln\left(4 + \sqrt{13}\right) - 1 + \frac{3}{2}\ln 3 \approx 4.816.$$

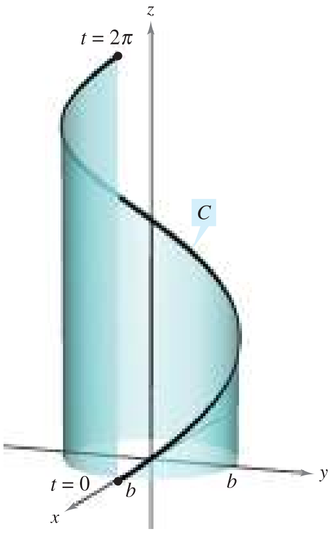
EXAMPLE 2 Finding the Arc Length of a Helix

Find the length of one turn of the helix given by

$$\mathbf{r}(t) = b\cos t\mathbf{i} + b\sin t\mathbf{j} + \sqrt{1-b^2}\,t\mathbf{k}$$

as shown in Figure 12.29.

Curve:
$\mathbf{r}(t) = b\cos t\mathbf{i} + b\sin t\mathbf{j} + \sqrt{1-b^2}\,t\mathbf{k}$



One turn of a helix
Figure 12.29

Solution Begin by finding the derivative.

$$\mathbf{r}'(t) = -b\sin t\mathbf{i} + b\cos t\mathbf{j} + \sqrt{1-b^2}\mathbf{k} \qquad \text{Derivative}$$

Now, using the formula for arc length, you can find the length of one turn of the helix by integrating $\|\mathbf{r}'(t)\|$ from 0 to 2π.

$$\begin{aligned} s &= \int_0^{2\pi} \|\mathbf{r}'(t)\|\,dt && \text{Formula for arc length} \\ &= \int_0^{2\pi} \sqrt{b^2(\sin^2 t + \cos^2 t) + (1-b^2)}\,dt \\ &= \int_0^{2\pi} dt \\ &= t\Big]_0^{2\pi} = 2\pi \end{aligned}$$

So, the length is 2π units. ■

Arc Length Parameter

You have seen that curves can be represented by vector-valued functions in different ways, depending on the choice of parameter. For *motion* along a curve, the convenient parameter is time t. However, for studying the *geometric properties* of a curve, the convenient parameter is often arc length s.

$s(t) = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\,du$

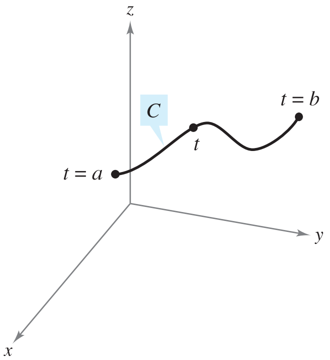


Figure 12.30

DEFINITION OF ARC LENGTH FUNCTION

Let C be a smooth curve given by $\mathbf{r}(t)$ defined on the closed interval $[a, b]$. For $a \le t \le b$, the **arc length function** is given by

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\,du = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\,du.$$

The arc length s is called the **arc length parameter.** (See Figure 12.30.)

NOTE The arc length function s is *nonnegative*. It measures the distance along C from the initial point $(x(a), y(a), z(a))$ to the point $(x(t), y(t), z(t))$. ■

Using the definition of the arc length function and the Second Fundamental Theorem of Calculus, you can conclude that

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|. \qquad \text{Derivative of arc length function}$$

In differential form, you can write

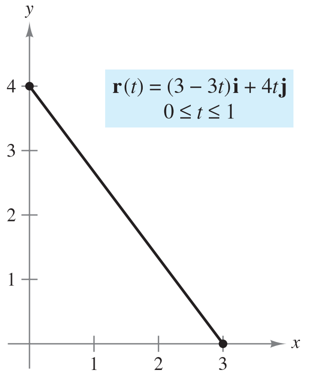
$$ds = \|\mathbf{r}'(t)\|\,dt.$$

EXAMPLE 3 Finding the Arc Length Function for a Line

Find the arc length function $s(t)$ for the line segment given by

$$\mathbf{r}(t) = (3 - 3t)\mathbf{i} + 4t\mathbf{j}, \quad 0 \le t \le 1$$

and write $\mathbf{r}$ as a function of the parameter s. (See Figure 12.31.)



The line segment from (3, 0) to (0, 4) can be parametrized using the arc length parameter s.
Figure 12.31

Solution Because $\mathbf{r}'(t) = -3\mathbf{i} + 4\mathbf{j}$ and

$$\|\mathbf{r}'(t)\| = \sqrt{(-3)^2 + 4^2} = 5$$

you have

$$\begin{aligned} s(t) &= \int_0^t \|\mathbf{r}'(u)\|\,du \\ &= \int_0^t 5\,du \\ &= 5t. \end{aligned}$$

Using $s = 5t$ (or $t = s/5$), you can rewrite $\mathbf{r}$ using the arc length parameter as follows.

$$\mathbf{r}(s) = \left(3 - \tfrac{3}{5}s\right)\mathbf{i} + \tfrac{4}{5}s\,\mathbf{j}, \quad 0 \le s \le 5$$ ■

One of the advantages of writing a vector-valued function in terms of the arc length parameter is that $\|\mathbf{r}'(s)\| = 1$. For instance, in Example 3, you have

$$\|\mathbf{r}'(s)\| = \sqrt{\left(-\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1.$$

So, for a smooth curve C represented by $\mathbf{r}(s)$, where s is the arc length parameter, the arc length between a and b is

$$\begin{aligned} \text{Length of arc} &= \int_a^b \|\mathbf{r}'(s)\|\,ds \\ &= \int_a^b ds \\ &= b - a \\ &= \text{length of interval.} \end{aligned}$$

Furthermore, if t is *any* parameter such that $\|\mathbf{r}'(t)\| = 1$, then t must be the arc length parameter. These results are summarized in the following theorem, which is stated without proof.

THEOREM 12.7 ARC LENGTH PARAMETER

If C is a smooth curve given by

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} \quad \text{or} \quad \mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k}$$

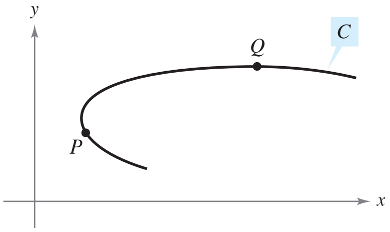
where s is the arc length parameter, then

$$\|\mathbf{r}'(s)\| = 1.$$

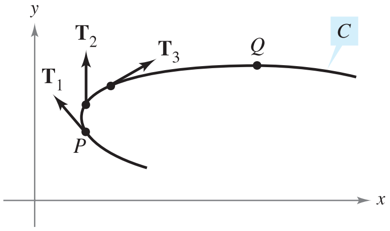
Moreover, if t is *any* parameter for the vector-valued function $\mathbf{r}$ such that $\|\mathbf{r}'(t)\| = 1$, then t must be the arc length parameter.

Curvature

An important use of the arc length parameter is to find **curvature**—the measure of how sharply a curve bends. For instance, in Figure 12.32 the curve bends more sharply at P than at Q, and you can say that the curvature is greater at P than at Q. You can calculate curvature by calculating the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to the arc length s, as shown in Figure 12.33.



Curvature at P is greater than at Q.
Figure 12.32



The magnitude of the rate of change of $\mathbf{T}$ with respect to the arc length is the curvature of a curve.
Figure 12.33

DEFINITION OF CURVATURE

Let C be a smooth curve (in the plane *or* in space) given by $\mathbf{r}(s)$, where s is the arc length parameter. The **curvature** K at s is given by

$$K = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{T}'(s)\|.$$

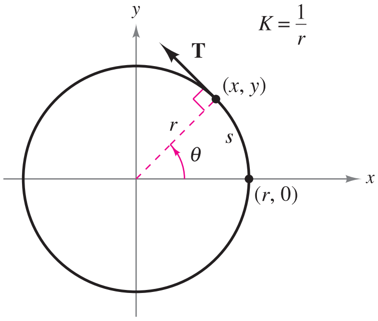
A circle has the same curvature at any point. Moreover, the curvature and the radius of the circle are inversely related. That is, a circle with a large radius has a small curvature, and a circle with a small radius has a large curvature. This inverse relationship is made explicit in the following example.

EXAMPLE 4 Finding the Curvature of a Circle

Show that the curvature of a circle of radius r is $K = 1/r$.

Solution Without loss of generality you can consider the circle to be centered at the origin. Let (x, y) be any point on the circle and let s be the length of the arc from $(r, 0)$ to (x, y), as shown in Figure 12.34. By letting θ be the central angle of the circle, you can represent the circle by

$$\mathbf{r}(\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}. \qquad \theta \text{ is the parameter.}$$



The curvature of a circle is constant.
Figure 12.34

Using the formula for the length of a circular arc $s = r\theta$, you can rewrite $\mathbf{r}(\theta)$ in terms of the arc length parameter as follows.

$$\mathbf{r}(s) = r\cos\frac{s}{r}\mathbf{i} + r\sin\frac{s}{r}\mathbf{j} \qquad \text{Arc length } s \text{ is the parameter.}$$

So, $\mathbf{r}'(s) = -\sin\frac{s}{r}\mathbf{i} + \cos\frac{s}{r}\mathbf{j}$, and it follows that $\|\mathbf{r}'(s)\| = 1$, which implies that the unit tangent vector is

$$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = -\sin\frac{s}{r}\mathbf{i} + \cos\frac{s}{r}\mathbf{j}$$

and the curvature is given by

$$K = \|\mathbf{T}'(s)\| = \left\| -\frac{1}{r}\cos\frac{s}{r}\mathbf{i} - \frac{1}{r}\sin\frac{s}{r}\mathbf{j} \right\| = \frac{1}{r}$$

at every point on the circle. ■

NOTE Because a straight line doesn't curve, you would expect its curvature to be 0. Try checking this by finding the curvature of the line given by

$$\mathbf{r}(s) = \left(3 - \frac{3}{5}s\right)\mathbf{i} + \frac{4}{5}s\mathbf{j}.$$ ■

In Example 4, the curvature was found by applying the definition directly. This requires that the curve be written in terms of the arc length parameter s. The following theorem gives two other formulas for finding the curvature of a curve written in terms of an arbitrary parameter t. The proof of this theorem is left as an exercise [see Exercise 100, parts (a) and (b)].

THEOREM 12.8 FORMULAS FOR CURVATURE

If C is a smooth curve given by $\mathbf{r}(t)$, then the curvature K of C at t is given by

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

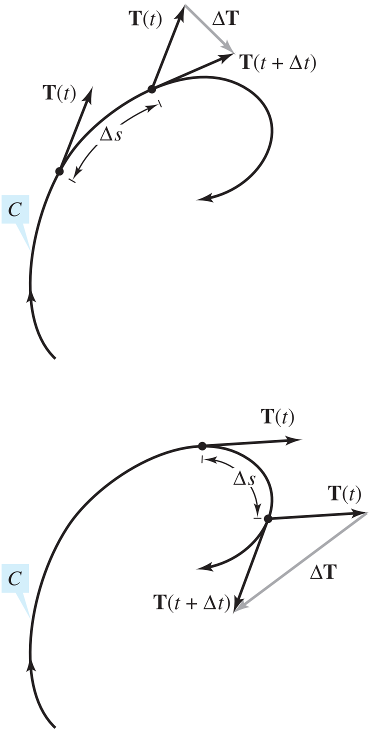


Figure 12.35

Because $\|\mathbf{r}'(t)\| = ds/dt$, the first formula implies that curvature is the ratio of the rate of change in the tangent vector $\mathbf{T}$ to the rate of change in arc length. To see that this is reasonable, let Δt be a "small number." Then,

$$\frac{\mathbf{T}'(t)}{ds/dt} \approx \frac{[\mathbf{T}(t+\Delta t) - \mathbf{T}(t)]/\Delta t}{[s(t+\Delta t) - s(t)]/\Delta t} = \frac{\mathbf{T}(t+\Delta t) - \mathbf{T}(t)}{s(t+\Delta t) - s(t)} = \frac{\Delta \mathbf{T}}{\Delta s}.$$

In other words, for a given Δs, the greater the length of $\Delta\mathbf{T}$, the more the curve bends at t, as shown in Figure 12.35.

EXAMPLE 5 Finding the Curvature of a Space Curve

Find the curvature of the curve given by $\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} - \frac{1}{3}t^3\mathbf{k}$.

Solution It is not apparent whether this parameter represents arc length, so you should use the formula $K = \|\mathbf{T}'(t)\|/\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 2\mathbf{i} + 2t\mathbf{j} - t^2\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2$$

Length of $\mathbf{r}'(t)$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2\mathbf{i} + 2t\mathbf{j} - t^2\mathbf{k}}{t^2 + 2}$$

$$\mathbf{T}'(t) = \frac{(t^2+2)(2\mathbf{j} - 2t\mathbf{k}) - (2t)(2\mathbf{i} + 2t\mathbf{j} - t^2\mathbf{k})}{(t^2+2)^2}$$

$$= \frac{-4t\mathbf{i} + (4 - 2t^2)\mathbf{j} - 4t\mathbf{k}}{(t^2+2)^2}$$

$$\|\mathbf{T}'(t)\| = \frac{\sqrt{16t^2 + 16 - 16t^2 + 4t^4 + 16t^2}}{(t^2+2)^2}$$

$$= \frac{2(t^2+2)}{(t^2+2)^2}$$

$$= \frac{2}{t^2+2}$$

Length of $\mathbf{T}'(t)$

Therefore,

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{2}{(t^2+2)^2}.$$

Curvature

■

The following theorem presents a formula for calculating the curvature of a plane curve given by $y = f(x)$.

THEOREM 12.9 CURVATURE IN RECTANGULAR COORDINATES

If C is the graph of a twice-differentiable function given by $y = f(x)$, then the curvature K at the point (x, y) is given by

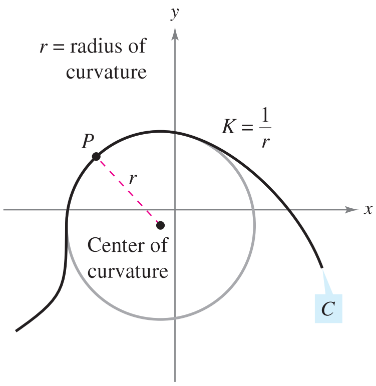
$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}}.$$

PROOF By representing the curve C by $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j} + 0\mathbf{k}$ (where x is the parameter), you obtain $\mathbf{r}'(x) = \mathbf{i} + f'(x)\mathbf{j}$,

$$\|\mathbf{r}'(x)\| = \sqrt{1 + [f'(x)]^2}$$

and $\mathbf{r}''(x) = f''(x)\mathbf{j}$. Because $\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x)\mathbf{k}$, it follows that the curvature is

$$\begin{aligned} K &= \frac{\|\mathbf{r}'(x) \times \mathbf{r}''(x)\|}{\|\mathbf{r}'(x)\|^3} \\ &= \frac{|f''(x)|}{\{1 + [f'(x)]^2\}^{3/2}} \\ &= \frac{|y''|}{[1 + (y')^2]^{3/2}}. \end{aligned}$$ ■

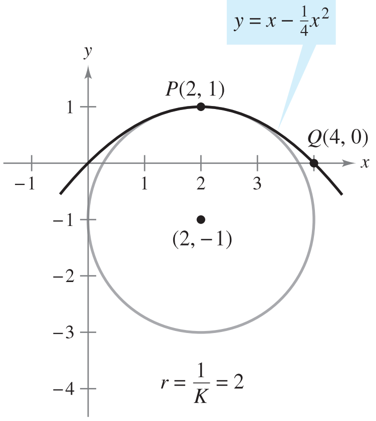


The circle of curvature
Figure 12.36

Let C be a curve with curvature K at point P. The circle passing through point P with radius $r = 1/K$ is called the **circle of curvature** if the circle lies on the concave side of the curve and shares a common tangent line with the curve at point P. The radius is called the **radius of curvature** at P, and the center of the circle is called the **center of curvature.**

The circle of curvature gives you a nice way to estimate graphically the curvature K at a point P on a curve. Using a compass, you can sketch a circle that lies against the concave side of the curve at point P, as shown in Figure 12.36. If the circle has a radius of r, you can estimate the curvature to be $K = 1/r$.

EXAMPLE 6 Finding Curvature in Rectangular Coordinates



The circle of curvature
Figure 12.37

Find the curvature of the parabola given by $y = x - \frac{1}{4}x^2$ at $x = 2$. Sketch the circle of curvature at $(2, 1)$.

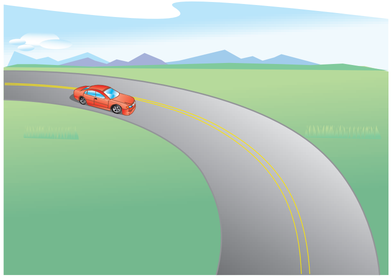
Solution The curvature at $x = 2$ is as follows.

$$y' = 1 - \frac{x}{2} \qquad\qquad y' = 0$$

$$y'' = -\frac{1}{2} \qquad\qquad y'' = -\frac{1}{2}$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} \qquad\qquad K = \frac{1}{2}$$

Because the curvature at $P(2, 1)$ is $\frac{1}{2}$, it follows that the radius of the circle of curvature at that point is 2. So, the center of curvature is $(2, -1)$, as shown in Figure 12.37. [In the figure, note that the curve has the greatest curvature at P. Try showing that the curvature at $Q(4, 0)$ is $1/2^{5/2} \approx 0.177$.] ■



The amount of thrust felt by passengers in a car that is turning depends on two things—the speed of the car and the sharpness of the turn.
Figure 12.38

Arc length and curvature are closely related to the tangential and normal components of acceleration. The tangential component of acceleration is the rate of change of the speed, which in turn is the rate of change of the arc length. This component is negative as a moving object slows down and positive as it speeds up—regardless of whether the object is turning or traveling in a straight line. So, the tangential component is solely a function of the arc length and is independent of the curvature.

On the other hand, the normal component of acceleration is a function of *both* speed and curvature. This component measures the acceleration acting perpendicular to the direction of motion. To see why the normal component is affected by both speed and curvature, imagine that you are driving a car around a turn, as shown in Figure 12.38. If your speed is high and the turn is sharp, you feel yourself thrown against the car door. By lowering your speed *or* taking a more gentle turn, you are able to lessen this sideways thrust.

The next theorem explicitly states the relationships among speed, curvature, and the components of acceleration.

NOTE Note that Theorem 12.10 gives additional formulas for $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$.

THEOREM 12.10 ACCELERATION, SPEED, AND CURVATURE

If $\mathbf{r}(t)$ is the position vector for a smooth curve C, then the acceleration vector is given by

$$\mathbf{a}(t) = \frac{d^2s}{dt^2}\mathbf{T} + K\left(\frac{ds}{dt}\right)^2\mathbf{N}$$

where K is the curvature of C and ds/dt is the speed.

PROOF For the position vector $\mathbf{r}(t)$, you have

$$\begin{aligned}\mathbf{a}(t) &= a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}\\ &= D_t[\|\mathbf{v}\|]\mathbf{T} + \|\mathbf{v}\|\,\|\mathbf{T}'\|\mathbf{N}\\ &= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}(\|\mathbf{v}\|K)\mathbf{N}\\ &= \frac{d^2s}{dt^2}\mathbf{T} + K\left(\frac{ds}{dt}\right)^2\mathbf{N}.\end{aligned}$$

■

EXAMPLE 7 Tangential and Normal Components of Acceleration

Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ for the curve given by

$$\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} - \tfrac{1}{3}t^3\mathbf{k}.$$

Solution From Example 5, you know that

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = t^2 + 2 \quad \text{and} \quad K = \frac{2}{(t^2+2)^2}.$$

Therefore,

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = 2t \qquad \text{Tangential component}$$

and

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{2}{(t^2+2)^2}(t^2+2)^2 = 2. \qquad \text{Normal component}$$

■

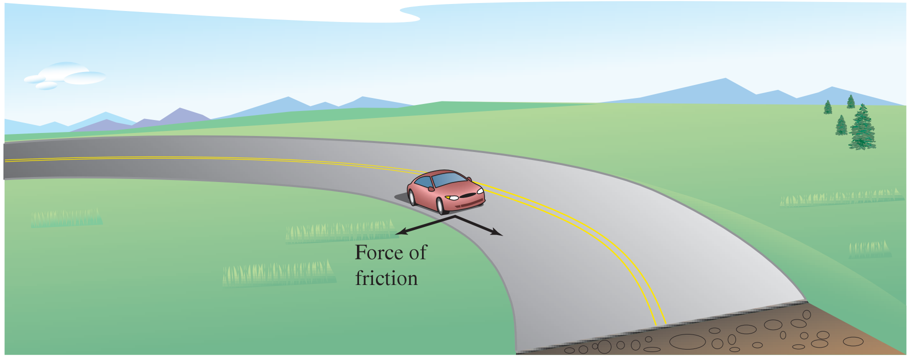
Application

There are many applications in physics and engineering dynamics that involve the relationships among speed, arc length, curvature, and acceleration. One such application concerns frictional force.

A moving object with mass m is in contact with a stationary object. The total force required to produce an acceleration **a** along a given path is

$$\begin{aligned}\mathbf{F} = m\mathbf{a} &= m\left(\frac{d^2s}{dt^2}\right)\mathbf{T} + mK\left(\frac{ds}{dt}\right)^2\mathbf{N}\\ &= ma_{\mathbf{T}}\mathbf{T} + ma_{\mathbf{N}}\mathbf{N}.\end{aligned}$$

The portion of this total force that is supplied by the stationary object is called the **force of friction.** For example, if a car moving with constant speed is rounding a turn, the roadway exerts a frictional force that keeps the car from sliding off the road. If the car is not sliding, the frictional force is perpendicular to the direction of motion and has magnitude equal to the normal component of acceleration, as shown in Figure 12.39. The potential frictional force of a road around a turn can be increased by banking the roadway.



The force of friction is perpendicular to the direction of motion.
Figure 12.39

EXAMPLE 8 Frictional Force

A 360-kilogram go-cart is driven at a speed of 60 kilometers per hour around a circular racetrack of radius 12 meters, as shown in Figure 12.40. To keep the cart from skidding off course, what frictional force must the track surface exert on the tires?

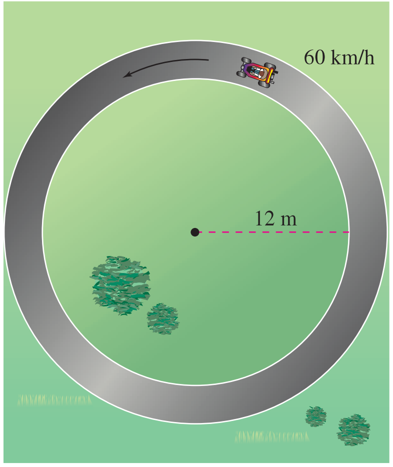


Figure 12.40

Solution The frictional force must equal the mass times the normal component of acceleration. For this circular path, you know that the curvature is

$$K = \frac{1}{12}. \qquad \text{Curvature of circular racetrack}$$

Therefore, the frictional force is

$$\begin{aligned}ma_{\mathbf{N}} &= mK\left(\frac{ds}{dt}\right)^2\\ &= (360 \text{ kg})\left(\frac{1}{12 \text{ m}}\right)\left(\frac{60{,}000 \text{ m}}{3600 \text{ sec}}\right)^2\\ &\approx 8333 \text{ (kg)(m)/sec}^2.\end{aligned}$$

■

SUMMARY OF VELOCITY, ACCELERATION, AND CURVATURE

Let C be a curve (in the plane or in space) given by the position function

$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ Curve in the plane

$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$ Curve in space

Velocity vector, speed, and acceleration vector:

$\mathbf{v}(t) = \mathbf{r}'(t)$ Velocity vector

$\|\mathbf{v}(t)\| = \dfrac{ds}{dt} = \|\mathbf{r}'(t)\|$ Speed

$\mathbf{a}(t) = \mathbf{r}''(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$ Acceleration vector

Unit tangent vector and principal unit normal vector:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \quad \text{and} \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

Components of acceleration:

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{d^2s}{dt^2}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = K\left(\frac{ds}{dt}\right)^2$$

Formulas for curvature in the plane:

$K = \dfrac{|y''|}{[1 + (y')^2]^{3/2}}$ C given by $y = f(x)$

$K = \dfrac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}}$ C given by $x = x(t)$, $y = y(t)$

Formulas for curvature in the plane or in space:

$K = \|\mathbf{T}'(s)\| = \|\mathbf{r}''(s)\|$ s is arc length parameter.

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$ t is general parameter.

$$K = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}(t)\|^2}$$

Cross product formulas apply only to curves in space.

12.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, sketch the plane curve and find its length over the given interval.

1. $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{j}, \quad [0, 3]$
2. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \quad [0, 4]$
3. $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}, \quad [0, 1]$
4. $\mathbf{r}(t) = (t + 1)\mathbf{i} + t^2\mathbf{j}, \quad [0, 6]$
5. $\mathbf{r}(t) = a\cos^3 t\,\mathbf{i} + a\sin^3 t\,\mathbf{j}, \quad [0, 2\pi]$
6. $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}, \quad [0, 2\pi]$

7. ***Projectile Motion*** A baseball is hit 3 feet above the ground at 100 feet per second and at an angle of 45° with respect to the ground.
   (a) Find the vector-valued function for the path of the baseball.
   (b) Find the maximum height.
   (c) Find the range.
   (d) Find the arc length of the trajectory.

8. ***Projectile Motion*** An object is launched from ground level. Determine the angle of the launch to obtain (a) the maximum height, (b) the maximum range, and (c) the maximum length of the trajectory. For part (c), let $v_0 = 96$ feet per second.

In Exercises 9–14, sketch the space curve and find its length over the given interval.

	Function	*Interval*
9.	$\mathbf{r}(t) = -t\mathbf{i} + 4t\mathbf{j} + 3t\mathbf{k}$	$[0, 1]$
10.	$\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$	$[0, 2]$
11.	$\mathbf{r}(t) = \langle 4t, -\cos t, \sin t\rangle$	$\left[0, \frac{3\pi}{2}\right]$
12.	$\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t\rangle$	$[0, \pi]$
13.	$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}$	$[0, 2\pi]$
14.	$\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t^2\rangle$	$\left[0, \frac{\pi}{2}\right]$

In Exercises 15 and 16, use the integration capabilities of a graphing utility to approximate the length of the space curve over the given interval.

	Function	*Interval*
15.	$\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \ln t\mathbf{k}$	$1 \le t \le 3$
16.	$\mathbf{r}(t) = \sin \pi t\,\mathbf{i} + \cos \pi t\,\mathbf{j} + t^3\mathbf{k}$	$0 \le t \le 2$

17. ***Investigation*** Consider the graph of the vector-valued function $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j} + t^3\mathbf{k}$ on the interval $[0, 2]$.

(a) Approximate the length of the curve by finding the length of the line segment connecting its endpoints.

(b) Approximate the length of the curve by summing the lengths of the line segments connecting the terminal points of the vectors $\mathbf{r}(0)$, $\mathbf{r}(0.5)$, $\mathbf{r}(1)$, $\mathbf{r}(1.5)$, and $\mathbf{r}(2)$.

(c) Describe how you could obtain a more accurate approximation by continuing the processes in parts (a) and (b).

(d) Use the integration capabilities of a graphing utility to approximate the length of the curve. Compare this result with the answers in parts (a) and (b).

18. ***Investigation*** Repeat Exercise 17 for the vector-valued function $\mathbf{r}(t) = 6\cos(\pi t/4)\mathbf{i} + 2\sin(\pi t/4)\mathbf{j} + t\mathbf{k}$.

19. ***Investigation*** Consider the helix represented by the vector-valued function $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$.

(a) Write the length of the arc s on the helix as a function of t by evaluating the integral

$$s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du.$$

(b) Solve for t in the relationship derived in part (a), and substitute the result into the original set of parametric equations. This yields a parametrization of the curve in terms of the arc length parameter s.

(c) Find the coordinates of the point on the helix for arc lengths $s = \sqrt{5}$ and $s = 4$.

(d) Verify that $\|\mathbf{r}'(s)\| = 1$.

20. ***Investigation*** Repeat Exercise 19 for the curve represented by the vector-valued function

$$\mathbf{r}(t) = \left\langle 4(\sin t - t\cos t), 4(\cos t + t\sin t), \tfrac{3}{2}t^2\right\rangle.$$

In Exercises 21–24, find the curvature *K* of the curve, where *s* is the arc length parameter.

21. $\mathbf{r}(s) = \left(1 + \frac{\sqrt{2}}{2}s\right)\mathbf{i} + \left(1 - \frac{\sqrt{2}}{2}s\right)\mathbf{j}$

22. $\mathbf{r}(s) = (3 + s)\mathbf{i} + \mathbf{j}$

23. Helix in Exercise 19: $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$

24. Curve in Exercise 20:

$\mathbf{r}(t) = \left\langle 4(\sin t - t\cos t), 4(\cos t + t\sin t), \frac{3}{2}t^2\right\rangle$

In Exercises 25–30, find the curvature *K* of the plane curve at the given value of the parameter.

25. $\mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}, \quad t = 1$

26. $\mathbf{r}(t) = t^2\mathbf{i} + \mathbf{j}, \quad t = 2$

27. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}, \quad t = 1$

28. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{9}t^3\mathbf{j}, \quad t = 2$

29. $\mathbf{r}(t) = \langle t, \sin t\rangle, \quad t = \frac{\pi}{2}$

30. $\mathbf{r}(t) = \langle 5\cos t, 4\sin t\rangle, \quad t = \frac{\pi}{3}$

In Exercises 31–40, find the curvature *K* of the curve.

31. $\mathbf{r}(t) = 4\cos 2\pi t\,\mathbf{i} + 4\sin 2\pi t\,\mathbf{j}$

32. $\mathbf{r}(t) = 2\cos \pi t\,\mathbf{i} + \sin \pi t\,\mathbf{j}$

33. $\mathbf{r}(t) = a\cos \omega t\,\mathbf{i} + a\sin \omega t\,\mathbf{j}$

34. $\mathbf{r}(t) = a\cos \omega t\,\mathbf{i} + b\sin \omega t\,\mathbf{j}$

35. $\mathbf{r}(t) = \langle a(\omega t - \sin \omega t), a(1 - \cos \omega t)\rangle$

36. $\mathbf{r}(t) = \langle \cos \omega t + \omega t\sin \omega t, \sin \omega t - \omega t\cos \omega t\rangle$

37. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$

38. $\mathbf{r}(t) = 2t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

39. $\mathbf{r}(t) = 4t\mathbf{i} + 3\cos t\,\mathbf{j} + 3\sin t\,\mathbf{k}$

40. $\mathbf{r}(t) = e^{2t}\mathbf{i} + e^{2t}\cos t\,\mathbf{j} + e^{2t}\sin t\,\mathbf{k}$

In Exercises 41–44, find the curvature *K* of the curve at the point *P*.

41. $\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j}, \quad P(-3, 2)$

42. $\mathbf{r}(t) = e^t\mathbf{i} + 4t\mathbf{j}, \quad P(1, 0)$

43. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^3}{4}\mathbf{k}, \quad P(2, 4, 2)$

44. $\mathbf{r}(t) = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j} + e^t\mathbf{k}, \quad P(1, 0, 1)$

In Exercises 45–54, find the curvature and radius of curvature of the plane curve at the given value of *x*.

45. $y = 3x - 2, \quad x = a$

46. $y = mx + b, \quad x = a$

47. $y = 2x^2 + 3, \quad x = -1$

48. $y = 2x + \frac{4}{x}, \quad x = 1$

49. $y = \cos 2x, \quad x = 2\pi$

50. $y = e^{3x}, \quad x = 0$

51. $y = \sqrt{a^2 - x^2}, \quad x = 0$

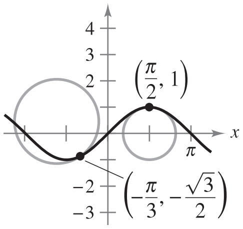
52. $y = \frac{3}{4}\sqrt{16 - x^2}, \quad x = 0$

53. $y = x^3, \quad x = 2$

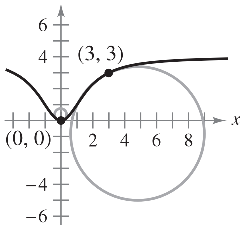
54. $y = x^n, \quad x = 1, \quad n \ge 2$

Writing **In Exercises 55 and 56, two circles of curvature to the graph of the function are given. (a) Find the equation of the smaller circle, and (b) write a short paragraph explaining why the circles have different radii.**

55. $f(x) = \sin x$



56. $f(x) = 4x^2/(x^2 + 3)$



In Exercises 57–60, use a graphing utility to graph the function. In the same viewing window, graph the circle of curvature to the graph at the given value of x.

57. $y = x + \dfrac{1}{x}, \quad x = 1$

58. $y = \ln x, \quad x = 1$

59. $y = e^x, \quad x = 0$

60. $y = \frac{1}{3}x^3, \quad x = 1$

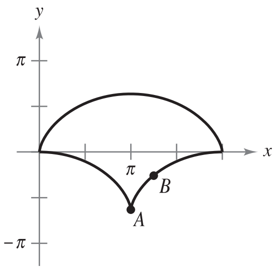
Evolute **An evolute is the curve formed by the set of centers of curvature of a curve. In Exercises 61 and 62, a curve and its evolute are given. Use a compass to sketch the circles of curvature with centers at points A and B. To print an enlarged copy of the graph, go to the website** ***www.mathgraphs.com*****.**

61. Cycloid: $x = t - \sin t$

$y = 1 - \cos t$

Evolute: $x = \sin t + t$

$y = \cos t - 1$

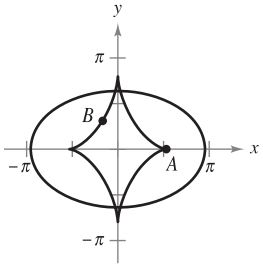


62. Ellipse: $x = 3\cos t$

$y = 2\sin t$

Evolute: $x = \frac{5}{3}\cos^3 t$

$y = \frac{5}{2}\sin^3 t$



In Exercises 63–70, (a) find the point on the curve at which the curvature K is a maximum and (b) find the limit of K as $x \to \infty$.

63. $y = (x - 1)^2 + 3$

64. $y = x^3$

65. $y = x^{2/3}$

66. $y = \dfrac{1}{x}$

67. $y = \ln x$

68. $y = e^x$

69. $y = \sinh x$

70. $y = \cosh x$

In Exercises 71–74, find all points on the graph of the function such that the curvature is zero.

71. $y = 1 - x^3$

72. $y = (x - 1)^3 + 3$

73. $y = \cos x$

74. $y = \sin x$

WRITING ABOUT CONCEPTS

75. (a) Give the formula for the arc length of a smooth curve in space.

(b) Give the formulas for curvature in the plane and in space.

76. Describe the graph of a vector-valued function for which the curvature is 0 for all values of t in its domain.

WRITING ABOUT CONCEPTS (continued)

77. Given a twice-differentiable function $y = f(x)$, determine its curvature at a relative extremum. Can the curvature ever be greater than it is at a relative extremum? Why or why not?

CAPSTONE

78. A particle moves along the plane curve C described by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$.

(a) Find the length of C on the interval $0 \le t \le 2$.

(b) Find the curvature K of the plane curve at $t = 0$, $t = 1$, and $t = 2$.

(c) Describe the curvature of C as t changes from $t = 0$ to $t = 2$.

79. Show that the curvature is greatest at the endpoints of the major axis, and is least at the endpoints of the minor axis, for the ellipse given by $x^2 + 4y^2 = 4$.

80. ***Investigation*** Find all a and b such that the two curves given by

$$y_1 = ax(b - x) \quad \text{and} \quad y_2 = \frac{x}{x + 2}$$

intersect at only one point and have a common tangent line and equal curvature at that point. Sketch a graph for each set of values of a and b.

CAS **81.** ***Investigation*** Consider the function $f(x) = x^4 - x^2$.

(a) Use a computer algebra system to find the curvature K of the curve as a function of x.

(b) Use the result of part (a) to find the circles of curvature to the graph of f when $x = 0$ and $x = 1$. Use a computer algebra system to graph the function and the two circles of curvature.

(c) Graph the function $K(x)$ and compare it with the graph of $f(x)$. For example, do the extrema of f and K occur at the same critical numbers? Explain your reasoning.

82. ***Investigation*** The surface of a goblet is formed by revolving the graph of the function

$$y = \tfrac{1}{4}x^{8/5}, \quad 0 \le x \le 5$$

about the y-axis. The measurements are given in centimeters.

CAS (a) Use a computer algebra system to graph the surface.

(b) Find the volume of the goblet.

(c) Find the curvature K of the generating curve as a function of x. Use a graphing utility to graph K.

(d) If a spherical object is dropped into the goblet, is it possible for it to touch the bottom? Explain.

83. A sphere of radius 4 is dropped into the paraboloid given by $z = x^2 + y^2$.

(a) How close will the sphere come to the vertex of the paraboloid?

(b) What is the radius of the largest sphere that will touch the vertex?

84. ***Speed*** The smaller the curvature of a bend in a road, the faster a car can travel. Assume that the maximum speed around a turn is inversely proportional to the square root of the curvature. A car moving on the path $y = \frac{1}{3}x^3$ (x and y are measured in miles) can safely go 30 miles per hour at $\left(1, \frac{1}{3}\right)$. How fast can it go at $\left(\frac{3}{2}, \frac{9}{8}\right)$?

85. Let C be a curve given by $y = f(x)$. Let K be the curvature ($K \neq 0$) at the point $P(x_0, y_0)$ and let

$$z = \frac{1 + f'(x_0)^2}{f''(x_0)}.$$

Show that the coordinates (α, β) of the center of curvature at P are $(\alpha, \beta) = (x_0 - f'(x_0)z, y_0 + z)$.

86. Use the result of Exercise 85 to find the center of curvature for the curve at the given point.

(a) $y = e^x, \quad (0, 1)$ (b) $y = \dfrac{x^2}{2}, \quad \left(1, \dfrac{1}{2}\right)$ (c) $y = x^2, \quad (0, 0)$

87. A curve C is given by the polar equation $r = f(\theta)$. Show that the curvature K at the point (r, θ) is

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}}$$

[*Hint:* Represent the curve by $\mathbf{r}(\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$.]

88. Use the result of Exercise 87 to find the curvature of each polar curve.

(a) $r = 1 + \sin\theta$ (b) $r = \theta$ (c) $r = a\sin\theta$ (d) $r = e^\theta$

89. Given the polar curve $r = e^{a\theta}$, $a > 0$, find the curvature K and determine the limit of K as (a) $\theta \to \infty$ and (b) $a \to \infty$.

90. Show that the formula for the curvature of a polar curve $r = f(\theta)$ given in Exercise 87 reduces to $K = 2/|r'|$ for the curvature *at the pole*.

In Exercises 91 and 92, use the result of Exercise 90 to find the curvature of the rose curve at the pole.

91. $r = 4\sin 2\theta$

92. $r = 6\cos 3\theta$

93. For a smooth curve given by the parametric equations $x = f(t)$ and $y = g(t)$, prove that the curvature is given by

$$K = \frac{\left|f'(t)g''(t) - g'(t)f''(t)\right|}{\{[f'(t)]^2 + [g'(t)]^2\}^{3/2}}.$$

94. Use the result of Exercise 93 to find the curvature K of the curve represented by the parametric equations $x(t) = t^3$ and $y(t) = \frac{1}{2}t^2$. Use a graphing utility to graph K and determine any horizontal asymptotes. Interpret the asymptotes in the context of the problem.

95. Use the result of Exercise 93 to find the curvature K of the cycloid represented by the parametric equations

$$x(\theta) = a(\theta - \sin\theta) \quad \text{and} \quad y(\theta) = a(1 - \cos\theta).$$

What are the minimum and maximum values of K?

96. Use Theorem 12.10 to find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ for each curve given by the vector-valued function.

(a) $\mathbf{r}(t) = 3t^2\mathbf{i} + (3t - t^3)\mathbf{j}$ (b) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

97. ***Frictional Force*** A 5500-pound vehicle is driven at a speed of 30 miles per hour on a circular interchange of radius 100 feet. To keep the vehicle from skidding off course, what frictional force must the road surface exert on the tires?

98. ***Frictional Force*** A 6400-pound vehicle is driven at a speed of 35 miles per hour on a circular interchange of radius 250 feet. To keep the vehicle from skidding off course, what frictional force must the road surface exert on the tires?

99. Verify that the curvature at any point (x, y) on the graph of $y = \cosh x$ is $1/y^2$.

100. Use the definition of curvature in space, $K = \|\mathbf{T}'(s)\| = \|\mathbf{r}''(s)\|$, to verify each formula.

(a) $K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$

(b) $K = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$

(c) $K = \dfrac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}(t)\|^2}$

True or False? **In Exercises 101–104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

101. The arc length of a space curve depends on the parametrization.

102. The curvature of a circle is the same as its radius.

103. The curvature of a line is 0.

104. The normal component of acceleration is a function of both speed and curvature.

Kepler's Laws **In Exercises 105–112, you are asked to verify Kepler's Laws of Planetary Motion. For these exercises, assume that each planet moves in an orbit given by the vector-valued function r. Let $r = \|\mathbf{r}\|$, let G represent the universal gravitational constant, let M represent the mass of the sun, and let m represent the mass of the planet.**

105. Prove that $\mathbf{r} \cdot \mathbf{r}' = r\dfrac{dr}{dt}$.

106. Using Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$, and Newton's Second Law of Gravitation, $\mathbf{F} = -(GmM/r^3)\mathbf{r}$, show that $\mathbf{a}$ and $\mathbf{r}$ are parallel, and that $\mathbf{r}(t) \times \mathbf{r}'(t) = \mathbf{L}$ is a constant vector. So, $\mathbf{r}(t)$ moves in a *fixed plane*, orthogonal to $\mathbf{L}$.

107. Prove that $\dfrac{d}{dt}\left[\dfrac{\mathbf{r}}{r}\right] = \dfrac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\}$.

108. Show that $\dfrac{\mathbf{r}'}{GM} \times \mathbf{L} - \dfrac{\mathbf{r}}{r} = \mathbf{e}$ is a constant vector.

109. Prove Kepler's First Law: Each planet moves in an elliptical orbit with the sun as a focus.

110. Assume that the elliptical orbit $r = ed/(1 + e\cos\theta)$ is in the xy-plane, with $\mathbf{L}$ along the z-axis. Prove that $\|\mathbf{L}\| = r^2\, d\theta/dt$.

111. Prove Kepler's Second Law: Each ray from the sun to a planet sweeps out equal areas of the ellipse in equal times.

112. Prove Kepler's Third Law: The square of the period of a planet's orbit is proportional to the cube of the mean distance between the planet and the sun.

12 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, (a) find the domain of r and (b) determine the values (if any) of t for which the function is continuous.

1. $\mathbf{r}(t) = \tan t\,\mathbf{i} + \mathbf{j} + t\,\mathbf{k}$

2. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \dfrac{1}{t-4}\mathbf{j} + \mathbf{k}$

3. $\mathbf{r}(t) = \ln t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}$

4. $\mathbf{r}(t) = (2t+1)\mathbf{i} + t^2\mathbf{j} + t\,\mathbf{k}$

In Exercises 5 and 6, evaluate (if possible) the vector-valued function at each given value of t.

5. $\mathbf{r}(t) = (2t+1)\mathbf{i} + t^2\mathbf{j} - \sqrt{t+2}\,\mathbf{k}$

(a) $\mathbf{r}(0)$ (b) $\mathbf{r}(-2)$ (c) $\mathbf{r}(c-1)$ (d) $\mathbf{r}(1+\Delta t) - \mathbf{r}(1)$

6. $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + (1-\sin t)\mathbf{j} - t\,\mathbf{k}$

(a) $\mathbf{r}(0)$ (b) $\mathbf{r}\left(\dfrac{\pi}{2}\right)$ (c) $\mathbf{r}(s-\pi)$ (d) $\mathbf{r}(\pi+\Delta t) - \mathbf{r}(\pi)$

In Exercises 7 and 8, sketch the plane curve represented by the vector-valued function and give the orientation of the curve.

7. $\mathbf{r}(t) = \langle \pi\cos t, \pi\sin t\rangle$

8. $\mathbf{r}(t) = \langle t+2, t^2-1\rangle$

CAS **In Exercises 9–14, use a computer algebra system to graph the space curve represented by the vector-valued function.**

9. $\mathbf{r}(t) = \mathbf{i} + t\,\mathbf{j} + t^2\mathbf{k}$

10. $\mathbf{r}(t) = t^2\mathbf{i} + 3t\,\mathbf{j} + t^3\mathbf{k}$

11. $\mathbf{r}(t) = \langle 1, \sin t, 1\rangle$

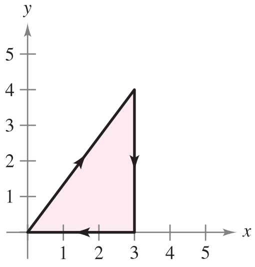
12. $\mathbf{r}(t) = \langle 2\cos t, t, 2\sin t\rangle$

13. $\mathbf{r}(t) = \left\langle t, \ln t, \tfrac{1}{2}t^2\right\rangle$

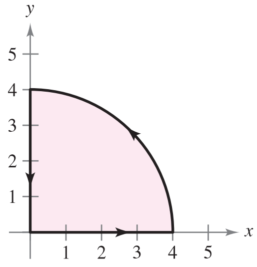
14. $\mathbf{r}(t) = \left\langle \tfrac{1}{2}t, \sqrt{t}, \tfrac{1}{4}t^3\right\rangle$

In Exercises 15 and 16, find vector-valued functions forming the boundaries of the region in the figure.

15.



16.



17. A particle moves on a straight-line path that passes through the points $(-2, -3, 8)$ and $(5, 1, -2)$. Find a vector-valued function for the path. (There are many correct answers.)

18. The outer edge of a spiral staircase is in the shape of a helix of radius 2 meters. The staircase has a height of 2 meters and is three-fourths of one complete revolution from bottom to top. Find a vector-valued function for the helix. (There are many correct answers.)

In Exercises 19 and 20, sketch the space curve represented by the intersection of the surfaces. Use the parameter $x = t$ to find a vector-valued function for the space curve.

19. $z = x^2 + y^2, \quad x + y = 0$

20. $x^2 + z^2 = 4, \quad x - y = 0$

In Exercises 21 and 22, evaluate the limit.

21. $\displaystyle\lim_{t\to 4^-} \left(t\,\mathbf{i} + \sqrt{4-t}\,\mathbf{j} + \mathbf{k}\right)$

22. $\displaystyle\lim_{t\to 0} \left(\frac{\sin 2t}{t}\mathbf{i} + e^{-t}\mathbf{j} + e^{t}\mathbf{k}\right)$

In Exercises 23 and 24, find the following.

(a) $\mathbf{r}'(t)$ **(b)** $\mathbf{r}''(t)$ **(c)** $D_t[\mathbf{r}(t)\cdot\mathbf{u}(t)]$

(d) $D_t[\mathbf{u}(t) - 2\mathbf{r}(t)]$ **(e)** $D_t[\|\mathbf{r}(t)\|],\ t > 0$ **(f)** $D_t[\mathbf{r}(t)\times\mathbf{u}(t)]$

23. $\mathbf{r}(t) = 3t\,\mathbf{i} + (t-1)\mathbf{j}, \quad \mathbf{u}(t) = t\,\mathbf{i} + t^2\mathbf{j} + \tfrac{2}{3}t^3\mathbf{k}$

24. $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\,\mathbf{k}, \quad \mathbf{u}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \dfrac{1}{t}\mathbf{k}$

25. ***Writing*** The x- and y-components of the derivative of the vector-valued function **u** are positive at $t = t_0$, and the z-component is negative. Describe the behavior of **u** at $t = t_0$.

26. ***Writing*** The x-component of the derivative of the vector-valued function **u** is 0 for t in the domain of the function. What does this information imply about the graph of **u**?

In Exercises 27–30, find the indefinite integral.

27. $\displaystyle\int (\cos t\,\mathbf{i} + t\cos t\,\mathbf{j})\,dt$

28. $\displaystyle\int (\ln t\,\mathbf{i} + t\ln t\,\mathbf{j} + \mathbf{k})\,dt$

29. $\displaystyle\int \|\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}\|\,dt$

30. $\displaystyle\int (t\,\mathbf{j} + t^2\mathbf{k}) \times (\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k})\,dt$

In Exercises 31–34, evaluate the definite integral.

31. $\displaystyle\int_{-2}^{2} (3t\,\mathbf{i} + 2t^2\mathbf{j} - t^3\mathbf{k})\,dt$

32. $\displaystyle\int_{0}^{1} \left(\sqrt{t}\,\mathbf{j} + t\sin t\,\mathbf{k}\right)dt$

33. $\displaystyle\int_{0}^{2} (e^{t/2}\,\mathbf{i} - 3t^2\mathbf{j} - \mathbf{k})\,dt$

34. $\displaystyle\int_{-1}^{1} (t^3\mathbf{i} + \arcsin t\,\mathbf{j} - t^2\mathbf{k})\,dt$

In Exercises 35 and 36, find r(t) for the given conditions.

35. $\mathbf{r}'(t) = 2t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}, \quad \mathbf{r}(0) = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$

36. $\mathbf{r}'(t) = \sec t\,\mathbf{i} + \tan t\,\mathbf{j} + t^2\mathbf{k}, \quad \mathbf{r}(0) = 3\mathbf{k}$

In Exercises 37–40, the position vector r describes the path of an object moving in space. Find the velocity, speed, and acceleration of the object.

37. $\mathbf{r}(t) = 4t\mathbf{i} + t^3\mathbf{j} - t\mathbf{k}$

38. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + 5t\mathbf{j} + 2t^2\mathbf{k}$

39. $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t, 3t\rangle$

40. $\mathbf{r}(t) = \langle t, -\tan t, e^t\rangle$

Linear Approximation **In Exercises 41 and 42, find a set of parametric equations for the tangent line to the graph of the vector-valued function at $t = t_0$. Use the equations for the line to approximate r($t_0 + 0.1$).**

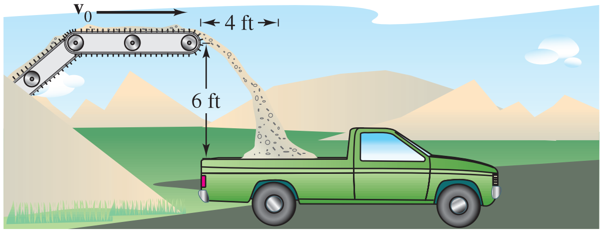
41. $\mathbf{r}(t) = \ln(t-3)\mathbf{i} + t^2\mathbf{j} + \tfrac{1}{2}t\,\mathbf{k}, \quad t_0 = 4$

42. $\mathbf{r}(t) = 3\cosh t\,\mathbf{i} + \sinh t\,\mathbf{j} - 2t\,\mathbf{k}, \quad t_0 = 0$

Projectile Motion **In Exercises 43–46, use the model for projectile motion, assuming there is no air resistance. [$a(t) = -32$ feet per second per second or $a(t) = -9.8$ meters per second per second.]**

43. A projectile is fired from ground level with an initial velocity of 84 feet per second at an angle of 30° with the horizontal. Find the range of the projectile.

44. The center of a truck bed is 6 feet below and 4 feet horizontally from the end of a horizontal conveyor that is discharging gravel (see figure). Determine the speed ds/dt at which the conveyor belt should be moving so that the gravel falls onto the center of the truck bed.



45. A projectile is fired from ground level at an angle of 20° with the horizontal. The projectile has a range of 95 meters. Find the minimum initial velocity.

46. Use a graphing utility to graph the paths of a projectile for $v_0 = 20$ meters per second, $h = 0$ and (a) $\theta = 30°$, (b) $\theta = 45°$, and (c) $\theta = 60°$. Use the graphs to approximate the maximum height and range of the projectile for each case.

In Exercises 47–54, find the velocity, speed, and acceleration at time t. Then find $\mathbf{a}\cdot\mathbf{T}$ and $\mathbf{a}\cdot\mathbf{N}$ at time t.

47. $\mathbf{r}(t) = (2 - t)\mathbf{i} + 3t\mathbf{j}$

48. $\mathbf{r}(t) = (1 + 4t)\mathbf{i} + (2 - 3t)\mathbf{j}$

49. $\mathbf{r}(t) = t\,\mathbf{i} + \sqrt{t}\,\mathbf{j}$

50. $\mathbf{r}(t) = 2(t + 1)\mathbf{i} + \dfrac{2}{t + 1}\mathbf{j}$

51. $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$

52. $\mathbf{r}(t) = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j}$

53. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + \dfrac{1}{2}t^2\,\mathbf{k}$

54. $\mathbf{r}(t) = (t - 1)\mathbf{i} + t\,\mathbf{j} + \dfrac{1}{t}\,\mathbf{k}$

In Exercises 55 and 56, find a set of parametric equations for the line tangent to the space curve at the given value of the parameter.

55. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\,\mathbf{k}, \quad t = \dfrac{\pi}{3}$

56. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{2}{3}t^3\,\mathbf{k}, \quad t = 2$

57. ***Satellite Orbit*** Find the speed necessary for a satellite to maintain a circular orbit 550 miles above the surface of Earth.

58. ***Centripetal Force*** An automobile in a circular traffic exchange is traveling at twice the posted speed. By what factor is the centripetal force increased over that which would occur at the posted speed?

In Exercises 59–62, sketch the plane curve and find its length over the given interval.

	Function	*Interval*
59.	$\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j}$	$[0, 5]$
60.	$\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{k}$	$[0, 3]$
61.	$\mathbf{r}(t) = 10\cos^3 t\,\mathbf{i} + 10\sin^3 t\,\mathbf{j}$	$[0, 2\pi]$
62.	$\mathbf{r}(t) = 10\cos t\,\mathbf{i} + 10\sin t\,\mathbf{j}$	$[0, 2\pi]$

In Exercises 63–66, sketch the space curve and find its length over the given interval.

	Function	*Interval*
63.	$\mathbf{r}(t) = -3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}$	$[0, 3]$
64.	$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$	$[0, 2]$
65.	$\mathbf{r}(t) = \langle 8\cos t, 8\sin t, t\rangle$	$[0, \pi/2]$
66.	$\mathbf{r}(t) = \langle 2(\sin t - t\cos t), 2(\cos t + t\sin t), t\rangle$	$[0, \pi/2]$

In Exercises 67–70, find the curvature K of the curve.

67. $\mathbf{r}(t) = 3t\mathbf{i} + 2t\mathbf{j}$

68. $\mathbf{r}(t) = 2\sqrt{t}\,\mathbf{i} + 3t\mathbf{j}$

69. $\mathbf{r}(t) = 2t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + t^2\mathbf{k}$

70. $\mathbf{r}(t) = 2t\mathbf{i} + 5\cos t\mathbf{j} + 5\sin t\mathbf{k}$

In Exercises 71 and 72, find the curvature K of the curve at point P.

71. $\mathbf{r}(t) = \dfrac{1}{2}t^2\mathbf{i} + t\mathbf{j} + \dfrac{1}{3}t^3\mathbf{k}, \quad P\left(\dfrac{1}{2}, 1, \dfrac{1}{3}\right)$

72. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{j} + t\mathbf{k}, \quad P(-4, 0, \pi)$

In Exercises 73–76, find the curvature and radius of curvature of the plane curve at the given value of x.

73. $y = \frac{1}{2}x^2 + 2, \quad x = 4$

74. $y = e^{-x/2}, \quad x = 0$

75. $y = \ln x, \quad x = 1$

76. $y = \tan x, \quad x = \dfrac{\pi}{4}$

77. ***Writing*** A civil engineer designs a highway as shown in the figure. BC is an arc of the circle. AB and CD are straight lines tangent to the circular arc. Criticize the design.

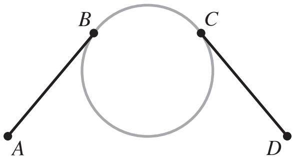


Figure for 77

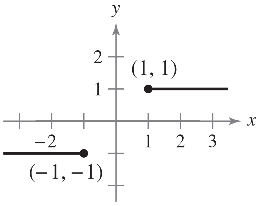


Figure for 78

78. A line segment extends horizontally to the left from the point $(-1, -1)$, and another line segment extends horizontally to the right from the point $(1, 1)$, as shown in the figure. Find a curve of the form

$$y = ax^5 + bx^3 + cx$$

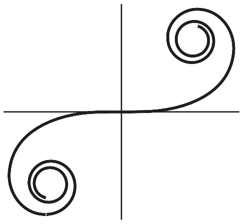
that connects the points $(-1, -1)$ and $(1, 1)$ so that the slope and curvature of the curve are zero at the endpoints.

P.S. PROBLEM SOLVING

1. The **cornu spiral** is given by

$$x(t) = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du \quad \text{and} \quad y(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du.$$

The spiral shown in the figure was plotted over the interval $-\pi \le t \le \pi$.



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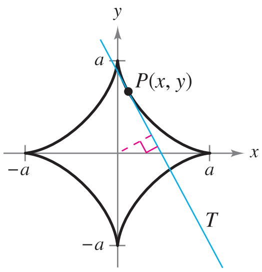
(a) Find the arc length of this curve from $t = 0$ to $t = a$.

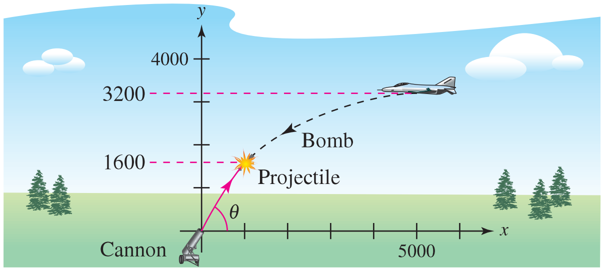
(b) Find the curvature of the graph when $t = a$.

(c) The cornu spiral was discovered by James Bernoulli. He found that the spiral has an amazing relationship between curvature and arc length. What is this relationship?

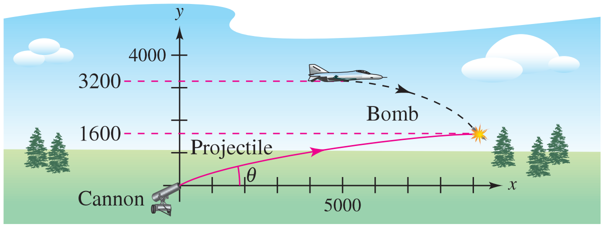
2. Let T be the tangent line at the point $P(x, y)$ to the graph of the curve $x^{2/3} + y^{2/3} = a^{2/3}$, $a > 0$, as shown in the figure. Show that the radius of curvature at P is three times the distance from the origin to the tangent line T.



3. A bomber is flying horizontally at an altitude of 3200 feet with a velocity of 400 feet per second when it releases a bomb. A projectile is launched 5 seconds later from a cannon at a site facing the bomber and 5000 feet from the point that was directly beneath the bomber when the bomb was released, as shown in the figure. The projectile is to intercept the bomb at an altitude of 1600 feet. Determine the required initial speed and angle of inclination of the projectile. (Ignore air resistance.)



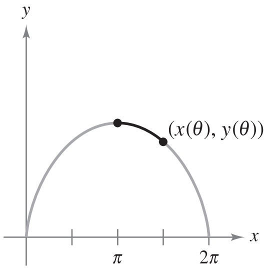
4. Repeat Exercise 3 for the case in which the bomber is facing *away* from the launch site, as shown in the figure.



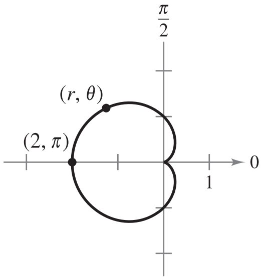
5. Consider one arch of the cycloid

$$\mathbf{r}(\theta) = (\theta - \sin\theta)\mathbf{i} + (1 - \cos\theta)\mathbf{j}, \quad 0 \le \theta \le 2\pi$$

as shown in the figure. Let $s(\theta)$ be the arc length from the highest point on the arch to the point $(x(\theta), y(\theta))$, and let $\rho(\theta) = \dfrac{1}{K}$ be the radius of curvature at the point $(x(\theta), y(\theta))$. Show that s and ρ are related by the equation $s^2 + \rho^2 = 16$. (This equation is called a *natural equation* for the curve.)



6. Consider the cardioid $r = 1 - \cos\theta$, $0 \le \theta \le 2\pi$, as shown in the figure. Let $s(\theta)$ be the arc length from the point $(2, \pi)$ on the cardioid to the point (r, θ), and let $\rho(\theta) = \dfrac{1}{K}$ be the radius of curvature at the point (r, θ). Show that s and ρ are related by the equation $s^2 + 9\rho^2 = 16$. (This equation is called a *natural equation* for the curve.)



7. If $\mathbf{r}(t)$ is a nonzero differentiable function of t, prove that

$$\frac{d}{dt}(\|\mathbf{r}(t)\|) = \frac{1}{\|\mathbf{r}(t)\|}\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

8. A communications satellite moves in a circular orbit around Earth at a distance of 42,000 kilometers from the center of Earth. The angular velocity

$$\frac{d\theta}{dt} = \omega = \frac{\pi}{12} \text{ radian per hour}$$

is constant.

(a) Use polar coordinates to show that the acceleration vector is given by

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u_r} + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{u}_\theta$$

where $\mathbf{u_r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ is the unit vector in the radial direction and $\mathbf{u}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$.

(b) Find the radial and angular components of acceleration for the satellite.

In Exercises 9–11, use the binormal vector defined by the equation $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

9. Find the unit tangent, unit normal, and binormal vectors for the helix $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 3t\mathbf{k}$ at $t = \frac{\pi}{2}$. Sketch the helix together with these three mutually orthogonal unit vectors.

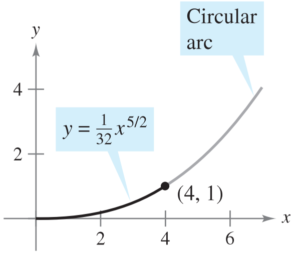
10. Find the unit tangent, unit normal, and binormal vectors for the curve $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} - \mathbf{k}$ at $t = \frac{\pi}{4}$. Sketch the curve together with these three mutually orthogonal unit vectors.

11. (a) Prove that there exists a scalar τ, called the **torsion,** such that $d\mathbf{B}/ds = -\tau\mathbf{N}$.

(b) Prove that $\frac{d\mathbf{N}}{ds} = -K\mathbf{T} + \tau\mathbf{B}$.

(The three equations $d\mathbf{T}/ds = K\mathbf{N}$, $d\mathbf{N}/ds = -K\mathbf{T} + \tau\mathbf{B}$, and $d\mathbf{B}/ds = -\tau\mathbf{N}$ are called the *Frenet-Serret formulas.*)

12. A highway has an exit ramp that begins at the origin of a coordinate system and follows the curve $y = \frac{1}{32}x^{5/2}$ to the point (4, 1) (see figure). Then it follows a circular path whose curvature is that given by the curve at (4, 1). What is the radius of the circular arc? Explain why the curve and the circular arc should have the same curvature at (4, 1).



13. Consider the vector-valued function $\mathbf{r}(t) = \langle t\cos\pi t, t\sin\pi t\rangle$, $0 \le t \le 2$.

(a) Use a graphing utility to graph the function.

(b) Find the length of the arc in part (a).

(c) Find the curvature K as a function of t. Find the curvatures for t-values of 0, 1, and 2.

(d) Use a graphing utility to graph the function K.

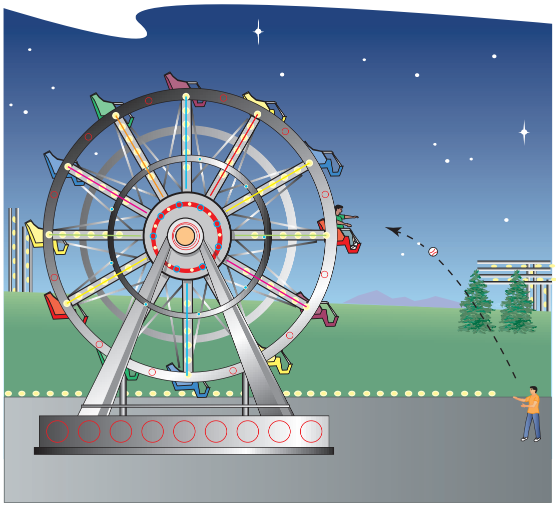
(e) Find (if possible) $\lim_{t\to\infty} K$.

(f) Using the result of part (e), make a conjecture about the graph of $\mathbf{r}$ as $t \to \infty$.

14. You want to toss an object to a friend who is riding a Ferris wheel (see figure). The following parametric equations give the path of the friend $\mathbf{r}_1(t)$ and the path of the object $\mathbf{r}_2(t)$. Distance is measured in meters and time is measured in seconds.

$$\mathbf{r}_1(t) = 15\left(\sin\frac{\pi t}{10}\right)\mathbf{i} + \left(16 - 15\cos\frac{\pi t}{10}\right)\mathbf{j}$$

$$\mathbf{r}_2(t) = [22 - 8.03(t - t_0)]\mathbf{i} + [1 + 11.47(t - t_0) - 4.9(t - t_0)^2]\mathbf{j}$$



(a) Locate your friend's position on the Ferris wheel at time $t = 0$.

(b) Determine the number of revolutions per minute of the Ferris wheel.

(c) What are the speed and angle of inclination (in degrees) at which the object is thrown at time $t = t_0$?

(d) Use a graphing utility to graph the vector-valued functions using a value of t_0 that allows your friend to be within reach of the object. (Do this by trial and error.) Explain the significance of t_0.

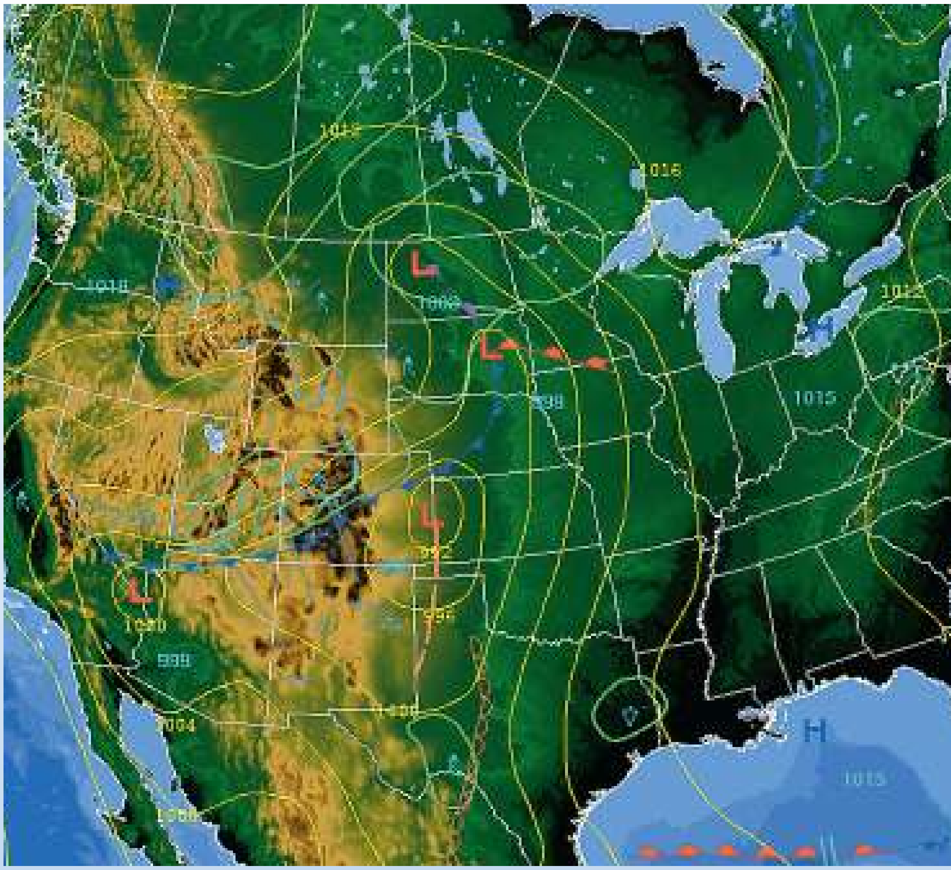
(e) Find the approximate time your friend should be able to catch the object. Approximate the speeds of your friend and the object at that time.

13 Functions of Several Variables

In this chapter, you will study functions of more than one independent variable. Many of the concepts presented are extensions of familiar ideas from earlier chapters.

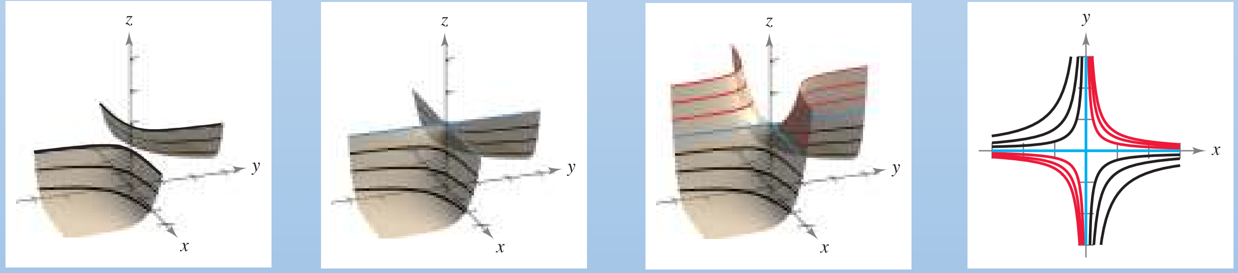
In this chapter, you should learn the following.

- How to sketch a graph, level curves, and level surfaces. (13.1)
- How to find a limit and determine continuity. (13.2)
- How to find and use a partial derivative. (13.3)
- How to find and use a total differential and determine differentiability. (13.4)
- How to use the Chain Rules and find a partial derivative implicitly. (13.5)
- How to find and use a directional derivative and a gradient. (13.6)
- How to find an equation of a tangent plane and an equation of a normal line to a surface, and how to find the angle of inclination of a plane. (13.7)
- How to find absolute and relative extrema. (13.8)
- How to solve an optimization problem, including constrained optimization using a Lagrange multiplier, and how to use the method of least squares. (13.9, 13.10)



NOAA

Meteorologists use maps that show curves of equal atmospheric pressure, called *isobars*, to predict weather patterns. How can you use pressure gradients to determine the area of the country that has the greatest wind speed? (See Section 13.6, Exercise 68.)



Many real-life quantities are functions of two or more variables. In Section 13.1, you will learn how to graph a function of two variables, like the one shown above. The first three graphs show cut-away views of the surface at various traces. Another way to visualize this surface is to project the traces onto the *xy*-plane, as shown in the fourth graph.

13.1 Introduction to Functions of Several Variables

- **Understand the notation for a function of several variables.**
- **Sketch the graph of a function of two variables.**
- **Sketch level curves for a function of two variables.**
- **Sketch level surfaces for a function of three variables.**
- **Use computer graphics to graph a function of two variables.**

Functions of Several Variables

So far in this text, you have dealt only with functions of a single (independent) variable. Many familiar quantities, however, are functions of two or more variables. For instance, the work done by a force $(W = FD)$ and the volume of a right circular cylinder $(V = \pi r^2 h)$ are both functions of two variables. The volume of a rectangular solid $(V = lwh)$ is a function of three variables. The notation for a function of two or more variables is similar to that for a function of a single variable. Here are two examples.

$$z = f\underbrace{(x, y)}_{\text{2 variables}} = x^2 + xy \qquad \text{Function of two variables}$$

and

$$w = f\underbrace{(x, y, z)}_{\text{3 variables}} = x + 2y - 3z \qquad \text{Function of three variables}$$

EXPLORATION

Comparing Dimensions
Without using a graphing utility, describe the graph of each function of two variables.

a. $z = x^2 + y^2$
b. $z = x + y$
c. $z = x^2 + y$
d. $z = \sqrt{x^2 + y^2}$
e. $z = \sqrt{1 - x^2 + y^2}$

DEFINITION OF A FUNCTION OF TWO VARIABLES

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is called a **function of x and y.** The set D is the **domain** of f, and the corresponding set of values for $f(x, y)$ is the **range** of f.

For the function given by $z = f(x, y)$, x and y are called the **independent variables** and z is called the **dependent variable.**

Similar definitions can be given for functions of three, four, or n variables, where the domains consist of ordered triples (x_1, x_2, x_3), quadruples (x_1, x_2, x_3, x_4), and n-tuples $(x_1, x_2, \ldots, x_n)$. In all cases, the range is a set of real numbers. In this chapter, you will study only functions of two or three variables.

As with functions of one variable, the most common way to describe a function of several variables is with an *equation*, and unless otherwise restricted, you can assume that the domain is the set of all points for which the equation is defined. For instance, the domain of the function given by

$$f(x, y) = x^2 + y^2$$

is assumed to be the entire xy-plane. Similarly, the domain of

$$f(x, y) = \ln xy$$

is the set of all points (x, y) in the plane for which $xy > 0$. This consists of all points in the first and third quadrants.

Archive Photos

MARY FAIRFAX SOMERVILLE (1780–1872)

Somerville was interested in the problem of creating geometric models for functions of several variables. Her most well-known book, *The Mechanics of the Heavens*, was published in 1831.

EXAMPLE 1 Domains of Functions of Several Variables

Find the domain of each function.

a. $f(x, y) = \dfrac{\sqrt{x^2 + y^2 - 9}}{x}$ **b.** $g(x, y, z) = \dfrac{x}{\sqrt{9 - x^2 - y^2 - z^2}}$

Solution

a. The function f is defined for all points (x, y) such that $x \neq 0$ and

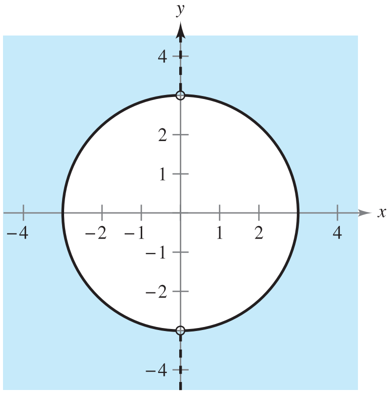
$$x^2 + y^2 \geq 9.$$

So, the domain is the set of all points lying on or outside the circle $x^2 + y^2 = 9$, *except* those points on the y-axis, as shown in Figure 13.1.

b. The function g is defined for all points (x, y, z) such that

$$x^2 + y^2 + z^2 < 9.$$

Consequently, the domain is the set of all points (x, y, z) lying inside a sphere of radius 3 that is centered at the origin. ■



Domain of
$f(x, y) = \dfrac{\sqrt{x^2 + y^2 - 9}}{x}$

Figure 13.1

Functions of several variables can be combined in the same ways as functions of single variables. For instance, you can form the sum, difference, product, and quotient of two functions of two variables as follows.

$$(f \pm g)(x, y) = f(x, y) \pm g(x, y)$$ Sum or difference

$$(fg)(x, y) = f(x, y)g(x, y)$$ Product

$$\frac{f}{g}(x, y) = \frac{f(x, y)}{g(x, y)}, \quad g(x, y) \neq 0$$ Quotient

You cannot form the composite of two functions of several variables. However, if h is a function of several variables and g is a function of a single variable, you can form the **composite** function $(g \circ h)(x, y)$ as follows.

$$(g \circ h)(x, y) = g(h(x, y))$$ Composition

The domain of this composite function consists of all (x, y) in the domain of h such that $h(x, y)$ is in the domain of g. For example, the function given by

$$f(x, y) = \sqrt{16 - 4x^2 - y^2}$$

can be viewed as the composite of the function of two variables given by $h(x, y) = 16 - 4x^2 - y^2$ and the function of a single variable given by $g(u) = \sqrt{u}$. The domain of this function is the set of all points lying on or inside the ellipse given by $4x^2 + y^2 = 16$.

A function that can be written as a sum of functions of the form cx^my^n (where c is a real number and m and n are nonnegative integers) is called a **polynomial function** of two variables. For instance, the functions given by

$$f(x, y) = x^2 + y^2 - 2xy + x + 2 \quad \text{and} \quad g(x, y) = 3xy^2 + x - 2$$

are polynomial functions of two variables. A **rational function** is the quotient of two polynomial functions. Similar terminology is used for functions of more than two variables.

The Graph of a Function of Two Variables

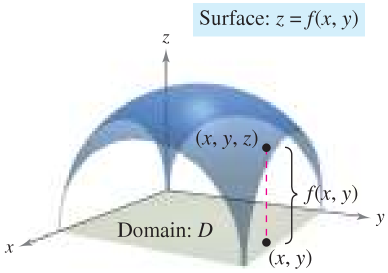


Figure 13.2

As with functions of a single variable, you can learn a lot about the behavior of a function of two variables by sketching its graph. The **graph** of a function f of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of f. This graph can be interpreted geometrically as a *surface in space*, as discussed in Sections 11.5 and 11.6. In Figure 13.2, note that the graph of $z = f(x, y)$ is a surface whose projection onto the xy-plane is D, the domain of f. To each point (x, y) in D there corresponds a point (x, y, z) on the surface, and, conversely, to each point (x, y, z) on the surface there corresponds a point (x, y) in D.

EXAMPLE 2 Describing the Graph of a Function of Two Variables

What is the range of $f(x, y) = \sqrt{16 - 4x^2 - y^2}$? Describe the graph of f.

Solution The domain D implied by the equation of f is the set of all points (x, y) such that $16 - 4x^2 - y^2 \geq 0$. So, D is the set of all points lying on or inside the ellipse given by

$$\frac{x^2}{4} + \frac{y^2}{16} = 1. \qquad \text{Ellipse in the } xy\text{-plane}$$

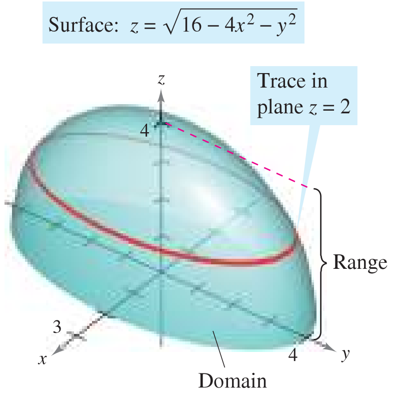
The range of f is all values $z = f(x, y)$ such that $0 \leq z \leq \sqrt{16}$ or

$$0 \leq z \leq 4. \qquad \text{Range of } f$$

A point (x, y, z) is on the graph of f if and only if

$$\begin{aligned} z &= \sqrt{16 - 4x^2 - y^2} \\ z^2 &= 16 - 4x^2 - y^2 \\ 4x^2 + y^2 + z^2 &= 16 \\ \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} &= 1, \qquad 0 \leq z \leq 4. \end{aligned}$$

From Section 11.6, you know that the graph of f is the upper half of an ellipsoid, as shown in Figure 13.3. ■



The graph of $f(x, y) = \sqrt{16 - 4x^2 - y^2}$ is the upper half of an ellipsoid.
Figure 13.3

To sketch a surface in space *by hand,* it helps to use traces in planes parallel to the coordinate planes, as shown in Figure 13.3. For example, to find the trace of the surface in the plane $z = 2$, substitute $z = 2$ in the equation $z = \sqrt{16 - 4x^2 - y^2}$ and obtain

$$2 = \sqrt{16 - 4x^2 - y^2} \implies \frac{x^2}{3} + \frac{y^2}{12} = 1.$$

So, the trace is an ellipse centered at the point $(0, 0, 2)$ with major and minor axes of lengths $4\sqrt{3}$ and $2\sqrt{3}$.

Traces are also used with most three-dimensional graphing utilities. For instance, Figure 13.4 shows a computer-generated version of the surface given in Example 2. For this graph, the computer took 25 traces parallel to the xy-plane and 12 traces in vertical planes.

If you have access to a three-dimensional graphing utility, use it to graph several surfaces.

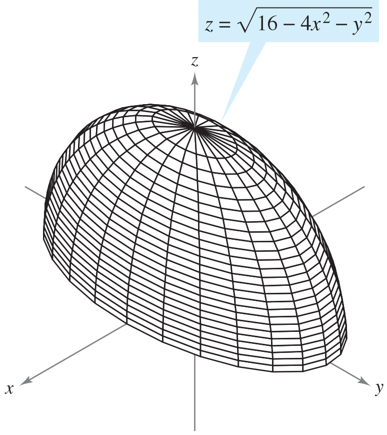
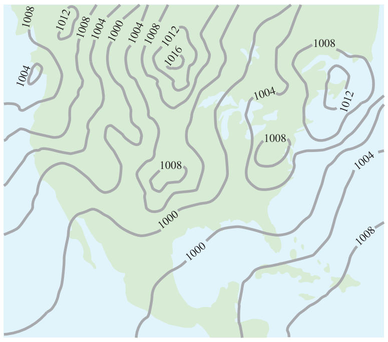


Figure 13.4

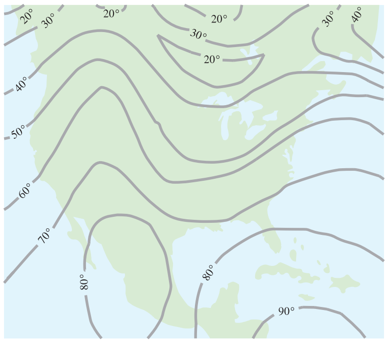
Level Curves

A second way to visualize a function of two variables is to use a **scalar field** in which the scalar $z = f(x, y)$ is assigned to the point (x, y). A scalar field can be characterized by **level curves** (or **contour lines**) along which the value of $f(x, y)$ is constant. For instance, the weather map in Figure 13.5 shows level curves of equal pressure called **isobars.** In weather maps for which the level curves represent points of equal temperature, the level curves are called **isotherms,** as shown in Figure 13.6. Another common use of level curves is in representing electric potential fields. In this type of map, the level curves are called **equipotential lines.**



Level curves show the lines of equal pressure (isobars) measured in millibars.

Figure 13.5



Level curves show the lines of equal temperature (isotherms) measured in degrees Fahrenheit.

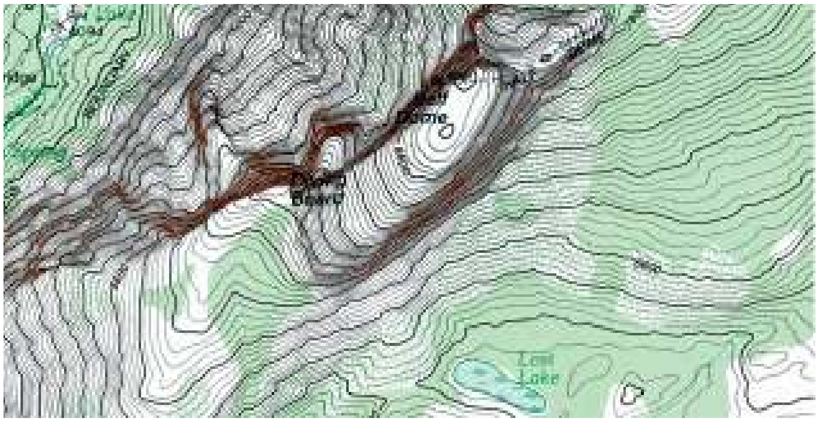
Figure 13.6

Contour maps are commonly used to show regions on Earth's surface, with the level curves representing the height above sea level. This type of map is called a **topographic map.** For example, the mountain shown in Figure 13.7 is represented by the topographic map in Figure 13.8.

A contour map depicts the variation of z with respect to x and y by the spacing between level curves. Much space between level curves indicates that z is changing slowly, whereas little space indicates a rapid change in z. Furthermore, to produce a good three-dimensional illusion in a contour map, it is important to choose c-values that are *evenly spaced*.

Alfred B. Thomas/Earth Scenes

Figure 13.7



USGS

Figure 13.8

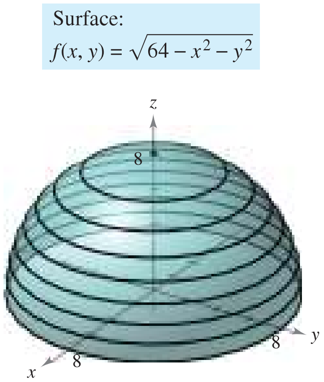
EXAMPLE 3 Sketching a Contour Map

The hemisphere given by $f(x, y) = \sqrt{64 - x^2 - y^2}$ is shown in Figure 13.9. Sketch a contour map of this surface using level curves corresponding to $c = 0, 1, 2, \ldots, 8$.

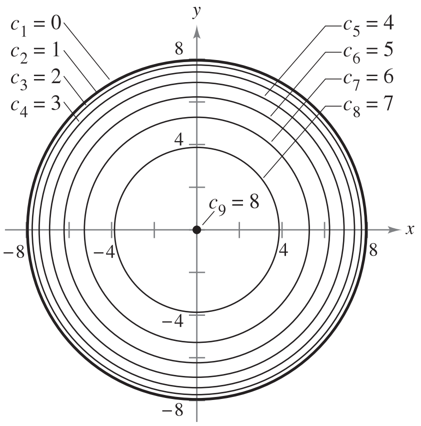
Solution For each value of c, the equation given by $f(x, y) = c$ is a circle (or point) in the xy-plane. For example, when $c_1 = 0$, the level curve is

$$x^2 + y^2 = 64 \qquad \text{Circle of radius 8}$$

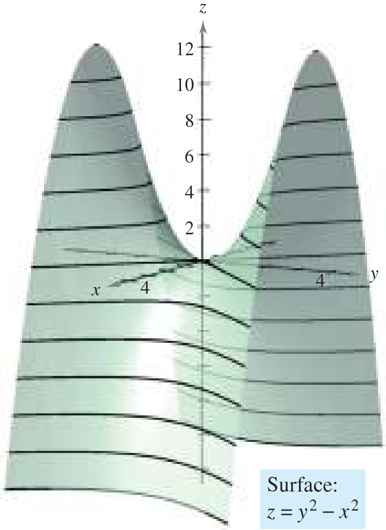
which is a circle of radius 8. Figure 13.10 shows the nine level curves for the hemisphere.



Hemisphere
Figure 13.9



Contour map
Figure 13.10



Hyperbolic paraboloid
Figure 13.11

EXAMPLE 4 Sketching a Contour Map

The hyperbolic paraboloid given by

$$z = y^2 - x^2$$

is shown in Figure 13.11. Sketch a contour map of this surface.

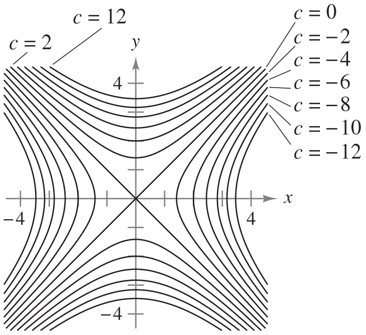
Solution For each value of c, let $f(x, y) = c$ and sketch the resulting level curve in the xy-plane. For this function, each of the level curves $(c \neq 0)$ is a hyperbola whose asymptotes are the lines $y = \pm x$. If $c < 0$, the transverse axis is horizontal. For instance, the level curve for $c = -4$ is given by

$$\frac{x^2}{2^2} - \frac{y^2}{2^2} = 1. \qquad \text{Hyperbola with horizontal transverse axis}$$

If $c > 0$, the transverse axis is vertical. For instance, the level curve for $c = 4$ is given by

$$\frac{y^2}{2^2} - \frac{x^2}{2^2} = 1. \qquad \text{Hyperbola with vertical transverse axis}$$

If $c = 0$, the level curve is the degenerate conic representing the intersecting asymptotes, as shown in Figure 13.12. ■



Hyperbolic level curves (at increments of 2)
Figure 13.12

The icon ↻ *indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems* Maple *and* Mathematica.

One example of a function of two variables used in economics is the **Cobb-Douglas production function.** This function is used as a model to represent the numbers of units produced by varying amounts of labor and capital. If x measures the units of labor and y measures the units of capital, the number of units produced is given by

$$f(x, y) = Cx^a y^{1-a}$$

where C and a are constants with $0 < a < 1$.

EXAMPLE 5 The Cobb-Douglas Production Function

A toy manufacturer estimates a production function to be $f(x, y) = 100x^{0.6}y^{0.4}$, where x is the number of units of labor and y is the number of units of capital. Compare the production level when $x = 1000$ and $y = 500$ with the production level when $x = 2000$ and $y = 1000$.

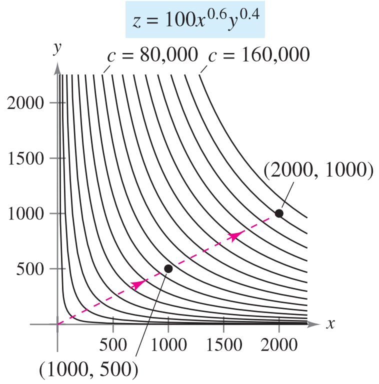
Solution When $x = 1000$ and $y = 500$, the production level is

$$f(1000, 500) = 100(1000^{0.6})(500^{0.4}) \approx 75{,}786.$$

When $x = 2000$ and $y = 1000$, the production level is

$$f(2000, 1000) = 100(2000^{0.6})(1000^{0.4}) = 151{,}572.$$

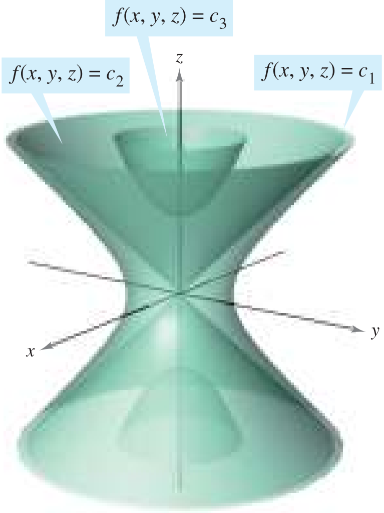
The level curves of $z = f(x, y)$ are shown in Figure 13.13. Note that by doubling both x and y, you double the production level (see Exercise 79). ■



Level curves (at increments of 10,000)
Figure 13.13

Level Surfaces

The concept of a level curve can be extended by one dimension to define a **level surface.** If f is a function of three variables and c is a constant, the graph of the equation $f(x, y, z) = c$ is a **level surface** of the function f, as shown in Figure 13.14.



Level surfaces of f
Figure 13.14

With computers, engineers and scientists have developed other ways to view functions of three variables. For instance, Figure 13.15 shows a computer simulation that uses color to represent the temperature distribution of fluid inside a pipe fitting.

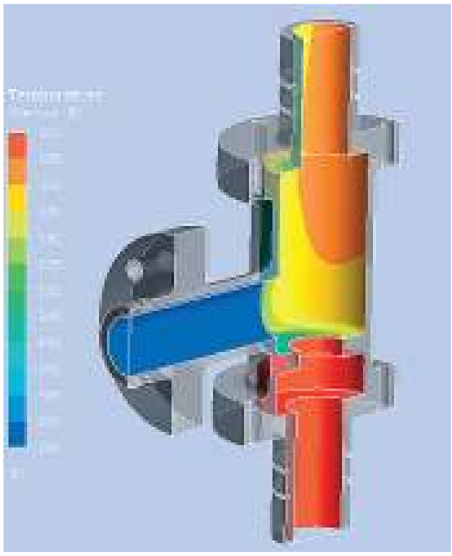


Image courtesy of CADFEM GmbH

One-way coupling of ANSYS CFX™ and ANSYS Mechanical™ for thermal stress analysis
Figure 13.15

EXAMPLE 6 Level Surfaces

Describe the level surfaces of the function

$$f(x, y, z) = 4x^2 + y^2 + z^2.$$

Solution Each level surface has an equation of the form

$$4x^2 + y^2 + z^2 = c. \quad \text{Equation of level surface}$$

So, the level surfaces are ellipsoids (whose cross sections parallel to the yz-plane are circles). As c increases, the radii of the circular cross sections increase according to the square root of c. For example, the level surfaces corresponding to the values $c = 0$, $c = 4$, and $c = 16$ are as follows.

$$4x^2 + y^2 + z^2 = 0 \quad \text{Level surface for } c = 0 \text{ (single point)}$$

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{4} = 1 \quad \text{Level surface for } c = 4 \text{ (ellipsoid)}$$

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1 \quad \text{Level surface for } c = 16 \text{ (ellipsoid)}$$

These level surfaces are shown in Figure 13.16. ■

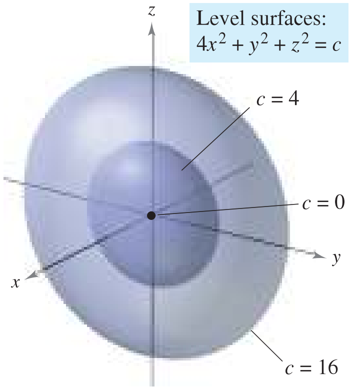


Figure 13.16

NOTE If the function in Example 6 represented the *temperature* at the point (x, y, z), the level surfaces shown in Figure 13.16 would be called **isothermal surfaces.** ■

Computer Graphics

The problem of sketching the graph of a surface in space can be simplified by using a computer. Although there are several types of three-dimensional graphing utilities, most use some form of trace analysis to give the illusion of three dimensions. To use such a graphing utility, you usually need to enter the equation of the surface, the region in the xy-plane over which the surface is to be plotted, and the number of traces to be taken. For instance, to graph the surface given by

$$f(x, y) = (x^2 + y^2)e^{1-x^2-y^2}$$

you might choose the following bounds for x, y, and z.

$$-3 \le x \le 3 \quad \text{Bounds for } x$$

$$-3 \le y \le 3 \quad \text{Bounds for } y$$

$$0 \le z \le 3 \quad \text{Bounds for } z$$

Figure 13.17 shows a computer-generated graph of this surface using 26 traces taken parallel to the yz-plane. To heighten the three-dimensional effect, the program uses a "hidden line" routine. That is, it begins by plotting the traces in the foreground (those corresponding to the largest x-values), and then, as each new trace is plotted, the program determines whether all or only part of the next trace should be shown.

The graphs on page 893 show a variety of surfaces that were plotted by computer. If you have access to a computer drawing program, use it to reproduce these surfaces. Remember also that the three-dimensional graphics in this text can be viewed and rotated. These rotatable graphs are available in the premium eBook for this text.

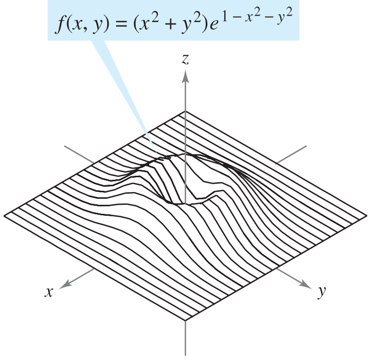
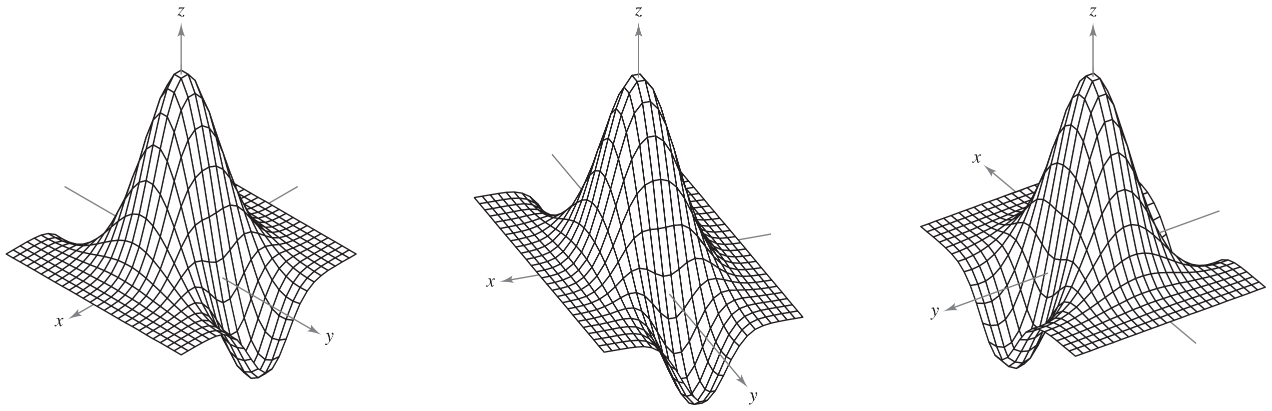
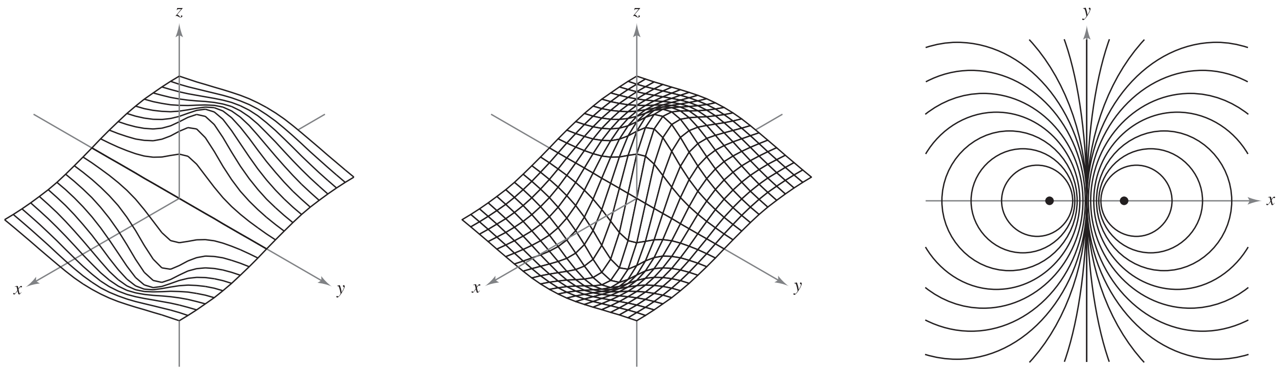


Figure 13.17



Three different views of the graph of $f(x, y) = (2 - y^2 + x^2)e^{1-x^2-(y^2/4)}$

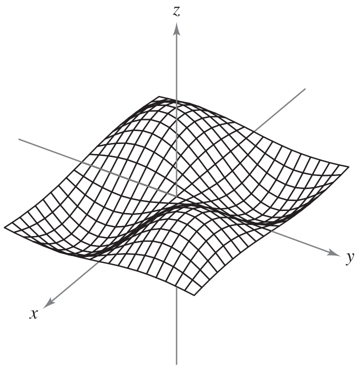


Single traces

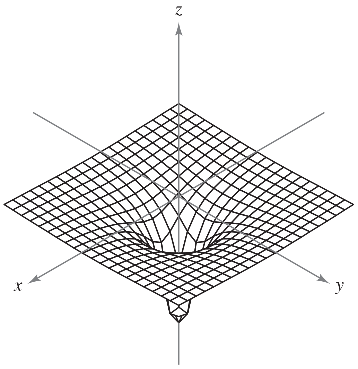
Double traces

Level curves

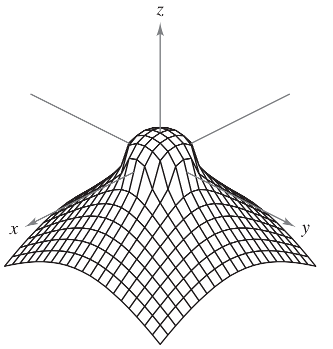
Traces and level curves of the graph of $f(x, y) = \dfrac{-4x}{x^2 + y^2 + 1}$



$f(x, y) = \sin x \sin y$



$f(x, y) = -\dfrac{1}{\sqrt{x^2 + y^2}}$



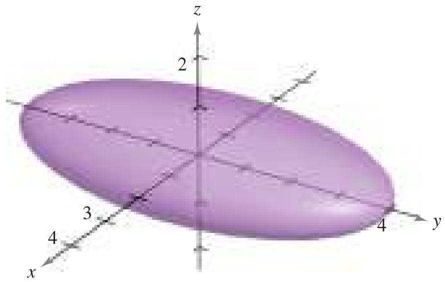
$f(x, y) = \dfrac{1 - x^2 - y^2}{\sqrt{|1 - x^2 - y^2|}}$

13.1 Exercises

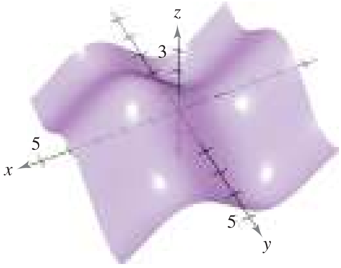
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use the graph to determine whether z is a function of x and y. Explain.

1.



2.



In Exercises 3–6, determine whether z is a function of x and y.

3. $x^2z + 3y^2 - xy = 10$

4. $xz^2 + 2xy - y^2 = 4$

5. $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$

6. $z + x \ln y - 8yz = 0$

In Exercises 7–18, find and simplify the function values.

7. $f(x, y) = xy$

(a) $(3, 2)$ (b) $(-1, 4)$ (c) $(30, 5)$
(d) $(5, y)$ (e) $(x, 2)$ (f) $(5, t)$

8. $f(x, y) = 4 - x^2 - 4y^2$

(a) $(0, 0)$ (b) $(0, 1)$ (c) $(2, 3)$
(d) $(1, y)$ (e) $(x, 0)$ (f) $(t, 1)$

9. $f(x, y) = xe^y$

(a) $(5, 0)$ (b) $(3, 2)$ (c) $(2, -1)$
(d) $(5, y)$ (e) $(x, 2)$ (f) (t, t)

10. $g(x, y) = \ln|x + y|$

(a) $(1, 0)$ (b) $(0, -1)$ (c) $(0, e)$
(d) $(1, 1)$ (e) $(e, e/2)$ (f) $(2, 5)$

11. $h(x, y, z) = \dfrac{xy}{z}$

(a) $(2, 3, 9)$ (b) $(1, 0, 1)$ (c) $(-2, 3, 4)$ (d) $(5, 4, -6)$

12. $f(x, y, z) = \sqrt{x + y + z}$

(a) $(0, 5, 4)$ (b) $(6, 8, -3)$
(c) $(4, 6, 2)$ (d) $(10, -4, -3)$

13. $f(x, y) = x \sin y$

(a) $(2, \pi/4)$ (b) $(3, 1)$ (c) $(-3, \pi/3)$ (d) $(4, \pi/2)$

14. $V(r, h) = \pi r^2 h$

(a) $(3, 10)$ (b) $(5, 2)$ (c) $(4, 8)$ (d) $(6, 4)$

15. $g(x, y) = \displaystyle\int_x^y (2t - 3)\, dt$

(a) $(4, 0)$ (b) $(4, 1)$ (c) $\left(4, \frac{3}{2}\right)$ (d) $\left(\frac{3}{2}, 0\right)$

16. $g(x, y) = \displaystyle\int_x^y \frac{1}{t}\, dt$

(a) $(4, 1)$ (b) $(6, 3)$ (c) $(2, 5)$ (d) $\left(\frac{1}{2}, 7\right)$

17. $f(x, y) = 2x + y^2$

(a) $\dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$

(b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

18. $f(x, y) = 3x^2 - 2y$

(a) $\dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$

(b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

In Exercises 19–30, describe the domain and range of the function.

19. $f(x, y) = x^2 + y^2$

20. $f(x, y) = e^{xy}$

21. $g(x, y) = x\sqrt{y}$

22. $g(x, y) = \dfrac{y}{\sqrt{x}}$

23. $z = \dfrac{x + y}{xy}$

24. $z = \dfrac{xy}{x - y}$

25. $f(x, y) = \sqrt{4 - x^2 - y^2}$

26. $f(x, y) = \sqrt{4 - x^2 - 4y^2}$

27. $f(x, y) = \arccos(x + y)$

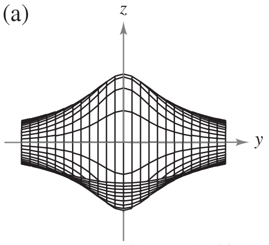
28. $f(x, y) = \arcsin(y/x)$

29. $f(x, y) = \ln(4 - x - y)$

30. $f(x, y) = \ln(xy - 6)$

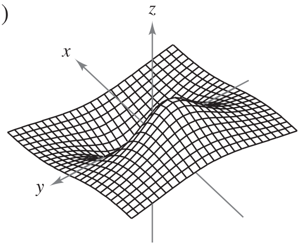
31. ***Think About It*** The graphs labeled (a), (b), (c), and (d) are graphs of the function $f(x, y) = -4x/(x^2 + y^2 + 1)$. Match the four graphs with the points in space from which the surface is viewed. The four points are $(20, 15, 25)$, $(-15, 10, 20)$, $(20, 20, 0)$, and $(20, 0, 0)$.

(a)



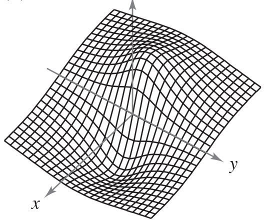
Generated by Maple

(b)



Generated by Maple

(c)



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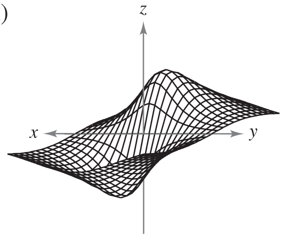
(d)



Generated by Maple

32. ***Think About It*** Use the function given in Exercise 31.

(a) Find the domain and range of the function.

(b) Identify the points in the xy-plane at which the function value is 0.

(c) Does the surface pass through all the octants of the rectangular coordinate system? Give reasons for your answer.

In Exercises 33–40, sketch the surface given by the function.

33. $f(x, y) = 4$

34. $f(x, y) = 6 - 2x - 3y$

35. $f(x, y) = y^2$

36. $g(x, y) = \frac{1}{2}y$

37. $z = -x^2 - y^2$

38. $z = \frac{1}{2}\sqrt{x^2 + y^2}$

39. $f(x, y) = e^{-x}$

40. $f(x, y) = \begin{cases} xy, & x \geq 0, y \geq 0 \\ 0, & x < 0 \text{ or } y < 0 \end{cases}$

CAS **In Exercises 41–44, use a computer algebra system to graph the function.**

41. $z = y^2 - x^2 + 1$

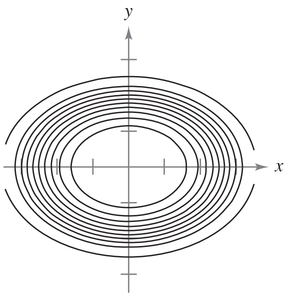
42. $z = \frac{1}{12}\sqrt{144 - 16x^2 - 9y^2}$

43. $f(x, y) = x^2e^{(-xy/2)}$

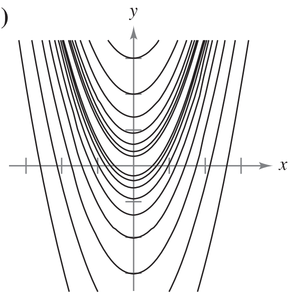
44. $f(x, y) = x \sin y$

In Exercises 45–48, match the graph of the surface with one of the contour maps. [The contour maps are labeled (a), (b), (c), and (d).]

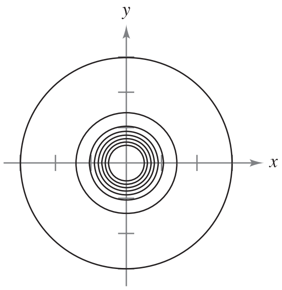
(a)



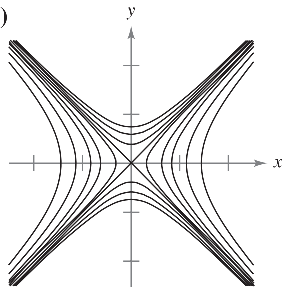
(b)



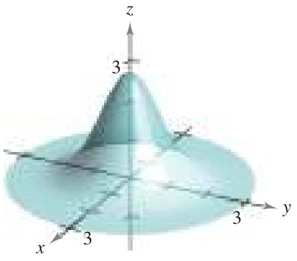
(c)



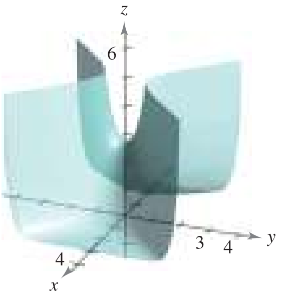
(d)



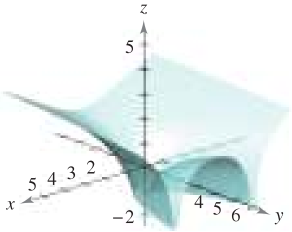
45. $f(x, y) = e^{1 - x^2 - y^2}$



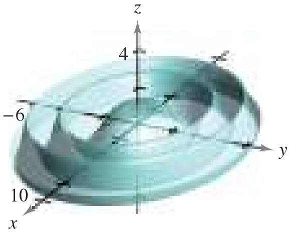
46. $f(x, y) = e^{1 - x^2 + y^2}$



47. $f(x, y) = \ln|y - x^2|$



48. $f(x, y) = \cos\left(\dfrac{x^2 + 2y^2}{4}\right)$



In Exercises 49–56, describe the level curves of the function. Sketch the level curves for the given c-values.

49. $z = x + y, \quad c = -1, 0, 2, 4$

50. $z = 6 - 2x - 3y, \quad c = 0, 2, 4, 6, 8, 10$

51. $z = x^2 + 4y^2, \quad c = 0, 1, 2, 3, 4$

52. $f(x, y) = \sqrt{9 - x^2 - y^2}, \quad c = 0, 1, 2, 3$

53. $f(x, y) = xy, \quad c = \pm 1, \pm 2, \ldots, \pm 6$

54. $f(x, y) = e^{xy/2}, \quad c = 2, 3, 4, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

55. $f(x, y) = x/(x^2 + y^2), \quad c = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2$

56. $f(x, y) = \ln(x - y), \quad c = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2$

In Exercises 57–60, use a graphing utility to graph six level curves of the function.

57. $f(x, y) = x^2 - y^2 + 2$

58. $f(x, y) = |xy|$

59. $g(x, y) = \dfrac{8}{1 + x^2 + y^2}$

60. $h(x, y) = 3 \sin(|x| + |y|)$

WRITING ABOUT CONCEPTS

61. What is a graph of a function of two variables? How is it interpreted geometrically? Describe level curves.

62. All of the level curves of the surface given by $z = f(x, y)$ are concentric circles. Does this imply that the graph of f is a hemisphere? Illustrate your answer with an example.

63. Construct a function whose level curves are lines passing through the origin.

CAPSTONE

64. Consider the function $f(x, y) = xy$, for $x \geq 0$ and $y \geq 0$.

(a) Sketch the graph of the surface given by f.

(b) Make a conjecture about the relationship between the graphs of f and $g(x, y) = f(x, y) - 3$. Explain your reasoning.

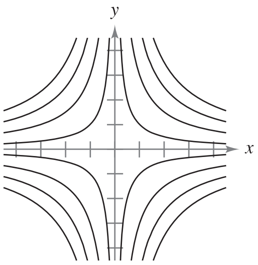
(c) Make a conjecture about the relationship between the graphs of f and $g(x, y) = -f(x, y)$. Explain your reasoning.

(d) Make a conjecture about the relationship between the graphs of f and $g(x, y) = \frac{1}{2}f(x, y)$. Explain your reasoning.

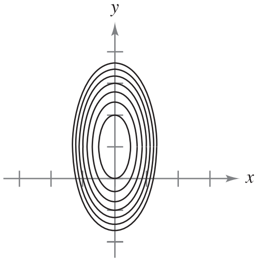
(e) On the surface in part (a), sketch the graph of $z = f(x, x)$.

Writing **In Exercises 65 and 66, use the graphs of the level curves (*c*-values evenly spaced) of the function *f* to write a description of a possible graph of *f*. Is the graph of *f* unique? Explain.**

65.



66.



67. ***Investment*** In 2009, an investment of \$1000 was made in a bond earning 6% compounded annually. Assume that the buyer pays tax at rate R and the annual rate of inflation is I. In the year 2019, the value V of the investment in constant 2009 dollars is

$$V(I, R) = 1000\left[\frac{1 + 0.06(1 - R)}{1 + I}\right]^{10}.$$

Use this function of two variables to complete the table.

	Inflation Rate		
Tax Rate	**0**	**0.03**	**0.05**
0			
0.28			
0.35			

68. ***Investment*** A principal of \$5000 is deposited in a savings account that earns interest at a rate of r (written as a decimal), compounded continuously. The amount $A(r, t)$ after t years is $A(r, t) = 5000e^{rt}$. Use this function of two variables to complete the table.

	Number of Years			
Rate	**5**	**10**	**15**	**20**
0.02				
0.03				
0.04				
0.05				

In Exercises 69–74, sketch the graph of the level surface $f(x, y, z) = c$ at the given value of c.

69. $f(x, y, z) = x - y + z, \quad c = 1$

70. $f(x, y, z) = 4x + y + 2z, \quad c = 4$

71. $f(x, y, z) = x^2 + y^2 + z^2, \quad c = 9$

72. $f(x, y, z) = x^2 + \frac{1}{4}y^2 - z, \quad c = 1$

73. $f(x, y, z) = 4x^2 + 4y^2 - z^2, \quad c = 0$

74. $f(x, y, z) = \sin x - z, \quad c = 0$

75. ***Forestry*** The **Doyle Log Rule** is one of several methods used to determine the lumber yield of a log (in board-feet) in terms of its diameter d (in inches) and its length L (in feet). The number of board-feet is

$$N(d, L) = \left(\frac{d - 4}{4}\right)^2 L.$$

(a) Find the number of board-feet of lumber in a log 22 inches in diameter and 12 feet in length.

(b) Find $N(30, 12)$.

76. ***Queuing Model*** The average length of time that a customer waits in line for service is

$$W(x, y) = \frac{1}{x - y}, \quad x > y$$

where y is the average arrival rate, written as the number of customers per unit of time, and x is the average service rate, written in the same units. Evaluate each of the following.

(a) $W(15, 9)$ (b) $W(15, 13)$ (c) $W(12, 7)$ (d) $W(5, 2)$

77. ***Temperature Distribution*** The temperature T (in degrees Celsius) at any point (x, y) in a circular steel plate of radius 10 meters is $T = 600 - 0.75x^2 - 0.75y^2$, where x and y are measured in meters. Sketch some of the isothermal curves.

78. ***Electric Potential*** The electric potential V at any point (x, y) is

$$V(x, y) = \frac{5}{\sqrt{25 + x^2 + y^2}}.$$

Sketch the equipotential curves for $V = \frac{1}{2}$, $V = \frac{1}{3}$, and $V = \frac{1}{4}$.

79. ***Cobb-Douglas Production Function*** Use the Cobb-Douglas production function (see Example 5) to show that if the number of units of labor and the number of units of capital are doubled, the production level is also doubled.

80. ***Cobb-Douglas Production Function*** Show that the Cobb-Douglas production function $z = Cx^a y^{1-a}$ can be rewritten as

$$\ln \frac{z}{y} = \ln C + a \ln \frac{x}{y}.$$

81. ***Construction Cost*** A rectangular box with an open top has a length of x feet, a width of y feet, and a height of z feet. It costs \$1.20 per square foot to build the base and \$0.75 per square foot to build the sides. Write the cost C of constructing the box as a function of x, y, and z.

82. ***Volume*** A propane tank is constructed by welding hemispheres to the ends of a right circular cylinder. Write the volume V of the tank as a function of r and l, where r is the radius of the cylinder and hemispheres, and l is the length of the cylinder.

83. ***Ideal Gas Law*** According to the Ideal Gas Law, $PV = kT$, where P is pressure, V is volume, T is temperature (in Kelvins), and k is a constant of proportionality. A tank contains 2000 cubic inches of nitrogen at a pressure of 26 pounds per square inch and a temperature of 300 K.

(a) Determine k.

(b) Write P as a function of V and T and describe the level curves.

84. ***Modeling Data*** The table shows the net sales x (in billions of dollars), the total assets y (in billions of dollars), and the shareholder's equity z (in billions of dollars) for Wal-Mart for the years 2002 through 2007. (*Source: 2007 Annual Report for Wal-Mart*)

Year	2002	2003	2004	2005	2006	2007
x	201.2	226.5	252.8	281.5	208.9	345.0
y	79.3	90.2	102.5	117.1	135.6	151.2
z	35.2	39.5	43.6	49.4	53.2	61.6

A model for these data is

$$z = f(x, y) = 0.026x + 0.316y + 5.04.$$

(a) Use a graphing utility and the model to approximate z for the given values of x and y.

(b) Which of the two variables in this model has the greater influence on shareholder's equity?

(c) Simplify the expression for $f(x, 95)$ and interpret its meaning in the context of the problem.

85. ***Meteorology*** Meteorologists measure the atmospheric pressure in millibars. From these observations they create weather maps on which the curves of equal atmospheric pressure (isobars) are drawn (see figure). On the map, the closer the isobars the higher the wind speed. Match points A, B, and C with (a) highest pressure, (b) lowest pressure, and (c) highest wind velocity.

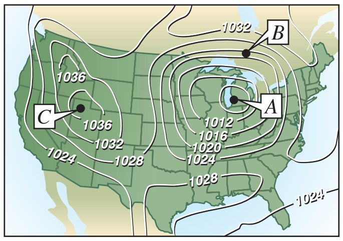


Figure for 85

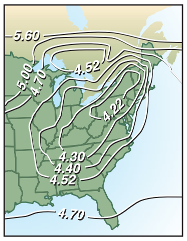
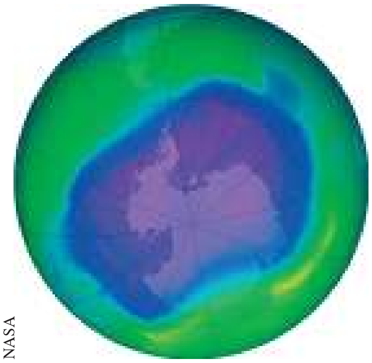


Figure for 86

86. ***Acid Rain*** The acidity of rainwater is measured in units called pH. A pH of 7 is neutral, smaller values are increasingly acidic, and larger values are increasingly alkaline. The map shows curves of equal pH and gives evidence that downwind of heavily industrialized areas the acidity has been increasing. Using the level curves on the map, determine the direction of the prevailing winds in the northeastern United States.

87. ***Atmosphere*** The contour map shown in the figure was computer generated using data collected by satellite instrumentation. Color is used to show the "ozone hole" in Earth's atmosphere. The purple and blue areas represent the lowest levels of ozone and the green areas represent the highest levels. (*Source: National Aeronautics and Space Administration*)



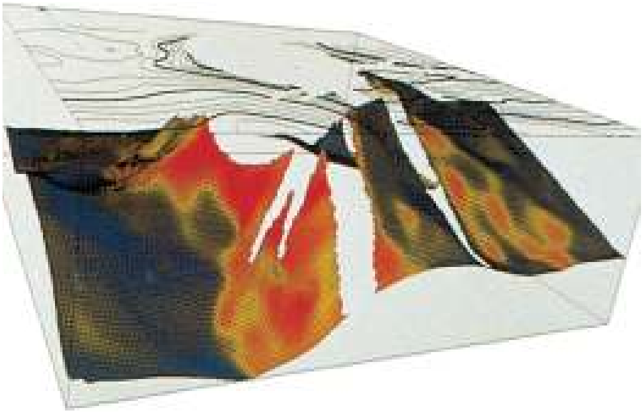
NASA

Figure for 87

(a) Do the level curves correspond to equally spaced ozone levels? Explain.

(b) Describe how to obtain a more detailed contour map.

88. ***Geology*** The contour map in the figure represents color-coded seismic amplitudes of a fault horizon and a projected contour map, which is used in earthquake studies. (*Source: Adapted from Shipman/Wilson/Todd,* An Introduction to Physical Science, *Tenth Edition*)



GeoQuest Systems, Inc.

(a) Discuss the use of color to represent the level curves.

(b) Do the level curves correspond to equally spaced amplitudes? Explain.

True or False? **In Exercises 89–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

89. If $f(x_0, y_0) = f(x_1, y_1)$, then $x_0 = x_1$ and $y_0 = y_1$.

90. If f is a function, then $f(ax, ay) = a^2 f(x, y)$.

91. A vertical line can intersect the graph of $z = f(x, y)$ at most once.

92. Two different level curves of the graph of $z = f(x, y)$ can intersect.

13.2 Limits and Continuity

- **Understand the definition of a neighborhood in the plane.**
- **Understand and use the definition of the limit of a function of two variables.**
- **Extend the concept of continuity to a function of two variables.**
- **Extend the concept of continuity to a function of three variables.**

Neighborhoods in the Plane

In this section, you will study limits and continuity involving functions of two or three variables. The section begins with functions of two variables. At the end of the section, the concepts are extended to functions of three variables.

We begin our discussion of the limit of a function of two variables by defining a two-dimensional analog to an interval on the real number line. Using the formula for the distance between two points (x, y) and (x_0, y_0) in the plane, you can define the **δ-neighborhood** about (x_0, y_0) to be the **disk** centered at (x_0, y_0) with radius $\delta > 0$

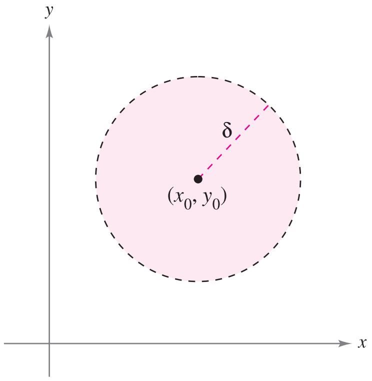
$$\left\{(x, y): \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\}$$ Open disk

as shown in Figure 13.18. When this formula contains the *less than* inequality sign, <, the disk is called **open,** and when it contains the *less than or equal to* inequality sign, ≤, the disk is called **closed.** This corresponds to the use of < and ≤ to define open and closed intervals.

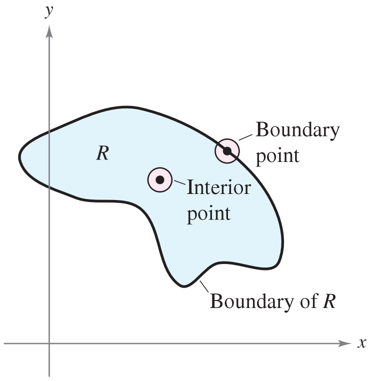
The Granger Collection

SONYA KOVALEVSKY (1850–1891)

Much of the terminology used to define limits and continuity of a function of two or three variables was introduced by the German mathematician Karl Weierstrass (1815–1897). Weierstrass's rigorous approach to limits and other topics in calculus gained him the reputation as the "father of modern analysis." Weierstrass was a gifted teacher. One of his best-known students was the Russian mathematician Sonya Kovalevsky, who applied many of Weierstrass's techniques to problems in mathematical physics and became one of the first women to gain acceptance as a research mathematician.



An open disk
Figure 13.18



The boundary and interior points of a region R
Figure 13.19

A point (x_0, y_0) in a plane region R is an **interior point** of R if there exists a δ-neighborhood about (x_0, y_0) that lies entirely in R, as shown in Figure 13.19. If every point in R is an interior point, then R is an **open region.** A point (x_0, y_0) is a **boundary point** of R if every open disk centered at (x_0, y_0) contains points inside R *and* points outside R. By definition, a region must contain its interior points, but it need not contain its boundary points. If a region contains all its boundary points, the region is **closed.** A region that contains some but not all of its boundary points is neither open nor closed.

■ **FOR FURTHER INFORMATION** For more information on Sonya Kovalevsky, see the article "S. Kovalevsky: A Mathematical Lesson" by Karen D. Rappaport in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*. ■

Limit of a Function of Two Variables

DEFINITION OF THE LIMIT OF A FUNCTION OF TWO VARIABLES

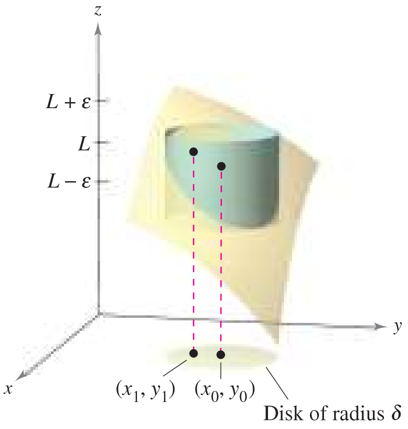
Let f be a function of two variables defined, except possibly at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L$$

if for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

NOTE Graphically, this definition of a limit implies that for any point $(x, y) \neq (x_0, y_0)$ in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$, as shown in Figure 13.20. ■



For any (x, y) in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$.
Figure 13.20

The definition of the limit of a function of two variables is similar to the definition of the limit of a function of a single variable, yet there is a critical difference. To determine whether a function of a single variable has a limit, you need only test the approach from two directions—from the right and from the left. If the function approaches the same limit from the right and from the left, you can conclude that the limit exists. However, for a function of two variables, the statement

$$(x, y) \to (x_0, y_0)$$

means that the point (x, y) is allowed to approach (x_0, y_0) from any direction. If the value of

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y)$$

is not the same for all possible approaches, or **paths,** to (x_0, y_0), the limit does not exist.

EXAMPLE 1 Verifying a Limit by the Definition

Show that

$$\lim_{(x, y)\to(a, b)} x = a.$$

Solution Let $f(x, y) = x$ and $L = a$. You need to show that for each $\varepsilon > 0$, there exists a δ-neighborhood about (a, b) such that

$$|f(x, y) - L| = |x - a| < \varepsilon$$

whenever $(x, y) \neq (a, b)$ lies in the neighborhood. You can first observe that from

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

it follows that

$$\begin{aligned} |f(x, y) - a| &= |x - a| \\ &= \sqrt{(x - a)^2} \\ &\leq \sqrt{(x - a)^2 + (y - b)^2} \\ &< \delta. \end{aligned}$$

So, you can choose $\delta = \varepsilon$, and the limit is verified. ■

Limits of functions of several variables have the same properties regarding sums, differences, products, and quotients as do limits of functions of single variables. (See Theorem 1.2 in Section 1.3.) Some of these properties are used in the next example.

EXAMPLE 2 Verifying a Limit

Evaluate $\displaystyle\lim_{(x,\,y)\to(1,\,2)} \frac{5x^2y}{x^2+y^2}$.

Solution By using the properties of limits of products and sums, you obtain

$$\begin{aligned}\lim_{(x,\,y)\to(1,\,2)} 5x^2y &= 5(1^2)(2)\\ &= 10\end{aligned}$$

and

$$\begin{aligned}\lim_{(x,\,y)\to(1,\,2)} (x^2+y^2) &= (1^2+2^2)\\ &= 5.\end{aligned}$$

Because the limit of a quotient is equal to the quotient of the limits (and the denominator is not 0), you have

$$\begin{aligned}\lim_{(x,\,y)\to(1,\,2)} \frac{5x^2y}{x^2+y^2} &= \frac{10}{5}\\ &= 2.\end{aligned}$$

EXAMPLE 3 Verifying a Limit

Evaluate $\displaystyle\lim_{(x,\,y)\to(0,\,0)} \frac{5x^2y}{x^2+y^2}$.

Solution In this case, the limits of the numerator and of the denominator are both 0, and so you cannot determine the existence (or nonexistence) of a limit by taking the limits of the numerator and denominator separately and then dividing. However, from the graph of f in Figure 13.21, it seems reasonable that the limit might be 0. So, you can try applying the definition to $L = 0$. First, note that

$$|y| \le \sqrt{x^2+y^2} \qquad \text{and} \qquad \frac{x^2}{x^2+y^2} \le 1.$$

Then, in a δ-neighborhood about $(0, 0)$, you have $0 < \sqrt{x^2+y^2} < \delta$, and it follows that, for $(x, y) \ne (0, 0)$,

$$\begin{aligned}|f(x,y) - 0| &= \left|\frac{5x^2y}{x^2+y^2}\right|\\ &= 5|y|\left(\frac{x^2}{x^2+y^2}\right)\\ &\le 5|y|\\ &\le 5\sqrt{x^2+y^2}\\ &< 5\delta.\end{aligned}$$

So, you can choose $\delta = \varepsilon/5$ and conclude that

$$\lim_{(x,\,y)\to(0,\,0)} \frac{5x^2y}{x^2+y^2} = 0. \qquad ■$$

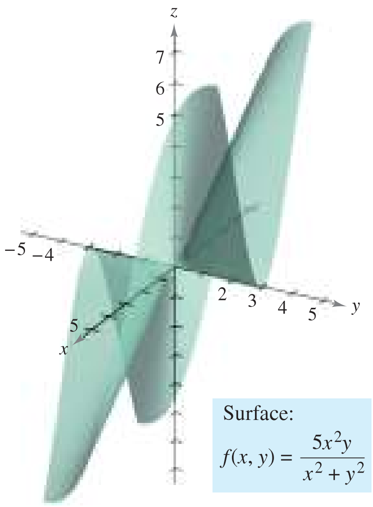
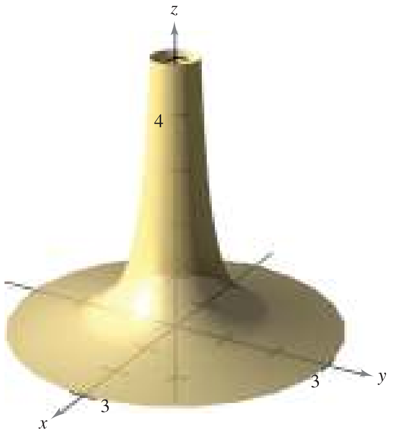


Figure 13.21

For some functions, it is easy to recognize that a limit does not exist. For instance, it is clear that the limit

$$\lim_{(x,\,y)\to(0,\,0)} \frac{1}{x^2+y^2}$$

does not exist because the values of $f(x, y)$ increase without bound as (x, y) approaches $(0, 0)$ along *any* path (see Figure 13.22).



$\lim_{(x,\,y)\to(0,\,0)} \frac{1}{x^2+y^2}$ does not exist.

Figure 13.22

For other functions, it is not so easy to recognize that a limit does not exist. For instance, the next example describes a limit that does not exist because the function approaches different values along different paths.

EXAMPLE 4 A Limit That Does Not Exist

Show that the following limit does not exist.

$$\lim_{(x,\,y)\to(0,\,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$$

Solution The domain of the function given by

$$f(x, y) = \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$$

consists of all points in the xy-plane except for the point $(0, 0)$. To show that the limit as (x, y) approaches $(0, 0)$ does not exist, consider approaching $(0, 0)$ along two different "paths," as shown in Figure 13.23. Along the x-axis, every point is of the form $(x, 0)$, and the limit along this approach is

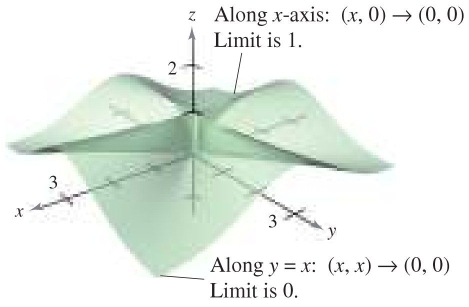
$$\lim_{(x,\,0)\to(0,\,0)} \left(\frac{x^2-0^2}{x^2+0^2}\right)^2 = \lim_{(x,\,0)\to(0,\,0)} 1^2 = 1. \qquad \text{Limit along } x\text{-axis}$$

However, if (x, y) approaches $(0, 0)$ along the line $y = x$, you obtain

$$\lim_{(x,\,x)\to(0,\,0)} \left(\frac{x^2-x^2}{x^2+x^2}\right)^2 = \lim_{(x,\,x)\to(0,\,0)} \left(\frac{0}{2x^2}\right)^2 = 0. \qquad \text{Limit along line } y = x$$

This means that in any open disk centered at $(0, 0)$, there are points (x, y) at which f takes on the value 1, and other points at which f takes on the value 0. For instance, $f(x, y) = 1$ at the points $(1, 0)$, $(0.1, 0)$, $(0.01, 0)$, and $(0.001, 0)$, and $f(x, y) = 0$ at the points $(1, 1)$, $(0.1, 0.1)$, $(0.01, 0.01)$, and $(0.001, 0.001)$. So, f does not have a limit as $(x, y) \to (0, 0)$.

NOTE In Example 4, you could conclude that the limit does not exist because you found two approaches that produced different limits. If two approaches had produced the same limit, you still could not have concluded that the limit exists. To form such a conclusion, you must show that the limit is the same along *all* possible approaches.



Along x-axis: $(x, 0) \to (0, 0)$
Limit is 1.

Along $y = x$: $(x, x) \to (0, 0)$
Limit is 0.

$\lim_{(x,\,y)\to(0,\,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$ does not exist.

Figure 13.23

■

Continuity of a Function of Two Variables

Notice in Example 2 that the limit of $f(x, y) = 5x^2y/(x^2 + y^2)$ as $(x, y) \to (1, 2)$ can be evaluated by direct substitution. That is, the limit is $f(1, 2) = 2$. In such cases the function f is said to be **continuous** at the point $(1, 2)$.

NOTE This definition of continuity can be extended to *boundary points* of the open region R by considering a special type of limit in which (x, y) is allowed to approach (x_0, y_0) along paths lying in the region R. This notion is similar to that of one-sided limits, as discussed in Chapter 1.

DEFINITION OF CONTINUITY OF A FUNCTION OF TWO VARIABLES

A function f of two variables is **continuous at a point (x_0, y_0)** in an open region R if $f(x_0, y_0)$ is equal to the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0). That is,

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = f(x_0, y_0).$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

In Example 3, it was shown that the function

$$f(x, y) = \frac{5x^2y}{x^2 + y^2}$$

is not continuous at $(0, 0)$. However, because the limit at this point exists, you can remove the discontinuity by defining f at $(0, 0)$ as being equal to its limit there. Such a discontinuity is called **removable.** In Example 4, the function

$$f(x, y) = \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$$

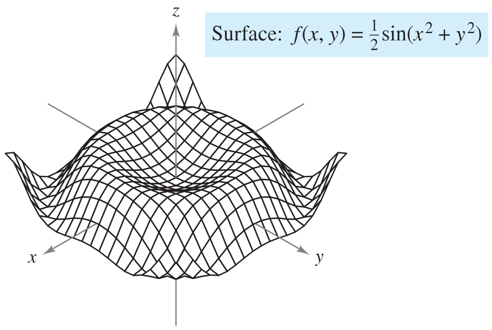
was also shown not to be continuous at $(0, 0)$, but this discontinuity is **nonremovable.**

THEOREM 13.1 CONTINUOUS FUNCTIONS OF TWO VARIABLES

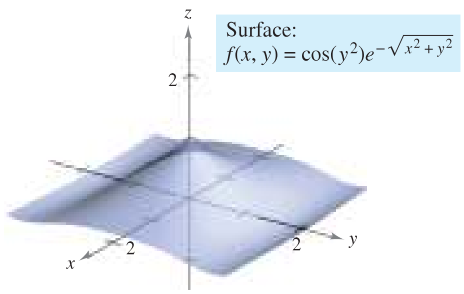
If k is a real number and f and g are continuous at (x_0, y_0), then the following functions are continuous at (x_0, y_0).

1. Scalar multiple: kf
2. Sum and difference: $f \pm g$
3. Product: fg
4. Quotient: f/g, if $g(x_0, y_0) \neq 0$

Theorem 13.1 establishes the continuity of *polynomial* and *rational* functions at every point in their domains. Furthermore, the continuity of other types of functions can be extended naturally from one to two variables. For instance, the functions whose graphs are shown in Figures 13.24 and 13.25 are continuous at every point in the plane.



The function f is continuous at every point in the plane.
Figure 13.24



The function f is continuous at every point in the plane.
Figure 13.25

EXPLORATION

Hold a spoon a foot or so from your eyes. Look at your image in the spoon. It should be upside down. Now, move the spoon closer and closer to one eye. At some point, your image will be right side up. Could it be that your image is being continuously deformed? Talk about this question and the general meaning of continuity with other members of your class. (This exploration was suggested by Irvin Roy Hentzel, Iowa State University.)

The next theorem states conditions under which a composite function is continuous.

THEOREM 13.2 CONTINUITY OF A COMPOSITE FUNCTION

If h is continuous at (x_0, y_0) and g is continuous at $h(x_0, y_0)$, then the composite function given by $(g \circ h)(x, y) = g(h(x, y))$ is continuous at (x_0, y_0). That is,

$$\lim_{(x, y)\to(x_0, y_0)} g(h(x, y)) = g(h(x_0, y_0)).$$

NOTE Note in Theorem 13.2 that h is a function of two variables and g is a function of one variable. ■

EXAMPLE 5 Testing for Continuity

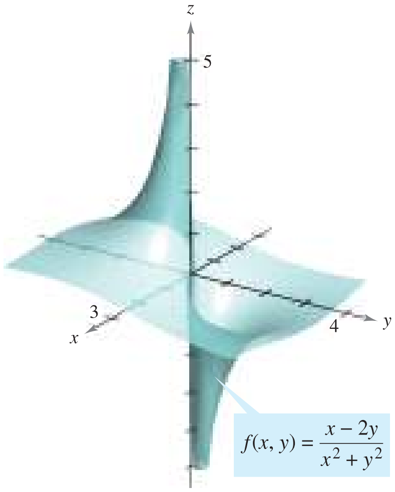
Discuss the continuity of each function.

a. $f(x, y) = \dfrac{x - 2y}{x^2 + y^2}$ **b.** $g(x, y) = \dfrac{2}{y - x^2}$

Solution

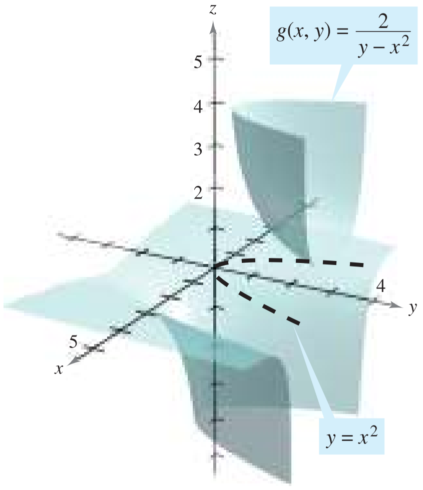
a. Because a rational function is continuous at every point in its domain, you can conclude that f is continuous at each point in the xy-plane except at $(0, 0)$, as shown in Figure 13.26.

b. The function given by $g(x, y) = 2/(y - x^2)$ is continuous except at the points at which the denominator is 0, $y - x^2 = 0$. So, you can conclude that the function is continuous at all points except those lying on the parabola $y = x^2$. Inside this parabola, you have $y > x^2$, and the surface represented by the function lies above the xy-plane, as shown in Figure 13.27. Outside the parabola, $y < x^2$, and the surface lies below the xy-plane.



The function f is not continuous at $(0, 0)$.

Figure 13.26



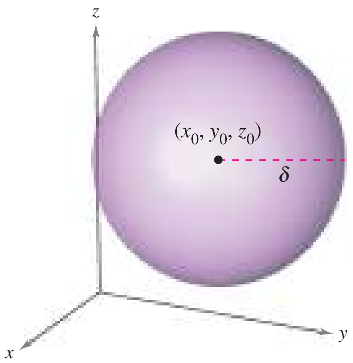
The function g is not continuous on the parabola $y = x^2$.

Figure 13.27 ■

Continuity of a Function of Three Variables

The preceding definitions of limits and continuity can be extended to functions of three variables by considering points (x, y, z) within the *open sphere*

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2. \qquad \text{Open sphere}$$



Open sphere in space
Figure 13.28

The radius of this sphere is δ, and the sphere is centered at (x_0, y_0, z_0), as shown in Figure 13.28. A point (x_0, y_0, z_0) in a region R in space is an **interior point** of R if there exists a δ-sphere about (x_0, y_0, z_0) that lies entirely in R. If every point in R is an interior point, then R is called **open.**

DEFINITION OF CONTINUITY OF A FUNCTION OF THREE VARIABLES

A function f of three variables is **continuous at a point (x_0, y_0, z_0)** in an open region R if $f(x_0, y_0, z_0)$ is defined and is equal to the limit of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0). That is,

$$\lim_{(x, y, z)\to(x_0, y_0, z_0)} f(x, y, z) = f(x_0, y_0, z_0).$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

EXAMPLE 6 Testing Continuity of a Function of Three Variables

The function

$$f(x, y, z) = \frac{1}{x^2 + y^2 - z}$$

is continuous at each point in space except at the points on the paraboloid given by $z = x^2 + y^2$. ■

13.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, use the definition of the limit of a function of two variables to verify the limit.

1. $\lim_{(x, y)\to(1, 0)} x = 1$
2. $\lim_{(x, y)\to(4, -1)} x = 4$
3. $\lim_{(x, y)\to(1, -3)} y = -3$
4. $\lim_{(x, y)\to(a, b)} y = b$

In Exercises 5–8, find the indicated limit by using the limits

$$\lim_{(x, y)\to(a, b)} f(x, y) = 4 \quad \text{and} \quad \lim_{(x, y)\to(a, b)} g(x, y) = 3.$$

5. $\lim_{(x, y)\to(a, b)} [f(x, y) - g(x, y)]$
6. $\lim_{(x, y)\to(a, b)} \left[\frac{5f(x, y)}{g(x, y)}\right]$
7. $\lim_{(x, y)\to(a, b)} [f(x, y)g(x, y)]$
8. $\lim_{(x, y)\to(a, b)} \left[\frac{f(x, y) + g(x, y)}{f(x, y)}\right]$

In Exercises 9–22, find the limit and discuss the continuity of the function.

9. $\lim_{(x, y)\to(2, 1)} (2x^2 + y)$
10. $\lim_{(x, y)\to(0, 0)} (x + 4y + 1)$
11. $\lim_{(x, y)\to(1, 2)} e^{xy}$
12. $\lim_{(x, y)\to(2, 4)} \frac{x + y}{x^2 + 1}$
13. $\lim_{(x, y)\to(0, 2)} \frac{x}{y}$
14. $\lim_{(x, y)\to(-1, 2)} \frac{x + y}{x - y}$
15. $\lim_{(x, y)\to(1, 1)} \frac{xy}{x^2 + y^2}$
16. $\lim_{(x, y)\to(1, 1)} \frac{x}{\sqrt{x + y}}$
17. $\lim_{(x, y)\to(\pi/4, 2)} y \cos xy$
18. $\lim_{(x, y)\to(2\pi, 4)} \sin \frac{x}{y}$
19. $\lim_{(x, y)\to(0, 1)} \frac{\arcsin xy}{1 - xy}$
20. $\lim_{(x, y)\to(0, 1)} \frac{\arccos (x/y)}{1 + xy}$
21. $\lim_{(x, y, z)\to(1, 3, 4)} \sqrt{x + y + z}$
22. $\lim_{(x, y, z)\to(-2, 1, 0)} xe^{yz}$

In Exercises 23–36, find the limit (if it exists). If the limit does not exist, explain why.

23. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy-1}{1+xy}$

24. $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^2y}{1+xy^2}$

25. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{x+y}$

26. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{x^2y^2}$

27. $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x^2-y^2}{x-y}$

28. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-4y^4}{x^2+2y^2}$

29. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x}-\sqrt{y}}$

30. $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1}$

31. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y}$

32. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x^2-y^2}$

33. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{(x^2+1)(y^2+1)}$

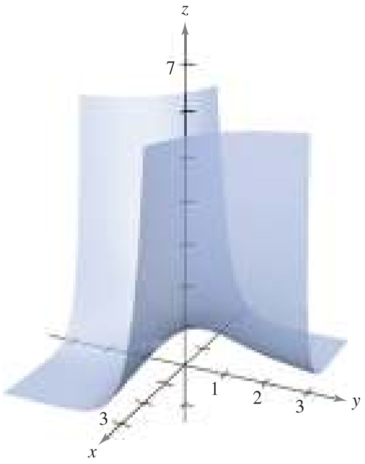
34. $\displaystyle\lim_{(x,y)\to(0,0)} \ln(x^2+y^2)$

35. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2}$

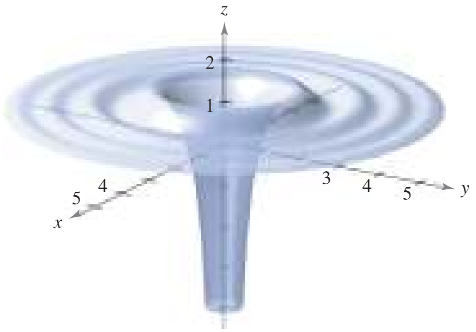
36. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2}$

In Exercises 37 and 38, discuss the continuity of the function and evaluate the limit of $f(x, y)$ (if it exists) as $(x, y) \to (0, 0)$.

37. $f(x, y) = e^{xy}$



38. $f(x, y) = 1 - \dfrac{\cos(x^2+y^2)}{x^2+y^2}$



In Exercises 39–42, use a graphing utility to make a table showing the values of $f(x, y)$ at the given points for each path. Use the result to make a conjecture about the limit of $f(x, y)$ as $(x, y) \to (0, 0)$. Determine whether the limit exists analytically and discuss the continuity of the function.

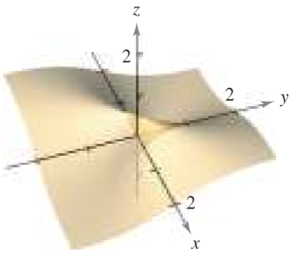
39. $f(x, y) = \dfrac{xy}{x^2+y^2}$

Path: $y = 0$

Points: (1, 0), (0.5, 0), (0.1, 0), (0.01, 0), (0.001, 0)

Path: $y = x$

Points: (1, 1), (0.5, 0.5), (0.1, 0.1), (0.01, 0.01), (0.001, 0.001)



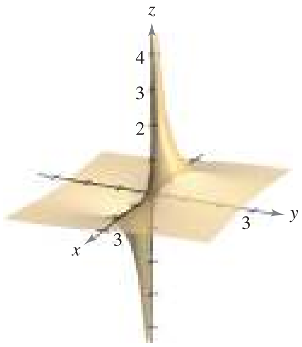
40. $f(x, y) = \dfrac{y}{x^2+y^2}$

Path: $y = 0$

Points: (1, 0), (0.5, 0), (0.1, 0), (0.01, 0), (0.001, 0)

Path: $y = x$

Points: (1, 1), (0.5, 0.5), (0.1, 0.1), (0.01, 0.01), (0.001, 0.001)



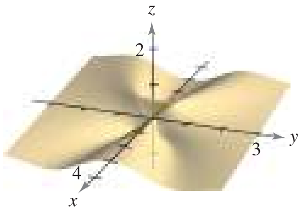
41. $f(x, y) = -\dfrac{xy^2}{x^2+y^4}$

Path: $x = y^2$

Points: (1, 1), (0.25, 0.5), (0.01, 0.1), (0.0001, 0.01), (0.000001, 0.001)

Path: $x = -y^2$

Points: $(-1, 1)$, $(-0.25, 0.5)$, $(-0.01, 0.1)$, $(-0.0001, 0.01)$, $(-0.000001, 0.001)$



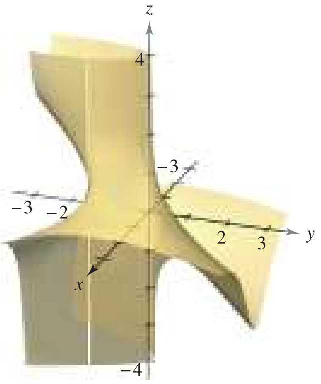
42. $f(x, y) = \dfrac{2x - y^2}{2x^2 + y}$

Path: $y = 0$

Points: $(1, 0)$, $(0.25, 0)$, $(0.01, 0)$, $(0.001, 0)$, $(0.000001, 0)$

Path: $y = x$

Points: $(1, 1)$, $(0.25, 0.25)$, $(0.01, 0.01)$, $(0.001, 0.001)$, $(0.0001, 0.0001)$



In Exercises 43–46, discuss the continuity of the functions *f* and *g*. Explain any differences.

43. $f(x, y) = \begin{cases} \dfrac{x^4 - y^4}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

$g(x, y) = \begin{cases} \dfrac{x^4 - y^4}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 1, & (x, y) = (0, 0) \end{cases}$

44. $f(x, y) = \begin{cases} \dfrac{4x^4 - y^4}{2x^2 + y^2}, & (x, y) \neq (0, 0) \\ -1, & (x, y) = (0, 0) \end{cases}$

$g(x, y) = \begin{cases} \dfrac{4x^4 - y^4}{2x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

45. $f(x, y) = \begin{cases} \dfrac{4x^2y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

$g(x, y) = \begin{cases} \dfrac{4x^2y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 2, & (x, y) = (0, 0) \end{cases}$

46. $f(x, y) = \begin{cases} \dfrac{x^2 + 2xy^2 + y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

$g(x, y) = \begin{cases} \dfrac{x^2 + 2xy^2 + y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 1, & (x, y) = (0, 0) \end{cases}$

CAS **In Exercises 47–52, use a computer algebra system to graph the function and find $\lim\limits_{(x, y)\to(0, 0)} f(x, y)$ (if it exists).**

47. $f(x, y) = \sin x + \sin y$

48. $f(x, y) = \sin \dfrac{1}{x} + \cos \dfrac{1}{x}$

49. $f(x, y) = \dfrac{x^2y}{x^4 + 2y^2}$

50. $f(x, y) = \dfrac{x^2 + y^2}{x^2y}$

51. $f(x, y) = \dfrac{5xy}{x^2 + 2y^2}$

52. $f(x, y) = \dfrac{6xy}{x^2 + y^2 + 1}$

In Exercises 53–58, use polar coordinates to find the limit. [*Hint:* Let $x = r \cos \theta$ and $y = r \sin \theta$, and note that $(x, y) \to (0, 0)$ implies $r \to 0$.]

53. $\lim\limits_{(x, y)\to(0, 0)} \dfrac{xy^2}{x^2 + y^2}$

54. $\lim\limits_{(x, y)\to(0, 0)} \dfrac{x^3 + y^3}{x^2 + y^2}$

55. $\lim\limits_{(x, y)\to(0, 0)} \dfrac{x^2y^2}{x^2 + y^2}$

56. $\lim\limits_{(x, y)\to(0, 0)} \dfrac{x^2 - y^2}{\sqrt{x^2 + y^2}}$

57. $\lim\limits_{(x, y)\to(0, 0)} \cos(x^2 + y^2)$

58. $\lim\limits_{(x, y)\to(0, 0)} \sin \sqrt{x^2 + y^2}$

In Exercises 59–62, use polar coordinates and L'Hôpital's Rule to find the limit.

59. $\lim\limits_{(x, y)\to(0, 0)} \dfrac{\sin \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}}$

60. $\lim\limits_{(x, y)\to(0, 0)} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$

61. $\lim\limits_{(x, y)\to(0, 0)} \dfrac{1 - \cos(x^2 + y^2)}{x^2 + y^2}$

62. $\lim\limits_{(x, y)\to(0, 0)} (x^2 + y^2)\ln(x^2 + y^2)$

In Exercises 63–68, discuss the continuity of the function.

63. $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

64. $f(x, y, z) = \dfrac{z}{x^2 + y^2 - 4}$

65. $f(x, y, z) = \dfrac{\sin z}{e^x + e^y}$

66. $f(x, y, z) = xy \sin z$

67. $f(x, y) = \begin{cases} \dfrac{\sin xy}{xy}, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$

68. $f(x, y) = \begin{cases} \dfrac{\sin(x^2 - y^2)}{x^2 - y^2}, & x^2 \neq y^2 \\ 1, & x^2 = y^2 \end{cases}$

In Exercises 69–72, discuss the continuity of the composite function $f \circ g$.

69. $f(t) = t^2$

$g(x, y) = 2x - 3y$

70. $f(t) = \dfrac{1}{t}$

$g(x, y) = x^2 + y^2$

71. $f(t) = \dfrac{1}{t}$

$g(x, y) = 2x - 3y$

72. $f(t) = \dfrac{1}{1 - t}$

$g(x, y) = x^2 + y^2$

In Exercises 73–78, find each limit.

(a) $\displaystyle\lim_{\Delta x\to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$

(b) $\displaystyle\lim_{\Delta y\to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

73. $f(x, y) = x^2 - 4y$

74. $f(x, y) = x^2 + y^2$

75. $f(x, y) = \dfrac{x}{y}$

76. $f(x, y) = \dfrac{1}{x + y}$

77. $f(x, y) = 3x + xy - 2y$

78. $f(x, y) = \sqrt{y}\,(y + 1)$

True or False? **In Exercises 79–82, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

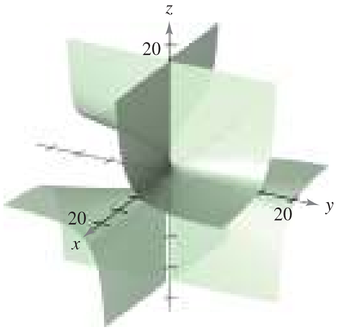
79. If $\lim_{(x, y)\to(0, 0)} f(x, y) = 0$, then $\lim_{x\to0} f(x, 0) = 0$.

80. If $\lim_{(x, y)\to(0, 0)} f(0, y) = 0$, then $\lim_{(x, y)\to(0, 0)} f(x, y) = 0$.

81. If f is continuous for all nonzero x and y, and $f(0, 0) = 0$, then $\lim_{(x, y)\to(0, 0)} f(x, y) = 0$.

82. If g and h are continuous functions of x and y, and $f(x, y) = g(x) + h(y)$, then f is continuous.

83. Consider $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{x^2 + y^2}{xy}$ (see figure).

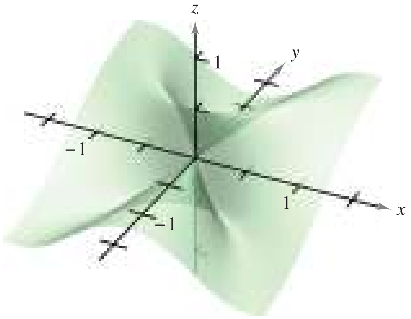


(a) Determine (if possible) the limit along any line of the form $y = ax$.

(b) Determine (if possible) the limit along the parabola $y = x^2$.

(c) Does the limit exist? Explain.

84. Consider $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{x^2y}{x^4 + y^2}$ (see figure).



(a) Determine (if possible) the limit along any line of the form $y = ax$.

(b) Determine (if possible) the limit along the parabola $y = x^2$.

(c) Does the limit exist? Explain.

In Exercises 85 and 86, use spherical coordinates to find the limit. [*Hint:* Let $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$, and note that $(x, y, z)\to(0, 0, 0)$ implies $\rho\to0^+$.]

85. $\displaystyle\lim_{(x, y, z)\to(0, 0, 0)} \frac{xyz}{x^2 + y^2 + z^2}$

86. $\displaystyle\lim_{(x, y, z)\to(0, 0, 0)} \tan^{-1}\left[\frac{1}{x^2 + y^2 + z^2}\right]$

87. Find the following limit.

$$\lim_{(x, y)\to(0, 1)} \tan^{-1}\left[\frac{x^2 + 1}{x^2 + (y - 1)^2}\right]$$

88. For the function

$$f(x, y) = xy\left(\frac{x^2 - y^2}{x^2 + y^2}\right)$$

define $f(0, 0)$ such that f is continuous at the origin.

89. Prove that

$$\lim_{(x, y)\to(a, b)} [f(x, y) + g(x, y)] = L_1 + L_2$$

where $f(x, y)$ approaches L_1 and $g(x, y)$ approaches L_2 as $(x, y) \to (a, b)$.

90. Prove that if f is continuous and $f(a, b) < 0$, there exists a δ-neighborhood about (a, b) such that $f(x, y) < 0$ for every point (x, y) in the neighborhood.

WRITING ABOUT CONCEPTS

91. Define the limit of a function of two variables. Describe a method for showing that

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y)$$

does not exist.

92. State the definition of continuity of a function of two variables.

93. Determine whether each of the following statements is true or false. Explain your reasoning.

(a) If $\lim_{(x, y)\to(2, 3)} f(x, y) = 4$, then $\lim_{x\to2} f(x, 3) = 4$.

(b) If $\lim_{x\to2} f(x, 3) = 4$, then $\lim_{(x, y)\to(2, 3)} f(x, y) = 4$.

(c) If $\lim_{x\to2} f(x, 3) = \lim_{y\to3} f(2, y) = 4$, then $\lim_{(x, y)\to(2, 3)} f(x, y) = 4$.

(d) If $\lim_{(x, y)\to(0, 0)} f(x, y) = 0$, then for any real number k, $\lim_{(x, y)\to(0, 0)} f(kx, y) = 0$.

CAPSTONE

94. (a) If $f(2, 3) = 4$, can you conclude anything about $\lim_{(x, y)\to(2, 3)} f(x, y)$? Give reasons for your answer.

(b) If $\lim_{(x, y)\to(2, 3)} f(x, y) = 4$, can you conclude anything about $f(2, 3)$? Give reasons for your answer.

13.3 Partial Derivatives

- **Find and use partial derivatives of a function of two variables.**
- **Find and use partial derivatives of a function of three or more variables.**
- **Find higher-order partial derivatives of a function of two or three variables.**

Partial Derivatives of a Function of Two Variables

In applications of functions of several variables, the question often arises, "How will the value of a function be affected by a change in one of its independent variables?" You can answer this by considering the independent variables one at a time. For example, to determine the effect of a catalyst in an experiment, a chemist could conduct the experiment several times using varying amounts of the catalyst, while keeping constant other variables such as temperature and pressure. You can use a similar procedure to determine the rate of change of a function f with respect to one of its several independent variables. This process is called **partial differentiation,** and the result is referred to as the **partial derivative** of f with respect to the chosen independent variable.

Mary Evans Picture Library

JEAN LE ROND D'ALEMBERT (1717–1783)

The introduction of partial derivatives followed Newton's and Leibniz's work in calculus by several years. Between 1730 and 1760, Leonhard Euler and Jean Le Rond d'Alembert separately published several papers on dynamics, in which they established much of the theory of partial derivatives. These papers used functions of two or more variables to study problems involving equilibrium, fluid motion, and vibrating strings.

DEFINITION OF PARTIAL DERIVATIVES OF A FUNCTION OF TWO VARIABLES

If $z = f(x, y)$, then the **first partial derivatives** of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided the limits exist.

This definition indicates that if $z = f(x, y)$, then to find f_x you *consider y constant* and differentiate with respect to x. Similarly, to find f_y, you *consider x constant* and differentiate with respect to y.

EXAMPLE 1 Finding Partial Derivatives

Find the partial derivatives f_x and f_y for the function

$$f(x, y) = 3x - x^2y^2 + 2x^3y.$$

Solution Considering y to be constant and differentiating with respect to x produces

$$f(x, y) = 3x - x^2y^2 + 2x^3y \qquad \text{Write original function.}$$

$$f_x(x, y) = 3 - 2xy^2 + 6x^2y. \qquad \text{Partial derivative with respect to } x$$

Considering x to be constant and differentiating with respect to y produces

$$f(x, y) = 3x - x^2y^2 + 2x^3y \qquad \text{Write original function.}$$

$$f_y(x, y) = -2x^2y + 2x^3. \qquad \text{Partial derivative with respect to } y$$

■

NOTATION FOR FIRST PARTIAL DERIVATIVES

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$

The first partials evaluated at the point (a, b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a,\, b)} = f_x(a, b) \qquad \text{and} \qquad \left.\frac{\partial z}{\partial y}\right|_{(a,\, b)} = f_y(a, b).$$

EXAMPLE 2 Finding and Evaluating Partial Derivatives

For $f(x, y) = xe^{x^2y}$, find f_x and f_y, and evaluate each at the point $(1, \ln 2)$.

Solution Because

$$f_x(x, y) = xe^{x^2y}(2xy) + e^{x^2y}$$

Partial derivative with respect to x

the partial derivative of f with respect to x at $(1, \ln 2)$ is

$$\begin{aligned} f_x(1, \ln 2) &= e^{\ln 2}(2 \ln 2) + e^{\ln 2} \\ &= 4 \ln 2 + 2. \end{aligned}$$

Because

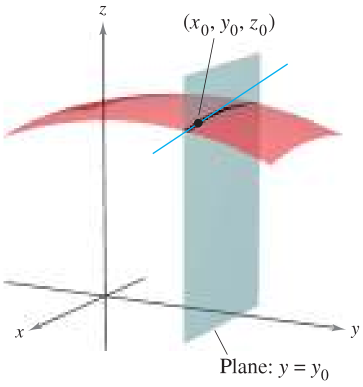
$$\begin{aligned} f_y(x, y) &= xe^{x^2y}(x^2) \\ &= x^3e^{x^2y} \end{aligned}$$

Partial derivative with respect to y

the partial derivative of f with respect to y at $(1, \ln 2)$ is

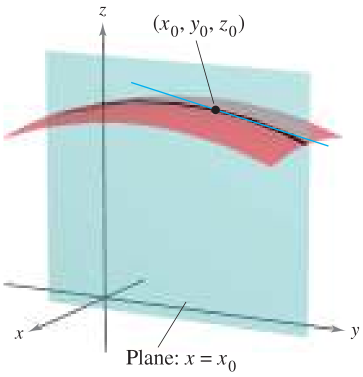
$$\begin{aligned} f_y(1, \ln 2) &= e^{\ln 2} \\ &= 2. \end{aligned}$$

■



$\dfrac{\partial f}{\partial x}$ = slope in x-direction

Figure 13.29



$\dfrac{\partial f}{\partial y}$ = slope in y-direction

Figure 13.30

The partial derivatives of a function of two variables, $z = f(x, y)$, have a useful geometric interpretation. If $y = y_0$, then $z = f(x, y_0)$ represents the curve formed by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$, as shown in Figure 13.29. Therefore,

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

represents the slope of this curve at the point $(x_0, y_0, f(x_0, y_0))$. Note that both the curve and the tangent line lie in the plane $y = y_0$. Similarly,

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

represents the slope of the curve given by the intersection of $z = f(x, y)$ and the plane $x = x_0$ at $(x_0, y_0, f(x_0, y_0))$, as shown in Figure 13.30.

Informally, the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point (x_0, y_0, z_0) denote the **slopes of the surface in the x- and y-directions,** respectively.

EXAMPLE 3 Finding the Slopes of a Surface in the *x*- and *y*-Directions

Find the slopes in the x-direction and in the y-direction of the surface given by

$$f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$$

at the point $\left(\frac{1}{2}, 1, 2\right)$.

Solution The partial derivatives of f with respect to x and y are

$$f_x(x, y) = -x \qquad \text{and} \qquad f_y(x, y) = -2y.$$

Partial derivatives

So, in the x-direction, the slope is

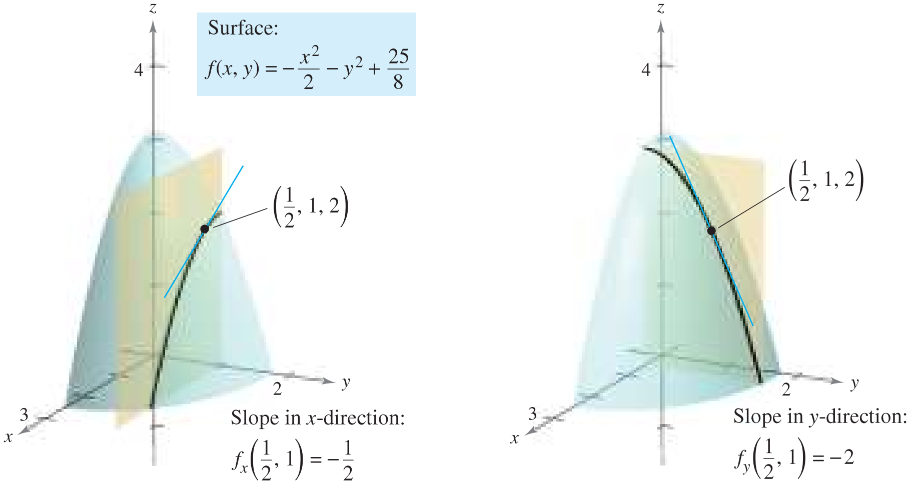
$$f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2}$$

Figure 13.31(a)

and in the y-direction, the slope is

$$f_y\left(\frac{1}{2}, 1\right) = -2.$$

Figure 13.31(b)


z
4
Surface:
$f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$
$\left(\frac{1}{2}, 1, 2\right)$
2
y
3
x
Slope in x-direction:
$f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2}$
z
4
$\left(\frac{1}{2}, 1, 2\right)$
2
y
3
x
Slope in y-direction:
$f_y\left(\frac{1}{2}, 1\right) = -2$

(a) **(b)**

Figure 13.31

EXAMPLE 4 Finding the Slopes of a Surface in the *x*- and *y*-Directions

Find the slopes of the surface given by

$$f(x, y) = 1 - (x - 1)^2 - (y - 2)^2$$

at the point $(1, 2, 1)$ in the x-direction and in the y-direction.

Solution The partial derivatives of f with respect to x and y are

$$f_x(x, y) = -2(x - 1) \qquad \text{and} \qquad f_y(x, y) = -2(y - 2).$$

Partial derivatives

So, at the point $(1, 2, 1)$, the slopes in the x- and y-directions are

$$f_x(1, 2) = -2(1 - 1) = 0 \qquad \text{and} \qquad f_y(1, 2) = -2(2 - 2) = 0$$

as shown in Figure 13.32. ■

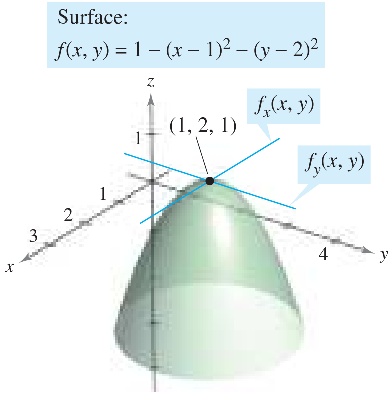
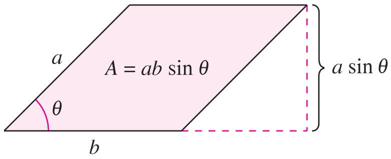

Surface:
$f(x, y) = 1 - (x - 1)^2 - (y - 2)^2$
z
1
(1, 2, 1)
$f_x(x, y)$
$f_y(x, y)$
1
2
3
x
4
y

Figure 13.32

No matter how many variables are involved, partial derivatives can be interpreted as *rates of change*.

EXAMPLE 5 Using Partial Derivatives to Find Rates of Change



The area of the parallelogram is $ab \sin \theta$.
Figure 13.33

The area of a parallelogram with adjacent sides a and b and included angle θ is given by $A = ab \sin \theta$, as shown in Figure 13.33.

a. Find the rate of change of A with respect to a for $a = 10$, $b = 20$, and $\theta = \frac{\pi}{6}$.

b. Find the rate of change of A with respect to θ for $a = 10$, $b = 20$, and $\theta = \frac{\pi}{6}$.

Solution

a. To find the rate of change of the area with respect to a, hold b and θ constant and differentiate with respect to a to obtain

$$\frac{\partial A}{\partial a} = b \sin \theta \qquad \text{Find partial with respect to } a.$$

$$\frac{\partial A}{\partial a} = 20 \sin \frac{\pi}{6} = 10. \qquad \text{Substitute for } b \text{ and } \theta.$$

b. To find the rate of change of the area with respect to θ, hold a and b constant and differentiate with respect to θ to obtain

$$\frac{\partial A}{\partial \theta} = ab \cos \theta \qquad \text{Find partial with respect to } \theta.$$

$$\frac{\partial A}{\partial \theta} = 200 \cos \frac{\pi}{6} = 100\sqrt{3}. \qquad \text{Substitute for } a, b, \text{ and } \theta.$$

■

Partial Derivatives of a Function of Three or More Variables

The concept of a partial derivative can be extended naturally to functions of three or more variables. For instance, if $w = f(x, y, z)$, there are three partial derivatives, each of which is formed by holding two of the variables constant. That is, to define the partial derivative of w with respect to x, consider y and z to be constant and differentiate with respect to x. A similar process is used to find the derivatives of w with respect to y and with respect to z.

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

In general, if $w = f(x_1, x_2, \ldots, x_n)$, there are n partial derivatives denoted by

$$\frac{\partial w}{\partial x_k} = f_{x_k}(x_1, x_2, \ldots, x_n), \quad k = 1, 2, \ldots, n.$$

To find the partial derivative with respect to one of the variables, hold the other variables constant and differentiate with respect to the given variable.

EXAMPLE 6 Finding Partial Derivatives

a. To find the partial derivative of $f(x, y, z) = xy + yz^2 + xz$ with respect to z, consider x and y to be constant and obtain

$$\frac{\partial}{\partial z}[xy + yz^2 + xz] = 2yz + x.$$

b. To find the partial derivative of $f(x, y, z) = z\sin(xy^2 + 2z)$ with respect to z, consider x and y to be constant. Then, using the Product Rule, you obtain

$$\begin{aligned}\frac{\partial}{\partial z}[z\sin(xy^2 + 2z)] &= (z)\frac{\partial}{\partial z}[\sin(xy^2 + 2z)] + \sin(xy^2 + 2z)\frac{\partial}{\partial z}[z]\\ &= (z)[\cos(xy^2 + 2z)](2) + \sin(xy^2 + 2z)\\ &= 2z\cos(xy^2 + 2z) + \sin(xy^2 + 2z).\end{aligned}$$

c. To find the partial derivative of $f(x, y, z, w) = (x + y + z)/w$ with respect to w, consider x, y, and z to be constant and obtain

$$\frac{\partial}{\partial w}\left[\frac{x + y + z}{w}\right] = -\frac{x + y + z}{w^2}.$$

■

Higher-Order Partial Derivatives

As is true for ordinary derivatives, it is possible to take second, third, and higher-order partial derivatives of a function of several variables, provided such derivatives exist. Higher-order derivatives are denoted by the order in which the differentiation occurs. For instance, the function $z = f(x, y)$ has the following second partial derivatives.

1. Differentiate twice with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$

2. Differentiate twice with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

3. Differentiate first with respect to x and then with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$

4. Differentiate first with respect to y and then with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

The third and fourth cases are called **mixed partial derivatives.**

NOTE Note that the two types of notation for mixed partials have different conventions for indicating the order of differentiation.

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \quad \text{Right-to-left order}$$

$$(f_x)_y = f_{xy} \quad \text{Left-to-right order}$$

You can remember the order by observing that in both notations you differentiate first with respect to the variable "nearest" f.

EXAMPLE 7 Finding Second Partial Derivatives

Find the second partial derivatives of $f(x, y) = 3xy^2 - 2y + 5x^2y^2$, and determine the value of $f_{xy}(-1, 2)$.

Solution Begin by finding the first partial derivatives with respect to x and y.

$$f_x(x, y) = 3y^2 + 10xy^2 \qquad \text{and} \qquad f_y(x, y) = 6xy - 2 + 10x^2y$$

Then, differentiate each of these with respect to x and y.

$$\begin{aligned} f_{xx}(x, y) &= 10y^2 && \text{and} \qquad f_{yy}(x, y) = 6x + 10x^2 \\ f_{xy}(x, y) &= 6y + 20xy && \text{and} \qquad f_{yx}(x, y) = 6y + 20xy \end{aligned}$$

At $(-1, 2)$, the value of f_{xy} is $f_{xy}(-1, 2) = 12 - 40 = -28$. ■

NOTE Notice in Example 7 that the two mixed partials are equal. Sufficient conditions for this occurrence are given in Theorem 13.3. ■

THEOREM 13.3 EQUALITY OF MIXED PARTIAL DERIVATIVES

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then, for every (x, y) in R,

$$f_{xy}(x, y) = f_{yx}(x, y).$$

Theorem 13.3 also applies to a function f of *three or more variables* so long as all second partial derivatives are continuous. For example, if $w = f(x, y, z)$ and all the second partial derivatives are continuous in an open region R, then at each point in R the order of differentiation in the mixed second partial derivatives is irrelevant. If the third partial derivatives of f are also continuous, the order of differentiation of the mixed third partial derivatives is irrelevant.

EXAMPLE 8 Finding Higher-Order Partial Derivatives

Show that $f_{xz} = f_{zx}$ and $f_{xzz} = f_{zxz} = f_{zzx}$ for the function given by

$$f(x, y, z) = ye^x + x \ln z.$$

Solution

First partials:

$$f_x(x, y, z) = ye^x + \ln z, \qquad f_z(x, y, z) = \frac{x}{z}$$

Second partials (note that the first two are equal):

$$f_{xz}(x, y, z) = \frac{1}{z}, \qquad f_{zx}(x, y, z) = \frac{1}{z}, \qquad f_{zz}(x, y, z) = -\frac{x}{z^2}$$

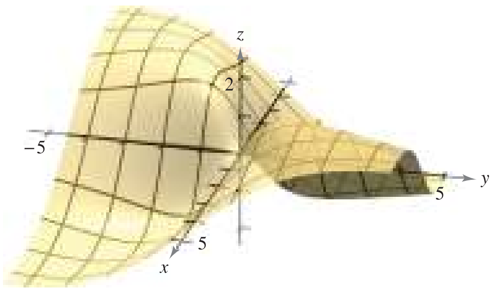
Third partials (note that all three are equal):

$$f_{xzz}(x, y, z) = -\frac{1}{z^2}, \qquad f_{zxz}(x, y, z) = -\frac{1}{z^2}, \qquad f_{zzx}(x, y, z) = -\frac{1}{z^2}$$ ■

13.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Think About It **In Exercises 1–4, use the graph of the surface to determine the sign of the indicated partial derivative.**



1. $f_x(4, 1)$

2. $f_y(-1, -2)$

3. $f_y(4, 1)$

4. $f_x(-1, -1)$

In Exercises 5–8, explain whether or not the Quotient Rule should be used to find the partial derivative. Do not differentiate.

5. $\dfrac{\partial}{\partial y}\left(\dfrac{x-y}{x^2+1}\right)$

6. $\dfrac{\partial}{\partial x}\left(\dfrac{x-y}{x^2+1}\right)$

7. $\dfrac{\partial}{\partial x}\left(\dfrac{xy}{x^2+1}\right)$

8. $\dfrac{\partial}{\partial y}\left(\dfrac{xy}{x^2+1}\right)$

In Exercises 9–40, find both first partial derivatives.

9. $f(x, y) = 2x - 5y + 3$

10. $f(x, y) = x^2 - 2y^2 + 4$

11. $f(x, y) = x^2y^3$

12. $f(x, y) = 4x^3y^{-2}$

13. $z = x\sqrt{y}$

14. $z = 2y^2\sqrt{x}$

15. $z = x^2 - 4xy + 3y^2$

16. $z = y^3 - 2xy^2 - 1$

17. $z = e^{xy}$

18. $z = e^{x/y}$

19. $z = x^2e^{2y}$

20. $z = ye^{y/x}$

21. $z = \ln\dfrac{x}{y}$

22. $z = \ln\sqrt{xy}$

23. $z = \ln(x^2 + y^2)$

24. $z = \ln\dfrac{x+y}{x-y}$

25. $z = \dfrac{x^2}{2y} + \dfrac{3y^2}{x}$

26. $z = \dfrac{xy}{x^2+y^2}$

27. $h(x, y) = e^{-(x^2+y^2)}$

28. $g(x, y) = \ln\sqrt{x^2+y^2}$

29. $f(x, y) = \sqrt{x^2+y^2}$

30. $f(x, y) = \sqrt{2x+y^3}$

31. $z = \cos xy$

32. $z = \sin(x + 2y)$

33. $z = \tan(2x - y)$

34. $z = \sin 5x \cos 5y$

35. $z = e^y \sin xy$

36. $z = \cos(x^2 + y^2)$

37. $z = \sinh(2x + 3y)$

38. $z = \cosh xy^2$

39. $f(x, y) = \displaystyle\int_x^y (t^2 - 1)\, dt$

40. $f(x, y) = \displaystyle\int_x^y (2t + 1)\, dt + \int_y^x (2t - 1)\, dt$

In Exercises 41–44, use the limit definition of partial derivatives to find $f_x(x, y)$ and $f_y(x, y)$.

41. $f(x, y) = 3x + 2y$

42. $f(x, y) = x^2 - 2xy + y^2$

43. $f(x, y) = \sqrt{x+y}$

44. $f(x, y) = \dfrac{1}{x+y}$

In Exercises 45–52, evaluate f_x and f_y at the given point.

45. $f(x, y) = e^y \sin x, \quad (\pi, 0)$

46. $f(x, y) = e^{-x} \cos y, \quad (0, 0)$

47. $f(x, y) = \cos(2x - y), \quad \left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$

48. $f(x, y) = \sin xy, \quad \left(2, \dfrac{\pi}{4}\right)$

49. $f(x, y) = \arctan\dfrac{y}{x}, \quad (2, -2)$

50. $f(x, y) = \arccos xy, \quad (1, 1)$

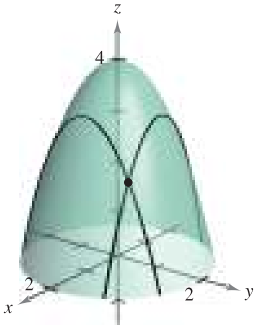
51. $f(x, y) = \dfrac{xy}{x-y}, \quad (2, -2)$

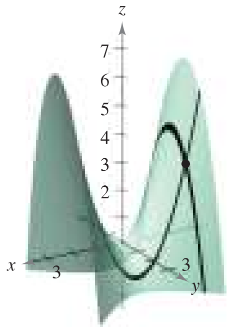
52. $f(x, y) = \dfrac{2xy}{\sqrt{4x^2+5y^2}}, \quad (1, 1)$

In Exercises 53 and 54, find the slopes of the surface in the *x*- and *y*-directions at the given point.

53. $g(x, y) = 4 - x^2 - y^2$
$(1, 1, 2)$

54. $h(x, y) = x^2 - y^2$
$(-2, 1, 3)$





CAS **In Exercises 55–58, use a computer algebra system to graph the curve formed by the intersection of the surface and the plane. Find the slope of the curve at the given point.**

	Surface	*Plane*	*Point*
55.	$z = \sqrt{49 - x^2 - y^2}$	$x = 2$	$(2, 3, 6)$
56.	$z = x^2 + 4y^2$	$y = 1$	$(2, 1, 8)$
57.	$z = 9x^2 - y^2$	$y = 3$	$(1, 3, 0)$
58.	$z = 9x^2 - y^2$	$x = 1$	$(1, 3, 0)$

In Exercises 59–64, find the first partial derivatives with respect to x, y, and z.

59. $H(x, y, z) = \sin(x + 2y + 3z)$

60. $f(x, y, z) = 3x^2y - 5xyz + 10yz^2$

61. $w = \sqrt{x^2 + y^2 + z^2}$

62. $w = \dfrac{7xz}{x + y}$

63. $F(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$

64. $G(x, y, z) = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

In Exercises 65–70, evaluate f_x, f_y, and f_z at the given point.

65. $f(x, y, z) = x^3yz^2, \quad (1, 1, 1)$

66. $f(x, y, z) = x^2y^3 + 2xyz - 3yz, \quad (-2, 1, 2)$

67. $f(x, y, z) = \dfrac{x}{yz}, \quad (1, -1, -1)$

68. $f(x, y, z) = \dfrac{xy}{x + y + z}, \quad (3, 1, -1)$

69. $f(x, y, z) = z\sin(x + y), \quad \left(0, \dfrac{\pi}{2}, -4\right)$

70. $f(x, y, z) = \sqrt{3x^2 + y^2 - 2z^2}, \quad (1, -2, 1)$

In Exercises 71–80, find the four second partial derivatives. Observe that the second mixed partials are equal.

71. $z = 3xy^2$

72. $z = x^2 + 3y^2$

73. $z = x^2 - 2xy + 3y^2$

74. $z = x^4 - 3x^2y^2 + y^4$

75. $z = \sqrt{x^2 + y^2}$

76. $z = \ln(x - y)$

77. $z = e^x \tan y$

78. $z = 2xe^y - 3ye^{-x}$

79. $z = \cos xy$

80. $z = \arctan \dfrac{y}{x}$

In Exercises 81–88, for $f(x, y)$, find all values of x and y such that $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously.

81. $f(x, y) = x^2 + xy + y^2 - 2x + 2y$

82. $f(x, y) = x^2 - xy + y^2 - 5x + y$

83. $f(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3$

84. $f(x, y) = x^2 - xy + y^2$

85. $f(x, y) = \dfrac{1}{x} + \dfrac{1}{y} + xy$

86. $f(x, y) = 3x^3 - 12xy + y^3$

87. $f(x, y) = e^{x^2 + xy + y^2}$

88. $f(x, y) = \ln(x^2 + y^2 + 1)$

CAS **In Exercises 89–92, use a computer algebra system to find the first and second partial derivatives of the function. Determine whether there exist values of x and y such that $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously.**

89. $f(x, y) = x\sec y$

90. $f(x, y) = \sqrt{25 - x^2 - y^2}$

91. $f(x, y) = \ln \dfrac{x}{x^2 + y^2}$

92. $f(x, y) = \dfrac{xy}{x - y}$

In Exercises 93–96, show that the mixed partial derivatives f_{xyy}, f_{yxy}, and f_{yyx} are equal.

93. $f(x, y, z) = xyz$

94. $f(x, y, z) = x^2 - 3xy + 4yz + z^3$

95. $f(x, y, z) = e^{-x}\sin yz$

96. $f(x, y, z) = \dfrac{2z}{x + y}$

Laplace's Equation **In Exercises 97–100, show that the function satisfies Laplace's equation $\partial^2z/\partial x^2 + \partial^2z/\partial y^2 = 0$.**

97. $z = 5xy$

98. $z = \frac{1}{2}(e^y - e^{-y})\sin x$

99. $z = e^x \sin y$

100. $z = \arctan \dfrac{y}{x}$

Wave Equation **In Exercises 101–104, show that the function satisfies the wave equation $\partial^2z/\partial t^2 = c^2(\partial^2z/\partial x^2)$.**

101. $z = \sin(x - ct)$

102. $z = \cos(4x + 4ct)$

103. $z = \ln(x + ct)$

104. $z = \sin \omega ct \sin \omega x$

Heat Equation **In Exercises 105 and 106, show that the function satisfies the heat equation $\partial z/\partial t = c^2(\partial^2z/\partial x^2)$.**

105. $z = e^{-t}\cos \dfrac{x}{c}$

106. $z = e^{-t}\sin \dfrac{x}{c}$

In Exercises 107 and 108, determine whether or not there exists a function $f(x, y)$ with the given partial derivatives. Explain your reasoning. If such a function exists, give an example.

107. $f_x(x, y) = -3\sin(3x - 2y), f_y(x, y) = 2\sin(3x - 2y)$

108. $f_x(x, y) = 2x + y, f_y(x, y) = x - 4y$

In Exercises 109 and 110, find the first partial derivative with respect to x.

109. $f(x, y, z) = (\tan y^2z)e^{z^2 + y^{-2}\sqrt{z}}$

110. $f(x, y, z) = x\left(\sinh \dfrac{y}{z}\right)^{(y^2 - 2\sqrt{y - 1})z}$

WRITING ABOUT CONCEPTS

111. Let f be a function of two variables x and y. Describe the procedure for finding the first partial derivatives.

112. Sketch a surface representing a function f of two variables x and y. Use the sketch to give geometric interpretations of $\partial f/\partial x$ and $\partial f/\partial y$.

113. Sketch the graph of a function $z = f(x, y)$ whose derivative f_x is always negative and whose derivative f_y is always positive.

114. Sketch the graph of a function $z = f(x, y)$ whose derivatives f_x and f_y are always positive.

115. If f is a function of x and y such that f_{xy} and f_{yx} are continuous, what is the relationship between the mixed partial derivatives? Explain.

CAPSTONE

116. Find the four second partial derivatives of the function given by $f(x, y) = \sin(x - 2y)$. Show that the second mixed partial derivatives f_{xy} and f_{yx} are equal.

117. ***Marginal Revenue*** A pharmaceutical corporation has two plants that produce the same over-the-counter medicine. If x_1 and x_2 are the numbers of units produced at plant 1 and plant 2, respectively, then the total revenue for the product is given by $R = 200x_1 + 200x_2 - 4x_1^2 - 8x_1x_2 - 4x_2^2$. When $x_1 = 4$ and $x_2 = 12$, find (a) the marginal revenue for plant 1, $\partial R/\partial x_1$, and (b) the marginal revenue for plant 2, $\partial R/\partial x_2$.

118. ***Marginal Costs*** A company manufactures two types of wood-burning stoves: a freestanding model and a fireplace-insert model. The cost function for producing x freestanding and y fireplace-insert stoves is

$$C = 32\sqrt{xy} + 175x + 205y + 1050.$$

(a) Find the marginal costs ($\partial C/\partial x$ and $\partial C/\partial y$) when $x = 80$ and $y = 20$.

(b) When additional production is required, which model of stove results in the cost increasing at a higher rate? How can this be determined from the cost model?

119. ***Psychology*** Early in the twentieth century, an intelligence test called the *Stanford-Binet Test* (more commonly known as the *IQ test*) was developed. In this test, an individual's mental age M is divided by the individual's chronological age C and the quotient is multiplied by 100. The result is the individual's IQ.

$$IQ(M, C) = \frac{M}{C} \times 100$$

Find the partial derivatives of IQ with respect to M and with respect to C. Evaluate the partial derivatives at the point $(12, 10)$ and interpret the result. *(Source: Adapted from Bernstein/Clark-Stewart/Roy/Wickens, Psychology, Fourth Edition)*

120. ***Marginal Productivity*** Consider the Cobb-Douglas production function $f(x, y) = 200x^{0.7}y^{0.3}$. When $x = 1000$ and $y = 500$, find (a) the marginal productivity of labor, $\partial f/\partial x$, and (b) the marginal productivity of capital, $\partial f/\partial y$.

121. ***Think About It*** Let N be the number of applicants to a university, p the charge for food and housing at the university, and t the tuition. N is a function of p and t such that $\partial N/\partial p < 0$ and $\partial N/\partial t < 0$. What information is gained by noticing that both partials are negative?

122. ***Investment*** The value of an investment of \$1000 earning 6% compounded annually is

$$V(I, R) = 1000\left[\frac{1 + 0.06(1 - R)}{1 + I}\right]^{10}$$

where I is the annual rate of inflation and R is the tax rate for the person making the investment. Calculate $V_I(0.03, 0.28)$ and $V_R(0.03, 0.28)$. Determine whether the tax rate or the rate of inflation is the greater "negative" factor in the growth of the investment.

123. ***Temperature Distribution*** The temperature at any point (x, y) in a steel plate is $T = 500 - 0.6x^2 - 1.5y^2$, where x and y are measured in meters. At the point $(2, 3)$, find the rates of change of the temperature with respect to the distances moved along the plate in the directions of the x- and y-axes.

124. ***Apparent Temperature*** A measure of how hot weather feels to an average person is the Apparent Temperature Index. A model for this index is

$$A = 0.885t - 22.4h + 1.20th - 0.544$$

where A is the apparent temperature in degrees Celsius, t is the air temperature, and h is the relative humidity in decimal form. *(Source: The UMAP Journal)*

(a) Find $\partial A/\partial t$ and $\partial A/\partial h$ when $t = 30°$ and $h = 0.80$.

(b) Which has a greater effect on A, air temperature or humidity? Explain.

125. ***Ideal Gas Law*** The Ideal Gas Law states that $PV = nRT$, where P is pressure, V is volume, n is the number of moles of gas, R is a fixed constant (the gas constant), and T is absolute temperature. Show that

$$\frac{\partial T}{\partial P}\frac{\partial P}{\partial V}\frac{\partial V}{\partial T} = -1.$$

126. ***Marginal Utility*** The utility function $U = f(x, y)$ is a measure of the utility (or satisfaction) derived by a person from the consumption of two products x and y. Suppose the utility function is $U = -5x^2 + xy - 3y^2$.

(a) Determine the marginal utility of product x.

(b) Determine the marginal utility of product y.

(c) When $x = 2$ and $y = 3$, should a person consume one more unit of product x or one more unit of product y? Explain your reasoning.

CAS (d) Use a computer algebra system to graph the function. Interpret the marginal utilities of products x and y graphically.

127. ***Modeling Data*** Per capita consumptions (in gallons) of different types of milk in the United States from 1999 through 2005 are shown in the table. Consumption of flavored milk, plain reduced-fat milk, and plain light and skim milks are represented by the variables x, y, and z, respectively. *(Source: U.S. Department of Agriculture)*

Year	1999	2000	2001	2002	2003	2004	2005
x	1.4	1.4	1.4	1.6	1.6	1.7	1.7
y	7.3	7.1	7.0	7.0	6.9	6.9	6.9
z	6.2	6.1	5.9	5.8	5.6	5.5	5.6

A model for the data is given by $z = -0.92x + 1.03y + 0.02$.

(a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

(b) Interpret the partial derivatives in the context of the problem.

128. ***Modeling Data*** The table shows the public medical expenditures (in billions of dollars) for worker's compensation x, public assistance y, and Medicare z from 2000 through 2005. *(Source: Centers for Medicare and Medicaid Services)*

Year	2000	2001	2002	2003	2004	2005
x	24.9	28.1	30.1	31.4	32.1	33.5
y	207.5	233.2	258.4	281.9	303.2	324.9
z	224.3	247.7	265.7	283.5	312.8	342.0

A model for the data is given by

$$z = -1.2225x^2 + 0.0096y^2 + 71.381x - 4.121y - 354.65.$$

(a) Find $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$.

(b) Determine the concavity of traces parallel to the xz-plane. Interpret the result in the context of the problem.

(c) Determine the concavity of traces parallel to the yz-plane. Interpret the result in the context of the problem.

True or False? **In Exercises 129–132, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

129. If $z = f(x, y)$ and $\partial z/\partial x = \partial z/\partial y$, then $z = c(x + y)$.

130. If $z = f(x)g(y)$, then $(\partial z/\partial x) + (\partial z/\partial y) = f'(x)g(y) + f(x)g'(y)$.

131. If $z = e^{xy}$, then $\dfrac{\partial^2 z}{\partial y \partial x} = (xy + 1)e^{xy}$.

132. If a cylindrical surface $z = f(x, y)$ has rulings parallel to the y-axis, then $\partial z/\partial y = 0$.

133. Consider the function defined by

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}.$$

(a) Find $f_x(x, y)$ and $f_y(x, y)$ for $(x, y) \neq (0, 0)$.

(b) Use the definition of partial derivatives to find $f_x(0, 0)$ and $f_y(0, 0)$.

$$\left[\textit{Hint: } f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x}.\right]$$

(c) Use the definition of partial derivatives to find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.

(d) Using Theorem 13.3 and the result of part (c), what can be said about f_{xy} or f_{yx}?

134. Let $f(x, y) = \displaystyle\int_x^y \sqrt{1 + t^3}\, dt$. Find $f_x(x, y)$ and $f_y(x, y)$.

135. Consider the function $f(x, y) = (x^3 + y^3)^{1/3}$.

(a) Find $f_x(0, 0)$ and $f_y(0, 0)$.

(b) Determine the points (if any) at which $f_x(x, y)$ or $f_y(x, y)$ fails to exist.

136. Consider the function $f(x, y) = (x^2 + y^2)^{2/3}$. Show that

$$f_x(x, y) = \begin{cases} \dfrac{4x}{3(x^2 + y^2)^{1/3}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}.$$

■ **FOR FURTHER INFORMATION** For more information about this problem, see the article "A Classroom Note on a Naturally Occurring Piecewise Defined Function" by Don Cohen in *Mathematics and Computer Education.*

SECTION PROJECT

Moiré Fringes

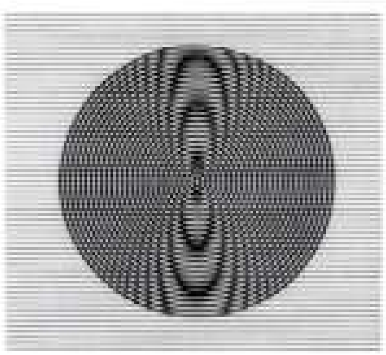
Read the article "Moiré Fringes and the Conic Sections" by Mike Cullen in *The College Mathematics Journal*. The article describes how two families of level curves given by

$$f(x, y) = a \qquad \text{and} \qquad g(x, y) = b$$

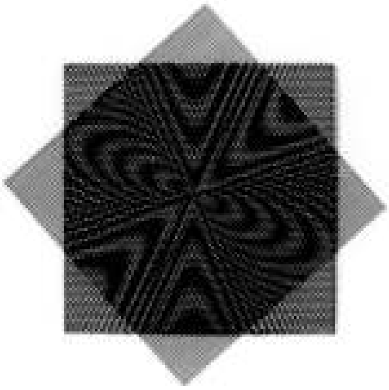
can form Moiré patterns. After reading the article, write a paper explaining how the expression

$$\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial y}$$

is related to the Moiré patterns formed by intersecting the two families of level curves. Use one of the following patterns as an example in your paper.



Mike Cullen



Mike Cullen

13.4 Differentials

- **Understand the concepts of increments and differentials.**
- **Extend the concept of differentiability to a function of two variables.**
- **Use a differential as an approximation.**

Increments and Differentials

In this section, the concepts of increments and differentials are generalized to functions of two or more variables. Recall from Section 3.9 that for $y = f(x)$, the differential of y was defined as

$$dy = f'(x)\,dx.$$

Similar terminology is used for a function of two variables, $z = f(x, y)$. That is, Δx and Δy are the **increments of x and y,** and the **increment of z** is given by

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$ Increment of z

DEFINITION OF TOTAL DIFFERENTIAL

If $z = f(x, y)$ and Δx and Δy are increments of x and y, then the **differentials** of the independent variables x and y are

$$dx = \Delta x \qquad \text{and} \qquad dy = \Delta y$$

and the **total differential** of the dependent variable z is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x, y)\,dx + f_y(x, y)\,dy.$$

This definition can be extended to a function of three or more variables. For instance, if $w = f(x, y, z, u)$, then $dx = \Delta x$, $dy = \Delta y$, $dz = \Delta z$, $du = \Delta u$, and the total differential of w is

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz + \frac{\partial w}{\partial u}du.$$

EXAMPLE 1 Finding the Total Differential

Find the total differential for each function.

a. $z = 2x \sin y - 3x^2y^2$ **b.** $w = x^2 + y^2 + z^2$

Solution

a. The total differential dz for $z = 2x \sin y - 3x^2y^2$ is

$$\begin{aligned} dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy && \text{Total differential } dz \\ &= (2 \sin y - 6xy^2)\,dx + (2x \cos y - 6x^2y)\,dy. \end{aligned}$$

b. The total differential dw for $w = x^2 + y^2 + z^2$ is

$$\begin{aligned} dw &= \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz && \text{Total differential } dw \\ &= 2x\,dx + 2y\,dy + 2z\,dz. \end{aligned}$$

■

Differentiability

In Section 3.9, you learned that for a *differentiable* function given by $y = f(x)$, you can use the differential $dy = f'(x)\,dx$ as an approximation (for small Δx) to the value $\Delta y = f(x + \Delta x) - f(x)$. When a similar approximation is possible for a function of two variables, the function is said to be **differentiable.** This is stated explicitly in the following definition.

DEFINITION OF DIFFERENTIABILITY

A function f given by $z = f(x, y)$ is **differentiable** at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. The function f is **differentiable in a region *R*** if it is differentiable at each point in R.

EXAMPLE 2 Showing That a Function Is Differentiable

Show that the function given by

$$f(x, y) = x^2 + 3y$$

is differentiable at every point in the plane.

Solution Letting $z = f(x, y)$, the increment of z at an arbitrary point (x, y) in the plane is

$$\begin{aligned}
\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) && \text{Increment of } z\\
&= (x^2 + 2x\Delta x + \Delta x^2) + 3(y + \Delta y) - (x^2 + 3y)\\
&= 2x\Delta x + \Delta x^2 + 3\Delta y\\
&= 2x(\Delta x) + 3(\Delta y) + \Delta x(\Delta x) + 0(\Delta y)\\
&= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y
\end{aligned}$$

where $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = 0$. Because $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$, it follows that f is differentiable at every point in the plane. The graph of f is shown in Figure 13.34. ■

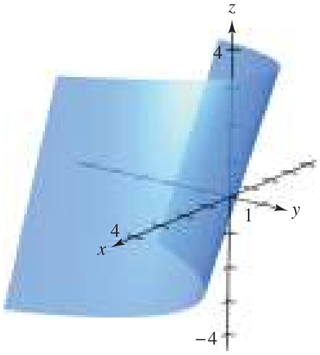


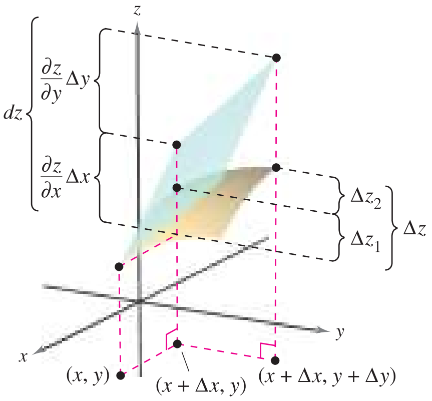
Figure 13.34

Be sure you see that the term "differentiable" is used differently for functions of two variables than for functions of one variable. A function of one variable is differentiable at a point if its derivative exists at the point. However, for a function of two variables, the existence of the partial derivatives f_x and f_y does not guarantee that the function is differentiable (see Example 5). The following theorem gives a *sufficient* condition for differentiability of a function of two variables. A proof of Theorem 13.4 is given in Appendix A.

THEOREM 13.4 SUFFICIENT CONDITION FOR DIFFERENTIABILITY

If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.

Approximation by Differentials


z
dz
$\frac{\partial z}{\partial y}\Delta y$
$\frac{\partial z}{\partial x}\Delta x$
Δz_2
Δz_1
Δz
y
x
(x, y)
$(x+\Delta x, y)$
$(x+\Delta x, y+\Delta y)$

The exact change in z is Δz. This change can be approximated by the differential dz.
Figure 13.35

Theorem 13.4 tells you that you can choose $(x + \Delta x, y + \Delta y)$ close enough to (x, y) to make $\varepsilon_1 \Delta x$ and $\varepsilon_2 \Delta y$ insignificant. In other words, for small Δx and Δy, you can use the approximation

$$\Delta z \approx dz.$$

This approximation is illustrated graphically in Figure 13.35. Recall that the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ can be interpreted as the slopes of the surface in the x- and y-directions. This means that

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

represents the change in height of a plane that is tangent to the surface at the point $(x, y, f(x, y))$. Because a plane in space is represented by a linear equation in the variables x, y, and z, the approximation of Δz by dz is called a **linear approximation.** You will learn more about this geometric interpretation in Section 13.7.

EXAMPLE 3 Using a Differential as an Approximation

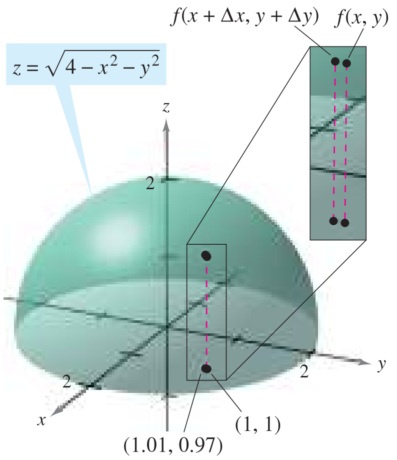
Use the differential dz to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x, y) moves from the point $(1, 1)$ to the point $(1.01, 0.97)$. Compare this approximation with the exact change in z.

Solution Letting $(x, y) = (1, 1)$ and $(x + \Delta x, y + \Delta y) = (1.01, 0.97)$ produces $dx = \Delta x = 0.01$ and $dy = \Delta y = -0.03$. So, the change in z can be approximated by

$$\Delta z \approx dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{-x}{\sqrt{4 - x^2 - y^2}}\Delta x + \frac{-y}{\sqrt{4 - x^2 - y^2}}\Delta y.$$

When $x = 1$ and $y = 1$, you have

$$\Delta z \approx -\frac{1}{\sqrt{2}}(0.01) - \frac{1}{\sqrt{2}}(-0.03) = \frac{0.02}{\sqrt{2}} = \sqrt{2}(0.01) \approx 0.0141.$$



As (x, y) moves from $(1, 1)$ to the point $(1.01, 0.97)$, the value of $f(x, y)$ changes by about 0.0137.
Figure 13.36

In Figure 13.36, you can see that the exact change corresponds to the difference in the heights of two points on the surface of a hemisphere. This difference is given by

$$\begin{aligned}\Delta z &= f(1.01, 0.97) - f(1, 1)\\ &= \sqrt{4 - (1.01)^2 - (0.97)^2} - \sqrt{4 - 1^2 - 1^2} \approx 0.0137.\end{aligned}$$

■

A function of three variables $w = f(x, y, z)$ is called **differentiable** at (x, y, z) provided that

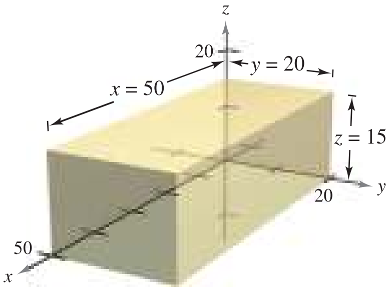
$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

can be written in the form

$$\Delta w = f_x \Delta x + f_y \Delta y + f_z \Delta z + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z$$

where ε_1, ε_2, and $\varepsilon_3 \to 0$ as $(\Delta x, \Delta y, \Delta z) \to (0, 0, 0)$. With this definition of differentiability, Theorem 13.4 has the following extension for functions of three variables: If f is a function of x, y, and z, where f, f_x, f_y, and f_z are continuous in an open region R, then f is differentiable on R.

In Section 3.9, you used differentials to approximate the propagated error introduced by an error in measurement. This application of differentials is further illustrated in Example 4.



Volume $= xyz$
Figure 13.37

EXAMPLE 4 Error Analysis

The possible error involved in measuring each dimension of a rectangular box is ±0.1 millimeter. The dimensions of the box are $x = 50$ centimeters, $y = 20$ centimeters, and $z = 15$ centimeters, as shown in Figure 13.37. Use dV to estimate the propagated error and the relative error in the calculated volume of the box.

Solution The volume of the box is given by $V = xyz$, and so

$$\begin{aligned} dV &= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \\ &= yz\,dx + xz\,dy + xy\,dz. \end{aligned}$$

Using 0.1 millimeter = 0.01 centimeter, you have $dx = dy = dz = \pm 0.01$, and the propagated error is approximately

$$\begin{aligned} dV &= (20)(15)(\pm 0.01) + (50)(15)(\pm 0.01) + (50)(20)(\pm 0.01) \\ &= 300(\pm 0.01) + 750(\pm 0.01) + 1000(\pm 0.01) \\ &= 2050(\pm 0.01) = \pm 20.5 \text{ cubic centimeters.} \end{aligned}$$

Because the measured volume is

$$V = (50)(20)(15) = 15{,}000 \text{ cubic centimeters,}$$

the relative error, $\Delta V/V$, is approximately

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{20.5}{15{,}000} \approx 0.14\%.$$ ■

As is true for a function of a single variable, if a function in two or more variables is differentiable at a point, it is also continuous there.

THEOREM 13.5 DIFFERENTIABILITY IMPLIES CONTINUITY

If a function of x and y is differentiable at (x_0, y_0), then it is continuous at (x_0, y_0).

PROOF Let f be differentiable at (x_0, y_0), where $z = f(x, y)$. Then

$$\Delta z = [f_x(x_0, y_0) + \varepsilon_1]\,\Delta x + [f_y(x_0, y_0) + \varepsilon_2]\,\Delta y$$

where both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. However, by definition, you know that Δz is given by

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

Letting $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$ produces

$$\begin{aligned} f(x, y) - f(x_0, y_0) &= [f_x(x_0, y_0) + \varepsilon_1]\,\Delta x + [f_y(x_0, y_0) + \varepsilon_2]\,\Delta y \\ &= [f_x(x_0, y_0) + \varepsilon_1](x - x_0) + [f_y(x_0, y_0) + \varepsilon_2](y - y_0). \end{aligned}$$

Taking the limit as $(x, y) \to (x_0, y_0)$, you have

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = f(x_0, y_0)$$

which means that f is continuous at (x_0, y_0). ■

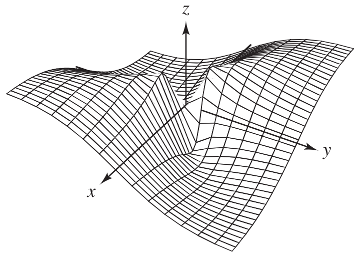
Remember that the existence of f_x and f_y is not sufficient to guarantee differentiability, as illustrated in the next example.

EXAMPLE 5 A Function That Is Not Differentiable

Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist, but that f is not differentiable at $(0, 0)$ where f is defined as

$$f(x, y) = \begin{cases} \dfrac{-3xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}.$$

TECHNOLOGY Use a graphing utility to graph the function given in Example 5. For instance, the graph shown below was generated by *Mathematica.*



Generated by Mathematica

Solution You can show that f is not differentiable at $(0, 0)$ by showing that it is not continuous at this point. To see that f is not continuous at $(0, 0)$, look at the values of $f(x, y)$ along two different approaches to $(0, 0)$, as shown in Figure 13.38. Along the line $y = x$, the limit is

$$\lim_{(x, x)\to(0, 0)} f(x, y) = \lim_{(x, x)\to(0, 0)} \frac{-3x^2}{2x^2} = -\frac{3}{2}$$

whereas along $y = -x$ you have

$$\lim_{(x, -x)\to(0, 0)} f(x, y) = \lim_{(x, -x)\to(0, 0)} \frac{3x^2}{2x^2} = \frac{3}{2}.$$

So, the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ does not exist, and you can conclude that f is not continuous at $(0, 0)$. Therefore, by Theorem 13.5, you know that f is not differentiable at $(0, 0)$. On the other hand, by the definition of the partial derivatives f_x and f_y, you have

$$f_x(0, 0) = \lim_{\Delta x\to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0 - 0}{\Delta x} = 0$$

and

$$f_y(0, 0) = \lim_{\Delta y\to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{0 - 0}{\Delta y} = 0.$$

So, the partial derivatives at $(0, 0)$ exist.

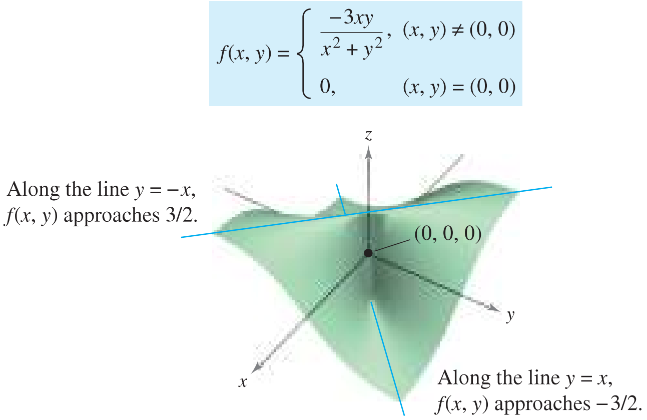


Figure 13.38 ■

13.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, find the total differential.

1. $z = 2x^2y^3$

2. $z = \dfrac{x^2}{y}$

3. $z = \dfrac{-1}{x^2 + y^2}$

4. $w = \dfrac{x + y}{z - 3y}$

5. $z = x \cos y - y \cos x$

6. $z = \frac{1}{2}(e^{x^2+y^2} - e^{-x^2-y^2})$

7. $z = e^x \sin y$

8. $w = e^y \cos x + z^2$

9. $w = 2z^3y \sin x$

10. $w = x^2yz^2 + \sin yz$

In Exercises 11–16, (a) evaluate $f(2, 1)$ and $f(2.1, 1.05)$ and calculate Δz, and (b) use the total differential dz to approximate Δz.

11. $f(x, y) = 2x - 3y$

12. $f(x, y) = x^2 + y^2$

13. $f(x, y) = 16 - x^2 - y^2$

14. $f(x, y) = \dfrac{y}{x}$

15. $f(x, y) = ye^x$

16. $f(x, y) = x \cos y$

In Exercises 17–20, find $z = f(x, y)$ and use the total differential to approximate the quantity.

17. $(2.01)^2(9.02) - 2^2 \cdot 9$

18. $\sqrt{(5.05)^2 + (3.1)^2} - \sqrt{5^2 + 3^2}$

19. $\dfrac{1 - (3.05)^2}{(5.95)^2} - \dfrac{1 - 3^2}{6^2}$

20. $\sin[(1.05)^2 + (0.95)^2] - \sin(1^2 + 1^2)$

WRITING ABOUT CONCEPTS

21. Define the total differential of a function of two variables.

22. Describe the change in accuracy of dz as an approximation of Δz as Δx and Δy increase.

23. What is meant by a linear approximation of $z = f(x, y)$ at the point $P(x_0, y_0)$?

24. When using differentials, what is meant by the terms *propagated error* and *relative error*?

25. ***Area*** The area of the shaded rectangle in the figure is $A = lh$. The possible errors in the length and height are Δl and Δh, respectively. Find dA and identify the regions in the figure whose areas are given by the terms of dA. What region represents the difference between ΔA and dA?

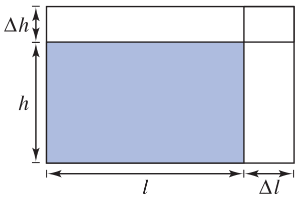


Figure for 25

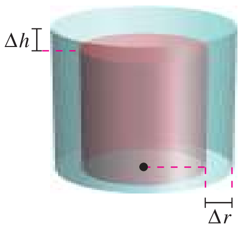


Figure for 26

26. ***Volume*** The volume of the red right circular cylinder in the figure is $V = \pi r^2h$. The possible errors in the radius and the height are Δr and Δh, respectively. Find dV and identify the solids in the figure whose volumes are given by the terms of dV. What solid represents the difference between ΔV and dV?

27. ***Numerical Analysis*** A right circular cone of height $h = 8$ and radius $r = 4$ is constructed, and in the process errors Δr and Δh are made in the radius and height, respectively. Complete the table to show the relationship between ΔV and dV for the indicated errors.

Δr	Δh	dV or dS	ΔV or ΔS	$\Delta V - dV$ or $\Delta S - dS$
0.1	0.1			
0.1	−0.1			
0.001	0.002			
−0.0001	0.0002			

28. ***Numerical Analysis*** The height and radius of a right circular cone are measured as $h = 16$ meters and $r = 6$ meters. In the process of measuring, errors Δr and Δh are made. S is the lateral surface area of the cone. Complete the table above to show the relationship between ΔS and dS for the indicated errors.

29. ***Modeling Data*** Per capita consumptions (in gallons) of different types of plain milk in the United States from 1999 through 2005 are shown in the table. Consumption of flavored milk, plain reduced-fat milk, and plain light and skim milks are represented by the variables x, y, and z, respectively. *(Source: U.S. Department of Agriculture)*

Year	1999	2000	2001	2002	2003	2004	2005
x	1.4	1.4	1.4	1.6	1.6	1.7	1.7
y	7.3	7.1	7.0	7.0	6.9	6.9	6.9
z	6.2	6.1	5.9	5.8	5.6	5.5	5.6

A model for the data is given by $z = -0.92x + 1.03y + 0.02$.

(a) Find the total differential of the model.

(b) A dairy industry forecast for a future year is that per capita consumption of flavored milk will be 1.9 ± 0.25 gallons and that per capita consumption of plain reduced-fat milk will be 7.5 ± 0.25 gallons. Use dz to estimate the maximum possible propagated error and relative error in the prediction for the consumption of plain light and skim milks.

30. ***Rectangular to Polar Coordinates*** A rectangular coordinate system is placed over a map, and the coordinates of a point of interest are $(7.2, 2.5)$. There is a possible error of 0.05 in each coordinate. Approximate the maximum possible error in measuring the polar coordinates of the point.

31. ***Volume*** The radius r and height h of a right circular cylinder are measured with possible errors of 4% and 2%, respectively. Approximate the maximum possible percent error in measuring the volume.

32. ***Area*** A triangle is measured and two adjacent sides are found to be 3 inches and 4 inches long, with an included angle of $\pi/4$. The possible errors in measurement are $\frac{1}{16}$ inch for the sides and 0.02 radian for the angle. Approximate the maximum possible error in the computation of the area.

33. ***Wind Chill*** The formula for wind chill C (in degrees Fahrenheit) is given by

$$C = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$$

where v is the wind speed in miles per hour and T is the temperature in degrees Fahrenheit. The wind speed is 23 ± 3 miles per hour and the temperature is $8° \pm 1°$. Use dC to estimate the maximum possible propagated error and relative error in calculating the wind chill. *(Source: National Oceanic and Atmospheric Administration)*

34. ***Acceleration*** The centripetal acceleration of a particle moving in a circle is $a = v^2/r$, where v is the velocity and r is the radius of the circle. Approximate the maximum percent error in measuring the acceleration due to errors of 3% in v and 2% in r.

35. ***Volume*** A trough is 16 feet long (see figure). Its cross sections are isosceles triangles with each of the two equal sides measuring 18 inches. The angle between the two equal sides is θ.

(a) Write the volume of the trough as a function of θ and determine the value of θ such that the volume is a maximum.

(b) The maximum error in the linear measurements is one-half inch and the maximum error in the angle measure is 2°. Approximate the change in the maximum volume.

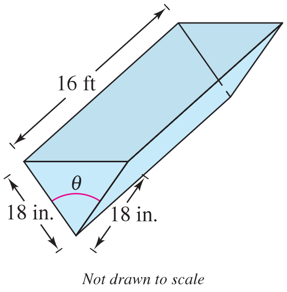


Figure for 35

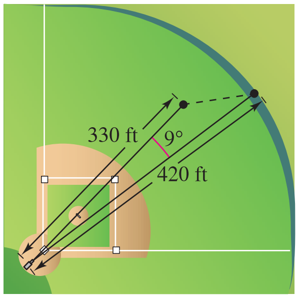


Figure for 36

36. ***Sports*** A baseball player in center field is playing approximately 330 feet from a television camera that is behind home plate. A batter hits a fly ball that goes to the wall 420 feet from the camera (see figure).

(a) The camera turns 9° to follow the play. Approximate the number of feet that the center fielder has to run to make the catch.

(b) The position of the center fielder could be in error by as much as 6 feet and the maximum error in measuring the rotation of the camera is 1°. Approximate the maximum possible error in the result of part (a).

37. ***Power*** Electrical power P is given by $P = E^2/R$, where E is voltage and R is resistance. Approximate the maximum percent error in calculating power if 120 volts is applied to a 2000-ohm resistor and the possible percent errors in measuring E and R are 3% and 4%, respectively.

38. ***Resistance*** The total resistance R of two resistors connected in parallel is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Approximate the change in R as R_1 is increased from 10 ohms to 10.5 ohms and R_2 is decreased from 15 ohms to 13 ohms.

39. ***Inductance*** The inductance L (in microhenrys) of a straight nonmagnetic wire in free space is

$$L = 0.00021\left(\ln\frac{2h}{r} - 0.75\right)$$

where h is the length of the wire in millimeters and r is the radius of a circular cross section. Approximate L when $r = 2 \pm \frac{1}{16}$ millimeters and $h = 100 \pm \frac{1}{100}$ millimeters.

40. ***Pendulum*** The period T of a pendulum of length L is $T = 2\pi\sqrt{L/g}$, where g is the acceleration due to gravity. A pendulum is moved from the Canal Zone, where $g = 32.09$ feet per second per second, to Greenland, where $g = 32.23$ feet per second per second. Because of the change in temperature, the length of the pendulum changes from 2.5 feet to 2.48 feet. Approximate the change in the period of the pendulum.

In Exercises 41–44, show that the function is differentiable by finding values of ε_1 and ε_2 as designated in the definition of differentiability, and verify that both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

41. $f(x, y) = x^2 - 2x + y$

42. $f(x, y) = x^2 + y^2$

43. $f(x, y) = x^2y$

44. $f(x, y) = 5x - 10y + y^3$

In Exercises 45 and 46, use the function to show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist, but that f is not differentiable at $(0, 0)$.

45. $$f(x, y) = \begin{cases} \dfrac{3x^2y}{x^4 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

46. $$f(x, y) = \begin{cases} \dfrac{5x^2y}{x^3 + y^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

47. Show that if $f(x, y)$ is differentiable at (x_0, y_0), then $f(x, y_0)$ is differentiable at $x = x_0$. Use this result to prove that $f(x, y) = \sqrt{x^2 + y^2}$ is not differentiable at $(0, 0)$.

CAPSTONE

48. Consider the function $f(x, y) = \sqrt{x^2 + y^2}$.

(a) Evaluate $f(3, 1)$ and $f(3.05, 1.1)$.

(b) Use the results of part (a) to calculate Δz.

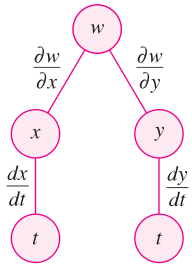
(c) Use the total differential dz to approximate Δz. Compare your result with that of part (b).

13.5 Chain Rules for Functions of Several Variables

- **Use the Chain Rules for functions of several variables.**
- **Find partial derivatives implicitly.**

Chain Rules for Functions of Several Variables

Your work with differentials in the preceding section provides the basis for the extension of the Chain Rule to functions of two variables. There are two cases—the first case involves w as a function of x and y, where x and y are functions of a single independent variable t, as shown in Theorem 13.6. (A proof of this theorem is given in Appendix A.)



w
$\frac{\partial w}{\partial x}$
$\frac{\partial w}{\partial y}$
x
y
$\frac{dx}{dt}$
$\frac{dy}{dt}$
t
t

Chain Rule: one independent variable w is a function of x and y, which are each functions of t. This diagram represents the derivative of w with respect to t.
Figure 13.39

THEOREM 13.6 CHAIN RULE: ONE INDEPENDENT VARIABLE

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

See Figure 13.39.

EXAMPLE 1 Using the Chain Rule with One Independent Variable

Let $w = x^2y - y^2$, where $x = \sin t$ and $y = e^t$. Find dw/dt when $t = 0$.

Solution By the Chain Rule for one independent variable, you have

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}\\ &= 2xy(\cos t) + (x^2 - 2y)e^t\\ &= 2(\sin t)(e^t)(\cos t) + (\sin^2 t - 2e^t)e^t\\ &= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}.\end{aligned}$$

When $t = 0$, it follows that

$$\frac{dw}{dt} = -2.$$

The Chain Rules presented in this section provide alternative techniques for solving many problems in single-variable calculus. For instance, in Example 1, you could have used single-variable techniques to find dw/dt by first writing w as a function of t,

$$\begin{aligned}w &= x^2y - y^2\\ &= (\sin t)^2(e^t) - (e^t)^2\\ &= e^t \sin^2 t - e^{2t}\end{aligned}$$

and then differentiating as usual.

$$\frac{dw}{dt} = 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}$$

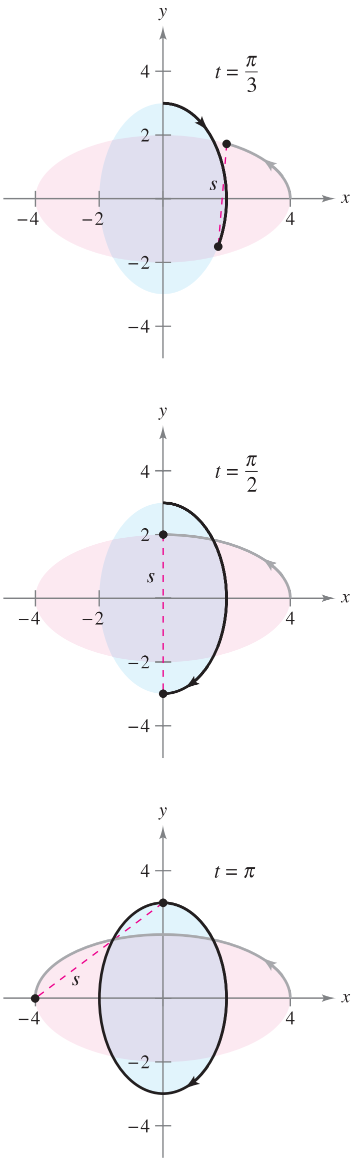
The Chain Rule in Theorem 13.6 can be extended to any number of variables. For example, if each x_i is a differentiable function of a single variable t, then for

$$w = f(x_1, x_2, \ldots, x_n)$$

you have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial w}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}.$$

EXAMPLE 2 An Application of a Chain Rule to Related Rates



Paths of two objects traveling in elliptical orbits
Figure 13.40

Two objects are traveling in elliptical paths given by the following parametric equations.

$x_1 = 4\cos t$ and $y_1 = 2\sin t$ First object

$x_2 = 2\sin 2t$ and $y_2 = 3\cos 2t$ Second object

At what rate is the distance between the two objects changing when $t = \pi$?

Solution From Figure 13.40, you can see that the distance s between the two objects is given by

$$s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

and that when $t = \pi$, you have $x_1 = -4$, $y_1 = 0$, $x_2 = 0$, $y_2 = 3$, and

$$s = \sqrt{(0 + 4)^2 + (3 - 0)^2} = 5.$$

When $t = \pi$, the partial derivatives of s are as follows.

$$\frac{\partial s}{\partial x_1} = \frac{-(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = -\frac{1}{5}(0 + 4) = -\frac{4}{5}$$

$$\frac{\partial s}{\partial y_1} = \frac{-(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = -\frac{1}{5}(3 - 0) = -\frac{3}{5}$$

$$\frac{\partial s}{\partial x_2} = \frac{(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = \frac{1}{5}(0 + 4) = \frac{4}{5}$$

$$\frac{\partial s}{\partial y_2} = \frac{(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = \frac{1}{5}(3 - 0) = \frac{3}{5}$$

When $t = \pi$, the derivatives of x_1, y_1, x_2, and y_2 are

$$\frac{dx_1}{dt} = -4\sin t = 0 \qquad \frac{dy_1}{dt} = 2\cos t = -2$$

$$\frac{dx_2}{dt} = 4\cos 2t = 4 \qquad \frac{dy_2}{dt} = -6\sin 2t = 0.$$

So, using the appropriate Chain Rule, you know that the distance is changing at a rate of

$$\begin{aligned}\frac{ds}{dt} &= \frac{\partial s}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial s}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial s}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial s}{\partial y_2}\frac{dy_2}{dt}\\ &= \left(-\frac{4}{5}\right)(0) + \left(-\frac{3}{5}\right)(-2) + \left(\frac{4}{5}\right)(4) + \left(\frac{3}{5}\right)(0)\\ &= \frac{22}{5}.\end{aligned}$$

■

In Example 2, note that s is the function of four *intermediate* variables, x_1, y_1, x_2, and y_2, each of which is a function of a single variable t. Another type of composite function is one in which the intermediate variables are themselves functions of more than one variable. For instance, if $w = f(x, y)$, where $x = g(s, t)$ and $y = h(s, t)$, it follows that w is a function of s and t, and you can consider the partial derivatives of w with respect to s and t. One way to find these partial derivatives is to write w as a function of s and t explicitly by substituting the equations $x = g(s, t)$ and $y = h(s, t)$ into the equation $w = f(x, y)$. Then you can find the partial derivatives in the usual way, as demonstrated in the next example.

EXAMPLE 3 Finding Partial Derivatives by Substitution

Find $\partial w/\partial s$ and $\partial w/\partial t$ for $w = 2xy$, where $x = s^2 + t^2$ and $y = s/t$.

Solution Begin by substituting $x = s^2 + t^2$ and $y = s/t$ into the equation $w = 2xy$ to obtain

$$w = 2xy = 2(s^2 + t^2)\left(\frac{s}{t}\right) = 2\left(\frac{s^3}{t} + st\right).$$

Then, to find $\partial w/\partial s$, hold t constant and differentiate with respect to s.

$$\begin{aligned}\frac{\partial w}{\partial s} &= 2\left(\frac{3s^2}{t} + t\right)\\ &= \frac{6s^2 + 2t^2}{t}\end{aligned}$$

Similarly, to find $\partial w/\partial t$, hold s constant and differentiate with respect to t to obtain

$$\begin{aligned}\frac{\partial w}{\partial t} &= 2\left(-\frac{s^3}{t^2} + s\right)\\ &= 2\left(\frac{-s^3 + st^2}{t^2}\right)\\ &= \frac{2st^2 - 2s^3}{t^2}.\end{aligned}$$

■

Theorem 13.7 gives an alternative method for finding the partial derivatives in Example 3, without explicitly writing w as a function of s and t.

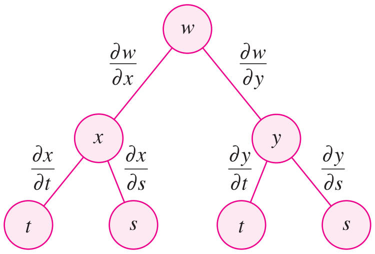
THEOREM 13.7 CHAIN RULE: TWO INDEPENDENT VARIABLES

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(s, t)$ and $y = h(s, t)$ such that the first partials $\partial x/\partial s$, $\partial x/\partial t$, $\partial y/\partial s$, and $\partial y/\partial t$ all exist, then $\partial w/\partial s$ and $\partial w/\partial t$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

PROOF To obtain $\partial w/\partial s$, hold t constant and apply Theorem 13.6 to obtain the desired result. Similarly, for $\partial w/\partial t$, hold s constant and apply Theorem 13.6. ■

NOTE The Chain Rule in Theorem 13.7 is shown schematically in Figure 13.41. ■



Chain Rule: two independent variables
Figure 13.41

EXAMPLE 4 The Chain Rule with Two Independent Variables

Use the Chain Rule to find $\partial w/\partial s$ and $\partial w/\partial t$ for

$$w = 2xy$$

where $x = s^2 + t^2$ and $y = s/t$.

Solution Note that these same partials were found in Example 3. This time, using Theorem 13.7, you can hold t constant and differentiate with respect to s to obtain

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}\\
&= 2y(2s) + 2x\left(\frac{1}{t}\right)\\
&= 2\left(\frac{s}{t}\right)(2s) + 2(s^2 + t^2)\left(\frac{1}{t}\right) && \text{Substitute } (s/t) \text{ for } y \text{ and } s^2 + t^2 \text{ for } x.\\
&= \frac{4s^2}{t} + \frac{2s^2 + 2t^2}{t}\\
&= \frac{6s^2 + 2t^2}{t}.
\end{aligned}$$

Similarly, holding s constant gives

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}\\
&= 2y(2t) + 2x\left(\frac{-s}{t^2}\right)\\
&= 2\left(\frac{s}{t}\right)(2t) + 2(s^2 + t^2)\left(\frac{-s}{t^2}\right) && \text{Substitute } (s/t) \text{ for } y \text{ and } s^2 + t^2 \text{ for } x.\\
&= 4s - \frac{2s^3 + 2st^2}{t^2}\\
&= \frac{4st^2 - 2s^3 - 2st^2}{t^2}\\
&= \frac{2st^2 - 2s^3}{t^2}.
\end{aligned}$$

■

The Chain Rule in Theorem 13.7 can also be extended to any number of variables. For example, if w is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$, where each x_i is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$, then for

$$w = f(x_1, x_2, \ldots, x_n)$$

you obtain the following.

$$\begin{aligned}
\frac{\partial w}{\partial t_1} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1}\\
\frac{\partial w}{\partial t_2} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_2}\\
&\vdots\\
\frac{\partial w}{\partial t_m} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}
\end{aligned}$$

EXAMPLE 5 The Chain Rule for a Function of Three Variables

Find $\partial w/\partial s$ and $\partial w/\partial t$ when $s = 1$ and $t = 2\pi$ for the function given by

$$w = xy + yz + xz$$

where $x = s\cos t$, $y = s\sin t$, and $z = t$.

Solution By extending the result of Theorem 13.7, you have

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}\\ &= (y + z)(\cos t) + (x + z)(\sin t) + (y + x)(0)\\ &= (y + z)(\cos t) + (x + z)(\sin t).\end{aligned}$$

When $s = 1$ and $t = 2\pi$, you have $x = 1$, $y = 0$, and $z = 2\pi$. So, $\partial w/\partial s = (0 + 2\pi)(1) + (1 + 2\pi)(0) = 2\pi$. Furthermore,

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}\\ &= (y + z)(-s\sin t) + (x + z)(s\cos t) + (y + x)(1)\end{aligned}$$

and for $s = 1$ and $t = 2\pi$, it follows that

$$\begin{aligned}\frac{\partial w}{\partial t} &= (0 + 2\pi)(0) + (1 + 2\pi)(1) + (0 + 1)(1)\\ &= 2 + 2\pi.\end{aligned}$$

■

Implicit Partial Differentiation

This section concludes with an application of the Chain Rule to determine the derivative of a function defined *implicitly*. Suppose that x and y are related by the equation $F(x, y) = 0$, where it is assumed that $y = f(x)$ is a differentiable function of x. To find dy/dx, you could use the techniques discussed in Section 2.5. However, you will see that the Chain Rule provides a convenient alternative. If you consider the function given by

$$w = F(x, y) = F(x, f(x))$$

you can apply Theorem 13.6 to obtain

$$\frac{dw}{dx} = F_x(x, y)\frac{dx}{dx} + F_y(x, y)\frac{dy}{dx}.$$

Because $w = F(x, y) = 0$ for all x in the domain of f, you know that $dw/dx = 0$ and you have

$$F_x(x, y)\frac{dx}{dx} + F_y(x, y)\frac{dy}{dx} = 0.$$

Now, if $F_y(x, y) \neq 0$, you can use the fact that $dx/dx = 1$ to conclude that

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}.$$

A similar procedure can be used to find the partial derivatives of functions of several variables that are defined implicitly.

THEOREM 13.8 CHAIN RULE: IMPLICIT DIFFERENTIATION

If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \qquad F_y(x, y) \neq 0.$$

If the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0.$$

This theorem can be extended to differentiable functions defined implicitly with any number of variables.

EXAMPLE 6 Finding a Derivative Implicitly

Find dy/dx, given $y^3 + y^2 - 5y - x^2 + 4 = 0$.

Solution Begin by defining a function F as

$$F(x, y) = y^3 + y^2 - 5y - x^2 + 4.$$

Then, using Theorem 13.8, you have

$$F_x(x, y) = -2x \quad \text{and} \quad F_y(x, y) = 3y^2 + 2y - 5$$

and it follows that

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = \frac{-(-2x)}{3y^2 + 2y - 5} = \frac{2x}{3y^2 + 2y - 5}.$$

■

NOTE Compare the solution of Example 6 with the solution of Example 2 in Section 2.5. ■

EXAMPLE 7 Finding Partial Derivatives Implicitly

Find $\partial z/\partial x$ and $\partial z/\partial y$, given $3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0$.

Solution To apply Theorem 13.8, let

$$F(x, y, z) = 3x^2z - x^2y^2 + 2z^3 + 3yz - 5.$$

Then

$$F_x(x, y, z) = 6xz - 2xy^2$$
$$F_y(x, y, z) = -2x^2y + 3z$$
$$F_z(x, y, z) = 3x^2 + 6z^2 + 3y$$

and you obtain

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = \frac{2xy^2 - 6xz}{3x^2 + 6z^2 + 3y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{2x^2y - 3z}{3x^2 + 6z^2 + 3y}.$$

■

13.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find dw/dt using the appropriate Chain Rule.

1. $w = x^2 + y^2$
$x = 2t, \quad y = 3t$

2. $w = \sqrt{x^2 + y^2}$
$x = \cos t, \quad y = e^t$

3. $w = x \sin y$
$x = e^t, \quad y = \pi - t$

4. $w = \ln \dfrac{y}{x}$
$x = \cos t, \quad y = \sin t$

In Exercises 5–10, find dw/dt (a) by using the appropriate Chain Rule and (b) by converting w to a function of t before differentiating.

5. $w = xy, \quad x = e^t, \quad y = e^{-2t}$

6. $w = \cos(x - y), \quad x = t^2, \quad y = 1$

7. $w = x^2 + y^2 + z^2, \quad x = \cos t, \quad y = \sin t, \quad z = e^t$

8. $w = xy \cos z, \quad x = t, \quad y = t^2, \quad z = \arccos t$

9. $w = xy + xz + yz, \quad x = t - 1, \quad y = t^2 - 1, \quad z = t$

10. $w = xy^2 + x^2z + yz^2, \quad x = t^2, \quad y = 2t, \quad z = 2$

Projectile Motion **In Exercises 11 and 12, the parametric equations for the paths of two projectiles are given. At what rate is the distance between the two objects changing at the given value of t?**

11. $x_1 = 10 \cos 2t, \ y_1 = 6 \sin 2t$ First object
$x_2 = 7 \cos t, \ y_2 = 4 \sin t$ Second object
$t = \pi/2$

12. $x_1 = 48\sqrt{2}\,t, \ y_1 = 48\sqrt{2}\,t - 16t^2$ First object
$x_2 = 48\sqrt{3}\,t, \ y_2 = 48t - 16t^2$ Second object
$t = 1$

In Exercises 13 and 14, find d^2w/dt^2 using the appropriate Chain Rule. Evaluate d^2w/dt^2 at the given value of t.

13. $w = \ln(x + y), \quad x = e^t, \quad y = e^{-t}, \quad t = 0$

14. $w = \dfrac{x^2}{y}, \quad x = t^2, \quad y = t + 1, \quad t = 1$

In Exercises 15–18, find $\partial w/\partial s$ and $\partial w/\partial t$ using the appropriate Chain Rule, and evaluate each partial derivative at the given values of s and t.

	Function	*Point*
15.	$w = x^2 + y^2$ $x = s + t, \quad y = s - t$	$s = 1, \quad t = 0$
16.	$w = y^3 - 3x^2y$ $x = e^s, \quad y = e^t$	$s = -1, \quad t = 2$
17.	$w = \sin(2x + 3y)$ $x = s + t, \quad y = s - t$	$s = 0, \quad t = \dfrac{\pi}{2}$
18.	$w = x^2 - y^2$ $x = s \cos t, \quad y = s \sin t$	$s = 3, \quad t = \dfrac{\pi}{4}$

In Exercises 19–22, find $\partial w/\partial r$ and $\partial w/\partial \theta$ (a) by using the appropriate Chain Rule and (b) by converting w to a function of r and θ before differentiating.

19. $w = \dfrac{yz}{x}, \quad x = \theta^2, \quad y = r + \theta, \quad z = r - \theta$

20. $w = x^2 - 2xy + y^2, \quad x = r + \theta, \quad y = r - \theta$

21. $w = \arctan \dfrac{y}{x}, \quad x = r \cos \theta, \quad y = r \sin \theta$

22. $w = \sqrt{25 - 5x^2 - 5y^2}, \quad x = r \cos \theta, \quad y = r \sin \theta$

In Exercises 23–26, find $\partial w/\partial s$ and $\partial w/\partial t$ by using the appropriate Chain Rule.

23. $w = xyz, \quad x = s + t, \quad y = s - t, \quad z = st^2$

24. $w = x^2 + y^2 + z^2, \quad x = t \sin s, \quad y = t \cos s, \quad z = st^2$

25. $w = ze^{xy}, \quad x = s - t, \quad y = s + t, \quad z = st$

26. $w = x \cos yz, \quad x = s^2, \quad y = t^2, \quad z = s - 2t$

In Exercises 27–30, differentiate implicitly to find dy/dx.

27. $x^2 - xy + y^2 - x + y = 0$

28. $\sec xy + \tan xy + 5 = 0$

29. $\ln\sqrt{x^2 + y^2} + x + y = 4$

30. $\dfrac{x}{x^2 + y^2} - y^2 = 6$

In Exercises 31–38, differentiate implicitly to find the first partial derivatives of z.

31. $x^2 + y^2 + z^2 = 1$

32. $xz + yz + xy = 0$

33. $x^2 + 2yz + z^2 = 1$

34. $x + \sin(y + z) = 0$

35. $\tan(x + y) + \tan(y + z) = 1$

36. $z = e^x \sin(y + z)$

37. $e^{xz} + xy = 0$

38. $x \ln y + y^2z + z^2 = 8$

In Exercises 39–42, differentiate implicitly to find the first partial derivatives of w.

39. $xy + yz - wz + wx = 5$

40. $x^2 + y^2 + z^2 - 5yw + 10w^2 = 2$

41. $\cos xy + \sin yz + wz = 20$

42. $w - \sqrt{x - y} - \sqrt{y - z} = 0$

Homogeneous Functions **A function f is *homogeneous of degree n* if $f(tx, ty) = t^n f(x, y)$. In Exercises 43–46, (a) show that the function is homogeneous and determine n, and (b) show that $xf_x(x, y) + yf_y(x, y) = nf(x, y)$.**

43. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

44. $f(x, y) = x^3 - 3xy^2 + y^3$

45. $f(x, y) = e^{x/y}$

46. $f(x, y) = \dfrac{x^2}{\sqrt{x^2 + y^2}}$

47. Let $w = f(x, y)$, $x = g(t)$, and $y = h(t)$, where f, g, and h are differentiable. Use the appropriate Chain Rule to find dw/dt when $t = 2$ given the following table of values.

$g(2)$	$h(2)$	$g'(2)$	$h'(2)$	$f_x(4, 3)$	$f_y(4, 3)$
4	3	-1	6	-5	7

48. Let $w = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$, where f, g, and h are differentiable. Use the appropriate Chain Rule to find $w_s(1, 2)$ and $w_t(1, 2)$ given the following table of values.

$g(1, 2)$	$h(1, 2)$	$g_s(1, 2)$	$h_s(1, 2)$
4	3	-3	5

$g_t(1, 2)$	$h_t(1, 2)$	$f_x(4, 3)$	$f_y(4, 3)$
-2	8	-5	7

WRITING ABOUT CONCEPTS

49. Let $w = f(x, y)$ be a function in which x and y are functions of a single variable t. Give the Chain Rule for finding dw/dt.

50. Let $w = f(x, y)$ be a function in which x and y are functions of two variables s and t. Give the Chain Rule for finding $\partial w/\partial s$ and $\partial w/\partial t$.

51. If $f(x, y) = 0$, give the rule for finding dy/dx implicitly. If $f(x, y, z) = 0$, give the rule for finding $\partial z/\partial x$ and $\partial z/\partial y$ implicitly.

CAPSTONE

52. Consider the function $f(x, y, z) = xyz$, where $x = t^2$, $y = 2t$, and $z = e^{-t}$.

(a) Use the appropriate Chain Rule to find df/dt.

(b) Write f as a function of t and then find df/dt. Explain why this result is the same as that of part (a).

53. ***Volume and Surface Area*** The radius of a right circular cylinder is increasing at a rate of 6 inches per minute, and the height is decreasing at a rate of 4 inches per minute. What are the rates of change of the volume and surface area when the radius is 12 inches and the height is 36 inches?

54. ***Volume and Surface Area*** Repeat Exercise 53 for a right circular cone.

55. ***Ideal Gas Law*** The Ideal Gas Law is $pV = mRT$, where R is a constant, m is a constant mass, and p and V are functions of time. Find dT/dt, the rate at which the temperature changes with respect to time.

56. ***Area*** Let θ be the angle between equal sides of an isosceles triangle and let x be the length of these sides. x is increasing at $\frac{1}{2}$ meter per hour and θ is increasing at $\pi/90$ radian per hour. Find the rate of increase of the area when $x = 6$ and $\theta = \pi/4$.

57. ***Moment of Inertia*** An annular cylinder has an inside radius of r_1 and an outside radius of r_2 (see figure). Its moment of inertia is $I = \frac{1}{2}m(r_1^2 + r_2^2)$, where m is the mass. The two radii are increasing at a rate of 2 centimeters per second. Find the rate at which I is changing at the instant the radii are 6 centimeters and 8 centimeters. (Assume mass is a constant.)

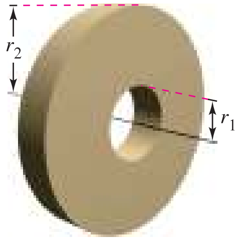


Figure for 57

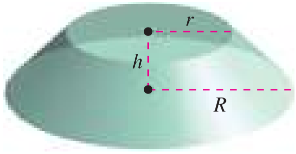


Figure for 58

58. ***Volume and Surface Area*** The two radii of the frustum of a right circular cone are increasing at a rate of 4 centimeters per minute, and the height is increasing at a rate of 12 centimeters per minute (see figure). Find the rates at which the volume and surface area are changing when the two radii are 15 centimeters and 25 centimeters, and the height is 10 centimeters.

59. Show that $(\partial w/\partial u) + (\partial w/\partial v) = 0$ for $w = f(x, y)$, $x = u - v$, and $y = v - u$.

60. Demonstrate the result of Exercise 59 for

$$w = (x - y)\sin(y - x).$$

61. Consider the function $w = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$. Verify each of the following.

(a) $$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\sin\theta + \frac{\partial w}{\partial \theta}\frac{\cos\theta}{r}$$

(b) $$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{1}{r^2}\right)\left(\frac{\partial w}{\partial \theta}\right)^2$$

62. Demonstrate the result of Exercise 61(b) for $w = \arctan(y/x)$.

63. ***Cauchy-Riemann Equations*** Given the functions $u(x, y)$ and $v(x, y)$, verify that the **Cauchy-Riemann differential equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

can be written in polar coordinate form as

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

64. Demonstrate the result of Exercise 63 for the functions

$$u = \ln\sqrt{x^2 + y^2} \quad \text{and} \quad v = \arctan\frac{y}{x}.$$

65. Show that if $f(x,y)$ is homogeneous of degree n, then

$$xf_x(x, y) + yf_y(x, y) = nf(x, y).$$

[*Hint:* Let $g(t) = f(tx, ty) = t^n f(x, y)$. Find $g'(t)$ and then let $t = 1$.]

13.6 Directional Derivatives and Gradients

- Find and use directional derivatives of a function of two variables.
- Find the gradient of a function of two variables.
- Use the gradient of a function of two variables in applications.
- Find directional derivatives and gradients of functions of three variables.

Directional Derivative

You are standing on the hillside pictured in Figure 13.42 and want to determine the hill's incline toward the z-axis. If the hill were represented by $z = f(x, y)$, you would already know how to determine the slopes in two different directions—the slope in the y-direction would be given by the partial derivative $f_y(x, y)$, and the slope in the x-direction would be given by the partial derivative $f_x(x, y)$. In this section, you will see that these two partial derivatives can be used to find the slope in *any* direction.

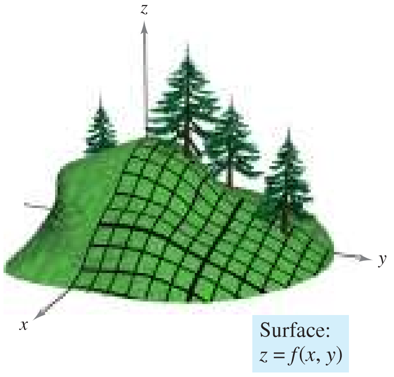


Figure 13.42

To determine the slope at a point on a surface, you will define a new type of derivative called a **directional derivative.** Begin by letting $z = f(x, y)$ be a *surface* and $P(x_0, y_0)$ be a *point* in the domain of f, as shown in Figure 13.43. The "direction" of the directional derivative is given by a unit vector

$$\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

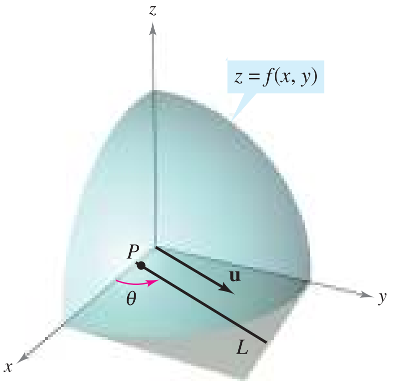


Figure 13.43

where θ is the angle the vector makes with the positive x-axis. To find the desired slope, reduce the problem to two dimensions by intersecting the surface with a vertical plane passing through the point P and parallel to $\mathbf{u}$, as shown in Figure 13.44. This vertical plane intersects the surface to form a curve C. The slope of the surface at $(x_0, y_0, f(x_0, y_0))$ in the direction of $\mathbf{u}$ is defined as the slope of the curve C at that point.

Informally, you can write the slope of the curve C as a limit that looks much like those used in single-variable calculus. The vertical plane used to form C intersects the xy-plane in a line L, represented by the parametric equations

$$x = x_0 + t\cos\theta$$

and

$$y = y_0 + t\sin\theta$$

so that for any value of t, the point $Q(x, y)$ lies on the line L. For each of the points P and Q, there is a corresponding point on the surface.

$(x_0, y_0, f(x_0, y_0))$ Point above P
$(x, y, f(x, y))$ Point above Q

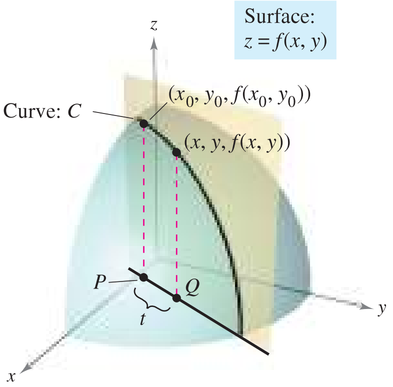


Figure 13.44

Moreover, because the distance between P and Q is

$$\begin{aligned}\sqrt{(x - x_0)^2 + (y - y_0)^2} &= \sqrt{(t\cos\theta)^2 + (t\sin\theta)^2}\\ &= |t|\end{aligned}$$

you can write the slope of the secant line through $(x_0, y_0, f(x_0, y_0))$ and $(x, y, f(x, y))$ as

$$\frac{f(x, y) - f(x_0, y_0)}{t} = \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}.$$

Finally, by letting t approach 0, you arrive at the following definition.

DEFINITION OF DIRECTIONAL DERIVATIVE

Let f be a function of two variables x and y and let $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ be a unit vector. Then the **directional derivative of f in the direction of u,** denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x, y) = \lim_{t\to 0}\frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

provided this limit exists.

Calculating directional derivatives by this definition is similar to finding the derivative of a function of one variable by the limit process (given in Section 2.1). A simpler "working" formula for finding directional derivatives involves the partial derivatives f_x and f_y.

THEOREM 13.9 DIRECTIONAL DERIVATIVE

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

PROOF For a fixed point (x_0, y_0), let $x = x_0 + t\cos\theta$ and let $y = y_0 + t\sin\theta$. Then, let $g(t) = f(x, y)$. Because f is differentiable, you can apply the Chain Rule given in Theorem 13.6 to obtain

$$g'(t) = f_x(x, y)x'(t) + f_y(x, y)y'(t) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

If $t = 0$, then $x = x_0$ and $y = y_0$, so

$$g'(0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta.$$

By the definition of $g'(t)$, it is also true that

$$\begin{aligned} g'(0) &= \lim_{t\to 0}\frac{g(t) - g(0)}{t} \\ &= \lim_{t\to 0}\frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}. \end{aligned}$$

Consequently, $D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$. ■

There are infinitely many directional derivatives of a surface at a given point—one for each direction specified by **u**, as shown in Figure 13.45. Two of these are the partial derivatives f_x and f_y.

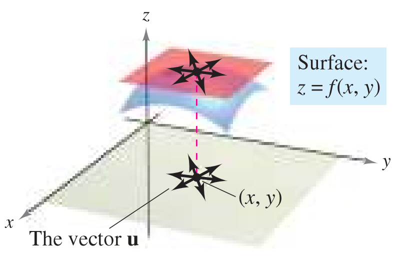


Figure 13.45

1. Direction of positive x-axis ($\theta = 0$): $\mathbf{u} = \cos 0\,\mathbf{i} + \sin 0\,\mathbf{j} = \mathbf{i}$

$$D_{\mathbf{i}}f(x, y) = f_x(x, y)\cos 0 + f_y(x, y)\sin 0 = f_x(x, y)$$

2. Direction of positive y-axis ($\theta = \pi/2$): $\mathbf{u} = \cos\frac{\pi}{2}\mathbf{i} + \sin\frac{\pi}{2}\mathbf{j} = \mathbf{j}$

$$D_{\mathbf{j}}f(x, y) = f_x(x, y)\cos\frac{\pi}{2} + f_y(x, y)\sin\frac{\pi}{2} = f_y(x, y)$$

EXAMPLE 1 Finding a Directional Derivative

Find the directional derivative of

$$f(x, y) = 4 - x^2 - \tfrac{1}{4}y^2 \qquad \text{Surface}$$

at (1, 2) in the direction of

$$\mathbf{u} = \left(\cos \frac{\pi}{3}\right)\mathbf{i} + \left(\sin \frac{\pi}{3}\right)\mathbf{j}. \qquad \text{Direction}$$

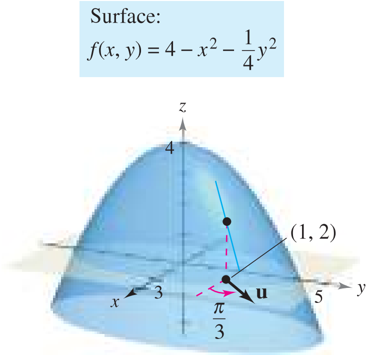


Figure 13.46

Solution Because f_x and f_y are continuous, f is differentiable, and you can apply Theorem 13.9.

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y) \cos \theta + f_y(x, y) \sin \theta \\ &= (-2x) \cos \theta + \left(-\frac{y}{2}\right) \sin \theta \end{aligned}$$

Evaluating at $\theta = \pi/3$, $x = 1$, and $y = 2$ produces

$$\begin{aligned} D_{\mathbf{u}} f(1, 2) &= (-2)\left(\frac{1}{2}\right) + (-1)\left(\frac{\sqrt{3}}{2}\right) \\ &= -1 - \frac{\sqrt{3}}{2} \\ &\approx -1.866. \end{aligned} \qquad \text{See Figure 13.46.}$$

■

NOTE Note in Figure 13.46 that you can interpret the directional derivative as giving the slope of the surface at the point (1, 2, 2) in the direction of the unit vector **u**. ■

You have been specifying direction by a unit vector **u**. If the direction is given by a vector whose length is not 1, you must normalize the vector before applying the formula in Theorem 13.9.

EXAMPLE 2 Finding a Directional Derivative

Find the directional derivative of

$$f(x, y) = x^2 \sin 2y \qquad \text{Surface}$$

at $(1, \pi/2)$ in the direction of

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}. \qquad \text{Direction}$$

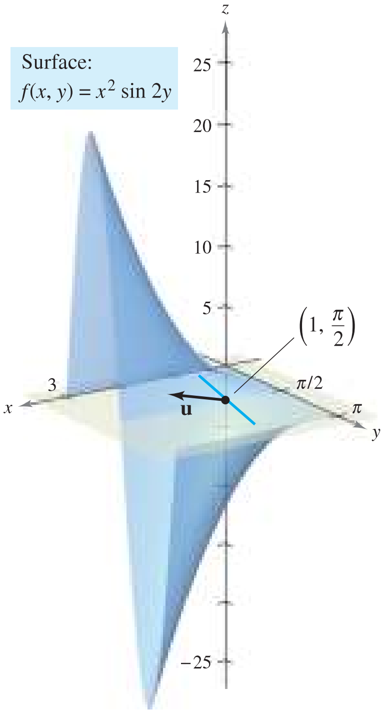


Figure 13.47

Solution Because f_x and f_y are continuous, f is differentiable, and you can apply Theorem 13.9. Begin by finding a unit vector in the direction of **v**.

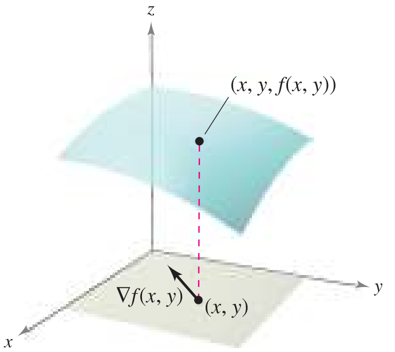
$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} = \cos \theta\mathbf{i} + \sin \theta\mathbf{j}$$

Using this unit vector, you have

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= (2x \sin 2y)(\cos \theta) + (2x^2 \cos 2y)(\sin \theta) \\ D_{\mathbf{u}} f\left(1, \frac{\pi}{2}\right) &= (2 \sin \pi)\left(\frac{3}{5}\right) + (2 \cos \pi)\left(-\frac{4}{5}\right) \\ &= (0)\left(\frac{3}{5}\right) + (-2)\left(-\frac{4}{5}\right) \\ &= \frac{8}{5}. \end{aligned} \qquad \text{See Figure 13.47.}$$

■

The Gradient of a Function of Two Variables



The gradient of f is a vector in the xy-plane.
Figure 13.48

The **gradient** of a function of two variables is a vector-valued function of two variables. This function has many important uses, some of which are described later in this section.

DEFINITION OF GRADIENT OF A FUNCTION OF TWO VARIABLES

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the **gradient of f,** denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

∇f is read as "del f." Another notation for the gradient is **grad** $f(x, y)$. In Figure 13.48, note that for each (x, y), the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).

NOTE No value is assigned to the symbol ∇ by itself. It is an operator in the same sense that d/dx is an operator. When ∇ operates on $f(x, y)$, it produces the vector $\nabla f(x, y)$. ■

EXAMPLE 3 Finding the Gradient of a Function

Find the gradient of $f(x, y) = y \ln x + xy^2$ at the point $(1, 2)$.

Solution Using

$$f_x(x, y) = \frac{y}{x} + y^2 \qquad \text{and} \qquad f_y(x, y) = \ln x + 2xy$$

you have

$$\nabla f(x, y) = \left(\frac{y}{x} + y^2\right)\mathbf{i} + (\ln x + 2xy)\mathbf{j}.$$

At the point $(1, 2)$, the gradient is

$$\begin{aligned}\nabla f(1, 2) &= \left(\frac{2}{1} + 2^2\right)\mathbf{i} + [\ln 1 + 2(1)(2)]\mathbf{j} \\ &= 6\mathbf{i} + 4\mathbf{j}.\end{aligned}$$

■

Because the gradient of f is a vector, you can write the directional derivative of f in the direction of $\mathbf{u}$ as

$$D_{\mathbf{u}}f(x, y) = [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] \cdot [\cos\theta\mathbf{i} + \sin\theta\mathbf{j}].$$

In other words, the directional derivative is the dot product of the gradient and the direction vector. This useful result is summarized in the following theorem.

THEOREM 13.10 ALTERNATIVE FORM OF THE DIRECTIONAL DERIVATIVE

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

EXAMPLE 4 Using $\nabla f(x, y)$ to Find a Directional Derivative

Find the directional derivative of

$$f(x, y) = 3x^2 - 2y^2$$

at $\left(-\frac{3}{4}, 0\right)$ in the direction from $P\left(-\frac{3}{4}, 0\right)$ to $Q(0, 1)$.

Solution Because the partials of f are continuous, f is differentiable and you can apply Theorem 13.10. A vector in the specified direction is

$$\begin{aligned}\overrightarrow{PQ} = \mathbf{v} &= \left(0 + \frac{3}{4}\right)\mathbf{i} + (1 - 0)\mathbf{j}\\ &= \frac{3}{4}\mathbf{i} + \mathbf{j}\end{aligned}$$

and a unit vector in this direction is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}.$$

Unit vector in direction of $\overrightarrow{PQ}$

Because $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 6x\mathbf{i} - 4y\mathbf{j}$, the gradient at $\left(-\frac{3}{4}, 0\right)$ is

$$\nabla f\left(-\frac{3}{4}, 0\right) = -\frac{9}{2}\mathbf{i} + 0\mathbf{j}.$$

Gradient at $\left(-\frac{3}{4}, 0\right)$

Consequently, at $\left(-\frac{3}{4}, 0\right)$ the directional derivative is

$$\begin{aligned}D_{\mathbf{u}}f\left(-\frac{3}{4}, 0\right) &= \nabla f\left(-\frac{3}{4}, 0\right) \cdot \mathbf{u}\\ &= \left(-\frac{9}{2}\mathbf{i} + 0\mathbf{j}\right) \cdot \left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right)\\ &= -\frac{27}{10}.\end{aligned}$$

Directional derivative at $\left(-\frac{3}{4}, 0\right)$

See Figure 13.49. ■

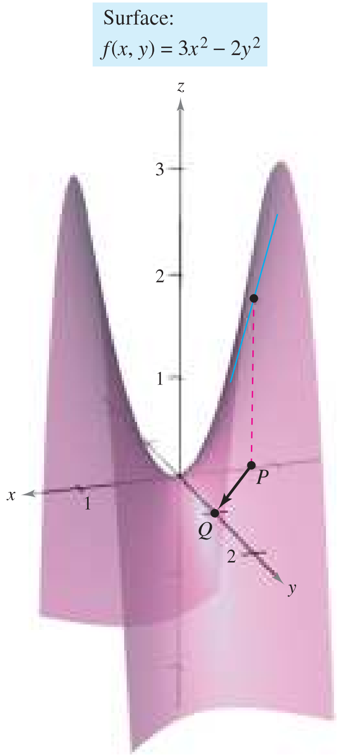


Figure 13.49

Applications of the Gradient

You have already seen that there are many directional derivatives at the point (x, y) on a surface. In many applications, you may want to know in which direction to move so that $f(x, y)$ increases most rapidly. This direction is called the direction of steepest ascent, and it is given by the gradient, as stated in the following theorem.

THEOREM 13.11 PROPERTIES OF THE GRADIENT

Let f be differentiable at the point (x, y).

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}}f(x, y) = 0$ for all $\mathbf{u}$.
2. The direction of *maximum* increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}}f(x, y)$ is $\|\nabla f(x, y)\|$.
3. The direction of *minimum* increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}}f(x, y)$ is $-\|\nabla f(x, y)\|$.

NOTE Part 2 of Theorem 13.11 says that at the point (x, y), f increases most rapidly in the direction of the gradient, $\nabla f(x, y)$.

PROOF If $\nabla f(x, y) = \mathbf{0}$, then for any direction (any $\mathbf{u}$), you have

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= \nabla f(x, y) \cdot \mathbf{u} \\ &= (0\mathbf{i} + 0\mathbf{j}) \cdot (\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \\ &= 0. \end{aligned}$$

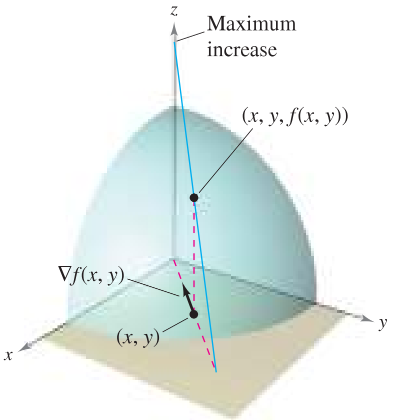
If $\nabla f(x, y) \neq \mathbf{0}$, then let ϕ be the angle between $\nabla f(x, y)$ and a unit vector $\mathbf{u}$. Using the dot product, you can apply Theorem 11.5 to conclude that

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= \nabla f(x, y) \cdot \mathbf{u} \\ &= \|\nabla f(x, y)\| \, \|\mathbf{u}\| \cos\phi \\ &= \|\nabla f(x, y)\| \cos\phi \end{aligned}$$

and it follows that the maximum value of $D_{\mathbf{u}} f(x, y)$ will occur when $\cos\phi = 1$. So, $\phi = 0$, and the maximum value of the directional derivative occurs when $\mathbf{u}$ has the same direction as $\nabla f(x, y)$. Moreover, this largest value of $D_{\mathbf{u}} f(x, y)$ is precisely

$$\|\nabla f(x, y)\| \cos\phi = \|\nabla f(x, y)\|.$$

Similarly, the minimum value of $D_{\mathbf{u}} f(x, y)$ can be obtained by letting $\phi = \pi$ so that $\mathbf{u}$ points in the direction opposite that of $\nabla f(x, y)$, as shown in Figure 13.50. ■



The gradient of f is a vector in the xy-plane that points in the direction of maximum increase on the surface given by $z = f(x, y)$.
Figure 13.50

To visualize one of the properties of the gradient, imagine a skier coming down a mountainside. If $f(x, y)$ denotes the altitude of the skier, then $-\nabla f(x, y)$ indicates the *compass direction* the skier should take to ski the path of steepest descent. (Remember that the gradient indicates direction in the xy-plane and does not itself point up or down the mountainside.)

As another illustration of the gradient, consider the temperature $T(x, y)$ at any point (x, y) on a flat metal plate. In this case, $\nabla T(x, y)$ gives the direction of greatest temperature increase at the point (x, y), as illustrated in the next example.

EXAMPLE 5 Finding the Direction of Maximum Increase

The temperature in degrees Celsius on the surface of a metal plate is

$$T(x, y) = 20 - 4x^2 - y^2$$

where x and y are measured in centimeters. In what direction from $(2, -3)$ does the temperature increase most rapidly? What is this rate of increase?

Solution The gradient is

$$\begin{aligned} \nabla T(x, y) &= T_x(x, y)\mathbf{i} + T_y(x, y)\mathbf{j} \\ &= -8x\,\mathbf{i} - 2y\,\mathbf{j}. \end{aligned}$$

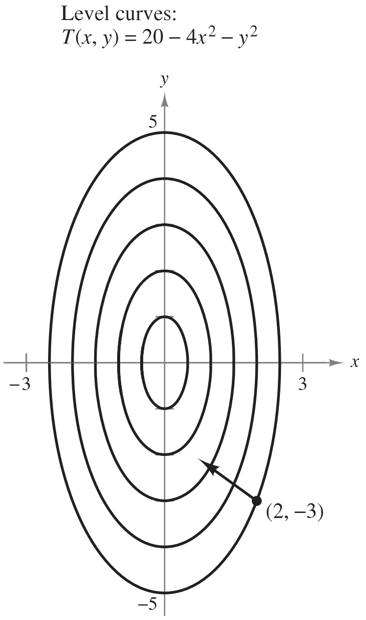
It follows that the direction of maximum increase is given by

$$\nabla T(2, -3) = -16\mathbf{i} + 6\mathbf{j}$$

as shown in Figure 13.51, and the rate of increase is

$$\begin{aligned} \|\nabla T(2, -3)\| &= \sqrt{256 + 36} \\ &= \sqrt{292} \\ &\approx 17.09^\circ \text{ per centimeter.} \end{aligned}$$

■



The direction of most rapid increase in temperature at $(2, -3)$ is given by $-16\mathbf{i} + 6\mathbf{j}$.
Figure 13.51

The solution presented in Example 5 can be misleading. Although the gradient points in the direction of maximum temperature increase, it does not necessarily point toward the hottest spot on the plate. In other words, the gradient provides a local solution to finding an increase relative to the temperature at the point $(2, -3)$. *Once you leave that position, the direction of maximum increase may change.*

EXAMPLE 6 Finding the Path of a Heat-Seeking Particle

A heat-seeking particle is located at the point $(2, -3)$ on a metal plate whose temperature at (x, y) is

$$T(x, y) = 20 - 4x^2 - y^2.$$

Find the path of the particle as it continuously moves in the direction of maximum temperature increase.

Solution Let the path be represented by the position function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}.$$

A tangent vector at each point $(x(t), y(t))$ is given by

$$\mathbf{r}'(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}.$$

Because the particle seeks maximum temperature increase, the directions of $\mathbf{r}'(t)$ and $\nabla T(x, y) = -8x\mathbf{i} - 2y\mathbf{j}$ are the same at each point on the path. So,

$$-8x = k\frac{dx}{dt} \quad \text{and} \quad -2y = k\frac{dy}{dt}$$

where k depends on t. By solving each equation for dt/k and equating the results, you obtain

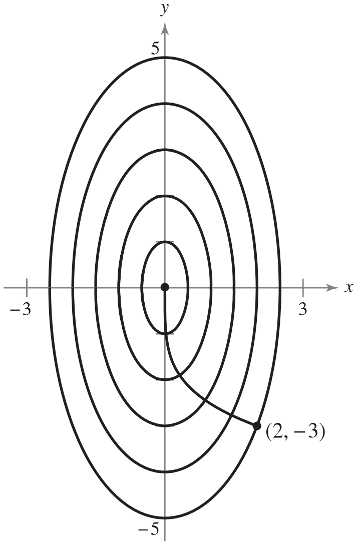
$$\frac{dx}{-8x} = \frac{dy}{-2y}.$$

The solution of this differential equation is $x = Cy^4$. Because the particle starts at the point $(2, -3)$, you can determine that $C = 2/81$. So, the path of the heat-seeking particle is

$$x = \frac{2}{81}y^4.$$

The path is shown in Figure 13.52. ■

Level curves:
$T(x, y) = 20 - 4x^2 - y^2$



Path followed by a heat-seeking particle
Figure 13.52

In Figure 13.52, the path of the particle (determined by the gradient at each point) appears to be orthogonal to each of the level curves. This becomes clear when you consider that the temperature $T(x, y)$ is constant along a given level curve. So, at any point (x, y) on the curve, the rate of change of T in the direction of a unit tangent vector $\mathbf{u}$ is 0, and you can write

$$\nabla f(x, y) \cdot \mathbf{u} = D_{\mathbf{u}}T(x, y) = 0. \qquad \mathbf{u} \text{ is a unit tangent vector.}$$

Because the dot product of $\nabla f(x, y)$ and $\mathbf{u}$ is 0, you can conclude that they must be orthogonal. This result is stated in the following theorem.

THEOREM 13.12 GRADIENT IS NORMAL TO LEVEL CURVES

If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

EXAMPLE 7 Finding a Normal Vector to a Level Curve

Sketch the level curve corresponding to $c = 0$ for the function given by

$$f(x, y) = y - \sin x$$

and find a normal vector at several points on the curve.

Solution The level curve for $c = 0$ is given by

$$\begin{aligned} 0 &= y - \sin x \\ y &= \sin x \end{aligned}$$

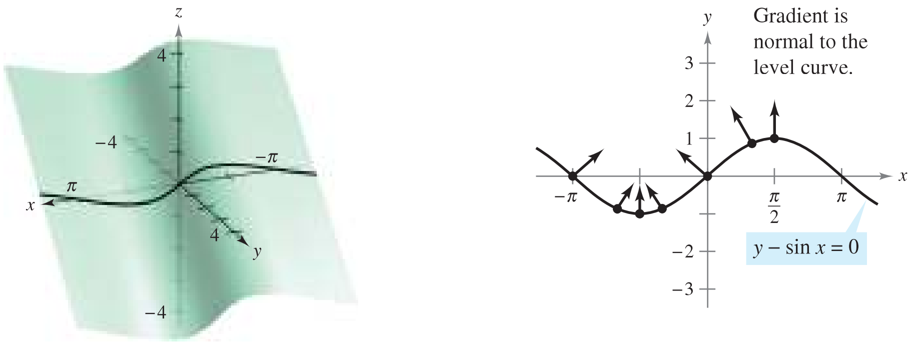
as shown in Figure 13.53(a). Because the gradient vector of f at (x, y) is

$$\begin{aligned} \nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \\ &= -\cos x\mathbf{i} + \mathbf{j} \end{aligned}$$

you can use Theorem 13.12 to conclude that $\nabla f(x, y)$ is normal to the level curve at the point (x, y). Some gradient vectors are

$$\begin{aligned} &\nabla f(-\pi, 0) = \mathbf{i} + \mathbf{j} \\ &\nabla f\left(-\frac{2\pi}{3}, -\frac{\sqrt{3}}{2}\right) = \frac{1}{2}\mathbf{i} + \mathbf{j} \\ &\nabla f\left(-\frac{\pi}{2}, -1\right) = \mathbf{j} \\ &\nabla f\left(-\frac{\pi}{3}, -\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j} \\ &\nabla f(0, 0) = -\mathbf{i} + \mathbf{j} \\ &\nabla f\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j} \\ &\nabla f\left(\frac{\pi}{2}, 1\right) = \mathbf{j}. \end{aligned}$$

These are shown in Figure 13.53(b).



(a) The surface is given by $f(x, y) = y - \sin x$.

(b) The level curve is given by $f(x, y) = 0$.

Figure 13.53

Functions of Three Variables

The definitions of the directional derivative and the gradient can be extended naturally to functions of three or more variables. As often happens, some of the geometric interpretation is lost in the generalization from functions of two variables to those of three variables. For example, you cannot interpret the directional derivative of a function of three variables to represent slope.

The definitions and properties of the directional derivative and the gradient of a function of three variables are given in the following summary.

DIRECTIONAL DERIVATIVE AND GRADIENT FOR THREE VARIABLES

Let f be a function of x, y, and z, with continuous first partial derivatives. The **directional derivative of f** in the direction of a unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by

$$D_{\mathbf{u}}f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The **gradient of f** is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Properties of the gradient are as follows.

1. $D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}}f(x, y, z) = 0$ for all $\mathbf{u}$.
3. The direction of *maximum* increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}}f(x, y, z)$ is

   $$\|\nabla f(x, y, z)\|.$$ Maximum value of $D_{\mathbf{u}}f(x, y, z)$

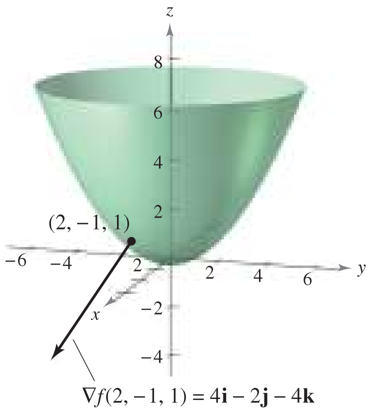
4. The direction of *minimum* increase of f is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}}f(x, y, z)$ is

   $$-\|\nabla f(x, y, z)\|.$$ Minimum value of $D_{\mathbf{u}}f(x, y, z)$

NOTE You can generalize Theorem 13.12 to functions of three variables. Under suitable hypotheses,

$$\nabla f(x_0, y_0, z_0)$$

is normal to the level surface through (x_0, y_0, z_0). ■



Level surface and gradient vector at $(2, -1, 1)$ for $f(x, y, z) = x^2 + y^2 - 4z$
Figure 13.54

EXAMPLE 8 Finding the Gradient for a Function of Three Variables

Find $\nabla f(x, y, z)$ for the function given by

$$f(x, y, z) = x^2 + y^2 - 4z$$

and find the direction of maximum increase of f at the point $(2, -1, 1)$.

Solution The gradient vector is given by

$$\begin{aligned}\nabla f(x, y, z) &= f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}\\ &= 2x\,\mathbf{i} + 2y\,\mathbf{j} - 4\mathbf{k}.\end{aligned}$$

So, it follows that the direction of maximum increase at $(2, -1, 1)$ is

$$\nabla f(2, -1, 1) = 4\,\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k}.$$ See Figure 13.54. ■

13.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–12, find the directional derivative of the function at P in the direction of v.

1. $f(x, y) = 3x - 4xy + 9y, \quad P(1, 2), \mathbf{v} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

2. $f(x, y) = x^3 - y^3, \quad P(4, 3), \mathbf{v} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$

3. $f(x, y) = xy, \quad P(0, -2), \mathbf{v} = \frac{1}{2}(\mathbf{i} + \sqrt{3}\mathbf{j})$

4. $f(x, y) = \frac{x}{y}, \quad P(1, 1), \mathbf{v} = -\mathbf{j}$

5. $h(x, y) = e^x \sin y, \quad P\left(1, \frac{\pi}{2}\right), \mathbf{v} = -\mathbf{i}$

6. $g(x, y) = \arccos xy, \quad P(1, 0), \mathbf{v} = \mathbf{j}$

7. $g(x, y) = \sqrt{x^2 + y^2}, \quad P(3, 4), \mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

8. $h(x, y) = e^{-(x^2+y^2)}, \quad P(0, 0), \mathbf{v} = \mathbf{i} + \mathbf{j}$

9. $f(x, y, z) = x^2 + y^2 + z^2, \quad P(1, 1, 1), \mathbf{v} = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$

10. $f(x, y, z) = xy + yz + xz, \quad P(1, 2, -1), \mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

11. $h(x, y, z) = xyz, \quad P(2, 1, 1), \mathbf{v} = \langle 2, 1, 2 \rangle$

12. $h(x, y, z) = x \arctan yz, \quad P(4, 1, 1), \mathbf{v} = \langle 1, 2, -1 \rangle$

In Exercises 13–16, find the directional derivative of the function in the direction of the unit vector u = cos θ i + sin θ j.

13. $f(x, y) = x^2 + y^2, \quad \theta = \frac{\pi}{4}$

14. $f(x, y) = \frac{y}{x + y}, \quad \theta = -\frac{\pi}{6}$

15. $f(x, y) = \sin(2x + y), \quad \theta = \frac{\pi}{3}$

16. $g(x, y) = xe^y, \quad \theta = \frac{2\pi}{3}$

In Exercises 17–20, find the directional derivative of the function at P in the direction of Q.

17. $f(x, y) = x^2 + 3y^2, \quad P(1, 1), Q(4, 5)$

18. $f(x, y) = \cos(x + y), \quad P(0, \pi), Q\left(\frac{\pi}{2}, 0\right)$

19. $g(x, y, z) = xye^z, \quad P(2, 4, 0), Q(0, 0, 0)$

20. $h(x, y, z) = \ln(x + y + z), \quad P(1, 0, 0), Q(4, 3, 1)$

In Exercises 21–26, find the gradient of the function at the given point.

21. $f(x, y) = 3x + 5y^2 + 1, \quad (2, 1)$

22. $g(x, y) = 2xe^{y/x}, \quad (2, 0)$

23. $z = \ln(x^2 - y), \quad (2, 3)$

24. $z = \cos(x^2 + y^2), \quad (3, -4)$

25. $w = 3x^2 - 5y^2 + 2z^2, \quad (1, 1, -2)$

26. $w = x \tan(y + z), \quad (4, 3, -1)$

In Exercises 27–30, use the gradient to find the directional derivative of the function at P in the direction of Q.

27. $g(x, y) = x^2 + y^2 + 1, \quad P(1, 2), \ Q(2, 3)$

28. $f(x, y) = 3x^2 - y^2 + 4, \quad P(-1, 4), \ Q(3, 6)$

29. $f(x, y) = e^y \sin x, \quad P(0, 0), \ Q(2, 1)$

30. $f(x, y) = \sin 2x \cos y, \quad P(\pi, 0), \ Q\left(\frac{\pi}{2}, \pi\right)$

In Exercises 31–40, find the gradient of the function and the maximum value of the directional derivative at the given point.

	Function	*Point*
31.	$f(x, y) = x^2 + 2xy$	$(1, 0)$
32.	$f(x, y) = \frac{x + y}{y + 1}$	$(0, 1)$
33.	$h(x, y) = x \tan y$	$\left(2, \frac{\pi}{4}\right)$
34.	$h(x, y) = y \cos(x - y)$	$\left(0, \frac{\pi}{3}\right)$
35.	$g(x, y) = ye^{-x}$	$(0, 5)$
36.	$g(x, y) = \ln \sqrt[3]{x^2 + y^2}$	$(1, 2)$
37.	$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$	$(1, 4, 2)$
38.	$w = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$	$(0, 0, 0)$
39.	$w = xy^2z^2$	$(2, 1, 1)$
40.	$f(x, y, z) = xe^{yz}$	$(2, 0, -4)$

In Exercises 41–46, consider the function $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$.

41. Sketch the graph of f in the first octant and plot the point $(3, 2, 1)$ on the surface.

42. Find $D_{\mathbf{u}} f(3, 2)$, where $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$, using each given value of θ.

(a) $\theta = \frac{\pi}{4}$ (b) $\theta = \frac{2\pi}{3}$

(c) $\theta = \frac{4\pi}{3}$ (d) $\theta = -\frac{\pi}{6}$

43. Find $D_{\mathbf{u}} f(3, 2)$, where $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$, using each given vector $\mathbf{v}$.

(a) $\mathbf{v} = \mathbf{i} + \mathbf{j}$

(b) $\mathbf{v} = -3\mathbf{i} - 4\mathbf{j}$

(c) $\mathbf{v}$ is the vector from $(1, 2)$ to $(-2, 6)$.

(d) $\mathbf{v}$ is the vector from $(3, 2)$ to $(4, 5)$.

44. Find $\nabla f(x, y)$.

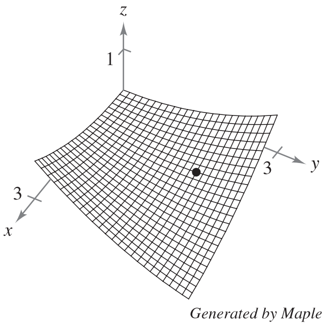
45. Find the maximum value of the directional derivative at $(3, 2)$.

46. Find a unit vector $\mathbf{u}$ orthogonal to $\nabla f(3, 2)$ and calculate $D_{\mathbf{u}} f(3, 2)$. Discuss the geometric meaning of the result.

Investigation **In Exercises 47 and 48, (a) use the graph to estimate the components of the vector in the direction of the maximum rate of increase in the function at the given point. (b) Find the gradient at the point and compare it with your estimate in part (a). (c) In what direction would the function be decreasing at the greatest rate? Explain.**

47. $f(x, y) = \frac{1}{10}(x^2 - 3xy + y^2)$, $(1, 2)$

48. $f(x, y) = \frac{1}{2}y\sqrt{x}$, $(1, 2)$



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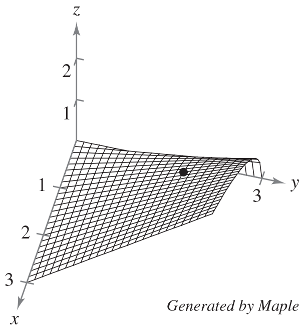

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CAS **49.** ***Investigation*** Consider the function

$f(x, y) = x^2 - y^2$

at the point $(4, -3, 7)$.

(a) Use a computer algebra system to graph the surface represented by the function.

(b) Determine the directional derivative $D_{\mathbf{u}}f(4, -3)$ as a function of θ, where $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$. Use a computer algebra system to graph the function on the interval $[0, 2\pi)$.

(c) Approximate the zeros of the function in part (b) and interpret each in the context of the problem.

(d) Approximate the critical numbers of the function in part (b) and interpret each in the context of the problem.

(e) Find $\|\nabla f(4, -3)\|$ and explain its relationship to your answers in part (d).

(f) Use a computer algebra system to graph the level curve of the function f at the level $c = 7$. On this curve, graph the vector in the direction of $\nabla f(4, -3)$, and state its relationship to the level curve.

50. ***Investigation*** Consider the function

$$f(x, y) = \frac{8y}{1 + x^2 + y^2}.$$

(a) Analytically verify that the level curve of $f(x, y)$ at the level $c = 2$ is a circle.

(b) At the point $(\sqrt{3}, 2)$ on the level curve for which $c = 2$, sketch the vector showing the direction of the greatest rate of increase of the function. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.)

(c) At the point $(\sqrt{3}, 2)$ on the level curve, sketch a vector such that the directional derivative is 0.

CAS (d) Use a computer algebra system to graph the surface to verify your answers in parts (a)–(c).

In Exercises 51–54, find a normal vector to the level curve $f(x, y) = c$ at P.

51. $f(x, y) = 6 - 2x - 3y$
$c = 6, \quad P(0, 0)$

52. $f(x, y) = x^2 + y^2$
$c = 25, \quad P(3, 4)$

53. $f(x, y) = xy$
$c = -3, \quad P(-1, 3)$

54. $f(x, y) = \dfrac{x}{x^2 + y^2}$
$c = \frac{1}{2}, \quad P(1, 1)$

In Exercises 55–58, (a) find the gradient of the function at P, (b) find a unit normal vector to the level curve $f(x, y) = c$ at P, (c) find the tangent line to the level curve $f(x, y) = c$ at P, and (d) sketch the level curve, the unit normal vector, and the tangent line in the xy-plane.

55. $f(x, y) = 4x^2 - y$
$c = 6, P(2, 10)$

56. $f(x, y) = x - y^2$
$c = 3, P(4, -1)$

57. $f(x, y) = 3x^2 - 2y^2$
$c = 1, P(1, 1)$

58. $f(x, y) = 9x^2 + 4y^2$
$c = 40, P(2, -1)$

WRITING ABOUT CONCEPTS

59. Define the derivative of the function $z = f(x, y)$ in the direction $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$.

60. Write a paragraph describing the directional derivative of the function f in the direction $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ when (a) $\theta = 0°$ and (b) $\theta = 90°$.

61. Define the gradient of a function of two variables. State the properties of the gradient.

62. Sketch the graph of a surface and select a point P on the surface. Sketch a vector in the xy-plane giving the direction of steepest ascent on the surface at P.

63. Describe the relationship of the gradient to the level curves of a surface given by $z = f(x, y)$.

CAPSTONE

64. Consider the function $f(x, y) = 9 - x^2 - y^2$.

(a) Sketch the graph of f in the first octant and plot the point $(1, 2, 4)$ on the surface.

(b) Find $D_{\mathbf{u}}f(1, 2)$, where $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, for $\theta = -\pi/4$.

(c) Repeat part (b) for $\theta = \pi/3$.

(d) Find $\nabla f(1, 2)$ and $\|\nabla f(1, 2)\|$.

(e) Find a unit vector $\mathbf{u}$ orthogonal to $\nabla f(1, 2)$ and calculate $D_{\mathbf{u}}f(1, 2)$. Discuss the geometric meaning of the result.

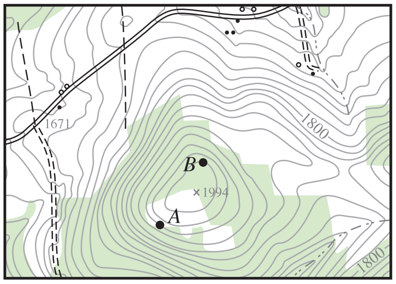
65. ***Temperature Distribution*** The temperature at the point (x, y) on a metal plate is

$$T = \frac{x}{x^2 + y^2}.$$

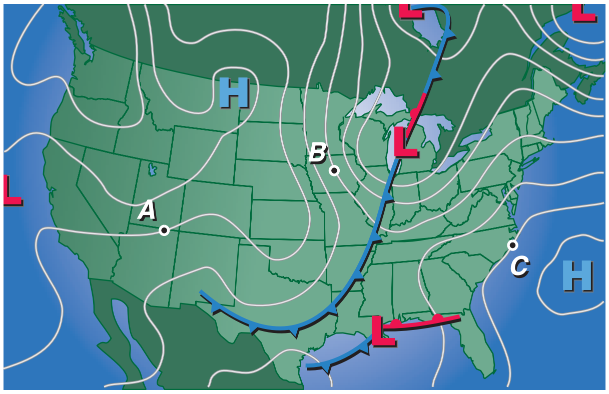
Find the direction of greatest increase in heat from the point $(3, 4)$.

66. ***Topography*** The surface of a mountain is modeled by the equation $h(x, y) = 5000 - 0.001x^2 - 0.004y^2$. A mountain climber is at the point $(500, 300, 4390)$. In what direction should the climber move in order to ascend at the greatest rate?

67. ***Topography*** The figure shows a topographic map carried by a group of hikers. Sketch the paths of steepest descent if the hikers start at point A and if they start at point B. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.)



68. ***Meteorology*** Meteorologists measure the atmospheric pressure in units called millibars. From these observations they create weather maps on which the curves of equal atmospheric pressure (isobars) are drawn (see figure). These are level curves to the function $P(x, y)$ yielding the pressure at any point. Sketch the gradients to the isobars at the points A, B, and C. Although the magnitudes of the gradients are unknown, their lengths relative to each other can be estimated. At which of the three points is the wind speed greatest if the speed increases as the pressure gradient increases? (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.)



Heat-Seeking Path **In Exercises 69 and 70, find the path of a heat-seeking particle placed at point P on a metal plate with a temperature field $T(x, y)$.**

	Temperature Field	*Point*
69.	$T(x, y) = 400 - 2x^2 - y^2$	$P(10, 10)$
70.	$T(x, y) = 100 - x^2 - 2y^2$	$P(4, 3)$

71. ***Temperature*** The temperature at the point (x, y) on a metal plate is modeled by $T(x, y) = 400e^{-(x^2+y)/2}$, $x \geq 0$, $y \geq 0$.

CAS (a) Use a computer algebra system to graph the temperature distribution function.

(b) Find the directions of no change in heat on the plate from the point $(3, 5)$.

(c) Find the direction of greatest increase in heat from the point $(3, 5)$.

CAS **72.** ***Investigation*** A team of oceanographers is mapping the ocean floor to assist in the recovery of a sunken ship. Using sonar, they develop the model

$$D = 250 + 30x^2 + 50 \sin \frac{\pi y}{2}, \quad 0 \leq x \leq 2, 0 \leq y \leq 2$$

where D is the depth in meters, and x and y are the distances in kilometers.

(a) Use a computer algebra system to graph the surface.

(b) Because the graph in part (a) is showing depth, it is not a map of the ocean floor. How could the model be changed so that the graph of the ocean floor could be obtained?

(c) What is the depth of the ship if it is located at the coordinates $x = 1$ and $y = 0.5$?

(d) Determine the steepness of the ocean floor in the positive x-direction from the position of the ship.

(e) Determine the steepness of the ocean floor in the positive y-direction from the position of the ship.

(f) Determine the direction of the greatest rate of change of depth from the position of the ship.

True or False? **In Exercises 73–76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

73. If $f(x, y) = \sqrt{1 - x^2 - y^2}$, then $D_{\mathbf{u}} f(0, 0) = 0$ for any unit vector $\mathbf{u}$.

74. If $f(x, y) = x + y$, then $-1 \leq D_{\mathbf{u}} f(x, y) \leq 1$.

75. If $D_{\mathbf{u}} f(x, y)$ exists, then $D_{\mathbf{u}} f(x, y) = -D_{-\mathbf{u}} f(x, y)$.

76. If $D_{\mathbf{u}} f(x_0, y_0) = c$ for any unit vector $\mathbf{u}$, then $c = 0$.

77. Find a function f such that $\nabla f = e^x \cos y\,\mathbf{i} - e^x \sin y\,\mathbf{j} + z\,\mathbf{k}$.

78. Consider the function

$$f(x, y) = \begin{cases} \dfrac{4xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

and the unit vector $\mathbf{u} = \dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$.

Does the directional derivative of f at $P(0, 0)$ in the direction of $\mathbf{u}$ exist? If $f(0, 0)$ were defined as 2 instead of 0, would the directional derivative exist?

79. Consider the function $f(x, y) = \sqrt[3]{xy}$.

(a) Show that f is continuous at the origin.

(b) Show that f_x and f_y exist at the origin, but that the directional derivatives at the origin in all other directions do not exist.

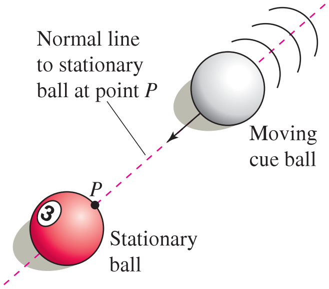
CAS (c) Use a computer algebra system to graph f near the origin to verify your answers in parts (a) and (b). Explain.

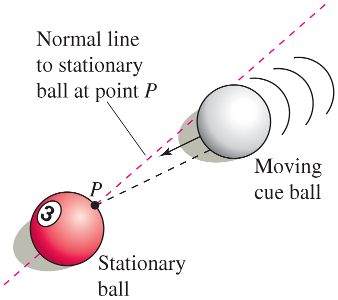
13.7 Tangent Planes and Normal Lines

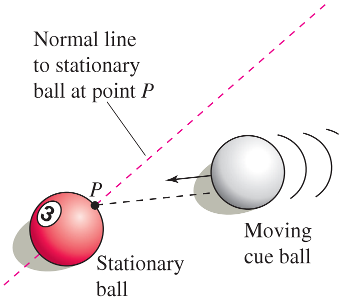
- **Find equations of tangent planes and normal lines to surfaces.**
- **Find the angle of inclination of a plane in space.**
- **Compare the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$.**

EXPLORATION

Billiard Balls and Normal Lines In each of the three figures below, the cue ball is about to strike a stationary ball at point P. Explain how you can use the normal line to the stationary ball at point P to describe the resulting motion of each of the two balls. Assuming that each cue ball has the same speed, which stationary ball will acquire the greatest speed? Which will acquire the least? Explain your reasoning.


Normal line to stationary ball at point P
Moving cue ball
P
3
Stationary ball


Normal line to stationary ball at point P
Moving cue ball
P
3
Stationary ball


Normal line to stationary ball at point P
P
3
Moving cue ball
Stationary ball

Tangent Plane and Normal Line to a Surface

So far you have represented surfaces in space primarily by equations of the form

$$z = f(x, y). \qquad \text{Equation of a surface } S$$

In the development to follow, however, it is convenient to use the more general representation $F(x, y, z) = 0$. For a surface S given by $z = f(x, y)$, you can convert to the general form by defining F as

$$F(x, y, z) = f(x, y) - z.$$

Because $f(x, y) - z = 0$, you can consider S to be the level surface of F given by

$$F(x, y, z) = 0. \qquad \text{Alternative equation of surface } S$$

EXAMPLE 1 Writing an Equation of a Surface

For the function given by

$$F(x, y, z) = x^2 + y^2 + z^2 - 4$$

describe the level surface given by $F(x, y, z) = 0$.

Solution The level surface given by $F(x, y, z) = 0$ can be written as

$$x^2 + y^2 + z^2 = 4$$

which is a sphere of radius 2 whose center is at the origin.

You have seen many examples of the usefulness of normal lines in applications involving curves. Normal lines are equally important in analyzing surfaces and solids. For example, consider the collision of two billiard balls. When a stationary ball is struck at a point P on its surface, it moves along the **line of impact** determined by P and the center of the ball. The impact can occur in *two* ways. If the cue ball is moving along the line of impact, it stops dead and imparts all of its momentum to the stationary ball, as shown in Figure 13.55. If the cue ball is not moving along the line of impact, it is deflected to one side or the other and retains part of its momentum. That part of the momentum that is transferred to the stationary ball occurs along the line of impact, *regardless* of the direction of the cue ball, as shown in Figure 13.56. This line of impact is called the **normal line** to the surface of the ball at the point P.

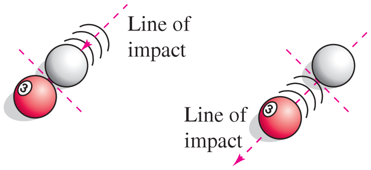

Line of impact
Line of impact

Figure 13.55

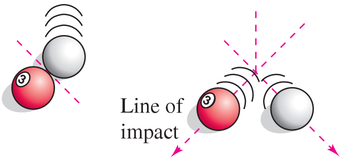

Line of impact

Figure 13.56

In the process of finding a normal line to a surface, you are also able to solve the problem of finding a **tangent plane** to the surface. Let S be a surface given by

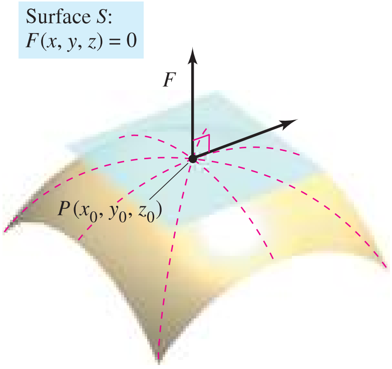
$$F(x, y, z) = 0$$

and let $P(x_0, y_0, z_0)$ be a point on S. Let C be a curve on S through P that is defined by the vector-valued function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

Then, for all t,

$$F(x(t),\ y(t),\ z(t)) = 0.$$



Tangent plane to surface S at P
Figure 13.57

If F is differentiable and $x'(t)$, $y'(t)$, and $z'(t)$ all exist, it follows from the Chain Rule that

$$\begin{aligned} 0 &= F'(t) \\ &= F_x(x, y, z)x'(t) + F_y(x, y, z)y'(t) + F_z(x, y, z)z'(t). \end{aligned}$$

At (x_0, y_0, z_0), the equivalent vector form is

$$0 = \underbrace{\nabla F(x_0, y_0, z_0)}_{\text{Gradient}} \cdot \underbrace{\mathbf{r}'(t_0)}_{\text{Tangent vector}}.$$

This result means that the gradient at P is orthogonal to the tangent vector of every curve on S through P. So, all tangent lines on S lie in a plane that is normal to $\nabla F(x_0, y_0, z_0)$ and contains P, as shown in Figure 13.57.

DEFINITIONS OF TANGENT PLANE AND NORMAL LINE

Let F be differentiable at the point $P(x_0, y_0, z_0)$ on the surface S given by $F(x, y, z) = 0$ such that $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$.

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane to *S* at *P*.**
2. The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the **normal line to *S* at *P*.**

NOTE In the remainder of this section, assume $\nabla F(x_0, y_0, z_0)$ to be nonzero unless stated otherwise. ■

To find an equation for the tangent plane to S at (x_0, y_0, z_0), let (x, y, z) be an arbitrary point in the tangent plane. Then the vector

$$\mathbf{v} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

lies in the tangent plane. Because $\nabla F(x_0, y_0, z_0)$ is normal to the tangent plane at (x_0, y_0, z_0), it must be orthogonal to every vector in the tangent plane, and you have $\nabla F(x_0, y_0, z_0) \cdot \mathbf{v} = 0$, which leads to the following theorem.

THEOREM 13.13 EQUATION OF TANGENT PLANE

If F is differentiable at (x_0, y_0, z_0), then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

EXAMPLE 2 Finding an Equation of a Tangent Plane

Find an equation of the tangent plane to the hyperboloid given by

$$z^2 - 2x^2 - 2y^2 = 12$$

at the point $(1, -1, 4)$.

Solution Begin by writing the equation of the surface as

$$z^2 - 2x^2 - 2y^2 - 12 = 0.$$

Then, considering

$$F(x, y, z) = z^2 - 2x^2 - 2y^2 - 12$$

you have

$$F_x(x, y, z) = -4x, \qquad F_y(x, y, z) = -4y, \quad \text{and} \quad F_z(x, y, z) = 2z.$$

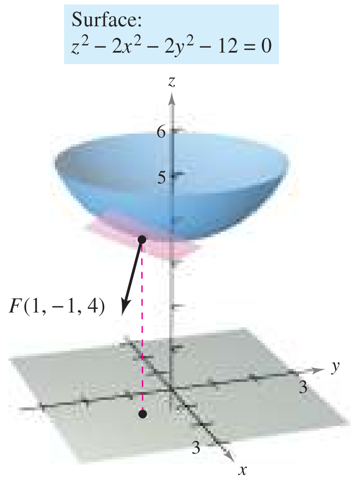
At the point $(1, -1, 4)$ the partial derivatives are

$$F_x(1, -1, 4) = -4, \qquad F_y(1, -1, 4) = 4, \quad \text{and} \quad F_z(1, -1, 4) = 8.$$

So, an equation of the tangent plane at $(1, -1, 4)$ is

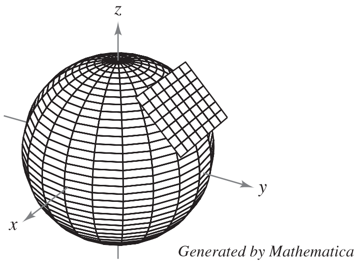
$$\begin{aligned} -4(x - 1) + 4(y + 1) + 8(z - 4) &= 0 \\ -4x + 4 + 4y + 4 + 8z - 32 &= 0 \\ -4x + 4y + 8z - 24 &= 0 \\ x - y - 2z + 6 &= 0. \end{aligned}$$

Figure 13.58 shows a portion of the hyperboloid and tangent plane. ■



Tangent plane to surface
Figure 13.58

TECHNOLOGY Some three-dimensional graphing utilities are capable of graphing tangent planes to surfaces. Two examples are shown below.



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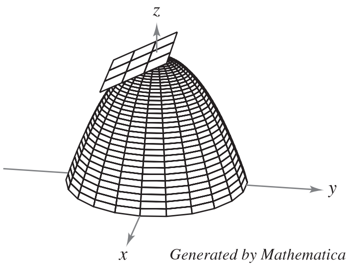
Sphere: $x^2 + y^2 + z^2 = 1$



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Paraboloid: $z = 2 - x^2 - y^2$

To find the equation of the tangent plane at a point on a surface given by $z = f(x, y)$, you can define the function F by

$$F(x, y, z) = f(x, y) - z.$$

Then S is given by the level surface $F(x, y, z) = 0$, and by Theorem 13.13 an equation of the tangent plane to S at the point (x_0, y_0, z_0) is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

EXAMPLE 3 Finding an Equation of the Tangent Plane

Find the equation of the tangent plane to the paraboloid

$$z = 1 - \frac{1}{10}(x^2 + 4y^2)$$

at the point $\left(1, 1, \frac{1}{2}\right)$.

Solution From $z = f(x, y) = 1 - \frac{1}{10}(x^2 + 4y^2)$, you obtain

$$f_x(x, y) = -\frac{x}{5} \quad \Rightarrow \quad f_x(1, 1) = -\frac{1}{5}$$

and

$$f_y(x, y) = -\frac{4y}{5} \quad \Rightarrow \quad f_y(1, 1) = -\frac{4}{5}.$$

So, an equation of the tangent plane at $\left(1, 1, \frac{1}{2}\right)$ is

$$\begin{aligned} f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) - \left(z - \frac{1}{2}\right) &= 0 \\ -\frac{1}{5}(x - 1) - \frac{4}{5}(y - 1) - \left(z - \frac{1}{2}\right) &= 0 \\ -\frac{1}{5}x - \frac{4}{5}y - z + \frac{3}{2} &= 0. \end{aligned}$$

This tangent plane is shown in Figure 13.59. ■

Surface:
$z = 1 - \frac{1}{10}(x^2 + 4y^2)$

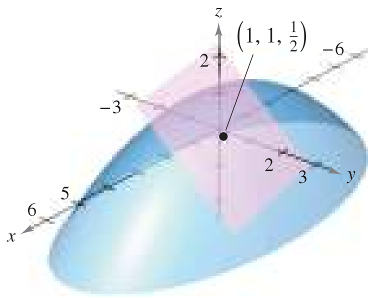


Figure 13.59

The gradient $\nabla F(x, y, z)$ provides a convenient way to find equations of normal lines, as shown in Example 4.

EXAMPLE 4 Finding an Equation of a Normal Line to a Surface

Find a set of symmetric equations for the normal line to the surface given by $xyz = 12$ at the point $(2, -2, -3)$.

Solution Begin by letting

$$F(x, y, z) = xyz - 12.$$

Then, the gradient is given by

$$\begin{aligned} \nabla F(x, y, z) &= F_x(x, y, z)\mathbf{i} + F_y(x, y, z)\mathbf{j} + F_z(x, y, z)\mathbf{k} \\ &= yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \end{aligned}$$

and at the point $(2, -2, -3)$ you have

$$\begin{aligned} \nabla F(2, -2, -3) &= (-2)(-3)\mathbf{i} + (2)(-3)\mathbf{j} + (2)(-2)\mathbf{k} \\ &= 6\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}. \end{aligned}$$

The normal line at $(2, -2, -3)$ has direction numbers 6, -6, and -4, and the corresponding set of symmetric equations is

$$\frac{x - 2}{6} = \frac{y + 2}{-6} = \frac{z + 3}{-4}.$$

See Figure 13.60. ■

Surface: $xyz = 12$

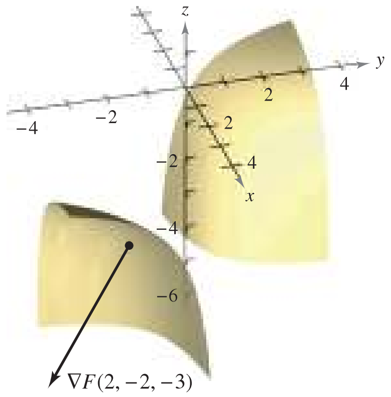


Figure 13.60

Knowing that the gradient $\nabla F(x, y, z)$ is normal to the surface given by $F(x, y, z) = 0$ allows you to solve a variety of problems dealing with surfaces and curves in space.

EXAMPLE 5 Finding the Equation of a Tangent Line to a Curve

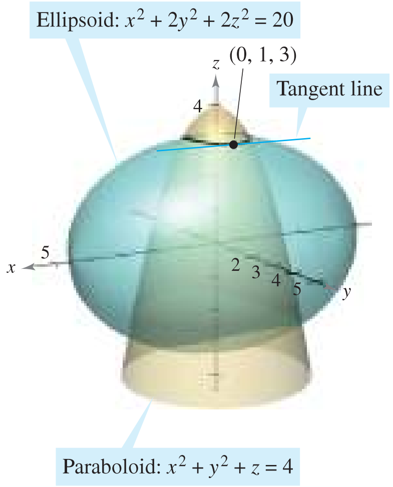

Ellipsoid: $x^2 + 2y^2 + 2z^2 = 20$
z
(0, 1, 3)
Tangent line
4
5
x
2
3
4
5
y
Paraboloid: $x^2 + y^2 + z = 4$

Figure 13.61

Describe the tangent line to the curve of intersection of the surfaces

$$x^2 + 2y^2 + 2z^2 = 20 \qquad \text{Ellipsoid}$$

$$x^2 + y^2 + z = 4 \qquad \text{Paraboloid}$$

at the point $(0, 1, 3)$, as shown in Figure 13.61.

Solution Begin by finding the gradients to both surfaces at the point $(0, 1, 3)$.

Ellipsoid	*Paraboloid*
$F(x, y, z) = x^2 + 2y^2 + 2z^2 - 20$	$G(x, y, z) = x^2 + y^2 + z - 4$
$\nabla F(x, y, z) = 2x\mathbf{i} + 4y\mathbf{j} + 4z\mathbf{k}$	$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$
$\nabla F(0, 1, 3) = 4\mathbf{j} + 12\mathbf{k}$	$\nabla G(0, 1, 3) = 2\mathbf{j} + \mathbf{k}$

The cross product of these two gradients is a vector that is tangent to both surfaces at the point $(0, 1, 3)$.

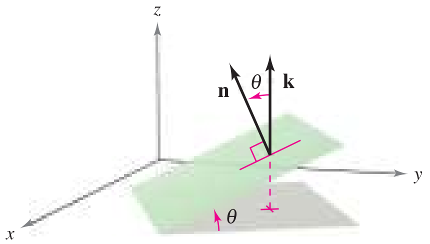
$$\nabla F(0, 1, 3) \times \nabla G(0, 1, 3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & 12 \\ 0 & 2 & 1 \end{vmatrix} = -20\mathbf{i}$$

So, the tangent line to the curve of intersection of the two surfaces at the point $(0, 1, 3)$ is a line that is parallel to the x-axis and passes through the point $(0, 1, 3)$. ■

The Angle of Inclination of a Plane

Another use of the gradient $\nabla F(x, y, z)$ is to determine the angle of inclination of the tangent plane to a surface. The **angle of inclination** of a plane is defined as the angle θ $(0 \le \theta \le \pi/2)$ between the given plane and the xy-plane, as shown in Figure 13.62. (The angle of inclination of a horizontal plane is defined as zero.) Because the vector $\mathbf{k}$ is normal to the xy-plane, you can use the formula for the cosine of the angle between two planes (given in Section 11.5) to conclude that the angle of inclination of a plane with normal vector $\mathbf{n}$ is given by

$$\cos\theta = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|\,\|\mathbf{k}\|} = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|}. \qquad \text{Angle of inclination of a plane}$$



The angle of inclination
Figure 13.62

EXAMPLE 6 Finding the Angle of Inclination of a Tangent Plane

Find the angle of inclination of the tangent plane to the ellipsoid given by

$$\frac{x^2}{12} + \frac{y^2}{12} + \frac{z^2}{3} = 1$$

at the point $(2, 2, 1)$.

Solution If you let

$$F(x, y, z) = \frac{x^2}{12} + \frac{y^2}{12} + \frac{z^2}{3} - 1$$

the gradient of F at the point $(2, 2, 1)$ is given by

$$\nabla F(x, y, z) = \frac{x}{6}\mathbf{i} + \frac{y}{6}\mathbf{j} + \frac{2z}{3}\mathbf{k}$$

$$\nabla F(2, 2, 1) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

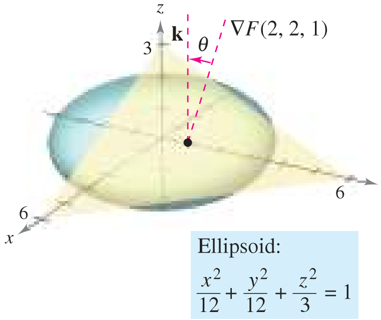


Figure 13.63

Because $\nabla F(2, 2, 1)$ is normal to the tangent plane and $\mathbf{k}$ is normal to the xy-plane, it follows that the angle of inclination of the tangent plane is given by

$$\cos \theta = \frac{|\nabla F(2, 2, 1) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 1)\|} = \frac{2/3}{\sqrt{(1/3)^2 + (1/3)^2 + (2/3)^2}} = \sqrt{\frac{2}{3}}$$

which implies that

$$\theta = \arccos \sqrt{\frac{2}{3}} \approx 35.3°,$$

as shown in Figure 13.63. ■

NOTE A special case of the procedure shown in Example 6 is worth noting. The angle of inclination θ of the tangent plane to the surface $z = f(x, y)$ at (x_0, y_0, z_0) is given by

$$\cos \theta = \frac{1}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + 1}}.$$

Alternative formula for angle of inclination (See Exercise 77.) ■

A Comparison of the Gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$

This section concludes with a comparison of the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$. In the preceding section, you saw that the gradient of a function f of two variables is normal to the level curves of f. Specifically, Theorem 13.12 states that if f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0). Having developed normal lines to surfaces, you can now extend this result to a function of three variables. The proof of Theorem 13.14 is left as an exercise (see Exercise 78).

THEOREM 13.14 GRADIENT IS NORMAL TO LEVEL SURFACES

If F is differentiable at (x_0, y_0, z_0) and $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\nabla F(x_0, y_0, z_0)$ is normal to the level surface through (x_0, y_0, z_0).

When working with the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$, be sure you remember that $\nabla f(x, y)$ is a vector in the xy-plane and $\nabla F(x, y, z)$ is a vector in space.

13.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, describe the level surface $F(x, y, z) = 0$.

1. $F(x, y, z) = 3x - 5y + 3z - 15$

2. $F(x, y, z) = x^2 + y^2 + z^2 - 25$

3. $F(x, y, z) = 4x^2 + 9y^2 - 4z^2$

4. $F(x, y, z) = 16x^2 - 9y^2 + 36z$

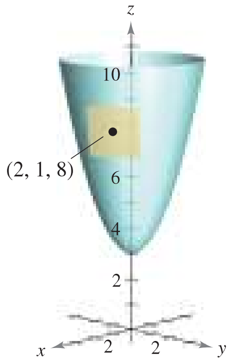
In Exercises 5–16, find a unit normal vector to the surface at the given point. [*Hint:* Normalize the gradient vector $\nabla F(x, y, z)$.]

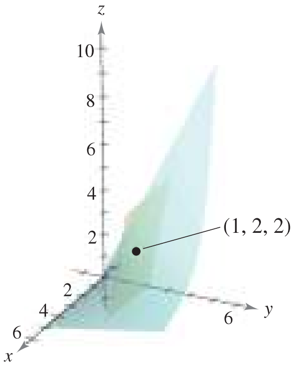
	Surface	*Point*
5.	$3x + 4y + 12z = 0$	$(0, 0, 0)$
6.	$x + y + z = 4$	$(2, 0, 2)$
7.	$x^2 + y^2 + z^2 = 6$	$(1, 1, 2)$
8.	$z = \sqrt{x^2 + y^2}$	$(3, 4, 5)$
9.	$z = x^3$	$(2, -1, 8)$
10.	$x^2y^4 - z = 0$	$(1, 2, 16)$
11.	$x^2 + 3y + z^3 = 9$	$(2, -1, 2)$
12.	$x^2y^3 - y^2z + 2xz^3 = 4$	$(-1, 1, -1)$
13.	$\ln\left(\dfrac{x}{y - z}\right) = 0$	$(1, 4, 3)$
14.	$ze^{x^2-y^2} - 3 = 0$	$(2, 2, 3)$
15.	$z - x\sin y = 4$	$\left(6, \dfrac{\pi}{6}, 7\right)$
16.	$\sin(x - y) - z = 2$	$\left(\dfrac{\pi}{3}, \dfrac{\pi}{6}, -\dfrac{3}{2}\right)$

In Exercises 17–30, find an equation of the tangent plane to the surface at the given point.

17. $z = x^2 + y^2 + 3$
$(2, 1, 8)$

18. $f(x, y) = \dfrac{y}{x}$
$(1, 2, 2)$





19. $z = \sqrt{x^2 + y^2}$, $(3, 4, 5)$

20. $g(x, y) = \arctan \dfrac{y}{x}$, $(1, 0, 0)$

21. $g(x, y) = x^2 + y^2$, $(1, -1, 2)$

22. $f(x, y) = x^2 - 2xy + y^2$, $(1, 2, 1)$

23. $z = 2 - \frac{2}{3}x - y$, $(3, -1, 1)$

24. $z = e^x(\sin y + 1)$, $\left(0, \dfrac{\pi}{2}, 2\right)$

25. $h(x, y) = \ln\sqrt{x^2 + y^2}$, $(3, 4, \ln 5)$

26. $h(x, y) = \cos y$, $\left(5, \dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\right)$

27. $x^2 + 4y^2 + z^2 = 36$, $(2, -2, 4)$

28. $x^2 + 2z^2 = y^2$, $(1, 3, -2)$

29. $xy^2 + 3x - z^2 = 8$, $(1, -3, 2)$

30. $x = y(2z - 3)$, $(4, 4, 2)$

In Exercises 31–40, find an equation of the tangent plane and find symmetric equations of the normal line to the surface at the given point.

31. $x + y + z = 9$, $(3, 3, 3)$

32. $x^2 + y^2 + z^2 = 9$, $(1, 2, 2)$

33. $x^2 + y^2 + z = 9$, $(1, 2, 4)$

34. $z = 16 - x^2 - y^2$, $(2, 2, 8)$

35. $z = x^2 - y^2$, $(3, 2, 5)$

36. $xy - z = 0$, $(-2, -3, 6)$

37. $xyz = 10$, $(1, 2, 5)$

38. $z = ye^{2xy}$, $(0, 2, 2)$

39. $z = \arctan \dfrac{y}{x}$, $\left(1, 1, \dfrac{\pi}{4}\right)$

40. $y \ln xz^2 = 2$, $(e, 2, 1)$

In Exercises 41–46, (a) find symmetric equations of the tangent line to the curve of intersection of the surfaces at the given point, and (b) find the cosine of the angle between the gradient vectors at this point. State whether or not the surfaces are orthogonal at the point of intersection.

41. $x^2 + y^2 = 2$, $z = x$, $(1, 1, 1)$

42. $z = x^2 + y^2$, $z = 4 - y$, $(2, -1, 5)$

43. $x^2 + z^2 = 25$, $y^2 + z^2 = 25$, $(3, 3, 4)$

44. $z = \sqrt{x^2 + y^2}$, $5x - 2y + 3z = 22$, $(3, 4, 5)$

45. $x^2 + y^2 + z^2 = 14$, $x - y - z = 0$, $(3, 1, 2)$

46. $z = x^2 + y^2$, $x + y + 6z = 33$, $(1, 2, 5)$

In Exercises 47–50, find the angle of inclination θ of the tangent plane to the surface at the given point.

47. $3x^2 + 2y^2 - z = 15$, $(2, 2, 5)$

48. $2xy - z^3 = 0$, $(2, 2, 2)$

49. $x^2 - y^2 + z = 0$, $(1, 2, 3)$

50. $x^2 + y^2 = 5$, $(2, 1, 3)$

In Exercises 51–56, find the point(s) on the surface at which the tangent plane is horizontal.

51. $z = 3 - x^2 - y^2 + 6y$

52. $z = 3x^2 + 2y^2 - 3x + 4y - 5$

53. $z = x^2 - xy + y^2 - 2x - 2y$

54. $z = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4$

55. $z = 5xy$

56. $z = xy + \dfrac{1}{x} + \dfrac{1}{y}$

In Exercises 57 and 58, show that the surfaces are tangent to each other at the given point by showing that the surfaces have the same tangent plane at this point.

57. $x^2 + 2y^2 + 3z^2 = 3,\ x^2 + y^2 + z^2 + 6x - 10y + 14 = 0,$
$(-1, 1, 0)$

58. $x^2 + y^2 + z^2 - 8x - 12y + 4z + 42 = 0,\ x^2 + y^2 + 2z = 7,$
$(2, 3, -3)$

In Exercises 59 and 60, (a) show that the surfaces intersect at the given point, and (b) show that the surfaces have perpendicular tangent planes at this point.

59. $z = 2xy^2,\ 8x^2 - 5y^2 - 8z = -13,\quad (1, 1, 2)$

60. $x^2 + y^2 + z^2 + 2x - 4y - 4z - 12 = 0,$
$4x^2 + y^2 + 16z^2 = 24,\quad (1, -2, 1)$

61. Find a point on the ellipsoid $x^2 + 4y^2 + z^2 = 9$ where the tangent plane is perpendicular to the line with parametric equations

$x = 2 - 4t,\ y = 1 + 8t,$ and $z = 3 - 2t.$

62. Find a point on the hyperboloid $x^2 + 4y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $x + 4y - z = 0$.

WRITING ABOUT CONCEPTS

63. Give the standard form of the equation of the tangent plane to a surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0).

64. For some surfaces, the normal lines at any point pass through the same geometric object. What is the common geometric object for a sphere? What is the common geometric object for a right circular cylinder? Explain.

65. Discuss the relationship between the tangent plane to a surface and approximation by differentials.

CAPSTONE

66. Consider the elliptic cone given by

$x^2 - y^2 + z^2 = 0.$

(a) Find an equation of the tangent plane at the point $(5, 13, -12)$.

(b) Find symmetric equations of the normal line at the point $(5, 13, -12)$.

67. ***Investigation*** Consider the function

$$f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}$$

on the intervals $-2 \le x \le 2$ and $0 \le y \le 3$.

(a) Find a set of parametric equations of the normal line and an equation of the tangent plane to the surface at the point $(1, 1, 1)$.

(b) Repeat part (a) for the point $\left(-1, 2, -\frac{4}{5}\right)$.

CAS (c) Use a computer algebra system to graph the surface, the normal lines, and the tangent planes found in parts (a) and (b).

68. ***Investigation*** Consider the function

$$f(x, y) = \frac{\sin y}{x}$$

on the intervals $-3 \le x \le 3$ and $0 \le y \le 2\pi$.

(a) Find a set of parametric equations of the normal line and an equation of the tangent plane to the surface at the point

$\left(2, \dfrac{\pi}{2}, \dfrac{1}{2}\right)$.

(b) Repeat part (a) for the point $\left(-\dfrac{2}{3}, \dfrac{3\pi}{2}, \dfrac{3}{2}\right)$.

CAS (c) Use a computer algebra system to graph the surface, the normal lines, and the tangent planes found in parts (a) and (b).

69. Consider the functions

$f(x, y) = 6 - x^2 - y^2/4$ and $g(x, y) = 2x + y.$

(a) Find a set of parametric equations of the tangent line to the curve of intersection of the surfaces at the point $(1, 2, 4)$, and find the angle between the gradient vectors.

CAS (b) Use a computer algebra system to graph the surfaces. Graph the tangent line found in part (a).

70. Consider the functions

$f(x, y) = \sqrt{16 - x^2 - y^2 + 2x - 4y}$

and

$g(x, y) = \dfrac{\sqrt{2}}{2}\sqrt{1 - 3x^2 + y^2 + 6x + 4y}.$

CAS (a) Use a computer algebra system to graph the first-octant portion of the surfaces represented by f and g.

(b) Find one first-octant point on the curve of intersection and show that the surfaces are orthogonal at this point.

(c) These surfaces are orthogonal along the curve of intersection. Does part (b) prove this fact? Explain.

In Exercises 71 and 72, show that the tangent plane to the quadric surface at the point (x_0, y_0, z_0) can be written in the given form.

71. Ellipsoid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Plane: $\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} + \dfrac{z_0z}{c^2} = 1$

72. Hyperboloid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

Plane: $\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} - \dfrac{z_0z}{c^2} = 1$

73. Show that any tangent plane to the cone

$$z^2 = a^2x^2 + b^2y^2$$

passes through the origin.

74. Let f be a differentiable function and consider the surface $z = xf(y/x)$. Show that the tangent plane at any point $P(x_0, y_0, z_0)$ on the surface passes through the origin.

75. ***Approximation*** Consider the following approximations for a function $f(x, y)$ centered at $(0, 0)$.

Linear approximation:

$$P_1(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y$$

Quadratic approximation:

$$P_2(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \tfrac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \tfrac{1}{2}f_{yy}(0, 0)y^2$$

[Note that the linear approximation is the tangent plane to the surface at $(0, 0, f(0, 0))$.]

(a) Find the linear approximation of $f(x, y) = e^{(x-y)}$ centered at $(0, 0)$.

(b) Find the quadratic approximation of $f(x, y) = e^{(x-y)}$ centered at $(0, 0)$.

(c) If $x = 0$ in the quadratic approximation, you obtain the second-degree Taylor polynomial for what function? Answer the same question for $y = 0$.

(d) Complete the table.

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0			
0	0.1			
0.2	0.1			
0.2	0.5			
1	0.5			

CAS (e) Use a computer algebra system to graph the surfaces $z = f(x, y)$, $z = P_1(x, y)$, and $z = P_2(x, y)$.

76. ***Approximation*** Repeat Exercise 75 for the function $f(x, y) = \cos(x + y)$.

77. Prove that the angle of inclination θ of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is given by

$$\cos\theta = \frac{1}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + 1}}.$$

78. Prove Theorem 13.14.

SECTION PROJECT

Wildflowers

The diversity of wildflowers in a meadow can be measured by counting the numbers of daisies, buttercups, shooting stars, and so on. If there are n types of wildflowers, each with a proportion p_i of the total population, it follows that $p_1 + p_2 + \cdots + p_n = 1$. The measure of diversity of the population is defined as

$$H = -\sum_{i=1}^{n} p_i \log_2 p_i.$$

In this definition, it is understood that $p_i \log_2 p_i = 0$ when $p_i = 0$. The tables show proportions of wildflowers in a meadow in May, June, August, and September.

May

Flower type	1	2	3	4
Proportion	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{1}{16}$

June

Flower type	1	2	3	4
Proportion	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

August

Flower type	1	2	3	4
Proportion	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$

September

Flower type	1	2	3	4
Proportion	0	0	0	1

(a) Determine the wildflower diversity for each month. How would you interpret September's diversity? Which month had the greatest diversity?

(b) If the meadow contains 10 types of wildflowers in roughly equal proportions, is the diversity of the population greater than or less than the diversity of a similar distribution of 4 types of flowers? What type of distribution (of 10 types of wildflowers) would produce maximum diversity?

(c) Let H_n represent the maximum diversity of n types of wildflowers. Does H_n approach a limit as $n \to \infty$?

■ **FOR FURTHER INFORMATION** Biologists use the concept of diversity to measure the proportions of different types of organisms within an environment. For more information on this technique, see the article "Information Theory and Biological Diversity" by Steven Kolmes and Kevin Mitchell in the *UMAP Modules*.

13.8 Extrema of Functions of Two Variables

- **Find absolute and relative extrema of a function of two variables.**
- **Use the Second Partials Test to find relative extrema of a function of two variables.**

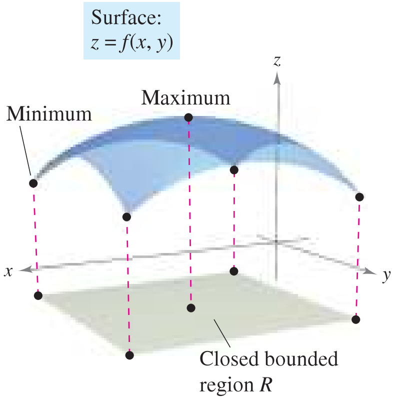
Absolute Extrema and Relative Extrema

In Chapter 3, you studied techniques for finding the extreme values of a function of a single variable. In this section, you will extend these techniques to functions of two variables. For example, in Theorem 13.15 below, the Extreme Value Theorem for a function of a single variable is extended to a function of two variables.

Consider the continuous function f of two variables, defined on a closed bounded region R. The values $f(a, b)$ and $f(c, d)$ such that

$$f(a, b) \le f(x, y) \le f(c, d) \qquad (a, b) \text{ and } (c, d) \text{ are in } R.$$

for all (x, y) in R are called the **minimum** and **maximum** of f in the region R, as shown in Figure 13.64. Recall from Section 13.2 that a region in the plane is *closed* if it contains all of its boundary points. The Extreme Value Theorem deals with a region in the plane that is both closed and *bounded*. A region in the plane is called **bounded** if it is a subregion of a closed disk in the plane.



R contains point(s) at which $f(x, y)$ is a minimum and point(s) at which $f(x, y)$ is a maximum.
Figure 13.64

THEOREM 13.15 EXTREME VALUE THEOREM

Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy-plane.

1. There is at least one point in R at which f takes on a minimum value.
2. There is at least one point in R at which f takes on a maximum value.

A minimum is also called an **absolute minimum** and a maximum is also called an **absolute maximum.** As in single-variable calculus, there is a distinction made between absolute extrema and **relative extrema.**

DEFINITION OF RELATIVE EXTREMA

Let f be a function defined on a region R containing (x_0, y_0).

1. The function f has a **relative minimum** at (x_0, y_0) if

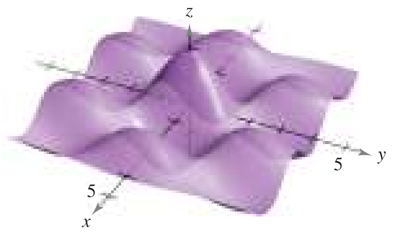
$$f(x, y) \ge f(x_0, y_0)$$

for all (x, y) in an *open* disk containing (x_0, y_0).

2. The function f has a **relative maximum** at (x_0, y_0) if

$$f(x, y) \le f(x_0, y_0)$$

for all (x, y) in an *open* disk containing (x_0, y_0).



Relative extrema
Figure 13.65

To say that f has a relative maximum at (x_0, y_0) means that the point (x_0, y_0, z_0) is at least as high as all nearby points on the graph of $z = f(x, y)$. Similarly, f has a relative minimum at (x_0, y_0) if (x_0, y_0, z_0) is at least as low as all nearby points on the graph. (See Figure 13.65.)

The Granger Collection

KARL WEIERSTRASS (1815–1897)

Although the Extreme Value Theorem had been used by earlier mathematicians, the first to provide a rigorous proof was the German mathematician Karl Weierstrass. Weierstrass also provided rigorous justifications for many other mathematical results already in common use. We are indebted to him for much of the logical foundation on which modern calculus is built.

To locate relative extrema of f, you can investigate the points at which the gradient of f is $\mathbf{0}$ or the points at which one of the partial derivatives does not exist. Such points are called **critical points** of f.

DEFINITION OF CRITICAL POINT

Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a **critical point** of f if one of the following is true.

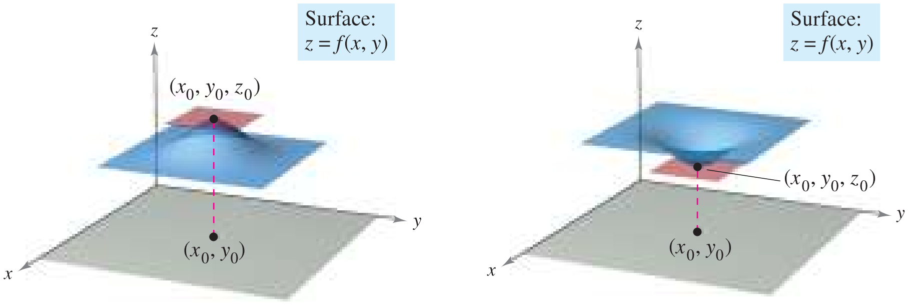
1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Recall from Theorem 13.11 that if f is differentiable and

$$\begin{aligned}\nabla f(x_0, y_0) &= f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}\\ &= 0\mathbf{i} + 0\mathbf{j}\end{aligned}$$

then every directional derivative at (x_0, y_0) must be 0. This implies that the function has a horizontal tangent plane at the point (x_0, y_0), as shown in Figure 13.66. It appears that such a point is a likely location of a relative extremum. This is confirmed by Theorem 13.16.



Relative maximum

Relative minimum

Figure 13.66

THEOREM 13.16 RELATIVE EXTREMA OCCUR ONLY AT CRITICAL POINTS

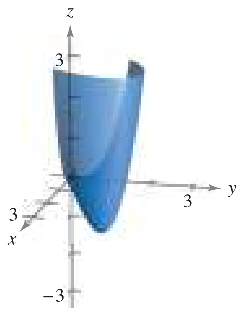
If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

EXPLORATION

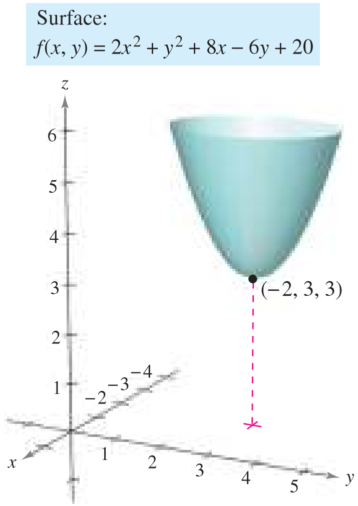
Use a graphing utility to graph

$$z = x^3 - 3xy + y^3$$

using the bounds $0 \le x \le 3$, $0 \le y \le 3$, and $-3 \le z \le 3$. This view makes it appear as though the surface has an absolute minimum. But does it?



EXAMPLE 1 Finding a Relative Extremum



The function $z = f(x, y)$ has a relative minimum at $(-2, 3)$.
Figure 13.67

Determine the relative extrema of

$$f(x, y) = 2x^2 + y^2 + 8x - 6y + 20.$$

Solution Begin by finding the critical points of f. Because

$$f_x(x, y) = 4x + 8 \qquad \text{Partial with respect to } x$$

and

$$f_y(x, y) = 2y - 6 \qquad \text{Partial with respect to } y$$

are defined for all x and y, the only critical points are those for which both first partial derivatives are 0. To locate these points, set $f_x(x, y)$ and $f_y(x, y)$ equal to 0, and solve the equations

$$4x + 8 = 0 \quad \text{and} \quad 2y - 6 = 0$$

to obtain the critical point $(-2, 3)$. By completing the square, you can conclude that for all $(x, y) \neq (-2, 3)$

$$f(x, y) = 2(x + 2)^2 + (y - 3)^2 + 3 > 3.$$

So, a relative *minimum* of f occurs at $(-2, 3)$. The value of the relative minimum is $f(-2, 3) = 3$, as shown in Figure 13.67. ■

Example 1 shows a relative minimum occurring at one type of critical point—the type for which both $f_x(x, y)$ and $f_y(x, y)$ are 0. The next example concerns a relative maximum that occurs at the other type of critical point—the type for which either $f_x(x, y)$ or $f_y(x, y)$ does not exist.

EXAMPLE 2 Finding a Relative Extremum

Determine the relative extrema of $f(x, y) = 1 - (x^2 + y^2)^{1/3}$.

Solution Because

$$f_x(x, y) = -\frac{2x}{3(x^2 + y^2)^{2/3}} \qquad \text{Partial with respect to } x$$

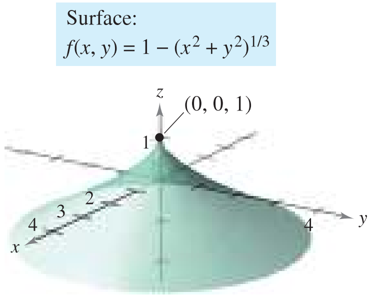
and

$$f_y(x, y) = -\frac{2y}{3(x^2 + y^2)^{2/3}} \qquad \text{Partial with respect to } y$$

it follows that both partial derivatives exist for all points in the xy-plane except for (0, 0). Moreover, because the partial derivatives cannot both be 0 unless both x and y are 0, you can conclude that (0, 0) is the only critical point. In Figure 13.68, note that $f(0, 0)$ is 1. For all other (x, y) it is clear that

$$f(x, y) = 1 - (x^2 + y^2)^{1/3} < 1.$$

So, f has a relative *maximum* at (0, 0). ■



$f_x(x, y)$ and $f_y(x, y)$ are undefined at (0, 0).
Figure 13.68

NOTE In Example 2, $f_x(x, y) = 0$ for every point on the y-axis other than (0, 0). However, because $f_y(x, y)$ is nonzero, these are not critical points. Remember that *one* of the partials must not exist or *both* must be 0 in order to yield a critical point. ■

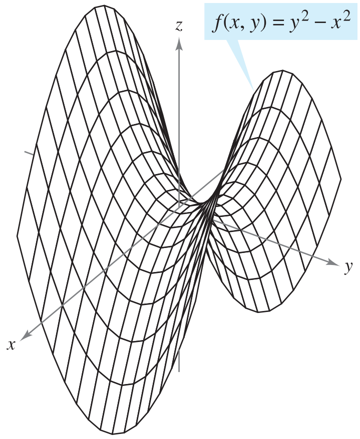
The Second Partials Test

Theorem 13.16 tells you that to find relative extrema you need only examine values of $f(x, y)$ at critical points. However, as is true for a function of one variable, the critical points of a function of two variables do not always yield relative maxima or minima. Some critical points yield **saddle points,** which are neither relative maxima nor relative minima.

As an example of a critical point that does not yield a relative extremum, consider the surface given by

$$f(x, y) = y^2 - x^2 \qquad \text{Hyperbolic paraboloid}$$

as shown in Figure 13.69. At the point $(0, 0)$, both partial derivatives are 0. The function f does not, however, have a relative extremum at this point because in any open disk centered at $(0, 0)$ the function takes on both negative values (along the x-axis) *and* positive values (along the y-axis). So, the point $(0, 0, 0)$ is a saddle point of the surface. (The term "saddle point" comes from the fact that surfaces such as the one shown in Figure 13.69 resemble saddles.)



Saddle point at $(0, 0, 0)$:
$f_x(0, 0) = f_y(0, 0) = 0$
Figure 13.69

For the functions in Examples 1 and 2, it was relatively easy to determine the relative extrema, because each function was either given, or able to be written, in completed square form. For more complicated functions, algebraic arguments are less convenient and it is better to rely on the analytic means presented in the following Second Partials Test. This is the two-variable counterpart of the Second Derivative Test for functions of one variable. The proof of this theorem is best left to a course in advanced calculus.

THEOREM 13.17 SECOND PARTIALS TEST

Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

To test for relative extrema of f, consider the quantity

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a **relative minimum** at (a, b).
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a **relative maximum** at (a, b).
3. If $d < 0$, then $(a, b, f(a, b))$ is a **saddle point.**
4. The test is inconclusive if $d = 0$.

NOTE If $d > 0$, then $f_{xx}(a, b)$ and $f_{yy}(a, b)$ must have the same sign. This means that $f_{xx}(a, b)$ can be replaced by $f_{yy}(a, b)$ in the first two parts of the test. ■

A convenient device for remembering the formula for d in the Second Partials Test is given by the 2×2 determinant

$$d = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix}$$

where $f_{xy}(a, b) = f_{yx}(a, b)$ by Theorem 13.3.

EXAMPLE 3 Using the Second Partials Test

Find the relative extrema of $f(x, y) = -x^3 + 4xy - 2y^2 + 1$.

Solution Begin by finding the critical points of f. Because

$$f_x(x, y) = -3x^2 + 4y \quad \text{and} \quad f_y(x, y) = 4x - 4y$$

exist for all x and y, the only critical points are those for which both first partial derivatives are 0. To locate these points, set $f_x(x, y)$ and $f_y(x, y)$ equal to 0 to obtain $-3x^2 + 4y = 0$ and $4x - 4y = 0$. From the second equation you know that $x = y$, and, by substitution into the first equation, you obtain two solutions: $y = x = 0$ and $y = x = \frac{4}{3}$. Because

$$f_{xx}(x, y) = -6x, \quad f_{yy}(x, y) = -4, \quad \text{and} \quad f_{xy}(x, y) = 4$$

it follows that, for the critical point $(0, 0)$,

$$d = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0 - 16 < 0$$

and, by the Second Partials Test, you can conclude that $(0, 0, 1)$ is a saddle point of f. Furthermore, for the critical point $\left(\frac{4}{3}, \frac{4}{3}\right)$,

$$\begin{aligned} d &= f_{xx}\left(\tfrac{4}{3}, \tfrac{4}{3}\right) f_{yy}\left(\tfrac{4}{3}, \tfrac{4}{3}\right) - \left[f_{xy}\left(\tfrac{4}{3}, \tfrac{4}{3}\right)\right]^2 \\ &= -8(-4) - 16 \\ &= 16 \\ &> 0 \end{aligned}$$

and because $f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) = -8 < 0$ you can conclude that f has a relative maximum at $\left(\frac{4}{3}, \frac{4}{3}\right)$, as shown in Figure 13.70. ■

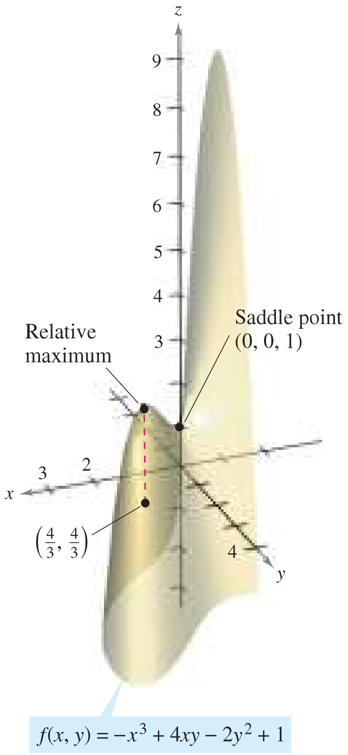

z
9
8
7
6
5
4
3
Relative maximum
Saddle point (0, 0, 1)
3
2
x
$\left(\frac{4}{3}, \frac{4}{3}\right)$
4
y
$f(x, y) = -x^3 + 4xy - 2y^2 + 1$

Figure 13.70

The Second Partials Test can fail to find relative extrema in two ways. If either of the first partial derivatives does not exist, you cannot use the test. Also, if

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 = 0$$

the test fails. In such cases, you can try a sketch or some other approach, as demonstrated in the next example.

EXAMPLE 4 Failure of the Second Partials Test

Find the relative extrema of $f(x, y) = x^2y^2$.

Solution Because $f_x(x, y) = 2xy^2$ and $f_y(x, y) = 2x^2y$, you know that both partial derivatives are 0 if $x = 0$ or $y = 0$. That is, every point along the x- or y-axis is a critical point. Moreover, because

$$f_{xx}(x, y) = 2y^2, \qquad f_{yy}(x, y) = 2x^2, \qquad \text{and} \qquad f_{xy}(x, y) = 4xy$$

you know that if either $x = 0$ or $y = 0$, then

$$\begin{aligned} d &= f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 \\ &= 4x^2y^2 - 16x^2y^2 = -12x^2y^2 = 0. \end{aligned}$$

So, the Second Partials Test fails. However, because $f(x, y) = 0$ for every point along the x- or y-axis and $f(x, y) = x^2y^2 > 0$ for all other points, you can conclude that each of these critical points yields an absolute minimum, as shown in Figure 13.71. ■

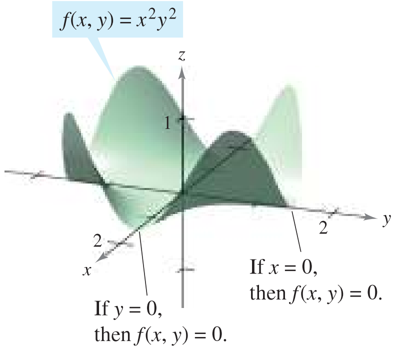

$f(x, y) = x^2y^2$
z
1
2
x
2
y
If $x = 0$, then $f(x, y) = 0$.
If $y = 0$, then $f(x, y) = 0$.

Figure 13.71

Absolute extrema of a function can occur in two ways. First, some relative extrema also happen to be absolute extrema. For instance, in Example 1, $f(-2, 3)$ is an absolute minimum of the function. (On the other hand, the relative maximum found in Example 3 is not an absolute maximum of the function.) Second, absolute extrema can occur at a boundary point of the domain. This is illustrated in Example 5.

EXAMPLE 5 Finding Absolute Extrema

Find the absolute extrema of the function

$$f(x, y) = \sin xy$$

on the closed region given by $0 \le x \le \pi$ and $0 \le y \le 1$.

Solution From the partial derivatives

$$f_x(x, y) = y \cos xy \quad \text{and} \quad f_y(x, y) = x \cos xy$$

you can see that each point lying on the hyperbola given by $xy = \pi/2$ is a critical point. These points each yield the value

$$f(x, y) = \sin \frac{\pi}{2} = 1$$

which you know is the absolute maximum, as shown in Figure 13.72. The only other critical point of *f lying in the given region* is $(0, 0)$. It yields an absolute minimum of 0, because

$$0 \le xy \le \pi$$

implies that

$$0 \le \sin xy \le 1.$$

To locate other absolute extrema, you should consider the four boundaries of the region formed by taking traces with the vertical planes $x = 0$, $x = \pi$, $y = 0$, and $y = 1$. In doing this, you will find that $\sin xy = 0$ at all points on the x-axis, at all points on the y-axis, and at the point $(\pi, 1)$. Each of these points yields an absolute minimum for the surface, as shown in Figure 13.72. ■

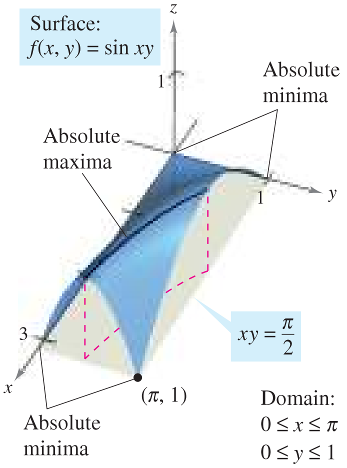


Figure 13.72

The concepts of relative extrema and critical points can be extended to functions of three or more variables. If all first partial derivatives of

$$w = f(x_1, x_2, x_3, \ldots, x_n)$$

exist, it can be shown that a relative maximum or minimum can occur at $(x_1, x_2, x_3, \ldots, x_n)$ only if every first partial derivative is 0 at that point. This means that the critical points are obtained by solving the following system of equations.

$$\begin{aligned} f_{x_1}(x_1, x_2, x_3, \ldots, x_n) &= 0 \\ f_{x_2}(x_1, x_2, x_3, \ldots, x_n) &= 0 \\ &\vdots \\ f_{x_n}(x_1, x_2, x_3, \ldots, x_n) &= 0 \end{aligned}$$

The extension of Theorem 13.17 to three or more variables is also possible, although you will not consider such an extension in this text.

13.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, identify any extrema of the function by recognizing its given form or its form after completing the square. Verify your results by using the partial derivatives to locate any critical points and test for relative extrema.

1. $g(x, y) = (x - 1)^2 + (y - 3)^2$
2. $g(x, y) = 5 - (x - 3)^2 - (y + 2)^2$
3. $f(x, y) = \sqrt{x^2 + y^2 + 1}$
4. $f(x, y) = \sqrt{25 - (x - 2)^2 - y^2}$
5. $f(x, y) = x^2 + y^2 + 2x - 6y + 6$
6. $f(x, y) = -x^2 - y^2 + 10x + 12y - 64$

In Exercises 7–16, examine the function for relative extrema.

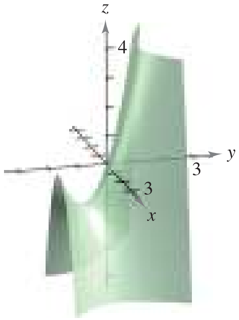
7. $f(x, y) = 3x^2 + 2y^2 - 6x - 4y + 16$
8. $f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5$
9. $f(x, y) = -x^2 - 5y^2 + 10x - 10y - 28$
10. $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$
11. $z = x^2 + xy + \frac{1}{2}y^2 - 2x + y$
12. $z = -5x^2 + 4xy - y^2 + 16x + 10$
13. $f(x, y) = \sqrt{x^2 + y^2}$
14. $h(x, y) = (x^2 + y^2)^{1/3} + 2$
15. $g(x, y) = 4 - |x| - |y|$
16. $f(x, y) = |x + y| - 2$

CAS **In Exercises 17–20, use a computer algebra system to graph the surface and locate any relative extrema and saddle points.**

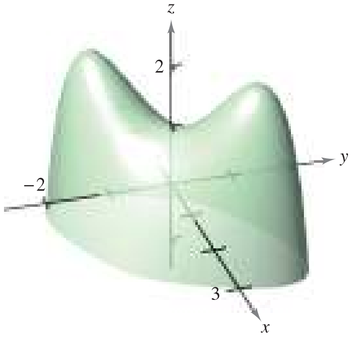
17. $z = \dfrac{-4x}{x^2 + y^2 + 1}$
18. $f(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$
19. $z = (x^2 + 4y^2)e^{1 - x^2 - y^2}$
20. $z = e^{xy}$

In Exercises 21–28, examine the function for relative extrema and saddle points.

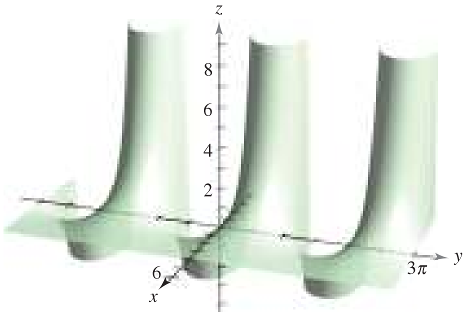
21. $h(x, y) = 80x + 80y - x^2 - y^2$
22. $g(x, y) = x^2 - y^2 - x - y$
23. $g(x, y) = xy$
24. $h(x, y) = x^2 - 3xy - y^2$
25. $f(x, y) = x^2 - xy - y^2 - 3x - y$



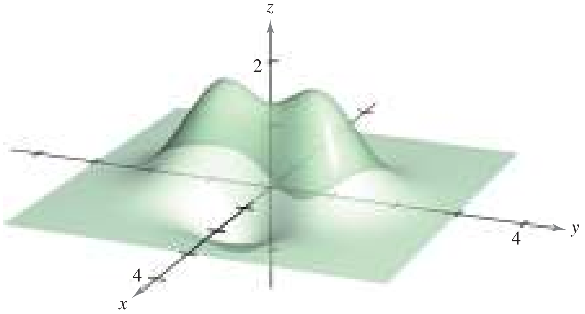
26. $f(x, y) = 2xy - \frac{1}{2}(x^4 + y^4) + 1$



27. $z = e^{-x} \sin y$



28. $z = \left(\frac{1}{2} - x^2 + y^2\right)e^{1 - x^2 - y^2}$



CAS **In Exercises 29 and 30, examine the function for extrema without using the derivative tests, and use a computer algebra system to graph the surface. (*Hint:* By observation, determine if it is possible for z to be negative. When is z equal to 0?)**

29. $z = \dfrac{(x - y)^4}{x^2 + y^2}$
30. $z = \dfrac{(x^2 - y^2)^2}{x^2 + y^2}$

Think About It **In Exercises 31–34, determine whether there is a relative maximum, a relative minimum, a saddle point, or insufficient information to determine the nature of the function $f(x, y)$ at the critical point (x_0, y_0).**

31. $f_{xx}(x_0, y_0) = 9, \quad f_{yy}(x_0, y_0) = 4, \quad f_{xy}(x_0, y_0) = 6$
32. $f_{xx}(x_0, y_0) = -3, \quad f_{yy}(x_0, y_0) = -8, \quad f_{xy}(x_0, y_0) = 2$
33. $f_{xx}(x_0, y_0) = -9, \quad f_{yy}(x_0, y_0) = 6, \quad f_{xy}(x_0, y_0) = 10$
34. $f_{xx}(x_0, y_0) = 25, \quad f_{yy}(x_0, y_0) = 8, \quad f_{xy}(x_0, y_0) = 10$

35. A function f has continuous second partial derivatives on an open region containing the critical point $(3, 7)$. The function has a minimum at $(3, 7)$, and $d > 0$ for the Second Partials Test. Determine the interval for $f_{xy}(3, 7)$ if $f_{xx}(3, 7) = 2$ and $f_{yy}(3, 7) = 8$.

36. A function f has continuous second partial derivatives on an open region containing the critical point (a, b). If $f_{xx}(a, b)$ and $f_{yy}(a, b)$ have opposite signs, what is implied? Explain.

CAS **In Exercises 37–42, (a) find the critical points, (b) test for relative extrema, (c) list the critical points for which the Second Partials Test fails, and (d) use a computer algebra system to graph the function, labeling any extrema and saddle points.**

37. $f(x, y) = x^3 + y^3$

38. $f(x, y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$

39. $f(x, y) = (x - 1)^2(y + 4)^2$

40. $f(x, y) = \sqrt{(x - 1)^2 + (y + 2)^2}$

41. $f(x, y) = x^{2/3} + y^{2/3}$

42. $f(x, y) = (x^2 + y^2)^{2/3}$

In Exercises 43 and 44, find the critical points of the function and, from the form of the function, determine whether a relative maximum or a relative minimum occurs at each point.

43. $f(x, y, z) = x^2 + (y - 3)^2 + (z + 1)^2$

44. $f(x, y, z) = 9 - [x(y - 1)(z + 2)]^2$

In Exercises 45–54, find the absolute extrema of the function over the region *R*. (In each case, *R* contains the boundaries.) Use a computer algebra system to confirm your results.

45. $f(x, y) = x^2 - 4xy + 5$

$R = \{(x, y) : 1 \le x \le 4,\ 0 \le y \le 2\}$

46. $f(x, y) = x^2 + xy, \quad R = \{(x, y) : |x| \le 2,\ |y| \le 1\}$

47. $f(x, y) = 12 - 3x - 2y$

R: The triangular region in the xy-plane with vertices $(2, 0)$, $(0, 1)$, and $(1, 2)$

48. $f(x, y) = (2x - y)^2$

R: The triangular region in the xy-plane with vertices $(2, 0)$, $(0, 1)$, and $(1, 2)$

49. $f(x, y) = 3x^2 + 2y^2 - 4y$

R: The region in the xy-plane bounded by the graphs of $y = x^2$ and $y = 4$

50. $f(x, y) = 2x - 2xy + y^2$

R: The region in the xy-plane bounded by the graphs of $y = x^2$ and $y = 1$

51. $f(x, y) = x^2 + 2xy + y^2, \quad R = \{(x, y) : |x| \le 2,\ |y| \le 1\}$

52. $f(x, y) = x^2 + 2xy + y^2, \quad R = \{(x, y) : x^2 + y^2 \le 8\}$

53. $f(x, y) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)}$

$R = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\}$

54. $f(x, y) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)}$

$R = \{(x, y) : x \ge 0,\ y \ge 0,\ x^2 + y^2 \le 1\}$

WRITING ABOUT CONCEPTS

55. The figure shows the level curves for an unknown function $f(x, y)$. What, if any, information can be given about f at the point A? Explain your reasoning.

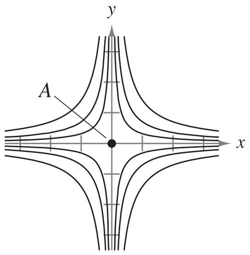


Figure for 55

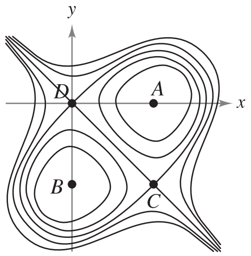


Figure for 56

56. The figure shows the level curves for an unknown function $f(x, y)$. What, if any, information can be given about f at the points A, B, C, and D? Explain your reasoning.

In Exercises 57–59, sketch the graph of an arbitrary function *f* satisfying the given conditions. State whether the function has any extrema or saddle points. (There are many correct answers.)

57. $f_x(x, y) > 0$ and $f_y(x, y) < 0$ for all (x, y).

58. All of the first and second partial derivatives of f are 0.

59. $f_x(0, 0) = 0,\ f_y(0, 0) = 0$

$$f_x(x, y)\begin{cases} < 0, & x < 0 \\ > 0, & x > 0 \end{cases}, \qquad f_y(x, y)\begin{cases} > 0, & y < 0 \\ < 0, & y > 0 \end{cases}$$

$f_{xx}(x, y) > 0,\ f_{yy}(x, y) < 0$, and $f_{xy}(x, y) = 0$ for all (x, y).

CAPSTONE

60. Consider the functions

$f(x, y) = x^2 - y^2$ and $g(x, y) = x^2 + y^2$.

(a) Show that both functions have a critical point at $(0, 0)$.

(b) Explain how f and g behave differently at this critical point.

True or False? **In Exercises 61–64, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

61. If f has a relative maximum at (x_0, y_0, z_0), then $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

62. If $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, then f has a relative maximum at (x_0, y_0, z_0).

63. Between any two relative minima of f, there must be at least one relative maximum of f.

64. If f is continuous for all x and y and has two relative minima, then f must have at least one relative maximum.

13.9 Applications of Extrema of Functions of Two Variables

- **Solve optimization problems involving functions of several variables.**
- **Use the method of least squares.**

Applied Optimization Problems

In this section, you will survey a few of the many applications of extrema of functions of two (or more) variables.

EXAMPLE 1 Finding Maximum Volume

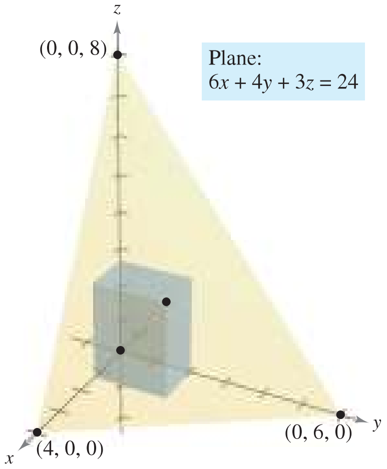


Figure 13.73

A rectangular box is resting on the xy-plane with one vertex at the origin. The opposite vertex lies in the plane

$$6x + 4y + 3z = 24$$

as shown in Figure 13.73. Find the maximum volume of such a box.

Solution Let x, y, and z represent the length, width, and height of the box. Because one vertex of the box lies in the plane $6x + 4y + 3z = 24$, you know that $z = \frac{1}{3}(24 - 6x - 4y)$, and you can write the volume xyz of the box as a function of two variables.

$$\begin{aligned} V(x, y) &= (x)(y)\left[\tfrac{1}{3}(24 - 6x - 4y)\right] \\ &= \tfrac{1}{3}(24xy - 6x^2y - 4xy^2) \end{aligned}$$

By setting the first partial derivatives equal to 0

$$V_x(x, y) = \frac{1}{3}(24y - 12xy - 4y^2) = \frac{y}{3}(24 - 12x - 4y) = 0$$

$$V_y(x, y) = \frac{1}{3}(24x - 6x^2 - 8xy) = \frac{x}{3}(24 - 6x - 8y) = 0$$

you obtain the critical points $(0, 0)$ and $\left(\frac{4}{3}, 2\right)$. At $(0, 0)$ the volume is 0, so that point does not yield a maximum volume. At the point $\left(\frac{4}{3}, 2\right)$, you can apply the Second Partials Test.

$$V_{xx}(x, y) = -4y$$

$$V_{yy}(x, y) = \frac{-8x}{3}$$

$$V_{xy}(x, y) = \frac{1}{3}(24 - 12x - 8y)$$

Because

$$V_{xx}\left(\tfrac{4}{3}, 2\right)V_{yy}\left(\tfrac{4}{3}, 2\right) - \left[V_{xy}\left(\tfrac{4}{3}, 2\right)\right]^2 = (-8)\left(-\tfrac{32}{9}\right) - \left(-\tfrac{8}{3}\right)^2 = \tfrac{64}{3} > 0$$

and

$$V_{xx}\left(\tfrac{4}{3}, 2\right) = -8 < 0$$

you can conclude from the Second Partials Test that the maximum volume is

$$\begin{aligned} V\left(\tfrac{4}{3}, 2\right) &= \tfrac{1}{3}\left[24\left(\tfrac{4}{3}\right)(2) - 6\left(\tfrac{4}{3}\right)^2(2) - 4\left(\tfrac{4}{3}\right)(2^2)\right] \\ &= \tfrac{64}{9} \text{ cubic units.} \end{aligned}$$

Note that the volume is 0 at the boundary points of the triangular domain of V. ■

NOTE In many applied problems, the domain of the function to be optimized is a closed bounded region. To find minimum or maximum points, you must not only test critical points, but also consider the values of the function at points on the boundary.

Applications of extrema in economics and business often involve more than one independent variable. For instance, a company may produce several models of one type of product. The price per unit and profit per unit are usually different for each model. Moreover, the demand for each model is often a function of the prices of the other models (as well as its own price). The next example illustrates an application involving two products.

EXAMPLE 2 Finding the Maximum Profit

An electronics manufacturer determines that the profit P (in dollars) obtained by producing and selling x units of a DVD player and y units of a DVD recorder is approximated by the model

$$P(x, y) = 8x + 10y - (0.001)(x^2 + xy + y^2) - 10{,}000.$$

Find the production level that produces a maximum profit. What is the maximum profit?

Solution The partial derivatives of the profit function are

$$P_x(x, y) = 8 - (0.001)(2x + y) \qquad \text{and} \qquad P_y(x, y) = 10 - (0.001)(x + 2y).$$

By setting these partial derivatives equal to 0, you obtain the following system of equations.

$$\begin{aligned} 8 - (0.001)(2x + y) &= 0 \\ 10 - (0.001)(x + 2y) &= 0 \end{aligned}$$

After simplifying, this system of linear equations can be written as

$$\begin{aligned} 2x + y &= 8000 \\ x + 2y &= 10{,}000. \end{aligned}$$

Solving this system produces $x = 2000$ and $y = 4000$. The second partial derivatives of P are

$$\begin{aligned} P_{xx}(2000, 4000) &= -0.002 \\ P_{yy}(2000, 4000) &= -0.002 \\ P_{xy}(2000, 4000) &= -0.001. \end{aligned}$$

Because $P_{xx} < 0$ and

$$P_{xx}(2000, 4000)P_{yy}(2000, 4000) - [P_{xy}(2000, 4000)]^2 = (-0.002)^2 - (-0.001)^2 > 0$$

you can conclude that the production level of $x = 2000$ units and $y = 4000$ units yields a *maximum* profit. The maximum profit is

$$\begin{aligned} P(2000, 4000) &= 8(2000) + 10(4000) - (0.001)[2000^2 + 2000(4000) + 4000^2)] - 10{,}000 \\ &= \$18{,}000. \end{aligned}$$

■ **FOR FURTHER INFORMATION** For more information on the use of mathematics in economics, see the article "Mathematical Methods of Economics" by Joel Franklin in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

NOTE In Example 2, it was assumed that the manufacturing plant is able to produce the required number of units to yield a maximum profit. In actual practice, the production would be bounded by physical constraints. You will study such constrained optimization problems in the next section.

The Method of Least Squares

Many of the examples in this text have involved **mathematical models.** For instance, Example 2 involves a quadratic model for profit. There are several ways to develop such models; one is called the **method of least squares.**

In constructing a model to represent a particular phenomenon, the goals are simplicity and accuracy. Of course, these goals often conflict. For instance, a simple linear model for the points in Figure 13.74 is

$$y = 1.8566x - 5.0246.$$

However, Figure 13.75 shows that by choosing the slightly more complicated quadratic model*

$$y = 0.1996x^2 - 0.7281x + 1.3749$$

you can achieve greater accuracy.

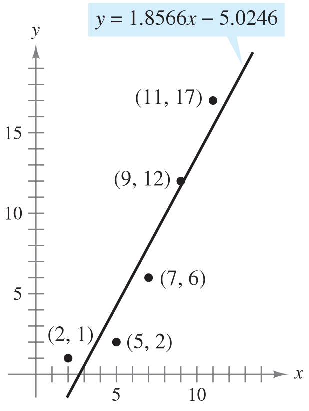


Figure 13.74

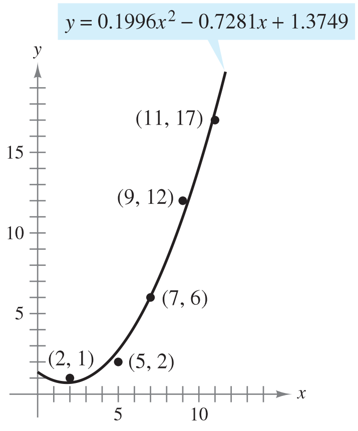


Figure 13.75

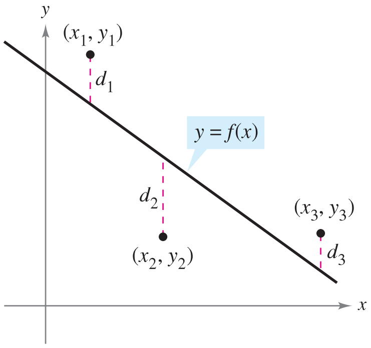
As a measure of how well the model $y = f(x)$ fits the collection of points

$$\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\}$$

you can add the squares of the differences between the actual y-values and the values given by the model to obtain the **sum of the squared errors**

$$\boxed{S = \sum_{i=1}^{n} [f(x_i) - y_i]^2.}$$

Sum of the squared errors



Sum of the squared errors:
$S = d_1^2 + d_2^2 + d_3^2$
Figure 13.76

Graphically, S can be interpreted as the sum of the squares of the vertical distances between the graph of f and the given points in the plane, as shown in Figure 13.76. If the model is perfect, then $S = 0$. However, when perfection is not feasible, you can settle for a model that minimizes S. For instance, the sum of the squared errors for the linear model in Figure 13.74 is $S \approx 17$. Statisticians call the *linear model* that minimizes S the **least squares regression line.** The proof that this line actually minimizes S involves the minimizing of a function of two variables.

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* *A method for finding the least squares quadratic model for a collection of data is described in Exercise 37.*

The Granger Collection

ADRIEN-MARIE LEGENDRE (1752–1833)

The method of least squares was introduced by the French mathematician Adrien-Marie Legendre. Legendre is best known for his work in geometry. In fact, his text Elements of Geometry was so popular in the United States that it continued to be used for 33 editions, spanning a period of more than 100 years.

THEOREM 13.18 LEAST SQUARES REGRESSION LINE

The **least squares regression line** for $\{(x_1, y_1), (x_2, y_2), . . . , (x_n, y_n)\}$ is given by $f(x) = ax + b$, where

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \quad \text{and} \quad b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right).$$

PROOF Let $S(a, b)$ represent the sum of the squared errors for the model $f(x) = ax + b$ and the given set of points. That is,

$$\begin{aligned} S(a, b) &= \sum_{i=1}^{n} [f(x_i) - y_i]^2 \\ &= \sum_{i=1}^{n} (ax_i + b - y_i)^2 \end{aligned}$$

where the points (x_i, y_i) represent constants. Because S is a function of a and b, you can use the methods discussed in the preceding section to find the minimum value of S. Specifically, the first partial derivatives of S are

$$\begin{aligned} S_a(a, b) &= \sum_{i=1}^{n} 2x_i(ax_i + b - y_i) \\ &= 2a\sum_{i=1}^{n} x_i^2 + 2b\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} x_i y_i \\ S_b(a, b) &= \sum_{i=1}^{n} 2(ax_i + b - y_i) \\ &= 2a\sum_{i=1}^{n} x_i + 2nb - 2\sum_{i=1}^{n} y_i. \end{aligned}$$

By setting these two partial derivatives equal to 0, you obtain the values of a and b that are listed in the theorem. It is left to you to apply the Second Partials Test (see Exercise 47) to verify that these values of a and b yield a minimum. ■

If the x-values are symmetrically spaced about the y-axis, then $\Sigma x_i = 0$ and the formulas for a and b simplify to

$$a = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

and

$$b = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

This simplification is often possible with a translation of the x-values. For instance, if the x-values in a data collection consist of the years 2005, 2006, 2007, 2008, and 2009, you could let 2007 be represented by 0.

EXAMPLE 3 Finding the Least Squares Regression Line

Find the least squares regression line for the points $(-3, 0)$, $(-1, 1)$, $(0, 2)$, and $(2, 3)$.

Solution The table shows the calculations involved in finding the least squares regression line using $n = 4$.

x	y	xy	x^2
-3	0	0	9
-1	1	-1	1
0	2	0	0
2	3	6	4
$\sum_{i=1}^{n} x_i = -2$	$\sum_{i=1}^{n} y_i = 6$	$\sum_{i=1}^{n} x_i y_i = 5$	$\sum_{i=1}^{n} x_i^2 = 14$

TECHNOLOGY Many calculators have "built-in" least squares regression programs. If your calculator has such a program, use it to duplicate the results of Example 3.

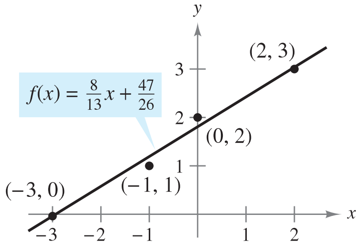
Applying Theorem 13.18 produces

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{4(5) - (-2)(6)}{4(14) - (-2)^2} = \frac{8}{13}$$

and

$$b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right) = \frac{1}{4}\left[6 - \frac{8}{13}(-2)\right] = \frac{47}{26}.$$

The least squares regression line is $f(x) = \frac{8}{13}x + \frac{47}{26}$, as shown in Figure 13.77. ■



Least squares regression line

Figure 13.77

13.9 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, find the minimum distance from the point to the plane $x - y + z = 3$. (*Hint:* To simplify the computations, minimize the square of the distance.)

1. $(0, 0, 0)$

2. $(1, 2, 3)$

In Exercises 3 and 4, find the minimum distance from the point to the surface $z = \sqrt{1 - 2x - 2y}$. (*Hint:* In Exercise 4, use the *root* feature of a graphing utility.)

3. $(-2, -2, 0)$

4. $(0, 0, 2)$

In Exercises 5–8, find three positive integers x, y, and z that satisfy the given conditions.

5. The product is 27 and the sum is a minimum.

6. The sum is 32 and $P = xy^2z$ is a maximum.

7. The sum is 30 and the sum of the squares is a minimum.

8. The product is 1 and the sum of the squares is a minimum.

9. ***Cost*** A home improvement contractor is painting the walls and ceiling of a rectangular room. The volume of the room is 668.25 cubic feet. The cost of wall paint is \$0.06 per square foot and the cost of ceiling paint is \$0.11 per square foot. Find the room dimensions that result in a minimum cost for the paint. What is the minimum cost for the paint?

10. ***Maximum Volume*** The material for constructing the base of an open box costs 1.5 times as much per unit area as the material for constructing the sides. For a fixed amount of money C, find the dimensions of the box of largest volume that can be made.

11. ***Maximum Volume*** The volume of an ellipsoid

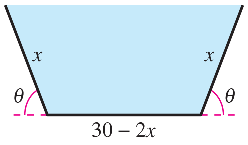
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $4\pi abc/3$. For a fixed sum $a + b + c$, show that the ellipsoid of maximum volume is a sphere.

12. ***Maximum Volume*** Show that the rectangular box of maximum volume inscribed in a sphere of radius r is a cube.

13. ***Volume and Surface Area*** Show that a rectangular box of given volume and minimum surface area is a cube.

14. ***Area*** A trough with trapezoidal cross sections is formed by turning up the edges of a 30-inch-wide sheet of aluminum (see figure). Find the cross section of maximum area.



15. ***Maximum Revenue*** A company manufactures two types of sneakers, running shoes and basketball shoes. The total revenue from x_1 units of running shoes and x_2 units of basketball shoes is $R = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$, where x_1 and x_2 are in thousands of units. Find x_1 and x_2 so as to maximize the revenue.

16. ***Maximum Profit*** A corporation manufactures candles at two locations. The cost of producing x_1 units at location 1 is

$$C_1 = 0.02x_1^2 + 4x_1 + 500$$

and the cost of producing x_2 units at location 2 is

$$C_2 = 0.05x_2^2 + 4x_2 + 275.$$

The candles sell for \$15 per unit. Find the quantity that should be produced at each location to maximize the profit $P = 15(x_1 + x_2) - C_1 - C_2$.

17. ***Hardy-Weinberg Law*** Common blood types are determined genetically by three alleles A, B, and O. (An allele is any of a group of possible mutational forms of a gene.) A person whose blood type is AA, BB, or OO is homozygous. A person whose blood type is AB, AO, or BO is heterozygous. The Hardy-Weinberg Law states that the proportion P of heterozygous individuals in any given population is

$$P(p, q, r) = 2pq + 2pr + 2qr$$

where p represents the percent of allele A in the population, q represents the percent of allele B in the population, and r represents the percent of allele O in the population. Use the fact that $p + q + r = 1$ to show that the maximum proportion of heterozygous individuals in any population is $\frac{2}{3}$.

18. ***Shannon Diversity Index*** One way to measure species diversity is to use the Shannon diversity index H. If a habitat consists of three species, A, B, and C, its Shannon diversity index is

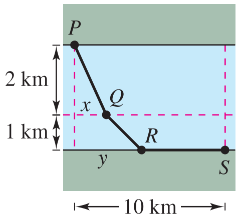
$$H = -x \ln x - y \ln y - z \ln z$$

where x is the percent of species A in the habitat, y is the percent of species B in the habitat, and z is the percent of species C in the habitat.

(a) Use the fact that $x + y + z = 1$ to show that the maximum value of H occurs when $x = y = z = \frac{1}{3}$.

(b) Use the results of part (a) to show that the maximum value of H in this habitat is ln 3.

19. ***Minimum Cost*** A water line is to be built from point P to point S and must pass through regions where construction costs differ (see figure). The cost per kilometer in dollars is $3k$ from P to Q, $2k$ from Q to R, and k from R to S. Find x and y such that the total cost C will be minimized.



20. ***Distance*** A company has retail outlets located at the points (0, 0), (2, 2), and (−2, 2) (see figure). Management plans to build a distribution center located such that the sum of the distances S from the center to the outlets is minimum. From the symmetry of the problem it is clear that the distribution center will be located on the y-axis, and therefore S is a function of the single variable y. Using techniques presented in Chapter 3, find the required value of y.

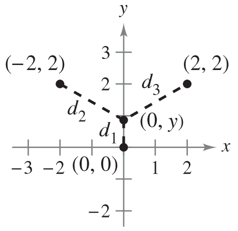


Figure for 20

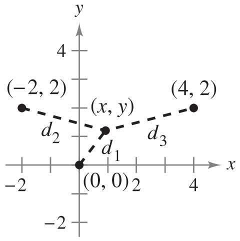


Figure for 21

CAS **21.** ***Investigation*** The retail outlets described in Exercise 20 are located at (0, 0), (4, 2), and (−2, 2) (see figure). The location of the distribution center is (x, y), and therefore the sum of the distances S is a function of x and y.

(a) Write the expression giving the sum of the distances S. Use a computer algebra system to graph S. Does the surface have a minimum?

(b) Use a computer algebra system to obtain S_x and S_y. Observe that solving the system $S_x = 0$ and $S_y = 0$ is very difficult. So, approximate the location of the distribution center.

(c) An initial estimate of the critical point is $(x_1, y_1) = (1, 1)$. Calculate $-\nabla S(1, 1)$ with components $-S_x(1, 1)$ and $-S_y(1, 1)$. What direction is given by the vector $-\nabla S(1, 1)$?

(d) The second estimate of the critical point is

$$(x_2, y_2) = (x_1 - S_x(x_1, y_1)t, y_1 - S_y(x_1, y_1)t).$$

If these coordinates are substituted into $S(x, y)$, then S becomes a function of the single variable t. Find the value of t that minimizes S. Use this value of t to estimate (x_2, y_2).

(e) Complete two more iterations of the process in part (d) to obtain (x_4, y_4). For this location of the distribution center, what is the sum of the distances to the retail outlets?

(f) Explain why $-\nabla S(x, y)$ was used to approximate the minimum value of S. In what types of problems would you use $\nabla S(x, y)$?

22. ***Investigation*** Repeat Exercise 21 for retail outlets located at the points $(-4, 0)$, $(1, 6)$, and $(12, 2)$.

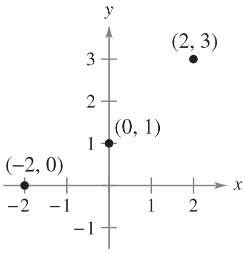
WRITING ABOUT CONCEPTS

23. In your own words, state the problem-solving strategy for applied minimum and maximum problems.

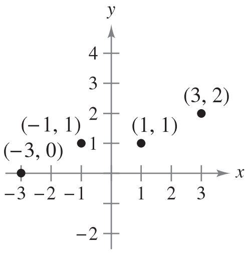
24. In your own words, describe the method of least squares for finding mathematical models.

In Exercises 25–28, (a) find the least squares regression line and (b) calculate *S*, the sum of the squared errors. Use the regression capabilities of a graphing utility to verify your results.

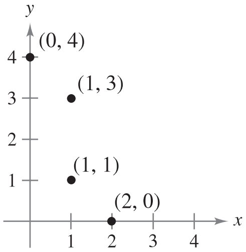
25.



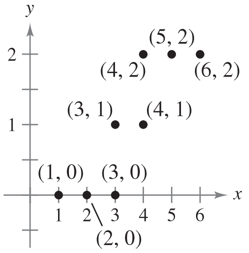
26.



27.



28.



In Exercises 29–32, find the least squares regression line for the points. Use the regression capabilities of a graphing utility to verify your results. Use the graphing utility to plot the points and graph the regression line.

29. $(0, 0), (1, 1), (3, 4), (4, 2), (5, 5)$

30. $(1, 0), (3, 3), (5, 6)$

31. $(0, 6), (4, 3), (5, 0), (8, -4), (10, -5)$

32. $(6, 4), (1, 2), (3, 3), (8, 6), (11, 8), (13, 8)$

33. ***Modeling Data*** The ages x (in years) and systolic blood pressures y of seven men are shown in the table.

Age, *x*	16	25	39	45	49	64	70
Systolic Blood Pressure, *y*	109	122	150	165	159	183	199

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model to approximate the change in systolic blood pressure for each one-year increase in age.

34. ***Modeling Data*** A store manager wants to know the demand y for an energy bar as a function of price x. The daily sales for three different prices of the energy bar are shown in the table.

Price, *x*	\$1.29	\$1.49	\$1.69
Demand, *y*	450	375	330

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use the model to estimate the demand when the price is \$1.59.

35. ***Modeling Data*** An agronomist used four test plots to determine the relationship between the wheat yield y (in bushels per acre) and the amount of fertilizer x (in hundreds of pounds per acre). The results are shown in the table.

Fertilizer, *x*	1.0	1.5	2.0	2.5
Yield, *y*	32	41	48	53

Use the regression capabilities of a graphing utility to find the least squares regression line for the data, and estimate the yield for a fertilizer application of 160 pounds per acre.

36. ***Modeling Data*** The table shows the percents x and numbers y (in millions) of women in the work force for selected years. *(Source: U.S. Bureau of Labor Statistics)*

Year	1970	1975	1980	1985
Percent, *x*	43.3	46.3	51.5	54.5
Number, *y*	31.5	37.5	45.5	51.1

Year	1990	1995	2000	2005
Percent, *x*	57.5	58.9	59.9	59.3
Number, *y*	56.8	60.9	66.3	69.3

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) According to this model, approximately how many women enter the labor force for each one-point increase in the percent of women in the labor force?

37. Find a system of equations whose solution yields the coefficients a, b, and c for the least squares regression quadratic $y = ax^2 + bx + c$ for the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ by minimizing the sum

$$S(a, b, c) = \sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c)^2.$$

CAPSTONE

38. The sum of the length and the girth (perimeter of a cross section) of a package carried by a delivery service cannot exceed 108 inches. Find the dimensions of the rectangular package of largest volume that may be sent.

In Exercises 39–42, use the result of Exercise 37 to find the least squares regression quadratic for the given points. Use the regression capabilities of a graphing utility to confirm your results. Use the graphing utility to plot the points and graph the least squares regression quadratic.

39. $(-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)$

40. $(-4, 5), (-2, 6), (2, 6), (4, 2)$

41. $(0, 0), (2, 2), (3, 6), (4, 12)$

42. $(0, 10), (1, 9), (2, 6), (3, 0)$

43. ***Modeling Data*** After a new turbocharger for an automobile engine was developed, the following experimental data were obtained for speed y in miles per hour at two-second time intervals x.

Time, x	0	2	4	6	8	10
Speed, y	0	15	30	50	65	70

(a) Find a least squares regression quadratic for the data. Use a graphing utility to confirm your results.

(b) Use a graphing utility to plot the points and graph the model.

44. ***Modeling Data*** The table shows the world populations y (in billions) for five different years. Let $x = 8$ represent the year 1998. *(Source: U.S. Census Bureau, International Data Base)*

Year, x	1998	2000	2002	2004	2006
Population, y	5.9	6.1	6.2	6.4	6.5

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use the regression capabilities of a graphing utility to find the least squares regression quadratic for the data.

(c) Use a graphing utility to plot the data and graph the models.

(d) Use both models to forecast the world population for the year 2014. How do the two models differ as you extrapolate into the future?

45. ***Modeling Data*** A meteorologist measures the atmospheric pressure P (in kilograms per square meter) at altitude h (in kilometers). The data are shown below.

Altitude, h	0	5	10	15	20
Pressure, P	10,332	5583	2376	1240	517

(a) Use the regression capabilities of a graphing utility to find a least squares regression line for the points $(h, \ln P)$.

(b) The result in part (a) is an equation of the form $\ln P = ah + b$. Write this logarithmic form in exponential form.

(c) Use a graphing utility to plot the original data and graph the exponential model in part (b).

(d) If your graphing utility can fit logarithmic models to data, use it to verify the result in part (b).

46. ***Modeling Data*** The endpoints of the interval over which distinct vision is possible are called the near point and far point of the eye. With increasing age, these points normally change. The table shows the approximate near points y (in inches) for various ages x (in years). *(Source: Ophthalmology & Physiological Optics)*

Age, x	16	32	44	50	60
Near Point, y	3.0	4.7	9.8	19.7	39.4

(a) Find a rational model for the data by taking the reciprocals of the near points to generate the points $(x, 1/y)$. Use the regression capabilities of a graphing utility to find a least squares regression line for the revised data. The resulting line has the form $1/y = ax + b$. Solve for y.

(b) Use a graphing utility to plot the data and graph the model.

(c) Do you think the model can be used to predict the near point for a person who is 70 years old? Explain.

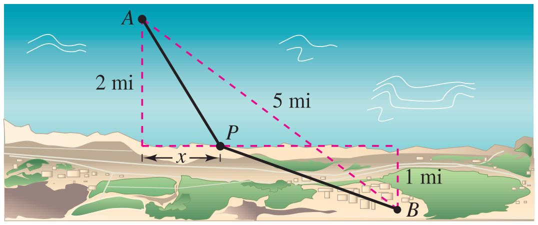
47. Use the Second Partials Test to verify that the formulas for a and b given in Theorem 13.18 yield a minimum.

$\left[\textit{Hint: } \text{Use the fact that } n\sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i\right)^2.\right]$

SECTION PROJECT

Building a Pipeline

An oil company wishes to construct a pipeline from its offshore facility A to its refinery B. The offshore facility is 2 miles from shore, and the refinery is 1 mile inland. Furthermore, A and B are 5 miles apart, as shown in the figure.



The cost of building the pipeline is \$3 million per mile in the water and \$4 million per mile on land. So, the cost of the pipeline depends on the location of point P, where it meets the shore. What would be the most economical route of the pipeline?

Imagine that you are to write a report to the oil company about this problem. Let x be the distance shown in the figure. Determine the cost of building the pipeline from A to P, and the cost from P to B. Analyze some sample pipeline routes and their corresponding costs. For instance, what is the cost of the most direct route? Then use calculus to determine the route of the pipeline that minimizes the cost. Explain all steps of your development and include any relevant graphs.

13.10 Lagrange Multipliers

- **Understand the Method of Lagrange Multipliers.**
- **Use Lagrange multipliers to solve constrained optimization problems.**
- **Use the Method of Lagrange Multipliers with two constraints.**

Lagrange Multipliers

Many optimization problems have restrictions, or **constraints,** on the values that can be used to produce the optimal solution. Such constraints tend to complicate optimization problems because the optimal solution can occur at a boundary point of the domain. In this section, you will study an ingenious technique for solving such problems. It is called the **Method of Lagrange Multipliers.**

JOSEPH-LOUIS LAGRANGE (1736–1813)

The Method of Lagrange Multipliers is named after the French mathematician Joseph-Louis Lagrange. Lagrange first introduced the method in his famous paper on mechanics, written when he was just 19 years old.

To see how this technique works, suppose you want to find the rectangle of maximum area that can be inscribed in the ellipse given by

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

Let (x, y) be the vertex of the rectangle in the first quadrant, as shown in Figure 13.78. Because the rectangle has sides of lengths $2x$ and $2y$, its area is given by

$$f(x, y) = 4xy. \qquad \text{Objective function}$$

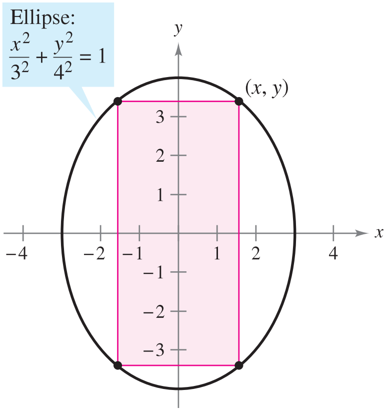
You want to find x and y such that $f(x, y)$ is a maximum. Your choice of (x, y) is restricted to first-quadrant points that lie on the ellipse

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1. \qquad \text{Constraint}$$

Now, consider the constraint equation to be a fixed level curve of

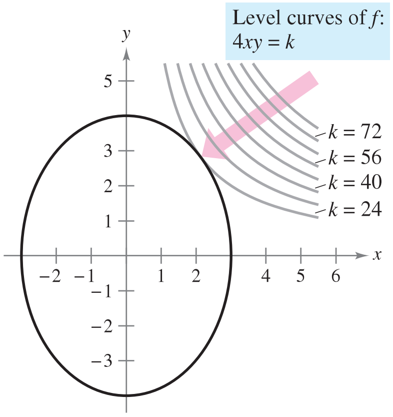
$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2}.$$

The level curves of f represent a family of hyperbolas $f(x, y) = 4xy = k$. In this family, the level curves that meet the given constraint correspond to the hyperbolas that intersect the ellipse. Moreover, to maximize $f(x, y)$, you want to find the hyperbola that just barely satisfies the constraint. The level curve that does this is the one that is *tangent* to the ellipse, as shown in Figure 13.79.



Objective function: $f(x, y) = 4xy$

Figure 13.78



Constraint: $g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$

Figure 13.79

To find the appropriate hyperbola, use the fact that two curves are tangent at a point if and only if their gradient vectors are parallel. This means that $\nabla f(x, y)$ must be a scalar multiple of $\nabla g(x, y)$ at the point of tangency. In the context of constrained optimization problems, this scalar is denoted by λ (the lowercase Greek letter lambda).

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

The scalar λ is called a **Lagrange multiplier.** Theorem 13.19 gives the necessary conditions for the existence of such multipliers.

THEOREM 13.19 LAGRANGE'S THEOREM

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

PROOF To begin, represent the smooth curve given by $g(x, y) = c$ by the vector-valued function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad \mathbf{r}'(t) \neq \mathbf{0}$$

where x' and y' are continuous on an open interval I. Define the function h as $h(t) = f(x(t), y(t))$. Then, because $f(x_0, y_0)$ is an extreme value of f, you know that

$$h(t_0) = f(x(t_0), y(t_0)) = f(x_0, y_0)$$

is an extreme value of h. This implies that $h'(t_0) = 0$, and, by the Chain Rule,

$$h'(t_0) = f_x(x_0, y_0)x'(t_0) + f_y(x_0, y_0)y'(t_0) = \nabla f(x_0, y_0) \cdot \mathbf{r}'(t_0) = 0.$$

So, $\nabla f(x_0, y_0)$ is orthogonal to $\mathbf{r}'(t_0)$. Moreover, by Theorem 13.12, $\nabla g(x_0, y_0)$ is also orthogonal to $\mathbf{r}'(t_0)$. Consequently, the gradients $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel, and there must exist a scalar λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$ ■

NOTE Lagrange's Theorem can be shown to be true for functions of three variables, using a similar argument with level surfaces and Theorem 13.14.

The Method of Lagrange Multipliers uses Theorem 13.19 to find the extreme values of a function f subject to a constraint.

METHOD OF LAGRANGE MULTIPLIERS

Let f and g satisfy the hypothesis of Lagrange's Theorem, and let f have a minimum or maximum subject to the constraint $g(x, y) = c$. To find the minimum or maximum of f, use the following steps.

1. Simultaneously solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = c$ by solving the following system of equations.

$$f_x(x, y) = \lambda g_x(x, y)$$
$$f_y(x, y) = \lambda g_y(x, y)$$
$$g(x, y) = c$$

2. Evaluate f at each solution point obtained in the first step. The largest value yields the maximum of f subject to the constraint $g(x, y) = c$, and the smallest value yields the minimum of f subject to the constraint $g(x, y) = c$.

NOTE As you will see in Examples 1 and 2, the Method of Lagrange Multipliers requires solving systems of nonlinear equations. This often can require some tricky algebraic manipulation.

Constrained Optimization Problems

In the problem at the beginning of this section, you wanted to maximize the area of a rectangle that is inscribed in an ellipse. Example 1 shows how to use Lagrange multipliers to solve this problem.

EXAMPLE 1 Using a Lagrange Multiplier with One Constraint

Find the maximum value of $f(x, y) = 4xy$ where $x > 0$ and $y > 0$, subject to the constraint $(x^2/3^2) + (y^2/4^2) = 1$.

Solution To begin, let

$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

By equating $\nabla f(x, y) = 4y\mathbf{i} + 4x\mathbf{j}$ and $\lambda\nabla g(x, y) = (2\lambda x/9)\mathbf{i} + (\lambda y/8)\mathbf{j}$, you can obtain the following system of equations.

$$4y = \frac{2}{9}\lambda x \qquad f_x(x, y) = \lambda g_x(x, y)$$

$$4x = \frac{1}{8}\lambda y \qquad f_y(x, y) = \lambda g_y(x, y)$$

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1 \qquad \text{Constraint}$$

From the first equation, you obtain $\lambda = 18y/x$, and substitution into the second equation produces

$$4x = \frac{1}{8}\left(\frac{18y}{x}\right)y \quad \Rightarrow \quad x^2 = \frac{9}{16}y^2.$$

Substituting this value for x^2 into the third equation produces

$$\frac{1}{9}\left(\frac{9}{16}y^2\right) + \frac{1}{16}y^2 = 1 \quad \Rightarrow \quad y^2 = 8.$$

So, $y = \pm 2\sqrt{2}$. Because it is required that $y > 0$, choose the positive value and find that

$$\begin{aligned} x^2 &= \frac{9}{16}y^2 \\ &= \frac{9}{16}(8) = \frac{9}{2} \\ x &= \frac{3}{\sqrt{2}}. \end{aligned}$$

So, the maximum value of f is

$$f\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right) = 4xy = 4\left(\frac{3}{\sqrt{2}}\right)\left(2\sqrt{2}\right) = 24.$$ ■

Note that writing the constraint as

$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1 \qquad \text{or} \qquad g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} - 1 = 0$$

does not affect the solution—the constant is eliminated when you form ∇g.

NOTE Example 1 can also be solved using the techniques you learned in Chapter 3. To see how, try to find the maximum value of $A = 4xy$ given that

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

To begin, solve the second equation for y to obtain

$$y = \tfrac{4}{3}\sqrt{9 - x^2}.$$

Then substitute into the first equation to obtain

$$A = 4x\left(\tfrac{4}{3}\sqrt{9 - x^2}\right).$$

Finally, use the techniques of Chapter 3 to maximize A.

EXAMPLE 2 A Business Application

The Cobb-Douglas production function (see Example 5, Section 13.1) for a software manufacturer is given by

$$f(x, y) = 100x^{3/4}y^{1/4} \qquad \text{Objective function}$$

where x represents the units of labor (at \$150 per unit) and y represents the units of capital (at \$250 per unit). The total cost of labor and capital is limited to \$50,000. Find the maximum production level for this manufacturer.

FOR FURTHER INFORMATION For more information on the use of Lagrange multipliers in economics, see the article "Lagrange Multiplier Problems in Economics" by John V. Baxley and John C. Moorhouse in *The American Mathematical Monthly.* To view this article, go to the website *www.matharticles.com*.

Solution From the given function, you have

$$\nabla f(x, y) = 75x^{-1/4}y^{1/4}\mathbf{i} + 25x^{3/4}y^{-3/4}\mathbf{j}.$$

The limit on the cost of labor and capital produces the constraint

$$g(x, y) = 150x + 250y = 50{,}000. \qquad \text{Constraint}$$

So, $\lambda\nabla g(x, y) = 150\lambda\mathbf{i} + 250\lambda\mathbf{j}$. This gives rise to the following system of equations.

$$\begin{aligned} 75x^{-1/4}y^{1/4} &= 150\lambda && f_x(x, y) = \lambda g_x(x, y) \\ 25x^{3/4}y^{-3/4} &= 250\lambda && f_y(x, y) = \lambda g_y(x, y) \\ 150x + 250y &= 50{,}000 && \text{Constraint} \end{aligned}$$

By solving for λ in the first equation

$$\lambda = \frac{75x^{-1/4}y^{1/4}}{150} = \frac{x^{-1/4}y^{1/4}}{2}$$

and substituting into the second equation, you obtain

$$\begin{aligned} 25x^{3/4}y^{-3/4} &= 250\left(\frac{x^{-1/4}y^{1/4}}{2}\right) \\ 25x &= 125y. && \text{Multiply by } x^{1/4}y^{3/4}. \end{aligned}$$

So, $x = 5y$. By substituting into the third equation, you have

$$\begin{aligned} 150(5y) + 250y &= 50{,}000 \\ 1000y &= 50{,}000 \\ y &= 50 \text{ units of capital} \\ x &= 250 \text{ units of labor.} \end{aligned}$$

So, the maximum production level is

$$\begin{aligned} f(250, 50) &= 100(250)^{3/4}(50)^{1/4} \\ &\approx 16{,}719 \text{ product units.} \end{aligned}$$

■

Economists call the Lagrange multiplier obtained in a production function the **marginal productivity of money.** For instance, in Example 2 the marginal productivity of money at $x = 250$ and $y = 50$ is

$$\lambda = \frac{x^{-1/4}y^{1/4}}{2} = \frac{(250)^{-1/4}(50)^{1/4}}{2} \approx 0.334$$

which means that for each additional dollar spent on production, an additional 0.334 unit of the product can be produced.

EXAMPLE 3 Lagrange Multipliers and Three Variables

Find the minimum value of

$$f(x, y, z) = 2x^2 + y^2 + 3z^2 \qquad \text{Objective function}$$

subject to the constraint $2x - 3y - 4z = 49$.

Solution Let $g(x, y, z) = 2x - 3y - 4z = 49$. Then, because

$$\nabla f(x, y, z) = 4x\mathbf{i} + 2y\mathbf{j} + 6z\mathbf{k} \qquad \text{and} \qquad \lambda\nabla g(x, y, z) = 2\lambda\mathbf{i} - 3\lambda\mathbf{j} - 4\lambda\mathbf{k}$$

you obtain the following system of equations.

$$\begin{aligned} 4x &= 2\lambda & & f_x(x, y, z) = \lambda g_x(x, y, z) \\ 2y &= -3\lambda & & f_y(x, y, z) = \lambda g_y(x, y, z) \\ 6z &= -4\lambda & & f_z(x, y, z) = \lambda g_z(x, y, z) \\ 2x - 3y - 4z &= 49 & & \text{Constraint} \end{aligned}$$

The solution of this system is $x = 3$, $y = -9$, and $z = -4$. So, the optimum value of f is

$$\begin{aligned} f(3, -9, -4) &= 2(3)^2 + (-9)^2 + 3(-4)^2 \\ &= 147. \end{aligned}$$

From the original function and constraint, it is clear that $f(x, y, z)$ has no maximum. So, the optimum value of f determined above is a minimum. ■

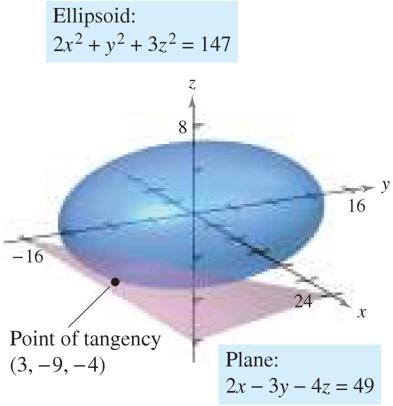


Figure 13.80

A graphical interpretation of constrained optimization problems in two variables was given at the beginning of this section. In three variables, the interpretation is similar, except that level surfaces are used instead of level curves. For instance, in Example 3, the level surfaces of f are ellipsoids centered at the origin, and the constraint

$$2x - 3y - 4z = 49$$

is a plane. The minimum value of f is represented by the ellipsoid that is tangent to the constraint plane, as shown in Figure 13.80.

EXAMPLE 4 Optimization Inside a Region

Find the extreme values of

$$f(x, y) = x^2 + 2y^2 - 2x + 3 \qquad \text{Objective function}$$

subject to the constraint $x^2 + y^2 \leq 10$.

Solution To solve this problem, you can break the constraint into two cases.

a. For points *on the circle* $x^2 + y^2 = 10$, you can use Lagrange multipliers to find that the maximum value of $f(x, y)$ is 24—this value occurs at $(-1, 3)$ and at $(-1, -3)$. In a similar way, you can determine that the minimum value of $f(x, y)$ is approximately 6.675—this value occurs at $(\sqrt{10}, 0)$.

b. For points *inside the circle*, you can use the techniques discussed in Section 13.8 to conclude that the function has a relative minimum of 2 at the point $(1, 0)$.

By combining these two results, you can conclude that f has a maximum of 24 at $(-1, \pm 3)$ and a minimum of 2 at $(1, 0)$, as shown in Figure 13.81. ■

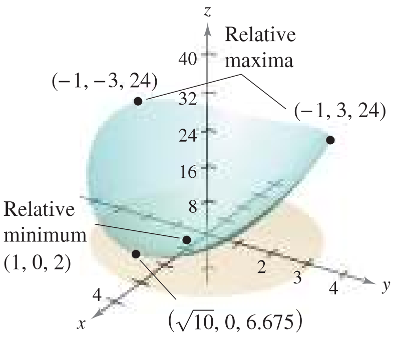


Figure 13.81

The Method of Lagrange Multipliers with Two Constraints

For optimization problems involving *two* constraint functions g and h, you can introduce a second Lagrange multiplier, μ (the lowercase Greek letter mu), and then solve the equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

where the gradient vectors are not parallel, as illustrated in Example 5.

EXAMPLE 5 Optimization with Two Constraints

Let $T(x, y, z) = 20 + 2x + 2y + z^2$ represent the temperature at each point on the sphere $x^2 + y^2 + z^2 = 11$. Find the extreme temperatures on the curve formed by the intersection of the plane $x + y + z = 3$ and the sphere.

Solution The two constraints are

$$g(x, y, z) = x^2 + y^2 + z^2 = 11 \quad \text{and} \quad h(x, y, z) = x + y + z = 3.$$

Using

$$\begin{aligned} \nabla T(x, y, z) &= 2\mathbf{i} + 2\mathbf{j} + 2z\mathbf{k} \\ \lambda \nabla g(x, y, z) &= 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j} + 2\lambda z\mathbf{k} \end{aligned}$$

and

$$\mu \nabla h(x, y, z) = \mu\mathbf{i} + \mu\mathbf{j} + \mu\mathbf{k}$$

you can write the following system of equations.

$$\begin{aligned} 2 &= 2\lambda x + \mu && T_x(x, y, z) = \lambda g_x(x, y, z) + \mu h_x(x, y, z) \\ 2 &= 2\lambda y + \mu && T_y(x, y, z) = \lambda g_y(x, y, z) + \mu h_y(x, y, z) \\ 2z &= 2\lambda z + \mu && T_z(x, y, z) = \lambda g_z(x, y, z) + \mu h_z(x, y, z) \\ x^2 + y^2 + z^2 &= 11 && \text{Constraint 1} \\ x + y + z &= 3 && \text{Constraint 2} \end{aligned}$$

By subtracting the second equation from the first, you can obtain the following system.

STUDY TIP The systems of equations that arise when the Method of Lagrange Multipliers is used are not, in general, linear systems, and finding the solutions often requires ingenuity.

$$\begin{aligned} \lambda(x - y) &= 0 \\ 2z(1 - \lambda) - \mu &= 0 \\ x^2 + y^2 + z^2 &= 11 \\ x + y + z &= 3 \end{aligned}$$

From the first equation, you can conclude that $\lambda = 0$ or $x = y$. If $\lambda = 0$, you can show that the critical points are $(3, -1, 1)$ and $(-1, 3, 1)$. (Try doing this—it takes a little work.) If $\lambda \neq 0$, then $x = y$ and you can show that the critical points occur when $x = y = \left(3 \pm 2\sqrt{3}\right)/3$ and $z = \left(3 \mp 4\sqrt{3}\right)/3$. Finally, to find the optimal solutions, compare the temperatures at the four critical points.

$$\begin{aligned} T(3, -1, 1) = T(-1, 3, 1) &= 25 \\ T\left(\frac{3 - 2\sqrt{3}}{3}, \frac{3 - 2\sqrt{3}}{3}, \frac{3 + 4\sqrt{3}}{3}\right) &= \frac{91}{3} \approx 30.33 \\ T\left(\frac{3 + 2\sqrt{3}}{3}, \frac{3 + 2\sqrt{3}}{3}, \frac{3 - 4\sqrt{3}}{3}\right) &= \frac{91}{3} \approx 30.33 \end{aligned}$$

So, $T = 25$ is the minimum temperature and $T = \frac{91}{3}$ is the maximum temperature on the curve. ■

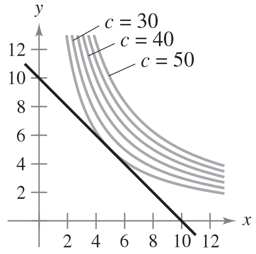
13.10 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, identify the constraint and level curves of the objective function shown in the figure. Use the figure to approximate the indicated extrema, assuming that x and y are positive. Use Lagrange multipliers to verify your result.

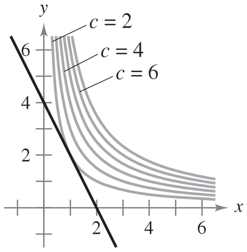
1. Maximize $z = xy$

Constraint: $x + y = 10$



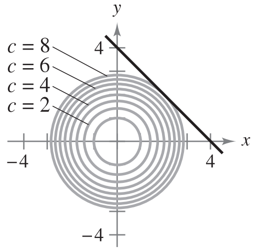
2. Maximize $z = xy$

Constraint: $2x + y = 4$



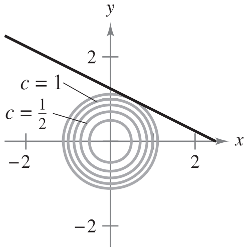
3. Minimize $z = x^2 + y^2$

Constraint: $x + y - 4 = 0$



4. Minimize $z = x^2 + y^2$

Constraint: $2x + 4y = 5$



In Exercises 5–10, use Lagrange multipliers to find the indicated extrema, assuming that x and y are positive.

5. Minimize $f(x, y) = x^2 + y^2$

Constraint: $x + 2y - 5 = 0$

6. Maximize $f(x, y) = x^2 - y^2$

Constraint: $2y - x^2 = 0$

7. Maximize $f(x, y) = 2x + 2xy + y$

Constraint: $2x + y = 100$

8. Minimize $f(x, y) = 3x + y + 10$

Constraint: $x^2y = 6$

9. Maximize $f(x, y) = \sqrt{6 - x^2 - y^2}$

Constraint: $x + y - 2 = 0$

10. Minimize $f(x, y) = \sqrt{x^2 + y^2}$

Constraint: $2x + 4y - 15 = 0$

In Exercises 11–14, use Lagrange multipliers to find the indicated extrema, assuming that x, y, and z are positive.

11. Minimize $f(x, y, z) = x^2 + y^2 + z^2$

Constraint: $x + y + z - 9 = 0$

12. Maximize $f(x, y, z) = xyz$

Constraint: $x + y + z - 3 = 0$

13. Minimize $f(x, y, z) = x^2 + y^2 + z^2$

Constraint: $x + y + z = 1$

14. Minimize $f(x, y) = x^2 - 10x + y^2 - 14y + 28$

Constraint: $x + y = 10$

In Exercises 15 and 16, use Lagrange multipliers to find any extrema of the function subject to the constraint $x^2 + y^2 \le 1$.

15. $f(x, y) = x^2 + 3xy + y^2$

16. $f(x, y) = e^{-xy/4}$

In Exercises 17 and 18, use Lagrange multipliers to find the indicated extrema of f subject to two constraints. In each case, assume that x, y, and z are nonnegative.

17. Maximize $f(x, y, z) = xyz$

Constraints: $x + y + z = 32, \quad x - y + z = 0$

18. Minimize $f(x, y, z) = x^2 + y^2 + z^2$

Constraints: $x + 2z = 6, \quad x + y = 12$

In Exercises 19–28, use Lagrange multipliers to find the minimum distance from the curve or surface to the indicated point. [*Hints:* In Exercise 19, minimize $f(x, y) = x^2 + y^2$ subject to the constraint $x + y = 1$. In Exercise 25, use the *root* feature of a graphing utility.]

Curve	*Point*
19. Line: $x + y = 1$	$(0, 0)$
20. Line: $2x + 3y = -1$	$(0, 0)$
21. Line: $x - y = 4$	$(0, 2)$
22. Line: $x + 4y = 3$	$(1, 0)$
23. Parabola: $y = x^2$	$(0, 3)$
24. Parabola: $y = x^2$	$(-3, 0)$
25. Parabola: $y = x^2 + 1$	$(\frac{1}{2}, 1)$
26. Circle: $(x - 4)^2 + y^2 = 4$	$(0, 10)$
Surface	*Point*
27. Plane: $x + y + z = 1$	$(2, 1, 1)$
28. Cone: $z = \sqrt{x^2 + y^2}$	$(4, 0, 0)$

In Exercises 29 and 30, find the highest point on the curve of intersection of the surfaces.

29. Cone: $x^2 + y^2 - z^2 = 0$, Plane: $x + 2z = 4$

30. Sphere: $x^2 + y^2 + z^2 = 36$, Plane: $2x + y - z = 2$

WRITING ABOUT CONCEPTS

31. Explain what is meant by constrained optimization problems.

32. Explain the Method of Lagrange Multipliers for solving constrained optimization problems.

In Exercises 33–42, use Lagrange multipliers to solve the indicated exercise in Section 13.9.

33. Exercise 1

34. Exercise 2

35. Exercise 5

36. Exercise 6

37. Exercise 9

38. Exercise 10

39. Exercise 11

40. Exercise 12

41. Exercise 17

42. Exercise 18

43. ***Maximum Volume*** Use Lagrange multipliers to find the dimensions of a rectangular box of maximum volume that can be inscribed (with edges parallel to the coordinate axes) in the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$.

CAPSTONE

44. The sum of the length and the girth (perimeter of a cross section) of a package carried by a delivery service cannot exceed 108 inches.

(a) Determine whether Lagrange multipliers can be used to find the dimensions of the rectangular package of largest volume that may be sent. Explain your reasoning.

(b) If Lagrange multipliers can be used, find the dimensions. Compare your answer with that obtained in Exercise 38, Section 13.9.

45. ***Minimum Cost*** A cargo container (in the shape of a rectangular solid) must have a volume of 480 cubic feet. The bottom will cost \$5 per square foot to construct and the sides and the top will cost \$3 per square foot to construct. Use Lagrange multipliers to find the dimensions of the container of this size that has minimum cost.

46. ***Geometric and Arithmetic Means***

(a) Use Lagrange multipliers to prove that the product of three positive numbers x, y, and z, whose sum has the constant value S, is a maximum when the three numbers are equal. Use this result to prove that $\sqrt[3]{xyz} \le (x + y + z)/3$.

(b) Generalize the result of part (a) to prove that the product $x_1 x_2 x_3 \cdot \cdot \cdot x_n$ is a maximum when $x_1 = x_2 = x_3 = \cdot \cdot \cdot = x_n$, $\sum_{i=1}^{n} x_i = S$, and all $x_i \ge 0$. Then prove that

$$\sqrt[n]{x_1 x_2 x_3 \cdot \cdot \cdot x_n} \le \frac{x_1 + x_2 + x_3 + \cdot \cdot \cdot + x_n}{n}.$$

This shows that the geometric mean is never greater than the arithmetic mean.

47. ***Minimum Surface Area*** Use Lagrange multipliers to find the dimensions of a right circular cylinder with volume V_0 cubic units and minimum surface area.

48. ***Temperature Distribution*** Let $T(x, y, z) = 100 + x^2 + y^2$ represent the temperature at each point on the sphere $x^2 + y^2 + z^2 = 50$. Find the maximum temperature on the curve formed by the intersection of the sphere and the plane $x - z = 0$.

49. ***Refraction of Light*** When light waves traveling in a transparent medium strike the surface of a second transparent medium, they tend to "bend" in order to follow the path of minimum time. This tendency is called *refraction* and is described by **Snell's Law of Refraction,**

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

where θ_1 and θ_2 are the magnitudes of the angles shown in the figure, and v_1 and v_2 are the velocities of light in the two media. Use Lagrange multipliers to derive this law using $x + y = a$.

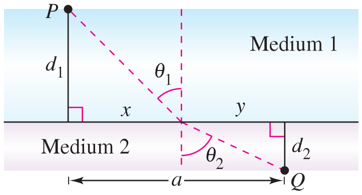


Figure for 49

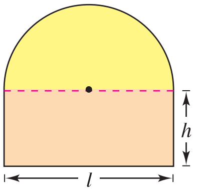


Figure for 50

50. ***Area and Perimeter*** A semicircle is on top of a rectangle (see figure). If the area is fixed and the perimeter is a minimum, or if the perimeter is fixed and the area is a maximum, use Lagrange multipliers to verify that the length of the rectangle is twice its height.

Production Level **In Exercises 51 and 52, find the maximum production level P if the total cost of labor (at \$72 per unit) and capital (at \$60 per unit) is limited to \$250,000, where x is the number of units of labor and y is the number of units of capital.**

51. $P(x, y) = 100x^{0.25}y^{0.75}$

52. $P(x, y) = 100x^{0.4}y^{0.6}$

Cost **In Exercises 53 and 54, find the minimum cost of producing 50,000 units of a product, where x is the number of units of labor (at \$72 per unit) and y is the number of units of capital (at \$60 per unit).**

53. $P(x, y) = 100x^{0.25}y^{0.75}$

54. $P(x, y) = 100x^{0.6}y^{0.4}$

55. ***Investigation*** Consider the objective function $g(\alpha, \beta, \gamma) = \cos \alpha \cos \beta \cos \gamma$ subject to the constraint that α, β, and γ are the angles of a triangle.

(a) Use Lagrange multipliers to maximize g.

CAS (b) Use the constraint to reduce the function g to a function of two independent variables. Use a computer algebra system to graph the surface represented by g. Identify the maximum values on the graph.

PUTNAM EXAM CHALLENGE

56. A can buoy is to be made of three pieces, namely, a cylinder and two equal cones, the altitude of each cone being equal to the altitude of the cylinder. For a given area of surface, what shape will have the greatest volume?

This problem was composed by the Committee on the Putnam Prize Competition.
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13 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, sketch the graph of the level surface $f(x, y, z) = c$ at the given value of c.

1. $f(x, y, z) = x^2 - y + z^2, \quad c = 2$

2. $f(x, y, z) = 4x^2 - y^2 + 4z^2, \quad c = 0$

3. ***Conjecture*** Consider the function $f(x, y) = x^2 + y^2$.

(a) Sketch the graph of the surface given by f.

(b) Make a conjecture about the relationship between the graphs of f and $g(x, y) = f(x, y) + 2$. Explain your reasoning.

(c) Make a conjecture about the relationship between the graphs of f and $g(x, y) = f(x, y - 2)$. Explain your reasoning.

(d) On the surface in part (a), sketch the graphs of $z = f(1, y)$ and $z = f(x, 1)$.

4. ***Conjecture*** Consider the function

$f(x, y) = \sqrt{1 - x^2 - y^2}$.

(a) Sketch the graph of the surface given by f.

(b) Make a conjecture about the relationship between the graphs of f and $g(x, y) = f(x + 2, y)$. Explain your reasoning.

(c) Make a conjecture about the relationship between the graphs of f and $g(x, y) = 4 - f(x, y)$. Explain your reasoning.

(d) On the surface in part (a), sketch the graphs of $z = f(0, y)$ and $z = f(x, 0)$.

CAS **In Exercises 5–8, use a computer algebra system to graph several level curves of the function.**

5. $f(x, y) = e^{x^2 + y^2}$ **6.** $f(x, y) = \ln xy$

7. $f(x, y) = x^2 - y^2$ **8.** $f(x, y) = \dfrac{x}{x + y}$

CAS **In Exercises 9 and 10, use a computer algebra system to graph the function.**

9. $f(x, y) = e^{-(x^2 + y^2)}$ **10.** $g(x, y) = |y|^{1 + |x|}$

In Exercises 11–14, find the limit and discuss the continuity of the function (if it exists).

11. $\displaystyle\lim_{(x, y) \to (1, 1)} \frac{xy}{x^2 + y^2}$ **12.** $\displaystyle\lim_{(x, y) \to (1, 1)} \frac{xy}{x^2 - y^2}$

13. $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{y + xe^{-y^2}}{1 + x^2}$ **14.** $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{x^2 y}{x^4 + y^2}$

In Exercises 15–24, find all first partial derivatives.

15. $f(x, y) = e^x \cos y$ **16.** $f(x, y) = \dfrac{xy}{x + y}$

17. $z = e^{-y} + e^{-x}$ **18.** $z = \ln(x^2 + y^2 + 1)$

19. $g(x, y) = \dfrac{xy}{x^2 + y^2}$ **20.** $w = \sqrt{x^2 - y^2 - z^2}$

21. $f(x, y, z) = z \arctan \dfrac{y}{x}$

22. $f(x, y, z) = \dfrac{1}{\sqrt{1 + x^2 + y^2 + z^2}}$

23. $u(x, t) = ce^{-n^2 t} \sin nx$ **24.** $u(x, t) = c \sin(akx) \cos kt$

25. ***Think About It*** Sketch a graph of a function $z = f(x, y)$ whose derivative f_x is always negative and whose derivative f_y is always negative.

26. Find the slopes of the surface $z = x^2 \ln(y + 1)$ in the x- and y-directions at the point $(2, 0, 0)$.

In Exercises 27–30, find all second partial derivatives and verify that the second mixed partials are equal.

27. $f(x, y) = 3x^2 - xy + 2y^3$ **28.** $h(x, y) = \dfrac{x}{x + y}$

29. $h(x, y) = x \sin y + y \cos x$ **30.** $g(x, y) = \cos(x - 2y)$

Laplace's Equation **In Exercises 31–34, show that the function satisfies Laplace's equation**

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

31. $z = x^2 - y^2$ **32.** $z = x^3 - 3xy^2$

33. $z = \dfrac{y}{x^2 + y^2}$ **34.** $z = e^y \sin x$

In Exercises 35 and 36, find the total differential.

35. $z = x \sin xy$ **36.** $z = \dfrac{xy}{\sqrt{x^2 + y^2}}$

37. ***Error Analysis*** The legs of a right triangle are measured to be 5 centimeters and 12 centimeters, with a possible error of $\frac{1}{2}$ centimeter. Approximate the maximum possible error in computing the length of the hypotenuse. Approximate the maximum percent error.

38. ***Error Analysis*** To determine the height of a tower, the angle of elevation to the top of the tower is measured from a point 100 feet $\pm\, \frac{1}{2}$ foot from the base. The angle is measured at 33°, with a possible error of 1°. Assuming that the ground is horizontal, approximate the maximum error in determining the height of the tower.

39. ***Volume*** A right circular cone is measured, and the radius and height are found to be 2 inches and 5 inches, respectively. The possible error in measurement is $\frac{1}{8}$ inch. Approximate the maximum possible error in the computation of the volume.

40. ***Lateral Surface Area*** Approximate the error in the computation of the lateral surface area of the cone in Exercise 39. (The lateral surface area is given by $A = \pi r \sqrt{r^2 + h^2}$.)

In Exercises 41–44, find the indicated derivatives (a) using the appropriate Chain Rule and (b) using substitution before differentiating.

41. $w = \ln(x^2 + y), \quad \dfrac{dw}{dt}$

$x = 2t, \quad y = 4 - t$

42. $u = y^2 - x, \quad \dfrac{du}{dt}$

$x = \cos t, \quad y = \sin t$

43. $w = \dfrac{xy}{z}, \quad \dfrac{\partial w}{\partial r}, \quad \dfrac{\partial w}{\partial t}$

$x = 2r + t, \quad y = rt, \quad z = 2r - t$

44. $u = x^2 + y^2 + z^2, \quad \dfrac{\partial u}{\partial r}, \quad \dfrac{\partial u}{\partial t}$

$x = r\cos t, \quad y = r\sin t, \quad z = t$

In Exercises 45 and 46, differentiate implicitly to find the first partial derivatives of *z*.

45. $x^2 + xy + y^2 + yz + z^2 = 0$ **46.** $xz^2 - y\sin z = 0$

In Exercises 47–50, find the directional derivative of the function at *P* in the direction of v.

47. $f(x, y) = x^2y, \quad (-5, 5), \quad \mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

48. $f(x, y) = \frac{1}{4}y^2 - x^2, \quad (1, 4), \quad \mathbf{v} = 2\mathbf{i} + \mathbf{j}$

49. $w = y^2 + xz, \quad (1, 2, 2), \quad \mathbf{v} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

50. $w = 5x^2 + 2xy - 3y^2z, \quad (1, 0, 1), \quad \mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

In Exercises 51–54, find the gradient of the function and the maximum value of the directional derivative at the given point.

51. $z = x^2y, \quad (2, 1)$ **52.** $z = e^{-x}\cos y, \quad \left(0, \dfrac{\pi}{4}\right)$

53. $z = \dfrac{y}{x^2 + y^2}, \quad (1, 1)$ **54.** $z = \dfrac{x^2}{x - y}, \quad (2, 1)$

In Exercises 55 and 56, (a) find the gradient of the function at *P*, (b) find a unit normal vector to the level curve $f(x, y) = c$ at *P*, (c) find the tangent line to the level curve $f(x, y) = c$ at *P*, and (d) sketch the level curve, the unit normal vector, and the tangent line in the *xy*-plane.

55. $f(x, y) = 9x^2 - 4y^2$ **56.** $f(x, y) = 4y\sin x - y$

$c = 65, \quad P(3, 2)$ $c = 3, \quad P\left(\dfrac{\pi}{2}, 1\right)$

In Exercises 57–60, find an equation of the tangent plane and parametric equations of the normal line to the surface at the given point.

	Surface	*Point*
57.	$f(x, y) = x^2y$	$(2, 1, 4)$
58.	$f(x, y) = \sqrt{25 - y^2}$	$(2, 3, 4)$
59.	$z = -9 + 4x - 6y - x^2 - y^2$	$(2, -3, 4)$
60.	$z = \sqrt{9 - x^2 - y^2}$	$(1, 2, 2)$

In Exercises 61 and 62, find symmetric equations of the tangent line to the curve of intersection of the surfaces at the given point.

	Surfaces	*Point*
61.	$z = 9 - y^2, \quad y = x$	$(2, 2, 5)$
62.	$z = x^2 - y^2, \quad z = 3$	$(2, 1, 3)$

63. Find the angle of inclination θ of the tangent plane to the surface $x^2 + y^2 + z^2 = 14$ at the point $(2, 1, 3)$.

64. ***Approximation*** Consider the following approximations for a function $f(x, y)$ centered at $(0, 0)$.

Linear approximation:

$$P_1(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y$$

Quadratic approximation:

$$P_2(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \tfrac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \tfrac{1}{2}f_{yy}(0, 0)y^2$$

[Note that the linear approximation is the tangent plane to the surface at $(0, 0, f(0, 0))$.]

(a) Find the linear approximation of $f(x, y) = \cos x + \sin y$ centered at $(0, 0)$.

(b) Find the quadratic approximation of $f(x, y) = \cos x + \sin y$ centered at $(0, 0)$.

(c) If $y = 0$ in the quadratic approximation, you obtain the second-degree Taylor polynomial for what function?

(d) Complete the table.

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0			
0	0.1			
0.2	0.1			
0.5	0.3			
1	0.5			

CAS (e) Use a computer algebra system to graph the surfaces $z = f(x, y)$, $z = P_1(x, y)$, and $z = P_2(x, y)$. How does the accuracy of the approximations change as the distance from $(0, 0)$ increases?

CAS **In Exercises 65–68, examine the function for relative extrema and saddle points. Use a computer algebra system to graph the function and confirm your results.**

65. $f(x, y) = 2x^2 + 6xy + 9y^2 + 8x + 14$

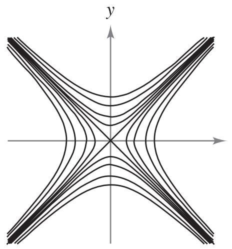
66. $f(x, y) = x^2 + 3xy + y^2 - 5x$

67. $f(x, y) = xy + \dfrac{1}{x} + \dfrac{1}{y}$

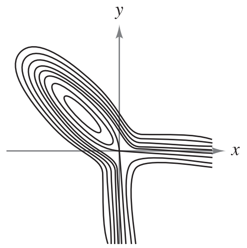
68. $z = 50(x + y) - (0.1x^3 + 20x + 150) - (0.05y^3 + 20.6y + 125)$

Writing **In Exercises 69 and 70, write a short paragraph about the surface whose level curves (*c*-values evenly spaced) are shown. Comment on possible extrema, saddle points, the magnitude of the gradient, etc.**

69.



70.



71. ***Maximum Profit*** A corporation manufactures digital cameras at two locations. The cost functions for producing x_1 units at location 1 and x_2 units at location 2 are

$$C_1 = 0.05x_1^2 + 15x_1 + 5400$$
$$C_2 = 0.03x_2^2 + 15x_2 + 6100$$

and the total revenue function is

$$R = [225 - 0.4(x_1 + x_2)](x_1 + x_2).$$

Find the production levels at the two locations that will maximize the profit $P(x_1, x_2) = R - C_1 - C_2$.

72. ***Minimum Cost*** A manufacturer has an order for 1000 units of wooden benches that can be produced at two locations. Let x_1 and x_2 be the numbers of units produced at the two locations. The cost function is

$$C = 0.25x_1^2 + 10x_1 + 0.15x_2^2 + 12x_2.$$

Find the number that should be produced at each location to meet the order and minimize cost.

73. ***Production Level*** The production function for a candy manufacturer is

$$f(x, y) = 4x + xy + 2y$$

where x is the number of units of labor and y is the number of units of capital. Assume that the total amount available for labor and capital is \$2000, and that units of labor and capital cost \$20 and \$4, respectively. Find the maximum production level for this manufacturer.

74. Find the minimum distance from the point $(2, 2, 0)$ to the surface $z = x^2 + y^2$.

75. ***Modeling Data*** The table shows the drag force y in kilograms for a motor vehicle at indicated speeds x in kilometers per hour.

Speed, *x*	25	50	75	100	125
Drag, *y*	24	34	50	71	98

(a) Use the regression capabilities of a graphing utility to find the least squares regression quadratic for the data.

(b) Use the model to estimate the total drag when the vehicle is moving at 80 kilometers per hour.

76. ***Modeling Data*** The data in the table show the yield y (in milligrams) of a chemical reaction after t minutes.

Minutes, *t*	1	2	3	4
Yield, *y*	1.2	7.1	9.9	13.1

Minutes, *t*	5	6	7	8
Yield, *y*	15.5	16.0	17.9	18.0

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data. Then use the graphing utility to plot the data and graph the model.

(b) Use a graphing utility to plot the points $(\ln t, y)$. Do these points appear to follow a linear pattern more closely than the plot of the given data in part (a)?

(c) Use the regression capabilities of a graphing utility to find the least squares regression line for the points $(\ln t, y)$ and obtain the logarithmic model $y = a + b \ln t$.

(d) Use a graphing utility to plot the data and graph the linear and logarithmic models. Which is a better model? Explain.

In Exercises 77 and 78, use Lagrange multipliers to locate and classify any extrema of the function.

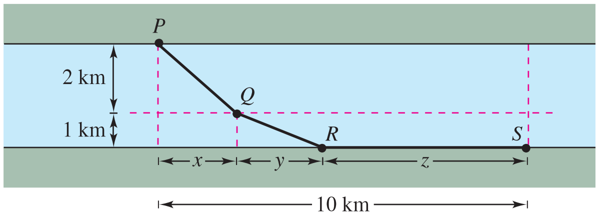
77. $w = xy + yz + xz$

Constraint: $x + y + z = 1$

78. $z = x^2y$

Constraint: $x + 2y = 2$

79. ***Minimum Cost*** A water line is to be built from point P to point S and must pass through regions where construction costs differ (see figure). The cost per kilometer in dollars is $3k$ from P to Q, $2k$ from Q to R, and k from R to S. For simplicity, let $k = 1$. Use Lagrange multipliers to find x, y, and z such that the total cost C will be minimized.



CAS **80.** ***Investigation*** Consider the objective function $f(x, y) = ax + by$ subject to the constraint $x^2/64 + y^2/36 = 1$. Assume that x and y are positive.

(a) Use a computer algebra system to graph the constraint. For $a = 4$ and $b = 3$, use the computer algebra system to graph the level curves of the objective function. By trial and error, find the level curve that appears to be tangent to the ellipse. Use the result to approximate the maximum of f subject to the constraint.

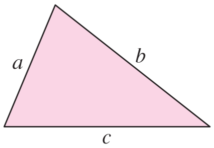
(b) Repeat part (a) for $a = 4$ and $b = 9$.

P.S. PROBLEM SOLVING

1. **Heron's Formula** states that the area of a triangle with sides of lengths a, b, and c is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \dfrac{a+b+c}{2}$, as shown in the figure.

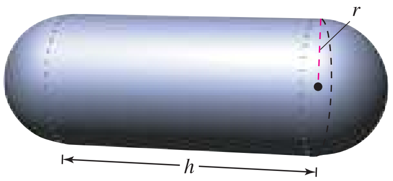


(a) Use Heron's Formula to find the area of the triangle with vertices $(0, 0)$, $(3, 4)$, and $(6, 0)$.

(b) Show that among all triangles having a fixed perimeter, the triangle with the largest area is an equilateral triangle.

(c) Show that among all triangles having a fixed area, the triangle with the smallest perimeter is an equilateral triangle.

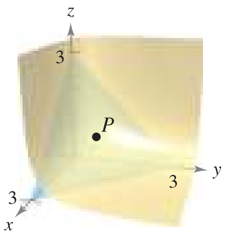
2. An industrial container is in the shape of a cylinder with hemispherical ends, as shown in the figure. The container must hold 1000 liters of fluid. Determine the radius r and length h that minimize the amount of material used in the construction of the tank.



3. Let $P(x_0, y_0, z_0)$ be a point in the first octant on the surface $xyz = 1$.

(a) Find the equation of the tangent plane to the surface at the point P.

(b) Show that the volume of the tetrahedron formed by the three coordinate planes and the tangent plane is constant, independent of the point of tangency (see figure).



4. Use a graphing utility to graph the functions $f(x) = \sqrt[3]{x^3 - 1}$ and $g(x) = x$ in the same viewing window.

(a) Show that

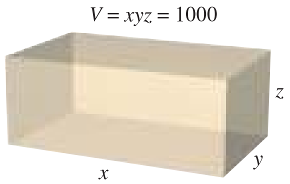
$$\lim_{x\to\infty}[f(x) - g(x)] = 0 \text{ and } \lim_{x\to-\infty}[f(x) - g(x)] = 0.$$

(b) Find the point on the graph of f that is farthest from the graph of g.

5. (a) Let $f(x, y) = x - y$ and $g(x, y) = x^2 + y^2 = 4$. Graph various level curves of f and the constraint g in the xy-plane. Use the graph to determine the maximum value of f subject to the constraint $g = 4$. Then verify your answer using Lagrange multipliers.

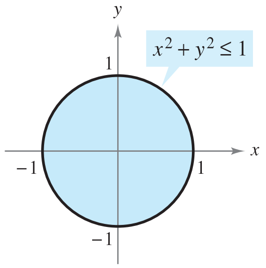
(b) Let $f(x, y) = x - y$ and $g(x, y) = x^2 + y^2 = 0$. Find the maximum and minimum values of f subject to the constraint $g = 0$. Does the method of Lagrange Multipliers work in this case? Explain.

6. A heated storage room has the shape of a rectangular box and has a volume of 1000 cubic feet, as shown in the figure. Because warm air rises, the heat loss per unit of area through the ceiling is five times as great as the heat loss through the floor. The heat loss through the four walls is three times as great as the heat loss through the floor. Determine the room dimensions that will minimize heat loss and therefore minimize heating costs.



7. Repeat Exercise 6 assuming that the heat loss through the walls and ceiling remain the same, but the floor is insulated so that there is no heat loss through the floor.

8. Consider a circular plate of radius 1 given by $x^2 + y^2 \le 1$, as shown in the figure. The temperature at any point $P(x, y)$ on the plate is $T(x, y) = 2x^2 + y^2 - y + 10$.



(a) Sketch the isotherm $T(x, y) = 10$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(b) Find the hottest and coldest points on the plate.

9. Consider the Cobb-Douglas production function

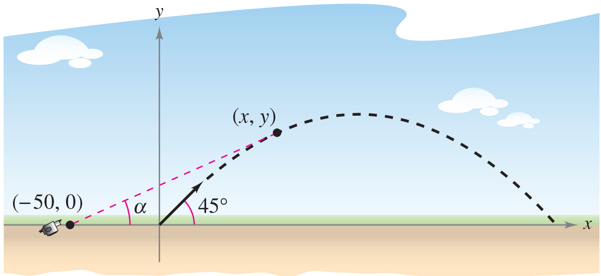
$$f(x, y) = Cx^a y^{1-a}, \quad 0 < a < 1.$$

(a) Show that f satisfies the equation $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = f$.

(b) Show that $f(tx, ty) = tf(x, y)$.

10. Rewrite Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ in cylindrical coordinates.

11. A projectile is launched at an angle of 45° with the horizontal and with an initial velocity of 64 feet per second. A television camera is located in the plane of the path of the projectile 50 feet behind the launch site (see figure).



(a) Find parametric equations for the path of the projectile in terms of the parameter t representing time.

(b) Write the angle α that the camera makes with the horizontal in terms of x and y and in terms of t.

(c) Use the results of part (b) to find $d\alpha/dt$.

(d) Use a graphing utility to graph α in terms of t. Is the graph symmetric to the axis of the parabolic arch of the projectile? At what time is the rate of change of α greatest?

(e) At what time is the angle α maximum? Does this occur when the projectile is at its greatest height?

12. Consider the distance d between the launch site and the projectile in Exercise 11.

(a) Write the distance d in terms of x and y and in terms of the parameter t.

(b) Use the results of part (a) to find the rate of change of d.

(c) Find the rate of change of the distance when $t = 2$.

(d) When is the rate of change of d minimum during the flight of the projectile? Does this occur at the time when the projectile reaches its maximum height?

CAS **13.** Consider the function

$$f(x, y) = (\alpha x^2 + \beta y^2)e^{-(x^2+y^2)}, \quad 0 < |\alpha| < \beta.$$

(a) Use a computer algebra system to graph the function for $\alpha = 1$ and $\beta = 2$, and identify any extrema or saddle points.

(b) Use a computer algebra system to graph the function for $\alpha = -1$ and $\beta = 2$, and identify any extrema or saddle points.

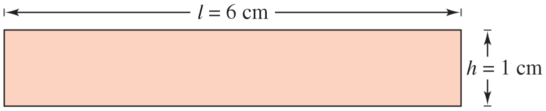
(c) Generalize the results in parts (a) and (b) for the function f.

14. Prove that if f is a differentiable function such that

$$\nabla f(x_0, y_0) = \mathbf{0}$$

then the tangent plane at (x_0, y_0) is horizontal.

15. The figure shows a rectangle that is approximately $l = 6$ centimeters long and $h = 1$ centimeter high.



(a) Draw a rectangular strip along the rectangular region showing a small increase in length.

(b) Draw a rectangular strip along the rectangular region showing a small increase in height.

(c) Use the results in parts (a) and (b) to identify the measurement that has more effect on the area A of the rectangle.

(d) Verify your answer in part (c) analytically by comparing the value of dA when $dl = 0.01$ and when $dh = 0.01$.

16. Consider converting a point $(5 \pm 0.05, \pi/18 \pm 0.05)$ in polar coordinates to rectangular coordinates (x, y).

(a) Use a geometric argument to determine whether the accuracy in x is more dependent on the accuracy in r or on the accuracy in θ. Explain. Verify your answer analytically.

(b) Use a geometric argument to determine whether the accuracy in y is more dependent on the accuracy in r or on the accuracy in θ. Explain. Verify your answer analytically.

17. Let f be a differentiable function of one variable. Show that all tangent planes to the surface $z = y\,f(x/y)$ intersect in a common point.

18. Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

that encloses the circle $x^2 + y^2 = 2x$. Find values of a and b that minimize the area of the ellipse.

19. Show that

$$u(x, t) = \frac{1}{2}[\sin(x - t) + \sin(x + t)]$$

is a solution to the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

20. Show that

$$u(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)]$$

is a solution to the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

(This equation describes the small transverse vibration of an elastic string such as those on certain musical instruments.)

14 Multiple Integration

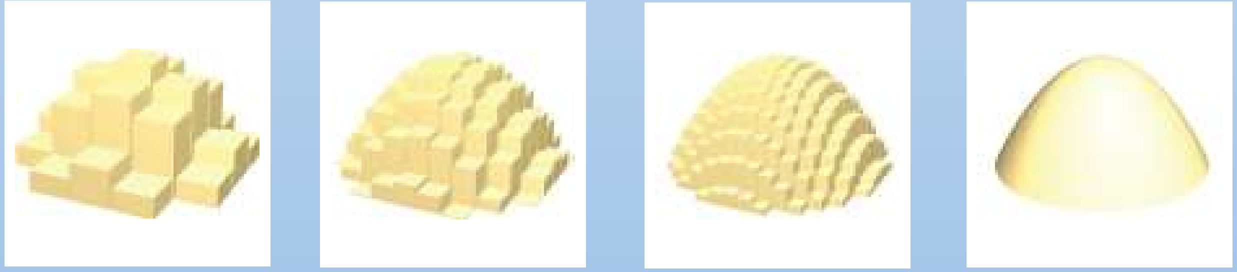
This chapter introduces the concepts of double integrals over regions in the plane and triple integrals over regions in space.

In this chapter, you should learn the following.

- How to evaluate an iterated integral and find the area of a plane region. (14.1)
- How to use a double integral to find the volume of a solid region. (14.2)
- How to write and evaluate double integrals in polar coordinates. (14.3)
- How to find the mass of a planar lamina, the center of mass of a planar lamina, and moments of inertia using double integrals. (14.4)
- How to use a double integral to find the area of a surface. (14.5)
- How to use a triple integral to find the volume, center of mass, and moments of inertia of a solid region. (14.6)
- How to write and evaluate triple integrals in cylindrical and spherical coordinates. (14.7)
- How to use a Jacobian to change variables in a double integral. (14.8)

Langley Photography/Getty Images

The center of pressure on a sail is that point at which the total aerodynamic force may be assumed to act. Letting the sail be represented by a plane region, how can you use double integrals to find the center of pressure on a sail? (See Section 14.4, Section Project.)



You can approximate the volume of a solid region by finding the sum of the volumes of representative rectangular prisms. As you increase the number of rectangular prisms, the approximation tends to become more and more accurate. In Chapter 14, you will learn how to use multiple integrals to find the volume of a solid region.

14.1 Iterated Integrals and Area in the Plane

- **Evaluate an iterated integral.**
- **Use an iterated integral to find the area of a plane region.**

Iterated Integrals

NOTE In Chapters 14 and 15, you will study several applications of integration involving functions of several variables. Chapter 14 is much like Chapter 7 in that it surveys the use of integration to find plane areas, volumes, surface areas, moments, and centers of mass.

In Chapter 13, you saw that it is meaningful to differentiate functions of several variables with respect to one variable while holding the other variables constant. You can *integrate* functions of several variables by a similar procedure. For example, if you are given the partial derivative

$$f_x(x, y) = 2xy$$

then, by considering y constant, you can integrate with respect to x to obtain

$$\begin{aligned}
f(x, y) &= \int f_x(x, y)\,dx && \text{Integrate with respect to } x.\\
&= \int 2xy\,dx && \text{Hold } y \text{ constant.}\\
&= y\int 2x\,dx && \text{Factor out constant } y.\\
&= y(x^2) + C(y) && \text{Antiderivative of } 2x \text{ is } x^2.\\
&= x^2y + C(y). && C(y) \text{ is a function of } y.
\end{aligned}$$

The "constant" of integration, $C(y)$, is a function of y. In other words, by integrating with respect to x, you are able to recover $f(x, y)$ only partially. The total recovery of a function of x and y from its partial derivatives is a topic you will study in Chapter 15. For now, we are more concerned with extending definite integrals to functions of several variables. For instance, by considering y constant, you can apply the Fundamental Theorem of Calculus to evaluate

$$\int_1^{2y} 2xy\,\underbrace{dx}_{\substack{x \text{ is the variable}\\ \text{of integration}\\ \text{and } y \text{ is fixed.}}} = x^2y\Big]_1^{2y} = \underbrace{(2y)^2y - (1)^2y}_{\substack{\text{Replace } x \text{ by}\\ \text{the limits of}\\ \text{integration.}}} = \underbrace{4y^3 - y}_{\substack{\text{The result is}\\ \text{a function}\\ \text{of } y.}}.$$

Similarly, you can integrate with respect to y by holding x fixed. Both procedures are summarized as follows.

$$\int_{h_1(y)}^{h_2(y)} f_x(x, y)\,dx = f(x, y)\Big]_{h_1(y)}^{h_2(y)} = f(h_2(y), y) - f(h_1(y), y) \qquad \text{With respect to } x$$

$$\int_{g_1(x)}^{g_2(x)} f_y(x, y)\,dy = f(x, y)\Big]_{g_1(x)}^{g_2(x)} = f(x, g_2(x)) - f(x, g_1(x)) \qquad \text{With respect to } y$$

Note that the variable of integration cannot appear in either limit of integration. For instance, it makes no sense to write

$$\int_0^x y\,dx.$$

EXAMPLE 1 Integrating with Respect to y

Evaluate $\displaystyle\int_1^x (2x^2y^{-2} + 2y)\,dy$.

Solution Considering x to be constant and integrating with respect to y produces

$$\begin{aligned}\int_1^x (2x^2y^{-2} + 2y)\,dy &= \left[\frac{-2x^2}{y} + y^2\right]_1^x && \text{Integrate with respect to } y.\\ &= \left(\frac{-2x^2}{x} + x^2\right) - \left(\frac{-2x^2}{1} + 1\right)\\ &= 3x^2 - 2x - 1.\end{aligned}$$

■

Notice in Example 1 that the integral defines a function of x and can *itself* be integrated, as shown in the next example.

EXAMPLE 2 The Integral of an Integral

Evaluate $\displaystyle\int_1^2 \left[\int_1^x (2x^2y^{-2} + 2y)\,dy\right] dx$.

Solution Using the result of Example 1, you have

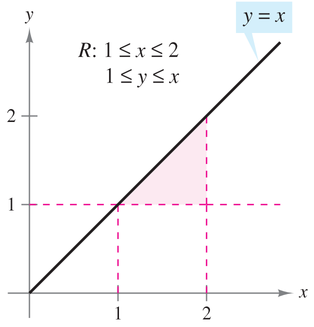
$$\begin{aligned}\int_1^2 \left[\int_1^x (2x^2y^{-2} + 2y)\,dy\right] dx &= \int_1^2 (3x^2 - 2x - 1)\,dx\\ &= \Big[x^3 - x^2 - x\Big]_1^2 && \text{Integrate with respect to } x.\\ &= 2 - (-1)\\ &= 3.\end{aligned}$$

■

The integral in Example 2 is an **iterated integral.** The brackets used in Example 2 are normally not written. Instead, iterated integrals are usually written simply as

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx \quad \text{and} \quad \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

The **inside limits of integration** can be variable with respect to the outer variable of integration. However, the **outside limits of integration** *must be* constant with respect to both variables of integration. After performing the inside integration, you obtain a "standard" definite integral, and the second integration produces a real number. The limits of integration for an iterated integral identify two sets of boundary intervals for the variables. For instance, in Example 2, the outside limits indicate that x lies in the interval $1 \le x \le 2$ and the inside limits indicate that y lies in the interval $1 \le y \le x$. Together, these two intervals determine the **region of integration R** of the iterated integral, as shown in Figure 14.1.

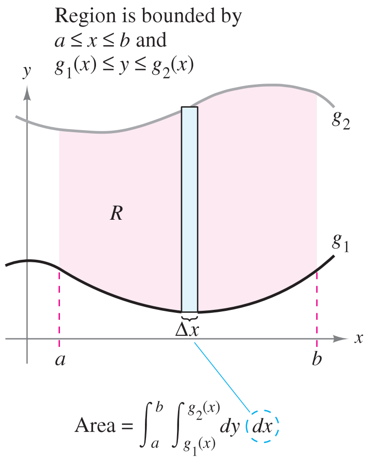


The region of integration for $\displaystyle\int_1^2 \int_1^x f(x, y)\,dy\,dx$

Figure 14.1

Because an iterated integral is just a special type of definite integral—one in which the integrand is also an integral—you can use the properties of definite integrals to evaluate iterated integrals.

Area of a Plane Region



Vertically simple region
Figure 14.2

In the remainder of this section, you will take a new look at an old problem—that of finding the area of a plane region. Consider the plane region R bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, as shown in Figure 14.2. The area of R is given by the definite integral

$$\int_a^b [g_2(x) - g_1(x)]\, dx. \qquad \text{Area of } R$$

Using the Fundamental Theorem of Calculus, you can rewrite the integrand $g_2(x) - g_1(x)$ as a definite integral. Specifically, if you consider x to be fixed and let y vary from $g_1(x)$ to $g_2(x)$, you can write

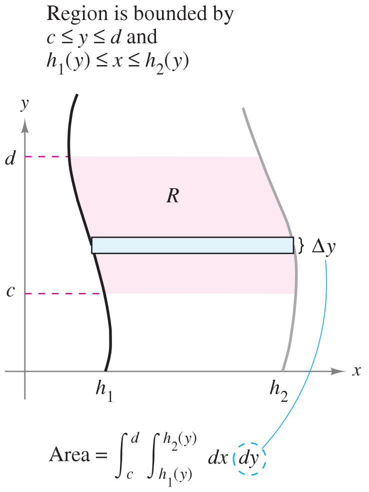
$$\int_{g_1(x)}^{g_2(x)} dy = y\Big]_{g_1(x)}^{g_2(x)} = g_2(x) - g_1(x).$$

Combining these two integrals, you can write the area of the region R as an iterated integral

$$\begin{aligned}\int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx &= \int_a^b y\Big]_{g_1(x)}^{g_2(x)} dx \qquad \text{Area of } R\\ &= \int_a^b [g_2(x) - g_1(x)]\, dx.\end{aligned}$$

Placing a representative rectangle in the region R helps determine both the order and the limits of integration. A vertical rectangle implies the order $dy\, dx$, with the inside limits corresponding to the upper and lower bounds of the rectangle, as shown in Figure 14.2. This type of region is called **vertically simple,** because the outside limits of integration represent the vertical lines $x = a$ and $x = b$.

Similarly, a horizontal rectangle implies the order $dx\, dy$, with the inside limits determined by the left and right bounds of the rectangle, as shown in Figure 14.3. This type of region is called **horizontally simple,** because the outside limits represent the horizontal lines $y = c$ and $y = d$. The iterated integrals used for these two types of simple regions are summarized as follows.



Horizontally simple region
Figure 14.3

AREA OF A REGION IN THE PLANE

1. If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then the area of R is given by

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx. \qquad \text{Figure 14.2 (vertically simple)}$$

2. If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then the area of R is given by

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy. \qquad \text{Figure 14.3 (horizontally simple)}$$

NOTE Be sure you see that the orders of integration of these two integrals are different—the order $dy\, dx$ corresponds to a vertically simple region, and the order $dx\, dy$ corresponds to a horizontally simple region. ■

If all four limits of integration happen to be constants, the region of integration is rectangular, as shown in Example 3.

EXAMPLE 3 The Area of a Rectangular Region

Use an iterated integral to represent the area of the rectangle shown in Figure 14.4.

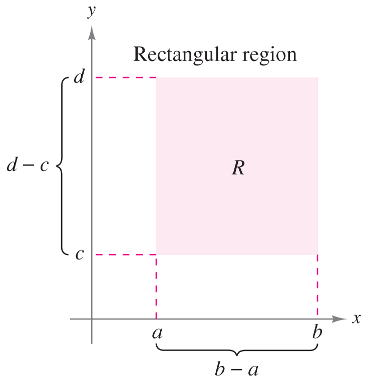


Figure 14.4

Solution The region shown in Figure 14.4 is both vertically simple and horizontally simple, so you can use either order of integration. By choosing the order *dy dx*, you obtain the following.

$$\begin{aligned}\int_a^b\int_c^d dy\,dx &= \int_a^b y\Big]_c^d dx && \text{Integrate with respect to } y.\\ &= \int_a^b (d-c)\,dx\\ &= \Big[(d-c)x\Big]_a^b && \text{Integrate with respect to } x.\\ &= (d-c)(b-a)\end{aligned}$$

Notice that this answer is consistent with what you know from geometry.

EXAMPLE 4 Finding Area by an Iterated Integral

Use an iterated integral to find the area of the region bounded by the graphs of

$$\begin{aligned}f(x) &= \sin x && \text{Sine curve forms upper boundary.}\\ g(x) &= \cos x && \text{Cosine curve forms lower boundary.}\end{aligned}$$

between $x = \pi/4$ and $x = 5\pi/4$.

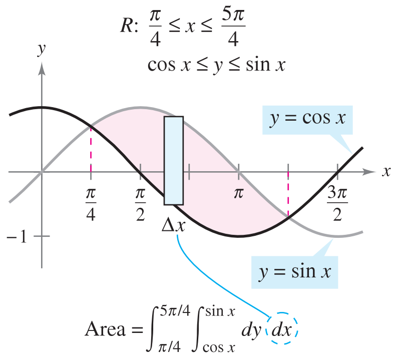


Figure 14.5

Solution Because f and g are given as functions of x, a vertical representative rectangle is convenient, and you can choose *dy dx* as the order of integration, as shown in Figure 14.5. The outside limits of integration are $\pi/4 \le x \le 5\pi/4$. Moreover, because the rectangle is bounded above by $f(x) = \sin x$ and below by $g(x) = \cos x$, you have

$$\begin{aligned}\text{Area of } R &= \int_{\pi/4}^{5\pi/4}\int_{\cos x}^{\sin x} dy\,dx\\ &= \int_{\pi/4}^{5\pi/4} y\Big]_{\cos x}^{\sin x} dx && \text{Integrate with respect to } y.\\ &= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx\\ &= \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4} && \text{Integrate with respect to } x.\\ &= 2\sqrt{2}.\end{aligned}$$

■

NOTE The region of integration of an iterated integral need not have any straight lines as boundaries. For instance, the region of integration shown in Figure 14.5 is *vertically simple* even though it has no vertical lines as left and right boundaries. The quality that makes the region vertically simple is that it is bounded above and below by the graphs of *functions of x*. ■

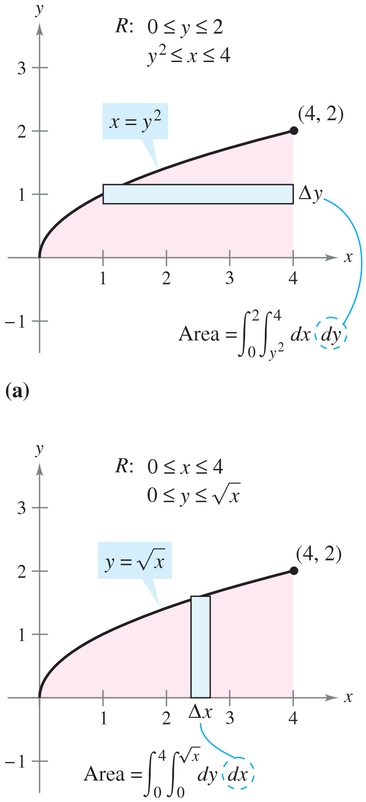
One order of integration will often produce a simpler integration problem than the other order. For instance, try reworking Example 4 with the order *dx dy*—you may be surprised to see that the task is formidable. However, if you succeed, you will see that the answer is the same. In other words, the order of integration affects the ease of integration, but not the value of the integral.

EXAMPLE 5 Comparing Different Orders of Integration

Sketch the region whose area is represented by the integral

$$\int_0^2 \int_{y^2}^4 dx\, dy.$$

Then find another iterated integral using the order *dy dx* to represent the same area and show that both integrals yield the same value.



(a)

(b)

Figure 14.6

Solution From the given limits of integration, you know that

$$y^2 \le x \le 4 \qquad \text{Inner limits of integration}$$

which means that the region R is bounded on the left by the parabola $x = y^2$ and on the right by the line $x = 4$. Furthermore, because

$$0 \le y \le 2 \qquad \text{Outer limits of integration}$$

you know that R is bounded below by the x-axis, as shown in Figure 14.6(a). The value of this integral is

$$\begin{aligned}
\int_0^2 \int_{y^2}^4 dx\, dy &= \int_0^2 x\Big]_{y^2}^4 dy && \text{Integrate with respect to } x. \\
&= \int_0^2 (4 - y^2)\, dy \\
&= \left[4y - \frac{y^3}{3}\right]_0^2 = \frac{16}{3}. && \text{Integrate with respect to } y.
\end{aligned}$$

To change the order of integration to *dy dx*, place a vertical rectangle in the region, as shown in Figure 14.6(b). From this you can see that the constant bounds $0 \le x \le 4$ serve as the outer limits of integration. By solving for y in the equation $x = y^2$, you can conclude that the inner bounds are $0 \le y \le \sqrt{x}$. So, the area of the region can also be represented by

$$\int_0^4 \int_0^{\sqrt{x}} dy\, dx.$$

By evaluating this integral, you can see that it has the same value as the original integral.

$$\begin{aligned}
\int_0^4 \int_0^{\sqrt{x}} dy\, dx &= \int_0^4 y\Big]_0^{\sqrt{x}} dx && \text{Integrate with respect to } y. \\
&= \int_0^4 \sqrt{x}\, dx \\
&= \frac{2}{3} x^{3/2}\Big]_0^4 = \frac{16}{3} && \text{Integrate with respect to } x.
\end{aligned}$$

■

The icon ↻ *indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems* Maple *and* Mathematica.

Sometimes it is not possible to calculate the area of a region with a single iterated integral. In these cases you can divide the region into subregions such that the area of each subregion can be calculated by an iterated integral. The total area is then the sum of the iterated integrals.

TECHNOLOGY Some computer software can perform symbolic integration for integrals such as those in Example 6. If you have access to such software, use it to evaluate the integrals in the exercises and examples given in this section.

EXAMPLE 6 An Area Represented by Two Iterated Integrals

Find the area of the region R that lies below the parabola

$$y = 4x - x^2 \qquad \text{Parabola forms upper boundary.}$$

above the x-axis, and above the line

$$y = -3x + 6. \qquad \text{Line and } x\text{-axis form lower boundary.}$$

Solution Begin by dividing R into the two subregions R_1 and R_2 shown in Figure 14.7.

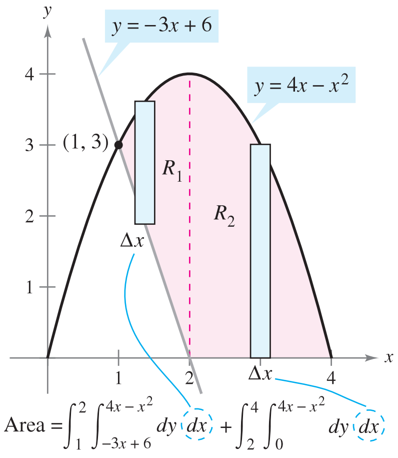


Figure 14.7

In both regions, it is convenient to use vertical rectangles, and you have

$$\begin{aligned}
\text{Area} &= \int_1^2 \int_{-3x+6}^{4x-x^2} dy\, dx + \int_2^4 \int_0^{4x-x^2} dy\, dx \\
&= \int_1^2 (4x - x^2 + 3x - 6)\, dx + \int_2^4 (4x - x^2)\, dx \\
&= \left[\frac{7x^2}{2} - \frac{x^3}{3} - 6x\right]_1^2 + \left[2x^2 - \frac{x^3}{3}\right]_2^4 \\
&= \left(14 - \frac{8}{3} - 12 - \frac{7}{2} + \frac{1}{3} + 6\right) + \left(32 - \frac{64}{3} - 8 + \frac{8}{3}\right) = \frac{15}{2}.
\end{aligned}$$

The area of the region is $15/2$ square units. Try checking this using the procedure for finding the area between two curves, as presented in Section 7.1. ■

At this point you may be wondering why you would need iterated integrals. After all, you already know how to use conventional integration to find the area of a region in the plane. (For instance, compare the solution of Example 4 in this section with that given in Example 3 in Section 7.1.) The need for iterated integrals will become clear in the next section. In this section, primary attention is given to procedures for finding the limits of integration of the region of an iterated integral, and the following exercise set is designed to develop skill in this important procedure.

NOTE In Examples 3 to 6, be sure you see the benefit of sketching the region of integration. You should develop the habit of making sketches to help you determine the limits of integration for all iterated integrals in this chapter.

14.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, evaluate the integral.

1. $\displaystyle\int_0^x (x + 2y)\, dy$

2. $\displaystyle\int_x^{x^2} \frac{y}{x}\, dy$

3. $\displaystyle\int_1^{2y} \frac{y}{x}\, dx, \quad y > 0$

4. $\displaystyle\int_0^{\cos y} y\, dx$

5. $\displaystyle\int_0^{\sqrt{4-x^2}} x^2 y\, dy$

6. $\displaystyle\int_{x^3}^{\sqrt{x}} (x^2 + 3y^2)\, dy$

7. $\displaystyle\int_{e^y}^{y} \frac{y \ln x}{x}\, dx, \quad y > 0$

8. $\displaystyle\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2)\, dx$

9. $\displaystyle\int_0^{x^3} ye^{-y/x}\, dy$

10. $\displaystyle\int_y^{\pi/2} \sin^3 x \cos y\, dx$

In Exercises 11–30, evaluate the iterated integral.

11. $\displaystyle\int_0^1\int_0^2 (x + y)\, dy\, dx$

12. $\displaystyle\int_{-1}^1\int_{-2}^2 (x^2 - y^2)\, dy\, dx$

13. $\displaystyle\int_1^2\int_0^4 (x^2 - 2y^2)\, dx\, dy$

14. $\displaystyle\int_{-1}^2\int_1^3 (x + y^2)\, dx\, dy$

15. $\displaystyle\int_0^{\pi/2}\int_0^1 y \cos x\, dy\, dx$

16. $\displaystyle\int_0^{\ln 4}\int_0^{\ln 3} e^{x+y}\, dy\, dx$

17. $\displaystyle\int_0^{\pi}\int_0^{\sin x} (1 + \cos x)\, dy\, dx$

18. $\displaystyle\int_1^4\int_1^{\sqrt{x}} 2ye^{-x}\, dy\, dx$

19. $\displaystyle\int_0^1\int_0^x \sqrt{1 - x^2}\, dy\, dx$

20. $\displaystyle\int_{-4}^4\int_0^{x^2} \sqrt{64 - x^3}\, dy\, dx$

21. $\displaystyle\int_{-1}^5\int_0^{3y} \left(3 + x^2 + \frac{1}{4}y^2\right) dx\, dy$

22. $\displaystyle\int_0^2\int_y^{2y} (10 + 2x^2 + 2y^2)\, dx\, dy$

23. $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}} (x + y)\, dx\, dy$

24. $\displaystyle\int_0^2\int_{3y^2-6y}^{2y-y^2} 3y\, dx\, dy$

25. $\displaystyle\int_0^2\int_0^{\sqrt{4-y^2}} \frac{2}{\sqrt{4 - y^2}}\, dx\, dy$

26. $\displaystyle\int_1^3\int_0^y \frac{4}{x^2 + y^2}\, dx\, dy$

27. $\displaystyle\int_0^{\pi/2}\int_0^{2\cos\theta} r\, dr\, d\theta$

28. $\displaystyle\int_0^{\pi/4}\int_{\sqrt{3}}^{\sqrt{3}\cos\theta} r\, dr\, d\theta$

29. $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} \theta r\, dr\, d\theta$

30. $\displaystyle\int_0^{\pi/4}\int_0^{\cos\theta} 3r^2 \sin\theta\, dr\, d\theta$

In Exercises 31–34, evaluate the improper iterated integral.

31. $\displaystyle\int_1^{\infty}\int_0^{1/x} y\, dy\, dx$

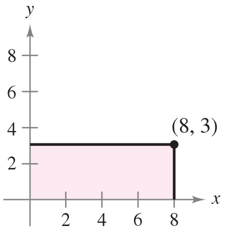
32. $\displaystyle\int_0^3\int_0^{\infty} \frac{x^2}{1 + y^2}\, dy\, dx$

33. $\displaystyle\int_1^{\infty}\int_1^{\infty} \frac{1}{xy}\, dx\, dy$

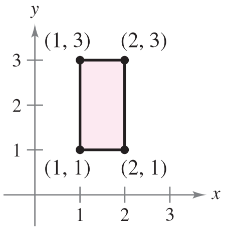
34. $\displaystyle\int_0^{\infty}\int_0^{\infty} xye^{-(x^2+y^2)}\, dx\, dy$

In Exercises 35–38, use an iterated integral to find the area of the region.

35.



36.



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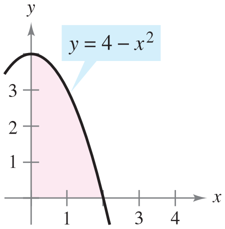

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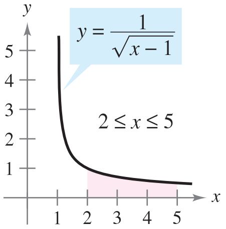


In Exercises 39–46, use an iterated integral to find the area of the region bounded by the graphs of the equations.

39. $\sqrt{x} + \sqrt{y} = 2, \quad x = 0, \quad y = 0$

40. $y = x^{3/2}, \quad y = 2x$

41. $2x - 3y = 0, \quad x + y = 5, \quad y = 0$

42. $xy = 9, \quad y = x, \quad y = 0, \quad x = 9$

43. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

44. $y = x, \quad y = 2x, \quad x = 2$

45. $y = 4 - x^2, \quad y = x + 2$

46. $x^2 + y^2 = 4, \quad x = 0, \quad y = 0$

In Exercises 47–54, sketch the region *R* of integration and switch the order of integration.

47. $\displaystyle\int_0^4\int_0^y f(x, y)\, dx\, dy$

48. $\displaystyle\int_0^4\int_{\sqrt{y}}^2 f(x, y)\, dx\, dy$

49. $\displaystyle\int_{-2}^2\int_0^{\sqrt{4-x^2}} f(x, y)\, dy\, dx$

50. $\displaystyle\int_0^2\int_0^{4-x^2} f(x, y)\, dy\, dx$

51. $\displaystyle\int_1^{10}\int_0^{\ln y} f(x, y)\, dx\, dy$

52. $\displaystyle\int_{-1}^2\int_0^{e^{-x}} f(x, y)\, dy\, dx$

53. $\displaystyle\int_{-1}^1\int_{x^2}^1 f(x, y)\, dy\, dx$

54. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^{\cos x} f(x, y)\, dy\, dx$

In Exercises 55–64, sketch the region *R* whose area is given by the iterated integral. Then switch the order of integration and show that both orders yield the same area.

55. $\displaystyle\int_0^1\int_0^2 dy\, dx$

56. $\displaystyle\int_1^2\int_2^4 dx\, dy$

57. $\displaystyle\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\, dy$

58. $\displaystyle\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\, dx$

59. $\displaystyle\int_0^2\int_0^x dy\,dx + \int_2^4\int_0^{4-x} dy\,dx$

60. $\displaystyle\int_0^4\int_0^{x/2} dy\,dx + \int_4^6\int_0^{6-x} dy\,dx$

61. $\displaystyle\int_0^2\int_{x/2}^1 dy\,dx$

62. $\displaystyle\int_0^9\int_{\sqrt{x}}^3 dy\,dx$

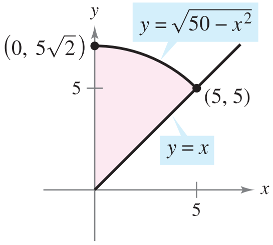
63. $\displaystyle\int_0^1\int_{y^2}^{\sqrt[3]{y}} dx\,dy$

64. $\displaystyle\int_{-2}^2\int_0^{4-y^2} dx\,dy$

65. ***Think About It*** Give a geometric argument for the equality. Verify the equality analytically.

$$\int_0^5\int_x^{\sqrt{50-x^2}} x^2y^2\,dy\,dx =$$

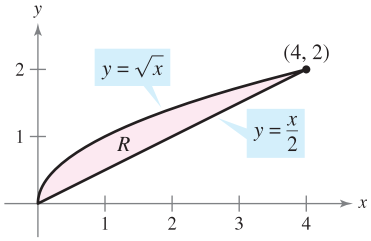
$$\int_0^5\int_0^y x^2y^2\,dx\,dy + \int_5^{5\sqrt{2}}\int_0^{\sqrt{50-y^2}} x^2y^2\,dx\,dy$$



CAPSTONE

66. ***Think About It*** Complete the iterated integrals so that each one represents the area of the region R (see figure). Then show that both integrals yield the same area.

(a) Area $= \displaystyle\int\int dx\,dy$ (b) Area $= \displaystyle\int\int dy\,dx$



In Exercises 67–72, sketch the region of integration. Then evaluate the iterated integral. (Note that it is necessary to switch the order of integration.)

67. $\displaystyle\int_0^2\int_x^2 x\sqrt{1+y^3}\,dy\,dx$

68. $\displaystyle\int_0^4\int_{\sqrt{x}}^2 \frac{3}{2+y^3}\,dy\,dx$

69. $\displaystyle\int_0^1\int_{2x}^2 4e^{y^2}\,dy\,dx$

70. $\displaystyle\int_0^2\int_x^2 e^{-y^2}\,dy\,dx$

71. $\displaystyle\int_0^1\int_y^1 \sin x^2\,dx\,dy$

72. $\displaystyle\int_0^2\int_{y^2}^4 \sqrt{x}\sin x\,dx\,dy$

CAS **In Exercises 73–76, use a computer algebra system to evaluate the iterated integral.**

73. $\displaystyle\int_0^2\int_{x^2}^{2x} (x^3+3y^2)\,dy\,dx$

74. $\displaystyle\int_0^1\int_y^{2y} \sin(x+y)\,dx\,dy$

75. $\displaystyle\int_0^4\int_0^y \frac{2}{(x+1)(y+1)}\,dx\,dy$

76. $\displaystyle\int_0^a\int_0^{a-x} (x^2+y^2)\,dy\,dx$

CAS **In Exercises 77 and 78, (a) sketch the region of integration, (b) switch the order of integration, and (c) use a computer algebra system to show that both orders yield the same value.**

77. $\displaystyle\int_0^2\int_{y^3}^{4\sqrt{2y}} (x^2y-xy^2)\,dx\,dy$

78. $\displaystyle\int_0^2\int_{\sqrt{4-x^2}}^{4-x^2/4} \frac{xy}{x^2+y^2+1}\,dy\,dx$

CAS **In Exercises 79–82, use a computer algebra system to approximate the iterated integral.**

79. $\displaystyle\int_0^2\int_0^{4-x^2} e^{xy}\,dy\,dx$

80. $\displaystyle\int_0^2\int_x^2 \sqrt{16-x^3-y^3}\,dy\,dx$

81. $\displaystyle\int_0^{2\pi}\int_0^{1+\cos\theta} 6r^2\cos\theta\,dr\,d\theta$

82. $\displaystyle\int_0^{\pi/2}\int_0^{1+\sin\theta} 15\theta r\,dr\,d\theta$

WRITING ABOUT CONCEPTS

83. Explain what is meant by an iterated integral. How is it evaluated?

84. Describe regions that are vertically simple and regions that are horizontally simple.

85. Give a geometric description of the region of integration if the inside and outside limits of integration are constants.

86. Explain why it is sometimes an advantage to change the order of integration.

True or False? **In Exercises 87 and 88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

87. $\displaystyle\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$

88. $\displaystyle\int_0^1\int_0^x f(x,y)\,dy\,dx = \int_0^1\int_0^y f(x,y)\,dx\,dy$

14.2 Double Integrals and Volume

- **Use a double integral to represent the volume of a solid region.**
- **Use properties of double integrals.**
- **Evaluate a double integral as an iterated integral.**
- **Find the average value of a function over a region.**

Double Integrals and Volume of a Solid Region

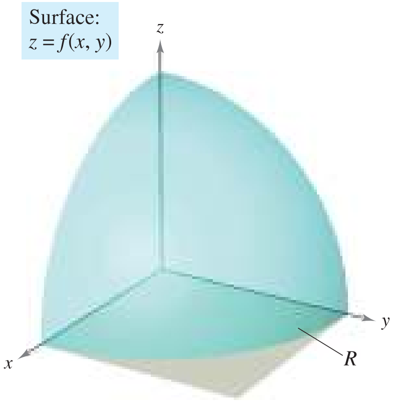


Figure 14.8

You already know that a definite integral over an *interval* uses a limit process to assign measures to quantities such as area, volume, arc length, and mass. In this section, you will use a similar process to define the **double integral** of a function of two variables over a *region in the plane*.

Consider a continuous function f such that $f(x, y) \geq 0$ for all (x, y) in a region R in the xy-plane. The goal is to find the volume of the solid region lying between the surface given by

$$z = f(x, y)$$

Surface lying above the xy-plane

and the xy-plane, as shown in Figure 14.8. You can begin by superimposing a rectangular grid over the region, as shown in Figure 14.9. The rectangles lying entirely within R form an **inner partition** Δ, whose **norm** $\|\Delta\|$ is defined as the length of the longest diagonal of the n rectangles. Next, choose a point (x_i, y_i) in each rectangle and form the rectangular prism whose height is $f(x_i, y_i)$, as shown in Figure 14.10. Because the area of the ith rectangle is

$$\Delta A_i$$

Area of ith rectangle

it follows that the volume of the ith prism is

$$f(x_i, y_i)\,\Delta A_i$$

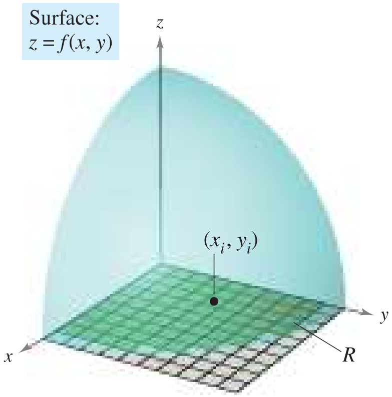
Volume of ith prism

and you can approximate the volume of the solid region by the Riemann sum of the volumes of all n prisms,

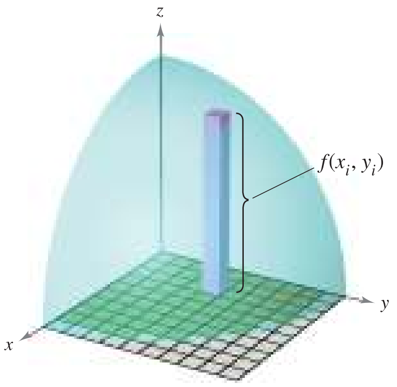
$$\sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i$$

Riemann sum

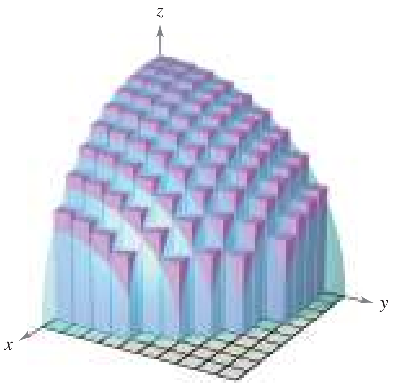
as shown in Figure 14.11. This approximation can be improved by tightening the mesh of the grid to form smaller and smaller rectangles, as shown in Example 1.



The rectangles lying within R form an inner partition of R.
Figure 14.9



Rectangular prism whose base has an area of ΔA_i and whose height is $f(x_i, y_i)$
Figure 14.10



Volume approximated by rectangular prisms
Figure 14.11

EXAMPLE 1 Approximating the Volume of a Solid

Approximate the volume of the solid lying between the paraboloid

$$f(x, y) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$$

and the square region R given by $0 \le x \le 1$, $0 \le y \le 1$. Use a partition made up of squares whose sides have a length of $\frac{1}{4}$.

Solution Begin by forming the specified partition of R. For this partition, it is convenient to choose the centers of the subregions as the points at which to evaluate $f(x, y)$.

$\left(\frac{1}{8}, \frac{1}{8}\right)$	$\left(\frac{1}{8}, \frac{3}{8}\right)$	$\left(\frac{1}{8}, \frac{5}{8}\right)$	$\left(\frac{1}{8}, \frac{7}{8}\right)$
$\left(\frac{3}{8}, \frac{1}{8}\right)$	$\left(\frac{3}{8}, \frac{3}{8}\right)$	$\left(\frac{3}{8}, \frac{5}{8}\right)$	$\left(\frac{3}{8}, \frac{7}{8}\right)$
$\left(\frac{5}{8}, \frac{1}{8}\right)$	$\left(\frac{5}{8}, \frac{3}{8}\right)$	$\left(\frac{5}{8}, \frac{5}{8}\right)$	$\left(\frac{5}{8}, \frac{7}{8}\right)$
$\left(\frac{7}{8}, \frac{1}{8}\right)$	$\left(\frac{7}{8}, \frac{3}{8}\right)$	$\left(\frac{7}{8}, \frac{5}{8}\right)$	$\left(\frac{7}{8}, \frac{7}{8}\right)$

Because the area of each square is $\Delta A_i = \frac{1}{16}$, you can approximate the volume by the sum

$$\sum_{i=1}^{16} f(x_i\, y_i)\,\Delta A_i = \sum_{i=1}^{16}\left(1 - \frac{1}{2}x_i^2 - \frac{1}{2}y_i^2\right)\left(\frac{1}{16}\right)$$
$$\approx 0.672.$$

This approximation is shown graphically in Figure 14.12. The exact volume of the solid is $\frac{2}{3}$ (see Example 2). You can obtain a better approximation by using a finer partition. For example, with a partition of squares with sides of length $\frac{1}{10}$, the approximation is 0.668. ■

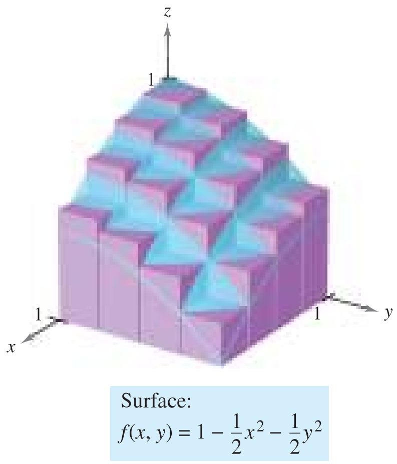


Figure 14.12

TECHNOLOGY Some three-dimensional graphing utilities are capable of graphing figures such as that shown in Figure 14.12. For instance, the graph shown in Figure 14.13 was drawn with a computer program. In this graph, note that each of the rectangular prisms lies within the solid region.

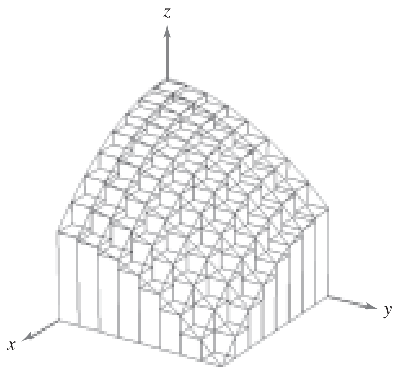


Figure 14.13

In Example 1, note that by using finer partitions, you obtain better approximations of the volume. This observation suggests that you could obtain the exact volume by taking a limit. That is,

$$\text{Volume} = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i.$$

The precise meaning of this limit is that the limit is equal to L if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| L - \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i \right| < \varepsilon$$

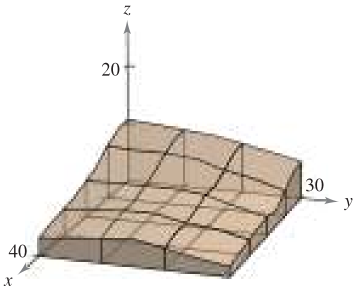
for all partitions Δ of the plane region R (that satisfy $\|\Delta\| < \delta$) and for all possible choices of x_i and y_i in the ith region.

Using the limit of a Riemann sum to define volume is a special case of using the limit to define a **double integral.** The general case, however, does not require that the function be positive or continuous.

EXPLORATION

The entries in the table represent the depths (in 10-yard units) of earth at the centers of the squares in the figure below.

x \ y	1	2	3
1	10	9	7
2	7	7	4
3	5	5	4
4	4	5	3



Approximate the number of cubic yards of earth in the first octant. (This exploration was submitted by Robert Vojack, Ridgewood High School, Ridgewood, NJ.)

DEFINITION OF DOUBLE INTEGRAL

If f is defined on a closed, bounded region R in the xy-plane, then the **double integral of f over R** is given by

$$\int\!\!\int_R f(x, y)\, dA = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i$$

provided the limit exists. If the limit exists, then f is **integrable** over R.

NOTE Having defined a double integral, you will see that a definite integral is occasionally referred to as a **single integral.** ■

Sufficient conditions for the double integral of f on the region R to exist are that R can be written as a union of a finite number of nonoverlapping subregions (see Figure 14.14) that are vertically or horizontally simple *and* that f is continuous on the region R.

A double integral can be used to find the volume of a solid region that lies between the xy-plane and the surface given by $z = f(x, y)$.

VOLUME OF A SOLID REGION

If f is integrable over a plane region R and $f(x, y) \geq 0$ for all (x, y) in R, then the volume of the solid region that lies above R and below the graph of f is defined as

$$V = \int\!\!\int_R f(x, y)\, dA.$$

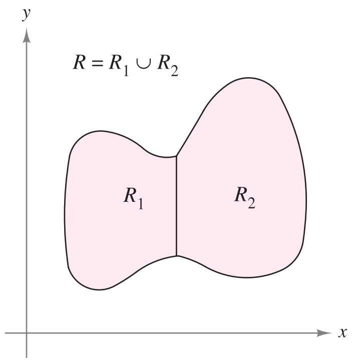
Properties of Double Integrals

Double integrals share many properties of single integrals.

THEOREM 14.1 PROPERTIES OF DOUBLE INTEGRALS

Let f and g be continuous over a closed, bounded plane region R, and let c be a constant.

1. $\displaystyle\int\!\!\int_R cf(x, y)\, dA = c\int\!\!\int_R f(x, y)\, dA$
2. $\displaystyle\int\!\!\int_R [f(x, y) \pm g(x, y)]\, dA = \int\!\!\int_R f(x, y)\, dA \pm \int\!\!\int_R g(x, y)\, dA$
3. $\displaystyle\int\!\!\int_R f(x, y)\, dA \geq 0, \quad \text{if } f(x, y) \geq 0$
4. $\displaystyle\int\!\!\int_R f(x, y)\, dA \geq \int\!\!\int_R g(x, y)\, dA, \quad \text{if } f(x, y) \geq g(x, y)$
5. $\displaystyle\int\!\!\int_R f(x, y)\, dA = \int\!\!\int_{R_1} f(x, y)\, dA + \int\!\!\int_{R_2} f(x, y)\, dA$, where R is the union of two nonoverlapping subregions R_1 and R_2.



Two regions are nonoverlapping if their intersection is a set that has an area of 0. In this figure, the area of the line segment that is common to R_1 and R_2 is 0.

Figure 14.14

Evaluation of Double Integrals

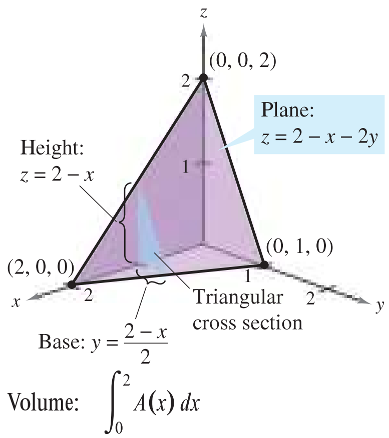


Figure 14.15

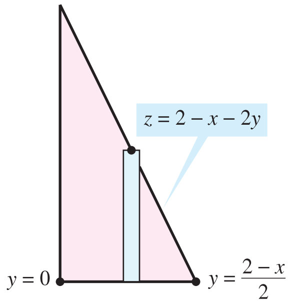
Normally, the first step in evaluating a double integral is to rewrite it as an iterated integral. To show how this is done, a geometric model of a double integral is used as the volume of a solid.

Consider the solid region bounded by the plane $z = f(x, y) = 2 - x - 2y$ and the three coordinate planes, as shown in Figure 14.15. Each vertical cross section taken parallel to the yz-plane is a triangular region whose base has a length of $y = (2 - x)/2$ and whose height is $z = 2 - x$. This implies that for a fixed value of x, the area of the triangular cross section is

$$A(x) = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}\left(\frac{2-x}{2}\right)(2-x) = \frac{(2-x)^2}{4}.$$

By the formula for the volume of a solid with known cross sections (Section 7.2), the volume of the solid is

$$\begin{aligned}\text{Volume} &= \int_a^b A(x)\,dx \\ &= \int_0^2 \frac{(2-x)^2}{4}\,dx \\ &= -\frac{(2-x)^3}{12}\bigg]_0^2 = \frac{2}{3}.\end{aligned}$$



Triangular cross section
Figure 14.16

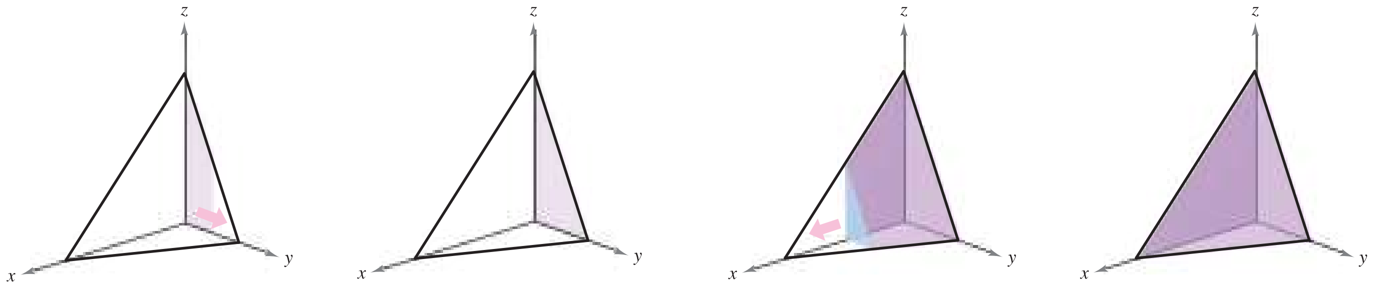
This procedure works no matter how $A(x)$ is obtained. In particular, you can find $A(x)$ by integration, as shown in Figure 14.16. That is, you consider x to be constant, and integrate $z = 2 - x - 2y$ from 0 to $(2 - x)/2$ to obtain

$$\begin{aligned}A(x) &= \int_0^{(2-x)/2} (2 - x - 2y)\,dy \\ &= \Big[(2-x)y - y^2\Big]_0^{(2-x)/2} \\ &= \frac{(2-x)^2}{4}.\end{aligned}$$

Combining these results, you have the *iterated integral*

$$\text{Volume} = \int_R\!\!\int f(x, y)\,dA = \int_0^2\!\int_0^{(2-x)/2} (2 - x - 2y)\,dy\,dx.$$

To understand this procedure better, it helps to imagine the integration as two sweeping motions. For the inner integration, a vertical line sweeps out the area of a cross section. For the outer integration, the triangular cross section sweeps out the volume, as shown in Figure 14.17.



Integrate with respect to y to obtain the area of the cross section.
Integrate with respect to x to obtain the volume of the solid.
Figure 14.17

The following theorem was proved by the Italian mathematician Guido Fubini (1879–1943). The theorem states that if R is a vertically or horizontally simple region and f is continuous on R, the double integral of f on R is equal to an iterated integral.

THEOREM 14.2 FUBINI'S THEOREM

Let f be continuous on a plane region R.

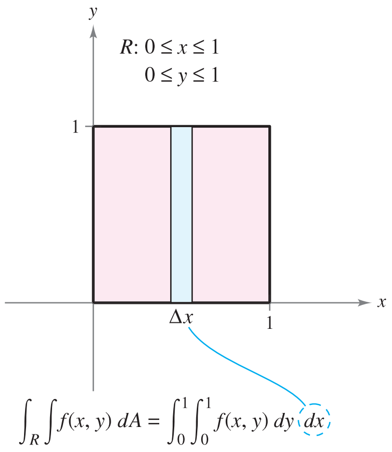
1. If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then
$$\int\!\!\int_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$
2. If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then
$$\int\!\!\int_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

EXAMPLE 2 Evaluating a Double Integral as an Iterated Integral

Evaluate

$$\int\!\!\int_R \left(1 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) dA$$

where R is the region given by $0 \le x \le 1$, $0 \le y \le 1$.


y
R: 0 ≤ x ≤ 1
0 ≤ y ≤ 1
1
Δx
1
x
$\int_R\int f(x, y)\, dA = \int_0^1\int_0^1 f(x, y)\, dy\, dx$

The volume of the solid region is $\frac{2}{3}$.
Figure 14.18

Solution Because the region R is a square, it is both vertically and horizontally simple, and you can use either order of integration. Choose $dy\,dx$ by placing a vertical representative rectangle in the region, as shown in Figure 14.18. This produces the following.

$$\begin{aligned}
\int\!\!\int_R \left(1 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) dA &= \int_0^1 \int_0^1 \left(1 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) dy\, dx \\
&= \int_0^1 \left[\left(1 - \frac{1}{2}x^2\right) y - \frac{y^3}{6}\right]_0^1 dx \\
&= \int_0^1 \left(\frac{5}{6} - \frac{1}{2}x^2\right) dx \\
&= \left[\frac{5}{6}x - \frac{x^3}{6}\right]_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

■

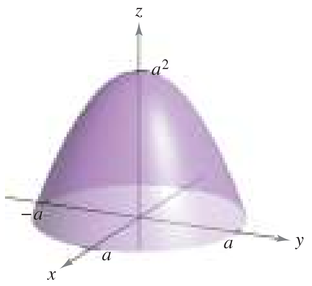
The double integral evaluated in Example 2 represents the volume of the solid region approximated in Example 1. Note that the approximation obtained in Example 1 is quite good $\left(0.672 \text{ vs. } \frac{2}{3}\right)$, even though you used a partition consisting of only 16 squares. The error resulted because the centers of the square subregions were used as the points in the approximation. This is comparable to the Midpoint Rule approximation of a single integral.

EXPLORATION

Volume of a Paraboloid Sector
The solid in Example 3 has an elliptical (not a circular) base. Consider the region bounded by the circular paraboloid

$$z = a^2 - x^2 - y^2, \quad a > 0$$

and the xy-plane. How many ways of finding the volume of this solid do you now know? For instance, you could use the disk method to find the volume as a solid of revolution. Does each method involve integration?



The difficulty of evaluating a single integral $\int_a^b f(x)\,dx$ usually depends on the function f, and not on the interval $[a, b]$. This is a major difference between single and double integrals. In the next example, you will integrate a function similar to the one in Examples 1 and 2. Notice that a change in the region R produces a much more difficult integration problem.

EXAMPLE 3 Finding Volume by a Double Integral

Find the volume of the solid region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy-plane.

Solution By letting $z = 0$, you can see that the base of the region in the xy-plane is the ellipse $x^2 + 2y^2 = 4$, as shown in Figure 14.19(a). This plane region is both vertically and horizontally simple, so the order $dy\,dx$ is appropriate.

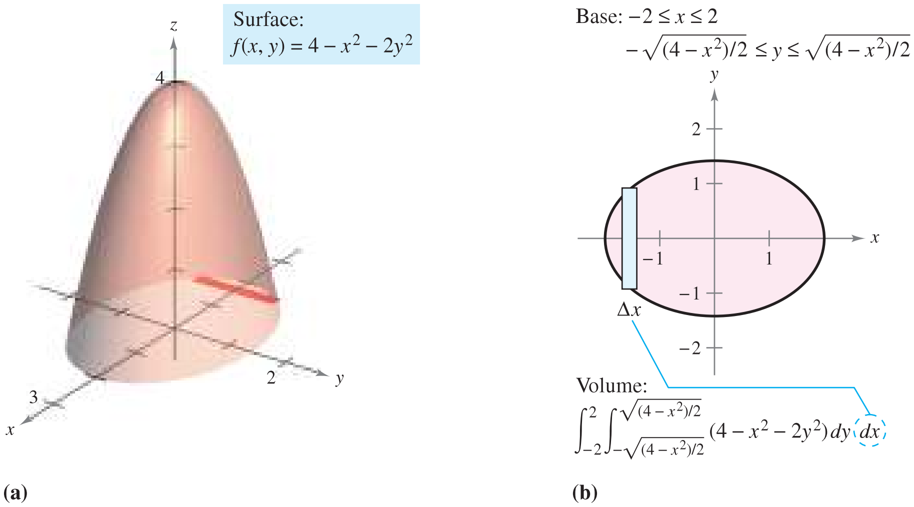
Variable bounds for y: $-\sqrt{\dfrac{(4 - x^2)}{2}} \le y \le \sqrt{\dfrac{(4 - x^2)}{2}}$

Constant bounds for x: $-2 \le x \le 2$

The volume is given by

$$\begin{aligned}
V &= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (4 - x^2 - 2y^2)\,dy\,dx && \text{See Figure 14.19(b).}\\
&= \int_{-2}^{2} \left[(4 - x^2)y - \frac{2y^3}{3}\right]_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} dx\\
&= \frac{4}{3\sqrt{2}} \int_{-2}^{2} (4 - x^2)^{3/2}\,dx\\
&= \frac{4}{3\sqrt{2}} \int_{-\pi/2}^{\pi/2} 16\cos^4\theta\,d\theta && x = 2\sin\theta\\
&= \frac{64}{3\sqrt{2}}(2) \int_{0}^{\pi/2} \cos^4\theta\,d\theta\\
&= \frac{128}{3\sqrt{2}}\left(\frac{3\pi}{16}\right) && \text{Wallis's Formula}\\
&= 4\sqrt{2}\pi.
\end{aligned}$$

NOTE In Example 3, note the usefulness of Wallis's Formula to evaluate $\int_0^{\pi/2} \cos^n\theta\,d\theta$. You may want to review this formula in Section 8.3.

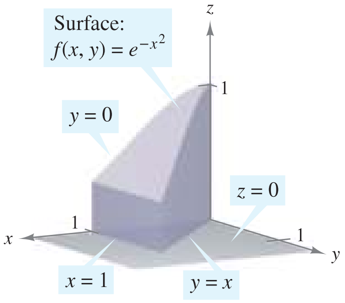

Surface:
$f(x, y) = 4 - x^2 - 2y^2$
z
4
3
x
2
y
Base: $-2 \le x \le 2$
$-\sqrt{(4-x^2)/2} \le y \le \sqrt{(4-x^2)/2}$
y
2
1
-1
1
x
-1
-2
Δx
Volume:
$\int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (4 - x^2 - 2y^2)\,dy\,dx$

(a) **(b)**

Figure 14.19

In Examples 2 and 3, the problems could be solved with either order of integration because the regions were both vertically and horizontally simple. Moreover, had you used the order $dx\,dy$, you would have obtained integrals of comparable difficulty. There are, however, some occasions in which one order of integration is much more convenient than the other. Example 4 shows such a case.

EXAMPLE 4 Comparing Different Orders of Integration


Surface: $f(x, y) = e^{-x^2}$
z
1
y = 0
z = 0
x
1
1
y
x = 1
y = x

Base is bounded by $y = 0$, $y = x$, and $x = 1$.
Figure 14.20

Find the volume of the solid region R bounded by the surface

$$f(x, y) = e^{-x^2} \qquad \text{Surface}$$

and the planes $z = 0$, $y = 0$, $y = x$, and $x = 1$, as shown in Figure 14.20.

Solution The base of R in the xy-plane is bounded by the lines $y = 0$, $x = 1$, and $y = x$. The two possible orders of integration are shown in Figure 14.21.

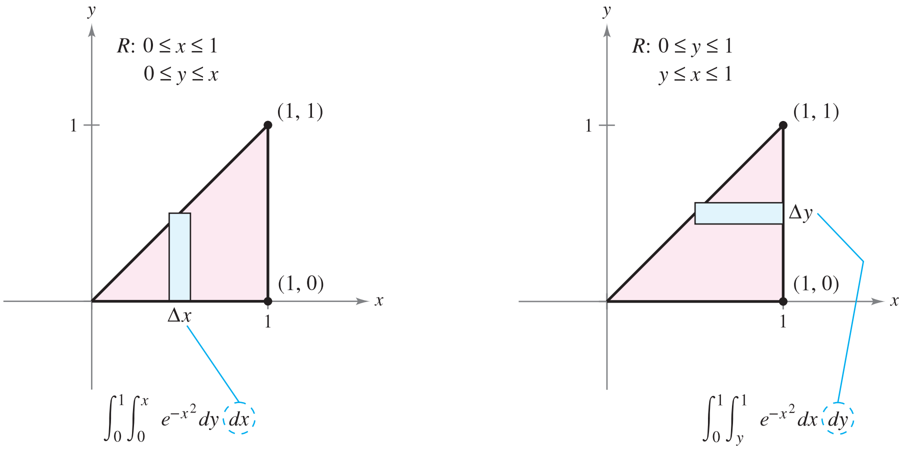

y
R: $0 \le x \le 1$
$0 \le y \le x$
1
(1, 1)
(1, 0)
Δx
1
x
$\int_0^1 \int_0^x e^{-x^2}\,dy\,dx$
y
R: $0 \le y \le 1$
$y \le x \le 1$
1
(1, 1)
Δy
(1, 0)
1
x
$\int_0^1 \int_y^1 e^{-x^2}\,dx\,dy$

Figure 14.21

By setting up the corresponding iterated integrals, you can see that the order $dx\,dy$ requires the antiderivative $\int e^{-x^2}\,dx$, which is not an elementary function. On the other hand, the order $dy\,dx$ produces the integral

$$\begin{aligned}\int_0^1 \int_0^x e^{-x^2}\,dy\,dx &= \int_0^1 e^{-x^2}y\Big]_0^x\,dx\\ &= \int_0^1 xe^{-x^2}\,dx\\ &= -\frac{1}{2}e^{-x^2}\Big]_0^1\\ &= -\frac{1}{2}\left(\frac{1}{e} - 1\right)\\ &= \frac{e-1}{2e}\\ &\approx 0.316.\end{aligned}$$

■

NOTE Try using a symbolic integration utility to evaluate the integral in Example 4. ■

EXAMPLE 5 Volume of a Region Bounded by Two Surfaces

Find the volume of the solid region R bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 1 - y$, as shown in Figure 14.22.

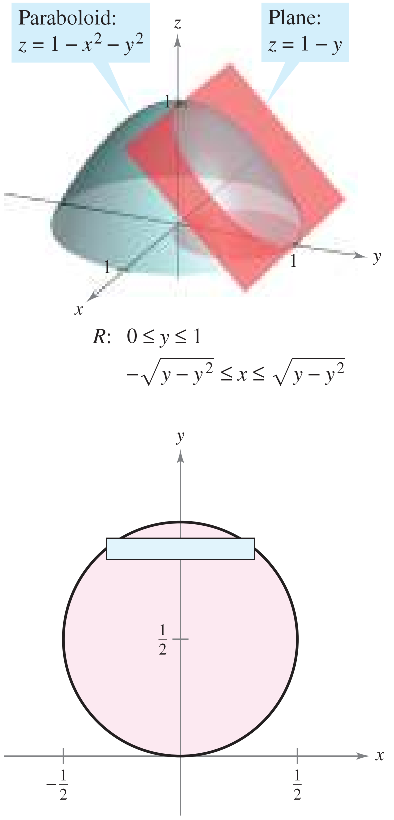


Figure 14.22

Solution Equating z-values, you can determine that the intersection of the two surfaces occurs on the right circular cylinder given by

$$1 - y = 1 - x^2 - y^2 \quad \Rightarrow \quad x^2 = y - y^2.$$

Because the volume of R is the difference between the volume under the paraboloid and the volume under the plane, you have

$$\begin{aligned}
\text{Volume} &= \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (1 - x^2 - y^2)\,dx\,dy - \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (1 - y)\,dx\,dy \\
&= \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (y - y^2 - x^2)\,dx\,dy \\
&= \int_0^1 \left[(y - y^2)x - \frac{x^3}{3} \right]_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} dy \\
&= \frac{4}{3} \int_0^1 (y - y^2)^{3/2}\,dy \\
&= \left(\frac{4}{3}\right)\left(\frac{1}{8}\right) \int_0^1 [1 - (2y - 1)^2]^{3/2}\,dy \\
&= \frac{1}{6} \int_{-\pi/2}^{\pi/2} \frac{\cos^4 \theta}{2}\,d\theta && 2y - 1 = \sin\theta \\
&= \frac{1}{6} \int_0^{\pi/2} \cos^4 d\theta \\
&= \left(\frac{1}{6}\right)\left(\frac{3\pi}{16}\right) && \text{Wallis's Formula} \\
&= \frac{\pi}{32}.
\end{aligned}$$

■

Average Value of a Function

Recall from Section 4.4 that for a function f in one variable, the average value of f on $[a, b]$ is

$$\frac{1}{b - a} \int_a^b f(x)\,dx.$$

Given a function f in two variables, you can find the average value of f over the region R as shown in the following definition.

DEFINITION OF THE AVERAGE VALUE OF A FUNCTION OVER A REGION

If f is integrable over the plane region R, then the **average value** of f over R is

$$\frac{1}{A} \int_R \int f(x, y)\,dA$$

where A is the area of R.

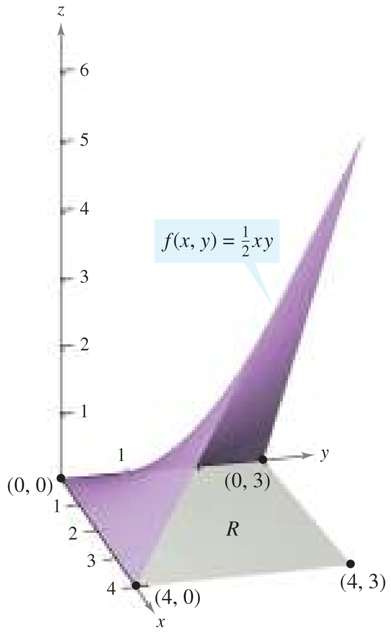


Figure 14.23

EXAMPLE 6 Finding the Average Value of a Function

Find the average value of $f(x, y) = \frac{1}{2}xy$ over the region R, where R is a rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 3)$, and $(0, 3)$.

Solution The area of the rectangular region R is $A = 12$ (see Figure 14.23). The average value is given by

$$\begin{aligned}
\frac{1}{A}\int_R\int f(x, y)\, dA &= \frac{1}{12}\int_0^4\int_0^3 \frac{1}{2}xy\, dy\, dx \\
&= \frac{1}{12}\int_0^4 \frac{1}{4}xy^2\Big]_0^3\, dx \\
&= \left(\frac{1}{12}\right)\left(\frac{9}{4}\right)\int_0^4 x\, dx \\
&= \frac{3}{16}\left[\frac{1}{2}x^2\right]_0^4 \\
&= \left(\frac{3}{16}\right)(8) \\
&= \frac{3}{2}.
\end{aligned}$$

14.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Approximation **In Exercises 1–4, approximate the integral $\int_R\int f(x, y)\, dA$ by dividing the rectangle R with vertices $(0, 0)$, $(4, 0)$, $(4, 2)$, and $(0, 2)$ into eight equal squares and finding the sum $\sum_{i=1}^{8} f(x_i, y_i)\,\Delta A_i$ where (x_i, y_i) is the center of the ith square. Evaluate the iterated integral and compare it with the approximation.**

1. $\displaystyle\int_0^4\int_0^2 (x + y)\, dy\, dx$

2. $\displaystyle\frac{1}{2}\int_0^4\int_0^2 x^2y\, dy\, dx$

3. $\displaystyle\int_0^4\int_0^2 (x^2 + y^2)\, dy\, dx$

4. $\displaystyle\int_0^4\int_0^2 \frac{1}{(x + 1)(y + 1)}\, dy\, dx$

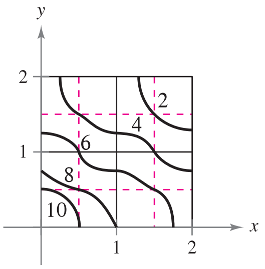
5. ***Approximation*** The table shows values of a function f over a square region R. Divide the region into 16 equal squares and select (x_i, y_i) to be the point in the ith square closest to the origin. Compare this approximation with that obtained by using the point in the ith square farthest from the origin.

$\int_0^4\int_0^4 f(x, y)\, dy\, dx$

x \ y	0	1	2	3	4
0	32	31	28	23	16
1	31	30	27	22	15
2	28	27	24	19	12
3	23	22	19	14	7
4	16	15	12	7	0

6. ***Approximation*** The figure shows the level curves for a function f over a square region R. Approximate the integral using four squares, selecting the midpoint of each square as (x_i, y_i).

$\displaystyle\int_0^2\int_0^2 f(x, y)\, dy\, dx$



In Exercises 7–12, sketch the region R and evaluate the iterated integral $\int_R\int f(x, y)\, dA$.

7. $\displaystyle\int_0^2\int_0^1 (1 + 2x + 2y)\, dy\, dx$

8. $\displaystyle\int_0^{\pi}\int_0^{\pi/2} \sin^2 x\cos^2 y\, dy\, dx$

9. $\displaystyle\int_0^6\int_{y/2}^3 (x + y)\, dx\, dy$

10. $\displaystyle\int_0^4\int_{\frac{1}{2}y}^{\sqrt{y}} x^2y^2\, dx\, dy$

11. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x + y)\, dy\, dx$

12. $\displaystyle\int_0^1\int_{y-1}^0 e^{x+y}\, dx\, dy + \int_0^1\int_0^{1-y} e^{x+y}\, dx\, dy$

In Exercises 13–20, set up integrals for both orders of integration, and use the more convenient order to evaluate the integral over the region R.

13. $\displaystyle\iint_R xy\,dA$

R: rectangle with vertices $(0, 0)$, $(0, 5)$, $(3, 5)$, $(3, 0)$

14. $\displaystyle\iint_R \sin x \sin y\,dA$

R: rectangle with vertices $(-\pi, 0)$, $(\pi, 0)$, $(\pi, \pi/2)$, $(-\pi, \pi/2)$

15. $\displaystyle\iint_R \frac{y}{x^2 + y^2}\,dA$

R: trapezoid bounded by $y = x$, $y = 2x$, $x = 1$, $x = 2$

16. $\displaystyle\iint_R xe^y\,dA$

R: triangle bounded by $y = 4 - x$, $y = 0$, $x = 0$

17. $\displaystyle\iint_R -2y\,dA$

R: region bounded by $y = 4 - x^2$, $y = 4 - x$

18. $\displaystyle\iint_R \frac{y}{1 + x^2}\,dA$

R: region bounded by $y = 0$, $y = \sqrt{x}$, $x = 4$

19. $\displaystyle\iint_R x\,dA$

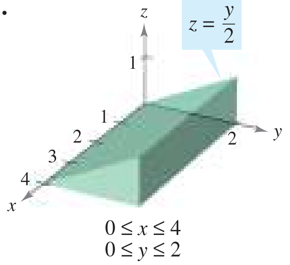
R: sector of a circle in the first quadrant bounded by $y = \sqrt{25 - x^2}$, $3x - 4y = 0$, $y = 0$

20. $\displaystyle\iint_R (x^2 + y^2)\,dA$

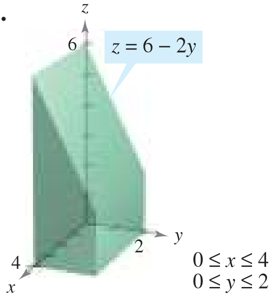
R: semicircle bounded by $y = \sqrt{4 - x^2}$, $y = 0$

In Exercises 21–30, use a double integral to find the volume of the indicated solid.

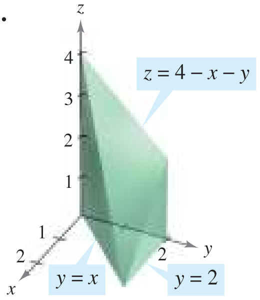
21.


z
$z = \frac{y}{2}$
1
1
2
3
4
2
y
x
$0 \le x \le 4$
$0 \le y \le 2$

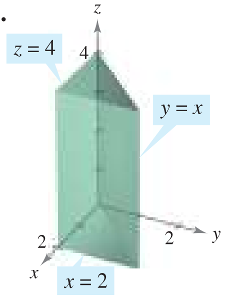
22.


z
6
$z = 6 - 2y$
2
y
4
x
$0 \le x \le 4$
$0 \le y \le 2$

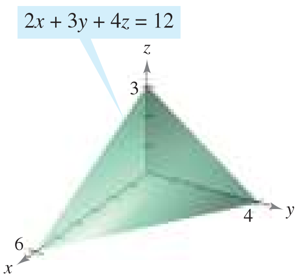
23.


z
4
$z = 4 - x - y$
3
2
1
1
2
2
y
x
$y = x$
$y = 2$

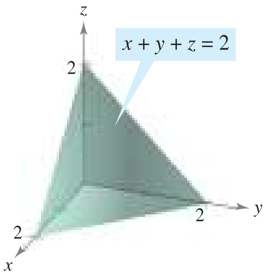
24.


z
$z = 4$
4
$y = x$
2
2
y
x
$x = 2$

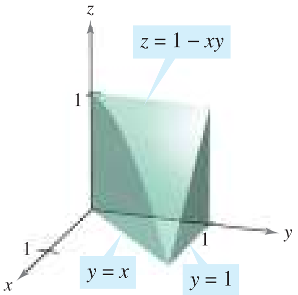
25.


$2x + 3y + 4z = 12$
z
3
4
y
6
x

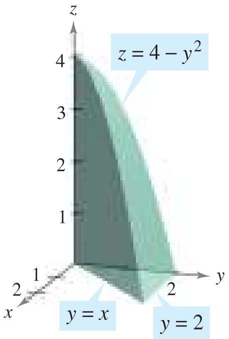
26.


z
$x + y + z = 2$
2
2
y
2
x

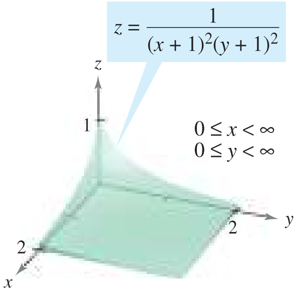
27.


z
$z = 1 - xy$
1
1
1
y
x
$y = x$
$y = 1$

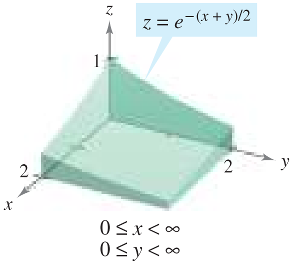
28.


z
4
$z = 4 - y^2$
3
2
1
1
2
2
y
x
$y = x$
$y = 2$

29. Improper integral

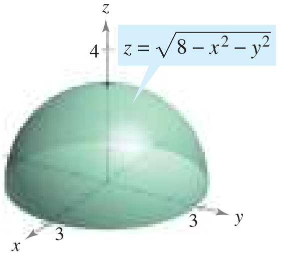

$z = \frac{1}{(x + 1)^2(y + 1)^2}$
z
1
$0 \le x < \infty$
$0 \le y < \infty$
2
y
2
x

30. Improper integral

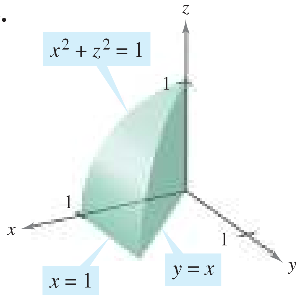

z
$z = e^{-(x+y)/2}$
1
2
2
y
x
$0 \le x < \infty$
$0 \le y < \infty$

CAS **In Exercises 31 and 32, use a computer algebra system to find the volume of the solid.**

31.


z
4
$z = \sqrt{8 - x^2 - y^2}$
3
3
y
x

32.


z
$x^2 + z^2 = 1$
1
1
x
1
y
$x = 1$
$y = x$

In Exercises 33–40, set up and evaluate a double integral to find the volume of the solid bounded by the graphs of the equations.

33. $z = xy$, $z = 0$, $y = x$, $x = 1$, first octant

34. $y = 0$, $z = 0$, $y = x$, $z = x$, $x = 0$, $x = 5$

35. $z = 0$, $z = x^2$, $x = 0$, $x = 2$, $y = 0$, $y = 4$

36. $x^2 + y^2 + z^2 = r^2$

37. $x^2 + z^2 = 1,\ y^2 + z^2 = 1,$ first octant

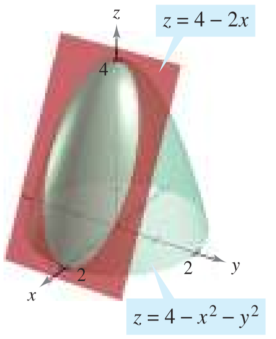
38. $y = 4 - x^2,\ z = 4 - x^2,$ first octant

39. $z = x + y,\ x^2 + y^2 = 4,$ first octant

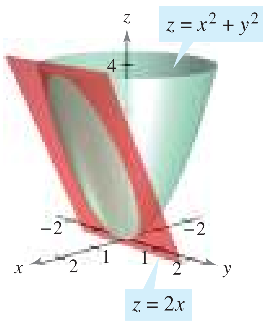
40. $z = \dfrac{1}{1 + y^2},\ x = 0,\ x = 2,\ y \geq 0$

In Exercises 41–46, set up a double integral to find the volume of the solid region bounded by the graphs of the equations. Do not evaluate the integral.

41.



42.



43. $z = x^2 + y^2,\ x^2 + y^2 = 4,\ z = 0$

44. $z = \sin^2 x,\ z = 0,\ 0 \leq x \leq \pi,\ 0 \leq y \leq 5$

45. $z = x^2 + 2y^2,\ z = 4y$

46. $z = x^2 + y^2,\ z = 18 - x^2 - y^2$

CAS **In Exercises 47–50, use a computer algebra system to find the volume of the solid bounded by the graphs of the equations.**

47. $z = 9 - x^2 - y^2,\ z = 0$

48. $x^2 = 9 - y,\ z^2 = 9 - y,$ first octant

49. $z = \dfrac{2}{1 + x^2 + y^2},\ z = 0,\ y = 0,\ x = 0,\ y = -0.5x + 1$

50. $z = \ln(1 + x + y),\ z = 0,\ y = 0,\ x = 0,\ x = 4 - \sqrt{y}$

51. If f is a continuous function such that $0 \leq f(x, y) \leq 1$ over a region R of area 1, prove that $0 \leq \int_R\int f(x, y)\, dA \leq 1$.

52. Find the volume of the solid in the first octant bounded by the coordinate planes and the plane $(x/a) + (y/b) + (z/c) = 1$, where $a > 0$, $b > 0$, and $c > 0$.

In Exercises 53–58, sketch the region of integration. Then evaluate the iterated integral, switching the order of integration if necessary.

53. $\displaystyle\int_0^1 \int_{y/2}^{1/2} e^{-x^2}\, dx\, dy$

54. $\displaystyle\int_0^{\ln 10} \int_{e^x}^{10} \frac{1}{\ln y}\, dy\, dx$

55. $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4 - y^2}\, dy\, dx$

56. $\displaystyle\int_0^3 \int_{y/3}^{1} \frac{1}{1 + x^4}\, dx\, dy$

57. $\displaystyle\int_0^1 \int_0^{\arccos y} \sin x \sqrt{1 + \sin^2 x}\, dx\, dy$

58. $\displaystyle\int_0^2 \int_{(1/2)x^2}^{2} \sqrt{y} \cos y\, dy\, dx$

Average Value **In Exercises 59–64, find the average value of $f(x, y)$ over the region R.**

59. $f(x, y) = x$

R: rectangle with vertices $(0, 0), (4, 0), (4, 2), (0, 2)$

60. $f(x, y) = 2xy$

R: rectangle with vertices $(0, 0), (5, 0), (5, 3), (0, 3)$

61. $f(x, y) = x^2 + y^2$

R: square with vertices $(0, 0), (2, 0), (2, 2), (0, 2)$

62. $f(x, y) = \dfrac{1}{x + y}$

R: triangle with vertices $(0, 0), (1, 0), (1, 1)$

63. $f(x, y) = e^{x+y}$

R: triangle with vertices $(0, 0), (0, 1), (1, 1)$

64. $f(x, y) = \sin(x + y)$

R: rectangle with vertices $(0, 0), (\pi, 0), (\pi, \pi), (0, \pi)$

65. ***Average Production*** The Cobb-Douglas production function for an automobile manufacturer is $f(x, y) = 100x^{0.6}y^{0.4}$, where x is the number of units of labor and y is the number of units of capital. Estimate the average production level if the number of units of labor x varies between 200 and 250 and the number of units of capital y varies between 300 and 325.

66. ***Average Temperature*** The temperature in degrees Celsius on the surface of a metal plate is $T(x, y) = 20 - 4x^2 - y^2$, where x and y are measured in centimeters. Estimate the average temperature if x varies between 0 and 2 centimeters and y varies between 0 and 4 centimeters.

WRITING ABOUT CONCEPTS

67. State the definition of a double integral. If the integrand is a nonnegative function over the region of integration, give the geometric interpretation of a double integral.

68. Let R be a region in the xy-plane whose area is B. If $f(x, y) = k$ for every point (x, y) in R, what is the value of $\int_R\int f(x, y)\, dA$? Explain.

69. Let R represent a county in the northern part of the United States, and let $f(x, y)$ represent the total annual snowfall at the point (x, y) in R. Interpret each of the following.

(a) $\displaystyle\int_R\int f(x, y)\, dA$

(b) $\displaystyle\frac{\int_R\int f(x, y)\, dA}{\int_R\int dA}$

70. Identify the expression that is invalid. Explain your reasoning.

(a) $\displaystyle\int_0^2 \int_0^3 f(x, y)\, dy\, dx$

(b) $\displaystyle\int_0^2 \int_0^y f(x, y)\, dy\, dx$

(c) $\displaystyle\int_0^2 \int_x^3 f(x, y)\, dy\, dx$

(d) $\displaystyle\int_0^2 \int_0^x f(x, y)\, dy\, dx$

71. Let the plane region R be a unit circle and let the maximum value of f on R be 6. Is the greatest possible value of $\int_R\int f(x, y)\, dy\, dx$ equal to 6? Why or why not? If not, what is the greatest possible value?

CAPSTONE

72. The following iterated integrals represent the solution to the same problem. Which iterated integral is easier to evaluate? Explain your reasoning.

$$\int_0^4 \int_{x/2}^2 \sin y^2 \, dy \, dx = \int_0^2 \int_0^{2y} \sin y^2 \, dx \, dy$$

Probability **A joint density function of the continuous random variables x and y is a function $f(x, y)$ satisfying the following properties.**

(a) $f(x, y) \geq 0$ **for all** (x, y) **(b)** $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dA = 1$

(c) $\displaystyle P[(x, y) \in R] = \int_R\int f(x, y)\, dA$

In Exercises 73–76, show that the function is a joint density function and find the required probability.

73. $f(x, y) = \begin{cases} \frac{1}{10}, & 0 \leq x \leq 5,\ 0 \leq y \leq 2 \\ 0, & \text{elsewhere} \end{cases}$

$P(0 \leq x \leq 2,\ 1 \leq y \leq 2)$

74. $f(x, y) = \begin{cases} \frac{1}{4}xy, & 0 \leq x \leq 2,\ 0 \leq y \leq 2 \\ 0, & \text{elsewhere} \end{cases}$

$P(0 \leq x \leq 1,\ 1 \leq y \leq 2)$

75. $f(x, y) = \begin{cases} \frac{1}{27}(9 - x - y), & 0 \leq x \leq 3,\ 3 \leq y \leq 6 \\ 0, & \text{elsewhere} \end{cases}$

$P(0 \leq x \leq 1,\ 4 \leq y \leq 6)$

76. $f(x, y) = \begin{cases} e^{-x-y}, & x \geq 0,\ y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

$P(0 \leq x \leq 1,\ x \leq y \leq 1)$

77. *Approximation* The base of a pile of sand at a cement plant is rectangular with approximate dimensions of 20 meters by 30 meters. If the base is placed on the xy-plane with one vertex at the origin, the coordinates on the surface of the pile are $(5, 5, 3)$, $(15, 5, 6)$, $(25, 5, 4)$, $(5, 15, 2)$, $(15, 15, 7)$, and $(25, 15, 3)$. Approximate the volume of sand in the pile.

78. *Programming* Consider a continuous function $f(x, y)$ over the rectangular region R with vertices (a, c), (b, c), (a, d), and (b, d), where $a < b$ and $c < d$. Partition the intervals $[a, b]$ and $[c, d]$ into m and n subintervals, so that the subintervals in a given direction are of equal length. Write a program for a graphing utility to compute the sum

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\,\Delta A_i \approx \int_a^b\int_c^d f(x, y)\, dA$$

where (x_i, y_j) is the center of a representative rectangle in R.

CAS ***Approximation*** **In Exercises 79–82, (a) use a computer algebra system to approximate the iterated integral, and (b) use the program in Exercise 78 to approximate the iterated integral for the given values of m and n.**

79. $\displaystyle\int_0^1\int_0^2 \sin\sqrt{x + y}\, dy\, dx$

$m = 4,\ n = 8$

80. $\displaystyle\int_0^2\int_0^4 20e^{-x^3/8} dy\, dx$

$m = 10,\ n = 20$

81. $\displaystyle\int_4^6\int_0^2 y\cos\sqrt{x}\, dx\, dy$

$m = 4,\ n = 8$

82. $\displaystyle\int_1^4\int_1^2 \sqrt{x^3 + y^3}\, dx\, dy$

$m = 6,\ n = 4$

Approximation **In Exercises 83 and 84, determine which value best approximates the volume of the solid between the *xy*-plane and the function over the region. (Make your selection on the basis of a sketch of the solid and *not* by performing any calculations.)**

83. $f(x, y) = 4x$

R: square with vertices $(0, 0)$, $(4, 0)$, $(4, 4)$, $(0, 4)$

(a) -200 (b) 600 (c) 50 (d) 125 (e) 1000

84. $f(x, y) = \sqrt{x^2 + y^2}$

R: circle bounded by $x^2 + y^2 = 9$

(a) 50 (b) 500 (c) -500 (d) 5 (e) 5000

True or False? **In Exercises 85 and 86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

85. The volume of the sphere $x^2 + y^2 + z^2 = 1$ is given by the integral

$$V = 8\int_0^1\int_0^1 \sqrt{1 - x^2 - y^2}\, dx\, dy.$$

86. If $f(x, y) \leq g(x, y)$ for all (x, y) in R, and both f and g are continuous over R, then $\int_R\int f(x, y)\, dA \leq \int_R\int g(x, y)\, dA$.

87. Let $f(x) = \int_1^x e^{t^2}\, dt$. Find the average value of f on the interval $[0, 1]$.

88. Find $\displaystyle\int_0^{\infty} \frac{e^{-x} - e^{-2x}}{x}\, dx.$ $\left(\textit{Hint: } \text{Evaluate } \displaystyle\int_1^2 e^{-xy}\, dy.\right)$

89. Determine the region R in the xy-plane that maximizes the value of $\int_R\int (9 - x^2 - y^2)\, dA$.

90. Determine the region R in the xy-plane that minimizes the value of $\int_R\int (x^2 + y^2 - 4)\, dA$.

91. Find $\int_0^2 [\arctan(\pi x) - \arctan x]\, dx$. (*Hint:* Convert the integral to a double integral.)

92. Use a geometric argument to show that

$$\int_0^3\int_0^{\sqrt{9-y^2}} \sqrt{9 - x^2 - y^2}\, dx\, dy = \frac{9\pi}{2}.$$

PUTNAM EXAM CHALLENGE

93. Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2,\, a^2y^2\}}\, dy\, dx$, where a and b are positive.

94. Show that if $\lambda > \frac{1}{2}$ there does not exist a real-valued function u such that for all x in the closed interval $0 \leq x \leq 1$, $u(x) = 1 + \lambda\int_x^1 u(y)u(y - x)\, dy$.

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14.3 Change of Variables: Polar Coordinates

- **Write and evaluate double integrals in polar coordinates.**

Double Integrals in Polar Coordinates

Some double integrals are *much* easier to evaluate in polar form than in rectangular form. This is especially true for regions such as circles, cardioids, and rose curves, and for integrands that involve $x^2 + y^2$.

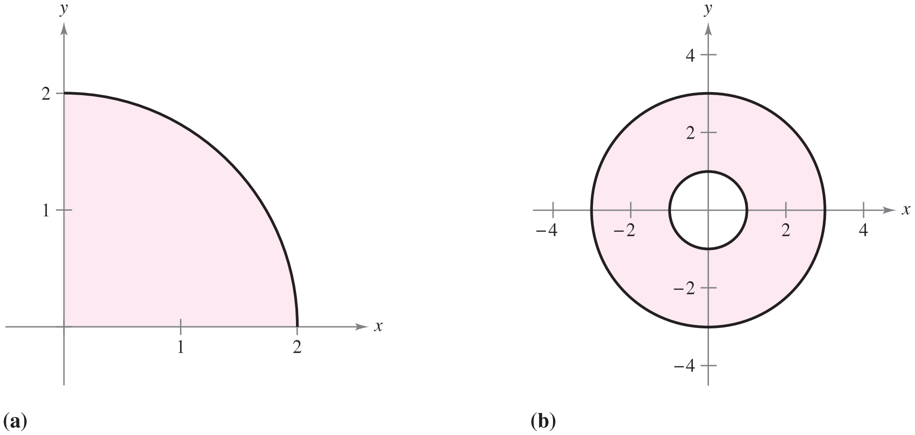
In Section 10.4, you learned that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

EXAMPLE 1 Using Polar Coordinates to Describe a Region

Use polar coordinates to describe each region shown in Figure 14.24.



(a) **(b)**

Figure 14.24

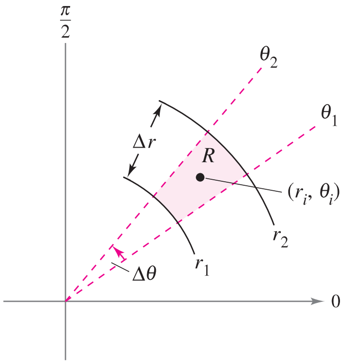
Solution

a. The region R is a quarter circle of radius 2. It can be described in polar coordinates as

$$R = \{(r, \theta): 0 \le r \le 2, \quad 0 \le \theta \le \pi/2\}.$$

b. The region R consists of all points between concentric circles of radii 1 and 3. It can be described in polar coordinates as

$$R = \{(r, \theta): 1 \le r \le 3, \quad 0 \le \theta \le 2\pi\}.$$ ■

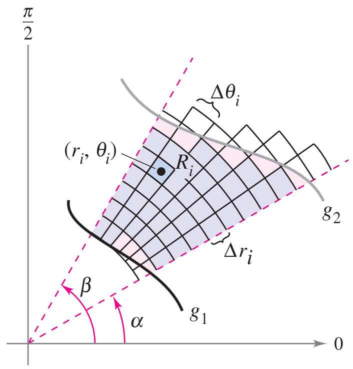


Polar sector
Figure 14.25

The regions in Example 1 are special cases of **polar sectors**

$$R = \{(r, \theta): r_1 \le r \le r_2, \quad \theta_1 \le \theta \le \theta_2\}$$ Polar sector

as shown in Figure 14.25.



Polar grid superimposed over region R
Figure 14.26

To define a double integral of a continuous function $z = f(x, y)$ in polar coordinates, consider a region R bounded by the graphs of $r = g_1(\theta)$ and $r = g_2(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$. Instead of partitioning R into small rectangles, use a partition of small polar sectors. On R, superimpose a polar grid made of rays and circular arcs, as shown in Figure 14.26. The polar sectors R_i lying entirely within R form an **inner polar partition** Δ, whose **norm** $\|\Delta\|$ is the length of the longest diagonal of the n polar sectors.

Consider a specific polar sector R_i, as shown in Figure 14.27. It can be shown (see Exercise 75) that the area of R_i is

$$\Delta A_i = r_i \Delta r_i \Delta\theta_i \qquad \text{Area of } R_i$$

where $\Delta r_i = r_2 - r_1$ and $\Delta\theta_i = \theta_2 - \theta_1$. This implies that the volume of the solid of height $f(r_i \cos\theta_i, r_i \sin\theta_i)$ above R_i is approximately

$$f(r_i \cos\theta_i, r_i \sin\theta_i) r_i \Delta r_i \Delta\theta_i$$

and you have

$$\int_R\!\!\int f(x, y)\, dA \approx \sum_{i=1}^{n} f(r_i \cos\theta_i, r_i \sin\theta_i) r_i \Delta r_i \Delta\theta_i.$$

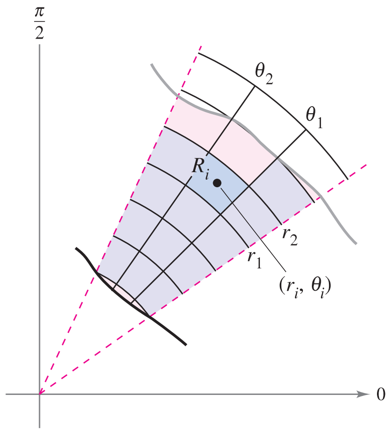
The sum on the right can be interpreted as a Riemann sum for $f(r\cos\theta, r\sin\theta)r$. The region R corresponds to a *horizontally simple* region S in the $r\theta$-plane, as shown in Figure 14.28. The polar sectors R_i correspond to rectangles S_i, and the area ΔA_i of S_i is $\Delta r_i \Delta\theta_i$. So, the right-hand side of the equation corresponds to the double integral

$$\int_S\!\!\int f(r\cos\theta, r\sin\theta) r\, dA.$$

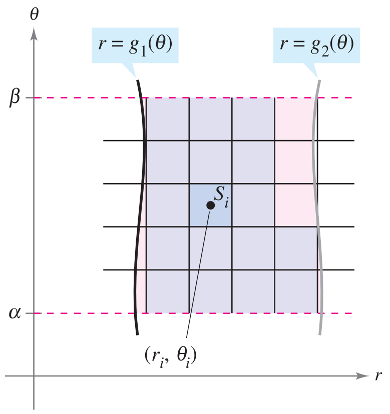
From this, you can apply Theorem 14.2 to write

$$\begin{aligned}\int_R\!\!\int f(x, y)\, dA &= \int_S\!\!\int f(r\cos\theta, r\sin\theta) r\, dA \\ &= \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.\end{aligned}$$

This suggests the following theorem, the proof of which is discussed in Section 14.8.



The polar sector R_i is the set of all points (r, θ) such that $r_1 \le r \le r_2$ and $\theta_1 \le \theta \le \theta_2$.
Figure 14.27



Horizontally simple region S
Figure 14.28

THEOREM 14.3 CHANGE OF VARIABLES TO POLAR FORM

Let R be a plane region consisting of all points $(x, y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le (\beta - \alpha) \le 2\pi$. If g_1 and g_2 are continuous on $[\alpha, \beta]$ and f is continuous on R, then

$$\int_R\!\!\int f(x, y)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

EXPLORATION

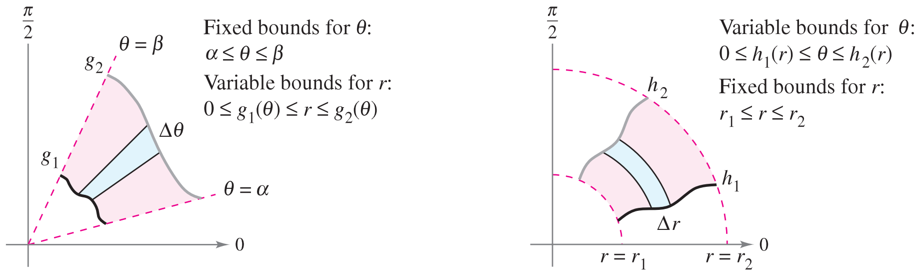
Volume of a Paraboloid Sector
In the Exploration feature on page 997, you were asked to summarize the different ways you know of finding the volume of the solid bounded by the paraboloid

$$z = a^2 - x^2 - y^2, \quad a > 0$$

and the xy-plane. You now know another way. Use it to find the volume of the solid.

NOTE If $z = f(x, y)$ is nonnegative on R, then the integral in Theorem 14.3 can be interpreted as the volume of the solid region between the graph of f and the region R. When using the integral in Theorem 14.3, be certain not to omit the extra factor of r in the integrand. ■

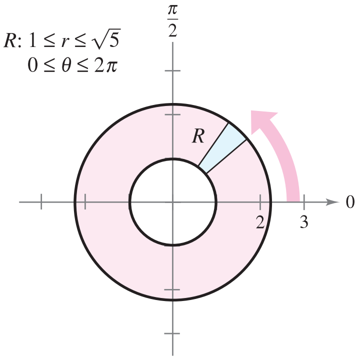
The region R is restricted to two basic types, ***r*-simple** regions and ***θ*-simple** regions, as shown in Figure 14.29.


$\frac{\pi}{2}$
$\theta = \beta$
g_2
$\Delta\theta$
g_1
$\theta = \alpha$
0
Fixed bounds for θ:
$\alpha \le \theta \le \beta$
Variable bounds for r:
$0 \le g_1(\theta) \le r \le g_2(\theta)$
$\frac{\pi}{2}$
h_2
h_1
Δr
0
$r = r_1$
$r = r_2$
Variable bounds for θ:
$0 \le h_1(r) \le \theta \le h_2(r)$
Fixed bounds for r:
$r_1 \le r \le r_2$

r-Simple region θ-Simple region
Figure 14.29

EXAMPLE 2 Evaluating a Double Polar Integral

Let R be the annular region lying between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 5$. Evaluate the integral $\int_R\int (x^2 + y)\, dA$.


$R: 1 \le r \le \sqrt{5}$
$0 \le \theta \le 2\pi$
$\frac{\pi}{2}$
R
2
3
0

r-Simple region
Figure 14.30

Solution The polar boundaries are $1 \le r \le \sqrt{5}$ and $0 \le \theta \le 2\pi$, as shown in Figure 14.30. Furthermore, $x^2 = (r\cos\theta)^2$ and $y = r\sin\theta$. So, you have

$$\begin{aligned}
\int_R\!\!\int (x^2 + y)\, dA &= \int_0^{2\pi}\int_1^{\sqrt{5}} (r^2\cos^2\theta + r\sin\theta) r\, dr\, d\theta \\
&= \int_0^{2\pi}\int_1^{\sqrt{5}} (r^3\cos^2\theta + r^2\sin\theta)\, dr\, d\theta \\
&= \int_0^{2\pi} \left(\frac{r^4}{4}\cos^2\theta + \frac{r^3}{3}\sin\theta\right)\Bigg]_1^{\sqrt{5}} d\theta \\
&= \int_0^{2\pi} \left(6\cos^2\theta + \frac{5\sqrt{5} - 1}{3}\sin\theta\right) d\theta \\
&= \int_0^{2\pi} \left(3 + 3\cos 2\theta + \frac{5\sqrt{5} - 1}{3}\sin\theta\right) d\theta \\
&= \left(3\theta + \frac{3\sin 2\theta}{2} - \frac{5\sqrt{5} - 1}{3}\cos\theta\right)\Bigg]_0^{2\pi} \\
&= 6\pi.
\end{aligned}$$

■

In Example 2, be sure to notice the extra factor of r in the integrand. This comes from the formula for the area of a polar sector. In differential notation, you can write

$$dA = r\,dr\,d\theta$$

which indicates that the area of a polar sector increases as you move away from the origin.

EXAMPLE 3 Change of Variables to Polar Coordinates

Use polar coordinates to find the volume of the solid region bounded above by the hemisphere

$$z = \sqrt{16 - x^2 - y^2} \qquad \text{Hemisphere forms upper surface.}$$

and below by the circular region R given by

$$x^2 + y^2 \le 4 \qquad \text{Circular region forms lower surface.}$$

as shown in Figure 14.31.

Surface: $z = \sqrt{16 - x^2 - y^2}$

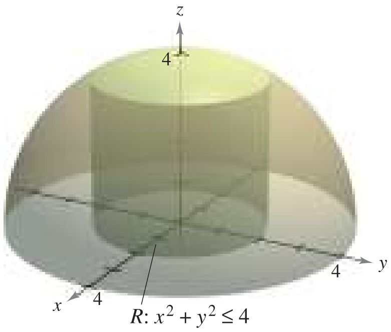


Figure 14.31

Solution In Figure 14.31, you can see that R has the bounds

$$-\sqrt{4 - y^2} \le x \le \sqrt{4 - y^2}, \quad -2 \le y \le 2$$

and that $0 \le z \le \sqrt{16 - x^2 - y^2}$. In polar coordinates, the bounds are

$$0 \le r \le 2 \quad \text{and} \quad 0 \le \theta \le 2\pi$$

with height $z = \sqrt{16 - x^2 - y^2} = \sqrt{16 - r^2}$. Consequently, the volume V is given by

$$\begin{aligned} V = \int\!\!\int_R f(x, y)\,dA &= \int_0^{2\pi}\int_0^2 \sqrt{16 - r^2}\,r\,dr\,d\theta \\ &= -\frac{1}{3}\int_0^{2\pi} (16 - r^2)^{3/2}\Big]_0^2\,d\theta \\ &= -\frac{1}{3}\int_0^{2\pi} \left(24\sqrt{3} - 64\right)d\theta \\ &= -\frac{8}{3}\left(3\sqrt{3} - 8\right)\theta\Big]_0^{2\pi} \\ &= \frac{16\pi}{3}\left(8 - 3\sqrt{3}\right) \approx 46.979. \end{aligned}$$

■

NOTE To see the benefit of polar coordinates in Example 3, you should try to evaluate the corresponding rectangular iterated integral

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{16 - x^2 - y^2}\,dx\,dy.$$

TECHNOLOGY Any computer algebra system that can handle double integrals in rectangular coordinates can also handle double integrals in polar coordinates. The reason this is true is that once you have formed the iterated integral, its value is not changed by using different variables. In other words, if you use a computer algebra system to evaluate

$$\int_0^{2\pi}\int_0^2 \sqrt{16 - x^2}\,x\,dx\,dy$$

you should obtain the same value as that obtained in Example 3.

Just as with rectangular coordinates, the double integral

$$\int\!\!\int_R dA$$

can be used to find the area of a region in the plane.

EXAMPLE 4 Finding Areas of Polar Regions

Use a double integral to find the area enclosed by the graph of $r = 3\cos 3\theta$.

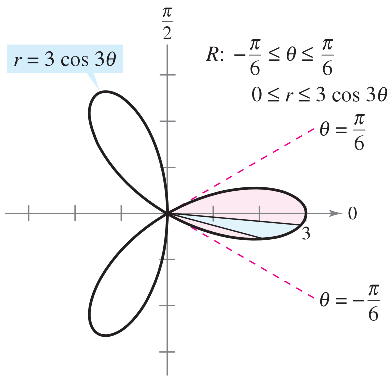

$\frac{\pi}{2}$
$r = 3\cos 3\theta$
R: $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$
$0 \le r \le 3\cos 3\theta$
$\theta = \frac{\pi}{6}$
0
3
$\theta = -\frac{\pi}{6}$

Figure 14.32

Solution Let R be one petal of the curve shown in Figure 14.32. This region is r-simple, and the boundaries are as follows.

$$-\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \qquad \text{Fixed bounds on } \theta$$

$$0 \le r \le 3\cos 3\theta \qquad \text{Variable bounds on } r$$

So, the area of one petal is

$$\begin{aligned}
\frac{1}{3}A = \int\!\!\int_R dA &= \int_{-\pi/6}^{\pi/6}\int_0^{3\cos 3\theta} r\,dr\,d\theta \\
&= \int_{-\pi/6}^{\pi/6} \frac{r^2}{2}\Big]_0^{3\cos 3\theta} d\theta \\
&= \frac{9}{2}\int_{-\pi/6}^{\pi/6} \cos^2 3\theta\,d\theta \\
&= \frac{9}{4}\int_{-\pi/6}^{\pi/6} (1 + \cos 6\theta)\,d\theta = \frac{9}{4}\Big[\theta + \frac{1}{6}\sin 6\theta\Big]_{-\pi/6}^{\pi/6} = \frac{3\pi}{4}.
\end{aligned}$$

So, the total area is $A = 9\pi/4$. ■

As illustrated in Example 4, the area of a region in the plane can be represented by

$$A = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} r\,dr\,d\theta.$$

If $g_1(\theta) = 0$, you obtain

$$A = \int_\alpha^\beta \int_0^{g_2(\theta)} r\,dr\,d\theta = \int_\alpha^\beta \frac{r^2}{2}\Big]_0^{g_2(\theta)} d\theta = \int_\alpha^\beta \frac{1}{2}(g_2(\theta))^2\,d\theta$$

which agrees with Theorem 10.13.

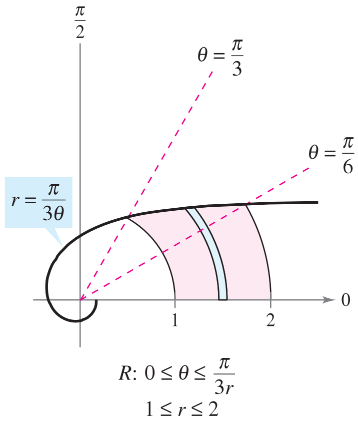
So far in this section, all of the examples of iterated integrals in polar form have been of the form

$$\int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$$

in which the order of integration is with respect to r first. Sometimes you can obtain a simpler integration problem by switching the order of integration, as illustrated in the next example.

EXAMPLE 5 Changing the Order of Integration

Find the area of the region bounded above by the spiral $r = \pi/(3\theta)$ and below by the polar axis, between $r = 1$ and $r = 2$.



θ-Simple region
Figure 14.33

Solution The region is shown in Figure 14.33. The polar boundaries for the region are

$$1 \le r \le 2 \quad \text{and} \quad 0 \le \theta \le \frac{\pi}{3r}.$$

So, the area of the region can be evaluated as follows.

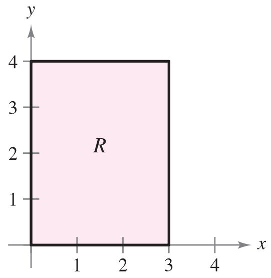
$$A = \int_1^2 \int_0^{\pi/(3r)} r\,d\theta\,dr = \int_1^2 r\theta\Big]_0^{\pi/(3r)} dr = \int_1^2 \frac{\pi}{3}\,dr = \frac{\pi r}{3}\Big]_1^2 = \frac{\pi}{3}$$ ■

14.3 Exercises

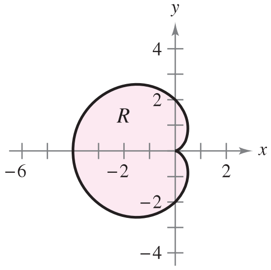
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, the region R for the integral $\int_R\int f(x, y)\,dA$ is shown. State whether you would use rectangular or polar coordinates to evaluate the integral.

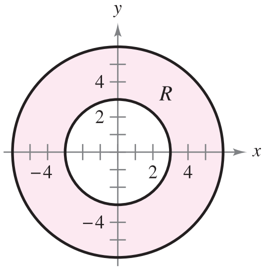
1.



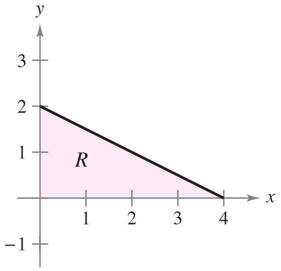
2.



3.

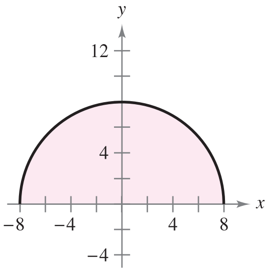


4.

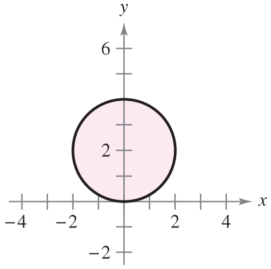


In Exercises 5–8, use polar coordinates to describe the region shown.

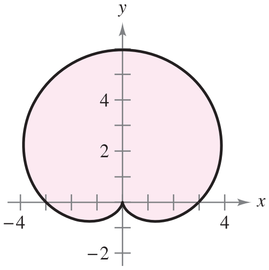
5.



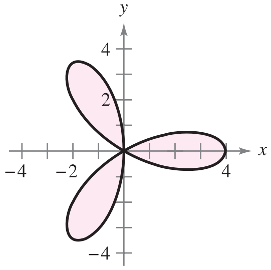
6.



7.



8.



In Exercises 9–16, evaluate the double integral $\int_R\int f(r, \theta)\,dA$, and sketch the region R.

9. $\displaystyle\int_0^{\pi}\int_0^{\cos\theta} r\,dr\,d\theta$

10. $\displaystyle\int_0^{\pi}\int_0^{\sin\theta} r^2\,dr\,d\theta$

11. $\displaystyle\int_0^{2\pi}\int_0^{6} 3r^2\sin\theta\,dr\,d\theta$

12. $\displaystyle\int_0^{\pi/4}\int_0^{4} r^2\sin\theta\cos\theta\,dr\,d\theta$

13. $\displaystyle\int_0^{\pi/2}\int_2^{3} \sqrt{9-r^2}\,r\,dr\,d\theta$

14. $\displaystyle\int_0^{\pi/2}\int_0^{3} re^{-r^2}\,dr\,d\theta$

15. $\displaystyle\int_0^{\pi/2}\int_0^{1+\sin\theta} \theta r\,dr\,d\theta$

16. $\displaystyle\int_0^{\pi/2}\int_0^{1-\cos\theta} (\sin\theta)r\,dr\,d\theta$

In Exercises 17–26, evaluate the iterated integral by converting to polar coordinates.

17. $\displaystyle\int_0^{a}\int_0^{\sqrt{a^2-y^2}} y\,dx\,dy$

18. $\displaystyle\int_0^{a}\int_0^{\sqrt{a^2-x^2}} x\,dy\,dx$

19. $\displaystyle\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$

20. $\displaystyle\int_0^{1}\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} (x^2+y^2)\,dy\,dx$

21. $\displaystyle\int_0^{3}\int_0^{\sqrt{9-x^2}} (x^2+y^2)^{3/2}\,dy\,dx$

22. $\displaystyle\int_0^{2}\int_y^{\sqrt{8-y^2}} \sqrt{x^2+y^2}\,dx\,dy$

23. $\displaystyle\int_0^{2}\int_0^{\sqrt{2x-x^2}} xy\,dy\,dx$

24. $\displaystyle\int_0^{4}\int_0^{\sqrt{4y-y^2}} x^2\,dx\,dy$

25. $\displaystyle\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} \cos(x^2+y^2)\,dy\,dx$

26. $\displaystyle\int_0^{2}\int_0^{\sqrt{4-x^2}} \sin\sqrt{x^2+y^2}\,dy\,dx$

In Exercises 27 and 28, combine the sum of the two iterated integrals into a single iterated integral by converting to polar coordinates. Evaluate the resulting iterated integral.

27. $\displaystyle\int_0^{2}\int_0^{x} \sqrt{x^2+y^2}\,dy\,dx + \int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

28. $\displaystyle\int_0^{5\sqrt{2}/2}\int_0^{x} xy\,dy\,dx + \int_{5\sqrt{2}/2}^{5}\int_0^{\sqrt{25-x^2}} xy\,dy\,dx$

In Exercises 29–32, use polar coordinates to set up and evaluate the double integral $\int_R\int f(x, y)\,dA$.

29. $f(x, y) = x + y$, R: $x^2 + y^2 \le 4$, $x \ge 0$, $y \ge 0$

30. $f(x, y) = e^{-(x^2+y^2)/2}$, R: $x^2 + y^2 \le 25$, $x \ge 0$

31. $f(x, y) = \arctan\dfrac{y}{x}$, R: $x^2 + y^2 \ge 1$, $x^2 + y^2 \le 4$, $0 \le y \le x$

32. $f(x, y) = 9 - x^2 - y^2$, R: $x^2 + y^2 \le 9$, $x \ge 0$, $y \ge 0$

Volume **In Exercises 33–38, use a double integral in polar coordinates to find the volume of the solid bounded by the graphs of the equations.**

33. $z = xy$, $x^2 + y^2 = 1$, first octant

34. $z = x^2 + y^2 + 3$, $z = 0$, $x^2 + y^2 = 1$

35. $z = \sqrt{x^2 + y^2}$, $z = 0$, $x^2 + y^2 = 25$

36. $z = \ln(x^2 + y^2)$, $z = 0$, $x^2 + y^2 \ge 1$, $x^2 + y^2 \le 4$

37. Inside the hemisphere $z = \sqrt{16 - x^2 - y^2}$ and inside the cylinder $x^2 + y^2 - 4x = 0$

38. Inside the hemisphere $z = \sqrt{16 - x^2 - y^2}$ and outside the cylinder $x^2 + y^2 = 1$

39. ***Volume*** Find a such that the volume inside the hemisphere $z = \sqrt{16 - x^2 - y^2}$ and outside the cylinder $x^2 + y^2 = a^2$ is one-half the volume of the hemisphere.

40. ***Volume*** Use a double integral in polar coordinates to find the volume of a sphere of radius a.

41. ***Volume*** Determine the diameter of a hole that is drilled vertically through the center of the solid bounded by the graphs of the equations $z = 25e^{-(x^2+y^2)/4}$, $z = 0$, and $x^2 + y^2 = 16$ if one-tenth of the volume of the solid is removed.

CAS **42.** ***Machine Design*** The surfaces of a double-lobed cam are modeled by the inequalities $\frac{1}{4} \le r \le \frac{1}{2}(1 + \cos^2 \theta)$ and

$$\frac{-9}{4(x^2 + y^2 + 9)} \le z \le \frac{9}{4(x^2 + y^2 + 9)}$$

where all measurements are in inches.

(a) Use a computer algebra system to graph the cam.

(b) Use a computer algebra system to approximate the perimeter of the polar curve

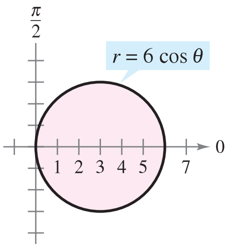
$$r = \tfrac{1}{2}(1 + \cos^2 \theta).$$

This is the distance a roller must travel as it runs against the cam through one revolution of the cam.

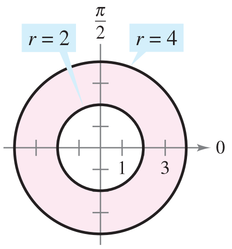
(c) Use a computer algebra system to find the volume of steel in the cam.

Area **In Exercises 43–48, use a double integral to find the area of the shaded region.**

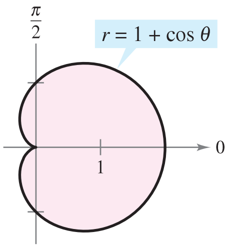
43.



44.



45.



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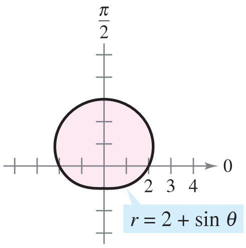

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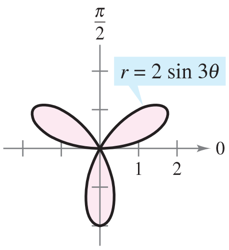
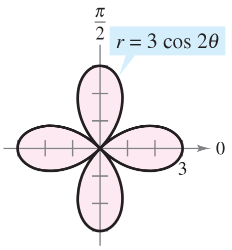


48.



Area **In Exercises 49–54, sketch a graph of the region bounded by the graphs of the equations. Then use a double integral to find the area of the region.**

49. Inside the circle $r = 2\cos\theta$ and outside the circle $r = 1$

50. Inside the cardioid $r = 2 + 2\cos\theta$ and outside the circle $r = 1$

51. Inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$

52. Inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$

53. Inside the rose curve $r = 4\sin 3\theta$ and outside the circle $r = 2$

54. Inside the circle $r = 2$ and outside the cardioid $r = 2 - 2\cos\theta$

WRITING ABOUT CONCEPTS

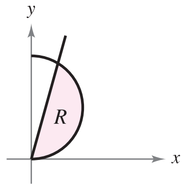
55. Describe the partition of the region R of integration in the xy-plane when polar coordinates are used to evaluate a double integral.

56. Explain how to change from rectangular coordinates to polar coordinates in a double integral.

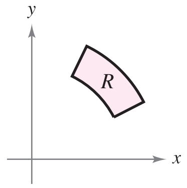
57. In your own words, describe r-simple regions and θ-simple regions.

58. Each figure shows a region of integration for the double integral $\int_R\int f(x, y)\, dA$. For each region, state whether horizontal representative elements, vertical representative elements, or polar sectors would yield the easiest method for obtaining the limits of integration. Explain your reasoning.

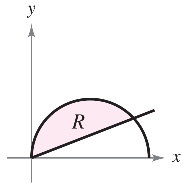
(a)



(b)



(c)



59. Let R be the region bounded by the circle $x^2 + y^2 = 9$.

(a) Set up the integral $\int_R\int f(x, y)\, dA$.

(b) Convert the integral in part (a) to polar coordinates.

(c) Which integral would you choose to evaluate? Why?

CAPSTONE

60. ***Think About It*** Without performing any calculations, identify the double integral that represents the integral of $f(x) = x^2 + y^2$ over a circle of radius 4. Explain your reasoning.

(a) $\displaystyle\int_0^{2\pi}\int_0^4 r^2\, dr\, d\theta$ (b) $\displaystyle\int_0^4\int_0^{2\pi} r^3\, dr\, d\theta$

(c) $\displaystyle\int_0^{2\pi}\int_0^4 r^3\, dr\, d\theta$ (d) $\displaystyle\int_0^{2\pi}\int_{-4}^4 r^3\, dr\, d\theta$

61. ***Think About It*** Consider the program you wrote to approximate double integrals in rectangular coordinates in Exercise 78, in Section 14.2. If the program is used to approximate the double integral

$$\int_R\int f(r, \theta)\, dA$$

in polar coordinates, how will you modify f when it is entered into the program? Because the limits of integration are constants, describe the plane region of integration.

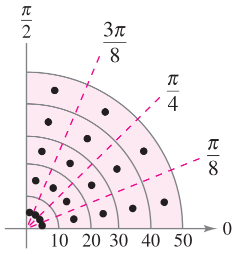
62. ***Approximation*** Horizontal cross sections of a piece of ice that broke from a glacier are in the shape of a quarter of a circle with a radius of approximately 50 feet. The base is divided into 20 subregions, as shown in the figure. At the center of each subregion, the height of the ice is measured, yielding the following points in cylindrical coordinates.

$$\left(5, \tfrac{\pi}{16}, 7\right), \left(15, \tfrac{\pi}{16}, 8\right), \left(25, \tfrac{\pi}{16}, 10\right), \left(35, \tfrac{\pi}{16}, 12\right), \left(45, \tfrac{\pi}{16}, 9\right),$$
$$\left(5, \tfrac{3\pi}{16}, 9\right), \left(15, \tfrac{3\pi}{16}, 10\right), \left(25, \tfrac{3\pi}{16}, 14\right), \left(35, \tfrac{3\pi}{16}, 15\right), \left(45, \tfrac{3\pi}{16}, 10\right),$$
$$\left(5, \tfrac{5\pi}{16}, 9\right), \left(15, \tfrac{5\pi}{16}, 11\right), \left(25, \tfrac{5\pi}{16}, 15\right), \left(35, \tfrac{5\pi}{16}, 18\right), \left(45, \tfrac{5\pi}{16}, 14\right),$$
$$\left(5, \tfrac{7\pi}{16}, 5\right), \left(15, \tfrac{7\pi}{16}, 8\right), \left(25, \tfrac{7\pi}{16}, 11\right), \left(35, \tfrac{7\pi}{16}, 16\right), \left(45, \tfrac{7\pi}{16}, 12\right)$$

(a) Approximate the volume of the solid.

(b) Ice weighs approximately 57 pounds per cubic foot. Approximate the weight of the solid.

(c) There are 7.48 gallons of water per cubic foot. Approximate the number of gallons of water in the solid.



CAS ***Approximation*** **In Exercises 63 and 64, use a computer algebra system to approximate the iterated integral.**

63. $\displaystyle\int_{\pi/4}^{\pi/2}\int_0^5 r\sqrt{1 + r^3}\sin\sqrt{\theta}\, dr\, d\theta$

64. $\displaystyle\int_0^{\pi/4}\int_0^4 5re^{\sqrt{r\theta}}\, dr\, d\theta$

Approximation **In Exercises 65 and 66, determine which value best approximates the volume of the solid between the *xy*-plane and the function over the region. (Make your selection on the basis of a sketch of the solid and *not* by performing any calculations.)**

65. $f(x, y) = 15 - 2y$; R: semicircle: $x^2 + y^2 = 16$, $y \geq 0$

(a) 100 (b) 200 (c) 300 (d) -200 (e) 800

66. $f(x, y) = xy + 2$; R: quarter circle: $x^2 + y^2 = 9$, $x \geq 0$, $y \geq 0$

(a) 25 (b) 8 (c) 100 (d) 50 (e) -30

True or False? **In Exercises 67 and 68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

67. If $\int_R\int f(r, \theta)\, dA > 0$, then $f(r, \theta) > 0$ for all (r, θ) in R.

68. If $f(r, \theta)$ is a constant function and the area of the region S is twice that of the region R, then $2\int_R\int f(r, \theta)\, dA = \int_S\int f(r, \theta)\, dA$.

69. ***Probability*** The value of the integral $I = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx$ is required in the development of the normal probability density function.

(a) Use polar coordinates to evaluate the improper integral.

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2/2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\, dy\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dA \end{aligned}$$

(b) Use the result of part (a) to determine I.

■ **FOR FURTHER INFORMATION** For more information on this problem, see the article "Integrating e^{-x^2} Without Polar Coordinates" by William Dunham in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

70. Use the result of Exercise 69 and a change of variables to evaluate each integral. No integration is required.

(a) $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx$ (b) $\displaystyle\int_{-\infty}^{\infty} e^{-4x^2}\, dx$

71. ***Population*** The population density of a city is approximated by the model $f(x, y) = 4000e^{-0.01(x^2+y^2)}$, $x^2 + y^2 \leq 49$, where x and y are measured in miles. Integrate the density function over the indicated circular region to approximate the population of the city.

72. ***Probability*** Find k such that the function

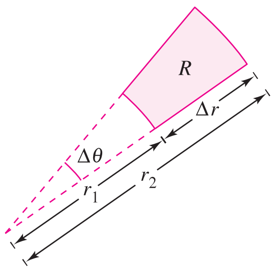
$$f(x, y) = \begin{cases} ke^{-(x^2+y^2)}, & x \geq 0, y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

is a probability density function.

73. ***Think About It*** Consider the region bounded by the graphs of $y = 2$, $y = 4$, $y = x$, and $y = \sqrt{3}x$ and the double integral $\int_R\int f\, dA$. Determine the limits of integration if the region R is divided into (a) horizontal representative elements, (b) vertical representative elements, and (c) polar sectors.

74. Repeat Exercise 73 for a region R bounded by the graph of the equation $(x - 2)^2 + y^2 = 4$.

75. Show that the area A of the polar sector R (see figure) is $A = r\Delta r\Delta\theta$, where $r = (r_1 + r_2)/2$ is the average radius of R.



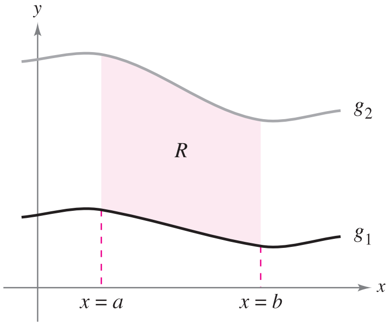
14.4 Center of Mass and Moments of Inertia

- **Find the mass of a planar lamina using a double integral.**
- **Find the center of mass of a planar lamina using double integrals.**
- **Find moments of inertia using double integrals.**

Mass

Section 7.6 discussed several applications of integration involving a lamina of *constant* density ρ. For example, if the lamina corresponding to the region R, as shown in Figure 14.34, has a constant density ρ, then the mass of the lamina is given by

$$\text{Mass} = \rho A = \rho \int_R\int dA = \int_R\int \rho \, dA. \qquad \text{Constant density}$$



Lamina of constant density ρ
Figure 14.34

If not otherwise stated, a lamina is assumed to have a constant density. In this section, however, you will extend the definition of the term *lamina* to include thin plates of *variable* density. Double integrals can be used to find the mass of a lamina of variable density, where the density at (x, y) is given by the **density function ρ.**

DEFINITION OF MASS OF A PLANAR LAMINA OF VARIABLE DENSITY

If ρ is a continuous density function on the lamina corresponding to a plane region R, then the mass m of the lamina is given by

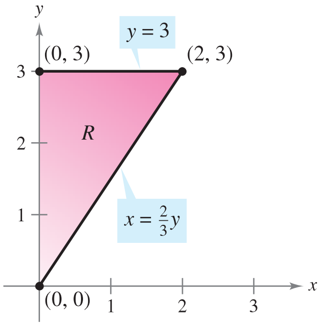
$$m = \int_R\int \rho(x, y)\, dA. \qquad \text{Variable density}$$

NOTE Density is normally expressed as mass per unit volume. For a planar lamina, however, density is mass per unit surface area.

EXAMPLE 1 Finding the Mass of a Planar Lamina

Find the mass of the triangular lamina with vertices (0, 0), (0, 3), and (2, 3), given that the density at (x, y) is $\rho(x, y) = 2x + y$.

Solution As shown in Figure 14.35, region R has the boundaries $x = 0$, $y = 3$, and $y = 3x/2$ (or $x = 2y/3$). Therefore, the mass of the lamina is



Lamina of variable density $\rho(x, y) = 2x + y$
Figure 14.35

$$\begin{aligned} m = \int_R\int (2x + y)\, dA &= \int_0^3 \int_0^{2y/3} (2x + y)\, dx\, dy \\ &= \int_0^3 \Big[x^2 + xy\Big]_0^{2y/3} dy \\ &= \frac{10}{9}\int_0^3 y^2\, dy \\ &= \frac{10}{9}\left[\frac{y^3}{3}\right]_0^3 \\ &= 10. \end{aligned}$$

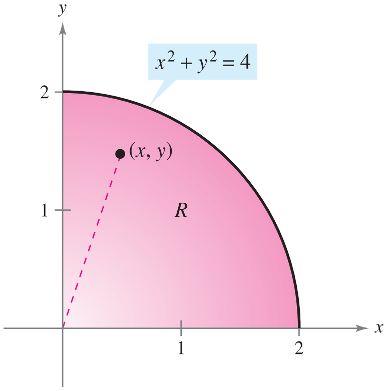
NOTE In Figure 14.35, note that the planar lamina is shaded so that the darkest shading corresponds to the densest part.

EXAMPLE 2 Finding Mass by Polar Coordinates

Find the mass of the lamina corresponding to the first-quadrant portion of the circle

$$x^2 + y^2 = 4$$

where the density at the point (x, y) is proportional to the distance between the point and the origin, as shown in Figure 14.36.



Density at (x, y): $\rho(x, y) = k\sqrt{x^2 + y^2}$
Figure 14.36

Solution At any point (x, y), the density of the lamina is

$$\begin{aligned}\rho(x, y) &= k\sqrt{(x-0)^2 + (y-0)^2}\\ &= k\sqrt{x^2 + y^2}.\end{aligned}$$

Because $0 \le x \le 2$ and $0 \le y \le \sqrt{4 - x^2}$, the mass is given by

$$\begin{aligned}m &= \int\!\!\int_R k\sqrt{x^2 + y^2}\, dA\\ &= \int_0^2 \int_0^{\sqrt{4-x^2}} k\sqrt{x^2 + y^2}\, dy\, dx.\end{aligned}$$

To simplify the integration, you can convert to polar coordinates, using the bounds $0 \le \theta \le \pi/2$ and $0 \le r \le 2$. So, the mass is

$$\begin{aligned}m = \int\!\!\int_R k\sqrt{x^2 + y^2}\, dA &= \int_0^{\pi/2} \int_0^2 k\sqrt{r^2}\, r\, dr\, d\theta\\ &= \int_0^{\pi/2} \int_0^2 kr^2\, dr\, d\theta\\ &= \int_0^{\pi/2} \frac{kr^3}{3}\bigg]_0^2 d\theta\\ &= \frac{8k}{3}\int_0^{\pi/2} d\theta\\ &= \frac{8k}{3}\bigg[\theta\bigg]_0^{\pi/2}\\ &= \frac{4\pi k}{3}.\end{aligned}$$

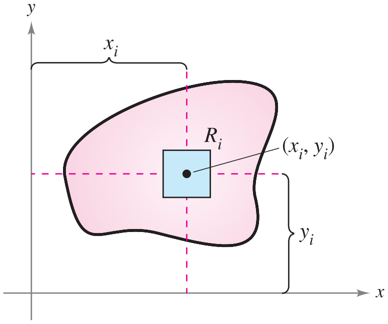
■

TECHNOLOGY On many occasions, this text has mentioned the benefits of computer programs that perform symbolic integration. Even if you use such a program regularly, you should remember that its greatest benefit comes only in the hands of a knowledgeable user. For instance, notice how much simpler the integral in Example 2 becomes when it is converted to polar form.

Rectangular Form	*Polar Form*
$\int_0^2 \int_0^{\sqrt{4-x^2}} k\sqrt{x^2 + y^2}\, dy\, dx$	$\int_0^{\pi/2} \int_0^2 kr^2\, dr\, d\theta$

If you have access to software that performs symbolic integration, use it to evaluate both integrals. Some software programs cannot handle the first integral, but any program that can handle double integrals can evaluate the second integral.

Moments and Center of Mass



$M_x = (\text{mass})(y_i)$
$M_y = (\text{mass})(x_i)$
Figure 14.37

For a lamina of variable density, moments of mass are defined in a manner similar to that used for the uniform density case. For a partition Δ of a lamina corresponding to a plane region R, consider the ith rectangle R_i of one area ΔA_i, as shown in Figure 14.37. Assume that the mass of R_i is concentrated at one of its interior points (x_i, y_i). The moment of mass of R_i with respect to the x-axis can be approximated by

$$(\text{Mass})(y_i) \approx [\rho(x_i, y_i)\,\Delta A_i](y_i).$$

Similarly, the moment of mass with respect to the y-axis can be approximated by

$$(\text{Mass})(x_i) \approx [\rho(x_i, y_i)\,\Delta A_i](x_i).$$

By forming the Riemann sum of all such products and taking the limits as the norm of Δ approaches 0, you obtain the following definitions of moments of mass with respect to the x- and y-axes.

MOMENTS AND CENTER OF MASS OF A VARIABLE DENSITY PLANAR LAMINA

Let ρ be a continuous density function on the planar lamina R. The **moments of mass** with respect to the x- and y-axes are

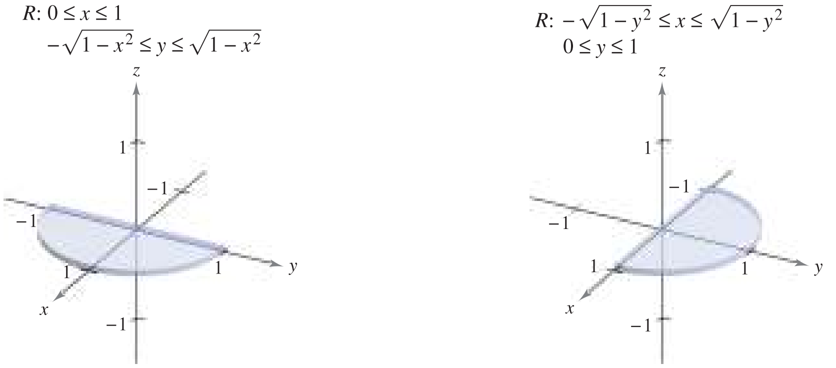
$$M_x = \int_R\!\!\int y\rho(x, y)\,dA \quad \text{and} \quad M_y = \int_R\!\!\int x\rho(x, y)\,dA.$$

If m is the mass of the lamina, then the **center of mass** is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right).$$

If R represents a simple plane region rather than a lamina, the point $(\bar{x}, \bar{y})$ is called the **centroid** of the region.

For some planar laminas with a constant density ρ, you can determine the center of mass (or one of its coordinates) using symmetry rather than using integration. For instance, consider the laminas of constant density shown in Figure 14.38. Using symmetry, you can see that $\bar{y} = 0$ for the first lamina and $\bar{x} = 0$ for the second lamina.

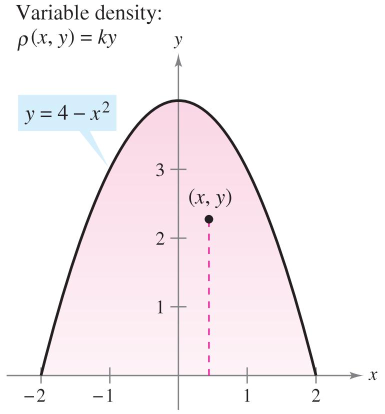


$R: 0 \le x \le 1$
$-\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$
z
1
−1
−1
1
1
y
x
−1
$R: -\sqrt{1 - y^2} \le x \le \sqrt{1 - y^2}$
$0 \le y \le 1$
z
1
−1
−1
1
1
y
x
−1

Lamina of constant density that is symmetric with respect to the x-axis

Lamina of constant density that is symmetric with respect to the y-axis

Figure 14.38



Parabolic region of variable density

Figure 14.39

EXAMPLE 3 Finding the Center of Mass

Find the center of mass of the lamina corresponding to the parabolic region

$$0 \le y \le 4 - x^2 \qquad \text{Parabolic region}$$

where the density at the point (x, y) is proportional to the distance between (x, y) and the x-axis, as shown in Figure 14.39.

Solution Because the lamina is symmetric with respect to the y-axis and

$$\rho(x, y) = ky$$

the center of mass lies on the y-axis. So, $\bar{x} = 0$. To find $\bar{y}$, first find the mass of the lamina.

$$\begin{aligned}
\text{Mass} = \int_{-2}^{2}\int_{0}^{4-x^2} ky\,dy\,dx &= \frac{k}{2}\int_{-2}^{2} y^2\Big]_0^{4-x^2} dx \\
&= \frac{k}{2}\int_{-2}^{2} (16 - 8x^2 + x^4)\,dx \\
&= \frac{k}{2}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2} \\
&= k\left(32 - \frac{64}{3} + \frac{32}{5}\right) \\
&= \frac{256k}{15}
\end{aligned}$$

Next, find the moment about the x-axis.

$$\begin{aligned}
M_x = \int_{-2}^{2}\int_{0}^{4-x^2} (y)(ky)\,dy\,dx &= \frac{k}{3}\int_{-2}^{2} y^3\Big]_0^{4-x^2} dx \\
&= \frac{k}{3}\int_{-2}^{2} (64 - 48x^2 + 12x^4 - x^6)\,dx \\
&= \frac{k}{3}\left[64x - 16x^3 + \frac{12x^5}{5} - \frac{x^7}{7}\right]_{-2}^{2} \\
&= \frac{4096k}{105}
\end{aligned}$$

So,

$$\bar{y} = \frac{M_x}{m} = \frac{4096k/105}{256k/15} = \frac{16}{7}$$

and the center of mass is $\left(0, \frac{16}{7}\right)$. ■

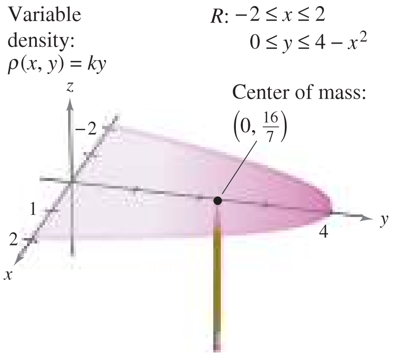


Figure 14.40

Although you can think of the moments M_x and M_y as measuring the tendency to rotate about the x- or y-axis, the calculation of moments is usually an intermediate step toward a more tangible goal. The use of the moments M_x and M_y is typical—to find the center of mass. Determination of the center of mass is useful in a variety of applications that allow you to treat a lamina as if its mass were concentrated at just one point. Intuitively, you can think of the center of mass as the balancing point of the lamina. For instance, the lamina in Example 3 should balance on the point of a pencil placed at $\left(0, \frac{16}{7}\right)$, as shown in Figure 14.40.

Moments of Inertia

The moments of M_x and M_y used in determining the center of mass of a lamina are sometimes called the **first moments** about the x- and y-axes. In each case, the moment is the product of a mass times a distance.

$$M_x = \int_R\!\!\int \underset{\substack{\uparrow\\ \text{Distance}\\ \text{to } x\text{-axis}}}{(y)} \underbrace{\rho(x, y)\,dA}_{\substack{\uparrow\\ \text{Mass}}} \qquad M_y = \int_R\!\!\int \underset{\substack{\uparrow\\ \text{Distance}\\ \text{to } y\text{-axis}}}{(x)} \underbrace{\rho(x, y)\,dA}_{\substack{\uparrow\\ \text{Mass}}}$$

You will now look at another type of moment—the **second moment,** or the **moment of inertia** of a lamina about a line. In the same way that mass is a measure of the tendency of matter to resist a change in straight-line motion, the moment of inertia about a line is a *measure of the tendency of matter to resist a change in rotational motion*. For example, if a particle of mass m is a distance d from a fixed line, its moment of inertia about the line is defined as

$$I = md^2 = (\text{mass})(\text{distance})^2.$$

As with moments of mass, you can generalize this concept to obtain the moments of inertia about the x- and y-axes of a lamina of variable density. These second moments are denoted by I_x and I_y, and in each case the moment is the product of a mass times the square of a distance.

$$I_x = \int_R\!\!\int \underset{\substack{\uparrow\\ \text{Square of distance}\\ \text{to } x\text{-axis}}}{(y^2)} \underbrace{\rho(x, y)\,dA}_{\substack{\uparrow\\ \text{Mass}}} \qquad I_y = \int_R\!\!\int \underset{\substack{\uparrow\\ \text{Square of distance}\\ \text{to } y\text{-axis}}}{(x^2)} \underbrace{\rho(x, y)\,dA}_{\substack{\uparrow\\ \text{Mass}}}$$

The sum of the moments I_x and I_y is called the **polar moment of inertia** and is denoted by I_0.

NOTE For a lamina in the xy-plane, I_0 represents the moment of inertia of the lamina about the z-axis. The term "polar moment of inertia" stems from the fact that the square of the polar distance r is used in the calculation.

$$\begin{aligned} I_0 &= \int_R\!\!\int (x^2 + y^2)\rho(x, y)\,dA \\ &= \int_R\!\!\int r^2\rho(x, y)\,dA \end{aligned}$$

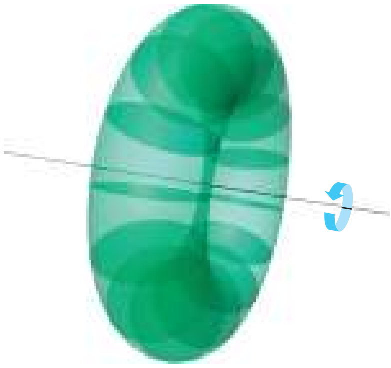
EXAMPLE 4 Finding the Moment of Inertia

Find the moment of inertia about the x-axis of the lamina in Example 3.

Solution From the definition of moment of inertia, you have

$$\begin{aligned} I_x &= \int_{-2}^{2}\int_{0}^{4-x^2} y^2(ky)\,dy\,dx \\ &= \frac{k}{4}\int_{-2}^{2} y^4\Big]_0^{4-x^2}\,dx \\ &= \frac{k}{4}\int_{-2}^{2} (256 - 256x^2 + 96x^4 - 16x^6 + x^8)\,dx \\ &= \frac{k}{4}\left[256x - \frac{256x^3}{3} + \frac{96x^5}{5} - \frac{16x^7}{7} + \frac{x^9}{9}\right]_{-2}^{2} \\ &= \frac{32{,}768k}{315}. \end{aligned}$$

■



Planar lamina revolving at ω radians per second

Figure 14.41

The moment of inertia I of a revolving lamina can be used to measure its kinetic energy. For example, suppose a planar lamina is revolving about a line with an **angular speed** of ω radians per second, as shown in Figure 14.41. The kinetic energy E of the revolving lamina is

$$E = \frac{1}{2} I\omega^2. \qquad \text{Kinetic energy for rotational motion}$$

On the other hand, the kinetic energy E of a mass m moving in a straight line at a velocity v is

$$E = \frac{1}{2} mv^2. \qquad \text{Kinetic energy for linear motion}$$

So, the kinetic energy of a mass moving in a straight line is proportional to its mass, but the kinetic energy of a mass revolving about an axis is proportional to its moment of inertia.

The **radius of gyration** $\bar{\bar{r}}$ of a revolving mass m with moment of inertia I is defined as

$$\bar{\bar{r}} = \sqrt{\frac{I}{m}}. \qquad \text{Radius of gyration}$$

If the entire mass were located at a distance $\bar{\bar{r}}$ from its axis of revolution, it would have the same moment of inertia and, consequently, the same kinetic energy. For instance, the radius of gyration of the lamina in Example 4 about the x-axis is given by

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32{,}768k/315}{256k/15}} = \sqrt{\frac{128}{21}} \approx 2.469.$$

EXAMPLE 5 Finding the Radius of Gyration

Find the radius of gyration about the y-axis for the lamina corresponding to the region R: $0 \le y \le \sin x$, $0 \le x \le \pi$, where the density at (x, y) is given by $\rho(x, y) = x$.

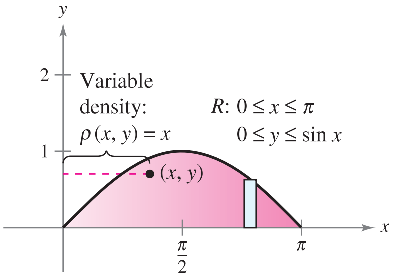


Figure 14.42

Solution The region R is shown in Figure 14.42. By integrating $\rho(x, y) = x$ over the region R, you can determine that the mass of the region is π. The moment of inertia about the y-axis is

$$\begin{aligned} I_y &= \int_0^{\pi} \int_0^{\sin x} x^3 \, dy \, dx \\ &= \int_0^{\pi} x^3 y \Big]_0^{\sin x} dx \\ &= \int_0^{\pi} x^3 \sin x \, dx \\ &= \Big[(3x^2 - 6)(\sin x) - (x^3 - 6x)(\cos x)\Big]_0^{\pi} \\ &= \pi^3 - 6\pi. \end{aligned}$$

So, the radius of gyration about the y-axis is

$$\begin{aligned} \bar{\bar{x}} &= \sqrt{\frac{I_y}{m}} \\ &= \sqrt{\frac{\pi^3 - 6\pi}{\pi}} \\ &= \sqrt{\pi^2 - 6} \approx 1.967. \end{aligned}$$

■

14.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find the mass of the lamina described by the inequalities, given that its density is $\rho(x, y) = xy$. (*Hint:* Some of the integrals are simpler in polar coordinates.)

1. $0 \le x \le 2,\ 0 \le y \le 2$

2. $0 \le x \le 3,\ 0 \le y \le 9 - x^2$

3. $0 \le x \le 1,\ 0 \le y \le \sqrt{1 - x^2}$

4. $x \ge 0,\ 3 \le y \le 3 + \sqrt{9 - x^2}$

In Exercises 5–8, find the mass and center of mass of the lamina for each density.

5. R: square with vertices $(0, 0), (a, 0), (0, a), (a, a)$

(a) $\rho = k$ (b) $\rho = ky$ (c) $\rho = kx$

6. R: rectangle with vertices $(0, 0), (a, 0), (0, b), (a, b)$

(a) $\rho = kxy$ (b) $\rho = k(x^2 + y^2)$

7. R: triangle with vertices $(0, 0), (0, a), (a, a)$

(a) $\rho = k$ (b) $\rho = ky$ (c) $\rho = kx$

8. R: triangle with vertices $(0, 0), (a/2, a), (a, 0)$

(a) $\rho = k$ (b) $\rho = kxy$

9. ***Translations in the Plane*** Translate the lamina in Exercise 5 to the right five units and determine the resulting center of mass.

10. ***Conjecture*** Use the result of Exercise 9 to make a conjecture about the change in the center of mass when a lamina of constant density is translated c units horizontally or d units vertically. Is the conjecture true if the density is not constant? Explain.

In Exercises 11–22, find the mass and center of mass of the lamina bounded by the graphs of the equations for the given density or densities. (*Hint:* Some of the integrals are simpler in polar coordinates.)

11. $y = \sqrt{x},\ y = 0,\ x = 1,\ \rho = ky$

12. $y = x^2,\ y = 0,\ x = 2,\ \rho = kxy$

13. $y = 4/x,\ y = 0,\ x = 1,\ x = 4,\ \rho = kx^2$

14. $y = \dfrac{1}{1 + x^2},\ y = 0,\ x = -1,\ x = 1,\ \rho = k$

15. $y = e^x,\ y = 0,\ x = 0,\ x = 1$

(a) $\rho = k$ (b) $\rho = ky$

16. $y = e^{-x},\ y = 0,\ x = 0,\ x = 1$

(a) $\rho = ky$ (b) $\rho = ky^2$

17. $y = 4 - x^2,\ y = 0,\ \rho = ky$

18. $x = 9 - y^2,\ x = 0,\ \rho = kx$

19. $y = \sin \dfrac{\pi x}{L},\ y = 0,\ x = 0,\ x = L,\ \rho = k$

20. $y = \cos \dfrac{\pi x}{L},\ y = 0,\ x = 0,\ x = \dfrac{L}{2},\ \rho = ky$

21. $y = \sqrt{a^2 - x^2},\ 0 \le y \le x,\ \rho = k$

22. $x^2 + y^2 = a^2,\ 0 \le x,\ 0 \le y,\ \rho = k(x^2 + y^2)$

CAS **In Exercises 23–26, use a computer algebra system to find the mass and center of mass of the lamina bounded by the graphs of the equations for the given density.**

23. $y = e^{-x},\ y = 0,\ x = 0,\ x = 2,\ \rho = kxy$

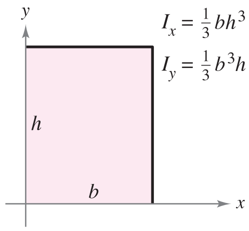
24. $y = \ln x,\ y = 0,\ x = 1,\ x = e,\ \rho = k/x$

25. $r = 2 \cos 3\theta,\ -\pi/6 \le \theta \le \pi/6,\ \rho = k$

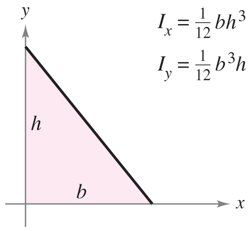
26. $r = 1 + \cos \theta,\ \rho = k$

In Exercises 27–32, verify the given moment(s) of inertia and find $\bar{\bar{x}}$ and $\bar{\bar{y}}$. Assume that each lamina has a density of $\rho = 1$ gram per square centimeter. (These regions are common shapes used in engineering.)

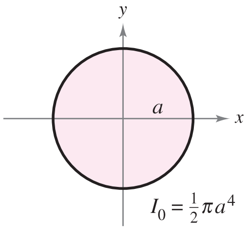
27. Rectangle



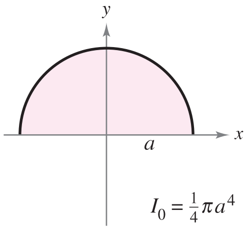
28. Right triangle



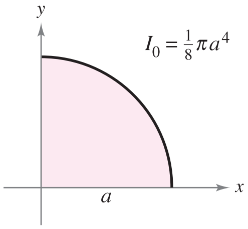
29. Circle



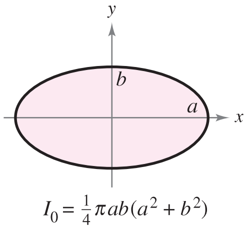
30. Semicircle



31. Quarter circle



32. Ellipse



CAS **In Exercises 33–40, find $I_x, I_y, I_0, \bar{\bar{x}}$, and $\bar{\bar{y}}$ for the lamina bounded by the graphs of the equations. Use a computer algebra system to evaluate the double integrals.**

33. $y = 0,\ y = b,\ x = 0,\ x = a,\ \rho = ky$

34. $y = \sqrt{a^2 - x^2},\ y = 0,\ \rho = ky$

35. $y = 4 - x^2,\ y = 0,\ x > 0,\ \rho = kx$

36. $y = x,\ y = x^2,\ \rho = kxy$

37. $y = \sqrt{x},\ y = 0,\ x = 4,\ \rho = kxy$

38. $y = x^2,\ y^2 = x,\ \rho = x^2 + y^2$

39. $y = x^2,\ y^2 = x,\ \rho = kx$

40. $y = x^3,\ y = 4x,\ \rho = k|y|$

CAS **In Exercises 41–46, set up the double integral required to find the moment of inertia I, about the given line, of the lamina bounded by the graphs of the equations. Use a computer algebra system to evaluate the double integral.**

41. $x^2 + y^2 = b^2,\ \rho = k,$ line: $x = a\ (a > b)$

42. $y = 0,\ y = 2,\ x = 0,\ x = 4,\ \rho = k,$ line: $x = 6$

43. $y = \sqrt{x},\ y = 0,\ x = 4,\ \rho = kx,$ line: $x = 6$

44. $y = \sqrt{a^2 - x^2},\ y = 0,\ \rho = ky,$ line: $y = a$

45. $y = \sqrt{a^2 - x^2},\ y = 0,\ x \geq 0,\ \rho = k(a - y),$ line: $y = a$

46. $y = 4 - x^2,\ y = 0,\ \rho = k,$ line: $y = 2$

WRITING ABOUT CONCEPTS

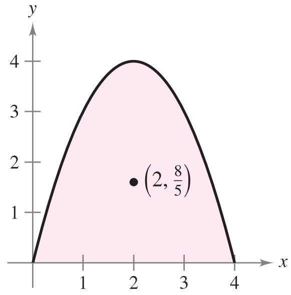
47. Give the formulas for finding the moments and center of mass of a variable density planar lamina.

48. Give the formulas for finding the moments of inertia about the x- and y-axes for a variable density planar lamina.

49. In your own words, describe what the radius of gyration measures.

CAPSTONE

50. The center of mass of the lamina of constant density shown in the figure is $\left(2, \frac{8}{5}\right)$. Make a conjecture about how the center of mass $(\bar{x}, \bar{y})$ changes for each given nonconstant density $\rho(x, y)$. Explain. (Make your conjecture *without* performing any calculations.)



(a) $\rho(x, y) = ky$

(b) $\rho(x, y) = k|2 - x|$

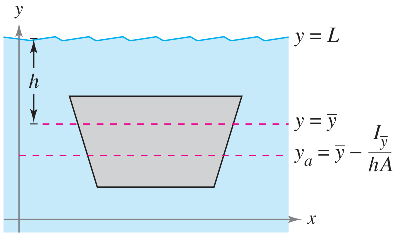
(c) $\rho(x, y) = kxy$

(d) $\rho(x, y) = k(4 - x)(4 - y)$

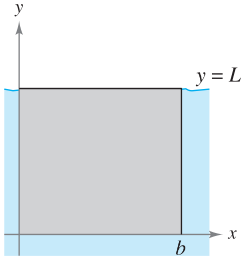
Hydraulics **In Exercises 51–54, determine the location of the horizontal axis y_a at which a vertical gate in a dam is to be hinged so that there is no moment causing rotation under the indicated loading (see figure). The model for y_a is**

$$y_a = \bar{y} - \frac{I_{\bar{y}}}{hA}$$

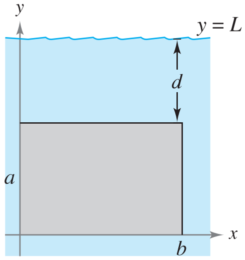
where $\bar{y}$ is the y-coordinate of the centroid of the gate, $I_{\bar{y}}$ is the moment of inertia of the gate about the line $y = \bar{y}$, h is the depth of the centroid below the surface, and A is the area of the gate.



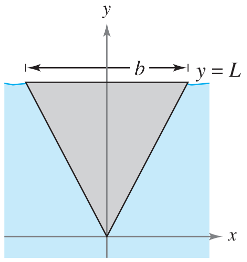
51.



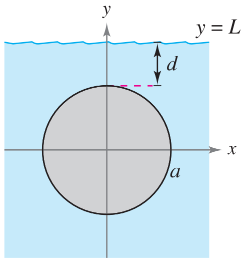
52.



53.



54.



55. Prove the following Theorem of Pappus: Let R be a region in a plane and let L be a line in the same plane such that L does not intersect the interior of R. If r is the distance between the centroid of R and the line, then the volume V of the solid of revolution formed by revolving R about the line is given by $V = 2\pi rA$, where A is the area of R.

SECTION PROJECT

Center of Pressure on a Sail

The center of pressure on a sail is that point (x_p, y_p) at which the total aerodynamic force may be assumed to act. If the sail is represented by a plane region R, the center of pressure is

$$x_p = \frac{\int_R\int xy\, dA}{\int_R\int y\, dA} \quad \text{and} \quad y_p = \frac{\int_R\int y^2\, dA}{\int_R\int y\, dA}.$$

Consider a triangular sail with vertices at $(0, 0)$, $(2, 1)$, and $(0, 5)$. Verify the value of each integral.

(a) $\displaystyle\int_R\int y\, dA = 10$ (b) $\displaystyle\int_R\int xy\, dA = \frac{35}{6}$ (c) $\displaystyle\int_R\int y^2\, dA = \frac{155}{6}$

Calculate the coordinates (x_p, y_p) of the center of pressure. Sketch a graph of the sail and indicate the location of the center of pressure.

14.5 Surface Area

- **Use a double integral to find the area of a surface.**

Surface Area

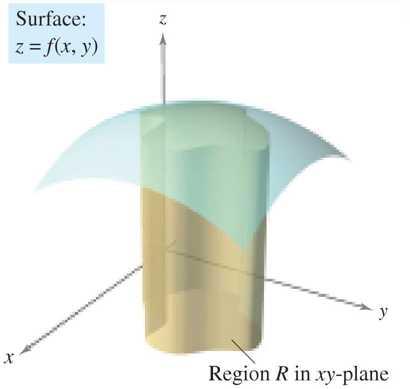


Figure 14.43

At this point you know a great deal about the solid region lying between a surface and a closed and bounded region R in the xy-plane, as shown in Figure 14.43. For example, you know how to find the extrema of f on R (Section 13.8), the area of the base R of the solid (Section 14.1), the volume of the solid (Section 14.2), and the centroid of the base R (Section 14.4).

In this section, you will learn how to find the upper **surface area** of the solid. Later, you will learn how to find the centroid of the solid (Section 14.6) and the lateral surface area (Section 15.2).

To begin, consider a surface S given by

$$z = f(x, y) \qquad \text{Surface defined over a region } R$$

defined over a region R. Assume that R is closed and bounded and that f has continuous first partial derivatives. To find the surface area, construct an inner partition of R consisting of n rectangles, where the area of the ith rectangle R_i is $\Delta A_i = \Delta x_i \Delta y_i$, as shown in Figure 14.44. In each R_i let (x_i, y_i) be the point that is closest to the origin. At the point $(x_i, y_i, z_i) = (x_i, y_i, f(x_i, y_i))$ on the surface S, construct a tangent plane T_i. The area of the portion of the tangent plane that lies directly above R_i is approximately equal to the area of the surface lying directly above R_i. That is, $\Delta T_i \approx \Delta S_i$. So, the surface area of S is given by

$$\sum_{i=1}^{n} \Delta S_i \approx \sum_{i=1}^{n} \Delta T_i.$$

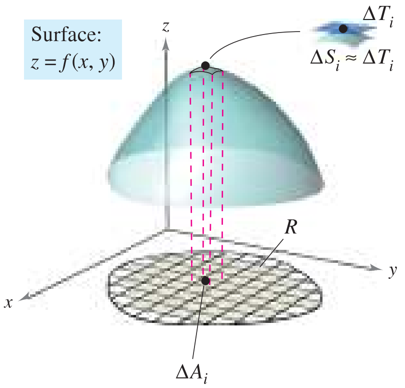


Figure 14.44

To find the area of the parallelogram ΔT_i, note that its sides are given by the vectors

$$\mathbf{u} = \Delta x_i \mathbf{i} + f_x(x_i, y_i)\,\Delta x_i \mathbf{k}$$

and

$$\mathbf{v} = \Delta y_i \mathbf{j} + f_y(x_i, y_i)\,\Delta y_i \mathbf{k}.$$

From Theorem 11.8, the area of ΔT_i is given by $\|\mathbf{u} \times \mathbf{v}\|$, where

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_i & 0 & f_x(x_i, y_i)\,\Delta x_i \\ 0 & \Delta y_i & f_y(x_i, y_i)\,\Delta y_i \end{vmatrix} \\ &= -f_x(x_i, y_i)\,\Delta x_i \Delta y_i \mathbf{i} - f_y(x_i, y_i)\,\Delta x_i \Delta y_i \mathbf{j} + \Delta x_i \Delta y_i \mathbf{k} \\ &= (-f_x(x_i, y_i)\mathbf{i} - f_y(x_i, y_i)\mathbf{j} + \mathbf{k})\,\Delta A_i.\end{aligned}$$

So, the area of ΔT_i is $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{[f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2 + 1}\,\Delta A_i$, and

$$\begin{aligned}\text{Surface area of } S &\approx \sum_{i=1}^{n} \Delta S_i \\ &\approx \sum_{i=1}^{n} \sqrt{1 + [f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2}\,\Delta A_i.\end{aligned}$$

This suggests the following definition of surface area.

DEFINITION OF SURFACE AREA

If f and its first partial derivatives are continuous on the closed region R in the xy-plane, then the **area of the surface S** given by $z = f(x, y)$ over R is defined as

$$\begin{aligned}\text{Surface area} &= \iint_R dS \\ &= \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA.\end{aligned}$$

As an aid to remembering the double integral for surface area, it is helpful to note its similarity to the integral for arc length.

Length on x-axis: $\displaystyle\int_a^b dx$

Arc length in xy-plane: $\displaystyle\int_a^b ds = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

Area in xy-plane: $\displaystyle\iint_R dA$

Surface area in space: $\displaystyle\iint_R dS = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA$

Like integrals for arc length, integrals for surface area are often very difficult to evaluate. However, one type that is easily evaluated is demonstrated in the next example.

EXAMPLE 1 The Surface Area of a Plane Region

Find the surface area of the portion of the plane

$$z = 2 - x - y$$

that lies above the circle $x^2 + y^2 \le 1$ in the first quadrant, as shown in Figure 14.45.

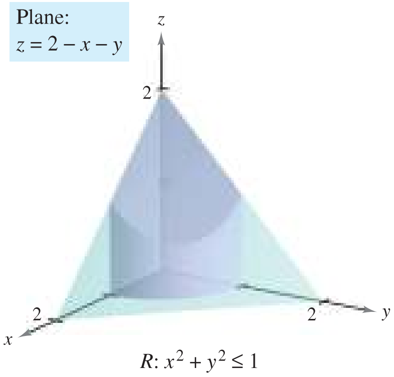


Figure 14.45

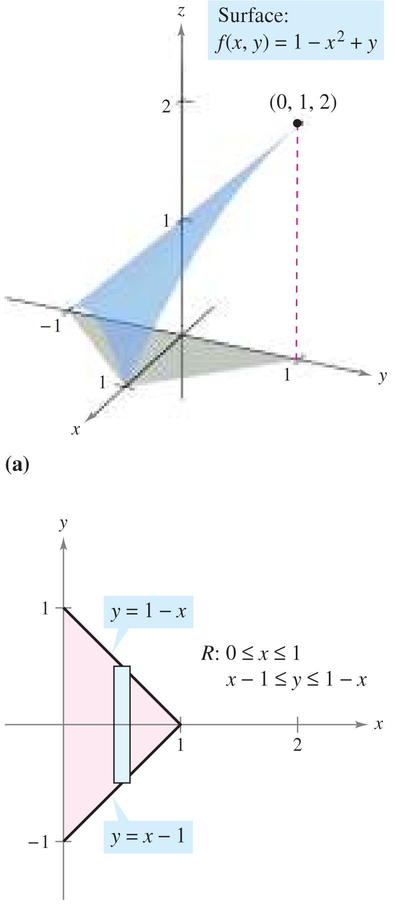
Solution Because $f_x(x, y) = -1$ and $f_y(x, y) = -1$, the surface area is given by

$$\begin{aligned}S &= \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA && \text{Formula for surface area} \\ &= \iint_R \sqrt{1 + (-1)^2 + (-1)^2}\, dA && \text{Substitute.} \\ &= \iint_R \sqrt{3}\, dA \\ &= \sqrt{3} \iint_R dA.\end{aligned}$$

Note that the last integral is simply $\sqrt{3}$ times the area of the region R. R is a quarter circle of radius 1, with an area of $\frac{1}{4}\pi(1^2)$ or $\pi/4$. So, the area of S is

$$\begin{aligned}S &= \sqrt{3}\,(\text{area of } R) \\ &= \sqrt{3}\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{3}\,\pi}{4}.\end{aligned}$$

■



(a)

(b)

Figure 14.46

EXAMPLE 2 Finding Surface Area

Find the area of the portion of the surface

$$f(x, y) = 1 - x^2 + y$$

that lies above the triangular region with vertices (1, 0, 0), (0, −1, 0), and (0, 1, 0), as shown in Figure 14.46(a).

Solution Because $f_x(x, y) = -2x$ and $f_y(x, y) = 1$, you have

$$S = \int_R\int \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\,dA = \int_R\int \sqrt{1 + 4x^2 + 1}\,dA.$$

In Figure 14.46(b), you can see that the bounds for R are $0 \le x \le 1$ and $x - 1 \le y \le 1 - x$. So, the integral becomes

$$\begin{aligned} S &= \int_0^1 \int_{x-1}^{1-x} \sqrt{2 + 4x^2}\,dy\,dx \\ &= \int_0^1 y\sqrt{2 + 4x^2}\,\Big]_{x-1}^{1-x}\,dx \\ &= \int_0^1 \Big[(1 - x)\sqrt{2 + 4x^2} - (x - 1)\sqrt{2 + 4x^2}\Big]\,dx \\ &= \int_0^1 \left(2\sqrt{2 + 4x^2} - 2x\sqrt{2 + 4x^2}\right)dx \end{aligned}$$

Integration tables (Appendix B), Formula 26 and Power Rule

$$\begin{aligned} &= \left[x\sqrt{2 + 4x^2} + \ln\left(2x + \sqrt{2 + 4x^2}\right) - \frac{(2 + 4x^2)^{3/2}}{6}\right]_0^1 \\ &= \sqrt{6} + \ln\left(2 + \sqrt{6}\right) - \sqrt{6} - \ln\sqrt{2} + \frac{1}{3}\sqrt{2} \approx 1.618. \end{aligned}$$

EXAMPLE 3 Change of Variables to Polar Coordinates

Find the surface area of the paraboloid $z = 1 + x^2 + y^2$ that lies above the unit circle, as shown in Figure 14.47.

Solution Because $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, you have

$$S = \int_R\int \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\,dA = \int_R\int \sqrt{1 + 4x^2 + 4y^2}\,dA.$$

You can convert to polar coordinates by letting $x = r\cos\theta$ and $y = r\sin\theta$. Then, because the region R is bounded by $0 \le r \le 1$ and $0 \le \theta \le 2\pi$, you have

$$\begin{aligned} S &= \int_0^{2\pi}\int_0^1 \sqrt{1 + 4r^2}\,r\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{1}{12}(1 + 4r^2)^{3/2}\Big]_0^1\,d\theta \\ &= \int_0^{2\pi} \frac{5\sqrt{5} - 1}{12}\,d\theta \\ &= \frac{5\sqrt{5} - 1}{12}\,\theta\Big]_0^{2\pi} \\ &= \frac{\pi\left(5\sqrt{5} - 1\right)}{6} \\ &\approx 5.33. \end{aligned}$$

■

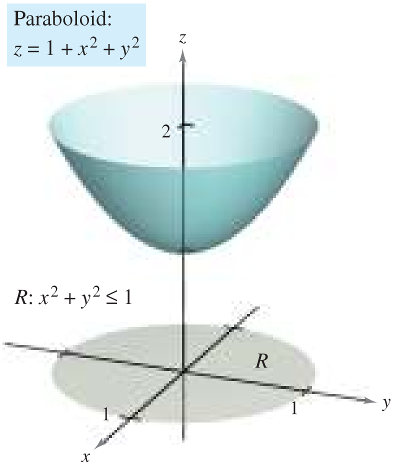


Figure 14.47

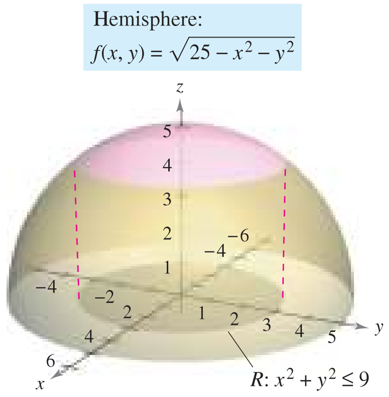


Figure 14.48

EXAMPLE 4 Finding Surface Area

Find the surface area S of the portion of the hemisphere

$$f(x, y) = \sqrt{25 - x^2 - y^2} \qquad \text{Hemisphere}$$

that lies above the region R bounded by the circle $x^2 + y^2 \le 9$, as shown in Figure 14.48.

Solution The first partial derivatives of f are

$$f_x(x, y) = \frac{-x}{\sqrt{25 - x^2 - y^2}} \quad \text{and} \quad f_y(x, y) = \frac{-y}{\sqrt{25 - x^2 - y^2}}$$

and, from the formula for surface area, you have

$$\begin{aligned} dS &= \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA \\ &= \sqrt{1 + \left(\frac{-x}{\sqrt{25 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{25 - x^2 - y^2}}\right)^2}\, dA \\ &= \frac{5}{\sqrt{25 - x^2 - y^2}}\, dA. \end{aligned}$$

So, the surface area is

$$S = \iint_R \frac{5}{\sqrt{25 - x^2 - y^2}}\, dA.$$

You can convert to polar coordinates by letting $x = r\cos\theta$ and $y = r\sin\theta$. Then, because the region R is bounded by $0 \le r \le 3$ and $0 \le \theta \le 2\pi$, you obtain

$$\begin{aligned} S &= \int_0^{2\pi}\int_0^3 \frac{5}{\sqrt{25 - r^2}}\, r\, dr\, d\theta \\ &= 5\int_0^{2\pi} -\sqrt{25 - r^2}\Big]_0^3\, d\theta \\ &= 5\int_0^{2\pi} d\theta \\ &= 10\pi. \end{aligned}$$

■

The procedure used in Example 4 can be extended to find the surface area of a sphere by using the region R bounded by the circle $x^2 + y^2 \le a^2$, where $0 < a < 5$, as shown in Figure 14.49. The surface area of the portion of the hemisphere

$$f(x, y) = \sqrt{25 - x^2 - y^2}$$

lying above the circular region can be shown to be

$$\begin{aligned} S &= \iint_R \frac{5}{\sqrt{25 - x^2 - y^2}}\, dA \\ &= \int_0^{2\pi}\int_0^a \frac{5}{\sqrt{25 - r^2}}\, r\, dr\, d\theta \\ &= 10\pi\left(5 - \sqrt{25 - a^2}\right). \end{aligned}$$

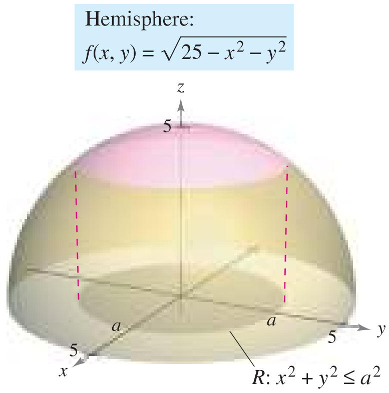


Figure 14.49

By taking the limit as a approaches 5 and doubling the result, you obtain a total area of 100π. (The surface area of a sphere of radius r is $S = 4\pi r^2$.)

You can use Simpson's Rule or the Trapezoidal Rule to approximate the value of a double integral, *provided* you can get through the first integration. This is demonstrated in the next example.

EXAMPLE 5 Approximating Surface Area by Simpson's Rule

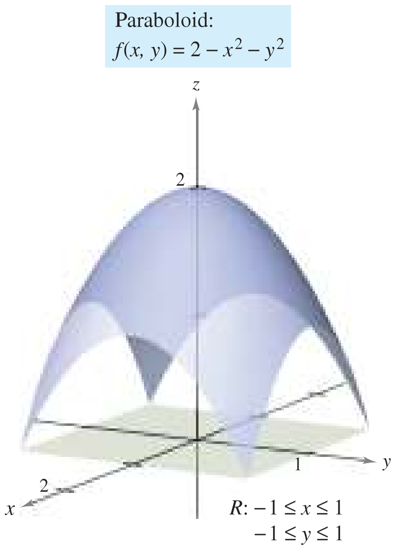


Figure 14.50

Find the area of the surface of the paraboloid

$$f(x, y) = 2 - x^2 - y^2 \qquad \text{Paraboloid}$$

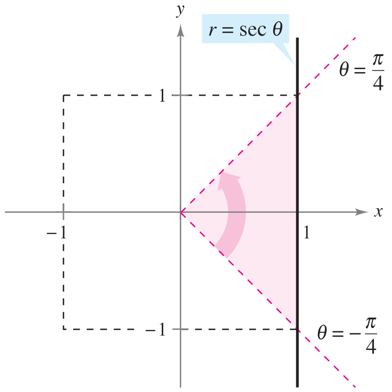
that lies above the square region bounded by $-1 \le x \le 1$ and $-1 \le y \le 1$, as shown in Figure 14.50.

Solution Using the partial derivatives

$$f_x(x, y) = -2x \quad \text{and} \quad f_y(x, y) = -2y$$

you have a surface area of

$$\begin{aligned} S &= \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA \\ &= \iint_R \sqrt{1 + (-2x)^2 + (-2y)^2}\, dA \\ &= \iint_R \sqrt{1 + 4x^2 + 4y^2}\, dA. \end{aligned}$$



One-fourth of the region R is bounded by $0 \le r \le \sec\theta$ and $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

Figure 14.51

In polar coordinates, the line $x = 1$ is given by $r \cos\theta = 1$ or $r = \sec\theta$, and you can determine from Figure 14.51 that one-fourth of the region R is bounded by

$$0 \le r \le \sec\theta \quad \text{and} \quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}.$$

Letting $x = r\cos\theta$ and $y = r\sin\theta$ produces

$$\begin{aligned} \frac{1}{4}S &= \frac{1}{4}\iint_R \sqrt{1 + 4x^2 + 4y^2}\, dA \\ &= \int_{-\pi/4}^{\pi/4}\int_0^{\sec\theta} \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\ &= \int_{-\pi/4}^{\pi/4} \frac{1}{12}(1 + 4r^2)^{3/2}\Big]_0^{\sec\theta}\, d\theta \\ &= \frac{1}{12}\int_{-\pi/4}^{\pi/4} \left[(1 + 4\sec^2\theta)^{3/2} - 1\right] d\theta. \end{aligned}$$

Finally, using Simpson's Rule with $n = 10$, you can approximate this single integral to be

$$\begin{aligned} S &= \frac{1}{3}\int_{-\pi/4}^{\pi/4} \left[(1 + 4\sec^2\theta)^{3/2} - 1\right] d\theta \\ &\approx 7.450. \end{aligned}$$

■

TECHNOLOGY Most computer programs that are capable of performing symbolic integration for multiple integrals are also capable of performing numerical approximation techniques. If you have access to such software, use it to approximate the value of the integral in Example 5.

14.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–14, find the area of the surface given by $z = f(x, y)$ over the region R. (*Hint:* Some of the integrals are simpler in polar coordinates.)

1. $f(x, y) = 2x + 2y$

R: triangle with vertices $(0, 0)$, $(4, 0)$, $(0, 4)$

2. $f(x, y) = 15 + 2x - 3y$

R: square with vertices $(0, 0)$, $(3, 0)$, $(0, 3)$, $(3, 3)$

3. $f(x, y) = 7 + 2x + 2y$

$R = \{(x, y): x^2 + y^2 \le 4\}$

4. $f(x, y) = 12 + 2x - 3y$

$R = \{(x, y): x^2 + y^2 \le 9\}$

5. $f(x, y) = 9 - x^2$

R: square with vertices $(0, 0)$, $(2, 0)$, $(0, 2)$, $(2, 2)$

6. $f(x, y) = y^2$

R: square with vertices $(0, 0)$, $(3, 0)$, $(0, 3)$, $(3, 3)$

7. $f(x, y) = 3 + x^{3/2}$

R: rectangle with vertices $(0, 0)$, $(0, 4)$, $(3, 4)$, $(3, 0)$

8. $f(x, y) = 2 + \frac{2}{3}y^{3/2}$

$R = \{(x, y): 0 \le x \le 2, 0 \le y \le 2 - x\}$

9. $f(x, y) = \ln|\sec x|$

$R = \left\{(x, y): 0 \le x \le \dfrac{\pi}{4}, 0 \le y \le \tan x\right\}$

10. $f(x, y) = 13 + x^2 - y^2$

$R = \{(x, y): x^2 + y^2 \le 4\}$

11. $f(x, y) = \sqrt{x^2 + y^2}$, $R = \{(x, y): 0 \le f(x, y) \le 1\}$

12. $f(x, y) = xy$, $R = \{(x, y): x^2 + y^2 \le 16\}$

13. $f(x, y) = \sqrt{a^2 - x^2 - y^2}$

$R = \{(x, y): x^2 + y^2 \le b^2, 0 < b < a\}$

14. $f(x, y) = \sqrt{a^2 - x^2 - y^2}$

$R = \{(x, y): x^2 + y^2 \le a^2\}$

In Exercises 15–18, find the area of the surface.

15. The portion of the plane $z = 24 - 3x - 2y$ in the first octant

16. The portion of the paraboloid $z = 16 - x^2 - y^2$ in the first octant

17. The portion of the sphere $x^2 + y^2 + z^2 = 25$ inside the cylinder $x^2 + y^2 = 9$

18. The portion of the cone $z = 2\sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 4$

CAS **In Exercises 19–24, write a double integral that represents the surface area of $z = f(x, y)$ over the region R. Use a computer algebra system to evaluate the double integral.**

19. $f(x, y) = 2y + x^2$

R: triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$

20. $f(x, y) = 2x + y^2$

R: triangle with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$

21. $f(x, y) = 9 - x^2 - y^2$

$R = \{(x, y): 0 \le f(x, y)\}$

22. $f(x, y) = x^2 + y^2$

$R = \{(x, y): 0 \le f(x, y) \le 16\}$

23. $f(x, y) = 4 - x^2 - y^2$

$R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

24. $f(x, y) = \frac{2}{3}x^{3/2} + \cos x$

$R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

Approximation **In Exercises 25 and 26, determine which value best approximates the surface area of $z = f(x, y)$ over the region R. (Make your selection on the basis of a sketch of the surface and *not* by performing any calculations.)**

25. $f(x, y) = 10 - \frac{1}{2}y^2$

R: square with vertices $(0, 0)$, $(4, 0)$, $(4, 4)$, $(0, 4)$

(a) 16 (b) 200 (c) -100 (d) 72 (e) 36

26. $f(x, y) = \frac{1}{4}\sqrt{x^2 + y^2}$

R: circle bounded by $x^2 + y^2 = 9$

(a) -100 (b) 150 (c) 9π (d) 55 (e) 500

CAS **In Exercises 27 and 28, use a computer algebra system to approximate the double integral that gives the surface area of the graph of f over the region $R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$.**

27. $f(x, y) = e^x$

28. $f(x, y) = \frac{2}{5}y^{5/2}$

In Exercises 29–34, set up a double integral that gives the area of the surface on the graph of f over the region R.

29. $f(x, y) = x^3 - 3xy + y^3$

R: square with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$

30. $f(x, y) = x^2 - 3xy - y^2$

$R = \{(x, y): 0 \le x \le 4, 0 \le y \le x\}$

31. $f(x, y) = e^{-x} \sin y$

$R = \{(x, y): x^2 + y^2 \le 4\}$

32. $f(x, y) = \cos(x^2 + y^2)$

$R = \left\{(x, y): x^2 + y^2 \le \dfrac{\pi}{2}\right\}$

33. $f(x, y) = e^{xy}$

$R = \{(x, y): 0 \le x \le 4, 0 \le y \le 10\}$

34. $f(x, y) = e^{-x} \sin y$

$R = \{(x, y): 0 \le x \le 4, 0 \le y \le x\}$

WRITING ABOUT CONCEPTS

35. State the double integral definition of the area of a surface S given by $z = f(x, y)$ over the region R in the xy-plane.

36. Consider the surface $f(x, y) = x^2 + y^2$ and the surface area of f over each region R. Without integrating, order the surface areas from least to greatest. Explain your reasoning.

(a) R: rectangle with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$

(b) R: triangle with vertices $(0, 0)$, $(2, 0)$, $(0, 2)$

(c) $R = \{(x, y): x^2 + y^2 \le 4$, first quadrant only$\}$

37. Will the surface area of the graph of a function $z = f(x, y)$ over a region R increase if the graph is shifted k units vertically? Why or why not?

CAPSTONE

38. Answer the following questions about the surface area S on a surface given by a positive function $z = f(x, y)$ over a region R in the xy-plane. Explain each answer.

(a) Is it possible for S to equal the area of R?

(b) Can S be greater than the area of R?

(c) Can S be less than the area of R?

39. Find the surface area of the solid of intersection of the cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ (see figure).

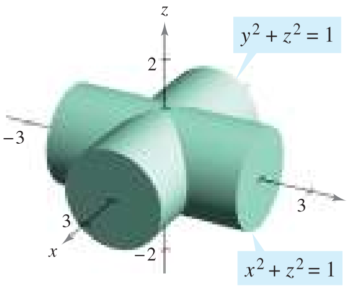


Figure for 39

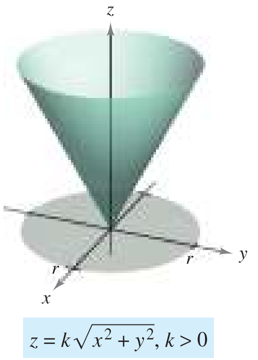
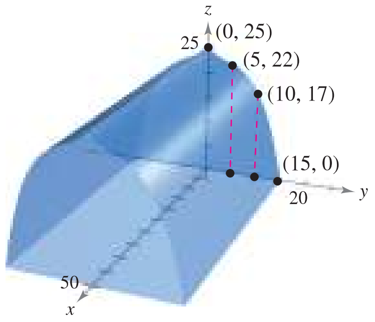


Figure for 40

40. Show that the surface area of the cone $z = k\sqrt{x^2 + y^2}$, $k > 0$, over the circular region $x^2 + y^2 \le r^2$ in the xy-plane is $\pi r^2\sqrt{k^2 + 1}$ (see figure).

41. ***Product Design*** A company produces a spherical object of radius 25 centimeters. A hole of radius 4 centimeters is drilled through the center of the object. Find (a) the volume of the object and (b) the outer surface area of the object.

42. ***Modeling Data*** A rancher builds a barn with dimensions 30 feet by 50 feet. The symmetrical shape and selected heights of the roof are shown in the figure.



(a) Use the regression capabilities of a graphing utility to find a model of the form $z = ay^3 + by^2 + cy + d$ for the roof line.

(b) Use the numerical integration capabilities of a graphing utility and the model in part (a) to approximate the volume of storage space in the barn.

(c) Use the numerical integration capabilities of a graphing utility and the model in part (a) to approximate the surface area of the roof.

(d) Approximate the arc length of the roof line and find the surface area of the roof by multiplying the arc length by the length of the barn. Compare the results and the integrations with those found in part (c).

SECTION PROJECT

Capillary Action

A well-known property of liquids is that they will rise in narrow vertical channels—this property is called "capillary action." The figure shows two plates, which form a narrow wedge, in a container of liquid. The upper surface of the liquid follows a hyperbolic shape given by

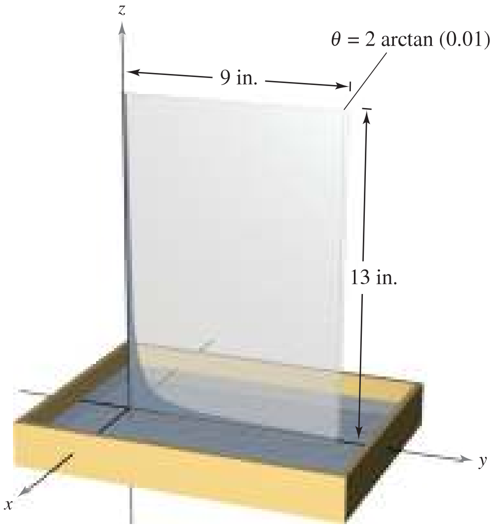
$$z = \frac{k}{\sqrt{x^2 + y^2}}$$

where x, y, and z are measured in inches. The constant k depends on the angle of the wedge, the type of liquid, and the material that comprises the flat plates.

(a) Find the volume of the liquid that has risen in the wedge. (Assume $k = 1$.)

(b) Find the horizontal surface area of the liquid that has risen in the wedge.

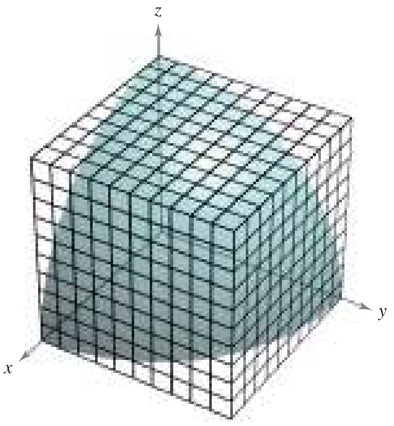
Adaptation of Capillary Action problem from "Capillary Phenomena" by Thomas B. Greenslade, Jr., *Physics Teacher*, May 1992. By permission of the author.



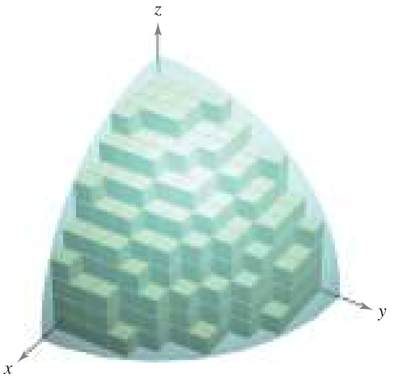
14.6 Triple Integrals and Applications

- **Use a triple integral to find the volume of a solid region.**
- **Find the center of mass and moments of inertia of a solid region.**

Triple Integrals



Solid region Q



Volume of $Q \approx \sum_{i=1}^{n} \Delta V_i$

Figure 14.52

The procedure used to define a **triple integral** follows that used for double integrals. Consider a function f of three variables that is continuous over a bounded solid region Q. Then, encompass Q with a network of boxes and form the **inner partition** consisting of all boxes lying entirely within Q, as shown in Figure 14.52. The volume of the ith box is

$$\Delta V_i = \Delta x_i \Delta y_i \Delta z_i. \qquad \text{Volume of } i\text{th box}$$

The **norm** $\|\Delta\|$ of the partition is the length of the longest diagonal of the n boxes in the partition. Choose a point (x_i, y_i, z_i) in each box and form the Riemann sum

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta V_i.$$

Taking the limit as $\|\Delta\| \to 0$ leads to the following definition.

DEFINITION OF TRIPLE INTEGRAL

If f is continuous over a bounded solid region Q, then the **triple integral of f over Q** is defined as

$$\iiint_Q f(x, y, z)\,dV = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta V_i$$

provided the limit exists. The **volume** of the solid region Q is given by

$$\text{Volume of } Q = \iiint_Q dV.$$

Some of the properties of double integrals in Theorem 14.1 can be restated in terms of triple integrals.

1. $\displaystyle \iiint_Q cf(x, y, z)\,dV = c\iiint_Q f(x, y, z)\,dV$

2. $\displaystyle \iiint_Q [f(x, y, z) \pm g(x, y, z)]\,dV = \iiint_Q f(x, y, z)\,dV \pm \iiint_Q g(x, y, z)\,dV$

3. $\displaystyle \iiint_Q f(x, y, z)\,dV = \iiint_{Q_1} f(x, y, z)\,dV + \iiint_{Q_2} f(x, y, z)\,dV$

In the properties above, Q is the union of two nonoverlapping solid subregions Q_1 and Q_2. If the solid region Q is simple, the triple integral $\iiint f(x, y, z)\,dV$ can be evaluated with an iterated integral using one of the six possible orders of integration:

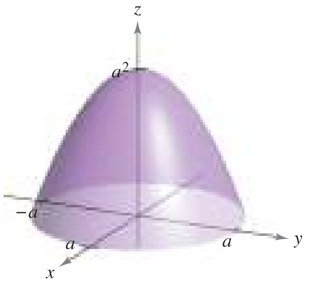
$$dx\,dy\,dz \quad dy\,dx\,dz \quad dz\,dx\,dy \quad dx\,dz\,dy \quad dy\,dz\,dx \quad dz\,dy\,dx.$$

EXPLORATION

Volume of a Paraboloid Sector
On pages 997 and 1006, you were asked to summarize the different ways you know of finding the volume of the solid bounded by the paraboloid

$$z = a^2 - x^2 - y^2, \quad a > 0$$

and the *xy*-plane. You now know one more way. Use it to find the volume of the solid.



The following version of Fubini's Theorem describes a region that is considered simple with respect to the order $dz\,dy\,dx$. Similar descriptions can be given for the other five orders.

THEOREM 14.4 EVALUATION BY ITERATED INTEGRALS

Let f be continuous on a solid region Q defined by

$$a \le x \le b, \quad h_1(x) \le y \le h_2(x), \quad g_1(x, y) \le z \le g_2(x, y)$$

where h_1, h_2, g_1, and g_2 are continuous functions. Then,

$$\iiint\limits_Q f(x, y, z)\,dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z)\,dz\,dy\,dx.$$

To evaluate a triple iterated integral in the order $dz\,dy\,dx$, hold *both* x and y constant for the innermost integration. Then, hold x constant for the second integration.

EXAMPLE 1 Evaluating a Triple Iterated Integral

Evaluate the triple iterated integral

$$\int_0^2 \int_0^x \int_0^{x+y} e^x(y + 2z)\,dz\,dy\,dx.$$

Solution For the first integration, hold x and y constant and integrate with respect to z.

$$\begin{aligned}\int_0^2 \int_0^x \int_0^{x+y} e^x(y + 2z)\,dz\,dy\,dx &= \int_0^2 \int_0^x e^x(yz + z^2)\Big]_0^{x+y}\,dy\,dx \\ &= \int_0^2 \int_0^x e^x(x^2 + 3xy + 2y^2)\,dy\,dx\end{aligned}$$

For the second integration, hold x constant and integrate with respect to y.

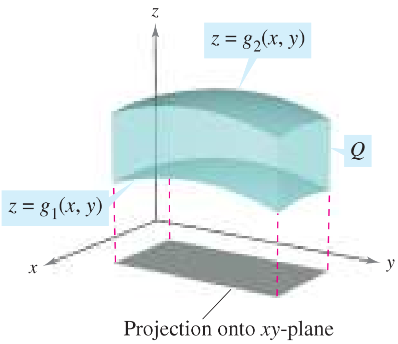
$$\begin{aligned}\int_0^2 \int_0^x e^x(x^2 + 3xy + 2y^2)\,dy\,dx &= \int_0^2 \left[e^x\left(x^2y + \frac{3xy^2}{2} + \frac{2y^3}{3}\right)\right]_0^x dx \\ &= \frac{19}{6}\int_0^2 x^3e^x\,dx\end{aligned}$$

Finally, integrate with respect to x.

$$\begin{aligned}\frac{19}{6}\int_0^2 x^3e^x\,dx &= \frac{19}{6}\Big[e^x(x^3 - 3x^2 + 6x - 6)\Big]_0^2 \\ &= 19\left(\frac{e^2}{3} + 1\right) \\ &\approx 65.797\end{aligned}$$

■

Example 1 demonstrates the integration order $dz\,dy\,dx$. For other orders, you can follow a similar procedure. For instance, to evaluate a triple iterated integral in the order $dx\,dy\,dz$, hold both y and z constant for the innermost integration and integrate with respect to x. Then, for the second integration, hold z constant and integrate with respect to y. Finally, for the third integration, integrate with respect to z.



Solid region Q lies between two surfaces.
Figure 14.53

To find the limits for a particular order of integration, it is generally advisable first to determine the innermost limits, which may be functions of the outer two variables. Then, by projecting the solid Q onto the coordinate plane of the outer two variables, you can determine their limits of integration by the methods used for double integrals. For instance, to evaluate

$$\iiint_Q f(x, y, z)\, dz\, dy\, dx$$

first determine the limits for z, and then the integral has the form

$$\iint \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\, dz \right] dy\, dx.$$

By projecting the solid Q onto the xy-plane, you can determine the limits for x and y as you did for double integrals, as shown in Figure 14.53.

EXAMPLE 2 Using a Triple Integral to Find Volume

Find the volume of the ellipsoid given by $4x^2 + 4y^2 + z^2 = 16$.

Solution Because x, y, and z play similar roles in the equation, the order of integration is probably immaterial, and you can arbitrarily choose $dz\, dy\, dx$. Moreover, you can simplify the calculation by considering only the portion of the ellipsoid lying in the first octant, as shown in Figure 14.54. From the order $dz\, dy\, dx$, you first determine the bounds for z.

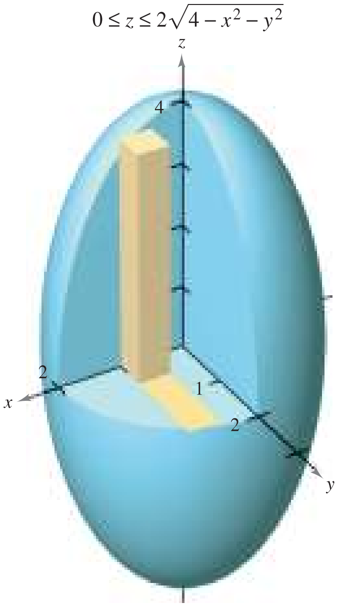
$$0 \le z \le 2\sqrt{4 - x^2 - y^2}$$

In Figure 14.55, you can see that the boundaries for x and y are $0 \le x \le 2$ and $0 \le y \le \sqrt{4 - x^2}$, so the volume of the ellipsoid is

$$\begin{aligned}
V &= \iiint_Q dV \\
&= 8\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{2\sqrt{4-x^2-y^2}} dz\, dy\, dx \\
&= 8\int_0^2 \int_0^{\sqrt{4-x^2}} z \Big]_0^{2\sqrt{4-x^2-y^2}} dy\, dx \\
&= 16\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{(4 - x^2) - y^2}\, dy\, dx \\
&= 8\int_0^2 \left[y\sqrt{4 - x^2 - y^2} + (4 - x^2)\arcsin\left(\frac{y}{\sqrt{4 - x^2}}\right) \right]_0^{\sqrt{4-x^2}} dx \\
&= 8\int_0^2 [0 + (4 - x^2)\arcsin(1) - 0 - 0]\, dx \\
&= 8\int_0^2 (4 - x^2)\left(\frac{\pi}{2}\right) dx \\
&= 4\pi\left[4x - \frac{x^3}{3}\right]_0^2 \\
&= \frac{64\pi}{3}.
\end{aligned}$$

Integration tables (Appendix B), Formula 37

■



Ellipsoid: $4x^2 + 4y^2 + z^2 = 16$

Figure 14.54

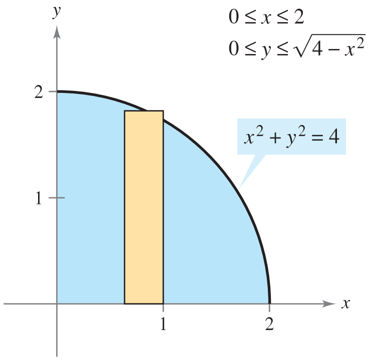


Figure 14.55

Example 2 is unusual in that all six possible orders of integration produce integrals of comparable difficulty. Try setting up some other possible orders of integration to find the volume of the ellipsoid. For instance, the order *dx dy dz* yields the integral

$$V = 8\int_0^4 \int_0^{\sqrt{16-z^2}/2} \int_0^{\sqrt{16-4y^2-z^2}/2} dx\,dy\,dz.$$

If you solve this integral, you will obtain the same volume obtained in Example 2. This is always the case—the order of integration does not affect the value of the integral. However, the order of integration often does affect the complexity of the integral. In Example 3, the given order of integration is not convenient, so you can change the order to simplify the problem.

EXAMPLE 3 Changing the Order of Integration

Evaluate $\displaystyle\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \int_1^3 \sin(y^2)\,dz\,dy\,dx.$

Solution Note that after one integration in the given order, you would encounter the integral $2\int \sin(y^2)\,dy$, which is not an elementary function. To avoid this problem, change the order of integration to *dz dx dy*, so that *y* is the outer variable. From Figure 14.56, you can see that the solid region *Q* is given by

$$0 \le x \le \sqrt{\frac{\pi}{2}}, \quad x \le y \le \sqrt{\frac{\pi}{2}}, \quad 1 \le z \le 3$$

and the projection of *Q* in the *xy*-plane yields the bounds

$$0 \le y \le \sqrt{\frac{\pi}{2}} \quad \text{and} \quad 0 \le x \le y.$$

So, evaluating the triple integral using the order *dz dx dy* produces

$$\begin{aligned}
\int_0^{\sqrt{\pi/2}} \int_0^y \int_1^3 \sin(y^2)\,dz\,dx\,dy &= \int_0^{\sqrt{\pi/2}} \int_0^y z\sin(y^2)\Big]_1^3\,dx\,dy \\
&= 2\int_0^{\sqrt{\pi/2}} \int_0^y \sin(y^2)\,dx\,dy \\
&= 2\int_0^{\sqrt{\pi/2}} x\sin(y^2)\Big]_0^y\,dy \\
&= 2\int_0^{\sqrt{\pi/2}} y\sin(y^2)\,dy \\
&= -\cos(y^2)\Big]_0^{\sqrt{\pi/2}} \\
&= 1.
\end{aligned}$$

■

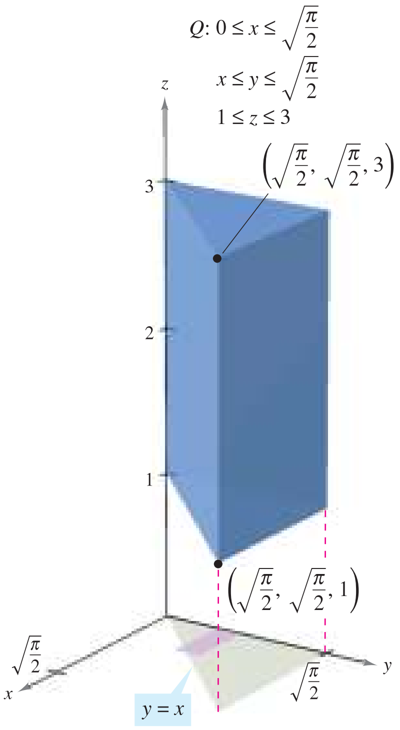

$Q: 0 \le x \le \sqrt{\frac{\pi}{2}}$
$x \le y \le \sqrt{\frac{\pi}{2}}$
$1 \le z \le 3$
z
$\left(\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}, 3\right)$
3
2
1
$\left(\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}, 1\right)$
$\sqrt{\frac{\pi}{2}}$
x
$\sqrt{\frac{\pi}{2}}$
y
y = x

Figure 14.56

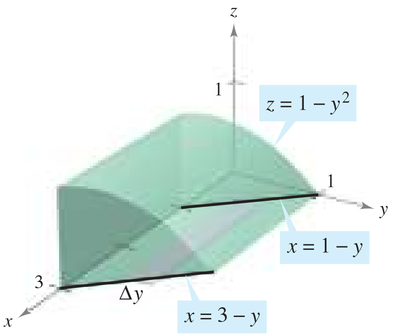
EXAMPLE 4 Determining the Limits of Integration

Set up a triple integral for the volume of each solid region.

a. The region in the first octant bounded above by the cylinder $z = 1 - y^2$ and lying between the vertical planes $x + y = 1$ and $x + y = 3$

b. The upper hemisphere given by $z = \sqrt{1 - x^2 - y^2}$

c. The region bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 6$



Q: $0 \le z \le 1 - y^2$
$1 - y \le x \le 3 - y$
$0 \le y \le 1$

Figure 14.57

Solution

a. In Figure 14.57, note that the solid is bounded below by the xy-plane $(z = 0)$ and above by the cylinder $z = 1 - y^2$. So,

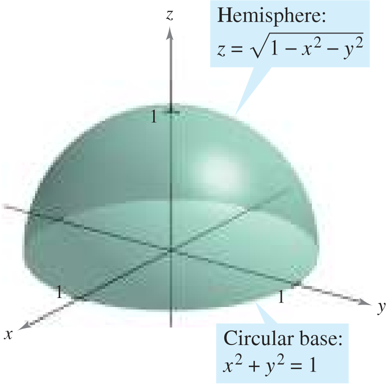
$$0 \le z \le 1 - y^2. \qquad \text{Bounds for } z$$

Projecting the region onto the xy-plane produces a parallelogram. Because two sides of the parallelogram are parallel to the x-axis, you have the following bounds:

$$1 - y \le x \le 3 - y \quad \text{and} \quad 0 \le y \le 1.$$

So, the volume of the region is given by

$$V = \iiint_Q dV = \int_0^1 \int_{1-y}^{3-y} \int_0^{1-y^2} dz\,dx\,dy.$$



Q: $0 \le z \le \sqrt{1 - x^2 - y^2}$
$-\sqrt{1 - y^2} \le x \le \sqrt{1 - y^2}$
$-1 \le y \le 1$

Figure 14.58

b. For the upper hemisphere given by $z = \sqrt{1 - x^2 - y^2}$, you have

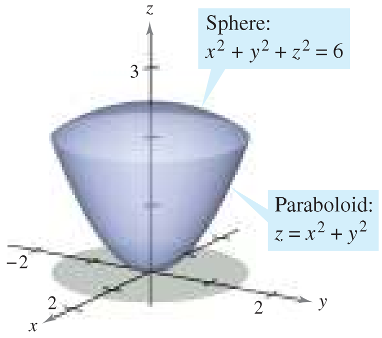
$$0 \le z \le \sqrt{1 - x^2 - y^2}. \qquad \text{Bounds for } z$$

In Figure 14.58, note that the projection of the hemisphere onto the xy-plane is the circle given by $x^2 + y^2 = 1$, and you can use either order $dx\,dy$ or $dy\,dx$. Choosing the first produces

$$-\sqrt{1 - y^2} \le x \le \sqrt{1 - y^2} \quad \text{and} \quad -1 \le y \le 1$$

which implies that the volume of the region is given by

$$V = \iiint_Q dV = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\,dx\,dy.$$



Q: $x^2 + y^2 \le z \le \sqrt{6 - x^2 - y^2}$
$-\sqrt{2 - x^2} \le y \le \sqrt{2 - x^2}$
$-\sqrt{2} \le x \le \sqrt{2}$

Figure 14.59

c. For the region bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 6$, you have

$$x^2 + y^2 \le z \le \sqrt{6 - x^2 - y^2}. \qquad \text{Bounds for } z$$

The sphere and the paraboloid intersect at $z = 2$. Moreover, you can see in Figure 14.59 that the projection of the solid region onto the xy-plane is the circle given by $x^2 + y^2 = 2$. Using the order $dy\,dx$ produces

$$-\sqrt{2 - x^2} \le y \le \sqrt{2 - x^2} \quad \text{and} \quad -\sqrt{2} \le x \le \sqrt{2}$$

which implies that the volume of the region is given by

$$V = \iiint_Q dV = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} dz\,dy\,dx. \qquad ■$$

Center of Mass and Moments of Inertia

EXPLORATION

Sketch the solid (of uniform density) bounded by $z = 0$ and

$$z = \frac{1}{1 + x^2 + y^2}$$

where $x^2 + y^2 \le 1$. From your sketch, estimate the coordinates of the center of mass of the solid. Now use a computer algebra system to verify your estimate. What do you observe?

In the remainder of this section, two important engineering applications of triple integrals are discussed. Consider a solid region Q whose density is given by the **density function ρ.** The **center of mass** of a solid region Q of mass m is given by $(\bar{x}, \bar{y}, \bar{z})$, where

$$m = \iiint\limits_Q \rho(x, y, z)\, dV \qquad \text{Mass of the solid}$$

$$M_{yz} = \iiint\limits_Q x\rho(x, y, z)\, dV \qquad \text{First moment about } yz\text{-plane}$$

$$M_{xz} = \iiint\limits_Q y\rho(x, y, z)\, dV \qquad \text{First moment about } xz\text{-plane}$$

$$M_{xy} = \iiint\limits_Q z\rho(x, y, z)\, dV \qquad \text{First moment about } xy\text{-plane}$$

and

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}.$$

NOTE In engineering and physics, the moment of inertia of a mass is used to find the time required for the mass to reach a given speed of rotation about an axis, as shown in Figure 14.60. The greater the moment of inertia, the longer a force must be applied for the mass to reach the given speed.

The quantities M_{yz}, M_{xz}, and M_{xy} are called the **first moments** of the region Q about the yz-, xz-, and xy-planes, respectively.

The first moments for solid regions are taken about a plane, whereas the second moments for solids are taken about a line. The **second moments** (or **moments of inertia**) about the x-, y-, and z-axes are as follows.

$$I_x = \iiint\limits_Q (y^2 + z^2)\rho(x, y, z)\, dV \qquad \text{Moment of inertia about } x\text{-axis}$$

$$I_y = \iiint\limits_Q (x^2 + z^2)\rho(x, y, z)\, dV \qquad \text{Moment of inertia about } y\text{-axis}$$

$$I_z = \iiint\limits_Q (x^2 + y^2)\rho(x, y, z)\, dV \qquad \text{Moment of inertia about } z\text{-axis}$$

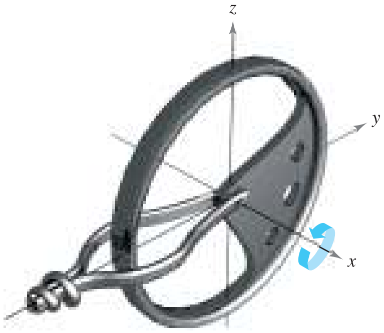


Figure 14.60

For problems requiring the calculation of all three moments, considerable effort can be saved by applying the additive property of triple integrals and writing

$$I_x = I_{xz} + I_{xy}, \quad I_y = I_{yz} + I_{xy}, \quad \text{and} \quad I_z = I_{yz} + I_{xz}$$

where I_{xy}, I_{xz}, and I_{yz} are as follows.

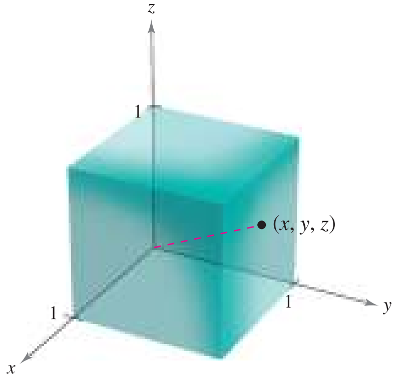
$$I_{xy} = \iiint\limits_Q z^2\rho(x, y, z)\, dV$$

$$I_{xz} = \iiint\limits_Q y^2\rho(x, y, z)\, dV$$

$$I_{yz} = \iiint\limits_Q x^2\rho(x, y, z)\, dV$$

EXAMPLE 5 Finding the Center of Mass of a Solid Region

Find the center of mass of the unit cube shown in Figure 14.61, given that the density at the point (x, y, z) is proportional to the square of its distance from the origin.



Variable density:
$\rho(x, y, z) = k(x^2 + y^2 + z^2)$
Figure 14.61

Solution Because the density at (x, y, z) is proportional to the square of the distance between $(0, 0, 0)$ and (x, y, z), you have

$$\rho(x, y, z) = k(x^2 + y^2 + z^2).$$

You can use this density function to find the mass of the cube. Because of the symmetry of the region, any order of integration will produce an integral of comparable difficulty.

$$\begin{aligned} m &= \int_0^1\int_0^1\int_0^1 k(x^2 + y^2 + z^2)\,dz\,dy\,dx \\ &= k\int_0^1\int_0^1 \left[(x^2 + y^2)z + \frac{z^3}{3}\right]_0^1 dy\,dx \\ &= k\int_0^1\int_0^1 \left(x^2 + y^2 + \frac{1}{3}\right) dy\,dx \\ &= k\int_0^1 \left[\left(x^2 + \frac{1}{3}\right)y + \frac{y^3}{3}\right]_0^1 dx \\ &= k\int_0^1 \left(x^2 + \frac{2}{3}\right) dx \\ &= k\left[\frac{x^3}{3} + \frac{2x}{3}\right]_0^1 = k \end{aligned}$$

The first moment about the yz-plane is

$$\begin{aligned} M_{yz} &= k\int_0^1\int_0^1\int_0^1 x(x^2 + y^2 + z^2)\,dz\,dy\,dx \\ &= k\int_0^1 x\left[\int_0^1\int_0^1 (x^2 + y^2 + z^2)\,dz\,dy\right]dx. \end{aligned}$$

Note that x can be factored out of the two inner integrals, because it is constant with respect to y and z. After factoring, the two inner integrals are the same as for the mass m. Therefore, you have

$$\begin{aligned} M_{yz} &= k\int_0^1 x\left(x^2 + \frac{2}{3}\right) dx \\ &= k\left[\frac{x^4}{4} + \frac{x^2}{3}\right]_0^1 \\ &= \frac{7k}{12}. \end{aligned}$$

So,

$$\bar{x} = \frac{M_{yz}}{m} = \frac{7k/12}{k} = \frac{7}{12}.$$

Finally, from the nature of ρ and the symmetry of x, y, and z in this solid region, you have $\bar{x} = \bar{y} = \bar{z}$, and the center of mass is $\left(\frac{7}{12}, \frac{7}{12}, \frac{7}{12}\right)$. ■

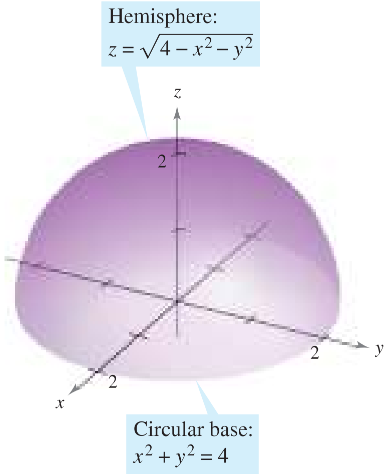
EXAMPLE 6 Moments of Inertia for a Solid Region

Find the moments of inertia about the x- and y-axes for the solid region lying between the hemisphere

$$z = \sqrt{4 - x^2 - y^2}$$

and the xy-plane, given that the density at (x, y, z) is proportional to the distance between (x, y, z) and the xy-plane.

$0 \le z \le \sqrt{4 - x^2 - y^2}$
$-\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}$
$-2 \le x \le 2$



Variable density: $\rho(x, y, z) = kz$
Figure 14.62

Solution The density of the region is given by $\rho(x, y, z) = kz$. Considering the symmetry of this problem, you know that $I_x = I_y$, and you need to compute only one moment, say I_x. From Figure 14.62, choose the order $dz\,dy\,dx$ and write

$$\begin{aligned}
I_x &= \iiint_Q (y^2 + z^2)\rho(x, y, z)\, dV \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} (y^2 + z^2)(kz)\, dz\, dy\, dx \\
&= k\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[\frac{y^2z^2}{2} + \frac{z^4}{4}\right]_0^{\sqrt{4-x^2-y^2}} dy\, dx \\
&= k\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[\frac{y^2(4 - x^2 - y^2)}{2} + \frac{(4 - x^2 - y^2)^2}{4}\right] dy\, dx \\
&= \frac{k}{4}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[(4 - x^2)^2 - y^4\right] dy\, dx \\
&= \frac{k}{4}\int_{-2}^{2} \left[(4 - x^2)^2 y - \frac{y^5}{5}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\
&= \frac{k}{4}\int_{-2}^{2} \frac{8}{5}(4 - x^2)^{5/2}\, dx \\
&= \frac{4k}{5}\int_{0}^{2} (4 - x^2)^{5/2}\, dx && x = 2\sin\theta \\
&= \frac{4k}{5}\int_{0}^{\pi/2} 64\cos^6\theta\, d\theta \\
&= \left(\frac{256k}{5}\right)\left(\frac{5\pi}{32}\right) && \text{Wallis's Formula} \\
&= 8k\pi.
\end{aligned}$$

So, $I_x = 8k\pi = I_y$. ■

In Example 6, notice that the moments of inertia about the x- and y-axes are equal to each other. The moment about the z-axis, however, is different. Does it seem that the moment of inertia about the z-axis should be less than or greater than the moments calculated in Example 6? By performing the calculations, you can determine that

$$I_z = \frac{16}{3}k\pi.$$

This tells you that the solid shown in Figure 14.62 has a greater resistance to rotation about the x- or y-axis than about the z-axis.

14.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, evaluate the iterated integral.

1. $\int_0^3\int_0^2\int_0^1 (x+y+z)\,dx\,dz\,dy$

2. $\int_{-1}^1\int_{-1}^1\int_{-1}^1 x^2y^2z^2\,dx\,dy\,dz$

3. $\int_0^1\int_0^x\int_0^{xy} x\,dz\,dy\,dx$

4. $\int_0^9\int_0^{y/3}\int_0^{\sqrt{y^2-9x^2}} z\,dz\,dx\,dy$

5. $\int_1^4\int_0^1\int_0^x 2ze^{-x^2}\,dy\,dx\,dz$

6. $\int_1^4\int_1^{e^2}\int_0^{1/xz} \ln z\,dy\,dz\,dx$

7. $\int_0^4\int_0^{\pi/2}\int_0^{1-x} x\cos y\,dz\,dy\,dx$

8. $\int_0^{\pi/2}\int_0^{y/2}\int_0^{1/y} \sin y\,dz\,dx\,dy$

CAS **In Exercises 9 and 10, use a computer algebra system to evaluate the iterated integral.**

9. $\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_0^{y^2} y\,dz\,dx\,dy$

10. $\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{2x^2+y^2}^{4-y^2} y\,dz\,dy\,dx$

CAS **In Exercises 11 and 12, use a computer algebra system to approximate the iterated integral.**

11. $\int_0^2\int_0^{\sqrt{4-x^2}}\int_1^4 \frac{x^2\sin y}{z}\,dz\,dy\,dx$

12. $\int_0^3\int_0^{2-(2y/3)}\int_0^{6-2y-3z} ze^{-x^2y^2}\,dx\,dz\,dy$

In Exercises 13–18, set up a triple integral for the volume of the solid.

13. The solid in the first octant bounded by the coordinate planes and the plane $z = 5 - x - y$

14. The solid bounded by $z = 9 - x^2$, $z = 0$, $y = 0$, and $y = 2x$

15. The solid bounded by $z = 6 - x^2 - y^2$ and $z = 0$

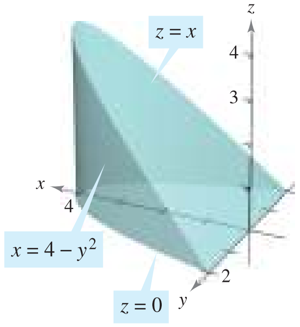
16. The solid bounded by $z = \sqrt{16 - x^2 - y^2}$ and $z = 0$

17. The solid that is the common interior below the sphere $x^2 + y^2 + z^2 = 80$ and above the paraboloid $z = \frac{1}{2}(x^2 + y^2)$

18. The solid bounded above by the cylinder $z = 4 - x^2$ and below by the paraboloid $z = x^2 + 3y^2$

Volume **In Exercises 19–22, use a triple integral to find the volume of the solid shown in the figure.**

19.



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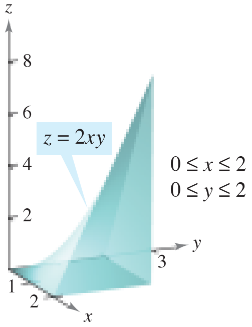

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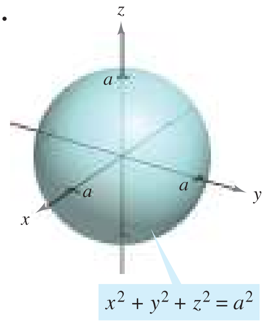

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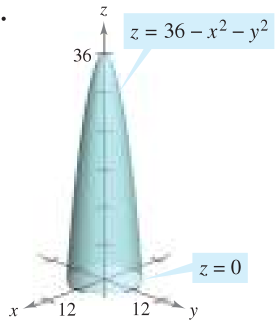


Volume **In Exercises 23–26, use a triple integral to find the volume of the solid bounded by the graphs of the equations.**

23. $z = 4 - x^2$, $y = 4 - x^2$, first octant

24. $z = 9 - x^3$, $y = -x^2 + 2$, $y = 0$, $z = 0$, $x \ge 0$

25. $z = 2 - y$, $z = 4 - y^2$, $x = 0$, $x = 3$, $y = 0$

26. $z = x$, $y = x + 2$, $y = x^2$, first octant

In Exercises 27–32, sketch the solid whose volume is given by the iterated integral and rewrite the integral using the indicated order of integration.

27. $\int_0^1\int_{-1}^0\int_0^{y^2} dz\,dy\,dx$

Rewrite using the order $dy\,dz\,dx$.

28. $\int_{-1}^1\int_{y^2}^1\int_0^{1-x} dz\,dx\,dy$

Rewrite using the order $dx\,dz\,dy$.

29. $\int_0^4\int_0^{(4-x)/2}\int_0^{(12-3x-6y)/4} dz\,dy\,dx$

Rewrite using the order $dy\,dx\,dz$.

30. $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{6-x-y} dz\,dy\,dx$

Rewrite using the order $dz\,dx\,dy$.

31. $\int_0^1\int_y^1\int_0^{\sqrt{1-y^2}} dz\,dx\,dy$

Rewrite using the order $dz\,dy\,dx$.

32. $\int_0^2\int_{2x}^4\int_0^{\sqrt{y^2-4x^2}} dz\,dy\,dx$

Rewrite using the order $dx\,dy\,dz$.

In Exercises 33–36, list the six possible orders of integration for the triple integral over the solid region Q, $\iiint_Q xyz\,dV$.

33. $Q = \{(x, y, z): 0 \le x \le 1, 0 \le y \le x, 0 \le z \le 3\}$

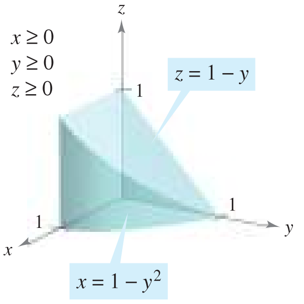
34. $Q = \{(x, y, z): 0 \le x \le 2, x^2 \le y \le 4, 0 \le z \le 2 - x\}$

35. $Q = \{(x, y, z): x^2 + y^2 \le 9, 0 \le z \le 4\}$

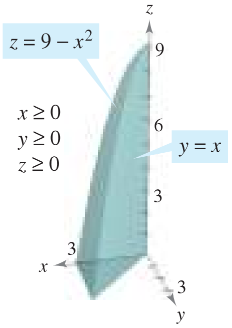
36. $Q = \{(x, y, z): 0 \le x \le 1, y \le 1 - x^2, 0 \le z \le 6\}$

In Exercises 37 and 38, the figure shows the region of integration for the given integral. Rewrite the integral as an equivalent iterated integral in the five other orders.

37. $\int_0^1 \int_0^{1-y^2} \int_0^{1-y} dz\,dx\,dy$



38. $\int_0^3 \int_0^x \int_0^{9-x^2} dz\,dy\,dx$



Mass and Center of Mass **In Exercises 39–42, find the mass and the indicated coordinates of the center of mass of the solid of given density bounded by the graphs of the equations.**

39. Find $\bar{x}$ using $\rho(x, y, z) = k$.

Q: $2x + 3y + 6z = 12, x = 0, y = 0, z = 0$

40. Find $\bar{y}$ using $\rho(x, y, z) = ky$.

Q: $3x + 3y + 5z = 15, x = 0, y = 0, z = 0$

41. Find $\bar{z}$ using $\rho(x, y, z) = kx$.

Q: $z = 4 - x, z = 0, y = 0, y = 4, x = 0$

42. Find $\bar{y}$ using $\rho(x, y, z) = k$.

Q: $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1\ (a, b, c > 0), x = 0, y = 0, z = 0$

Mass and Center of Mass **In Exercises 43 and 44, set up the triple integrals for finding the mass and the center of mass of the solid bounded by the graphs of the equations.**

43. $x = 0, x = b, y = 0, y = b, z = 0, z = b$

$\rho(x, y, z) = kxy$

44. $x = 0, x = a, y = 0, y = b, z = 0, z = c$

$\rho(x, y, z) = kz$

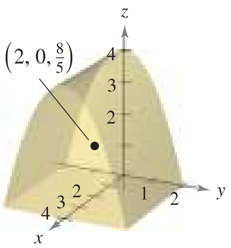
Think About It **The center of mass of a solid of constant density is shown in the figure. In Exercises 45–48, make a conjecture about how the center of mass $(\bar{x}, \bar{y}, \bar{z})$ will change for the nonconstant density $\rho(x, y, z)$. Explain.**

45. $\rho(x, y, z) = kx$

46. $\rho(x, y, z) = kz$

47. $\rho(x, y, z) = k(y + 2)$

48. $\rho(x, y, z) = kxz^2(y + 2)^2$



CAS ***Centroid*** **In Exercises 49–54, find the centroid of the solid region bounded by the graphs of the equations or described by the figure. Use a computer algebra system to evaluate the triple integrals. (Assume uniform density and find the center of mass.)**

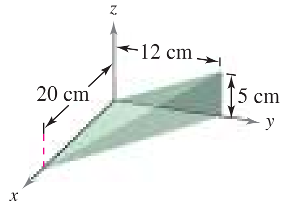
49. $z = \dfrac{h}{r}\sqrt{x^2 + y^2}, z = h$

50. $y = \sqrt{9 - x^2}, z = y, z = 0$

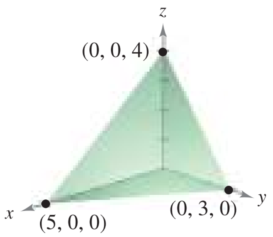
51. $z = \sqrt{16 - x^2 - y^2}, z = 0$

52. $z = \dfrac{1}{y^2 + 1}, z = 0, x = -2, x = 2, y = 0, y = 1$

53.



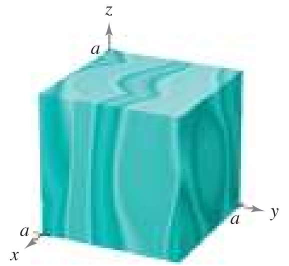
54.



CAS ***Moments of Inertia*** **In Exercises 55–58, find I_x, I_y, and I_z for the solid of given density. Use a computer algebra system to evaluate the triple integrals.**

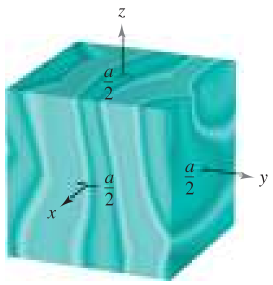
55. (a) $\rho = k$

(b) $\rho = kxyz$



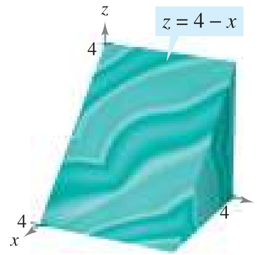
56. (a) $\rho(x, y, z) = k$

(b) $\rho(x, y, z) = k(x^2 + y^2)$



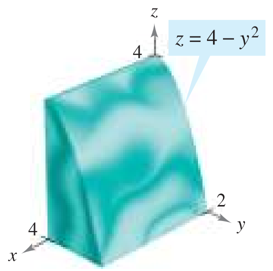
57. (a) $\rho(x, y, z) = k$

(b) $\rho = ky$



58. (a) $\rho = kz$

(b) $\rho = k(4 - z)$

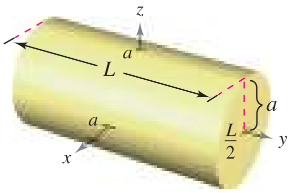


CAS ***Moments of Inertia*** **In Exercises 59 and 60, verify the moments of inertia for the solid of uniform density. Use a computer algebra system to evaluate the triple integrals.**

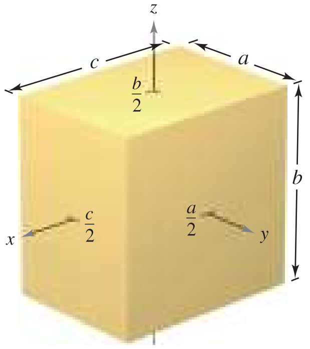
59. $I_x = \frac{1}{12}m(3a^2 + L^2)$

$I_y = \frac{1}{2}ma^2$

$I_z = \frac{1}{12}m(3a^2 + L^2)$



60. $I_x = \frac{1}{12}m(a^2 + b^2)$
$I_y = \frac{1}{12}m(b^2 + c^2)$
$I_z = \frac{1}{12}m(a^2 + c^2)$



Moments of Inertia **In Exercises 61 and 62, set up a triple integral that gives the moment of inertia about the z-axis of the solid region Q of density ρ.**

61. $Q = \{(x, y, z): -1 \le x \le 1, -1 \le y \le 1, 0 \le z \le 1 - x\}$
$\rho = \sqrt{x^2 + y^2 + z^2}$

62. $Q = \{(x, y, z): x^2 + y^2 \le 1, 0 \le z \le 4 - x^2 - y^2\}$
$\rho = kx^2$

In Exercises 63 and 64, using the description of the solid region, set up the integral for (a) the mass, (b) the center of mass, and (c) the moment of inertia about the z-axis.

63. The solid bounded by $z = 4 - x^2 - y^2$ and $z = 0$ with density function $\rho = kz$

64. The solid in the first octant bounded by the coordinate planes and $x^2 + y^2 + z^2 = 25$ with density function $\rho = kxy$

WRITING ABOUT CONCEPTS

65. Define a triple integral and describe a method of evaluating a triple integral.

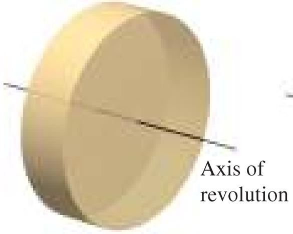
66. Determine whether the moment of inertia about the y-axis of the cylinder in Exercise 59 will increase or decrease for the nonconstant density $\rho(x, y, z) = \sqrt{x^2 + z^2}$ and $a = 4$.

67. Consider two solids, solid A and solid B, of equal weight as shown below.

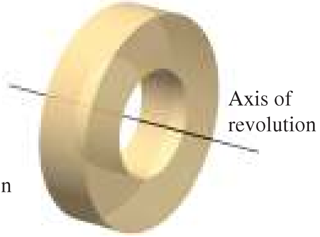
(a) Because the solids have the same weight, which has the greater density?

(b) Which solid has the greater moment of inertia? Explain.

(c) The solids are rolled down an inclined plane. They are started at the same time and at the same height. Which will reach the bottom first? Explain.



Solid A



Solid B

CAPSTONE

68. ***Think About It*** Of the integrals (a)–(c), which one is equal to $\int_1^3 \int_0^2 \int_{-1}^1 f(x, y, z)\, dz\, dy\, dx$? Explain.

(a) $\int_1^3 \int_0^2 \int_{-1}^1 f(x, y, z)\, dz\, dx\, dy$

(b) $\int_{-1}^1 \int_0^2 \int_1^3 f(x, y, z)\, dx\, dy\, dz$

(c) $\int_0^2 \int_1^3 \int_{-1}^1 f(x, y, z)\, dy\, dx\, dz$

Average Value **In Exercises 69–72, find the average value of the function over the given solid. The average value of a continuous function $f(x, y, z)$ over a solid region Q is**

$$\frac{1}{V}\iiint_Q f(x, y, z)\, dV$$

where V is the volume of the solid region Q.

69. $f(x, y, z) = z^2 + 4$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$

70. $f(x, y, z) = xyz$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 4$, $y = 4$, and $z = 4$

71. $f(x, y, z) = x + y + z$ over the tetrahedron in the first octant with vertices (0, 0, 0), (2, 0, 0), (0, 2, 0) and (0, 0, 2)

72. $f(x, y, z) = x + y$ over the solid bounded by the sphere $x^2 + y^2 + z^2 = 3$

CAS **73.** Find the solid region Q where the triple integral

$$\iiint_Q (1 - 2x^2 - y^2 - 3z^2)\, dV$$

is a maximum. Use a computer algebra system to approximate the maximum value. What is the exact maximum value?

CAS **74.** Find the solid region Q where the triple integral

$$\iiint_Q (1 - x^2 - y^2 - z^2)\, dV$$

is a maximum. Use a computer algebra system to approximate the maximum value. What is the exact maximum value?

75. Solve for a in the triple integral.

$$\int_0^1 \int_0^{3-a-y^2} \int_a^{4-x-y^2} dz\, dx\, dy = \frac{14}{15}$$

76. Determine the value of b such that the volume of the ellipsoid $x^2 + (y^2/b^2) + (z^2/9) = 1$ is 16π.

PUTNAM EXAM CHALLENGE

77. Evaluate

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2\left\{\frac{\pi}{2n}(x_1 + x_2 + \cdots + x_n)\right\} dx_1\, dx_2 \cdots dx_n.$$

This problem was composed by the Committee on the Putnam Prize Competition.
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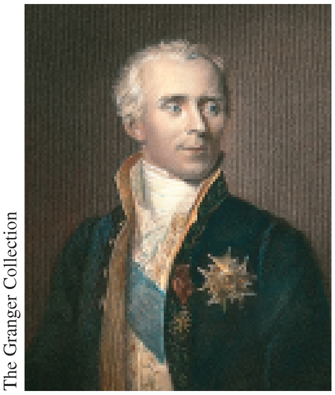
14.7 Triple Integrals in Cylindrical and Spherical Coordinates

- **Write and evaluate a triple integral in cylindrical coordinates.**
- **Write and evaluate a triple integral in spherical coordinates.**



The Granger Collection

PIERRE SIMON DE LAPLACE (1749–1827)

One of the first to use a cylindrical coordinate system was the French mathematician Pierre Simon de Laplace. Laplace has been called the "Newton of France," and he published many important works in mechanics, differential equations, and probability.

Triple Integrals in Cylindrical Coordinates

Many common solid regions such as spheres, ellipsoids, cones, and paraboloids can yield difficult triple integrals in rectangular coordinates. In fact, it is precisely this difficulty that led to the introduction of nonrectangular coordinate systems. In this section, you will learn how to use *cylindrical* and *spherical* coordinates to evaluate triple integrals.

Recall from Section 11.7 that the rectangular conversion equations for cylindrical coordinates are

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z.$$

STUDY TIP An easy way to remember these conversions is to note that the equations for x and y are the same as in polar coordinates and z is unchanged. ■

In this coordinate system, the simplest solid region is a cylindrical block determined by

$$r_1 \le r \le r_2, \quad \theta_1 \le \theta \le \theta_2, \quad z_1 \le z \le z_2$$

as shown in Figure 14.63. To obtain the cylindrical coordinate form of a triple integral, suppose that Q is a solid region whose projection R onto the xy-plane can be described in polar coordinates. That is,

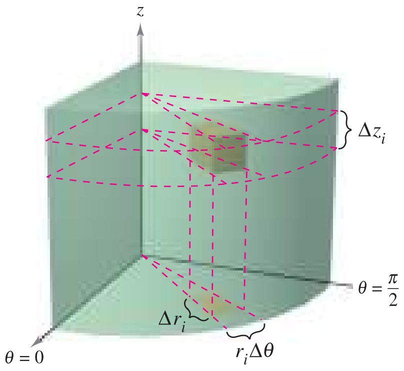
$$Q = \{(x, y, z): (x, y) \text{ is in } R, \quad h_1(x, y) \le z \le h_2(x, y)\}$$

and

$$R = \{(r, \theta): \theta_1 \le \theta \le \theta_2, \quad g_1(\theta) \le r \le g_2(\theta)\}.$$

If f is a continuous function on the solid Q, you can write the triple integral of f over Q as

$$\iiint\limits_Q f(x, y, z)\, dV = \iint\limits_R \left[\int_{h_1(x, y)}^{h_2(x, y)} f(x, y, z)\, dz\right] dA$$



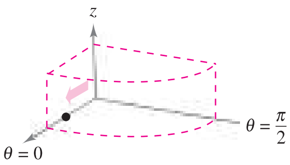
Volume of cylindrical block:
$\Delta V_i = r_i\,\Delta r_i\,\Delta\theta_i\,\Delta z_i$
Figure 14.63

where the double integral over R is evaluated in polar coordinates. That is, R is a plane region that is either r-simple or θ-simple. If R is r-simple, the iterated form of the triple integral in cylindrical form is

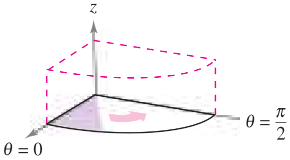
$$\iiint\limits_Q f(x, y, z)\, dV = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta,\, r\sin\theta)}^{h_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta.$$

NOTE This is only one of six possible orders of integration. The other five are $dz\,d\theta\,dr$, $dr\,dz\,d\theta$, $dr\,d\theta\,dz$, $d\theta\,dz\,dr$, and $d\theta\,dr\,dz$. ■

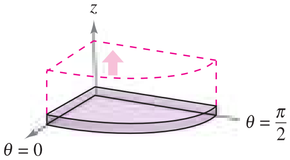
To visualize a particular order of integration, it helps to view the iterated integral in terms of three sweeping motions—each adding another dimension to the solid. For instance, in the order $dr\,d\theta\,dz$, the first integration occurs in the r-direction as a point sweeps out a ray. Then, as θ increases, the line sweeps out a sector. Finally, as z increases, the sector sweeps out a solid wedge, as shown in Figure 14.64.



Integrate with respect to r.



Integrate with respect to θ.



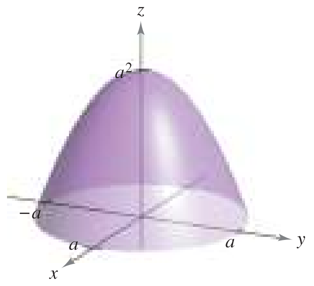
Integrate with respect to z.
Figure 14.64

EXPLORATION

Volume of a Paraboloid Sector On pages 997, 1006, and 1028, you were asked to summarize the different ways you know of finding the volume of the solid bounded by the paraboloid

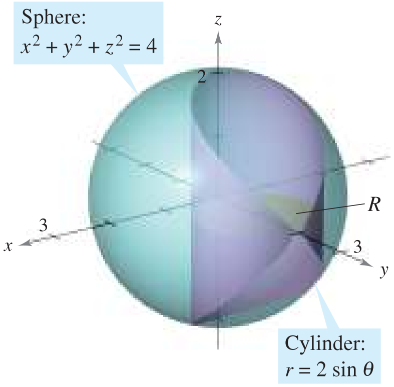
$$z = a^2 - x^2 - y^2, \quad a > 0$$

and the xy-plane. You now know one more way. Use it to find the volume of the solid. Compare the different methods. What are the advantages and disadvantages of each?



EXAMPLE 1 Finding Volume in Cylindrical Coordinates

Find the volume of the solid region Q cut from the sphere $x^2 + y^2 + z^2 = 4$ by the cylinder $r = 2\sin\theta$, as shown in Figure 14.65.



Sphere: $x^2 + y^2 + z^2 = 4$

Cylinder: $r = 2\sin\theta$

Figure 14.65

Solution Because $x^2 + y^2 + z^2 = r^2 + z^2 = 4$, the bounds on z are

$$-\sqrt{4 - r^2} \le z \le \sqrt{4 - r^2}.$$

Let R be the circular projection of the solid onto the $r\theta$-plane. Then the bounds on R are $0 \le r \le 2\sin\theta$ and $0 \le \theta \le \pi$. So, the volume of Q is

$$\begin{aligned}
V &= \int_0^{\pi}\int_0^{2\sin\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta \\
&= 2\int_0^{\pi/2}\int_0^{2\sin\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta \\
&= 2\int_0^{\pi/2}\int_0^{2\sin\theta} 2r\sqrt{4 - r^2}\,dr\,d\theta \\
&= 2\int_0^{\pi/2} -\frac{2}{3}(4 - r^2)^{3/2}\Big]_0^{2\sin\theta}\,d\theta \\
&= \frac{4}{3}\int_0^{\pi/2} (8 - 8\cos^3\theta)\,d\theta \\
&= \frac{32}{3}\int_0^{\pi/2} [1 - (\cos\theta)(1 - \sin^2\theta)]\,d\theta \\
&= \frac{32}{3}\left[\theta - \sin\theta + \frac{\sin^3\theta}{3}\right]_0^{\pi/2} \\
&= \frac{16}{9}(3\pi - 4) \\
&\approx 9.644.
\end{aligned}$$

■

EXAMPLE 2 Finding Mass in Cylindrical Coordinates

Find the mass of the ellipsoid Q given by $4x^2 + 4y^2 + z^2 = 16$, lying above the xy-plane. The density at a point in the solid is proportional to the distance between the point and the xy-plane.

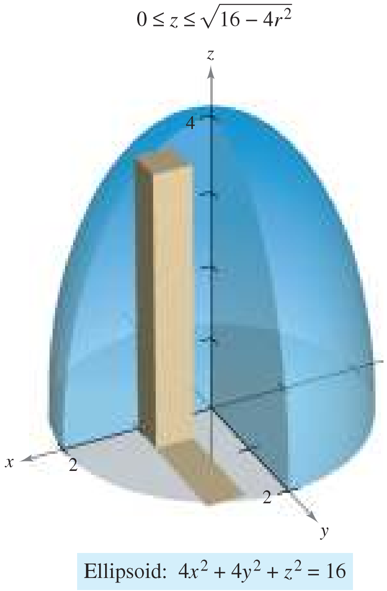

$0 \le z \le \sqrt{16 - 4r^2}$
z
4
x
2
2
y
Ellipsoid: $4x^2 + 4y^2 + z^2 = 16$

Figure 14.66

Solution The density function is $\rho(r, \theta, z) = kz$. The bounds on z are

$$0 \le z \le \sqrt{16 - 4x^2 - 4y^2} = \sqrt{16 - 4r^2}$$

where $0 \le r \le 2$ and $0 \le \theta \le 2\pi$, as shown in Figure 14.66. The mass of the solid is

$$\begin{aligned}
m &= \int_0^{2\pi}\int_0^2\int_0^{\sqrt{16-4r^2}} kzr\,dz\,dr\,d\theta \\
&= \frac{k}{2}\int_0^{2\pi}\int_0^2 z^2r\Big]_0^{\sqrt{16-4r^2}} dr\,d\theta \\
&= \frac{k}{2}\int_0^{2\pi}\int_0^2 (16r - 4r^3)\,dr\,d\theta \\
&= \frac{k}{2}\int_0^{2\pi}\Big[8r^2 - r^4\Big]_0^2 d\theta \\
&= 8k\int_0^{2\pi} d\theta = 16\pi k.
\end{aligned}$$

■

Integration in cylindrical coordinates is useful when factors involving $x^2 + y^2$ appear in the integrand, as illustrated in Example 3.

EXAMPLE 3 Finding a Moment of Inertia

Find the moment of inertia about the axis of symmetry of the solid Q bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$, as shown in Figure 14.67. The density at each point is proportional to the distance between the point and the z-axis.

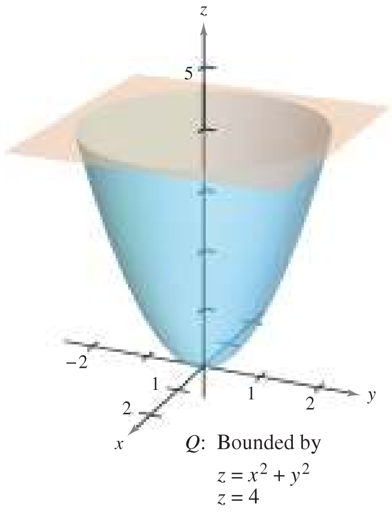


Figure 14.67

Solution Because the z-axis is the axis of symmetry, and $\rho(x, y, z) = k\sqrt{x^2 + y^2}$, it follows that

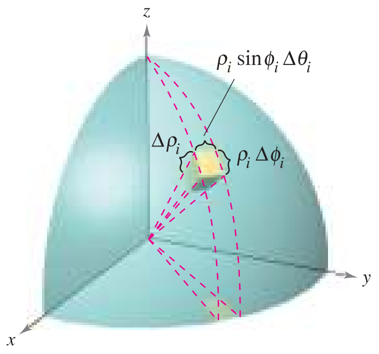
$$I_z = \iiint\limits_Q k(x^2 + y^2)\sqrt{x^2 + y^2}\,dV.$$

In cylindrical coordinates, $0 \le r \le \sqrt{x^2 + y^2} = \sqrt{z}$. So, you have

$$\begin{aligned}
I_z &= k\int_0^4\int_0^{2\pi}\int_0^{\sqrt{z}} r^2(r)r\,dr\,d\theta\,dz \\
&= k\int_0^4\int_0^{2\pi} \frac{r^5}{5}\Big]_0^{\sqrt{z}} d\theta\,dz \\
&= k\int_0^4\int_0^{2\pi} \frac{z^{5/2}}{5}\,d\theta\,dz \\
&= \frac{k}{5}\int_0^4 z^{5/2}(2\pi)\,dz \\
&= \frac{2\pi k}{5}\Big[\frac{2}{7}z^{7/2}\Big]_0^4 = \frac{512k\pi}{35}.
\end{aligned}$$

■

Triple Integrals in Spherical Coordinates



Spherical block:
$\Delta V_i \approx \rho_i^2 \sin\phi_i\,\Delta\rho_i\,\Delta\phi_i\,\Delta\theta_i$
Figure 14.68

Triple integrals involving spheres or cones are often easier to evaluate by converting to spherical coordinates. Recall from Section 11.7 that the rectangular conversion equations for spherical coordinates are

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi.$$

In this coordinate system, the simplest region is a spherical block determined by

$$\{(\rho, \theta, \phi): \rho_1 \le \rho \le \rho_2, \quad \theta_1 \le \theta \le \theta_2, \quad \phi_1 \le \phi \le \phi_2\}$$

where $\rho_1 \ge 0$, $\theta_2 - \theta_1 \le 2\pi$, and $0 \le \phi_1 \le \phi_2 \le \pi$, as shown in Figure 14.68. If (ρ, θ, ϕ) is a point in the interior of such a block, then the volume of the block can be approximated by $\Delta V \approx \rho^2 \sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$ (see Exercise 18 in the Problem Solving exercises at the end of this chapter).

Using the usual process involving an inner partition, summation, and a limit, you can develop the following version of a triple integral in spherical coordinates for a continuous function f defined on the solid region Q.

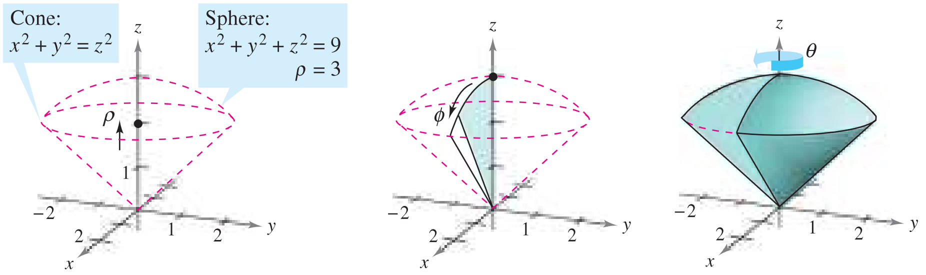
$$\iiint_Q f(x, y, z)\,dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

This formula can be modified for different orders of integration and generalized to include regions with variable boundaries.

Like triple integrals in cylindrical coordinates, triple integrals in spherical coordinates are evaluated with iterated integrals. As with cylindrical coordinates, you can visualize a particular order of integration by viewing the iterated integral in terms of three sweeping motions—each adding another dimension to the solid. For instance, the iterated integral

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

(which is used in Example 4) is illustrated in Figure 14.69.



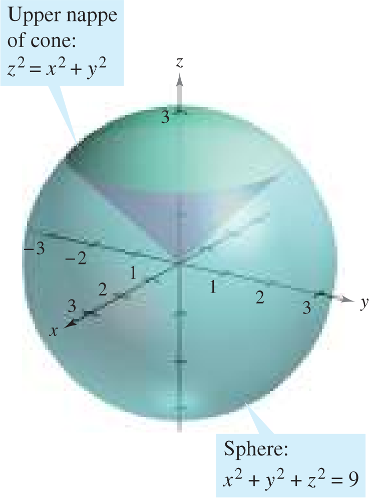
ρ varies from 0 to 3 with ϕ and θ held constant.

ϕ varies from 0 to $\pi/4$ with θ held constant.

θ varies from 0 to 2π.

Figure 14.69

NOTE The Greek letter ρ used in spherical coordinates is not related to density. Rather, it is the three-dimensional analog of the r used in polar coordinates. For problems involving spherical coordinates and a density function, this text uses a different symbol to denote density. ■



Upper nappe of cone: $z^2 = x^2 + y^2$

Sphere: $x^2 + y^2 + z^2 = 9$

Figure 14.70

EXAMPLE 4 Finding Volume in Spherical Coordinates

Find the volume of the solid region Q bounded below by the upper nappe of the cone $z^2 = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 9$, as shown in Figure 14.70.

Solution In spherical coordinates, the equation of the sphere is

$$\rho^2 = x^2 + y^2 + z^2 = 9 \quad \Rightarrow \quad \rho = 3.$$

Furthermore, the sphere and cone intersect when

$$(x^2 + y^2) + z^2 = (z^2) + z^2 = 9 \quad \Rightarrow \quad z = \frac{3}{\sqrt{2}}$$

and, because $z = \rho \cos \phi$, it follows that

$$\left(\frac{3}{\sqrt{2}}\right)\left(\frac{1}{3}\right) = \cos \phi \quad \Rightarrow \quad \phi = \frac{\pi}{4}.$$

Consequently, you can use the integration order $d\rho\, d\phi\, d\theta$, where $0 \le \rho \le 3$, $0 \le \phi \le \pi/4$, and $0 \le \theta \le 2\pi$. The volume is

$$\begin{aligned}
V = \iiint_Q dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} 9 \sin \phi \, d\phi \, d\theta \\
&= 9 \int_0^{2\pi} -\cos \phi \Big]_0^{\pi/4} d\theta \\
&= 9 \int_0^{2\pi} \left(1 - \frac{\sqrt{2}}{2}\right) d\theta = 9\pi\left(2 - \sqrt{2}\right) \approx 16.563.
\end{aligned}$$

EXAMPLE 5 Finding the Center of Mass of a Solid Region

Find the center of mass of the solid region Q of uniform density, bounded below by the upper nappe of the cone $z^2 = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 9$.

Solution Because the density is uniform, you can consider the density at the point (x, y, z) to be k. By symmetry, the center of mass lies on the z-axis, and you need only calculate $\bar{z} = M_{xy}/m$, where $m = kV = 9k\pi\left(2 - \sqrt{2}\right)$ from Example 4. Because $z = \rho \cos \phi$, it follows that

$$\begin{aligned}
M_{xy} = \iiint_Q kz \, dV &= k \int_0^3 \int_0^{2\pi} \int_0^{\pi/4} (\rho \cos \phi)\rho^2 \sin \phi \, d\phi \, d\theta \, d\rho \\
&= k \int_0^3 \int_0^{2\pi} \rho^3 \frac{\sin^2 \phi}{2} \Big]_0^{\pi/4} d\theta \, d\rho \\
&= \frac{k}{4} \int_0^3 \int_0^{2\pi} \rho^3 \, d\theta \, d\rho = \frac{k\pi}{2} \int_0^3 \rho^3 \, d\rho = \frac{81k\pi}{8}.
\end{aligned}$$

So,

$$\bar{z} = \frac{M_{xy}}{m} = \frac{81k\pi/8}{9k\pi\left(2 - \sqrt{2}\right)} = \frac{9\left(2 + \sqrt{2}\right)}{16} \approx 1.920$$

and the center of mass is approximately (0, 0, 1.92). ■

14.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, evaluate the iterated integral.

1. $\displaystyle\int_{-1}^{5}\int_{0}^{\pi/2}\int_{0}^{3} r\cos\theta\,dr\,d\theta\,dz$

2. $\displaystyle\int_{0}^{\pi/4}\int_{0}^{6}\int_{0}^{6-r} rz\,dz\,dr\,d\theta$

3. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{2\cos^2\theta}\int_{0}^{4-r^2} r\sin\theta\,dz\,dr\,d\theta$

4. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{\pi}\int_{0}^{2} e^{-\rho^3}\rho^2\,d\rho\,d\theta\,d\phi$

5. $\displaystyle\int_{0}^{2\pi}\int_{0}^{\pi/4}\int_{0}^{\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

6. $\displaystyle\int_{0}^{\pi/4}\int_{0}^{\pi/4}\int_{0}^{\cos\theta} \rho^2\sin\phi\cos\phi\,d\rho\,d\theta\,d\phi$

CAS **In Exercises 7 and 8, use a computer algebra system to evaluate the iterated integral.**

7. $\displaystyle\int_{0}^{4}\int_{0}^{z}\int_{0}^{\pi/2} re^{r}\,d\theta\,dr\,dz$

8. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{\pi}\int_{0}^{\sin\theta} (2\cos\phi)\rho^2\,d\rho\,d\theta\,d\phi$

In Exercises 9–12, sketch the solid region whose volume is given by the iterated integral, and evaluate the iterated integral.

9. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{3}\int_{0}^{e^{-r^2}} r\,dz\,dr\,d\theta$

10. $\displaystyle\int_{0}^{2\pi}\int_{0}^{\sqrt{5}}\int_{0}^{5-r^2} r\,dz\,dr\,d\theta$

11. $\displaystyle\int_{0}^{2\pi}\int_{\pi/6}^{\pi/2}\int_{0}^{4} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

12. $\displaystyle\int_{0}^{2\pi}\int_{0}^{\pi}\int_{2}^{5} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

In Exercises 13–16, convert the integral from rectangular coordinates to both cylindrical and spherical coordinates, and evaluate the simplest iterated integral.

13. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} x\,dz\,dy\,dx$

14. $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2}\,dz\,dy\,dx$

15. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{a}^{a+\sqrt{a^2-x^2-y^2}} x\,dz\,dy\,dx$

16. $\displaystyle\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$

Volume **In Exercises 17–22, use cylindrical coordinates to find the volume of the solid.**

17. Solid inside both $x^2 + y^2 + z^2 = a^2$ and $(x - a/2)^2 + y^2 = (a/2)^2$

18. Solid inside $x^2 + y^2 + z^2 = 16$ and outside $z = \sqrt{x^2 + y^2}$

19. Solid bounded above by $z = 2x$ and below by $z = 2x^2 + 2y^2$

20. Solid bounded above by $z = 2 - x^2 - y^2$ and below by $z = x^2 + y^2$

21. Solid bounded by the graphs of the sphere $r^2 + z^2 = a^2$ and the cylinder $r = a\cos\theta$

22. Solid inside the sphere $x^2 + y^2 + z^2 = 4$ and above the upper nappe of the cone $z^2 = x^2 + y^2$

Mass **In Exercises 23 and 24, use cylindrical coordinates to find the mass of the solid Q.**

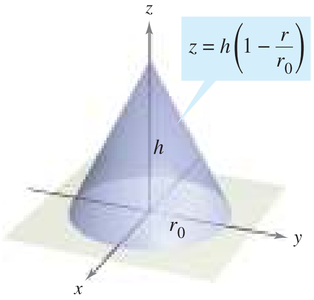
23. $Q = \{(x, y, z): 0 \le z \le 9 - x - 2y, x^2 + y^2 \le 4\}$

$\rho(x, y, z) = k\sqrt{x^2 + y^2}$

24. $Q = \{(x, y, z): 0 \le z \le 12e^{-(x^2+y^2)}, x^2 + y^2 \le 4, x \ge 0, y \ge 0\}$

$\rho(x, y, z) = k$

In Exercises 25–30, use cylindrical coordinates to find the indicated characteristic of the cone shown in the figure.



25. ***Volume*** Find the volume of the cone.

26. ***Centroid*** Find the centroid of the cone.

CAS **27.** ***Center of Mass*** Find the center of mass of the cone assuming that its density at any point is proportional to the distance between the point and the axis of the cone. Use a computer algebra system to evaluate the triple integral.

CAS **28.** ***Center of Mass*** Find the center of mass of the cone assuming that its density at any point is proportional to the distance between the point and the base. Use a computer algebra system to evaluate the triple integral.

29. ***Moment of Inertia*** Assume that the cone has uniform density and show that the moment of inertia about the z-axis is

$$I_z = \tfrac{3}{10}mr_0^2.$$

30. ***Moment of Inertia*** Assume that the density of the cone is $\rho(x, y, z) = k\sqrt{x^2 + y^2}$ and find the moment of inertia about the z-axis.

Moment of Inertia **In Exercises 31 and 32, use cylindrical coordinates to verify the given formula for the moment of inertia of the solid of uniform density.**

31. Cylindrical shell: $I_z = \frac{1}{2}m(a^2 + b^2)$

$0 < a \le r \le b, \quad 0 \le z \le h$

CAS **32.** Right circular cylinder: $I_z = \frac{3}{2}ma^2$

$r = 2a \sin \theta, \quad 0 \le z \le h$

Use a computer algebra system to evaluate the triple integral.

Volume **In Exercises 33–36, use spherical coordinates to find the volume of the solid.**

33. Solid inside $x^2 + y^2 + z^2 = 9$, outside $z = \sqrt{x^2 + y^2}$, and above the xy-plane

34. Solid bounded above by $x^2 + y^2 + z^2 = z$ and below by $z = \sqrt{x^2 + y^2}$

CAS **35.** The torus given by $\rho = 4 \sin \phi$ (Use a computer algebra system to evaluate the triple integral.)

36. The solid between the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, $b > a$, and inside the cone $z^2 = x^2 + y^2$

Mass **In Exercises 37 and 38, use spherical coordinates to find the mass of the sphere $x^2 + y^2 + z^2 = a^2$ with the given density.**

37. The density at any point is proportional to the distance between the point and the origin.

38. The density at any point is proportional to the distance of the point from the z-axis.

Center of Mass **In Exercises 39 and 40, use spherical coordinates to find the center of mass of the solid of uniform density.**

39. Hemispherical solid of radius r

40. Solid lying between two concentric hemispheres of radii r and R, where $r < R$

Moment of Inertia **In Exercises 41 and 42, use spherical coordinates to find the moment of inertia about the z-axis of the solid of uniform density.**

41. Solid bounded by the hemisphere $\rho = \cos \phi$, $\pi/4 \le \phi \le \pi/2$, and the cone $\phi = \pi/4$

42. Solid lying between two concentric hemispheres of radii r and R, where $r < R$

WRITING ABOUT CONCEPTS

43. Give the equations for conversion from rectangular to cylindrical coordinates and vice versa.

44. Give the equations for conversion from rectangular to spherical coordinates and vice versa.

45. Give the iterated form of the triple integral $\iiint\limits_Q f(x, y, z)\, dV$ in cylindrical form.

46. Give the iterated form of the triple integral $\iiint\limits_Q f(x, y, z)\, dV$ in spherical form.

47. Describe the surface whose equation is a coordinate equal to a constant for each of the coordinates in (a) the cylindrical coordinate system and (b) the spherical coordinate system.

CAPSTONE

48. Convert the integral from rectangular coordinates to both (a) cylindrical and (b) spherical coordinates. Without calculating, which integral appears to be the simplest to evaluate? Why?

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx$$

49. Find the "volume" of the "four-dimensional sphere"

$x^2 + y^2 + z^2 + w^2 = a^2$

by evaluating

$$16 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2-z^2}} dw\, dz\, dy\, dx.$$

50. Use spherical coordinates to show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{x^2 + y^2 + z^2}\, e^{-(x^2+y^2+z^2)}\, dx\, dy\, dz = 2\pi.$$

PUTNAM EXAM CHALLENGE

51. Find the volume of the region of points (x, y, z) such that $(x^2 + y^2 + z^2 + 8)^2 \le 36(x^2 + y^2)$.

This problem was composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.

SECTION PROJECT

Wrinkled and Bumpy Spheres

In parts (a) and (b), find the volume of the wrinkled sphere or bumpy sphere. These solids are used as models for tumors.

(a) Wrinkled sphere

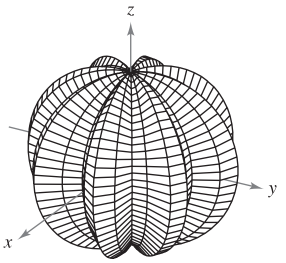
$\rho = 1 + 0.2 \sin 8\theta \sin \phi$

$0 \le \theta \le 2\pi, 0 \le \phi \le \pi$

(b) Bumpy sphere

$\rho = 1 + 0.2 \sin 8\theta \sin 4\phi$

$0 \le \theta \le 2\pi, 0 \le \phi \le \pi$



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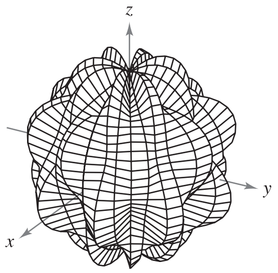

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FOR FURTHER INFORMATION For more information on these types of spheres, see the article "Heat Therapy for Tumors" by Leah Edelstein-Keshet in *The UMAP Journal*.

14.8 Change of Variables: Jacobians

- **Understand the concept of a Jacobian.**
- **Use a Jacobian to change variables in a double integral.**

CARL GUSTAV JACOBI (1804–1851)

The Jacobian is named after the German mathematician Carl Gustav Jacobi. Jacobi is known for his work in many areas of mathematics, but his interest in integration stemmed from the problem of finding the circumference of an ellipse.

Jacobians

For the single integral

$$\int_a^b f(x)\,dx$$

you can change variables by letting $x = g(u)$, so that $dx = g'(u)\,du$, and obtain

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

where $a = g(c)$ and $b = g(d)$. Note that the change of variables process introduces an additional factor $g'(u)$ into the integrand. This also occurs in the case of double integrals

$$\int_R\int f(x, y)\,dA = \int_S\int f(g(u, v), h(u, v))\underbrace{\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right|}_{\text{Jacobian}}du\,dv$$

where the change of variables $x = g(u, v)$ and $y = h(u, v)$ introduces a factor called the **Jacobian** of x and y with respect to u and v. In defining the Jacobian, it is convenient to use the following determinant notation.

DEFINITION OF THE JACOBIAN

If $x = g(u, v)$ and $y = h(u, v)$, then the **Jacobian** of x and y with respect to u and v, denoted by $\partial(x, y)/\partial(u, v)$, is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

EXAMPLE 1 The Jacobian for Rectangular-to-Polar Conversion

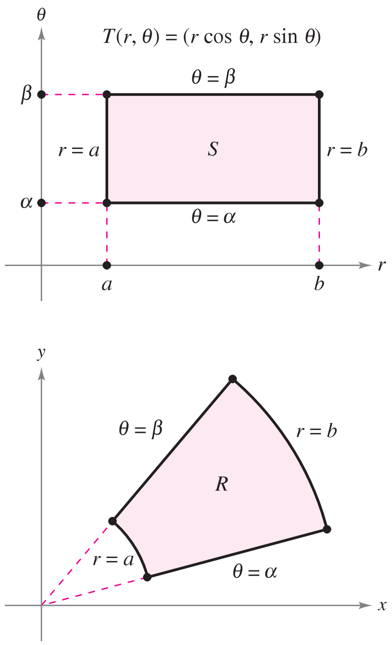
Find the Jacobian for the change of variables defined by

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

Solution From the definition of the Jacobian, you obtain

$$\begin{aligned}\frac{\partial(x, y)}{\partial(r, \theta)} &= \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r.\end{aligned}$$

■



θ
$T(r, \theta) = (r\cos\theta, r\sin\theta)$
$\theta = \beta$
β
$r = a$
S
$r = b$
α
$\theta = \alpha$
a
b
r
y
$\theta = \beta$
$r = b$
R
$r = a$
$\theta = \alpha$
x

S is the region in the $r\theta$-plane that corresponds to R in the xy-plane.
Figure 14.71

Example 1 points out that the change of variables from rectangular to polar coordinates for a double integral can be written as

$$\int\!\!\int_R f(x, y)\, dA = \int\!\!\int_S f(r\cos\theta, r\sin\theta) r\, dr\, d\theta, \quad r > 0$$
$$= \int\!\!\int_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| dr\, d\theta$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane, as shown in Figure 14.71. This formula is similar to that found in Theorem 14.3 on page 1006.

In general, a change of variables is given by a one-to-one **transformation** T from a region S in the uv-plane to a region R in the xy-plane, to be given by

$$T(u, v) = (x, y) = (g(u, v), h(u, v))$$

where g and h have continuous first partial derivatives in the region S. Note that the point (u, v) lies in S and the point (x, y) lies in R. In most cases, you are hunting for a transformation in which the region S is simpler than the region R.

EXAMPLE 2 Finding a Change of Variables to Simplify a Region

Let R be the region bounded by the lines

$$x - 2y = 0, \quad x - 2y = -4, \quad x + y = 4, \quad \text{and} \quad x + y = 1$$

as shown in Figure 14.72. Find a transformation T from a region S to R such that S is a rectangular region (with sides parallel to the u- or v-axis).

Solution To begin, let $u = x + y$ and $v = x - 2y$. Solving this system of equations for x and y produces $T(u, v) = (x, y)$, where

$$x = \frac{1}{3}(2u + v) \quad \text{and} \quad y = \frac{1}{3}(u - v).$$

The four boundaries for R in the xy-plane give rise to the following bounds for S in the uv-plane.

Bounds in the xy-Plane		*Bounds in the uv-Plane*
$x + y = 1$	⇨	$u = 1$
$x + y = 4$	⇨	$u = 4$
$x - 2y = 0$	⇨	$v = 0$
$x - 2y = -4$	⇨	$v = -4$

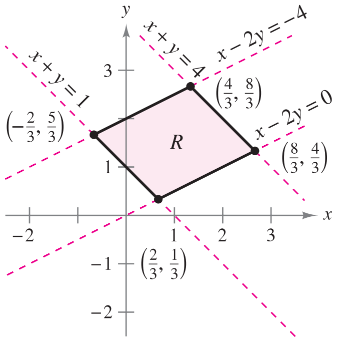
The region S is shown in Figure 14.73. Note that the transformation

$$T(u, v) = (x, y) = \left(\frac{1}{3}[2u + v], \frac{1}{3}[u - v]\right)$$

maps the vertices of the region S onto the vertices of the region R. For instance,

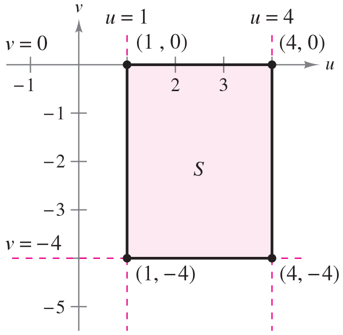
$$T(1, 0) = \left(\tfrac{1}{3}[2(1) + 0], \tfrac{1}{3}[1 - 0]\right) = \left(\tfrac{2}{3}, \tfrac{1}{3}\right)$$
$$T(4, 0) = \left(\tfrac{1}{3}[2(4) + 0], \tfrac{1}{3}[4 - 0]\right) = \left(\tfrac{8}{3}, \tfrac{4}{3}\right)$$
$$T(4, -4) = \left(\tfrac{1}{3}[2(4) - 4], \tfrac{1}{3}[4 - (-4)]\right) = \left(\tfrac{4}{3}, \tfrac{8}{3}\right)$$
$$T(1, -4) = \left(\tfrac{1}{3}[2(1) - 4], \tfrac{1}{3}[1 - (-4)]\right) = \left(-\tfrac{2}{3}, \tfrac{5}{3}\right).$$

■



y
$x + y = 4$
$x - 2y = -4$
$x + y = 1$
3
$\left(\frac{4}{3}, \frac{8}{3}\right)$
$x - 2y = 0$
$\left(-\frac{2}{3}, \frac{5}{3}\right)$
R
$\left(\frac{8}{3}, \frac{4}{3}\right)$
1
x
-2
1
2
3
-1
$\left(\frac{2}{3}, \frac{1}{3}\right)$
-2

Region R in the xy-plane
Figure 14.72



v
$u = 1$
$u = 4$
$v = 0$
$(1, 0)$
$(4, 0)$
u
-1
2
3
-1
-2
S
-3
$v = -4$
$(1, -4)$
$(4, -4)$
-5

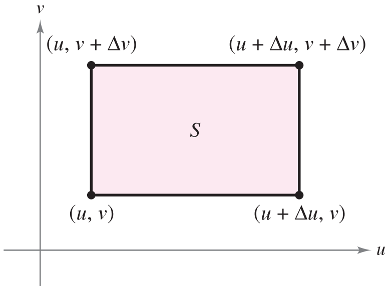
Region S in the uv-plane
Figure 14.73

Change of Variables for Double Integrals

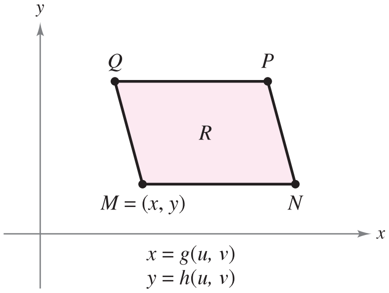
THEOREM 14.5 CHANGE OF VARIABLES FOR DOUBLE INTEGRALS

Let R be a vertically or horizontally simple region in the xy-plane, and let S be a vertically or horizontally simple region in the uv-plane. Let T from S to R be given by $T(u, v) = (x, y) = (g(u, v), h(u, v))$, where g and h have continuous first partial derivatives. Assume that T is one-to-one except possibly on the boundary of S. If f is continuous on R, and $\partial(x, y)/\partial(u, v)$ is nonzero on S, then

$$\int_R\int f(x, y)\,dx\,dy = \int_S\int f(g(u, v), h(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv.$$



Area of $S = \Delta u\,\Delta v$
$\Delta u > 0,\ \Delta v > 0$
Figure 14.74



The vertices in the xy-plane are $M(g(u, v), h(u, v))$, $N(g(u + \Delta u, v), h(u + \Delta u, v))$, $P(g(u + \Delta u, v + \Delta v), h(u + \Delta u, v + \Delta v))$, and $Q(g(u, v + \Delta v), h(u, v + \Delta v))$.
Figure 14.75

PROOF Consider the case in which S is a rectangular region in the uv-plane with vertices (u, v), $(u + \Delta u, v)$, $(u + \Delta u, v + \Delta v)$, and $(u, v + \Delta v)$, as shown in Figure 14.74. The images of these vertices in the xy-plane are shown in Figure 14.75. If Δu and Δv are small, the continuity of g and h implies that R is approximately a parallelogram determined by the vectors $\overrightarrow{MN}$ and $\overrightarrow{MQ}$. So, the area of R is

$$\Delta A \approx \|\overrightarrow{MN} \times \overrightarrow{MQ}\|.$$

Moreover, for small Δu and Δv, the partial derivatives of g and h with respect to u can be approximated by

$$g_u(u, v) \approx \frac{g(u + \Delta u, v) - g(u, v)}{\Delta u} \quad \text{and} \quad h_u(u, v) \approx \frac{h(u + \Delta u, v) - h(u, v)}{\Delta u}.$$

Consequently,

$$\begin{aligned}\overrightarrow{MN} &= [g(u + \Delta u, v) - g(u, v)]\mathbf{i} + [h(u + \Delta u, v) - h(u, v)]\mathbf{j}\\ &\approx [g_u(u, v)\,\Delta u]\mathbf{i} + [h_u(u, v)\,\Delta u]\mathbf{j}\\ &= \frac{\partial x}{\partial u}\Delta u\mathbf{i} + \frac{\partial y}{\partial u}\Delta u\mathbf{j}.\end{aligned}$$

Similarly, you can approximate $\overrightarrow{MQ}$ by $\frac{\partial x}{\partial v}\Delta v\mathbf{i} + \frac{\partial y}{\partial v}\Delta v\mathbf{j}$, which implies that

$$\overrightarrow{MN} \times \overrightarrow{MQ} \approx \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u & 0\\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v & 0\end{vmatrix} = \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}\Delta u\,\Delta v\mathbf{k}.$$

It follows that, in Jacobian notation,

$$\Delta A \approx \|\overrightarrow{MN} \times \overrightarrow{MQ}\| \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right|\Delta u\,\Delta v.$$

Because this approximation improves as Δu and Δv approach 0, the limiting case can be written as

$$dA \approx \|\overrightarrow{MN} \times \overrightarrow{MQ}\| \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv.$$

So,

$$\int_R\int f(x, y)\,dx\,dy = \int_S\int f(g(u, v), h(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv. \quad ■$$

The next two examples show how a change of variables can simplify the integration process. The simplification can occur in various ways. You can make a change of variables to simplify either the *region* R or the *integrand* $f(x, y)$, or both.

EXAMPLE 3 Using a Change of Variables to Simplify a Region

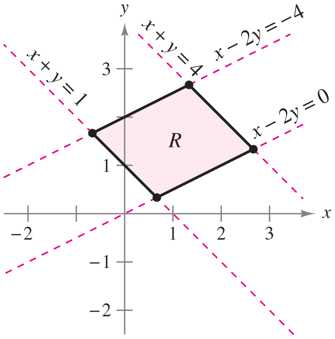


Figure 14.76

Let R be the region bounded by the lines

$$x - 2y = 0, \quad x - 2y = -4, \quad x + y = 4, \quad \text{and} \quad x + y = 1$$

as shown in Figure 14.76. Evaluate the double integral

$$\int\!\!\int_R 3xy\, dA.$$

Solution From Example 2, you can use the following change of variables.

$$x = \frac{1}{3}(2u + v) \quad \text{and} \quad y = \frac{1}{3}(u - v)$$

The partial derivatives of x and y are

$$\frac{\partial x}{\partial u} = \frac{2}{3}, \quad \frac{\partial x}{\partial v} = \frac{1}{3}, \quad \frac{\partial y}{\partial u} = \frac{1}{3}, \quad \text{and} \quad \frac{\partial y}{\partial v} = -\frac{1}{3}$$

which implies that the Jacobian is

$$\begin{aligned}
\frac{\partial(x, y)}{\partial(u, v)} &= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{2}{3} & \dfrac{1}{3} \\[2mm] \dfrac{1}{3} & -\dfrac{1}{3} \end{vmatrix} \\
&= -\frac{2}{9} - \frac{1}{9} \\
&= -\frac{1}{3}.
\end{aligned}$$

So, by Theorem 14.5, you obtain

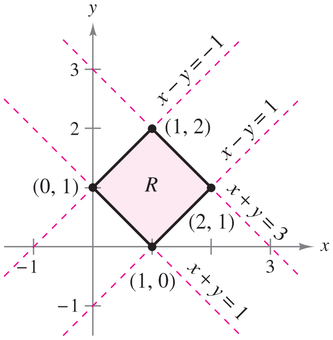
$$\begin{aligned}
\int\!\!\int_R 3xy\, dA &= \int\!\!\int_S 3\left[\frac{1}{3}(2u + v)\frac{1}{3}(u - v)\right]\left|\frac{\partial(x, y)}{\partial(u, v)}\right| dv\, du \\
&= \int_1^4 \int_{-4}^0 \frac{1}{9}(2u^2 - uv - v^2)\, dv\, du \\
&= \frac{1}{9}\int_1^4 \left[2u^2v - \frac{uv^2}{2} - \frac{v^3}{3}\right]_{-4}^0 du \\
&= \frac{1}{9}\int_1^4 \left(8u^2 + 8u - \frac{64}{3}\right) du \\
&= \frac{1}{9}\left[\frac{8u^3}{3} + 4u^2 - \frac{64}{3}u\right]_1^4 \\
&= \frac{164}{9}.
\end{aligned}$$

■

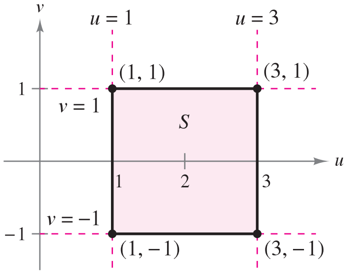
EXAMPLE 4 Using a Change of Variables to Simplify an Integrand

Let R be the region bounded by the square with vertices $(0, 1)$, $(1, 2)$, $(2, 1)$, and $(1, 0)$. Evaluate the integral

$$\int_R\int (x+y)^2 \sin^2(x-y)\, dA.$$



Region R in the xy-plane
Figure 14.77



Region S in the uv-plane
Figure 14.78

Solution Note that the sides of R lie on the lines $x+y=1$, $x-y=1$, $x+y=3$, and $x-y=-1$, as shown in Figure 14.77. Letting $u=x+y$ and $v=x-y$, you can determine the bounds for region S in the uv-plane to be

$$1 \le u \le 3 \qquad \text{and} \qquad -1 \le v \le 1$$

as shown in Figure 14.78. Solving for x and y in terms of u and v produces

$$x = \frac{1}{2}(u+v) \qquad \text{and} \qquad y = \frac{1}{2}(u-v).$$

The partial derivatives of x and y are

$$\frac{\partial x}{\partial u} = \frac{1}{2}, \quad \frac{\partial x}{\partial v} = \frac{1}{2}, \quad \frac{\partial y}{\partial u} = \frac{1}{2}, \quad \text{and} \quad \frac{\partial y}{\partial v} = -\frac{1}{2}$$

which implies that the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

By Theorem 14.5, it follows that

$$\begin{aligned}
\int_R\int (x+y)^2 \sin^2(x-y)\, dA &= \int_{-1}^{1}\int_{1}^{3} u^2 \sin^2 v\left(\frac{1}{2}\right) du\, dv \\
&= \frac{1}{2}\int_{-1}^{1} (\sin^2 v)\, \frac{u^3}{3}\Bigg]_1^3 dv \\
&= \frac{13}{3}\int_{-1}^{1} \sin^2 v\, dv \\
&= \frac{13}{6}\int_{-1}^{1} (1 - \cos 2v)\, dv \\
&= \frac{13}{6}\left[v - \frac{1}{2}\sin 2v\right]_{-1}^{1} \\
&= \frac{13}{6}\left[2 - \frac{1}{2}\sin 2 + \frac{1}{2}\sin(-2)\right] \\
&= \frac{13}{6}(2 - \sin 2) \\
&\approx 2.363.
\end{aligned}$$

In each of the change of variables examples in this section, the region S has been a rectangle with sides parallel to the u- or v-axis. Occasionally, a change of variables can be used for other types of regions. For instance, letting $T(u, v) = \left(x, \frac{1}{2}y\right)$ changes the circular region $u^2 + v^2 = 1$ to the elliptical region $x^2 + (y^2/4) = 1$.

14.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, find the Jacobian $\partial(x, y)/\partial(u, v)$ for the indicated change of variables.

1. $x = -\frac{1}{2}(u - v),\ y = \frac{1}{2}(u + v)$

2. $x = au + bv,\ y = cu + dv$

3. $x = u - v^2,\ y = u + v$

4. $x = uv - 2u,\ y = uv$

5. $x = u\cos\theta - v\sin\theta,\ y = u\sin\theta + v\cos\theta$

6. $x = u + a,\ y = v + a$

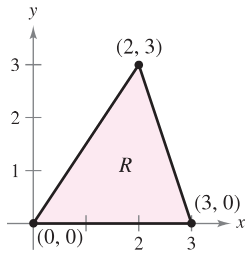
7. $x = e^u \sin v,\ y = e^u \cos v$

8. $x = \dfrac{u}{v},\ y = u + v$

In Exercises 9–12, sketch the image S in the uv-plane of the region R in the xy-plane using the given transformations.

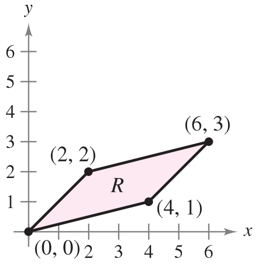
9. $x = 3u + 2v$

$y = 3v$


y
x
3
2
1
2
3
(2, 3)
R
(3, 0)
(0, 0)

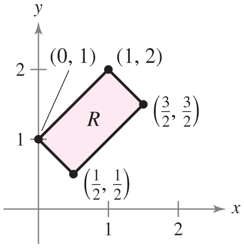
10. $x = \frac{1}{3}(4u - v)$

$y = \frac{1}{3}(u - v)$


y
x
6
5
4
3
2
1
2
3
4
5
6
(6, 3)
(2, 2)
R
(4, 1)
(0, 0)

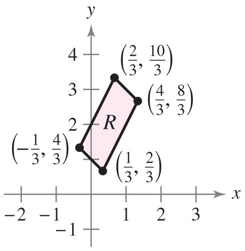
11. $x = \frac{1}{2}(u + v)$

$y = \frac{1}{2}(u - v)$


y
x
2
1
1
2
(0, 1)
(1, 2)
R
(3/2, 3/2)
(1/2, 1/2)

12. $x = \frac{1}{3}(v - u)$

$y = \frac{1}{3}(2v + u)$


y
x
4
3
2
−1
−2
−1
1
2
3
(2/3, 10/3)
(4/3, 8/3)
R
(−1/3, 4/3)
(1/3, 2/3)

CAS **In Exercises 13 and 14, verify the result of the indicated example by setting up the integral using $dy\,dx$ or $dx\,dy$ for dA. Then use a computer algebra system to evaluate the integral.**

13. Example 3

14. Example 4

In Exercises 15–20, use the indicated change of variables to evaluate the double integral.

15. $\displaystyle\int_R\int 4(x^2 + y^2)\,dA$

$x = \frac{1}{2}(u + v)$

$y = \frac{1}{2}(u - v)$

16. $\displaystyle\int_R\int 60xy\,dA$

$x = \frac{1}{2}(u + v)$

$y = -\frac{1}{2}(u - v)$

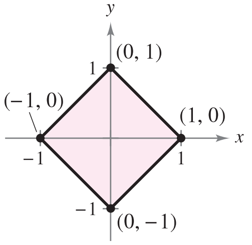

y
x
1
−1
1
−1
(0, 1)
(−1, 0)
(1, 0)
(0, −1)

Figure for 15

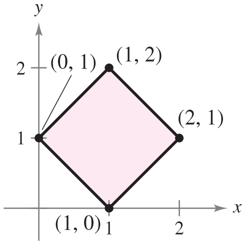
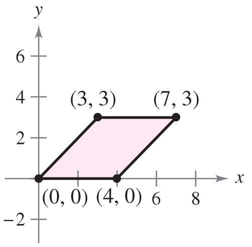

y
x
2
1
1
2
(0, 1)
(1, 2)
(2, 1)
(1, 0)

Figure for 16

17. $\displaystyle\int_R\int y(x - y)\,dA$

$x = u + v$

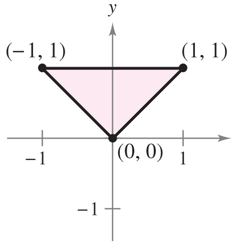
$y = u$


y
x
6
4
2
−2
6
8
(3, 3)
(7, 3)
(0, 0)
(4, 0)

18. $\displaystyle\int_R\int 4(x + y)e^{x-y}\,dA$

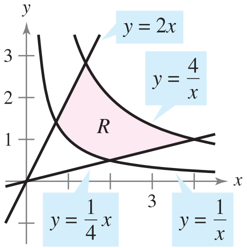
$x = \frac{1}{2}(u + v)$

$y = \frac{1}{2}(u - v)$


y
x
(−1, 1)
(1, 1)
−1
1
(0, 0)
−1

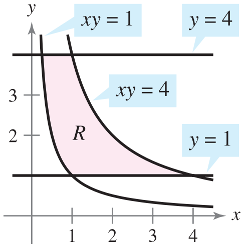
19. $\displaystyle\int_R\int e^{-xy/2}\,dA$

$x = \sqrt{\dfrac{v}{u}},\quad y = \sqrt{uv}$


y
x
3
2
1
3
y = 2x
y = 4/x
R
y = 1/4 x
y = 1/x

20. $\displaystyle\int_R\int y\sin xy\,dA$

$x = \dfrac{u}{v},\quad y = v$


y
x
xy = 1
y = 4
xy = 4
3
2
R
y = 1
1
2
3
4

In Exercises 21–28, use a change of variables to find the volume of the solid region lying below the surface $z = f(x, y)$ and above the plane region R.

21. $f(x, y) = 48xy$

R: region bounded by the square with vertices $(1, 0), (0, 1), (1, 2), (2, 1)$

22. $f(x, y) = (3x + 2y)^2\sqrt{2y - x}$

R: region bounded by the parallelogram with vertices $(0, 0), (-2, 3), (2, 5), (4, 2)$

23. $f(x, y) = (x + y)e^{x-y}$

R: region bounded by the square with vertices $(4, 0), (6, 2), (4, 4), (2, 2)$

24. $f(x, y) = (x + y)^2 \sin^2(x - y)$

R: region bounded by the square with vertices $(\pi, 0)$, $(3\pi/2, \pi/2)$, (π, π), $(\pi/2, \pi/2)$

25. $f(x, y) = \sqrt{(x - y)(x + 4y)}$

R: region bounded by the parallelogram with vertices $(0, 0)$, $(1, 1)$, $(5, 0)$, $(4, -1)$

26. $f(x, y) = (3x + 2y)(2y - x)^{3/2}$

R: region bounded by the parallelogram with vertices $(0, 0)$, $(-2, 3)$, $(2, 5)$, $(4, 2)$

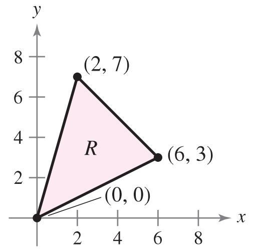
27. $f(x, y) = \sqrt{x + y}$

R: region bounded by the triangle with vertices $(0, 0)$, $(a, 0)$, $(0, a)$, where $a > 0$

28. $f(x, y) = \dfrac{xy}{1 + x^2y^2}$

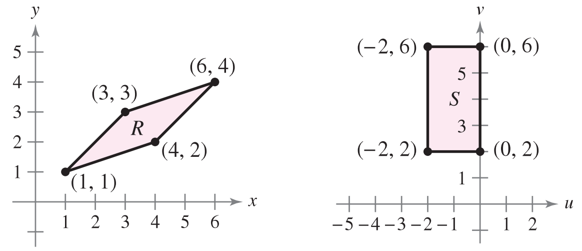
R: region bounded by the graphs of $xy = 1$, $xy = 4$, $x = 1$, $x = 4$ (*Hint:* Let $x = u$, $y = v/u$.)

29. The substitutions $u = 2x - y$ and $v = x + y$ make the region R (see figure) into a simpler region S in the uv-plane. Determine the total number of sides of S that are parallel to either the u-axis or the v-axis.



CAPSTONE

30. Find a transformation $T(u, v) = (x, y) = (g(u, v), h(u, v))$ that when applied to the region R will result in the image S (see figure). Explain your reasoning.



31. Consider the region R in the xy-plane bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the transformations $x = au$ and $y = bv$.

(a) Sketch the graph of the region R and its image S under the given transformation.

(b) Find $\dfrac{\partial(x, y)}{\partial(u, v)}$.

(c) Find the area of the ellipse.

32. Use the result of Exercise 31 to find the volume of each dome-shaped solid lying below the surface $z = f(x, y)$ and above the elliptical region R. (*Hint:* After making the change of variables given by the results in Exercise 31, make a second change of variables to polar coordinates.)

(a) $f(x, y) = 16 - x^2 - y^2$

R: $\dfrac{x^2}{16} + \dfrac{y^2}{9} \le 1$

(b) $f(x, y) = A \cos\left(\dfrac{\pi}{2}\sqrt{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}\right)$

R: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$

WRITING ABOUT CONCEPTS

33. State the definition of the Jacobian.

34. Describe how to use the Jacobian to change variables in double integrals.

In Exercises 35–40, find the Jacobian $\partial(x, y, z)/\partial(u, v, w)$ for the indicated change of variables. If $x = f(u, v, w)$, $y = g(u, v, w)$, and $z = h(u, v, w)$, then the Jacobian of x, y, and z with respect to u, v, and w is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

35. $x = u(1 - v)$, $y = uv(1 - w)$, $z = uvw$

36. $x = 4u - v$, $y = 4v - w$, $z = u + w$

37. $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$, $z = 2uvw$

38. $x = u - v + w$, $y = 2uv$, $z = u + v + w$

39. ***Spherical Coordinates***

$x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$

40. ***Cylindrical Coordinates***

$x = r \cos\theta$, $y = r \sin\theta$, $z = z$

PUTNAM EXAM CHALLENGE

41. Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

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14 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, evaluate the integral.

1. $\int_1^{x^2} x \ln y \, dy$

2. $\int_y^{2y} (x^2 + y^2) \, dx$

In Exercises 3–6, sketch the region of integration. Then evaluate the iterated integral. Change the coordinate system when convenient.

3. $\int_0^1 \int_0^{1+x} (3x + 2y) \, dy \, dx$

4. $\int_0^2 \int_{x^2}^{2x} (x^2 + 2y) \, dy \, dx$

5. $\int_0^3 \int_0^{\sqrt{9-x^2}} 4x \, dy \, dx$

6. $\int_0^{\sqrt{3}} \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx \, dy$

Area **In Exercises 7–14, write the limits for the double integral**

$$\int_R\int f(x, y) \, dA$$

for both orders of integration. Compute the area of R by letting $f(x, y) = 1$ and integrating.

7. Triangle: vertices $(0, 0)$, $(3, 0)$, $(0, 1)$

8. Triangle: vertices $(0, 0)$, $(3, 0)$, $(2, 2)$

9. The larger area between the graphs of $x^2 + y^2 = 25$ and $x = 3$

10. Region bounded by the graphs of $y = 6x - x^2$ and $y = x^2 - 2x$

11. Region enclosed by the graph of $y^2 = x^2 - x^4$

12. Region bounded by the graphs of $x = y^2 + 1$, $x = 0$, $y = 0$, and $y = 2$

13. Region bounded by the graphs of $x = y + 3$ and $x = y^2 + 1$

14. Region bounded by the graphs of $x = -y$ and $x = 2y - y^2$

Think About It **In Exercises 15 and 16, give a geometric argument for the given equality. Verify the equality analytically.**

15. $$\int_0^1 \int_{2y}^{2\sqrt{2-y^2}} (x + y) \, dx \, dy = \int_0^2 \int_0^{x/2} (x + y) \, dy \, dx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}/2} (x + y) \, dy \, dx$$

16. $$\int_0^2 \int_{3y/2}^{5-y} e^{x+y} \, dx \, dy = \int_0^3 \int_0^{2x/3} e^{x+y} \, dy \, dx + \int_3^5 \int_0^{5-x} e^{x+y} \, dy \, dx$$

Volume **In Exercises 17 and 18, use a multiple integral and a convenient coordinate system to find the volume of the solid.**

17. Solid bounded by the graphs of $z = x^2 - y + 4$, $z = 0$, $y = 0$, $x = 0$, and $x = 4$

18. Solid bounded by the graphs of $z = x + y$, $z = 0$, $y = 0$, $x = 3$, and $y = x$

Average Value **In Exercises 19 and 20, find the average of $f(x, y)$ over the region R.**

19. $f(x) = 16 - x^2 - y^2$

R: rectangle with vertices $(2, 2)$, $(-2, 2)$, $(-2, -2)$, $(2, -2)$

20. $f(x) = 2x^2 + y^2$

R: square with vertices $(0, 0)$, $(3, 0)$, $(3, 3)$, $(0, 3)$

21. ***Average Temperature*** The temperature in degrees Celsius on the surface of a metal plate is

$T(x, y) = 40 - 6x^2 - y^2$

where x and y are measured in centimeters. Estimate the average temperature if x varies between 0 and 3 centimeters and y varies between 0 and 5 centimeters.

CAS **22.** ***Average Profit*** A firm's profit P from marketing two soft drinks is

$P = 192x + 576y - x^2 - 5y^2 - 2xy - 5000$

where x and y represent the numbers of units of the two soft drinks. Use a computer algebra system to evaluate the double integral yielding the average weekly profit if x varies between 40 and 50 units and y varies between 45 and 60 units.

Probability **In Exercises 23 and 24, find k such that the function is a joint density function and find the required probability, where**

$$P(a \le x \le b, c \le y \le d) = \int_c^d \int_a^b f(x, y) \, dx \, dy.$$

23. $f(x, y) = \begin{cases} kxye^{-(x+y)}, & x \ge 0, y \ge 0 \\ 0, & \text{elsewhere} \end{cases}$

$P(0 \le x \le 1, 0 \le y \le 1)$

24. $f(x, y) = \begin{cases} kxy, & 0 \le x \le 1, 0 \le y \le x \\ 0, & \text{elsewhere} \end{cases}$

$P(0 \le x \le 0.5, 0 \le y \le 0.25)$

Approximation **In Exercises 25 and 26, determine which value best approximates the volume of the solid between the xy-plane and the function over the region. (Make your selection on the basis of a sketch of the solid and *not* by performing any calculations.)**

25. $f(x, y) = x + y$

R: triangle with vertices $(0, 0)$, $(3, 0)$, $(3, 3)$

(a) $\frac{9}{2}$ (b) 5 (c) 13 (d) 100 (e) -100

26. $f(x, y) = 10x^2y^2$

R: circle bounded by $x^2 + y^2 = 1$

(a) π (b) -15 (c) $\frac{2}{3}$ (d) 3 (e) 15

True or False? **In Exercises 27–30, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

27. $\displaystyle\int_a^b\int_c^d f(x)g(y)\,dy\,dx = \left[\int_a^b f(x)\,dx\right]\left[\int_c^d g(y)\,dy\right]$

28. If f is continuous over R_1 and R_2, and

$$\int_{R_1}\int dA = \int_{R_2}\int dA$$

then

$$\int_{R_1}\int f(x, y)\,dA = \int_{R_2}\int f(x, y)\,dA.$$

29. $\displaystyle\int_{-1}^1\int_{-1}^1 \cos(x^2 + y^2)\,dx\,dy = 4\int_0^1\int_0^1 \cos(x^2 + y^2)\,dx\,dy$

30. $\displaystyle\int_0^1\int_0^1 \frac{1}{1 + x^2 + y^2}\,dx\,dy < \frac{\pi}{4}$

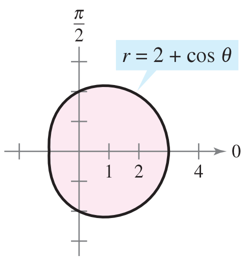
In Exercises 31 and 32, evaluate the iterated integral by converting to polar coordinates.

31. $\displaystyle\int_0^h\int_0^x \sqrt{x^2 + y^2}\,dy\,dx$

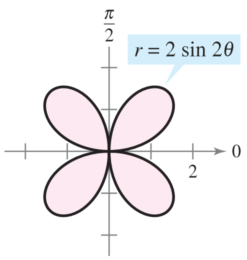
32. $\displaystyle\int_0^4\int_0^{\sqrt{16-y^2}} (x^2 + y^2)\,dx\,dy$

Area **In Exercises 33 and 34, use a double integral to find the area of the shaded region.**

33.



34.



Volume **In Exercises 35 and 36, use a multiple integral and a convenient coordinate system to find the volume of the solid.**

35. Solid bounded by the graphs of $z = 0$ and $z = h$, outside the cylinder $x^2 + y^2 = 1$ and inside the hyperboloid $x^2 + y^2 - z^2 = 1$

36. Solid that remains after drilling a hole of radius b through the center of a sphere of radius R $(b < R)$

37. Consider the region R in the xy-plane bounded by the graph of the equation

$$(x^2 + y^2)^2 = 9(x^2 - y^2).$$

(a) Convert the equation to polar coordinates. Use a graphing utility to graph the equation.

(b) Use a double integral to find the area of the region R.

CAS (c) Use a computer algebra system to determine the volume of the solid over the region R and beneath the hemisphere $z = \sqrt{9 - x^2 - y^2}$.

38. Combine the sum of the two iterated integrals into a single iterated integral by converting to polar coordinates. Evaluate the resulting iterated integral.

$$\int_0^{8/\sqrt{13}}\int_0^{3x/2} xy\,dy\,dx + \int_{8/\sqrt{13}}^4\int_0^{\sqrt{16-x^2}} xy\,dy\,dx$$

CAS ***Mass and Center of Mass*** **In Exercises 39 and 40, find the mass and center of mass of the lamina bounded by the graphs of the equations for the given density or densities. Use a computer algebra system to evaluate the multiple integrals.**

39. $y = 2x$, $y = 2x^3$, first quadrant

(a) $\rho = kxy$

(b) $\rho = k(x^2 + y^2)$

40. $y = \dfrac{h}{2}\left(2 - \dfrac{x}{L} - \dfrac{x^2}{L^2}\right)$, $\rho = k$, first quadrant

CAS **In Exercises 41 and 42, find I_x, I_y, I_0, $\bar{\bar{x}}$, and $\bar{\bar{y}}$ for the lamina bounded by the graphs of the equations. Use a computer algebra system to evaluate the double integrals.**

41. $y = 0$, $y = b$, $x = 0$, $x = a$, $\rho = kx$

42. $y = 4 - x^2$, $y = 0$, $x > 0$, $\rho = ky$

Surface Area **In Exercises 43–46, find the area of the surface given by $z = f(x, y)$ over the region R.**

43. $f(x, y) = 25 - x^2 - y^2$

$R = \{(x, y)\colon\ x^2 + y^2 \le 25\}$

CAS **44.** $f(x, y) = 16 - x - y^2$

$R = \{(x, y)\colon\ 0 \le x \le 2, 0 \le y \le x\}$

Use a computer algebra system to evaluate the integral.

45. $f(x, y) = 9 - y^2$

R: triangle bounded by the graphs of the equations $y = x$, $y = -x$, and $y = 3$

46. $f(x, y) = 4 - x^2$

R: triangle bounded by the graphs of the equations $y = x$, $y = -x$, and $y = 2$

47. ***Building Design*** A new auditorium is built with a foundation in the shape of one-fourth of a circle of radius 50 feet. So, it forms a region R bounded by the graph of

$$x^2 + y^2 = 50^2$$

with $x \ge 0$ and $y \ge 0$. The following equations are models for the floor and ceiling.

Floor: $z = \dfrac{x + y}{5}$

Ceiling: $z = 20 + \dfrac{xy}{100}$

(a) Calculate the volume of the room, which is needed to determine the heating and cooling requirements.

(b) Find the surface area of the ceiling.

CAS **48.** ***Surface Area*** The roof over the stage of an open air theater at a theme park is modeled by

$$f(x, y) = 25\left[1 + e^{-(x^2+y^2)/1000}\cos^2\left(\frac{x^2+y^2}{1000}\right)\right]$$

where the stage is a semicircle bounded by the graphs of $y = \sqrt{50^2 - x^2}$ and $y = 0$.

(a) Use a computer algebra system to graph the surface.

(b) Use a computer algebra system to approximate the number of square feet of roofing required to cover the surface.

In Exercises 49–52, evaluate the iterated integral.

49. $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{x^2+y^2}^{9}\sqrt{x^2+y^2}\,dz\,dy\,dx$

50. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{(x^2+y^2)/2}(x^2+y^2)\,dz\,dy\,dx$

51. $\displaystyle\int_{0}^{a}\int_{0}^{b}\int_{0}^{c}(x^2+y^2+z^2)\,dx\,dy\,dz$

52. $\displaystyle\int_{0}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}}\frac{1}{1+x^2+y^2+z^2}\,dz\,dy\,dx$

CAS **In Exercises 53 and 54, use a computer algebra system to evaluate the iterated integral.**

53. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}(x^2+y^2)\,dz\,dy\,dx$

54. $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}}xyz\,dz\,dy\,dx$

Volume **In Exercises 55 and 56, use a multiple integral to find the volume of the solid.**

55. Solid inside the graphs of $r = 2\cos\theta$ and $r^2 + z^2 = 4$

56. Solid inside the graphs of $r^2 + z = 16$, $z = 0$, and $r = 2\sin\theta$

Center of Mass **In Exercises 57–60, find the center of mass of the solid of uniform density bounded by the graphs of the equations.**

57. Solid inside the hemisphere $\rho = \cos\phi$, $\pi/4 \le \phi \le \pi/2$, and outside the cone $\phi = \pi/4$

58. Wedge: $x^2 + y^2 = a^2$, $z = cy\ (c > 0)$, $y \ge 0$, $z \ge 0$

59. $x^2 + y^2 + z^2 = a^2$, first octant

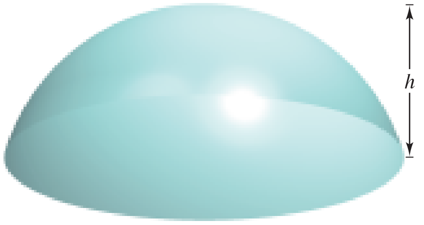
60. $x^2 + y^2 + z^2 = 25$, $z = 4$ (the larger solid)

Moment of Inertia **In Exercises 61 and 62, find the moment of inertia I_z of the solid of given density.**

61. The solid of uniform density inside the paraboloid $z = 16 - x^2 - y^2$, and outside the cylinder $x^2 + y^2 = 9$, $z \ge 0$.

62. $x^2 + y^2 + z^2 = a^2$, density proportional to the distance from the center

63. ***Investigation*** Consider a spherical segment of height h from a sphere of radius a, where $h \le a$, and constant density $\rho(x, y, z) = k$ (see figure).



(a) Find the volume of the solid.

(b) Find the centroid of the solid.

(c) Use the result of part (b) to find the centroid of a hemisphere of radius a.

(d) Find $\lim\limits_{h\to 0}\bar{z}$.

(e) Find I_z.

(f) Use the result of part (e) to find I_z for a hemisphere.

64. ***Moment of Inertia*** Find the moment of inertia about the z-axis of the ellipsoid $x^2 + y^2 + \dfrac{z^2}{a^2} = 1$, where $a > 0$.

In Exercises 65 and 66, give a geometric interpretation of the iterated integral.

65. $\displaystyle\int_{0}^{2\pi}\int_{0}^{\pi}\int_{0}^{6\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

66. $\displaystyle\int_{0}^{\pi}\int_{0}^{2}\int_{0}^{1+r^2}r\,dz\,dr\,d\theta$

In Exercises 67 and 68, find the Jacobian $\partial(x, y)/\partial(u, v)$ for the indicated change of variables.

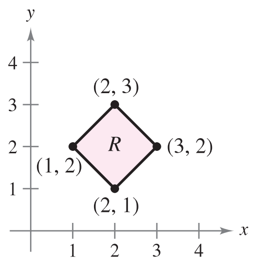
67. $x = u + 3v$, $y = 2u - 3v$

68. $x = u^2 + v^2$, $y = u^2 - v^2$

In Exercises 69 and 70, use the indicated change of variables to evaluate the double integral.

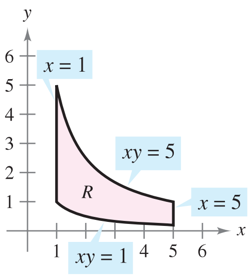
69. $\displaystyle\iint_R \ln(x+y)\,dA$

$x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$



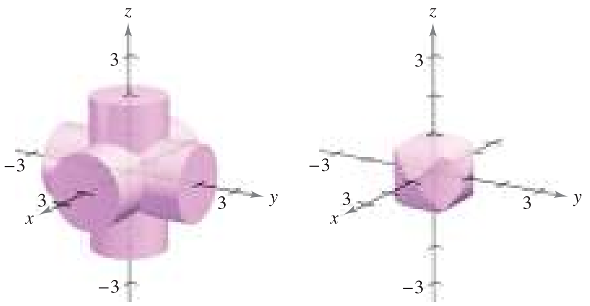
70. $\displaystyle\iint_R \frac{x}{1+x^2y^2}\,dA$

$x = u$, $y = \dfrac{v}{u}$



P.S. PROBLEM SOLVING

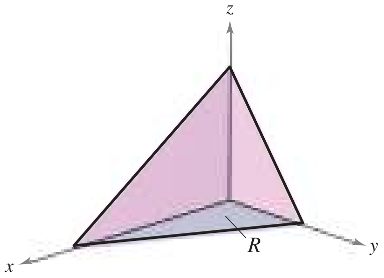
1. Find the volume of the solid of intersection of the three cylinders $x^2 + z^2 = 1$, $y^2 + z^2 = 1$, and $x^2 + y^2 = 1$ (see figure).



2. Let a, b, c, and d be positive real numbers. The first octant of the plane $ax + by + cz = d$ is shown in the figure. Show that the surface area of this portion of the plane is equal to

$$\frac{A(R)}{c}\sqrt{a^2 + b^2 + c^2}$$

where $A(R)$ is the area of the triangular region R in the xy-plane, as shown in the figure.



3. Derive Euler's famous result that was mentioned in Section 9.3, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, by completing each step.

(a) Prove that $\displaystyle\int \frac{dv}{2 - u^2 + v^2} = \frac{1}{\sqrt{2 - u^2}} \arctan \frac{v}{\sqrt{2 - u^2}} + C.$

(b) Prove that $\displaystyle I_1 = \int_0^{\sqrt{2}/2} \int_{-u}^{u} \frac{2}{2 - u^2 + v^2}\, dv\, du = \frac{\pi^2}{18}$ by using the substitution $u = \sqrt{2} \sin \theta$.

(c) Prove that

$$I_2 = \int_{\sqrt{2}/2}^{\sqrt{2}} \int_{u - \sqrt{2}}^{-u + \sqrt{2}} \frac{2}{2 - u^2 + v^2}\, dv\, du$$

$$= 4\int_{\pi/6}^{\pi/2} \arctan \frac{1 - \sin \theta}{\cos \theta}\, d\theta$$

by using the substitution $u = \sqrt{2} \sin \theta$.

(d) Prove the trigonometric identity $\displaystyle\frac{1 - \sin \theta}{\cos \theta} = \tan\left(\frac{(\pi/2) - \theta}{2}\right)$.

(e) Prove that $\displaystyle I_2 = \int_{\sqrt{2}/2}^{\sqrt{2}} \int_{u - \sqrt{2}}^{-u + \sqrt{2}} \frac{2}{2 - u^2 + v^2}\, dv\, du = \frac{\pi^2}{9}$.

(f) Use the formula for the sum of an infinite geometric series to verify that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \int_0^1 \int_0^1 \frac{1}{1 - xy}\, dx\, dy$.

(g) Use the change of variables $\displaystyle u = \frac{x + y}{\sqrt{2}}$ and $\displaystyle v = \frac{y - x}{\sqrt{2}}$ to prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \int_0^1 \int_0^1 \frac{1}{1 - xy}\, dx\, dy = I_1 + I_2 = \frac{\pi^2}{6}$.

4. Consider a circular lawn with a radius of 10 feet, as shown in the figure. Assume that a sprinkler distributes water in a radial fashion according to the formula

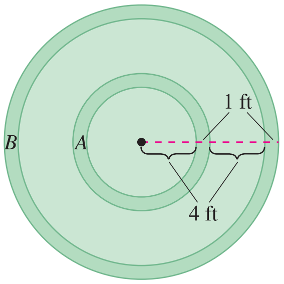
$$f(r) = \frac{r}{16} - \frac{r^2}{160}$$

(measured in cubic feet of water per hour per square foot of lawn), where r is the distance in feet from the sprinkler. Find the amount of water that is distributed in 1 hour in the following two annular regions.

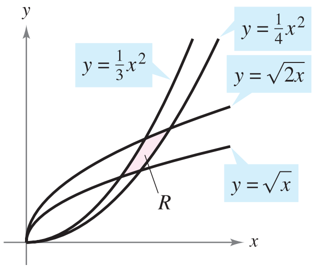
$A = \{(r, \theta)\colon 4 \le r \le 5,\ 0 \le \theta \le 2\pi\}$

$B = \{(r, \theta)\colon 9 \le r \le 10,\ 0 \le \theta \le 2\pi\}$

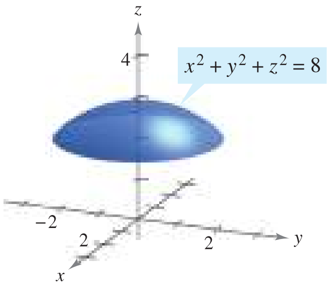
Is the distribution of water uniform? Determine the amount of water the entire lawn receives in 1 hour.



5. The figure shows the region R bounded by the curves $y = \sqrt{x}$, $y = \sqrt{2x}$, $\displaystyle y = \frac{x^2}{3}$, and $\displaystyle y = \frac{x^2}{4}$. Use the change of variables $x = u^{1/3}v^{2/3}$ and $y = u^{2/3}v^{1/3}$ to find the area of the region R.



6. The figure shows a solid bounded below by the plane $z = 2$ and above by the sphere $x^2 + y^2 + z^2 = 8$.



(a) Find the volume of the solid using cylindrical coordinates.

(b) Find the volume of the solid using spherical coordinates.

7. Sketch the solid whose volume is given by the sum of the iterated integrals

$$\int_0^6 \int_{z/2}^3 \int_{z/2}^y dx\,dy\,dz + \int_0^6 \int_3^{(12-z)/2} \int_{z/2}^{6-y} dx\,dy\,dz.$$

Then write the volume as a single iterated integral in the order $dy\,dz\,dx$.

8. Prove that $\lim_{n\to\infty} \int_0^1 \int_0^1 x^n y^n\,dx\,dy = 0$.

In Exercises 9 and 10, evaluate the integral. (*Hint:* See Exercise 69 in Section 14.3.)

9. $\int_0^\infty x^2 e^{-x^2}\,dx$

10. $\int_0^1 \sqrt{\ln \frac{1}{x}}\,dx$

11. Consider the function

$$f(x, y) = \begin{cases} ke^{-(x+y)/a}, & x \geq 0,\ y \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the relationship between the positive constants a and k such that f is a joint density function of the continuous random variables x and y.

12. Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = e^{-x^2}$ about the y-axis. Use this result to find

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

13. From 1963 to 1986, the volume of the Great Salt Lake approximately tripled while its top surface area approximately doubled. Read the article "Relations between Surface Area and Volume in Lakes" by Daniel Cass and Gerald Wildenberg in *The College Mathematics Journal*. Then give examples of solids that have "water levels" a and b such that $V(b) = 3V(a)$ and $A(b) = 2A(a)$ (see figure), where V is volume and A is area.

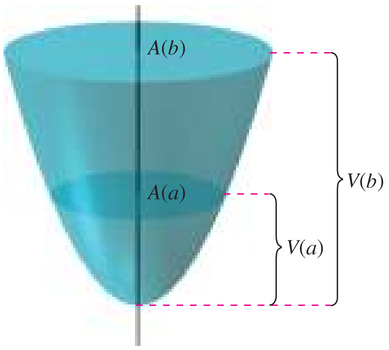
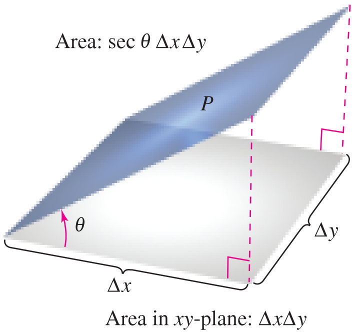


Figure for 13

14. The angle between a plane P and the xy-plane is θ, where $0 \leq \theta < \pi/2$. The projection of a rectangular region in P onto the xy-plane is a rectangle whose sides have lengths Δx and Δy, as shown in the figure. Prove that the area of the rectangular region in P is $\sec\theta\,\Delta x\,\Delta y$.


Area: $\sec\theta\,\Delta x \Delta y$
P
θ
Δy
Δx
Area in xy-plane: $\Delta x \Delta y$

15. Use the result of Exercise 14 to order the planes in ascending order of their surface areas for a fixed region R in the xy-plane. Explain your ordering without doing any calculations.

(a) $z_1 = 2 + x$

(b) $z_2 = 5$

(c) $z_3 = 10 - 5x + 9y$

(d) $z_4 = 3 + x - 2y$

16. Evaluate the integral $\int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2}\,dx\,dy$.

17. Evaluate the integrals

$$\int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\,dx\,dy$$

and

$$\int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\,dy\,dx.$$

Are the results the same? Why or why not?

18. Show that the volume of a spherical block can be approximated by

$$\Delta V \approx \rho^2 \sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta.$$

15 Vector Analysis

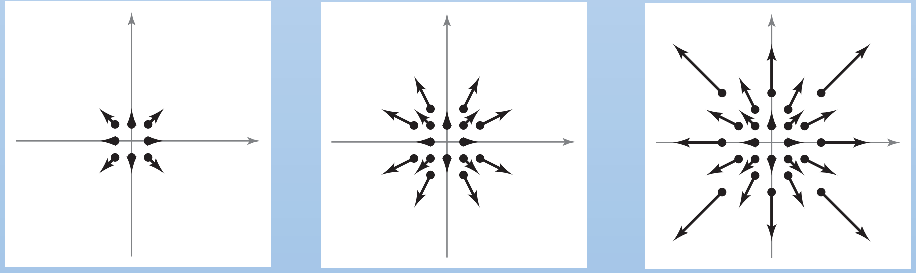
In this chapter, you will study vector fields, line integrals, and surface integrals. You will learn to use these to determine real-life quantities such as surface area, mass, flux, work, and energy.

In this chapter, you should learn the following.

- How to sketch a vector field, determine whether a vector field is conservative, find a potential function, find curl, and find divergence. (15.1)
- How to find a piecewise smooth parametrization, write and evaluate a line integral, and use Green's Theorem. (15.2, 15.4)
- How to use the Fundamental Theorem of Line Integrals, independence of path, and conservation of energy. (15.3)
- How to sketch a parametric surface, find a set of parametric equations to represent a surface, find a normal vector, find a tangent plane, and find the area of a parametric surface. (15.5)
- How to evaluate a surface integral, determine the orientation of a surface, evaluate a flux integral, and use the Divergence Theorem. (15.6, 15.7)
- How to use Stokes's Theorem to evaluate a line integral or a surface integral and how to use curl to analyze the motion of a rotating liquid. (15.8)

NASA

While on the ground awaiting liftoff, space shuttle astronauts have access to a basket and slide wire system that is designed to move them as far away from the shuttle as possible in an emergency situation. Does the amount of work done by the gravitational force field vary for different slide wire paths between two fixed points? (See Section 15.3, Exercise 39.)



In Chapter 15, you will combine your knowledge of vectors with your knowledge of integral calculus. Section 15.1 introduces *vector fields,* such as those shown above. Examples of vector fields include velocity fields, electromagnetic fields, and gravitational fields.

15.1 Vector Fields

- **Understand the concept of a vector field.**
- **Determine whether a vector field is conservative.**
- **Find the curl of a vector field.**
- **Find the divergence of a vector field.**

Vector Fields

In Chapter 12, you studied vector-valued functions—functions that assign a vector to a *real number*. There you saw that vector-valued functions of real numbers are useful in representing curves and motion along a curve. In this chapter, you will study two other types of vector-valued functions—functions that assign a vector to a *point in the plane* or a *point in space*. Such functions are called **vector fields,** and they are useful in representing various types of **force fields** and **velocity fields.**

DEFINITION OF VECTOR FIELD

A **vector field over a plane region R** is a function $\mathbf{F}$ that assigns a vector $\mathbf{F}(x, y)$ to each point in R.

A **vector field over a solid region Q in space** is a function $\mathbf{F}$ that assigns a vector $\mathbf{F}(x, y, z)$ to each point in Q.

NOTE Although a vector field consists of infinitely many vectors, you can get a good idea of what the vector field looks like by sketching several representative vectors $\mathbf{F}(x, y)$ whose initial points are (x, y). ■

The *gradient* is one example of a vector field. For example, if

$$f(x, y) = x^2y + 3xy^3$$

then the gradient of f

$$\begin{aligned}\nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}\\ &= (2xy + 3y^3)\mathbf{i} + (x^2 + 9xy^2)\mathbf{j}\end{aligned}$$

Vector field in the plane

is a vector field in the plane. From Chapter 13, the graphical interpretation of this field is a family of vectors, each of which points in the direction of maximum increase along the surface given by $z = f(x, y)$.

Similarly, if

$$f(x, y, z) = x^2 + y^2 + z^2$$

then the gradient of f

$$\begin{aligned}\nabla f(x, y, z) &= f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}\\ &= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}\end{aligned}$$

Vector field in space

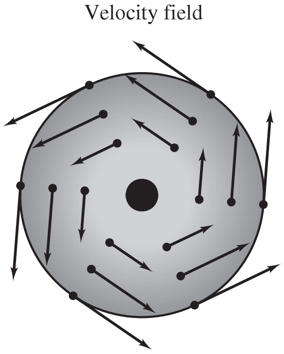
is a vector field in space. Note that the component functions for this particular vector field are $2x$, $2y$, and $2z$.

A vector field

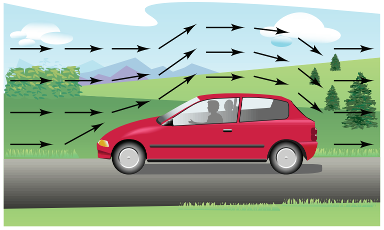
$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

is **continuous** at a point if and only if each of its component functions M, N, and P is continuous at that point.

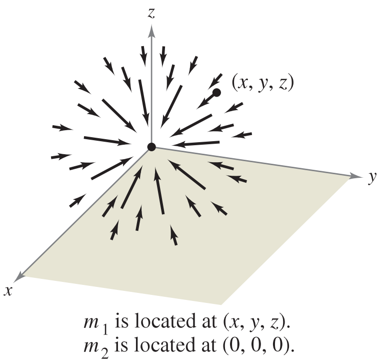
Some common *physical* examples of vector fields are **velocity fields, gravitational fields,** and **electric force fields.**



Rotating wheel
Figure 15.1



Air flow vector field
Figure 15.2


z
(x, y, z)
y
x
m_1 is located at (x, y, z).
m_2 is located at (0, 0, 0).

Gravitational force field
Figure 15.3

1. *Velocity fields* describe the motions of systems of particles in the plane or in space. For instance, Figure 15.1 shows the vector field determined by a wheel rotating on an axle. Notice that the velocity vectors are determined by the locations of their initial points—the farther a point is from the axle, the greater its velocity. Velocity fields are also determined by the flow of liquids through a container or by the flow of air currents around a moving object, as shown in Figure 15.2.
2. *Gravitational fields* are defined by **Newton's Law of Gravitation,** which states that the force of attraction exerted on a particle of mass m_1 located at (x, y, z) by a particle of mass m_2 located at $(0, 0, 0)$ is given by

   $$\mathbf{F}(x, y, z) = \frac{-Gm_1m_2}{x^2 + y^2 + z^2}\mathbf{u}$$

   where G is the gravitational constant and $\mathbf{u}$ is the unit vector in the direction from the origin to (x, y, z). In Figure 15.3, you can see that the gravitational field $\mathbf{F}$ has the properties that $\mathbf{F}(x, y, z)$ always points toward the origin, and that the magnitude of $\mathbf{F}(x, y, z)$ is the same at all points equidistant from the origin. A vector field with these two properties is called a **central force field.** Using the position vector

   $$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

   for the point (x, y, z), you can write the gravitational field $\mathbf{F}$ as

   $$\begin{aligned}\mathbf{F}(x, y, z) &= \frac{-Gm_1m_2}{\|\mathbf{r}\|^2}\left(\frac{\mathbf{r}}{\|\mathbf{r}\|}\right)\\ &= \frac{-Gm_1m_2}{\|\mathbf{r}\|^2}\mathbf{u}.\end{aligned}$$

3. *Electric force fields* are defined by **Coulomb's Law,** which states that the force exerted on a particle with electric charge q_1 located at (x, y, z) by a particle with electric charge q_2 located at $(0, 0, 0)$ is given by

   $$\mathbf{F}(x, y, z) = \frac{cq_1q_2}{\|\mathbf{r}\|^2}\mathbf{u}$$

   where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$, and c is a constant that depends on the choice of units for $\|\mathbf{r}\|$, q_1, and q_2.

   Note that an electric force field has the same form as a gravitational field. That is,

$$\mathbf{F}(x, y, z) = \frac{k}{\|\mathbf{r}\|^2}\mathbf{u}.$$

Such a force field is called an **inverse square field.**

DEFINITION OF INVERSE SQUARE FIELD

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a position vector. The vector field $\mathbf{F}$ is an **inverse square field** if

$$\mathbf{F}(x, y, z) = \frac{k}{\|\mathbf{r}\|^2}\mathbf{u}$$

where k is a real number and $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$ is a unit vector in the direction of $\mathbf{r}$.

Because vector fields consist of infinitely many vectors, it is not possible to create a sketch of the entire field. Instead, when you sketch a vector field, your goal is to sketch representative vectors that help you visualize the field.

EXAMPLE 1 Sketching a Vector Field

Sketch some vectors in the vector field given by

$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}.$$

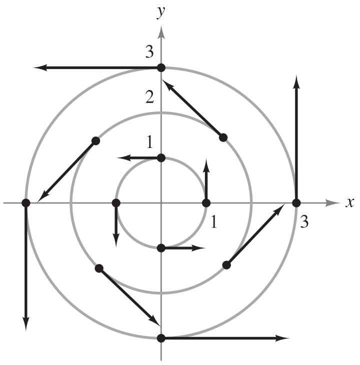
Solution You could plot vectors at several random points in the plane. However, it is more enlightening to plot vectors of equal magnitude. This corresponds to finding level curves in scalar fields. In this case, vectors of equal magnitude lie on circles.

$$\begin{aligned} \|\mathbf{F}\| &= c & &\text{Vectors of length } c \\ \sqrt{x^2 + y^2} &= c \\ x^2 + y^2 &= c^2 & &\text{Equation of circle} \end{aligned}$$

To begin making the sketch, choose a value for c and plot several vectors on the resulting circle. For instance, the following vectors occur on the unit circle.

Point	*Vector*
$(1, 0)$	$\mathbf{F}(1, 0) = \mathbf{j}$
$(0, 1)$	$\mathbf{F}(0, 1) = -\mathbf{i}$
$(-1, 0)$	$\mathbf{F}(-1, 0) = -\mathbf{j}$
$(0, -1)$	$\mathbf{F}(0, -1) = \mathbf{i}$

These and several other vectors in the vector field are shown in Figure 15.4. Note in the figure that this vector field is similar to that given by the rotating wheel shown in Figure 15.1.



Vector field:
$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$

Figure 15.4

EXAMPLE 2 Sketching a Vector Field

Sketch some vectors in the vector field given by

$$\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}.$$

Solution For this vector field, vectors of equal length lie on ellipses given by

$$\|\mathbf{F}\| = \sqrt{(2x)^2 + (y)^2} = c$$

which implies that

$$4x^2 + y^2 = c^2.$$

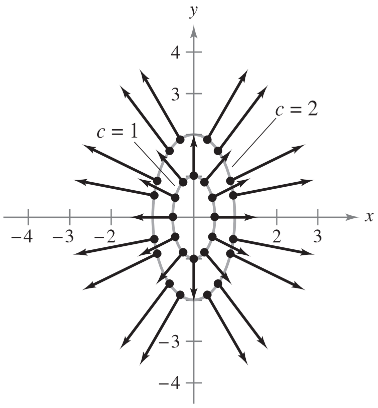
For $c = 1$, sketch several vectors $2x\mathbf{i} + y\mathbf{j}$ of magnitude 1 at points on the ellipse given by

$$4x^2 + y^2 = 1.$$

For $c = 2$, sketch several vectors $2x\mathbf{i} + y\mathbf{j}$ of magnitude 2 at points on the ellipse given by

$$4x^2 + y^2 = 4.$$

These vectors are shown in Figure 15.5. ■



Vector field:
$\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$

Figure 15.5

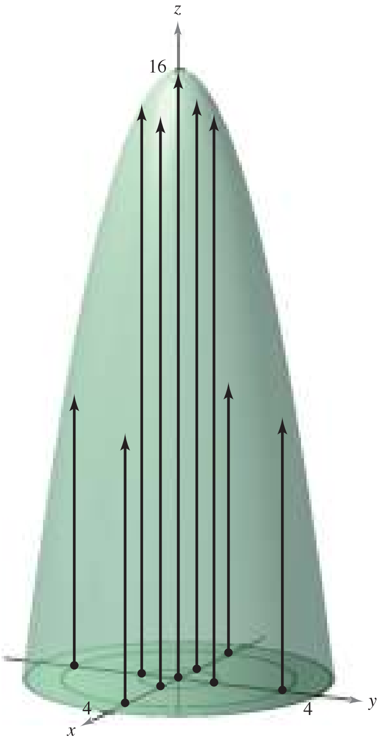
EXAMPLE 3 Sketching a Velocity Field

Sketch some vectors in the velocity field given by

$$\mathbf{v}(x, y, z) = (16 - x^2 - y^2)\mathbf{k}$$

where $x^2 + y^2 \le 16$.

Solution You can imagine that $\mathbf{v}$ describes the velocity of a liquid flowing through a tube of radius 4. Vectors near the z-axis are longer than those near the edge of the tube. For instance, at the point $(0, 0, 0)$, the velocity vector is $\mathbf{v}(0, 0, 0) = 16\mathbf{k}$, whereas at the point $(0, 3, 0)$, the velocity vector is $\mathbf{v}(0, 3, 0) = 7\mathbf{k}$. Figure 15.6 shows these and several other vectors for the velocity field. From the figure, you can see that the speed of the liquid is greater near the center of the tube than near the edges of the tube. ■



Velocity field:
$\mathbf{v}(x, y, z) = (16 - x^2 - y^2)\mathbf{k}$

Figure 15.6

Conservative Vector Fields

Notice in Figure 15.5 that all the vectors appear to be normal to the level curve from which they emanate. Because this is a property of gradients, it is natural to ask whether the vector field given by $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$ is the *gradient* of some differentiable function f. The answer is that some vector fields can be represented as the gradients of differentiable functions and some cannot—those that can are called **conservative** vector fields.

DEFINITION OF CONSERVATIVE VECTOR FIELD

A vector field $\mathbf{F}$ is called **conservative** if there exists a differentiable function f such that $\mathbf{F} = \nabla f$. The function f is called the **potential function** for $\mathbf{F}$.

EXAMPLE 4 Conservative Vector Fields

a. The vector field given by $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$ is conservative. To see this, consider the potential function $f(x, y) = x^2 + \frac{1}{2}y^2$. Because

$$\nabla f = 2x\mathbf{i} + y\mathbf{j} = \mathbf{F}$$

it follows that $\mathbf{F}$ is conservative.

b. Every inverse square field is conservative. To see this, let

$$\mathbf{F}(x, y, z) = \frac{k}{\|\mathbf{r}\|^2}\mathbf{u} \quad \text{and} \quad f(x, y, z) = \frac{-k}{\sqrt{x^2 + y^2 + z^2}}$$

where $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$. Because

$$\begin{aligned}\nabla f &= \frac{kx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{ky}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{kz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k} \\ &= \frac{k}{x^2 + y^2 + z^2}\left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}\right) \\ &= \frac{k}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} \\ &= \frac{k}{\|\mathbf{r}\|^2}\mathbf{u}\end{aligned}$$

it follows that $\mathbf{F}$ is conservative. ■

As can be seen in Example 4(b), many important vector fields, including gravitational fields and electric force fields, are conservative. Most of the terminology in this chapter comes from physics. For example, the term "conservative" is derived from the classic physical law regarding the conservation of energy. This law states that the sum of the kinetic energy and the potential energy of a particle moving in a conservative force field is constant. (The kinetic energy of a particle is the energy due to its motion, and the potential energy is the energy due to its position in the force field.)

The following important theorem gives a necessary and sufficient condition for a vector field *in the plane* to be conservative.

THEOREM 15.1 TEST FOR CONSERVATIVE VECTOR FIELD IN THE PLANE

Let M and N have continuous first partial derivatives on an open disk R. The vector field given by $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

PROOF To prove that the given condition is necessary for $\mathbf{F}$ to be conservative, suppose there exists a potential function f such that

$$\mathbf{F}(x, y) = \nabla f(x, y) = M\mathbf{i} + N\mathbf{j}.$$

Then you have

$$f_x(x, y) = M \quad \Rightarrow \quad f_{xy}(x, y) = \frac{\partial M}{\partial y}$$

$$f_y(x, y) = N \quad \Rightarrow \quad f_{yx}(x, y) = \frac{\partial N}{\partial x}$$

and, by the equivalence of the mixed partials f_{xy} and f_{yx}, you can conclude that $\partial N/\partial x = \partial M/\partial y$ for all (x, y) in R. The sufficiency of this condition is proved in Section 15.4. ■

NOTE Theorem 15.1 is valid on *simply connected* domains. A plane region R is simply connected if every simple closed curve in R encloses only points that are in R. See Figure 15.26 in Section 15.4. ■

EXAMPLE 5 Testing for Conservative Vector Fields in the Plane

Decide whether the vector field given by $\mathbf{F}$ is conservative.

a. $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy\mathbf{j}$ **b.** $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$

Solution

a. The vector field given by $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy\mathbf{j}$ is not conservative because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[x^2y] = x^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[xy] = y.$$

b. The vector field given by $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$ is conservative because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[2x] = 0 \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[y] = 0.$$

■

Theorem 15.1 tells you whether a vector field is conservative. It does not tell you how to find a potential function of **F**. The problem is comparable to antidifferentiation. Sometimes you will be able to find a potential function by simple inspection. For instance, in Example 4 you observed that

$$f(x, y) = x^2 + \frac{1}{2}y^2$$

has the property that $\nabla f(x, y) = 2x\mathbf{i} + y\mathbf{j}$.

EXAMPLE 6 Finding a Potential Function for F(*x*, *y*)

Find a potential function for

$$\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}.$$

Solution From Theorem 15.1 it follows that **F** is conservative because

$$\frac{\partial}{\partial y}[2xy] = 2x \quad \text{and} \quad \frac{\partial}{\partial x}[x^2 - y] = 2x.$$

If f is a function whose gradient is equal to $\mathbf{F}(x, y)$, then

$$\nabla f(x, y) = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}$$

which implies that

$$f_x(x, y) = 2xy$$

and

$$f_y(x, y) = x^2 - y.$$

To reconstruct the function f from these two partial derivatives, integrate $f_x(x, y)$ with respect to x and integrate $f_y(x, y)$ with respect to y, as follows.

$$f(x, y) = \int f_x(x, y)\,dx = \int 2xy\,dx = x^2y + g(y)$$

$$f(x, y) = \int f_y(x, y)\,dy = \int (x^2 - y)\,dy = x^2y - \frac{y^2}{2} + h(x)$$

Notice that $g(y)$ is constant with respect to x and $h(x)$ is constant with respect to y. To find a single expression that represents $f(x, y)$, let

$$g(y) = -\frac{y^2}{2} \quad \text{and} \quad h(x) = K.$$

Then, you can write

$$\begin{aligned} f(x, y) &= x^2y + g(y) + K \\ &= x^2y - \frac{y^2}{2} + K. \end{aligned}$$

You can check this result by forming the gradient of f. You will see that it is equal to the original function **F**. ■

NOTE Notice that the solution in Example 6 is comparable to that given by an indefinite integral. That is, the solution represents a family of potential functions, any two of which differ by a constant. To find a unique solution, you would have to be given an initial condition satisfied by the potential function. ■

Curl of a Vector Field

Theorem 15.1 has a counterpart for vector fields in space. Before stating that result, the definition of the **curl of a vector field** in space is given.

DEFINITION OF CURL OF A VECTOR FIELD

The curl of $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\begin{aligned}\text{curl } \mathbf{F}(x, y, z) &= \nabla \times \mathbf{F}(x, y, z)\\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.\end{aligned}$$

NOTE If curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is said to be **irrotational.** ■

The cross product notation used for curl comes from viewing the gradient ∇f as the result of the **differential operator** ∇ acting on the function f. In this context, you can use the following determinant form as an aid in remembering the formula for curl.

$$\begin{aligned}\text{curl } \mathbf{F}(x, y, z) &= \nabla \times \mathbf{F}(x, y, z)\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix}\\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}\end{aligned}$$

EXAMPLE 7 Finding the Curl of a Vector Field

Find curl $\mathbf{F}$ of the vector field given by

$$\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2yz\mathbf{k}.$$

Is $\mathbf{F}$ irrotational?

Solution The curl of $\mathbf{F}$ is given by

$$\begin{aligned}\text{curl } \mathbf{F}(x, y, z) &= \nabla \times \mathbf{F}(x, y, z)\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy & x^2 + z^2 & 2yz \end{vmatrix}\\ &= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + z^2 & 2yz \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ 2xy & 2yz \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ 2xy & x^2 + z^2 \end{vmatrix}\mathbf{k}\\ &= (2z - 2z)\mathbf{i} - (0 - 0)\mathbf{j} + (2x - 2x)\mathbf{k}\\ &= \mathbf{0}.\end{aligned}$$

Because curl $\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is irrotational. ■

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

Later in this chapter, you will assign a physical interpretation to the curl of a vector field. But for now, the primary use of curl is shown in the following test for conservative vector fields in space. The test states that for a vector field in space, the curl is **0** at every point in its domain if and only if **F** is conservative. The proof is similar to that given for Theorem 15.1.

THEOREM 15.2 TEST FOR CONSERVATIVE VECTOR FIELD IN SPACE

Suppose that M, N, and P have continuous first partial derivatives in an open sphere Q in space. The vector field given by $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative if and only if

$$\text{curl } \mathbf{F}(x, y, z) = \mathbf{0}.$$

That is, **F** is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

NOTE Theorem 15.2 is valid for *simply connected* domains in space. A simply connected domain in space is a domain D for which every simple closed curve in D (see Section 15.4) can be shrunk to a point in D without leaving D.

From Theorem 15.2, you can see that the vector field given in Example 7 is conservative because $\text{curl } \mathbf{F}(x, y, z) = \mathbf{0}$. Try showing that the vector field

$$\mathbf{F}(x, y, z) = x^3y^2z\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$$

is not conservative—you can do this by showing that its curl is

$$\text{curl } \mathbf{F}(x, y, z) = (x^3y^2 - 2xy)\mathbf{j} + (2xz - 2x^3yz)\mathbf{k} \neq \mathbf{0}.$$

For vector fields in space that pass the test for being conservative, you can find a potential function by following the same pattern used in the plane (as demonstrated in Example 6).

NOTE Examples 6 and 8 are illustrations of a type of problem called *recovering a function from its gradient*. If you go on to take a course in differential equations, you will study other methods for solving this type of problem. One popular method gives an interplay between successive "partial integrations" and partial differentiations.

EXAMPLE 8 Finding a Potential Function for F(*x*, *y*, *z*)

Find a potential function for $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2yz\mathbf{k}$.

Solution From Example 7, you know that the vector field given by **F** is conservative. If f is a function such that $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$, then

$$f_x(x, y, z) = 2xy, \quad f_y(x, y, z) = x^2 + z^2, \quad \text{and} \quad f_z(x, y, z) = 2yz$$

and integrating with respect to x, y, and z separately produces

$$f(x, y, z) = \int M\,dx = \int 2xy\,dx = x^2y + g(y, z)$$

$$f(x, y, z) = \int N\,dy = \int (x^2 + z^2)\,dy = x^2y + yz^2 + h(x, z)$$

$$f(x, y, z) = \int P\,dz = \int 2yz\,dz = yz^2 + k(x, y).$$

Comparing these three versions of $f(x, y, z)$, you can conclude that

$$g(y, z) = yz^2 + K, \quad h(x, z) = K, \quad \text{and} \quad k(x, y) = x^2y + K.$$

So, $f(x, y, z)$ is given by

$$f(x, y, z) = x^2y + yz^2 + K.$$

■

Divergence of a Vector Field

NOTE Divergence can be viewed as a type of derivative of **F** in that, for vector fields representing velocities of moving particles, the divergence measures the rate of particle flow per unit volume at a point. In hydrodynamics (the study of fluid motion), a velocity field that is divergence free is called **incompressible.** In the study of electricity and magnetism, a vector field that is divergence free is called **solenoidal.**

You have seen that the curl of a vector field **F** is itself a vector field. Another important function defined on a vector field is **divergence,** which is a scalar function.

DEFINITION OF DIVERGENCE OF A VECTOR FIELD

The **divergence** of $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is

$$\text{div } \mathbf{F}(x, y) = \nabla \cdot \mathbf{F}(x, y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \qquad \text{Plane}$$

The **divergence** of $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\text{div } \mathbf{F}(x, y, z) = \nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}. \qquad \text{Space}$$

If div $\mathbf{F} = 0$, then **F** is said to be **divergence free.**

The dot product notation used for divergence comes from considering ∇ as a **differential operator,** as follows.

$$\begin{aligned}\nabla \cdot \mathbf{F}(x, y, z) &= \left[\left(\frac{\partial}{\partial x}\right)\mathbf{i} + \left(\frac{\partial}{\partial y}\right)\mathbf{j} + \left(\frac{\partial}{\partial z}\right)\mathbf{k}\right] \cdot (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) \\ &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\end{aligned}$$

EXAMPLE 9 Finding the Divergence of a Vector Field

Find the divergence at $(2, 1, -1)$ for the vector field

$$\mathbf{F}(x, y, z) = x^3y^2z\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}.$$

Solution The divergence of **F** is

$$\text{div } \mathbf{F}(x, y, z) = \frac{\partial}{\partial x}[x^3y^2z] + \frac{\partial}{\partial y}[x^2z] + \frac{\partial}{\partial z}[x^2y] = 3x^2y^2z.$$

At the point $(2, 1, -1)$, the divergence is

$$\text{div } \mathbf{F}(2, 1, -1) = 3(2^2)(1^2)(-1) = -12.$$

■

There are many important properties of the divergence and curl of a vector field **F** (see Exercises 83–89). One that is used often is described in Theorem 15.3. You are asked to prove this theorem in Exercise 90.

THEOREM 15.3 DIVERGENCE AND CURL

If $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field and M, N, and P have continuous second partial derivatives, then

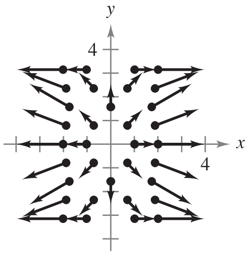
$$\text{div}(\text{curl } \mathbf{F}) = 0.$$

15.1 Exercises

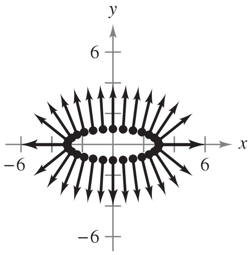
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, match the vector field with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

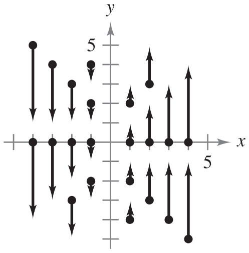
(a)



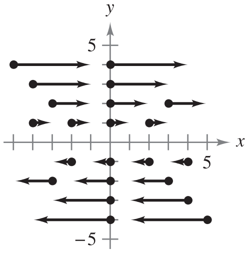
(b)



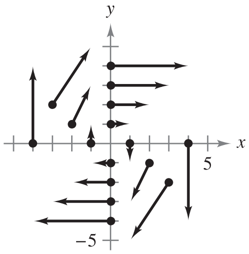
(c)



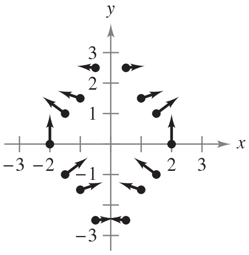
(d)



(e)



(f)



1. $\mathbf{F}(x, y) = y\mathbf{i}$

2. $\mathbf{F}(x, y) = x\mathbf{j}$

3. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

4. $\mathbf{F}(x, y) = x\mathbf{i} + 3y\mathbf{j}$

5. $\mathbf{F}(x, y) = \langle x, \sin y \rangle$

6. $\mathbf{F}(x, y) = \left\langle \frac{1}{2}xy, \frac{1}{4}x^2 \right\rangle$

In Exercises 7–16, compute $\|\mathbf{F}\|$ and sketch several representative vectors in the vector field.

7. $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$

8. $\mathbf{F}(x, y) = 2\mathbf{i}$

9. $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$

10. $\mathbf{F}(x, y) = y\mathbf{i} - 2x\mathbf{j}$

11. $\mathbf{F}(x, y, z) = 3y\mathbf{j}$

12. $\mathbf{F}(x, y) = x\mathbf{i}$

13. $\mathbf{F}(x, y) = 4x\mathbf{i} + y\mathbf{j}$

14. $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + \mathbf{j}$

15. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

16. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

CAS **In Exercises 17–20, use a computer algebra system to graph several representative vectors in the vector field.**

17. $\mathbf{F}(x, y) = \frac{1}{8}(2xy\mathbf{i} + y^2\mathbf{j})$

18. $\mathbf{F}(x, y) = (2y - 3x)\mathbf{i} + (2y + 3x)\mathbf{j}$

19. $\mathbf{F}(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

20. $\mathbf{F}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$

In Exercises 21–30, find the conservative vector field for the potential function by finding its gradient.

21. $f(x, y) = x^2 + 2y^2$

22. $f(x, y) = x^2 - \frac{1}{4}y^2$

23. $g(x, y) = 5x^2 + 3xy + y^2$

24. $g(x, y) = \sin 3x \cos 4y$

25. $f(x, y, z) = 6xyz$

26. $f(x, y, z) = \sqrt{x^2 + 4y^2 + z^2}$

27. $g(x, y, z) = z + ye^{x^2}$

28. $g(x, y, z) = \dfrac{y}{z} + \dfrac{z}{x} - \dfrac{xz}{y}$

29. $h(x, y, z) = xy \ln(x + y)$

30. $h(x, y, z) = x \arcsin yz$

In Exercises 31–34, verify that the vector field is conservative.

31. $\mathbf{F}(x, y) = xy^2\mathbf{i} + x^2y\mathbf{j}$

32. $\mathbf{F}(x, y) = \dfrac{1}{x^2}(y\mathbf{i} - x\mathbf{j})$

33. $\mathbf{F}(x, y) = \sin y\mathbf{i} + x \cos y\mathbf{j}$

34. $\mathbf{F}(x, y) = \dfrac{1}{xy}(y\mathbf{i} - x\mathbf{j})$

In Exercises 35–38, determine whether the vector field is conservative. Justify your answer.

35. $\mathbf{F}(x, y) = 5y^2(y\mathbf{i} + 3x\mathbf{j})$

36. $\mathbf{F}(x, y) = \dfrac{2}{y^2}e^{2x/y}(y\mathbf{i} - x\mathbf{j})$

37. $\mathbf{F}(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}(\mathbf{i} + \mathbf{j})$

38. $\mathbf{F}(x, y) = \dfrac{1}{\sqrt{1 + xy}}(y\mathbf{i} + x\mathbf{j})$

In Exercises 39–48, determine whether the vector field is conservative. If it is, find a potential function for the vector field.

39. $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$

40. $\mathbf{F}(x, y) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$

41. $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

42. $\mathbf{F}(x, y) = xe^{x^2y}(2y\mathbf{i} + x\mathbf{j})$

43. $\mathbf{F}(x, y) = 15y^3\mathbf{i} - 5xy^2\mathbf{j}$

44. $\mathbf{F}(x, y) = \dfrac{1}{y^2}(y\mathbf{i} - 2x\mathbf{j})$

45. $\mathbf{F}(x, y) = \dfrac{2y}{x}\mathbf{i} - \dfrac{x^2}{y^2}\mathbf{j}$

46. $\mathbf{F}(x, y) = \dfrac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$

47. $\mathbf{F}(x, y) = e^x(\cos y\mathbf{i} - \sin y\mathbf{j})$

48. $\mathbf{F}(x, y) = \dfrac{2x\mathbf{i} + 2y\mathbf{j}}{(x^2 + y^2)^2}$

In Exercises 49–52, find curl F for the vector field at the given point.

	Vector Field	*Point*
49.	$\mathbf{F}(x, y, z) = xyz\mathbf{i} + xyz\mathbf{j} + xyz\mathbf{k}$	$(2, 1, 3)$
50.	$\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$	$(2, -1, 3)$
51.	$\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j}$	$(0, 0, 1)$
52.	$\mathbf{F}(x, y, z) = e^{-xyz}(\mathbf{i} + \mathbf{j} + \mathbf{k})$	$(3, 2, 0)$

CAS **In Exercises 53–56, use a computer algebra system to find the curl F for the vector field.**

53. $\mathbf{F}(x, y, z) = \arctan\left(\dfrac{x}{y}\right)\mathbf{i} + \ln\sqrt{x^2 + y^2}\,\mathbf{j} + \mathbf{k}$

54. $\mathbf{F}(x, y, z) = \dfrac{yz}{y - z}\mathbf{i} + \dfrac{xz}{x - z}\mathbf{j} + \dfrac{xy}{x - y}\mathbf{k}$

55. $\mathbf{F}(x, y, z) = \sin(x - y)\mathbf{i} + \sin(y - z)\mathbf{j} + \sin(z - x)\mathbf{k}$

56. $\mathbf{F}(x, y, z) = \sqrt{x^2 + y^2 + z^2}\,(\mathbf{i} + \mathbf{j} + \mathbf{k})$

In Exercises 57–62, determine whether the vector field F is conservative. If it is, find a potential function for the vector field.

57. $\mathbf{F}(x, y, z) = xy^2z^2\mathbf{i} + x^2yz^2\mathbf{j} + x^2y^2z\mathbf{k}$

58. $\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$

59. $\mathbf{F}(x, y, z) = \sin z\,\mathbf{i} + \sin x\,\mathbf{j} + \sin y\,\mathbf{k}$

60. $\mathbf{F}(x, y, z) = ye^z\mathbf{i} + ze^x\mathbf{j} + xe^y\mathbf{k}$

61. $\mathbf{F}(x, y, z) = \dfrac{z}{y}\mathbf{i} - \dfrac{xz}{y^2}\mathbf{j} + \dfrac{x}{y}\mathbf{k}$

62. $\mathbf{F}(x, y, z) = \dfrac{x}{x^2 + y^2}\mathbf{i} + \dfrac{y}{x^2 + y^2}\mathbf{j} + \mathbf{k}$

In Exercises 63–66, find the divergence of the vector field F.

63. $\mathbf{F}(x, y) = x^2\mathbf{i} + 2y^2\mathbf{j}$

64. $\mathbf{F}(x, y) = xe^x\mathbf{i} + ye^y\mathbf{j}$

65. $\mathbf{F}(x, y, z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + z^2\mathbf{k}$

66. $\mathbf{F}(x, y, z) = \ln(x^2 + y^2)\mathbf{i} + xy\mathbf{j} + \ln(y^2 + z^2)\mathbf{k}$

In Exercises 67–70, find the divergence of the vector field F at the given point.

	Vector Field	*Point*
67.	$\mathbf{F}(x, y, z) = xyz\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$	$(2, 1, 1)$
68.	$\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$	$(2, -1, 3)$
69.	$\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j} + z^2\mathbf{k}$	$(3, 0, 0)$
70.	$\mathbf{F}(x, y, z) = \ln(xyz)(\mathbf{i} + \mathbf{j} + \mathbf{k})$	$(3, 2, 1)$

WRITING ABOUT CONCEPTS

71. Define a vector field in the plane and in space. Give some physical examples of vector fields.

72. What is a conservative vector field, and how do you test for it in the plane and in space?

73. Define the curl of a vector field.

74. Define the divergence of a vector field in the plane and in space.

In Exercises 75 and 76, find curl(F × G) = ∇ × (F × G).

75. $\mathbf{F}(x, y, z) = \mathbf{i} + 3x\mathbf{j} + 2y\mathbf{k}$
$\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$

76. $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$
$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$

In Exercises 77 and 78, find curl(curl F) = ∇ × (∇ × F).

77. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

78. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

In Exercises 79 and 80, find div(F × G) = ∇ · (F × G).

79. $\mathbf{F}(x, y, z) = \mathbf{i} + 3x\mathbf{j} + 2y\mathbf{k}$
$\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$

80. $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$
$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$

In Exercises 81 and 82, find div(curl F) = ∇ · (∇ × F).

81. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

82. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

In Exercises 83–90, prove the property for vector fields F and G and scalar function *f*. (Assume that the required partial derivatives are continuous.)

83. $\text{curl}(\mathbf{F} + \mathbf{G}) = \text{curl}\,\mathbf{F} + \text{curl}\,\mathbf{G}$

84. $\text{curl}(\nabla f) = \nabla \times (\nabla f) = \mathbf{0}$

85. $\text{div}(\mathbf{F} + \mathbf{G}) = \text{div}\,\mathbf{F} + \text{div}\,\mathbf{G}$

86. $\text{div}(\mathbf{F} \times \mathbf{G}) = (\text{curl}\,\mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\text{curl}\,\mathbf{G})$

87. $\nabla \times [\nabla f + (\nabla \times \mathbf{F})] = \nabla \times (\nabla \times \mathbf{F})$

88. $\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F}$

89. $\text{div}(f\mathbf{F}) = f\,\text{div}\,\mathbf{F} + \nabla f \cdot \mathbf{F}$

90. $\text{div}(\text{curl}\,\mathbf{F}) = 0$ (Theorem 15.3)

In Exercises 91–93, let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and let $f(x, y, z) = \|\mathbf{F}(x, y, z)\|$.

91. Show that $\nabla(\ln f) = \dfrac{\mathbf{F}}{f^2}$.

92. Show that $\nabla\left(\dfrac{1}{f}\right) = -\dfrac{\mathbf{F}}{f^3}$.

93. Show that $\nabla f^n = nf^{n-2}\mathbf{F}$.

CAPSTONE

94. (a) Sketch several representative vectors in the vector field given by

$$\mathbf{F}(x, y) = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}.$$

(b) Sketch several representative vectors in the vector field given by

$$\mathbf{G}(x, y) = \frac{x\mathbf{i} - y\mathbf{j}}{\sqrt{x^2 + y^2}}.$$

(c) Explain any similarities or differences in the vector fields $\mathbf{F}(x, y)$ and $\mathbf{G}(x, y)$.

True or False? **In Exercises 95–98, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

95. If $\mathbf{F}(x, y) = 4x\mathbf{i} - y^2\mathbf{j}$, then $\|\mathbf{F}(x, y)\| \to 0$ as $(x, y) \to (0, 0)$.

96. If $\mathbf{F}(x, y) = 4x\mathbf{i} - y^2\mathbf{j}$ and (x, y) is on the positive y-axis, then the vector points in the negative y-direction.

97. If f is a scalar field, then curl f is a meaningful expression.

98. If $\mathbf{F}$ is a vector field and $\text{curl}\,\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is irrotational but not conservative.

15.2 Line Integrals

- **Understand and use the concept of a piecewise smooth curve.**
- **Write and evaluate a line integral.**
- **Write and evaluate a line integral of a vector field.**
- **Write and evaluate a line integral in differential form.**

Piecewise Smooth Curves

A classic property of gravitational fields is that, subject to certain physical constraints, the work done by gravity on an object moving between two points in the field is independent of the path taken by the object. One of the constraints is that the **path** must be a piecewise smooth curve. Recall that a plane curve C given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad a \le t \le b$$

is **smooth** if

$$\frac{dx}{dt} \quad \text{and} \quad \frac{dy}{dt}$$

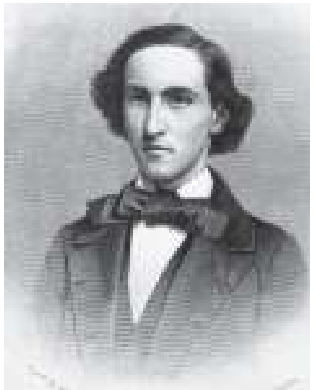
are continuous on $[a, b]$ and not simultaneously 0 on (a, b). Similarly, a space curve C given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b$$

is **smooth** if

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \text{and} \quad \frac{dz}{dt}$$

are continuous on $[a, b]$ and not simultaneously 0 on (a, b). A curve C is **piecewise smooth** if the interval $[a, b]$ can be partitioned into a finite number of subintervals, on each of which C is smooth.


The Granger Collection

JOSIAH WILLARD GIBBS (1839–1903)

Many physicists and mathematicians have contributed to the theory and applications described in this chapter—Newton, Gauss, Laplace, Hamilton, and Maxwell, among others. However, the use of vector analysis to describe these results is attributed primarily to the American mathematical physicist Josiah Willard Gibbs.

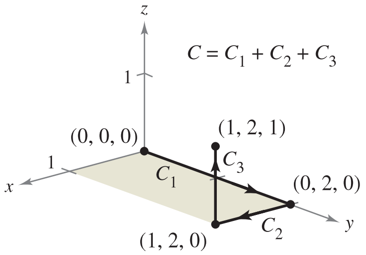


Figure 15.7

EXAMPLE 1 Finding a Piecewise Smooth Parametrization

Find a piecewise smooth parametrization of the graph of C shown in Figure 15.7.

Solution Because C consists of three line segments C_1, C_2, and C_3, you can construct a smooth parametrization for each segment and piece them together by making the last t-value in C_i correspond to the first t-value in C_{i+1}, as follows.

$$\begin{array}{llll} C_1\colon\ x(t) = 0, & y(t) = 2t, & z(t) = 0, & 0 \le t \le 1 \\ C_2\colon\ x(t) = t - 1, & y(t) = 2, & z(t) = 0, & 1 \le t \le 2 \\ C_3\colon\ x(t) = 1, & y(t) = 2, & z(t) = t - 2, & 2 \le t \le 3 \end{array}$$

So, C is given by

$$\mathbf{r}(t) = \begin{cases} 2t\mathbf{j}, & 0 \le t \le 1 \\ (t-1)\mathbf{i} + 2\mathbf{j}, & 1 \le t \le 2. \\ \mathbf{i} + 2\mathbf{j} + (t-2)\mathbf{k}, & 2 \le t \le 3 \end{cases}$$

Because C_1, C_2, and C_3 are smooth, it follows that C is piecewise smooth. ■

Recall that parametrization of a curve induces an **orientation** to the curve. For instance, in Example 1, the curve is oriented such that the positive direction is from $(0, 0, 0)$, following the curve to $(1, 2, 1)$. Try finding a parametrization that induces the opposite orientation.

Line Integrals

Up to this point in the text, you have studied various types of integrals. For a single integral

$$\int_a^b f(x)\,dx$$

Integrate over interval $[a, b]$.

you integrated over the interval $[a, b]$. Similarly, for a double integral

$$\int\int_R f(x, y)\,dA$$

Integrate over region R.

you integrated over the region R in the plane. In this section, you will study a new type of integral called a **line integral**

$$\int_C f(x, y)\,ds$$

Integrate over curve C.

for which you integrate over a piecewise smooth curve C. (The terminology is somewhat unfortunate—this type of integral might be better described as a "curve integral.")

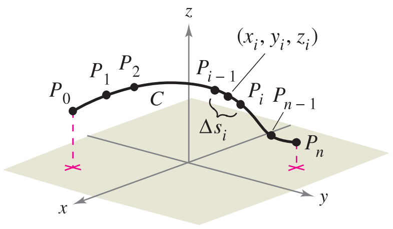
To introduce the concept of a line integral, consider the mass of a wire of finite length, given by a curve C in space. The density (mass per unit length) of the wire at the point (x, y, z) is given by $f(x, y, z)$. Partition the curve C by the points

$$P_0, P_1, \ldots, P_n$$

producing n subarcs, as shown in Figure 15.8. The length of the ith subarc is given by Δs_i. Next, choose a point (x_i, y_i, z_i) in each subarc. If the length of each subarc is small, the total mass of the wire can be approximated by the sum

$$\text{Mass of wire} \approx \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta s_i.$$

If you let $\|\Delta\|$ denote the length of the longest subarc and let $\|\Delta\|$ approach 0, it seems reasonable that the limit of this sum approaches the mass of the wire. This leads to the following definition.



Partitioning of curve C
Figure 15.8

DEFINITION OF LINE INTEGRAL

If f is defined in a region containing a smooth curve C of finite length, then the **line integral of f along C** is given by

$$\int_C f(x, y)\,ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta s_i$$

Plane

or

$$\int_C f(x, y, z)\,ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta s_i$$

Space

provided this limit exists.

As with the integrals discussed in Chapter 14, evaluation of a line integral is best accomplished by converting it to a definite integral. It can be shown that if f is *continuous*, the limit given above exists and is the same for all smooth parametrizations of C.

To evaluate a line integral over a plane curve C given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, use the fact that

$$ds = \|\mathbf{r}'(t)\|\,dt = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt.$$

A similar formula holds for a space curve, as indicated in Theorem 15.4.

THEOREM 15.4 EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL

Let f be continuous in a region containing a smooth curve C. If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where $a \le t \le b$, then

$$\int_C f(x, y)\,ds = \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt.$$

If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $a \le t \le b$, then

$$\int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt.$$

Note that if $f(x, y, z) = 1$, the line integral gives the arc length of the curve C, as defined in Section 12.5. That is,

$$\int_C 1\,ds = \int_a^b \|\mathbf{r}'(t)\|\,dt = \text{length of curve } C.$$

EXAMPLE 2 Evaluating a Line Integral

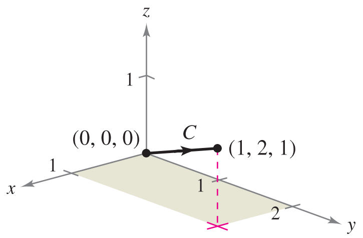


Figure 15.9

Evaluate

$$\int_C (x^2 - y + 3z)\,ds$$

where C is the line segment shown in Figure 15.9.

Solution Begin by writing a parametric form of the equation of the line segment:

$$x = t, \quad y = 2t, \quad \text{and} \quad z = t, \quad 0 \le t \le 1.$$

Therefore, $x'(t) = 1$, $y'(t) = 2$, and $z'(t) = 1$, which implies that

$$\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}.$$

So, the line integral takes the following form.

$$\begin{aligned}\int_C (x^2 - y + 3z)\,ds &= \int_0^1 (t^2 - 2t + 3t)\sqrt{6}\,dt \\ &= \sqrt{6}\int_0^1 (t^2 + t)\,dt \\ &= \sqrt{6}\left[\frac{t^3}{3} + \frac{t^2}{2}\right]_0^1 \\ &= \frac{5\sqrt{6}}{6}\end{aligned}$$

■

NOTE The value of the line integral in Example 2 does not depend on the parametrization of the line segment C (any smooth parametrization will produce the same value). To convince yourself of this, try some other parametrizations, such as $x = 1 + 2t$, $y = 2 + 4t$, $z = 1 + 2t$, $-\frac{1}{2} \le t \le 0$, or $x = -t$, $y = -2t$, $z = -t$, $-1 \le t \le 0$.

Suppose C is a path composed of smooth curves $C_1, C_2, \ldots, C_n$. If f is continuous on C, it can be shown that

$$\int_C f(x, y)\, ds = \int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds.$$

This property is used in Example 3.

EXAMPLE 3 Evaluating a Line Integral Over a Path

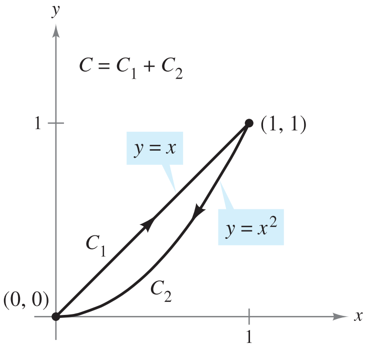


Figure 15.10

Evaluate $\int_C x\, ds$, where C is the piecewise smooth curve shown in Figure 15.10.

Solution Begin by integrating up the line $y = x$, using the following parametrization.

$$C_1\colon\ x = t,\ y = t,\quad 0 \le t \le 1$$

For this curve, $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$, which implies that $x'(t) = 1$ and $y'(t) = 1$. So,

$$\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{2}$$

and you have

$$\int_{C_1} x\, ds = \int_0^1 t\sqrt{2}\, dt = \frac{\sqrt{2}}{2}t^2\Big]_0^1 = \frac{\sqrt{2}}{2}.$$

Next, integrate down the parabola $y = x^2$, using the parametrization

$$C_2\colon\ x = 1 - t,\quad y = (1 - t)^2,\quad 0 \le t \le 1.$$

For this curve, $\mathbf{r}(t) = (1 - t)\mathbf{i} + (1 - t)^2\mathbf{j}$, which implies that $x'(t) = -1$ and $y'(t) = -2(1 - t)$. So,

$$\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{1 + 4(1 - t)^2}$$

and you have

$$\begin{aligned}\int_{C_2} x\, ds &= \int_0^1 (1 - t)\sqrt{1 + 4(1 - t)^2}\, dt\\ &= -\frac{1}{8}\left[\frac{2}{3}[1 + 4(1 - t)^2]^{3/2}\right]_0^1\\ &= \frac{1}{12}(5^{3/2} - 1).\end{aligned}$$

Consequently,

$$\int_C x\, ds = \int_{C_1} x\, ds + \int_{C_2} x\, ds = \frac{\sqrt{2}}{2} + \frac{1}{12}(5^{3/2} - 1) \approx 1.56.$$

■

For parametrizations given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, it is helpful to remember the form of ds as

$$ds = \|\mathbf{r}'(t)\|\, dt = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt.$$

This is demonstrated in Example 4.

EXAMPLE 4 Evaluating a Line Integral

Evaluate $\int_C (x + 2)\, ds$, where C is the curve represented by

$$\mathbf{r}(t) = t\mathbf{i} + \frac{4}{3}t^{3/2}\mathbf{j} + \frac{1}{2}t^2\mathbf{k}, \quad 0 \le t \le 2.$$

Solution Because $\mathbf{r}'(t) = \mathbf{i} + 2t^{1/2}\mathbf{j} + t\mathbf{k}$ and

$$\|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} = \sqrt{1 + 4t + t^2}$$

it follows that

$$\begin{aligned}\int_C (x + 2)\, ds &= \int_0^2 (t + 2)\sqrt{1 + 4t + t^2}\, dt \\ &= \frac{1}{2}\int_0^2 2(t + 2)(1 + 4t + t^2)^{1/2}\, dt \\ &= \frac{1}{3}\Big[(1 + 4t + t^2)^{3/2}\Big]_0^2 \\ &= \frac{1}{3}\left(13\sqrt{13} - 1\right) \\ &\approx 15.29.\end{aligned}$$

■

The next example shows how a line integral can be used to find the mass of a spring whose density varies. In Figure 15.11, note that the density of this spring increases as the spring spirals up the z-axis.

EXAMPLE 5 Finding the Mass of a Spring

Find the mass of a spring in the shape of the circular helix

$$\mathbf{r}(t) = \frac{1}{\sqrt{2}}(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}), \quad 0 \le t \le 6\pi$$

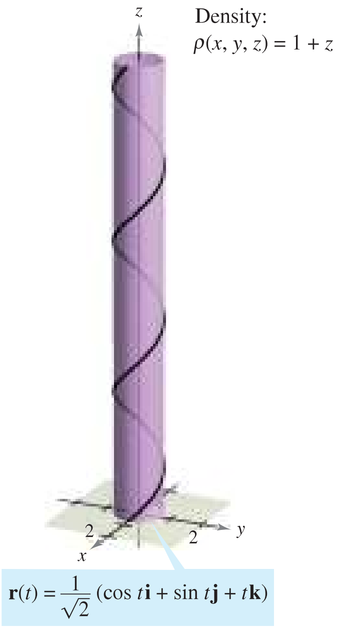
where the density of the spring is $\rho(x, y, z) = 1 + z$, as shown in Figure 15.11.

Solution Because

$$\|\mathbf{r}'(t)\| = \frac{1}{\sqrt{2}}\sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} = 1$$

it follows that the mass of the spring is

$$\begin{aligned}\text{Mass} = \int_C (1 + z)\, ds &= \int_0^{6\pi}\left(1 + \frac{t}{\sqrt{2}}\right) dt \\ &= \left[t + \frac{t^2}{2\sqrt{2}}\right]_0^{6\pi} \\ &= 6\pi\left(1 + \frac{3\pi}{\sqrt{2}}\right) \\ &\approx 144.47.\end{aligned}$$

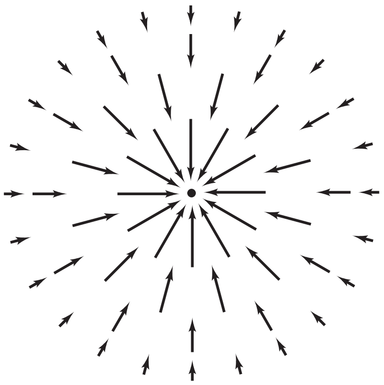


Density: $\rho(x, y, z) = 1 + z$

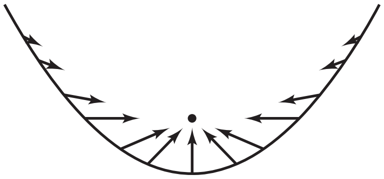
$\mathbf{r}(t) = \frac{1}{\sqrt{2}}(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k})$

Figure 15.11

The mass of the spring is approximately 144.47. ■



Inverse square force field **F**



Vectors along a parabolic path in the force field **F**

Figure 15.12

Line Integrals of Vector Fields

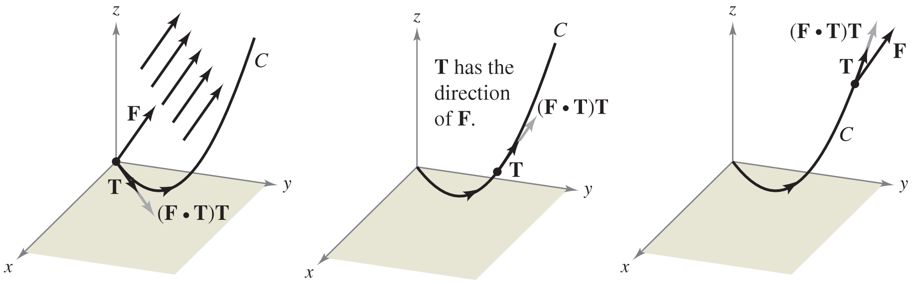
One of the most important physical applications of line integrals is that of finding the **work** done on an object moving in a force field. For example, Figure 15.12 shows an inverse square force field similar to the gravitational field of the sun. Note that the magnitude of the force along a circular path about the center is constant, whereas the magnitude of the force along a parabolic path varies from point to point.

To see how a line integral can be used to find work done in a force field $\mathbf{F}$, consider an object moving along a path C in the field, as shown in Figure 15.13. To determine the work done by the force, you need consider only that part of the force that is acting in the same direction as that in which the object is moving (or the opposite direction). This means that at each point on C, you can consider the projection $\mathbf{F}\cdot\mathbf{T}$ of the force vector $\mathbf{F}$ onto the unit tangent vector $\mathbf{T}$. On a small subarc of length Δs_i, the increment of work is

$$\begin{aligned}\Delta W_i &= (\text{force})(\text{distance})\\ &\approx [\mathbf{F}(x_i, y_i, z_i)\cdot\mathbf{T}(x_i, y_i, z_i)]\,\Delta s_i\end{aligned}$$

where (x_i, y_i, z_i) is a point in the ith subarc. Consequently, the total work done is given by the following integral.

$$W = \int_C \mathbf{F}(x, y, z)\cdot\mathbf{T}(x, y, z)\,ds$$



At each point on C, the force in the direction of motion is $(\mathbf{F}\cdot\mathbf{T})\mathbf{T}$.

Figure 15.13

This line integral appears in other contexts and is the basis of the following definition of the **line integral of a vector field.** Note in the definition that

$$\begin{aligned}\mathbf{F}\cdot\mathbf{T}\,ds &= \mathbf{F}\cdot\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\|\mathbf{r}'(t)\|\,dt\\ &= \mathbf{F}\cdot\mathbf{r}'(t)\,dt\\ &= \mathbf{F}\cdot d\mathbf{r}.\end{aligned}$$

DEFINITION OF THE LINE INTEGRAL OF A VECTOR FIELD

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t)$, $a \le t \le b$. The **line integral** of $\mathbf{F}$ on C is given by

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}(x(t),\ y(t), z(t))\cdot\mathbf{r}'(t)\,dt.$$

EXAMPLE 6 Work Done by a Force

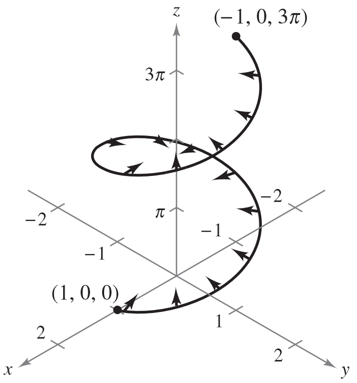


Figure 15.14

Find the work done by the force field

$$\mathbf{F}(x, y, z) = -\frac{1}{2}x\mathbf{i} - \frac{1}{2}y\mathbf{j} + \frac{1}{4}\mathbf{k} \qquad \text{Force field } \mathbf{F}$$

on a particle as it moves along the helix given by

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k} \qquad \text{Space curve } C$$

from the point $(1, 0, 0)$ to $(-1, 0, 3\pi)$, as shown in Figure 15.14.

Solution Because

$$\begin{aligned}\mathbf{r}(t) &= x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}\\ &= \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}\end{aligned}$$

it follows that $x(t) = \cos t$, $y(t) = \sin t$, and $z(t) = t$. So, the force field can be written as

$$\mathbf{F}(x(t), y(t), z(t)) = -\frac{1}{2}\cos t\,\mathbf{i} - \frac{1}{2}\sin t\,\mathbf{j} + \frac{1}{4}\mathbf{k}.$$

To find the work done by the force field in moving a particle along the curve C, use the fact that

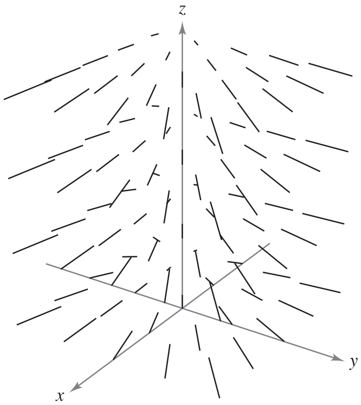
$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$$

and write the following.

$$\begin{aligned}W &= \int_C \mathbf{F} \cdot d\mathbf{r}\\ &= \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\,dt\\ &= \int_0^{3\pi} \left(-\frac{1}{2}\cos t\,\mathbf{i} - \frac{1}{2}\sin t\,\mathbf{j} + \frac{1}{4}\mathbf{k}\right) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k})\,dt\\ &= \int_0^{3\pi} \left(\frac{1}{2}\sin t\cos t - \frac{1}{2}\sin t\cos t + \frac{1}{4}\right) dt\\ &= \int_0^{3\pi} \frac{1}{4}\,dt\\ &= \frac{1}{4}t\Big]_0^{3\pi}\\ &= \frac{3\pi}{4}\end{aligned}$$

■

NOTE In Example 6, note that the x- and y-components of the force field end up contributing nothing to the total work. This occurs because *in this particular example* the z-component of the force field is the only portion of the force that is acting in the same (or opposite) direction in which the particle is moving (see Figure 15.15). ■



Generated by Mathematica

Figure 15.15

TECHNOLOGY The computer-generated view of the force field in Example 6 shown in Figure 15.15 indicates that each vector in the force field points toward the z-axis.

For line integrals of vector functions, the orientation of the curve C is important. If the orientation of the curve is reversed, the unit tangent vector $\mathbf{T}(t)$ is changed to $-\mathbf{T}(t)$, and you obtain

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_{C} \mathbf{F} \cdot d\mathbf{r}.$$

EXAMPLE 7 Orientation and Parametrization of a Curve

Let $\mathbf{F}(x, y) = y\mathbf{i} + x^2\mathbf{j}$ and evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for each parabolic curve shown in Figure 15.16.

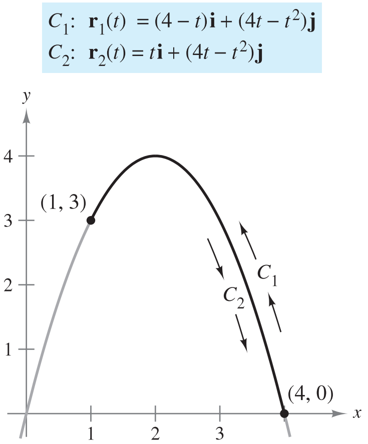


Figure 15.16

a. C_1: $\mathbf{r}_1(t) = (4-t)\mathbf{i} + (4t-t^2)\mathbf{j}, \quad 0 \le t \le 3$

b. C_2: $\mathbf{r}_2(t) = t\mathbf{i} + (4t-t^2)\mathbf{j}, \quad 1 \le t \le 4$

Solution

a. Because $\mathbf{r}_1'(t) = -\mathbf{i} + (4-2t)\mathbf{j}$ and

$$\mathbf{F}(x(t), y(t)) = (4t-t^2)\mathbf{i} + (4-t)^2\mathbf{j}$$

the line integral is

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^3 \left[(4t-t^2)\mathbf{i} + (4-t)^2\mathbf{j}\right] \cdot \left[-\mathbf{i} + (4-2t)\mathbf{j}\right] dt \\ &= \int_0^3 (-4t + t^2 + 64 - 64t + 20t^2 - 2t^3)\, dt \\ &= \int_0^3 (-2t^3 + 21t^2 - 68t + 64)\, dt \\ &= \left[-\frac{t^4}{2} + 7t^3 - 34t^2 + 64t\right]_0^3 \\ &= \frac{69}{2}.\end{aligned}$$

b. Because $\mathbf{r}_2'(t) = \mathbf{i} + (4-2t)\mathbf{j}$ and

$$\mathbf{F}(x(t), y(t)) = (4t-t^2)\mathbf{i} + t^2\mathbf{j}$$

the line integral is

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_1^4 \left[(4t-t^2)\mathbf{i} + t^2\mathbf{j}\right] \cdot \left[\mathbf{i} + (4-2t)\mathbf{j}\right] dt \\ &= \int_1^4 (4t - t^2 + 4t^2 - 2t^3)\, dt \\ &= \int_1^4 (-2t^3 + 3t^2 + 4t)\, dt \\ &= \left[-\frac{t^4}{2} + t^3 + 2t^2\right]_1^4 \\ &= -\frac{69}{2}.\end{aligned}$$

The answer in part (b) is the negative of that in part (a) because C_1 and C_2 represent opposite orientations of the same parabolic segment. ■

NOTE Although the value of the line integral in Example 7 depends on the orientation of C, it does not depend on the parametrization of C. To see this, let C_3 be represented by

$$\mathbf{r}_3 = (t+2)\mathbf{i} + (4-t^2)\mathbf{j}$$

where $-1 \le t \le 2$. The graph of this curve is the same parabolic segment shown in Figure 15.16. Does the value of the line integral over C_3 agree with the value over C_1 or C_2? Why or why not?

Line Integrals in Differential Form

A second commonly used form of line integrals is derived from the vector field notation used in the preceding section. If $\mathbf{F}$ is a vector field of the form $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$, and C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then $\mathbf{F} \cdot d\mathbf{r}$ is often written as $M\,dx + N\,dy$.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt\\ &= \int_a^b (M\mathbf{i} + N\mathbf{j}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j})\,dt\\ &= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt}\right) dt\\ &= \int_C (M\,dx + N\,dy)\end{aligned}$$

This **differential form** can be extended to three variables. The parentheses are often omitted, as follows.

$$\int_C M\,dx + N\,dy \qquad \text{and} \qquad \int_C M\,dx + N\,dy + P\,dz$$

Notice how this differential notation is used in Example 8.

EXAMPLE 8 Evaluating a Line Integral in Differential Form

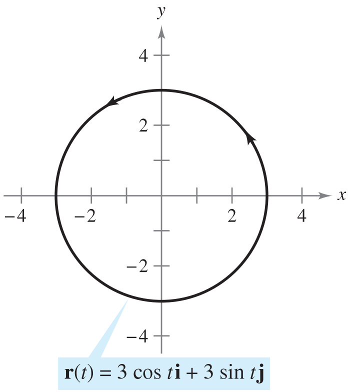


Figure 15.17

Let C be the circle of radius 3 given by

$$\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}, \quad 0 \le t \le 2\pi$$

as shown in Figure 15.17. Evaluate the line integral

$$\int_C y^3\,dx + (x^3 + 3xy^2)\,dy.$$

Solution Because $x = 3\cos t$ and $y = 3\sin t$, you have $dx = -3\sin t\,dt$ and $dy = 3\cos t\,dt$. So, the line integral is

$$\begin{aligned}&\int_C M\,dx + N\,dy\\ &= \int_C y^3\,dx + (x^3 + 3xy^2)\,dy\\ &= \int_0^{2\pi} \left[(27\sin^3 t)(-3\sin t) + (27\cos^3 t + 81\cos t\sin^2 t)(3\cos t)\right] dt\\ &= 81\int_0^{2\pi} (\cos^4 t - \sin^4 t + 3\cos^2 t\sin^2 t)\,dt\\ &= 81\int_0^{2\pi} \left(\cos^2 t - \sin^2 t + \frac{3}{4}\sin^2 2t\right) dt\\ &= 81\int_0^{2\pi} \left[\cos 2t + \frac{3}{4}\left(\frac{1 - \cos 4t}{2}\right)\right] dt\\ &= 81\left[\frac{\sin 2t}{2} + \frac{3}{8}t - \frac{3\sin 4t}{32}\right]_0^{2\pi}\\ &= \frac{243\pi}{4}.\end{aligned}$$

■

NOTE The orientation of C affects the value of the differential form of a line integral. Specifically, if $-C$ has the orientation opposite to that of C, then

$$\int_{-C} M\,dx + N\,dy = -\int_C M\,dx + N\,dy.$$

So, of the three line integral forms presented in this section, the orientation of C does not affect the form $\int_C f(x, y)\,ds$, but it does affect the vector form and the differential form.

For curves represented by $y = g(x)$, $a \le x \le b$, you can let $x = t$ and obtain the parametric form

$$x = t \quad \text{and} \quad y = g(t), \quad a \le t \le b.$$

Because $dx = dt$ for this form, you have the option of evaluating the line integral in the variable x or the variable t. This is demonstrated in Example 9.

EXAMPLE 9 Evaluating a Line Integral in Differential Form

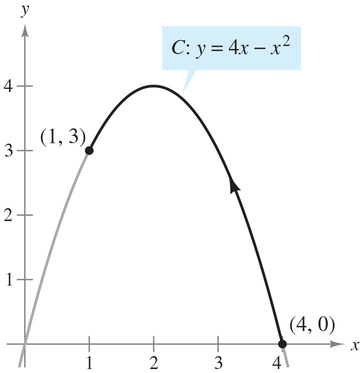


Figure 15.18

Evaluate

$$\int_C y\,dx + x^2\,dy$$

where C is the parabolic arc given by $y = 4x - x^2$ from (4, 0) to (1, 3), as shown in Figure 15.18.

Solution Rather than converting to the parameter t, you can simply retain the variable x and write

$$y = 4x - x^2 \quad \Rightarrow \quad dy = (4 - 2x)\,dx.$$

Then, in the direction from (4, 0) to (1, 3), the line integral is

$$\begin{aligned}\int_C y\,dx + x^2\,dy &= \int_4^1 [(4x - x^2)\,dx + x^2(4 - 2x)\,dx]\\ &= \int_4^1 (4x + 3x^2 - 2x^3)\,dx\\ &= \left[2x^2 + x^3 - \frac{x^4}{2}\right]_4^1 = \frac{69}{2}.\end{aligned}$$

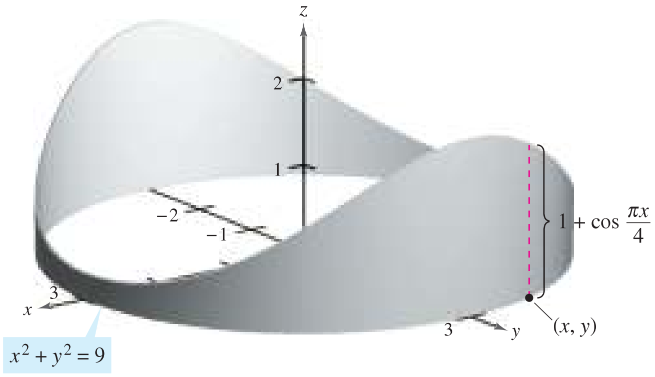
See Example 7. ■

EXPLORATION

Finding Lateral Surface Area The figure below shows a piece of tin that has been cut from a circular cylinder. The base of the circular cylinder is modeled by $x^2 + y^2 = 9$. At any point (x, y) on the base, the height of the object is given by

$$f(x, y) = 1 + \cos \frac{\pi x}{4}.$$

Explain how to use a line integral to find the surface area of the piece of tin.

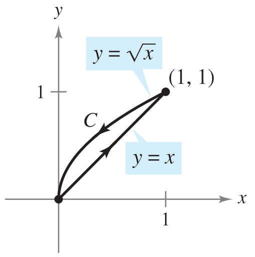


15.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find a piecewise smooth parametrization of the path C. (Note that there is more than one correct answer.)

1.



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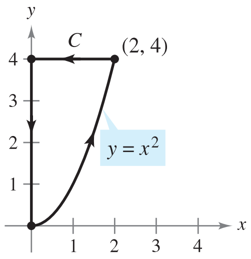

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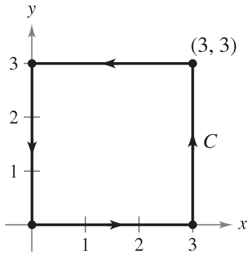

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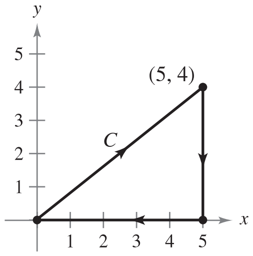

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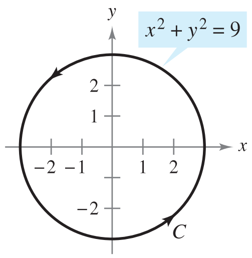

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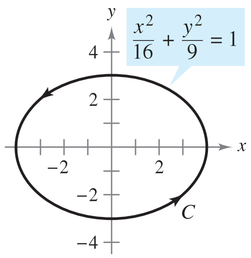


In Exercises 7–10, evaluate the line integral along the given path.

7. $\int_C xy\,ds$

C: $\mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j}$

$0 \le t \le 1$

8. $\int_C 3(x - y)\,ds$

C: $\mathbf{r}(t) = t\mathbf{i} + (2 - t)\mathbf{j}$

$0 \le t \le 2$

9. $\int_C (x^2 + y^2 + z^2)\,ds$

C: $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}$

$0 \le t \le \pi/2$

10. $\int_C 2xyz\,ds$

C: $\mathbf{r}(t) = 12t\mathbf{i} + 5t\mathbf{j} + 84t\mathbf{k}$

$0 \le t \le 1$

In Exercises 11–14, (a) find a parametrization of the path C, and (b) evaluate

$$\int_C (x^2 + y^2)\,ds$$

along C.

11. C: line segment from (0, 0) to (1, 1)

12. C: line segment from (0, 0) to (2, 4)

13. C: counterclockwise around the circle $x^2 + y^2 = 1$ from (1, 0) to (0, 1)

14. C: counterclockwise around the circle $x^2 + y^2 = 4$ from (2, 0) to (0, 2)

In Exercises 15–18, (a) find a parametrization of the path C, and (b) evaluate

$$\int_C (x + 4\sqrt{y})\,ds$$

along C.

15. C: x-axis from $x = 0$ to $x = 1$

16. C: y-axis from $y = 1$ to $y = 9$

17. C: counterclockwise around the triangle with vertices (0, 0), (1, 0), and (0, 1)

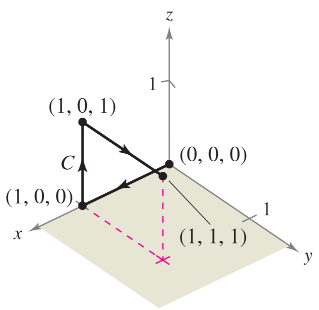
18. C: counterclockwise around the square with vertices (0, 0), (2, 0), (2, 2), and (0, 2)

In Exercises 19 and 20, (a) find a piecewise smooth parametrization of the path C shown in the figure, and (b) evaluate

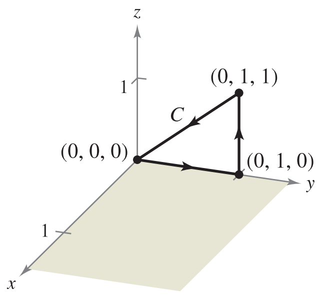
$$\int_C (2x + y^2 - z)\,ds$$

along C.

19.



20.



Mass **In Exercises 21 and 22, find the total mass of two turns of a spring with density ρ in the shape of the circular helix**

$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k},\ 0 \le t \le 4\pi.$

21. $\rho(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2)$

22. $\rho(x, y, z) = z$

Mass **In Exercises 23–26, find the total mass of the wire with density ρ.**

23. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\quad \rho(x, y) = x + y,\quad 0 \le t \le \pi$

24. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j},\quad \rho(x, y) = \frac{3}{4}y,\quad 0 \le t \le 1$

25. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + t\mathbf{k},\quad \rho(x, y, z) = kz\quad (k > 0),\quad 1 \le t \le 3$

26. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 3t\mathbf{k},\quad \rho(x, y, z) = k + z$ $(k > 0),\quad 0 \le t \le 2\pi$

In Exercises 27–32, evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is represented by $\mathbf{r}(t)$.

27. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1$

28. $\mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j}$

C: $\mathbf{r}(t) = 4 \cos t\mathbf{i} + 4 \sin t\mathbf{j}, \quad 0 \le t \le \pi/2$

29. $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j}$

C: $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad 0 \le t \le \pi/2$

30. $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} + \sqrt{4 - t^2}\mathbf{j}, \quad -2 \le t \le 2$

31. $\mathbf{F}(x, y, z) = xy\mathbf{i} + xz\mathbf{j} + yz\mathbf{k}$

C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}, \quad 0 \le t \le 1$

32. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

C: $\mathbf{r}(t) = 2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}, \quad 0 \le t \le \pi$

CAS **In Exercises 33 and 34, use a computer algebra system to evaluate the integral**

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is represented by $\mathbf{r}(t)$.

33. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + 6y\mathbf{j} + yz^2\mathbf{k}$

C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, \quad 1 \le t \le 3$

34. $\mathbf{F}(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

C: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k}, \quad 0 \le t \le 2$

Work **In Exercises 35–40, find the work done by the force field F on a particle moving along the given path.**

35. $\mathbf{F}(x, y) = x\mathbf{i} + 2y\mathbf{j}$

C: $x = t, y = t^3$ from $(0, 0)$ to $(2, 8)$

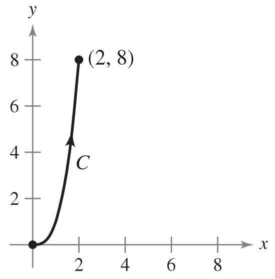


Figure for 35

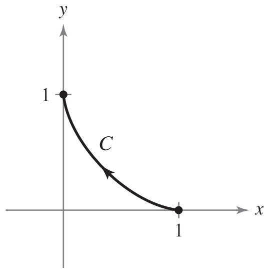


Figure for 36

36. $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$

C: $x = \cos^3 t, y = \sin^3 t$ from $(1, 0)$ to $(0, 1)$

37. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

C: counterclockwise around the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$ (*Hint:* See Exercise 17a.)

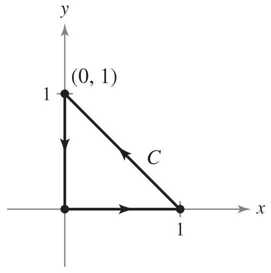


Figure for 37

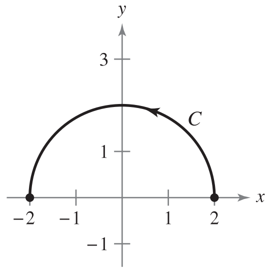


Figure for 38

38. $\mathbf{F}(x, y) = -y\mathbf{i} - x\mathbf{j}$

C: counterclockwise along the semicircle $y = \sqrt{4 - x^2}$ from $(2, 0)$ to $(-2, 0)$

39. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 5z\mathbf{k}$

C: $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 2\pi$

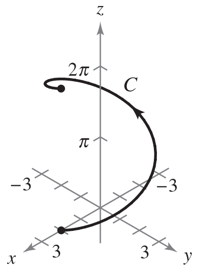


Figure for 39

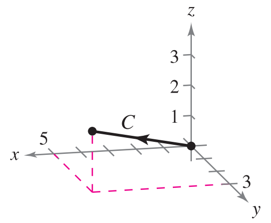


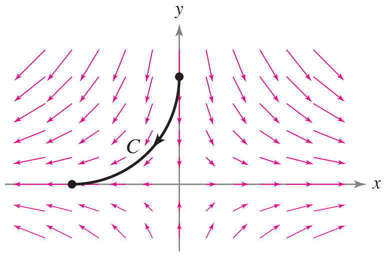
Figure for 40

40. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

C: line from $(0, 0, 0)$ to $(5, 3, 2)$

In Exercises 41–44, determine whether the work done along the path C is positive, negative, or zero. Explain.

41.



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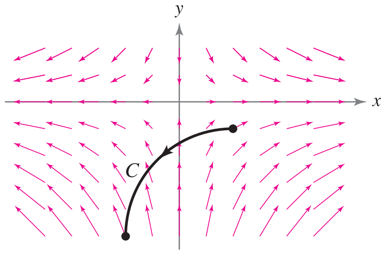

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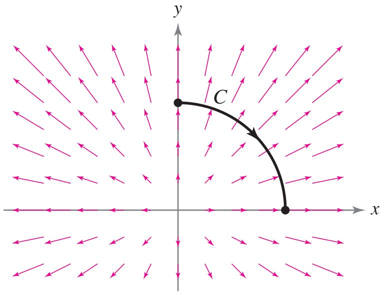
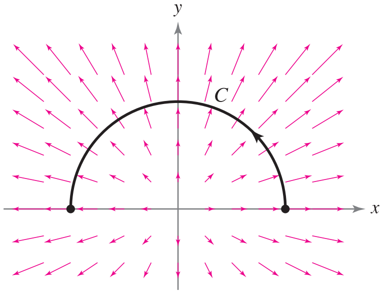


44.



In Exercises 45 and 46, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for each curve. Discuss the orientation of the curve and its effect on the value of the integral.

45. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = 2t\mathbf{i} + (t - 1)\mathbf{j}, \quad 1 \le t \le 3$

(b) $\mathbf{r}_2(t) = 2(3 - t)\mathbf{i} + (2 - t)\mathbf{j}, \quad 0 \le t \le 2$

46. $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy^{3/2}\mathbf{j}$

(a) $\mathbf{r}_1(t) = (t + 1)\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 2$

(b) $\mathbf{r}_2(t) = (1 + 2\cos t)\mathbf{i} + (4\cos^2 t)\mathbf{j}, \quad 0 \le t \le \pi/2$

In Exercises 47–50, demonstrate the property that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

regardless of the initial and terminal points of C, if the tangent vector $\mathbf{r}'(t)$ is orthogonal to the force field $\mathbf{F}$.

47. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} - 2t\mathbf{j}$

48. $\mathbf{F}(x, y) = -3y\mathbf{i} + x\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} - t^3\mathbf{j}$

49. $\mathbf{F}(x, y) = (x^3 - 2x^2)\mathbf{i} + \left(x - \dfrac{y}{2}\right)\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

50. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

C: $\mathbf{r}(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j}$

In Exercises 51–54, evaluate the line integral along the path C given by $x = 2t, y = 10t$, where $0 \le t \le 1$.

51. $\displaystyle\int_C (x + 3y^2)\, dy$

52. $\displaystyle\int_C (x + 3y^2)\, dx$

53. $\displaystyle\int_C xy\, dx + y\, dy$

54. $\displaystyle\int_C (3y - x)\, dx + y^2\, dy$

In Exercises 55–62, evaluate the integral

$$\int_C (2x - y)\, dx + (x + 3y)\, dy$$

along the path C.

55. C: x-axis from $x = 0$ to $x = 5$

56. C: y-axis from $y = 0$ to $y = 2$

57. C: line segments from $(0, 0)$ to $(3, 0)$ and $(3, 0)$ to $(3, 3)$

58. C: line segments from $(0, 0)$ to $(0, -3)$ and $(0, -3)$ to $(2, -3)$

59. C: arc on $y = 1 - x^2$ from $(0, 1)$ to $(1, 0)$

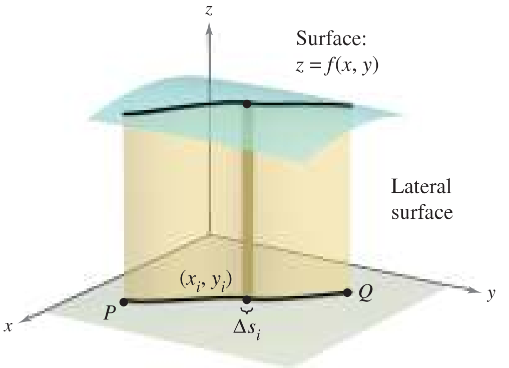
60. C: arc on $y = x^{3/2}$ from $(0, 0)$ to $(4, 8)$

61. C: parabolic path $x = t$, $y = 2t^2$, from $(0, 0)$ to $(2, 8)$

62. C: elliptic path $x = 4\sin t$, $y = 3\cos t$, from $(0, 3)$ to $(4, 0)$

Lateral Surface Area **In Exercises 63–70, find the area of the lateral surface (see figure) over the curve C in the xy-plane and under the surface $z = f(x, y)$, where**

$$\textbf{Lateral surface area} = \int_C f(x, y)\, ds.$$



C: Curve in xy-plane

63. $f(x, y) = h$, C: line from $(0, 0)$ to $(3, 4)$

64. $f(x, y) = y$, C: line from $(0, 0)$ to $(4, 4)$

65. $f(x, y) = xy$, C: $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$

66. $f(x, y) = x + y$, C: $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$

67. $f(x, y) = h$, C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

68. $f(x, y) = y + 1$, C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

69. $f(x, y) = xy$, C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

70. $f(x, y) = x^2 - y^2 + 4$, C: $x^2 + y^2 = 4$

71. ***Engine Design*** A tractor engine has a steel component with a circular base modeled by the vector-valued function $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$. Its height is given by $z = 1 + y^2$. (All measurements of the component are in centimeters.)

(a) Find the lateral surface area of the component.

(b) The component is in the form of a shell of thickness 0.2 centimeter. Use the result of part (a) to approximate the amount of steel used in its manufacture.

(c) Draw a sketch of the component.

72. ***Building Design*** The ceiling of a building has a height above the floor given by $z = 20 + \frac{1}{4}x$, and one of the walls follows a path modeled by $y = x^{3/2}$. Find the surface area of the wall if $0 \le x \le 40$. (All measurements are in feet.)

Moments of Inertia **Consider a wire of density $\rho(x, y)$ given by the space curve**

$$C:\ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad 0 \le t \le b.$$

The moments of inertia about the x- and y-axes are given by

$$I_x = \int_C y^2 \rho(x, y)\, ds$$

$$I_y = \int_C x^2 \rho(x, y)\, ds.$$

In Exercises 73 and 74, find the moments of inertia for the wire of density ρ.

73. A wire lies along $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$, $0 \le t \le 2\pi$ and $a > 0$, with density $\rho(x, y) = 1$.

74. A wire lies along $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$, $0 \le t \le 2\pi$ and $a > 0$, with density $\rho(x, y) = y$.

CAS **75.** ***Investigation*** The top outer edge of a solid with vertical sides and resting on the xy-plane is modeled by $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + (1 + \sin^2 2t)\mathbf{k}$, where all measurements are in centimeters. The intersection of the plane $y = b\ (-3 < b < 3)$ with the top of the solid is a horizontal line.

(a) Use a computer algebra system to graph the solid.

(b) Use a computer algebra system to approximate the lateral surface area of the solid.

(c) Find (if possible) the volume of the solid.

76. ***Work*** A particle moves along the path $y = x^2$ from the point $(0, 0)$ to the point $(1, 1)$. The force field $\mathbf{F}$ is measured at five points along the path, and the results are shown in the table. Use Simpson's Rule or a graphing utility to approximate the work done by the force field.

(x, y)	$(0, 0)$	$\left(\frac{1}{4}, \frac{1}{16}\right)$	$\left(\frac{1}{2}, \frac{1}{4}\right)$	$\left(\frac{3}{4}, \frac{9}{16}\right)$	$(1, 1)$
$\mathbf{F}(x, y)$	$\langle 5, 0\rangle$	$\langle 3.5, 1\rangle$	$\langle 2, 2\rangle$	$\langle 1.5, 3\rangle$	$\langle 1, 5\rangle$

77. ***Work*** Find the work done by a person weighing 175 pounds walking exactly one revolution up a circular helical staircase of radius 3 feet if the person rises 10 feet.

78. ***Investigation*** Determine the value of c such that the work done by the force field

$$\mathbf{F}(x, y) = 15[(4 - x^2 y)\mathbf{i} - xy\mathbf{j}]$$

on an object moving along the parabolic path $y = c(1 - x^2)$ between the points $(-1, 0)$ and $(1, 0)$ is a minimum. Compare the result with the work required to move the object along the straight-line path connecting the points.

WRITING ABOUT CONCEPTS

79. Define a line integral of a function f along a smooth curve C in the plane and in space. How do you evaluate the line integral as a definite integral?

80. Define a line integral of a continuous vector field $\mathbf{F}$ on a smooth curve C. How do you evaluate the line integral as a definite integral?

81. Order the surfaces in ascending order of the lateral surface area under the surface and over the curve $y = \sqrt{x}$ from $(0, 0)$ to $(4, 2)$ in the xy-plane. Explain your ordering without doing any calculations.

(a) $z_1 = 2 + x$ (b) $z_2 = 5 + x$

(c) $z_3 = 2$ (d) $z_4 = 10 + x + 2y$

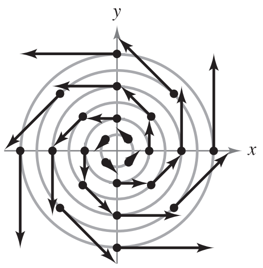
CAPSTONE

82. For each of the following, determine whether the work done in moving an object from the first to the second point through the force field shown in the figure is positive, negative, or zero. Explain your answer.

(a) From $(-3, -3)$ to $(3, 3)$

(b) From $(-3, 0)$ to $(0, 3)$

(c) From $(5, 0)$ to $(0, 3)$



True or False? **In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

83. If C is given by $x(t) = t$, $y(t) = t$, $0 \le t \le 1$, then

$$\int_C xy\, ds = \int_0^1 t^2\, dt.$$

84. If $C_2 = -C_1$, then $\displaystyle\int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds = 0.$

85. The vector functions $\mathbf{r}_1 = t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{r}_2 = (1 - t)\mathbf{i} + (1 - t)^2\mathbf{j}$, $0 \le t \le 1$, define the same curve.

86. If $\displaystyle\int_C \mathbf{F} \cdot \mathbf{T}\, ds = 0$, then $\mathbf{F}$ and $\mathbf{T}$ are orthogonal.

87. ***Work*** Consider a particle that moves through the force field $\mathbf{F}(x, y) = (y - x)\mathbf{i} + xy\mathbf{j}$ from the point $(0, 0)$ to the point $(0, 1)$ along the curve $x = kt(1 - t)$, $y = t$. Find the value of k such that the work done by the force field is 1.

15.3 Conservative Vector Fields and Independence of Path

- **Understand and use the Fundamental Theorem of Line Integrals.**
- **Understand the concept of independence of path.**
- **Understand the concept of conservation of energy.**

Fundamental Theorem of Line Integrals

The discussion at the beginning of the preceding section pointed out that in a gravitational field the work done by gravity on an object moving between two points in the field is independent of the path taken by the object. In this section, you will study an important generalization of this result—it is called the **Fundamental Theorem of Line Integrals.**

To begin, an example is presented in which the line integral of a *conservative vector field* is evaluated over three different paths.

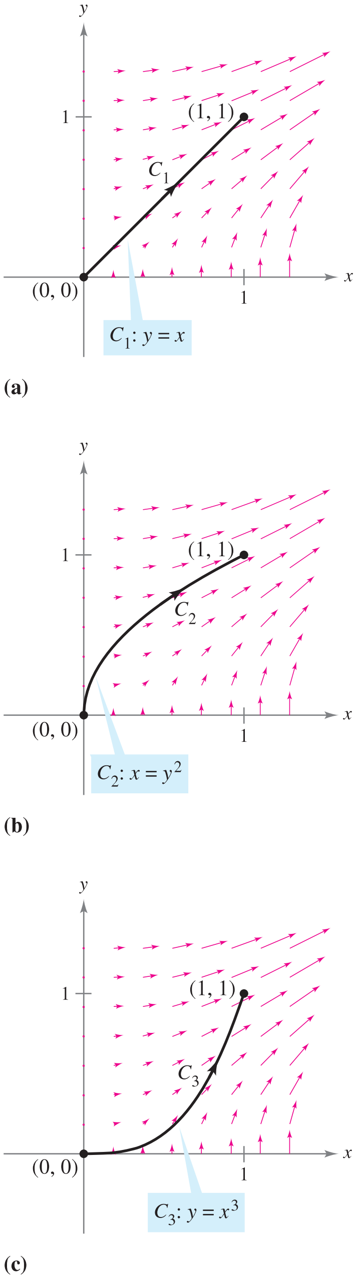


Figure 15.19

EXAMPLE 1 Line Integral of a Conservative Vector Field

Find the work done by the force field

$$\mathbf{F}(x, y) = \frac{1}{2}xy\mathbf{i} + \frac{1}{4}x^2\mathbf{j}$$

on a particle that moves from (0, 0) to (1, 1) along each path, as shown in Figure 15.19.

a. C_1: $y = x$ **b.** C_2: $x = y^2$ **c.** C_3: $y = x^3$

Solution

a. Let $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$ for $0 \leq t \leq 1$, so that

$$d\mathbf{r} = (\mathbf{i} + \mathbf{j})\,dt \qquad \text{and} \qquad \mathbf{F}(x, y) = \frac{1}{2}t^2\mathbf{i} + \frac{1}{4}t^2\mathbf{j}.$$

Then, the work done is

$$W = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \frac{3}{4}t^2\,dt = \frac{1}{4}t^3\Big]_0^1 = \frac{1}{4}.$$

b. Let $\mathbf{r}(t) = t\mathbf{i} + \sqrt{t}\,\mathbf{j}$ for $0 \leq t \leq 1$, so that

$$d\mathbf{r} = \left(\mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}\right)dt \qquad \text{and} \qquad \mathbf{F}(x, y) = \frac{1}{2}t^{3/2}\mathbf{i} + \frac{1}{4}t^2\mathbf{j}.$$

Then, the work done is

$$W = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \frac{5}{8}t^{3/2}\,dt = \frac{1}{4}t^{5/2}\Big]_0^1 = \frac{1}{4}.$$

c. Let $\mathbf{r}(t) = \frac{1}{2}t\mathbf{i} + \frac{1}{8}t^3\mathbf{j}$ for $0 \leq t \leq 2$, so that

$$d\mathbf{r} = \left(\frac{1}{2}\mathbf{i} + \frac{3}{8}t^2\mathbf{j}\right)dt \qquad \text{and} \qquad \mathbf{F}(x, y) = \frac{1}{32}t^4\mathbf{i} + \frac{1}{16}t^2\mathbf{j}.$$

Then, the work done is

$$W = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \frac{5}{128}t^4\,dt = \frac{1}{128}t^5\Big]_0^2 = \frac{1}{4}.$$

So, the work done by a conservative vector field is the same for all paths. ■

In Example 1, note that the vector field $\mathbf{F}(x, y) = \frac{1}{2}xy\mathbf{i} + \frac{1}{4}x^2\mathbf{j}$ is conservative because $\mathbf{F}(x, y) = \nabla f(x, y)$, where $f(x, y) = \frac{1}{4}x^2y$. In such cases, the following theorem states that the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is given by

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= f(x(1), y(1)) - f(x(0), y(0)) \\ &= \frac{1}{4} - 0 \\ &= \frac{1}{4}.\end{aligned}$$

NOTE Notice how the Fundamental Theorem of Line Integrals is similar to the Fundamental Theorem of Calculus (Section 4.4), which states that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F'(x) = f(x)$.

THEOREM 15.5 FUNDAMENTAL THEOREM OF LINE INTEGRALS

Let C be a piecewise smooth curve lying in an open region R and given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad a \le t \le b.$$

If $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is conservative in R, and M and N are continuous in R, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a))$$

where f is a potential function of $\mathbf{F}$. That is, $\mathbf{F}(x, y) = \nabla f(x, y)$.

PROOF A proof is provided only for a smooth curve. For piecewise smooth curves, the procedure is carried out separately on each smooth portion. Because $\mathbf{F}(x, y) = \nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$, it follows that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt \\ &= \int_a^b \left[f_x(x, y)\frac{dx}{dt} + f_y(x, y)\frac{dy}{dt}\right] dt\end{aligned}$$

and, by the Chain Rule (Theorem 13.6), you have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \frac{d}{dt}[f(x(t), y(t))]\,dt \\ &= f(x(b), y(b)) - f(x(a), y(a)).\end{aligned}$$

The last step is an application of the Fundamental Theorem of Calculus. ■

In space, the Fundamental Theorem of Line Integrals takes the following form. Let C be a piecewise smooth curve lying in an open region Q and given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b.$$

If $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative and M, N, and P are continuous, then

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} \\ &= f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))\end{aligned}$$

where $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$.

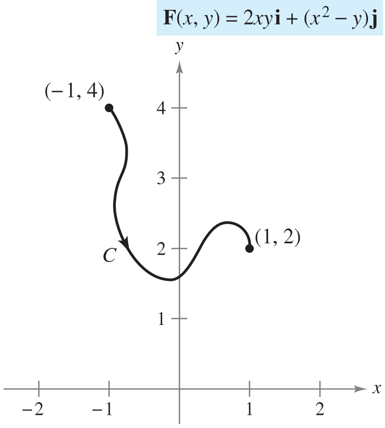
The Fundamental Theorem of Line Integrals states that if the vector field $\mathbf{F}$ is conservative, then the line integral between any two points is simply the difference in the values of the *potential* function f at these points.

EXAMPLE 2 Using the Fundamental Theorem of Line Integrals

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a piecewise smooth curve from $(-1, 4)$ to $(1, 2)$ and

$$\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}$$

as shown in Figure 15.20.



Using the Fundamental Theorem of Line Integrals, $\int_C \mathbf{F} \cdot d\mathbf{r}$.
Figure 15.20

Solution From Example 6 in Section 15.1, you know that $\mathbf{F}$ is the gradient of f where

$$f(x, y) = x^2y - \frac{y^2}{2} + K.$$

Consequently, $\mathbf{F}$ is conservative, and by the Fundamental Theorem of Line Integrals, it follows that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= f(1, 2) - f(-1, 4)\\ &= \left[1^2(2) - \frac{2^2}{2}\right] - \left[(-1)^2(4) - \frac{4^2}{2}\right]\\ &= 4.\end{aligned}$$

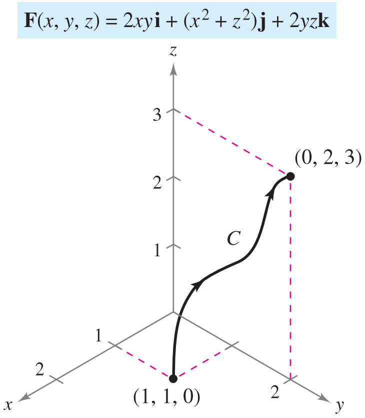
Note that it is unnecessary to include a constant K as part of f, because it is canceled by subtraction.

EXAMPLE 3 Using the Fundamental Theorem of Line Integrals

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a piecewise smooth curve from $(1, 1, 0)$ to $(0, 2, 3)$ and

$$\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2yz\mathbf{k}$$

as shown in Figure 15.21.


$\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2yz\mathbf{k}$
z
x
y
(0, 2, 3)
(1, 1, 0)
C
3
2
1
1
2
2

Using the Fundamental Theorem of Line Integrals, $\int_C \mathbf{F} \cdot d\mathbf{r}$.
Figure 15.21

Solution From Example 8 in Section 15.1, you know that $\mathbf{F}$ is the gradient of f where $f(x, y, z) = x^2y + yz^2 + K$. Consequently, $\mathbf{F}$ is conservative, and by the Fundamental Theorem of Line Integrals, it follows that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= f(0, 2, 3) - f(1, 1, 0)\\ &= [(0)^2(2) + (2)(3)^2] - [(1)^2(1) + (1)(0)^2]\\ &= 17.\end{aligned}$$

■

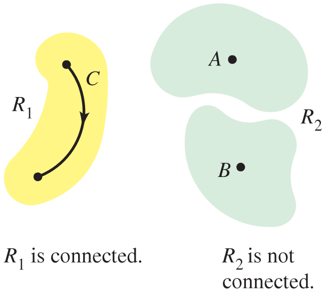
In Examples 2 and 3, be sure you see that the value of the line integral is the same for any smooth curve C that has the given initial and terminal points. For instance, in Example 3, try evaluating the line integral for the curve given by

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + (1 + t)\mathbf{j} + 3t\mathbf{k}.$$

You should obtain

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (30t^2 + 16t - 1)\, dt\\ &= 17.\end{aligned}$$

Independence of Path



R_1 is connected. R_2 is not connected.

Figure 15.22

From the Fundamental Theorem of Line Integrals it is clear that if **F** is continuous and conservative in an open region R, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the same for every piecewise smooth curve C from one fixed point in R to another fixed point in R. This result is described by saying that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** in the region R.

A region in the plane (or in space) is **connected** if any two points in the region can be joined by a piecewise smooth curve lying entirely within the region, as shown in Figure 15.22. In open regions that are *connected*, the path independence of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is equivalent to the condition that **F** is conservative.

THEOREM 15.6 INDEPENDENCE OF PATH AND CONSERVATIVE VECTOR FIELDS

If **F** is continuous on an open connected region, then the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is independent of path if and only if **F** is conservative.

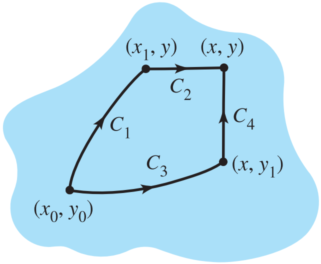


Figure 15.23

PROOF If **F** is conservative, then, by the Fundamental Theorem of Line Integrals, the line integral is independent of path. Now establish the converse for a plane region R. Let $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$, and let (x_0, y_0) be a fixed point in R. If (x, y) is any point in R, choose a piecewise smooth curve C running from (x_0, y_0) to (x, y), and define f by

$$f(x, y) = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy.$$

The existence of C in R is guaranteed by the fact that R is connected. You can show that f is a potential function of **F** by considering two different paths between (x_0, y_0) and (x, y). For the *first* path, choose (x_1, y) in R such that $x \neq x_1$. This is possible because R is open. Then choose C_1 and C_2, as shown in Figure 15.23. Using the independence of path, it follows that

$$\begin{aligned} f(x, y) &= \int_C M\,dx + N\,dy \\ &= \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy. \end{aligned}$$

Because the first integral does not depend on x, and because $dy = 0$ in the second integral, you have

$$f(x, y) = g(y) + \int_{C_2} M\,dx$$

and it follows that the partial derivative of f with respect to x is $f_x(x, y) = M$. For the *second* path, choose a point (x, y_1). Using reasoning similar to that used for the first path, you can conclude that $f_y(x, y) = N$. Therefore,

$$\begin{aligned} \nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \\ &= M\mathbf{i} + N\mathbf{j} \\ &= \mathbf{F}(x, y) \end{aligned}$$

and it follows that **F** is conservative. ■

EXAMPLE 4 Finding Work in a Conservative Force Field

For the force field given by

$$\mathbf{F}(x, y, z) = e^x \cos y\mathbf{i} - e^x \sin y\mathbf{j} + 2\mathbf{k}$$

show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, and calculate the work done by $\mathbf{F}$ on an object moving along a curve C from $(0, \pi/2, 1)$ to $(1, \pi, 3)$.

Solution Writing the force field in the form $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, you have $M = e^x \cos y$, $N = -e^x \sin y$, and $P = 2$, and it follows that

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$$

$$\frac{\partial P}{\partial x} = 0 = \frac{\partial M}{\partial z}$$

$$\frac{\partial N}{\partial x} = -e^x \sin y = \frac{\partial M}{\partial y}.$$

So, $\mathbf{F}$ is conservative. If f is a potential function of $\mathbf{F}$, then

$$f_x(x, y, z) = e^x \cos y$$

$$f_y(x, y, z) = -e^x \sin y$$

$$f_z(x, y, z) = 2.$$

By integrating with respect to x, y, and z separately, you obtain

$$f(x, y, z) = \int f_x(x, y, z)\,dx = \int e^x \cos y\,dx = e^x \cos y + g(y, z)$$

$$f(x, y, z) = \int f_y(x, y, z)\,dy = \int -e^x \sin y\,dy = e^x \cos y + h(x, z)$$

$$f(x, y, z) = \int f_z(x, y, z)\,dz = \int 2\,dz = 2z + k(x, y).$$

By comparing these three versions of $f(x, y, z)$, you can conclude that

$$f(x, y, z) = e^x \cos y + 2z + K.$$

Therefore, the work done by $\mathbf{F}$ along *any* curve C from $(0, \pi/2, 1)$ to $(1, \pi, 3)$ is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \Big[e^x \cos y + 2z\Big]_{(0, \pi/2, 1)}^{(1, \pi, 3)} \\ &= (-e + 6) - (0 + 2) \\ &= 4 - e. \end{aligned}$$

■

How much work would be done if the object in Example 4 moved from the point $(0, \pi/2, 1)$ to $(1, \pi, 3)$ and then back to the starting point $(0, \pi/2, 1)$? The Fundamental Theorem of Line Integrals states that there is zero work done. Remember that, by definition, work can be negative. So, by the time the object gets back to its starting point, the amount of work that registers positively is canceled out by the amount of work that registers negatively.

A curve C given by $\mathbf{r}(t)$ for $a \le t \le b$ is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$. By the Fundamental Theorem of Line Integrals, you can conclude that if $\mathbf{F}$ is continuous and conservative on an open region R, then the line integral over every closed curve C is 0.

NOTE Theorem 15.7 gives you options for evaluating a line integral involving a conservative vector field. You can use a potential function, or it might be more convenient to choose a particularly simple path, such as a straight line.

THEOREM 15.7 EQUIVALENT CONDITIONS

Let $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ have continuous first partial derivatives in an open connected region R, and let C be a piecewise smooth curve in R. The following conditions are equivalent.

1. $\mathbf{F}$ is conservative. That is, $\mathbf{F} = \nabla f$ for some function f.
2. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.
3. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every *closed* curve C in R.

EXAMPLE 5 Evaluating a Line Integral

Evaluate $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y) = (y^3 + 1)\mathbf{i} + (3xy^2 + 1)\mathbf{j}$$

and C_1 is the semicircular path from (0, 0) to (2, 0), as shown in Figure 15.24.

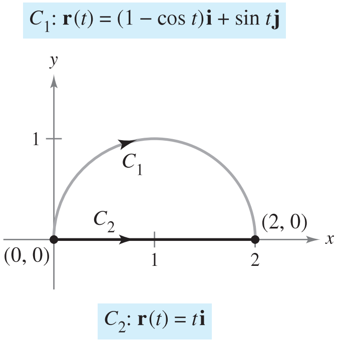

C_1: $\mathbf{r}(t) = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j}$
y
1
C_1
C_2
(2, 0)
x
(0, 0)
1
2
C_2: $\mathbf{r}(t) = t\mathbf{i}$

Figure 15.24

Solution You have the following three options.

a. You can use the method presented in the preceding section to evaluate the line integral along the *given curve*. To do this, you can use the parametrization $\mathbf{r}(t) = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j}$, where $0 \le t \le \pi$. For this parametrization, it follows that $d\mathbf{r} = \mathbf{r}'(t)\,dt = (\sin t\mathbf{i} + \cos t\mathbf{j})\,dt$, and

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} (\sin t + \sin^4 t + \cos t + 3\sin^2 t \cos t - 3\sin^2 t \cos^2 t)\,dt.$$

This integral should dampen your enthusiasm for this option.

b. You can try to find a *potential function* and evaluate the line integral by the Fundamental Theorem of Line Integrals. Using the technique demonstrated in Example 4, you can find the potential function to be $f(x, y) = xy^3 + x + y + K$, and, by the Fundamental Theorem,

$$W = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = f(2, 0) - f(0, 0) = 2.$$

c. Knowing that $\mathbf{F}$ is conservative, you have a third option. Because the value of the line integral is independent of path, you can replace the semicircular path with a *simpler path*. Suppose you choose the straight-line path C_2 from (0, 0) to (2, 0). Then, $\mathbf{r}(t) = t\mathbf{i}$, where $0 \le t \le 2$. So, $d\mathbf{r} = \mathbf{i}\,dt$ and $\mathbf{F}(x, y) = (y^3 + 1)\mathbf{i} + (3xy^2 + 1)\mathbf{j} = \mathbf{i} + \mathbf{j}$, so that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 1\,dt = t\Big]_0^2 = 2.$$

Of the three options, obviously the third one is the easiest. ■

Conservation of Energy

The Granger Collection

MICHAEL FARADAY (1791–1867)

Several philosophers of science have considered Faraday's Law of Conservation of Energy to be the greatest generalization ever conceived by humankind. Many physicists have contributed to our knowledge of this law. Two early and influential ones were James Prescott Joule (1818–1889) and Hermann Ludwig Helmholtz (1821–1894).

In 1840, the English physicist Michael Faraday wrote, "Nowhere is there a pure creation or production of power without a corresponding exhaustion of something to supply it." This statement represents the first formulation of one of the most important laws of physics—**the Law of Conservation of Energy.** In modern terminology, the law is stated as follows: *In a conservative force field, the sum of the potential and kinetic energies of an object remains constant from point to point.*

You can use the Fundamental Theorem of Line Integrals to derive this law. From physics, the **kinetic energy** of a particle of mass m and speed v is $k = \frac{1}{2}mv^2$. The **potential energy** p of a particle at point (x, y, z) in a conservative vector field $\mathbf{F}$ is defined as $p(x, y, z) = -f(x, y, z)$, where f is the potential function for $\mathbf{F}$. Consequently, the work done by $\mathbf{F}$ along a smooth curve C from A to B is

$$\begin{aligned} W = \int_C \mathbf{F} \cdot d\mathbf{r} &= f(x, y, z)\Big]_A^B \\ &= -p(x, y, z)\Big]_A^B \\ &= p(A) - p(B) \end{aligned}$$

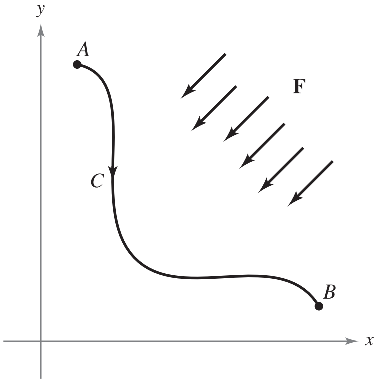
as shown in Figure 15.25. In other words, work W is equal to the difference in the potential energies of A and B. Now, suppose that $\mathbf{r}(t)$ is the position vector for a particle moving along C from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$. At any time t, the particle's velocity, acceleration, and speed are $\mathbf{v}(t) = \mathbf{r}'(t)$, $\mathbf{a}(t) = \mathbf{r}''(t)$, and $v(t) = \|\mathbf{v}(t)\|$, respectively. So, by Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}(t) = m(\mathbf{v}'(t))$, and the work done by $\mathbf{F}$ is

$$\begin{aligned} W = \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt \\ &= \int_a^b \mathbf{F} \cdot \mathbf{v}(t)\, dt = \int_a^b [m\mathbf{v}'(t)] \cdot \mathbf{v}(t)\, dt \\ &= \int_a^b m[\mathbf{v}'(t) \cdot \mathbf{v}(t)]\, dt \\ &= \frac{m}{2}\int_a^b \frac{d}{dt}[\mathbf{v}(t) \cdot \mathbf{v}(t)]\, dt \\ &= \frac{m}{2}\int_a^b \frac{d}{dt}[\|\mathbf{v}(t)\|^2]\, dt \\ &= \frac{m}{2}\Big[\|\mathbf{v}(t)\|^2\Big]_a^b \\ &= \frac{m}{2}\Big[[v(t)]^2\Big]_a^b \\ &= \frac{1}{2}m[v(b)]^2 - \frac{1}{2}m[v(a)]^2 \\ &= k(B) - k(A). \end{aligned}$$

Equating these two results for W produces

$$\begin{aligned} p(A) - p(B) &= k(B) - k(A) \\ p(A) + k(A) &= p(B) + k(B) \end{aligned}$$

which implies that the sum of the potential and kinetic energies remains constant from point to point.



The work done by $\mathbf{F}$ along C is

$W = \int_C \mathbf{F} \cdot d\mathbf{r} = p(A) - p(B).$

Figure 15.25

15.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, show that the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the same for each parametric representation of C.

1. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$

(b) $\mathbf{r}_2(\theta) = \sin\theta\mathbf{i} + \sin^2\theta\mathbf{j}, \quad 0 \le \theta \le \dfrac{\pi}{2}$

2. $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + \sqrt{t}\,\mathbf{j}, \quad 0 \le t \le 4$

(b) $\mathbf{r}_2(w) = w^2\mathbf{i} + w\mathbf{j}, \quad 0 \le w \le 2$

3. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(\theta) = \sec\theta\mathbf{i} + \tan\theta\mathbf{j}, \quad 0 \le \theta \le \dfrac{\pi}{3}$

(b) $\mathbf{r}_2(t) = \sqrt{t+1}\,\mathbf{i} + \sqrt{t}\,\mathbf{j}, \quad 0 \le t \le 3$

4. $\mathbf{F}(x, y) = y\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = (2 + t)\mathbf{i} + (3 - t)\mathbf{j}, \quad 0 \le t \le 3$

(b) $\mathbf{r}_2(w) = (2 + \ln w)\mathbf{i} + (3 - \ln w)\mathbf{j}, \quad 1 \le w \le e^3$

In Exercises 5–10, determine whether or not the vector field is conservative.

5. $\mathbf{F}(x, y) = e^x(\sin y\mathbf{i} + \cos y\mathbf{j})$

6. $\mathbf{F}(x, y) = 15x^2y^2\mathbf{i} + 10x^3y\mathbf{j}$

7. $\mathbf{F}(x, y) = \dfrac{1}{y^2}(y\mathbf{i} + x\mathbf{j})$

8. $\mathbf{F}(x, y, z) = y\ln z\mathbf{i} - x\ln z\mathbf{j} + \dfrac{xy}{z}\mathbf{k}$

9. $\mathbf{F}(x, y, z) = y^2z\mathbf{i} + 2xyz\mathbf{j} + xy^2\mathbf{k}$

10. $\mathbf{F}(x, y, z) = \sin yz\mathbf{i} + xz\cos yz\mathbf{j} + xy\sin yz\mathbf{k}$

In Exercises 11–24, find the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(*Hint:* If F is conservative, the integration may be easier on an alternative path.)

11. $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^3\mathbf{j}, \quad 0 \le t \le 1$

12. $\mathbf{F}(x, y) = ye^{xy}\mathbf{i} + xe^{xy}\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} - (t - 3)\mathbf{j}, \quad 0 \le t \le 3$

(b) The closed path consisting of line segments from $(0, 3)$ to $(0, 0)$, from $(0, 0)$ to $(3, 0)$, and then from $(3, 0)$ to $(0, 3)$

13. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$

(c) $\mathbf{r}_3(t) = t\mathbf{i} + t^3\mathbf{j}, \quad 0 \le t \le 1$

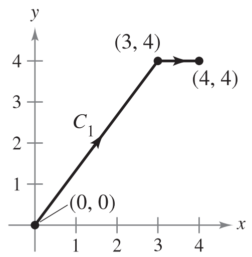
14. $\mathbf{F}(x, y) = xy^2\mathbf{i} + 2x^2y\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j}, \quad 1 \le t \le 3$

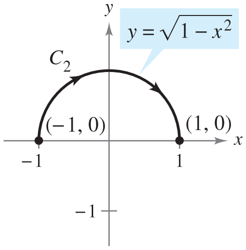
(b) $\mathbf{r}_2(t) = (t + 1)\mathbf{i} - \frac{1}{3}(t - 3)\mathbf{j}, \quad 0 \le t \le 2$

15. $\displaystyle\int_C y^2\,dx + 2xy\,dy$

(a)



(b)



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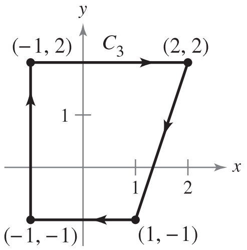
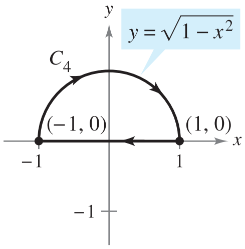
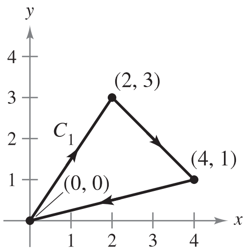


(d)



16. $\displaystyle\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy$

(a)



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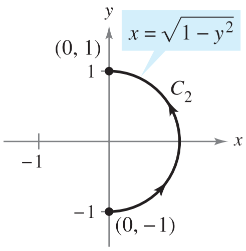

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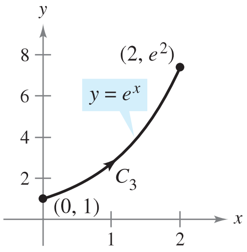
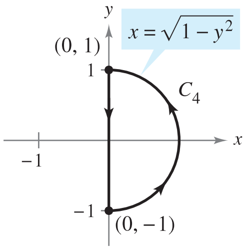


(d)



17. $\displaystyle\int_C 2xy\,dx + (x^2 + y^2)\,dy$

(a) C: ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ from $(5, 0)$ to $(0, 4)$

(b) C: parabola $y = 4 - x^2$ from $(2, 0)$ to $(0, 4)$

18. $\displaystyle\int_C (x^2 + y^2)\,dx + 2xy\,dy$

(a) $\mathbf{r}_1(t) = t^3\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 2$

(b) $\mathbf{r}_2(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad 0 \le t \le \dfrac{\pi}{2}$

19. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 4$

(b) $\mathbf{r}_2(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, \quad 0 \le t \le 2$

20. $\mathbf{F}(x, y, z) = \mathbf{i} + z\mathbf{j} + y\mathbf{k}$

(a) $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t^2\mathbf{k}, \quad 0 \le t \le \pi$

(b) $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi^2 t\mathbf{k}, \quad 0 \le t \le 1$

21. $\mathbf{F}(x, y, z) = (2y + x)\mathbf{i} + (x^2 - z)\mathbf{j} + (2y - 4z)\mathbf{k}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t\mathbf{j} + (2t - 1)^2\mathbf{k}, \quad 0 \le t \le 1$

22. $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + 3xz^2\mathbf{k}$

(a) $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le \pi$

(b) $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi t\mathbf{k}, \quad 0 \le t \le 1$

23. $\mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})$

(a) $\mathbf{r}_1(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + 3\mathbf{k}, \quad 0 \le t \le \pi$

(b) $\mathbf{r}_2(t) = (4 - 8t)\mathbf{i} + 3\mathbf{k}, \quad 0 \le t \le 1$

24. $\mathbf{F}(x, y, z) = y\sin z\,\mathbf{i} + x\sin z\,\mathbf{j} + xy\cos x\mathbf{k}$

(a) $\mathbf{r}_1(t) = t^2\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 2$

(b) $\mathbf{r}_2(t) = 4t\mathbf{i} + 4t\mathbf{j}, \quad 0 \le t \le 1$

In Exercises 25–34, evaluate the line integral using the Fundamental Theorem of Line Integrals. Use a computer algebra system to verify your results.

25. $\displaystyle\int_C (3y\mathbf{i} + 3x\mathbf{j}) \cdot d\mathbf{r}$

C: smooth curve from $(0, 0)$ to $(3, 8)$

26. $\displaystyle\int_C [2(x + y)\mathbf{i} + 2(x + y)\mathbf{j}] \cdot d\mathbf{r}$

C: smooth curve from $(-1, 1)$ to $(3, 2)$

27. $\displaystyle\int_C \cos x \sin y\,dx + \sin x \cos y\,dy$

C: line segment from $(0, -\pi)$ to $\left(\dfrac{3\pi}{2}, \dfrac{\pi}{2}\right)$

28. $\displaystyle\int_C \frac{y\,dx - x\,dy}{x^2 + y^2}$

C: line segment from $(1, 1)$ to $\left(2\sqrt{3}, 2\right)$

29. $\displaystyle\int_C e^x \sin y\,dx + e^x \cos y\,dy$

C: cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ from $(0, 0)$ to $(2\pi, 0)$

30. $\displaystyle\int_C \frac{2x}{(x^2 + y^2)^2}\,dx + \frac{2y}{(x^2 + y^2)^2}\,dy$

C: circle $(x - 4)^2 + (y - 5)^2 = 9$ clockwise from $(7, 5)$ to $(1, 5)$

31. $\displaystyle\int_C (z + 2y)\,dx + (2x - z)\,dy + (x - y)\,dz$

(a) C: line segment from $(0, 0, 0)$ to $(1, 1, 1)$

(b) C: line segments from $(0, 0, 0)$ to $(0, 0, 1)$ to $(1, 1, 1)$

(c) C: line segments from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 1, 0)$ to $(1, 1, 1)$

32. Repeat Exercise 31 using the integral

$$\int_C zy\,dx + xz\,dy + xy\,dz.$$

33. $\displaystyle\int_C -\sin x\,dx + z\,dy + y\,dz$

C: smooth curve from $(0, 0, 0)$ to $\left(\dfrac{\pi}{2}, 3, 4\right)$

34. $\displaystyle\int_C 6x\,dx - 4z\,dy - (4y - 20z)\,dz$

C: smooth curve from $(0, 0, 0)$ to $(3, 4, 0)$

Work **In Exercises 35 and 36, find the work done by the force field F in moving an object from *P* to *Q*.**

35. $\mathbf{F}(x, y) = 9x^2y^2\mathbf{i} + (6x^3y - 1)\mathbf{j};\ P(0, 0),\ Q(5, 9)$

36. $\mathbf{F}(x, y) = \dfrac{2x}{y}\mathbf{i} - \dfrac{x^2}{y^2}\mathbf{j};\ P(-1, 1),\ Q(3, 2)$

37. ***Work*** A stone weighing 1 pound is attached to the end of a two-foot string and is whirled horizontally with one end held fixed. It makes 1 revolution per second. Find the work done by the force **F** that keeps the stone moving in a circular path. [*Hint:* Use Force = (mass)(centripetal acceleration).]

38. ***Work*** If $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is a constant force vector field, show that the work done in moving a particle along any path from P to Q is $W = \mathbf{F} \cdot \overrightarrow{PQ}$.

39. ***Work*** To allow a means of escape for workers in a hazardous job 50 meters above ground level, a slide wire is installed. It runs from their position to a point on the ground 50 meters from the base of the installation where they are located. Show that the work done by the gravitational force field for a 175-pound worker moving the length of the slide wire is the same for each path.

(a) $\mathbf{r}(t) = t\mathbf{i} + (50 - t)\mathbf{j}$

(b) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{50}(50 - t)^2\mathbf{j}$

40. ***Work*** Can you find a path for the slide wire in Exercise 39 such that the work done by the gravitational force field would differ from the amounts of work done for the two paths given? Explain why or why not.

WRITING ABOUT CONCEPTS

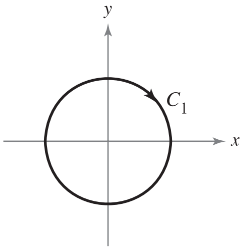
41. State the Fundamental Theorem of Line Integrals.

42. What does it mean that a line integral is independent of path? State the method for determining if a line integral is independent of path.

43. ***Think About It*** Let $\mathbf{F}(x, y) = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$. Find the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(a)



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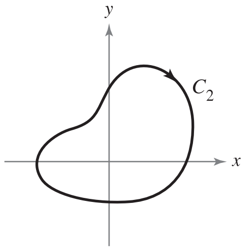

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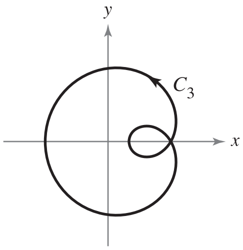

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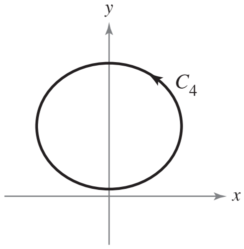
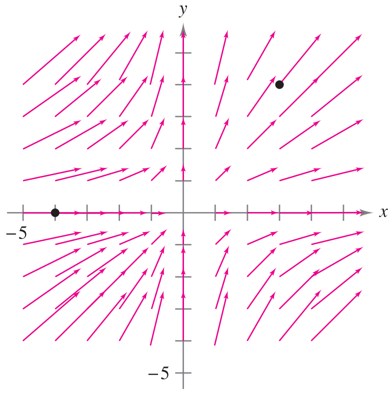


CAPSTONE

44. Consider the force field shown in the figure.



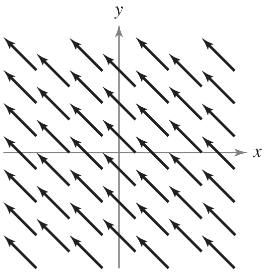
(a) Give a verbal argument that the force field is not conservative because you can identify two paths that require different amounts of work to move an object from $(-4, 0)$ to $(3, 4)$. Identify two paths and state which requires the greater amount of work. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(b) Give a verbal argument that the force field is not conservative because you can find a closed curve C such that

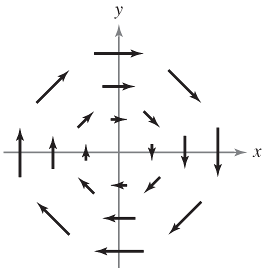
$$\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0.$$

In Exercises 45 and 46, consider the force field shown in the figure. Is the force field conservative? Explain why or why not.

45.



46.



True or False? **In Exercises 47–50, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

47. If C_1, C_2, and C_3 have the same initial and terminal points and $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2$, then $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = \int_{C_3} \mathbf{F} \cdot d\mathbf{r}_3$.

48. If $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$ and C is given by $\mathbf{r}(t) = (4 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j}$, $0 \leq t \leq \pi$, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

49. If $\mathbf{F}$ is conservative in a region R bounded by a simple closed path and C lies within R, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.

50. If $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ and $\partial M/\partial x = \partial N/\partial y$, then $\mathbf{F}$ is conservative.

51. A function f is called *harmonic* if $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. Prove that if f is harmonic, then

$$\int_C \left(\frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy\right) = 0$$

where C is a smooth closed curve in the plane.

52. ***Kinetic and Potential Energy*** The kinetic energy of an object moving through a conservative force field is decreasing at a rate of 15 units per minute. At what rate is the potential energy changing?

53. Let $\mathbf{F}(x, y) = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$.

(a) Show that

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

where

$$M = \frac{y}{x^2 + y^2} \text{ and } N = \frac{-x}{x^2 + y^2}.$$

(b) If $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ for $0 \leq t \leq \pi$, find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

(c) If $\mathbf{r}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j}$ for $0 \leq t \leq \pi$, find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

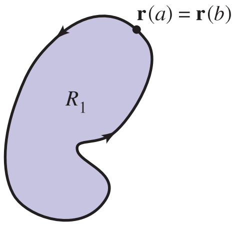
(d) If $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ for $0 \leq t \leq 2\pi$, find $\int_C \mathbf{F} \cdot d\mathbf{r}$. Why doesn't this contradict Theorem 15.7?

(e) Show that $\nabla\left(\arctan \dfrac{x}{y}\right) = \mathbf{F}$.

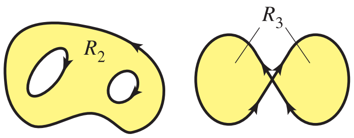
15.4 Green's Theorem

- **Use Green's Theorem to evaluate a line integral.**
- **Use alternative forms of Green's Theorem.**

Green's Theorem



Simply connected



Not simply connected

Figure 15.26

In this section, you will study **Green's Theorem,** named after the English mathematician George Green (1793–1841). This theorem states that the value of a double integral over a *simply connected* plane region R is determined by the value of a line integral around the boundary of R.

A curve C given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where $a \le t \le b$, is **simple** if it does not cross itself—that is, $\mathbf{r}(c) \ne \mathbf{r}(d)$ for all c and d in the open interval (a, b). A plane region R is **simply connected** if every simple closed curve in R encloses only points that are in R (see Figure 15.26).

THEOREM 15.8 GREEN'S THEOREM

Let R be a simply connected region with a piecewise smooth boundary C, oriented counterclockwise (that is, C is traversed *once* so that the region R always lies to the *left*). If M and N have continuous first partial derivatives in an open region containing R, then

$$\int_C M\,dx + N\,dy = \int_R\!\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

PROOF A proof is given only for a region that is both vertically simple and horizontally simple, as shown in Figure 15.27.

$$\begin{aligned}\int_C M\,dx &= \int_{C_1} M\,dx + \int_{C_2} M\,dx\\ &= \int_a^b M(x, f_1(x))\,dx + \int_b^a M(x, f_2(x))\,dx\\ &= \int_a^b [M(x, f_1(x)) - M(x, f_2(x))]\,dx\end{aligned}$$

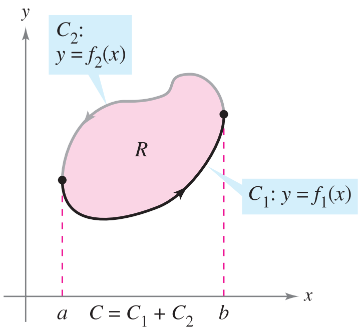
On the other hand,

$$\begin{aligned}\int_R\!\!\int \frac{\partial M}{\partial y}\,dA &= \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,dy\,dx\\ &= \int_a^b M(x, y)\Big]_{f_1(x)}^{f_2(x)}\,dx\\ &= \int_a^b [M(x, f_2(x)) - M(x, f_1(x))]\,dx.\end{aligned}$$

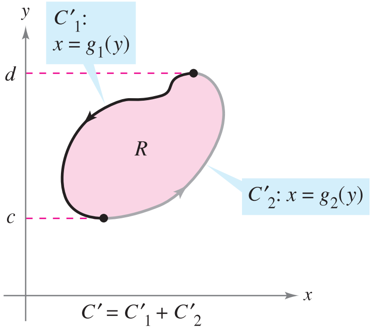
Consequently,

$$\int_C M\,dx = -\int_R\!\!\int \frac{\partial M}{\partial y}\,dA.$$

Similarly, you can use $g_1(y)$ and $g_2(y)$ to show that $\int_C N\,dy = \int_R\!\int \partial N/\partial x\,dA$. By adding the integrals $\int_C M\,dx$ and $\int_C N\,dy$, you obtain the conclusion stated in the theorem. ■



R is vertically simple.



R is horizontally simple.

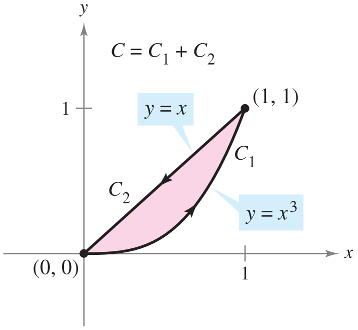
Figure 15.27

EXAMPLE 1 Using Green's Theorem

Use Green's Theorem to evaluate the line integral

$$\int_C y^3\,dx + (x^3 + 3xy^2)\,dy$$

where C is the path from $(0, 0)$ to $(1, 1)$ along the graph of $y = x^3$ and from $(1, 1)$ to $(0, 0)$ along the graph of $y = x$, as shown in Figure 15.28.



C is simple and closed, and the region R always lies to the left of C.
Figure 15.28

Solution Because $M = y^3$ and $N = x^3 + 3xy^2$, it follows that

$$\frac{\partial N}{\partial x} = 3x^2 + 3y^2 \quad \text{and} \quad \frac{\partial M}{\partial y} = 3y^2.$$

Applying Green's Theorem, you then have

$$\begin{aligned}
\int_C y^3\,dx + (x^3 + 3xy^2)\,dy &= \int_R\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\
&= \int_0^1 \int_{x^3}^{x} \left[(3x^2 + 3y^2) - 3y^2\right] dy\,dx \\
&= \int_0^1 \int_{x^3}^{x} 3x^2\,dy\,dx \\
&= \int_0^1 3x^2 y \Big]_{x^3}^{x}\,dx \\
&= \int_0^1 (3x^3 - 3x^5)\,dx \\
&= \left[\frac{3x^4}{4} - \frac{x^6}{2}\right]_0^1 \\
&= \frac{1}{4}.
\end{aligned}$$

■

GEORGE GREEN (1793–1841)

Green, a self-educated miller's son, first published the theorem that bears his name in 1828 in an essay on electricity and magnetism. At that time there was almost no mathematical theory to explain electrical phenomena. "Considering how desirable it was that a power of universal agency, like electricity, should, as far as possible, be submitted to calculation, . . . I was induced to try whether it would be possible to discover any general relations existing between this function and the quantities of electricity in the bodies producing it."

Green's Theorem cannot be applied to every line integral. Among other restrictions stated in Theorem 15.8, the curve C must be simple and closed. When Green's Theorem does apply, however, it can save time. To see this, try using the techniques described in Section 15.2 to evaluate the line integral in Example 1. To do this, you would need to write the line integral as

$$\int_C y^3\,dx + (x^3 + 3xy^2)\,dy =$$
$$\int_{C_1} y^3\,dx + (x^3 + 3xy^2)\,dy + \int_{C_2} y^3\,dx + (x^3 + 3xy^2)\,dy$$

where C_1 is the cubic path given by

$$\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}$$

from $t = 0$ to $t = 1$, and C_2 is the line segment given by

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + (1 - t)\mathbf{j}$$

from $t = 0$ to $t = 1$.

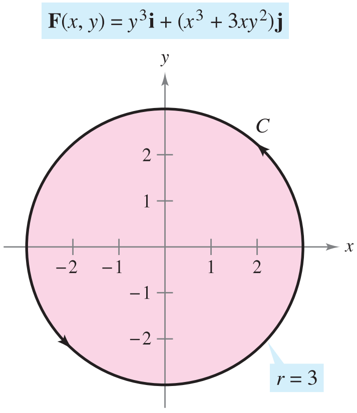

$\mathbf{F}(x, y) = y^3\mathbf{i} + (x^3 + 3xy^2)\mathbf{j}$
y
C
2
1
−2
−1
1
2
x
−1
−2
r = 3

Figure 15.29

EXAMPLE 2 Using Green's Theorem to Calculate Work

While subject to the force

$$\mathbf{F}(x, y) = y^3\mathbf{i} + (x^3 + 3xy^2)\mathbf{j}$$

a particle travels once around the circle of radius 3 shown in Figure 15.29. Use Green's Theorem to find the work done by **F**.

Solution From Example 1, you know by Green's Theorem that

$$\int_C y^3\,dx + (x^3 + 3xy^2)\,dy = \int\!\!\int_R 3x^2\,dA.$$

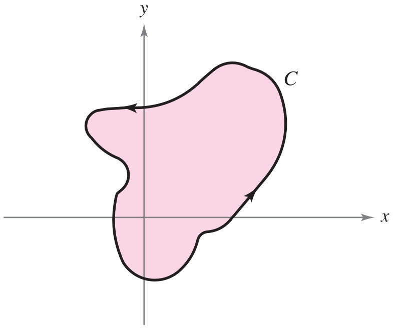
In polar coordinates, using $x = r\cos\theta$ and $dA = r\,dr\,d\theta$, the work done is

$$\begin{aligned}
W = \int\!\!\int_R 3x^2\,dA &= \int_0^{2\pi}\int_0^3 3(r\cos\theta)^2\,r\,dr\,d\theta \\
&= 3\int_0^{2\pi}\int_0^3 r^3\cos^2\theta\,dr\,d\theta \\
&= 3\int_0^{2\pi}\frac{r^4}{4}\cos^2\theta\Big]_0^3\,d\theta \\
&= 3\int_0^{2\pi}\frac{81}{4}\cos^2\theta\,d\theta \\
&= \frac{243}{8}\int_0^{2\pi}(1 + \cos 2\theta)\,d\theta \\
&= \frac{243}{8}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} \\
&= \frac{243\pi}{4}.
\end{aligned}$$

■

When evaluating line integrals over closed curves, remember that for conservative vector fields (those for which $\partial N/\partial x = \partial M/\partial y$), the value of the line integral is 0. This is easily seen from the statement of Green's Theorem:

$$\int_C M\,dx + N\,dy = \int\!\!\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0.$$



C is closed.
Figure 15.30

EXAMPLE 3 Green's Theorem and Conservative Vector Fields

Evaluate the line integral

$$\int_C y^3\,dx + 3xy^2\,dy$$

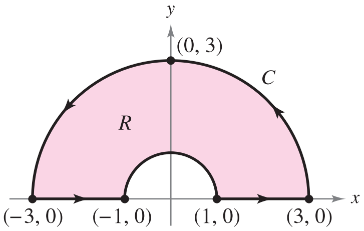
where C is the path shown in Figure 15.30.

Solution From this line integral, $M = y^3$ and $N = 3xy^2$. So, $\partial N/\partial x = 3y^2$ and $\partial M/\partial y = 3y^2$. This implies that the vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is conservative, and because C is closed, you can conclude that

$$\int_C y^3\,dx + 3xy^2\,dy = 0.$$

■

EXAMPLE 4 Using Green's Theorem for a Piecewise Smooth Curve



C is piecewise smooth.
Figure 15.31

Evaluate

$$\int_C (\arctan x + y^2)\,dx + (e^y - x^2)\,dy$$

where C is the path enclosing the annular region shown in Figure 15.31.

Solution In polar coordinates, R is given by $1 \le r \le 3$ for $0 \le \theta \le \pi$. Moreover,

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -2x - 2y = -2(r\cos\theta + r\sin\theta).$$

So, by Green's Theorem,

$$\begin{aligned}
\int_C (\arctan x + y^2)\,dx + (e^y - x^2)\,dy &= \int_R\!\!\int -2(x+y)\,dA \\
&= \int_0^{\pi}\!\!\int_1^3 -2r(\cos\theta + \sin\theta)r\,dr\,d\theta \\
&= \int_0^{\pi} -2(\cos\theta + \sin\theta)\frac{r^3}{3}\bigg]_1^3\,d\theta \\
&= \int_0^{\pi}\left(-\frac{52}{3}\right)(\cos\theta + \sin\theta)\,d\theta \\
&= -\frac{52}{3}\Big[\sin\theta - \cos\theta\Big]_0^{\pi} \\
&= -\frac{104}{3}.
\end{aligned}$$

In Examples 1, 2, and 4, Green's Theorem was used to evaluate line integrals as double integrals. You can also use the theorem to evaluate double integrals as line integrals. One useful application occurs when $\partial N/\partial x - \partial M/\partial y = 1$.

$$\begin{aligned}
\int_C M\,dx + N\,dy &= \int_R\!\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\
&= \int_R\!\!\int 1\,dA \qquad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 \\
&= \text{area of region } R
\end{aligned}$$

Among the many choices for M and N satisfying the stated condition, the choice of $M = -y/2$ and $N = x/2$ produces the following line integral for the area of region R.

THEOREM 15.9 LINE INTEGRAL FOR AREA

If R is a plane region bounded by a piecewise smooth simple closed curve C, oriented counterclockwise, then the area of R is given by

$$A = \frac{1}{2}\int_C x\,dy - y\,dx.$$

EXAMPLE 5 Finding Area by a Line Integral

Use a line integral to find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

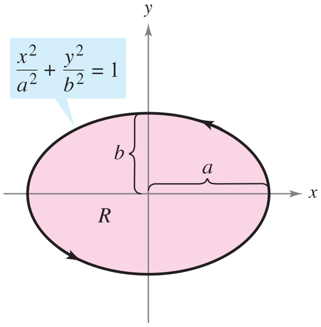


Figure 15.32

Solution Using Figure 15.32, you can induce a counterclockwise orientation to the elliptical path by letting

$$x = a \cos t \quad \text{and} \quad y = b \sin t, \quad 0 \le t \le 2\pi.$$

So, the area is

$$\begin{aligned} A = \frac{1}{2}\int_C x\,dy - y\,dx &= \frac{1}{2}\int_0^{2\pi} [(a \cos t)(b \cos t)\,dt - (b \sin t)(-a \sin t)\,dt] \\ &= \frac{ab}{2}\int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt \\ &= \frac{ab}{2}\Big[t\Big]_0^{2\pi} \\ &= \pi ab. \end{aligned}$$

Green's Theorem can be extended to cover some regions that are not simply connected. This is demonstrated in the next example.

EXAMPLE 6 Green's Theorem Extended to a Region with a Hole

Let R be the region inside the ellipse $(x^2/9) + (y^2/4) = 1$ and outside the circle $x^2 + y^2 = 1$. Evaluate the line integral

$$\int_C 2xy\,dx + (x^2 + 2x)\,dy$$

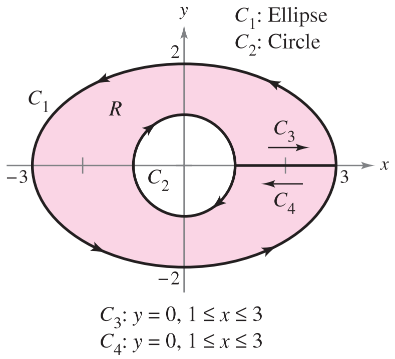


Figure 15.33

where $C = C_1 + C_2$ is the boundary of R, as shown in Figure 15.33.

Solution To begin, you can introduce the line segments C_3 and C_4, as shown in Figure 15.33. Note that because the curves C_3 and C_4 have opposite orientations, the line integrals over them cancel. Furthermore, you can apply Green's Theorem to the region R using the boundary $C_1 + C_4 + C_2 + C_3$ to obtain

$$\begin{aligned} \int_C 2xy\,dx + (x^2 + 2x)\,dy &= \int\!\!\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\ &= \int\!\!\int_R (2x + 2 - 2x)\,dA \\ &= 2\int\!\!\int_R dA \\ &= 2(\text{area of } R) \\ &= 2(\pi ab - \pi r^2) \\ &= 2[\pi(3)(2) - \pi(1^2)] \\ &= 10\pi. \end{aligned}$$

In Section 15.1, a necessary and sufficient condition for conservative vector fields was listed. There, only one direction of the proof was shown. You can now outline the other direction, using Green's Theorem. Let $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ be defined on an open disk R. You want to show that if M and N have continuous first partial derivatives and

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

then $\mathbf{F}$ is conservative. Suppose that C is a closed path forming the boundary of a connected region lying in R. Then, using the fact that $\partial M/\partial y = \partial N/\partial x$, you can apply Green's Theorem to conclude that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C M\,dx + N\,dy \\ &= \int_R\!\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\ &= 0.\end{aligned}$$

This, in turn, is equivalent to showing that $\mathbf{F}$ is conservative (see Theorem 15.7).

Alternative Forms of Green's Theorem

This section concludes with the derivation of two vector forms of Green's Theorem for regions in the plane. The extension of these vector forms to three dimensions is the basis for the discussion in the remaining sections of this chapter. If $\mathbf{F}$ is a vector field in the plane, you can write

$$\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + 0\mathbf{k}$$

so that the curl of $\mathbf{F}$, as described in Section 15.1, is given by

$$\begin{aligned}\text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} \\ &= -\frac{\partial N}{\partial z}\mathbf{i} + \frac{\partial M}{\partial z}\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.\end{aligned}$$

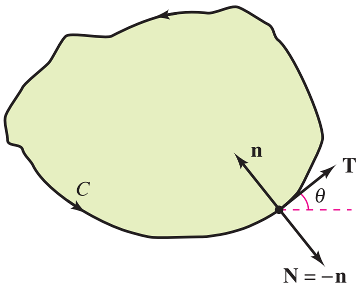
Consequently,

$$\begin{aligned}(\text{curl } \mathbf{F}) \cdot \mathbf{k} &= \left[-\frac{\partial N}{\partial z}\mathbf{i} + \frac{\partial M}{\partial z}\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}\right] \cdot \mathbf{k} \\ &= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.\end{aligned}$$

With appropriate conditions on $\mathbf{F}$, C, and R, you can write Green's Theorem in the vector form

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_R\!\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\ &= \int_R\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dA. \qquad \text{First alternative form}\end{aligned}$$

The extension of this vector form of Green's Theorem to surfaces in space produces **Stokes's Theorem,** discussed in Section 15.8.



$\mathbf{T} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$

$\mathbf{n} = \cos\left(\theta + \frac{\pi}{2}\right)\mathbf{i} + \sin\left(\theta + \frac{\pi}{2}\right)\mathbf{j}$

$\quad = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$

$\mathbf{N} = \sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j}$

Figure 15.34

For the second vector form of Green's Theorem, assume the same conditions for $\mathbf{F}$, C, and R. Using the arc length parameter s for C, you have $\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$. So, a unit tangent vector $\mathbf{T}$ to curve C is given by $\mathbf{r}'(s) = \mathbf{T} = x'(s)\mathbf{i} + y'(s)\mathbf{j}$. From Figure 15.34 you can see that the *outward* unit normal vector $\mathbf{N}$ can then be written as

$$\mathbf{N} = y'(s)\mathbf{i} - x'(s)\mathbf{j}.$$

Consequently, for $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$, you can apply Green's Theorem to obtain

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{N}\,ds &= \int_a^b (M\mathbf{i} + N\mathbf{j})\cdot(y'(s)\mathbf{i} - x'(s)\mathbf{j})\,ds\\ &= \int_a^b \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right)ds\\ &= \int_C M\,dy - N\,dx\\ &= \int_C -N\,dx + M\,dy\\ &= \int_R\!\!\int \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dA\\ &= \int_R\!\!\int \operatorname{div}\mathbf{F}\,dA.\end{aligned}$$

Green's Theorem

Therefore,

$$\int_C \mathbf{F}\cdot\mathbf{N}\,ds = \int_R\!\!\int \operatorname{div}\mathbf{F}\,dA.$$

Second alternative form

The extension of this form to three dimensions is called the **Divergence Theorem,** discussed in Section 15.7. The physical interpretations of divergence and curl will be discussed in Sections 15.7 and 15.8.

15.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, verify Green's Theorem by evaluating both integrals

$$\int_C y^2\,dx + x^2\,dy = \int_R\!\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$$

for the given path.

1. C: boundary of the region lying between the graphs of $y = x$ and $y = x^2$
2. C: boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$
3. C: square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$
4. C: rectangle with vertices $(0, 0)$, $(3, 0)$, $(3, 4)$, and $(0, 4)$

CAS **In Exercises 5 and 6, verify Green's Theorem by using a computer algebra system to evaluate both integrals**

$$\int_C xe^y\,dx + e^x\,dy = \int_R\!\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$$

for the given path.

5. C: circle given by $x^2 + y^2 = 4$
6. C: boundary of the region lying between the graphs of $y = x$ and $y = x^3$ in the first quadrant

In Exercises 7–10, use Green's Theorem to evaluate the integral

$$\int_C (y - x)\,dx + (2x - y)\,dy$$

for the given path.

7. C: boundary of the region lying between the graphs of $y = x$ and $y = x^2 - 2x$
8. C: $x = 2\cos\theta$, $y = \sin\theta$
9. C: boundary of the region lying inside the rectangle bounded by $x = -5$, $x = 5$, $y = -3$, and $y = 3$, and outside the square bounded by $x = -1$, $x = 1$, $y = -1$, and $y = 1$
10. C: boundary of the region lying inside the semicircle $y = \sqrt{25 - x^2}$ and outside the semicircle $y = \sqrt{9 - x^2}$

In Exercises 11–20, use Green's Theorem to evaluate the line integral.

11. $\int_C 2xy\,dx + (x + y)\,dy$

C: boundary of the region lying between the graphs of $y = 0$ and $y = 1 - x^2$

12. $\int_C y^2\,dx + xy\,dy$

C: boundary of the region lying between the graphs of $y = 0$, $y = \sqrt{x}$, and $x = 9$

13. $\int_C (x^2 - y^2)\,dx + 2xy\,dy$

C: $x^2 + y^2 = 16$

14. $\int_C (x^2 - y^2)\,dx + 2xy\,dy$

C: $r = 1 + \cos\theta$

15. $\int_C e^x \cos 2y\,dx - 2e^x \sin 2y\,dy$

C: $x^2 + y^2 = a^2$

16. $\int_C 2\arctan\frac{y}{x}\,dx + \ln(x^2 + y^2)\,dy$

C: $x = 4 + 2\cos\theta$, $y = 4 + \sin\theta$

17. $\int_C \cos y\,dx + (xy - x\sin y)\,dy$

C: boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$

18. $\int_C (e^{-x^2/2} - y)\,dx + (e^{-y^2/2} + x)\,dy$

C: boundary of the region lying between the graphs of the circle $x = 6\cos\theta$, $y = 6\sin\theta$ and the ellipse $x = 3\cos\theta$, $y = 2\sin\theta$

19. $\int_C (x - 3y)\,dx + (x + y)\,dy$

C: boundary of the region lying between the graphs of $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$

20. $\int_C 3x^2 e^y\,dx + e^y\,dy$

C: boundary of the region lying between the squares with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$, and $(2, 2)$, $(-2, 2)$, $(-2, -2)$, and $(2, -2)$

Work **In Exercises 21–24, use Green's Theorem to calculate the work done by the force F on a particle that is moving counterclockwise around the closed path *C*.**

21. $\mathbf{F}(x, y) = xy\mathbf{i} + (x + y)\mathbf{j}$

C: $x^2 + y^2 = 1$

22. $\mathbf{F}(x, y) = (e^x - 3y)\mathbf{i} + (e^y + 6x)\mathbf{j}$

C: $r = 2\cos\theta$

23. $\mathbf{F}(x, y) = (x^{3/2} - 3y)\mathbf{i} + \left(6x + 5\sqrt{y}\right)\mathbf{j}$

C: boundary of the triangle with vertices $(0, 0)$, $(5, 0)$, and $(0, 5)$

24. $\mathbf{F}(x, y) = (3x^2 + y)\mathbf{i} + 4xy^2\,\mathbf{j}$

C: boundary of the region lying between the graphs of $y = \sqrt{x}$, $y = 0$, and $x = 9$

Area **In Exercises 25–28, use a line integral to find the area of the region *R*.**

25. R: region bounded by the graph of $x^2 + y^2 = a^2$

26. R: triangle bounded by the graphs of $x = 0$, $3x - 2y = 0$, and $x + 2y = 8$

27. R: region bounded by the graphs of $y = 5x - 3$ and $y = x^2 + 1$

28. R: region inside the loop of the folium of Descartes bounded by the graph of

$$x = \frac{3t}{t^3 + 1}, \quad y = \frac{3t^2}{t^3 + 1}$$

WRITING ABOUT CONCEPTS

29. State Green's Theorem.

30. Give the line integral for the area of a region R bounded by a piecewise smooth simple curve C.

In Exercises 31 and 32, use Green's Theorem to verify the line integral formulas.

31. The centroid of the region having area A bounded by the simple closed path C is

$$\bar{x} = \frac{1}{2A}\int_C x^2\,dy, \quad \bar{y} = -\frac{1}{2A}\int_C y^2\,dx.$$

32. The area of a plane region bounded by the simple closed path C given in polar coordinates is $A = \frac{1}{2}\int_C r^2\,d\theta$.

CAS ***Centroid*** **In Exercises 33–36, use a computer algebra system and the results of Exercise 31 to find the centroid of the region.**

33. R: region bounded by the graphs of $y = 0$ and $y = 4 - x^2$

34. R: region bounded by the graphs of $y = \sqrt{a^2 - x^2}$ and $y = 0$

35. R: region bounded by the graphs of $y = x^3$ and $y = x$, $0 \le x \le 1$

36. R: triangle with vertices $(-a, 0)$, $(a, 0)$, and (b, c), where $-a \le b \le a$

CAS ***Area*** **In Exercises 37–40, use a computer algebra system and the results of Exercise 32 to find the area of the region bounded by the graph of the polar equation.**

37. $r = a(1 - \cos\theta)$

38. $r = a\cos 3\theta$

39. $r = 1 + 2\cos\theta$ (inner loop)

40. $r = \dfrac{3}{2 - \cos\theta}$

41. (a) Evaluate $\int_{C_1} y^3\,dx + (27x - x^3)\,dy$, where C_1 is the unit circle given by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$.

(b) Find the maximum value of $\int_C y^3\,dx + (27x - x^3)\,dy$, where C is any closed curve in the xy-plane, oriented counterclockwise.

CAPSTONE

42. For each given path, verify Green's Theorem by showing that

$$\int_C y^2\,dx + x^2\,dy = \int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA.$$

For each path, which integral is easier to evaluate? Explain.

(a) C: triangle with vertices $(0, 0)$, $(4, 0)$, $(4, 4)$

(b) C: circle given by $x^2 + y^2 = 1$

43. ***Think About It*** Let

$$I = \int_C \frac{y\,dx - x\,dy}{x^2 + y^2}$$

where C is a circle oriented counterclockwise. Show that $I = 0$ if C does not contain the origin. What is I if C does contain the origin?

44. (a) Let C be the line segment joining (x_1, y_1) and (x_2, y_2). Show that $\int_C -y\,dx + x\,dy = x_1 y_2 - x_2 y_1$.

(b) Let (x_1, y_1), (x_2, y_2), . . . , (x_n, y_n) be the vertices of a polygon. Prove that the area enclosed is

$$\tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)].$$

Area **In Exercises 45 and 46, use the result of Exercise 44(b) to find the area enclosed by the polygon with the given vertices.**

45. Pentagon: $(0, 0)$, $(2, 0)$, $(3, 2)$, $(1, 4)$, $(-1, 1)$

46. Hexagon: $(0, 0)$, $(2, 0)$, $(3, 2)$, $(2, 4)$, $(0, 3)$, $(-1, 1)$

In Exercises 47 and 48, prove the identity where R is a simply connected region with boundary C. Assume that the required partial derivatives of the scalar functions f and g are continuous. The expressions $D_{\mathbf{N}}f$ and $D_{\mathbf{N}}g$ are the derivatives in the direction of the outward normal vector N of C, and are defined by $D_{\mathbf{N}}f = \nabla f \cdot \mathbf{N}$, and $D_{\mathbf{N}}g = \nabla g \cdot \mathbf{N}$.

47. Green's first identity:

$$\int_R\int (f\nabla^2 g + \nabla f \cdot \nabla g)\,dA = \int_C f D_{\mathbf{N}}g\,ds$$

[*Hint:* Use the second alternative form of Green's Theorem and the property $\operatorname{div}(f\mathbf{G}) = f\operatorname{div}\mathbf{G} + \nabla f \cdot \mathbf{G}$.]

48. Green's second identity:

$$\int_R\int (f\nabla^2 g - g\nabla^2 f)\,dA = \int_C (fD_{\mathbf{N}}g - g\,D_{\mathbf{N}}f)\,ds$$

(*Hint:* Use Green's first identity, given in Exercise 47, twice.)

49. Use Green's Theorem to prove that

$$\int_C f(x)\,dx + g(y)\,dy = 0$$

if f and g are differentiable functions and C is a piecewise smooth simple closed path.

50. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, where M and N have continuous first partial derivatives in a simply connected region R. Prove that if C is simple, smooth, and closed, and $N_x = M_y$, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 0.$$

SECTION PROJECT

Hyperbolic and Trigonometric Functions

(a) Sketch the plane curve represented by the vector-valued function $\mathbf{r}(t) = \cosh t\,\mathbf{i} + \sinh t\,\mathbf{j}$ on the interval $0 \le t \le 5$. Show that the rectangular equation corresponding to $\mathbf{r}(t)$ is the hyperbola $x^2 - y^2 = 1$. Verify your sketch by using a graphing utility to graph the hyperbola.

(b) Let $P = (\cosh\phi, \sinh\phi)$ be the point on the hyperbola corresponding to $\mathbf{r}(\phi)$ for $\phi > 0$. Use the formula for area

$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

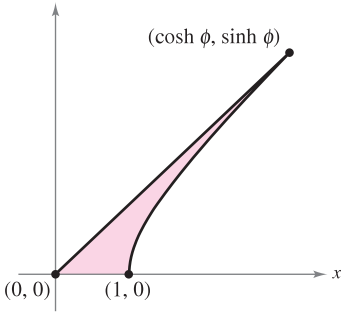
to verify that the area of the region shown in the figure is $\frac{1}{2}\phi$.

(c) Show that the area of the indicated region is also given by the integral

$$A = \int_0^{\sinh\phi}\left[\sqrt{1 + y^2} - (\coth\phi)y\right]dy.$$

Confirm your answer in part (b) by numerically approximating this integral for $\phi = 1, 2, 4,$ and 10.

(d) Consider the unit circle given by $x^2 + y^2 = 1$. Let θ be the angle formed by the x-axis and the radius to (x, y). The area of the corresponding sector is $\frac{1}{2}\theta$. That is, the trigonometric functions $f(\theta) = \cos\theta$ and $g(\theta) = \sin\theta$ could have been defined as the coordinates of that point $(\cos\theta, \sin\theta)$ on the unit circle that determines a sector of area $\frac{1}{2}\theta$. Write a short paragraph explaining how you could define the hyperbolic functions in a similar manner, using the "unit hyperbola" $x^2 - y^2 = 1$.



15.5 Parametric Surfaces

- **Understand the definition of a parametric surface, and sketch the surface.**
- **Find a set of parametric equations to represent a surface.**
- **Find a normal vector and a tangent plane to a parametric surface.**
- **Find the area of a parametric surface.**

Parametric Surfaces

You already know how to represent a curve in the plane or in space by a set of parametric equations—or, equivalently, by a vector-valued function.

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad \text{Plane curve}$$

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad \text{Space curve}$$

In this section, you will learn how to represent a surface in space by a set of parametric equations—or by a vector-valued function. For curves, note that the vector-valued function $\mathbf{r}$ is a function of a *single* parameter t. For surfaces, the vector-valued function is a function of *two* parameters u and v.

DEFINITION OF PARAMETRIC SURFACE

Let x, y, and z be functions of u and v that are continuous on a domain D in the uv-plane. The set of points (x, y, z) given by

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad \text{Parametric surface}$$

is called a **parametric surface.** The equations

$$x = x(u, v), \quad y = y(u, v), \quad \text{and} \quad z = z(u, v) \qquad \text{Parametric equations}$$

are the **parametric equations** for the surface.

If S is a parametric surface given by the vector-valued function $\mathbf{r}$, then S is traced out by the position vector $\mathbf{r}(u, v)$ as the point (u, v) moves throughout the domain D, as shown in Figure 15.35.

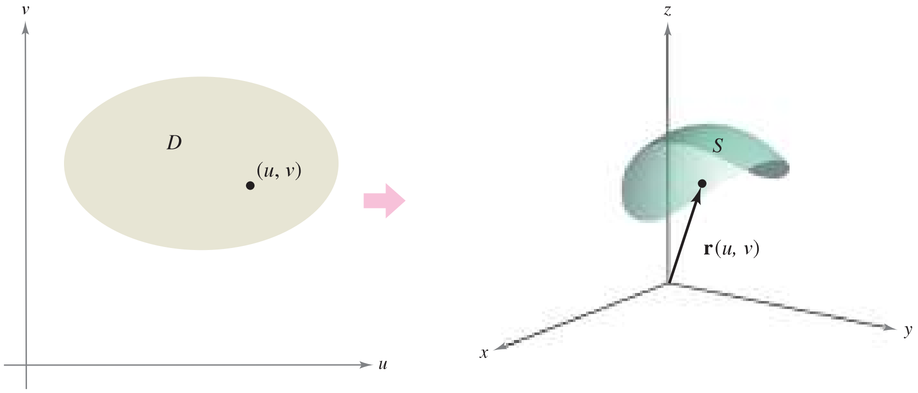


Figure 15.35

TECHNOLOGY Some computer algebra systems are capable of graphing surfaces that are represented parametrically. If you have access to such software, use it to graph some of the surfaces in the examples and exercises in this section.

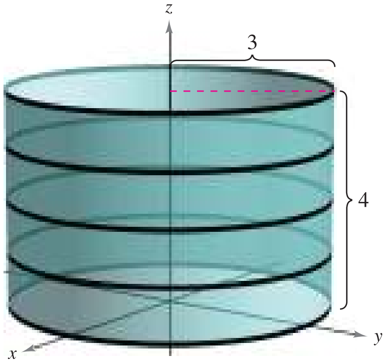


Figure 15.36

EXAMPLE 1 Sketching a Parametric Surface

Identify and sketch the parametric surface S given by

$$\mathbf{r}(u, v) = 3\cos u\mathbf{i} + 3\sin u\mathbf{j} + v\mathbf{k}$$

where $0 \leq u \leq 2\pi$ and $0 \leq v \leq 4$.

Solution Because $x = 3\cos u$ and $y = 3\sin u$, you know that for each point (x, y, z) on the surface, x and y are related by the equation $x^2 + y^2 = 3^2$. In other words, each cross section of S taken parallel to the xy-plane is a circle of radius 3, centered on the z-axis. Because $z = v$, where $0 \leq v \leq 4$, you can see that the surface is a right circular cylinder of height 4. The radius of the cylinder is 3, and the z-axis forms the axis of the cylinder, as shown in Figure 15.36. ■

As with parametric representations of curves, parametric representations of surfaces are not unique. That is, there are many other sets of parametric equations that could be used to represent the surface shown in Figure 15.36.

EXAMPLE 2 Sketching a Parametric Surface

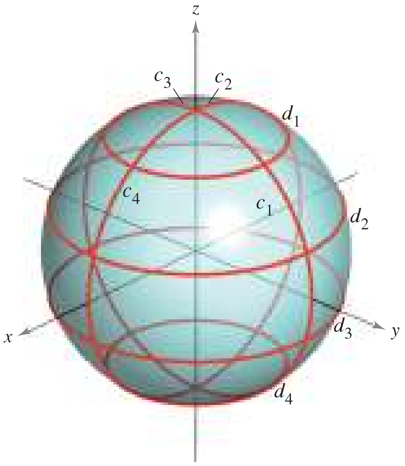


Figure 15.37

Identify and sketch the parametric surface S given by

$$\mathbf{r}(u, v) = \sin u\cos v\mathbf{i} + \sin u\sin v\mathbf{j} + \cos u\mathbf{k}$$

where $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$.

Solution To identify the surface, you can try to use trigonometric identities to eliminate the parameters. After some experimentation, you can discover that

$$\begin{aligned} x^2 + y^2 + z^2 &= (\sin u\cos v)^2 + (\sin u\sin v)^2 + (\cos u)^2 \\ &= \sin^2 u\cos^2 v + \sin^2 u\sin^2 v + \cos^2 u \\ &= \sin^2 u(\cos^2 v + \sin^2 v) + \cos^2 u \\ &= \sin^2 u + \cos^2 u \\ &= 1. \end{aligned}$$

So, each point on S lies on the unit sphere, centered at the origin, as shown in Figure 15.37. For fixed $u = d_i$, $\mathbf{r}(u, v)$ traces out latitude circles

$$x^2 + y^2 = \sin^2 d_i, \quad 0 \leq d_i \leq \pi$$

that are parallel to the xy-plane, and for fixed $v = c_i$, $\mathbf{r}(u, v)$ traces out longitude (or meridian) half-circles. ■

NOTE To convince yourself further that the vector-valued function in Example 2 traces out the entire unit sphere, recall that the parametric equations

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad \text{and} \quad z = \rho\cos\phi$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$, describe the conversion from spherical to rectangular coordinates, as discussed in Section 11.7. ■

Finding Parametric Equations for Surfaces

In Examples 1 and 2, you were asked to identify the surface described by a given set of parametric equations. The reverse problem—that of writing a set of parametric equations for a given surface—is generally more difficult. One type of surface for which this problem is straightforward, however, is a surface that is given by $z = f(x, y)$. You can parametrize such a surface as

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}.$$

EXAMPLE 3 Representing a Surface Parametrically

Write a set of parametric equations for the cone given by

$$z = \sqrt{x^2 + y^2}$$

as shown in Figure 15.38.

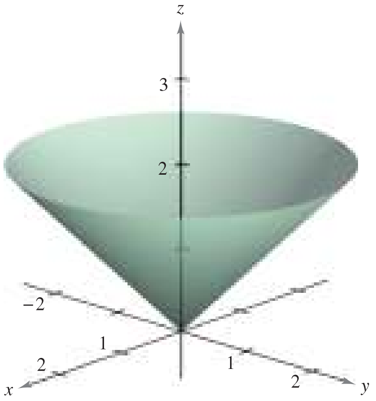


Figure 15.38

Solution Because this surface is given in the form $z = f(x, y)$, you can let x and y be the parameters. Then the cone is represented by the vector-valued function

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + \sqrt{x^2 + y^2}\,\mathbf{k}$$

where (x, y) varies over the entire xy-plane. ■

A second type of surface that is easily represented parametrically is a surface of revolution. For instance, to represent the surface formed by revolving the graph of $y = f(x)$, $a \le x \le b$, about the x-axis, use

$$x = u, \quad y = f(u)\cos v, \quad \text{and} \quad z = f(u)\sin v$$

where $a \le u \le b$ and $0 \le v \le 2\pi$.

EXAMPLE 4 Representing a Surface of Revolution Parametrically

Write a set of parametric equations for the surface of revolution obtained by revolving

$$f(x) = \frac{1}{x}, \quad 1 \le x \le 10$$

about the x-axis.

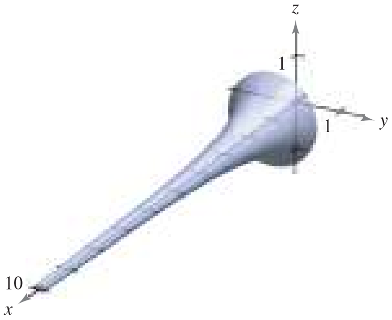


Figure 15.39

Solution Use the parameters u and v as described above to write

$$x = u, \quad y = f(u)\cos v = \frac{1}{u}\cos v, \quad \text{and} \quad z = f(u)\sin v = \frac{1}{u}\sin v$$

where $1 \le u \le 10$ and $0 \le v \le 2\pi$. The resulting surface is a portion of *Gabriel's Horn*, as shown in Figure 15.39. ■

The surface of revolution in Example 4 is formed by revolving the graph of $y = f(x)$ about the x-axis. For other types of surfaces of revolution, a similar parametrization can be used. For instance, to parametrize the surface formed by revolving the graph of $x = f(z)$ about the z-axis, you can use

$$z = u, \quad x = f(u)\cos v, \quad \text{and} \quad y = f(u)\sin v.$$

Normal Vectors and Tangent Planes

Let S be a parametric surface given by

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

over an open region D such that x, y, and z have continuous partial derivatives on D. The **partial derivatives of r** with respect to u and v are defined as

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u, v)\mathbf{i} + \frac{\partial y}{\partial u}(u, v)\mathbf{j} + \frac{\partial z}{\partial u}(u, v)\mathbf{k}$$

and

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u, v)\mathbf{i} + \frac{\partial y}{\partial v}(u, v)\mathbf{j} + \frac{\partial z}{\partial v}(u, v)\mathbf{k}.$$

Each of these partial derivatives is a vector-valued function that can be interpreted geometrically in terms of tangent vectors. For instance, if $v = v_0$ is held constant, then $\mathbf{r}(u, v_0)$ is a vector-valued function of a single parameter and defines a curve C_1 that lies on the surface S. The tangent vector to C_1 at the point $(x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$ is given by

$$\mathbf{r}_u(u_0, v_0) = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}$$

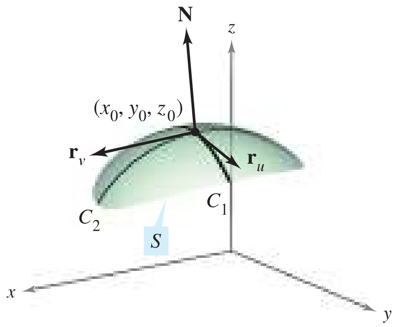


Figure 15.40

as shown in Figure 15.40. In a similar way, if $u = u_0$ is held constant, then $\mathbf{r}(u_0, v)$ is a vector-valued function of a single parameter and defines a curve C_2 that lies on the surface S. The tangent vector to C_2 at the point $(x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$ is given by

$$\mathbf{r}_v(u_0, v_0) = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

If the normal vector $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$ for any (u, v) in D, the surface S is called **smooth** and will have a tangent plane. Informally, a smooth surface is one that has no sharp points or cusps. For instance, spheres, ellipsoids, and paraboloids are smooth, whereas the cone given in Example 3 is not smooth.

NORMAL VECTOR TO A SMOOTH PARAMETRIC SURFACE

Let S be a smooth parametric surface

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

defined over an open region D in the uv-plane. Let (u_0, v_0) be a point in D. A normal vector at the point

$$(x_0, y_0, z_0) = (x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$$

is given by

$$\mathbf{N} = \mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}.$$

NOTE Figure 15.40 shows the normal vector $\mathbf{r}_u \times \mathbf{r}_v$. The vector $\mathbf{r}_v \times \mathbf{r}_u$ is also normal to S and points in the opposite direction. ■

EXAMPLE 5 Finding a Tangent Plane to a Parametric Surface

Find an equation of the tangent plane to the paraboloid given by

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (u^2 + v^2)\mathbf{k}$$

at the point (1, 2, 5).

Solution The point in the uv-plane that is mapped to the point $(x, y, z) = (1, 2, 5)$ is $(u, v) = (1, 2)$. The partial derivatives of $\mathbf{r}$ are

$$\mathbf{r}_u = \mathbf{i} + 2u\mathbf{k} \quad \text{and} \quad \mathbf{r}_v = \mathbf{j} + 2v\mathbf{k}.$$

The normal vector is given by

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\mathbf{i} - 2v\mathbf{j} + \mathbf{k}$$

which implies that the normal vector at (1, 2, 5) is $\mathbf{r}_u \times \mathbf{r}_v = -2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$. So, an equation of the tangent plane at (1, 2, 5) is

$$-2(x - 1) - 4(y - 2) + (z - 5) = 0$$
$$-2x - 4y + z = -5.$$

The tangent plane is shown in Figure 15.41. ■

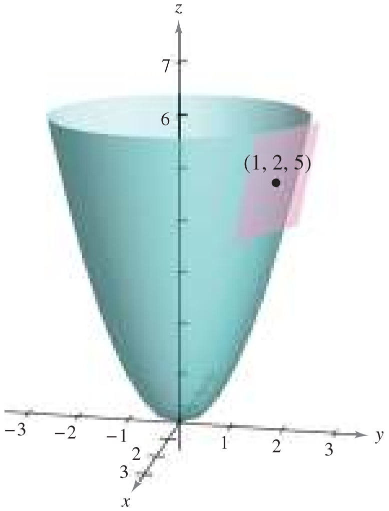


Figure 15.41

Area of a Parametric Surface

To define the area of a parametric surface, you can use a development that is similar to that given in Section 14.5. Begin by constructing an inner partition of D consisting of n rectangles, where the area of the ith rectangle D_i is $\Delta A_i = \Delta u_i \, \Delta v_i$, as shown in Figure 15.42. In each D_i let (u_i, v_i) be the point that is closest to the origin. At the point $(x_i, y_i, z_i) = (x(u_i, v_i), y(u_i, v_i), z(u_i, v_i))$ on the surface S, construct a tangent plane T_i. The area of the portion of S that corresponds to D_i, ΔT_i, can be approximated by a parallelogram in the tangent plane. That is, $\Delta T_i \approx \Delta S_i$. So, the surface of S is given by $\Sigma\, \Delta S_i \approx \Sigma\, \Delta T_i$. The area of the parallelogram in the tangent plane is

$$\|\Delta u_i \mathbf{r}_u \times \Delta v_i \mathbf{r}_v\| = \|\mathbf{r}_u \times \mathbf{r}_v\| \, \Delta u_i \, \Delta v_i$$

which leads to the following definition.

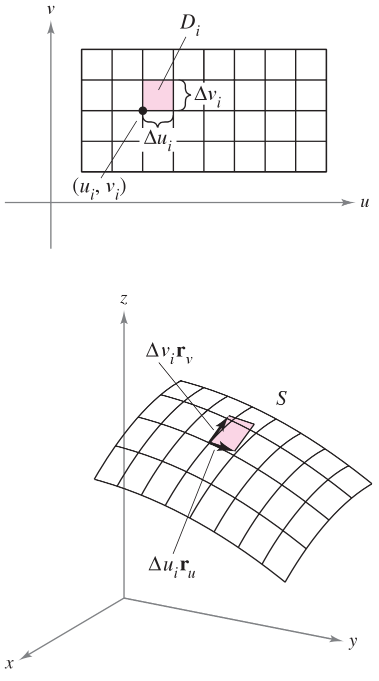


Figure 15.42

AREA OF A PARAMETRIC SURFACE

Let S be a smooth parametric surface

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

defined over an open region D in the uv-plane. If each point on the surface S corresponds to exactly one point in the domain D, then the **surface area** of S is given by

$$\text{Surface area} = \iint_S dS = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA$$

where $\mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k}$ and $\mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k}$.

For a surface S given by $z = f(x, y)$, this formula for surface area corresponds to that given in Section 14.5. To see this, you can parametrize the surface using the vector-valued function

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$$

defined over the region R in the xy-plane. Using

$$\mathbf{r}_x = \mathbf{i} + f_x(x, y)\mathbf{k} \quad \text{and} \quad \mathbf{r}_y = \mathbf{j} + f_y(x, y)\mathbf{k}$$

you have

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x, y) \\ 0 & 1 & f_y(x, y) \end{vmatrix} = -f_x(x, y)\mathbf{i} - f_y(x, y)\mathbf{j} + \mathbf{k}$$

and $\|\mathbf{r}_x \times \mathbf{r}_y\| = \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}$. This implies that the surface area of S is

$$\begin{aligned} \text{Surface area} &= \int\!\!\int_R \|\mathbf{r}_x \times \mathbf{r}_y\|\, dA \\ &= \int\!\!\int_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA. \end{aligned}$$

EXAMPLE 6 Finding Surface Area

NOTE The surface in Example 6 does not quite fulfill the hypothesis that each point on the surface corresponds to exactly one point in D. For this surface, $\mathbf{r}(u, 0) = \mathbf{r}(u, 2\pi)$ for any fixed value of u. However, because the overlap consists of only a semicircle (which has no area), you can still apply the formula for the area of a parametric surface.

Find the surface area of the unit sphere given by

$$\mathbf{r}(u, v) = \sin u \cos v\mathbf{i} + \sin u \sin v\mathbf{j} + \cos u\mathbf{k}$$

where the domain D is given by $0 \le u \le \pi$ and $0 \le v \le 2\pi$.

Solution Begin by calculating $\mathbf{r}_u$ and $\mathbf{r}_v$.

$$\begin{aligned} \mathbf{r}_u &= \cos u \cos v\mathbf{i} + \cos u \sin v\mathbf{j} - \sin u\mathbf{k} \\ \mathbf{r}_v &= -\sin u \sin v\mathbf{i} + \sin u \cos v\mathbf{j} \end{aligned}$$

The cross product of these two vectors is

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin u \sin v & \sin u \cos v & 0 \end{vmatrix} \\ &= \sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \sin u \cos u\mathbf{k} \end{aligned}$$

which implies that

$$\begin{aligned} \|\mathbf{r}_u \times \mathbf{r}_v\| &= \sqrt{(\sin^2 u \cos v)^2 + (\sin^2 u \sin v)^2 + (\sin u \cos u)^2} \\ &= \sqrt{\sin^4 u + \sin^2 u \cos^2 u} \\ &= \sqrt{\sin^2 u} \\ &= \sin u. \qquad \sin u > 0 \text{ for } 0 \le u \le \pi \end{aligned}$$

Finally, the surface area of the sphere is

$$\begin{aligned} A = \int\!\!\int_D \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA &= \int_0^{2\pi}\!\!\int_0^{\pi} \sin u\, du\, dv \\ &= \int_0^{2\pi} 2\, dv \\ &= 4\pi. \end{aligned}$$

■

EXPLORATION

For the torus in Example 7, describe the function $\mathbf{r}(u, v)$ for fixed u. Then describe the function $\mathbf{r}(u, v)$ for fixed v.

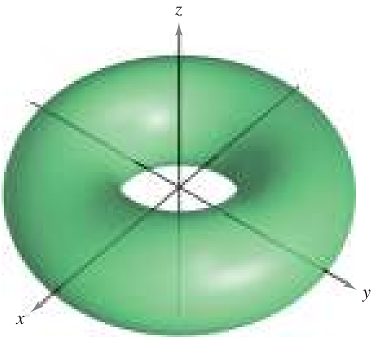


Figure 15.43

EXAMPLE 7 Finding Surface Area

Find the surface area of the torus given by

$$\mathbf{r}(u, v) = (2 + \cos u)\cos v\mathbf{i} + (2 + \cos u)\sin v\mathbf{j} + \sin u\mathbf{k}$$

where the domain D is given by $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$. (See Figure 15.43.)

Solution Begin by calculating $\mathbf{r}_u$ and $\mathbf{r}_v$.

$$\begin{aligned}\mathbf{r}_u &= -\sin u\cos v\mathbf{i} - \sin u\sin v\mathbf{j} + \cos u\mathbf{k}\\ \mathbf{r}_v &= -(2 + \cos u)\sin v\mathbf{i} + (2 + \cos u)\cos v\mathbf{j}\end{aligned}$$

The cross product of these two vectors is

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin u\cos v & -\sin u\sin v & \cos u\\ -(2 + \cos u)\sin v & (2 + \cos u)\cos v & 0\end{vmatrix}\\ &= -(2 + \cos u)(\cos v\cos u\mathbf{i} + \sin v\cos u\mathbf{j} + \sin u\mathbf{k})\end{aligned}$$

which implies that

$$\begin{aligned}\|\mathbf{r}_u \times \mathbf{r}_v\| &= (2 + \cos u)\sqrt{(\cos v\cos u)^2 + (\sin v\cos u)^2 + \sin^2 u}\\ &= (2 + \cos u)\sqrt{\cos^2 u(\cos^2 v + \sin^2 v) + \sin^2 u}\\ &= (2 + \cos u)\sqrt{\cos^2 u + \sin^2 u}\\ &= 2 + \cos u.\end{aligned}$$

Finally, the surface area of the torus is

$$\begin{aligned}A = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|\,dA &= \int_0^{2\pi}\int_0^{2\pi} (2 + \cos u)\,du\,dv\\ &= \int_0^{2\pi} 4\pi\,dv\\ &= 8\pi^2.\end{aligned}$$

■

If the surface S is a surface of revolution, you can show that the formula for surface area given in Section 7.4 is equivalent to the formula given in this section. For instance, suppose f is a nonnegative function such that f' is continuous over the interval $[a, b]$. Let S be the surface of revolution formed by revolving the graph of f, where $a \le x \le b$, about the x-axis. From Section 7.4, you know that the surface area is given by

$$\text{Surface area} = 2\pi\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

To represent S parametrically, let $x = u$, $y = f(u)\cos v$, and $z = f(u)\sin v$, where $a \le u \le b$ and $0 \le v \le 2\pi$. Then,

$$\mathbf{r}(u, v) = u\mathbf{i} + f(u)\cos v\mathbf{j} + f(u)\sin v\mathbf{k}.$$

Try showing that the formula

$$\text{Surface area} = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|\,dA$$

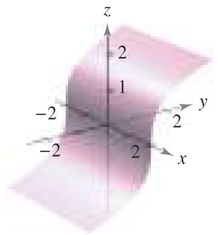
is equivalent to the formula given above (see Exercise 58).

15.5 Exercises

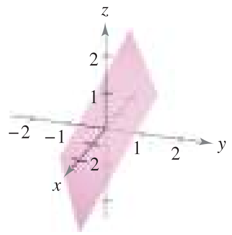
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, match the vector-valued function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

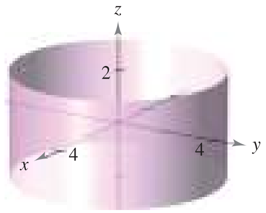
(a)



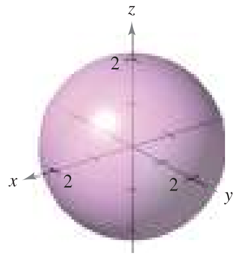
(b)



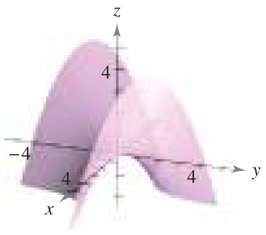
(c)



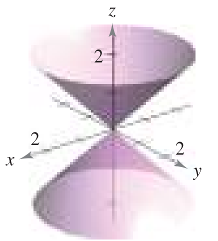
(d)



(e)



(f)



1. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + uv\mathbf{k}$
2. $\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u\mathbf{k}$
3. $\mathbf{r}(u, v) = u\mathbf{i} + \frac{1}{2}(u + v)\mathbf{j} + v\mathbf{k}$
4. $\mathbf{r}(u, v) = u\mathbf{i} + \frac{1}{4}v^3\mathbf{j} + v\mathbf{k}$
5. $\mathbf{r}(u, v) = 2\cos v\cos u\mathbf{i} + 2\cos v\sin u\mathbf{j} + 2\sin v\mathbf{k}$
6. $\mathbf{r}(u, v) = 4\cos u\mathbf{i} + 4\sin u\mathbf{j} + v\mathbf{k}$

In Exercises 7–10, find the rectangular equation for the surface by eliminating the parameters from the vector-valued function. Identify the surface and sketch its graph.

7. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \dfrac{v}{2}\mathbf{k}$
8. $\mathbf{r}(u, v) = 2u\cos v\mathbf{i} + 2u\sin v\mathbf{j} + \frac{1}{2}u^2\mathbf{k}$
9. $\mathbf{r}(u, v) = 2\cos u\mathbf{i} + v\mathbf{j} + 2\sin u\mathbf{k}$
10. $\mathbf{r}(u, v) = 3\cos v\cos u\mathbf{i} + 3\cos v\sin u\mathbf{j} + 5\sin v\mathbf{k}$

CAS **In Exercises 11–16, use a computer algebra system to graph the surface represented by the vector-valued function.**

11. $\mathbf{r}(u, v) = 2u\cos v\mathbf{i} + 2u\sin v\mathbf{j} + u^4\mathbf{k}$

    $0 \le u \le 1, \quad 0 \le v \le 2\pi$
12. $\mathbf{r}(u, v) = 2\cos v\cos u\mathbf{i} + 4\cos v\sin u\mathbf{j} + \sin v\mathbf{k}$

    $0 \le u \le 2\pi, \quad 0 \le v \le 2\pi$
13. $\mathbf{r}(u, v) = 2\sinh u\cos v\mathbf{i} + \sinh u\sin v\mathbf{j} + \cosh u\mathbf{k}$

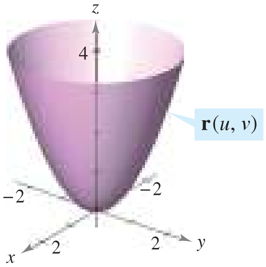
    $0 \le u \le 2, \quad 0 \le v \le 2\pi$
14. $\mathbf{r}(u, v) = 2u\cos v\mathbf{i} + 2u\sin v\mathbf{j} + v\mathbf{k}$

    $0 \le u \le 1, \quad 0 \le v \le 3\pi$
15. $\mathbf{r}(u, v) = (u - \sin u)\cos v\mathbf{i} + (1 - \cos u)\sin v\mathbf{j} + u\mathbf{k}$

    $0 \le u \le \pi, \quad 0 \le v \le 2\pi$
16. $\mathbf{r}(u, v) = \cos^3 u\cos v\mathbf{i} + \sin^3 u\sin v\mathbf{j} + u\mathbf{k}$

    $0 \le u \le \dfrac{\pi}{2}, \quad 0 \le v \le 2\pi$

Think About It **In Exercises 17–20, determine how the graph of the surface $\mathbf{s}(u, v)$ differs from the graph of $\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u^2\mathbf{k}$ (see figure), where $0 \le u \le 2$ and $0 \le v \le 2\pi$. (It is not necessary to graph s.)**



17. $\mathbf{s}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} - u^2\mathbf{k}$

    $0 \le u \le 2, \quad 0 \le v \le 2\pi$
18. $\mathbf{s}(u, v) = u\cos v\mathbf{i} + u^2\mathbf{j} + u\sin v\mathbf{k}$

    $0 \le u \le 2, \quad 0 \le v \le 2\pi$
19. $\mathbf{s}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u^2\mathbf{k}$

    $0 \le u \le 3, \quad 0 \le v \le 2\pi$
20. $\mathbf{s}(u, v) = 4u\cos v\mathbf{i} + 4u\sin v\mathbf{j} + u^2\mathbf{k}$

    $0 \le u \le 2, \quad 0 \le v \le 2\pi$

In Exercises 21–30, find a vector-valued function whose graph is the indicated surface.

21. The plane $z = y$
22. The plane $x + y + z = 6$
23. The cone $y = \sqrt{4x^2 + 9z^2}$
24. The cone $x = \sqrt{16y^2 + z^2}$
25. The cylinder $x^2 + y^2 = 25$
26. The cylinder $4x^2 + y^2 = 16$
27. The cylinder $z = x^2$
28. The ellipsoid $\dfrac{x^2}{9} + \dfrac{y^2}{4} + \dfrac{z^2}{1} = 1$
29. The part of the plane $z = 4$ that lies inside the cylinder $x^2 + y^2 = 9$
30. The part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 9$

Surface of Revolution **In Exercises 31–34, write a set of parametric equations for the surface of revolution obtained by revolving the graph of the function about the given axis.**

	Function	*Axis of Revolution*
31.	$y = \frac{x}{2}, \quad 0 \le x \le 6$	x-axis
32.	$y = \sqrt{x}, \quad 0 \le x \le 4$	x-axis
33.	$x = \sin z, \quad 0 \le z \le \pi$	z-axis
34.	$z = y^2 + 1, \quad 0 \le y \le 2$	y-axis

Tangent Plane **In Exercises 35–38, find an equation of the tangent plane to the surface represented by the vector-valued function at the given point.**

35. $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + v\mathbf{k}, \quad (1, -1, 1)$

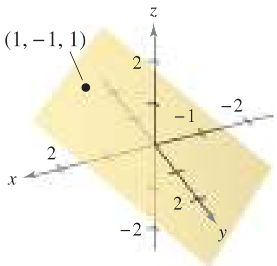


Figure for 35

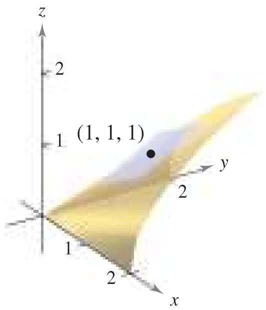
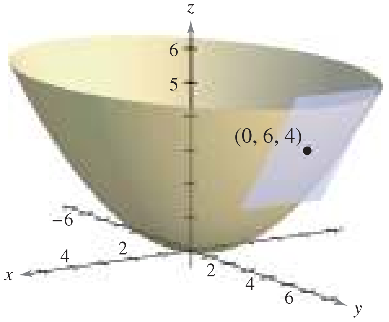


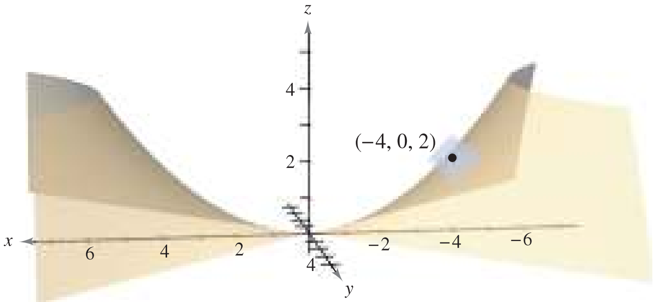
Figure for 36

36. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{uv}\,\mathbf{k}, \quad (1, 1, 1)$

37. $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 3u \sin v\mathbf{j} + u^2\mathbf{k}, \quad (0, 6, 4)$



38. $\mathbf{r}(u, v) = 2u \cosh v\mathbf{i} + 2u \sinh v\mathbf{j} + \frac{1}{2}u^2\mathbf{k}, \quad (-4, 0, 2)$



Area **In Exercises 39–46, find the area of the surface over the given region. Use a computer algebra system to verify your results.**

39. The part of the plane $\mathbf{r}(u, v) = 4u\mathbf{i} - v\mathbf{j} + v\mathbf{k}$, where $0 \le u \le 2$ and $0 \le v \le 1$

40. The part of the paraboloid $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + u^2\mathbf{k}$, where $0 \le u \le 2$ and $0 \le v \le 2\pi$

41. The part of the cylinder $\mathbf{r}(u, v) = a \cos u\mathbf{i} + a \sin u\mathbf{j} + v\mathbf{k}$, where $0 \le u \le 2\pi$ and $0 \le v \le b$

42. The sphere $\mathbf{r}(u, v) = a \sin u \cos v\mathbf{i} + a \sin u \sin v\mathbf{j} + a \cos u\mathbf{k}$, where $0 \le u \le \pi$ and $0 \le v \le 2\pi$

43. The part of the cone $\mathbf{r}(u, v) = au \cos v\mathbf{i} + au \sin v\mathbf{j} + u\mathbf{k}$, where $0 \le u \le b$ and $0 \le v \le 2\pi$

44. The torus $\mathbf{r}(u, v) = (a + b \cos v)\cos u\mathbf{i} + (a + b \cos v)\sin u\mathbf{j} + b \sin v\mathbf{k}$, where $a > b$, $0 \le u \le 2\pi$, and $0 \le v \le 2\pi$

45. The surface of revolution $\mathbf{r}(u, v) = \sqrt{u} \cos v\mathbf{i} + \sqrt{u} \sin v\mathbf{j} + u\mathbf{k}$, where $0 \le u \le 4$ and $0 \le v \le 2\pi$

46. The surface of revolution $\mathbf{r}(u, v) = \sin u \cos v\mathbf{i} + u\mathbf{j} + \sin u \sin v\mathbf{k}$, where $0 \le u \le \pi$ and $0 \le v \le 2\pi$

WRITING ABOUT CONCEPTS

47. Define a parametric surface.

48. Give the double integral that yields the surface area of a parametric surface over an open region D.

49. Show that the cone in Example 3 can be represented parametrically by $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u\mathbf{k}$, where $0 \le u$ and $0 \le v \le 2\pi$.

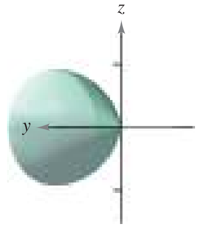
CAPSTONE

50. The four figures below are graphs of the surface $\mathbf{r}(u, v) = u\mathbf{i} + \sin u \cos v\mathbf{j} + \sin u \sin v\mathbf{k}$,

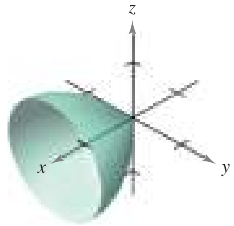
$$0 \le u \le \pi/2, \quad 0 \le v \le 2\pi.$$

Match each of the four graphs with the point in space from which the surface is viewed. The four points are $(10, 0, 0)$, $(-10, 10, 0)$, $(0, 10, 0)$, and $(10, 10, 10)$.

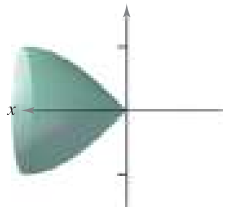
(a)



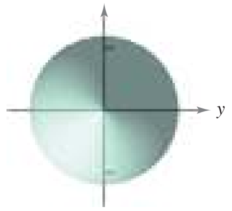
(b)



(c)



(d)



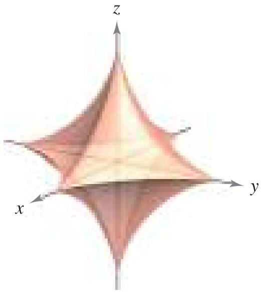
51. ***Astroidal Sphere*** An equation of an **astroidal sphere** in x, y, and z is

$x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$.

A graph of an astroidal sphere is shown below. Show that this surface can be represented parametrically by

$\mathbf{r}(u, v) = a \sin^3 u \cos^3 v\mathbf{i} + a \sin^3 u \sin^3 v\mathbf{j} + a \cos^3 u\mathbf{k}$

where $0 \le u \le \pi$ and $0 \le v \le 2\pi$.



CAS **52.** Use a computer algebra system to graph three views of the graph of the vector-valued function

$\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + v\mathbf{k}, \quad 0 \le u \le \pi, \quad 0 \le v \le \pi$

from the points $(10, 0, 0)$, $(0, 0, 10)$, and $(10, 10, 10)$.

CAS **53.** ***Investigation*** Use a computer algebra system to graph the torus

$$\mathbf{r}(u, v) = (a + b \cos v) \cos u\mathbf{i} + (a + b \cos v) \sin u\mathbf{j} + b \sin v\mathbf{k}$$

for each set of values of a and b, where $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$. Use the results to describe the effects of a and b on the shape of the torus.

(a) $a = 4, \quad b = 1$ (b) $a = 4, \quad b = 2$

(c) $a = 8, \quad b = 1$ (d) $a = 8, \quad b = 3$

54. ***Investigation*** Consider the function in Exercise 14.

(a) Sketch a graph of the function where u is held constant at $u = 1$. Identify the graph.

(b) Sketch a graph of the function where v is held constant at $v = 2\pi/3$. Identify the graph.

(c) Assume that a surface is represented by the vector-valued function $\mathbf{r} = \mathbf{r}(u, v)$. What generalization can you make about the graph of the function if one of the parameters is held constant?

55. ***Surface Area*** The surface of the dome on a new museum is given by

$\mathbf{r}(u, v) = 20 \sin u \cos v\mathbf{i} + 20 \sin u \sin v\mathbf{j} + 20 \cos u\mathbf{k}$

where $0 \le u \le \pi/3$, $0 \le v \le 2\pi$, and $\mathbf{r}$ is in meters. Find the surface area of the dome.

56. Find a vector-valued function for the hyperboloid

$x^2 + y^2 - z^2 = 1$

and determine the tangent plane at $(1, 0, 0)$.

57. Graph and find the area of one turn of the spiral ramp

$\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + 2v\mathbf{k}$

where $0 \le u \le 3$ and $0 \le v \le 2\pi$.

58. Let f be a nonnegative function such that f' is continuous over the interval $[a, b]$. Let S be the surface of revolution formed by revolving the graph of f, where $a \le x \le b$, about the x-axis. Let $x = u$, $y = f(u) \cos v$, and $z = f(u)\sin v$, where $a \le u \le b$ and $0 \le v \le 2\pi$. Then, S is represented parametrically by $\mathbf{r}(u, v) = u\mathbf{i} + f(u) \cos v\mathbf{j} + f(u) \sin v\mathbf{k}$. Show that the following formulas are equivalent.

$$\text{Surface area } = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

$$\text{Surface area } = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA$$

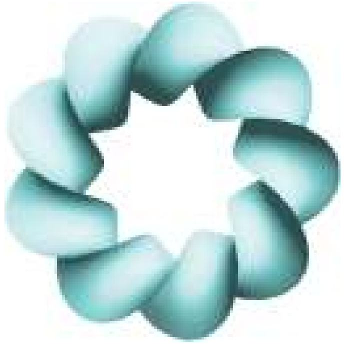
CAS **59.** ***Open-Ended Project*** The parametric equations

$x = 3 + \sin u[7 - \cos(3u - 2v) - 2 \cos(3u + v)]$

$y = 3 + \cos u[7 - \cos(3u - 2v) - 2 \cos(3u + v)]$

$z = \sin(3u - 2v) + 2 \sin(3u + v)$

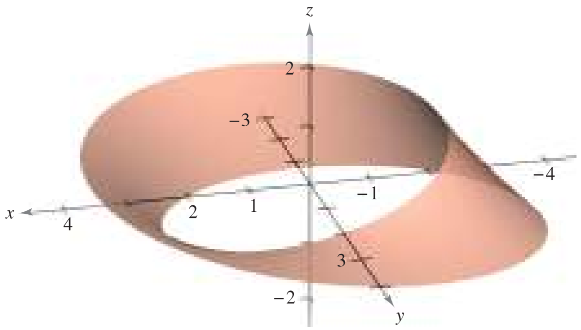
where $-\pi \le u \le \pi$ and $-\pi \le v \le \pi$, represent the surface shown below. Try to create your own parametric surface using a computer algebra system.



CAS **60.** ***Möbius Strip*** The surface shown in the figure is called a **Möbius strip** and can be represented by the parametric equations

$$x = \left(a + u \cos\frac{v}{2}\right) \cos v,\ y = \left(a + u \cos\frac{v}{2}\right) \sin v,\ z = u \sin\frac{v}{2}$$

where $-1 \le u \le 1$, $0 \le v \le 2\pi$, and $a = 3$. Try to graph other Möbius strips for different values of a using a computer algebra system.



15.6 Surface Integrals

- **Evaluate a surface integral as a double integral.**
- **Evaluate a surface integral for a parametric surface.**
- **Determine the orientation of a surface.**
- **Understand the concept of a flux integral.**

Surface Integrals

The remainder of this chapter deals primarily with **surface integrals.** You will first consider surfaces given by $z = g(x, y)$. Later in this section you will consider more general surfaces given in parametric form.

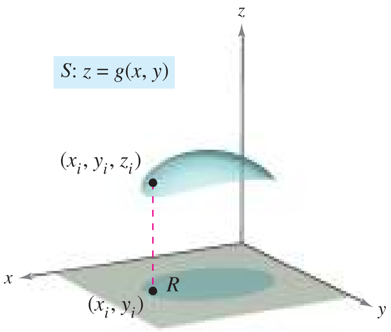
Let S be a surface given by $z = g(x, y)$ and let R be its projection onto the xy-plane, as shown in Figure 15.44. Suppose that g, g_x, and g_y are continuous at all points in R and that f is defined on S. Employing the procedure used to find surface area in Section 14.5, evaluate f at (x_i, y_i, z_i) and form the sum

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta S_i$$

where $\Delta S_i \approx \sqrt{1 + [g_x(x_i, y_i)]^2 + [g_y(x_i, y_i)]^2}\,\Delta A_i$. Provided the limit of this sum as $\|\Delta\|$ approaches 0 exists, the **surface integral of f over S** is defined as

$$\int_S\!\!\int f(x, y, z)\,dS = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta S_i.$$

This integral can be evaluated by a double integral.



Scalar function f assigns a number to each point of S.
Figure 15.44

THEOREM 15.10 EVALUATING A SURFACE INTEGRAL

Let S be a surface with equation $z = g(x, y)$ and let R be its projection onto the xy-plane. If g, g_x, and g_y are continuous on R and f is continuous on S, then the surface integral of f over S is

$$\int_S\!\!\int f(x, y, z)\,dS = \int_R\!\!\int f(x, y, g(x, y))\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\,dA.$$

For surfaces described by functions of x and z (or y and z), you can make the following adjustments to Theorem 15.10. If S is the graph of $y = g(x, z)$ and R is its projection onto the xz-plane, then

$$\int_S\!\!\int f(x, y, z)\,dS = \int_R\!\!\int f(x, g(x, z), z)\sqrt{1 + [g_x(x, z)]^2 + [g_z(x, z)]^2}\,dA.$$

If S is the graph of $x = g(y, z)$ and R is its projection onto the yz-plane, then

$$\int_S\!\!\int f(x, y, z)\,dS = \int_R\!\!\int f(g(y, z), y, z)\sqrt{1 + [g_y(y, z)]^2 + [g_z(y, z)]^2}\,dA.$$

If $f(x, y, z) = 1$, the surface integral over S yields the surface area of S. For instance, suppose the surface S is the plane given by $z = x$, where $0 \le x \le 1$ and $0 \le y \le 1$. The surface area of S is $\sqrt{2}$ square units. Try verifying that $\int_S\!\int f(x, y, z)\,dS = \sqrt{2}$.

EXAMPLE 1 Evaluating a Surface Integral

Evaluate the surface integral

$$\int_S\int (y^2 + 2yz)\, dS$$

where S is the first-octant portion of the plane $2x + y + 2z = 6$.

Solution Begin by writing S as

$$z = \frac{1}{2}(6 - 2x - y)$$

$$g(x, y) = \frac{1}{2}(6 - 2x - y).$$

Using the partial derivatives $g_x(x, y) = -1$ and $g_y(x, y) = -\frac{1}{2}$, you can write

$$\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2} = \sqrt{1 + 1 + \frac{1}{4}} = \frac{3}{2}.$$

Using Figure 15.45 and Theorem 15.10, you obtain

$$\begin{aligned}\int_S\int (y^2 + 2yz)\, dS &= \int_R\int f(x, y, g(x, y))\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\, dA\\ &= \int_R\int \left[y^2 + 2y\left(\frac{1}{2}\right)(6 - 2x - y)\right]\left(\frac{3}{2}\right) dA\\ &= 3\int_0^3\int_0^{2(3-x)} y(3 - x)\, dy\, dx\\ &= 6\int_0^3 (3 - x)^3\, dx\\ &= -\frac{3}{2}(3 - x)^4\Big]_0^3\\ &= \frac{243}{2}.\end{aligned}$$

■

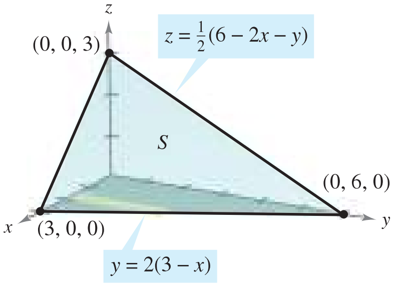


Figure 15.45

An alternative solution to Example 1 would be to project S onto the yz-plane, as shown in Figure 15.46. Then, $x = \frac{1}{2}(6 - y - 2z)$, and

$$\sqrt{1 + [g_y(y, z)]^2 + [g_z(y, z)]^2} = \sqrt{1 + \frac{1}{4} + 1} = \frac{3}{2}.$$

So, the surface integral is

$$\begin{aligned}\int_S\int (y^2 + 2yz)\, dS &= \int_R\int f(g(y, z), y, z)\sqrt{1 + [g_y(y, z)]^2 + [g_z(y, z)]^2}\, dA\\ &= \int_0^6\int_0^{(6-y)/2} (y^2 + 2yz)\left(\frac{3}{2}\right) dz\, dy\\ &= \frac{3}{8}\int_0^6 (36y - y^3)\, dy\\ &= \frac{243}{2}.\end{aligned}$$

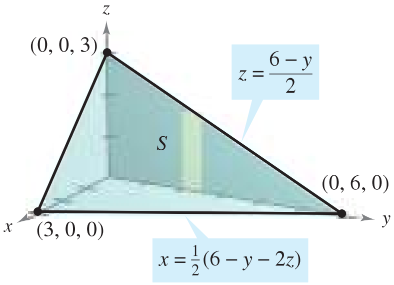


Figure 15.46

Try reworking Example 1 by projecting S onto the xz-plane.

In Example 1, you could have projected the surface S onto any one of the three coordinate planes. In Example 2, S is a portion of a cylinder centered about the x-axis, and you can project it onto either the xz-plane or the xy-plane.

EXAMPLE 2 Evaluating a Surface Integral

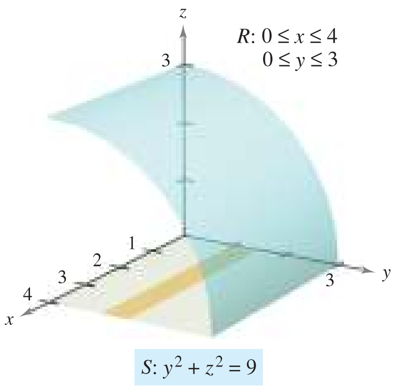


Figure 15.47

Evaluate the surface integral

$$\int\!\!\int_S (x + z)\, dS$$

where S is the first-octant portion of the cylinder $y^2 + z^2 = 9$ between $x = 0$ and $x = 4$, as shown in Figure 15.47.

Solution Project S onto the xy-plane, so that $z = g(x, y) = \sqrt{9 - y^2}$, and obtain

$$\begin{aligned}\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2} &= \sqrt{1 + \left(\frac{-y}{\sqrt{9 - y^2}}\right)^2}\\ &= \frac{3}{\sqrt{9 - y^2}}.\end{aligned}$$

Theorem 15.10 does not apply directly, because g_y is not continuous when $y = 3$. However, you can apply Theorem 15.10 for $0 \le b < 3$ and then take the limit as b approaches 3, as follows.

$$\begin{aligned}\int\!\!\int_S (x + z)\, dS &= \lim_{b \to 3^-} \int_0^b \int_0^4 \left(x + \sqrt{9 - y^2}\right) \frac{3}{\sqrt{9 - y^2}}\, dx\, dy\\ &= \lim_{b \to 3^-} 3 \int_0^b \int_0^4 \left(\frac{x}{\sqrt{9 - y^2}} + 1\right) dx\, dy\\ &= \lim_{b \to 3^-} 3 \int_0^b \left[\frac{x^2}{2\sqrt{9 - y^2}} + x\right]_0^4 dy\\ &= \lim_{b \to 3^-} 3 \int_0^b \left(\frac{8}{\sqrt{9 - y^2}} + 4\right) dy\\ &= \lim_{b \to 3^-} 3\left[4y + 8 \arcsin \frac{y}{3}\right]_0^b\\ &= \lim_{b \to 3^-} 3\left(4b + 8 \arcsin \frac{b}{3}\right)\\ &= 36 + 24\left(\frac{\pi}{2}\right)\\ &= 36 + 12\pi\end{aligned}$$

■

TECHNOLOGY Some computer algebra systems are capable of evaluating improper integrals. If you have access to such computer software, use it to evaluate the improper integral

$$\int_0^3 \int_0^4 \left(x + \sqrt{9 - y^2}\right) \frac{3}{\sqrt{9 - y^2}}\, dx\, dy.$$

Do you obtain the same result as in Example 2?

You have already seen that if the function f defined on the surface S is simply $f(x, y, z) = 1$, the surface integral yields the *surface area* of S.

$$\text{Area of surface} = \int\!\!\int_S 1 \, dS$$

On the other hand, if S is a lamina of variable density and $\rho(x, y, z)$ is the density at the point (x, y, z), then the *mass* of the lamina is given by

$$\text{Mass of lamina} = \int\!\!\int_S \rho(x, y, z)\, dS.$$

EXAMPLE 3 Finding the Mass of a Surface Lamina

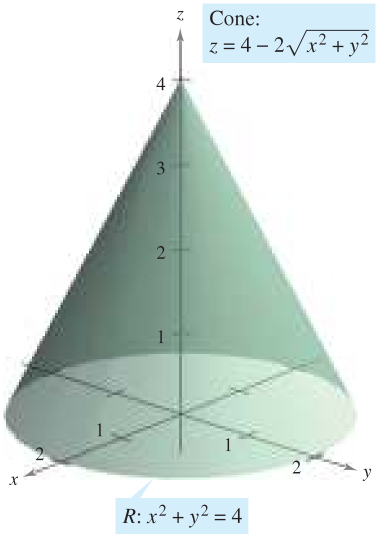


Figure 15.48

A cone-shaped surface lamina S is given by

$$z = 4 - 2\sqrt{x^2 + y^2}, \quad 0 \le z \le 4$$

as shown in Figure 15.48. At each point on S, the density is proportional to the distance between the point and the z-axis. Find the mass m of the lamina.

Solution Projecting S onto the xy-plane produces

$$S: z = 4 - 2\sqrt{x^2 + y^2} = g(x, y), \quad 0 \le z \le 4$$
$$R: x^2 + y^2 \le 4$$

with a density of $\rho(x, y, z) = k\sqrt{x^2 + y^2}$. Using a surface integral, you can find the mass to be

$$\begin{aligned}
m &= \int\!\!\int_S \rho(x, y, z)\, dS \\
&= \int\!\!\int_R k\sqrt{x^2 + y^2}\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\, dA \\
&= k\int\!\!\int_R \sqrt{x^2 + y^2}\sqrt{1 + \frac{4x^2}{x^2 + y^2} + \frac{4y^2}{x^2 + y^2}}\, dA \\
&= k\int\!\!\int_R \sqrt{5}\sqrt{x^2 + y^2}\, dA \\
&= k\int_0^{2\pi}\int_0^2 \left(\sqrt{5}r\right) r\, dr\, d\theta \qquad \text{Polar coordinates} \\
&= \frac{\sqrt{5}k}{3}\int_0^{2\pi} r^3 \Big]_0^2 \, d\theta \\
&= \frac{8\sqrt{5}k}{3}\int_0^{2\pi} d\theta \\
&= \frac{8\sqrt{5}k}{3}\Big[\theta\Big]_0^{2\pi} = \frac{16\sqrt{5}k\pi}{3}.
\end{aligned}$$

■

TECHNOLOGY Use a computer algebra system to confirm the result shown in Example 3. The computer algebra system *Maple* evaluated the integral as follows.

$$k\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{5}\sqrt{x^2 + y^2}\, dx\, dy = k\int_0^{2\pi}\int_0^2 \left(\sqrt{5}r\right) r\, dr\, d\theta = \frac{16\sqrt{5}k\pi}{3}$$

Parametric Surfaces and Surface Integrals

For a surface S given by the vector-valued function

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad \text{Parametric surface}$$

defined over a region D in the uv-plane, you can show that the surface integral of $f(x, y, z)$ over S is given by

$$\int\!\!\int_S f(x, y, z)\, dS = \int\!\!\int_D f(x(u, v), y(u, v), z(u, v))\, \|\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)\|\, dA.$$

Note the similarity to a line integral over a space curve C.

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\, \|\mathbf{r}'(t)\|\, dt \qquad \text{Line integral}$$

NOTE Notice that ds and dS can be written as $ds = \|\mathbf{r}'(t)\|\, dt$ and $dS = \|\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)\|\, dA$. ■

EXAMPLE 4 Evaluating a Surface Integral

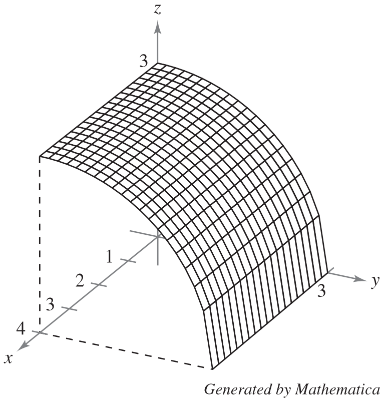


Figure 15.49

Example 2 demonstrated an evaluation of the surface integral

$$\int\!\!\int_S (x + z)\, dS$$

where S is the first-octant portion of the cylinder $y^2 + z^2 = 9$ between $x = 0$ and $x = 4$ (see Figure 15.49). Reevaluate this integral in parametric form.

Solution In parametric form, the surface is given by

$$\mathbf{r}(x, \theta) = x\mathbf{i} + 3\cos\theta\mathbf{j} + 3\sin\theta\mathbf{k}$$

where $0 \le x \le 4$ and $0 \le \theta \le \pi/2$. To evaluate the surface integral in parametric form, begin by calculating the following.

$$\begin{aligned}
\mathbf{r}_x &= \mathbf{i}\\
\mathbf{r}_\theta &= -3\sin\theta\mathbf{j} + 3\cos\theta\mathbf{k}\\
\mathbf{r}_x \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 0\\ 0 & -3\sin\theta & 3\cos\theta \end{vmatrix} = -3\cos\theta\mathbf{j} - 3\sin\theta\mathbf{k}\\
\|\mathbf{r}_x \times \mathbf{r}_\theta\| &= \sqrt{9\cos^2\theta + 9\sin^2\theta} = 3
\end{aligned}$$

So, the surface integral can be evaluated as follows.

$$\begin{aligned}
\int\!\!\int_D (x + 3\sin\theta)3\, dA &= \int_0^4 \int_0^{\pi/2} (3x + 9\sin\theta)\, d\theta\, dx\\
&= \int_0^4 \Big[3x\theta - 9\cos\theta\Big]_0^{\pi/2} dx\\
&= \int_0^4 \left(\frac{3\pi}{2}x + 9\right) dx\\
&= \left[\frac{3\pi}{4}x^2 + 9x\right]_0^4\\
&= 12\pi + 36
\end{aligned}$$

■

Orientation of a Surface

Unit normal vectors are used to induce an orientation to a surface S in space. A surface is called **orientable** if a unit normal vector $\mathbf{N}$ can be defined at every nonboundary point of S in such a way that the normal vectors vary continuously over the surface S. If this is possible, S is called an **oriented surface.**

An orientable surface S has two distinct sides. So, when you orient a surface, you are selecting one of the two possible unit normal vectors. If S is a closed surface such as a sphere, it is customary to choose the unit normal vector $\mathbf{N}$ to be the one that points outward from the sphere.

Most common surfaces, such as spheres, paraboloids, ellipses, and planes, are orientable. (See Exercise 43 for an example of a surface that is *not* orientable.) Moreover, for an orientable surface, the gradient vector provides a convenient way to find a unit normal vector. That is, for an orientable surface S given by

$$z = g(x, y) \qquad \text{Orientable surface}$$

let

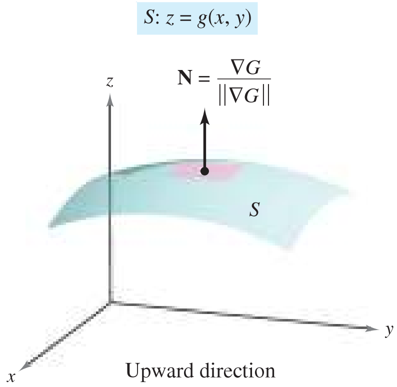
$$G(x, y, z) = z - g(x, y).$$

Then, S can be oriented by either the unit normal vector

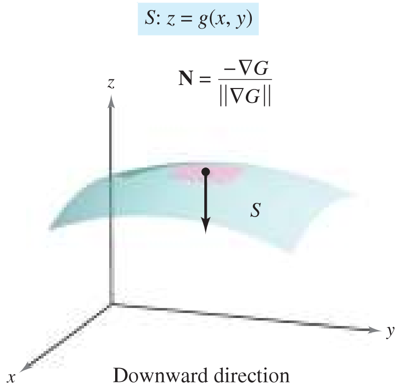
$$\begin{aligned}\mathbf{N} &= \frac{\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|}\\ &= \frac{-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}}{\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}} \qquad \text{Upward unit normal vector}\end{aligned}$$

or the unit normal vector

$$\begin{aligned}\mathbf{N} &= \frac{-\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|}\\ &= \frac{g_x(x, y)\mathbf{i} + g_y(x, y)\mathbf{j} - \mathbf{k}}{\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}} \qquad \text{Downward unit normal vector}\end{aligned}$$



S is oriented in an upward direction.



S is oriented in a downward direction.
Figure 15.50

as shown in Figure 15.50. If the smooth orientable surface S is given in parametric form by

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad \text{Parametric surface}$$

the unit normal vectors are given by

$$\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$$

and

$$\mathbf{N} = \frac{\mathbf{r}_v \times \mathbf{r}_u}{\|\mathbf{r}_v \times \mathbf{r}_u\|}.$$

NOTE Suppose that the orientable surface is given by $y = g(x, z)$ or $x = g(y, z)$. Then you can use the gradient vector

$$\nabla G(x, y, z) = -g_x(x, z)\mathbf{i} + \mathbf{j} - g_z(x, z)\mathbf{k} \qquad G(x, y, z) = y - g(x, z)$$

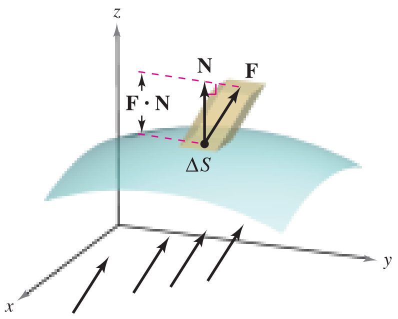
or

$$\nabla G(x, y, z) = \mathbf{i} - g_y(y, z)\mathbf{j} - g_z(y, z)\mathbf{k} \qquad G(x, y, z) = x - g(y, z)$$

to orient the surface. ■

Flux Integrals

One of the principal applications involving the vector form of a surface integral relates to the flow of a fluid through a surface S. Suppose an oriented surface S is submerged in a fluid having a continuous velocity field $\mathbf{F}$. Let ΔS be the area of a small patch of the surface S over which $\mathbf{F}$ is nearly constant. Then the amount of fluid crossing this region per unit of time is approximated by the volume of the column of height $\mathbf{F} \cdot \mathbf{N}$, as shown in Figure 15.51. That is,



The velocity field **F** indicates the direction of the fluid flow.
Figure 15.51

$$\Delta V = (\text{height})(\text{area of base}) = (\mathbf{F} \cdot \mathbf{N})\Delta S.$$

Consequently, the volume of fluid crossing the surface S per unit of time (called the **flux of F across *S***) is given by the surface integral in the following definition.

DEFINITION OF FLUX INTEGRAL

Let $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, where M, N, and P have continuous first partial derivatives on the surface S oriented by a unit normal vector $\mathbf{N}$. The **flux integral of F across *S*** is given by

$$\int\!\!\int_S \mathbf{F} \cdot \mathbf{N}\, dS.$$

Geometrically, a flux integral is the surface integral over S of the *normal component* of $\mathbf{F}$. If $\rho(x, y, z)$ is the density of the fluid at (x, y, z), the flux integral

$$\int\!\!\int_S \rho\, \mathbf{F} \cdot \mathbf{N}\, dS$$

represents the *mass* of the fluid flowing across S per unit of time.

To evaluate a flux integral for a surface given by $z = g(x, y)$, let

$$G(x, y, z) = z - g(x, y).$$

Then, $\mathbf{N}\, dS$ can be written as follows.

$$\begin{aligned}
\mathbf{N}\, dS &= \frac{\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|}\, dS \\
&= \frac{\nabla G(x, y, z)}{\sqrt{(g_x)^2 + (g_y)^2 + 1}} \sqrt{(g_x)^2 + (g_y)^2 + 1}\, dA \\
&= \nabla G(x, y, z)\, dA
\end{aligned}$$

THEOREM 15.11 EVALUATING A FLUX INTEGRAL

Let S be an oriented surface given by $z = g(x, y)$ and let R be its projection onto the xy-plane.

$$\int\!\!\int_S \mathbf{F} \cdot \mathbf{N}\, dS = \int\!\!\int_R \mathbf{F} \cdot [-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}]\, dA \qquad \text{Oriented upward}$$

$$\int\!\!\int_S \mathbf{F} \cdot \mathbf{N}\, dS = \int\!\!\int_R \mathbf{F} \cdot [g_x(x, y)\mathbf{i} + g_y(x, y)\mathbf{j} - \mathbf{k}]\, dA \qquad \text{Oriented downward}$$

For the first integral, the surface is oriented upward, and for the second integral, the surface is oriented downward.

EXAMPLE 5 Using a Flux Integral to Find the Rate of Mass Flow

Let S be the portion of the paraboloid

$$z = g(x, y) = 4 - x^2 - y^2$$

lying above the xy-plane, oriented by an upward unit normal vector, as shown in Figure 15.52. A fluid of constant density ρ is flowing through the surface S according to the vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Find the rate of mass flow through S.

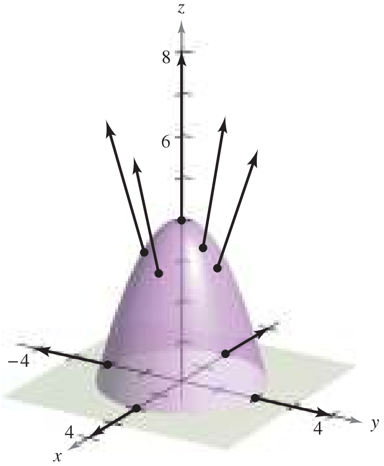


Figure 15.52

Solution Begin by computing the partial derivatives of g.

$$g_x(x, y) = -2x$$

and

$$g_y(x, y) = -2y$$

The rate of mass flow through the surface S is

$$\begin{aligned}\int\!\!\int_S \rho\mathbf{F}\cdot\mathbf{N}\,dS &= \rho\int\!\!\int_R \mathbf{F}\cdot[-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}]\,dA\\ &= \rho\int\!\!\int_R [x\mathbf{i} + y\mathbf{j} + (4 - x^2 - y^2)\mathbf{k}]\cdot(2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\,dA\\ &= \rho\int\!\!\int_R [2x^2 + 2y^2 + (4 - x^2 - y^2)]\,dA\\ &= \rho\int\!\!\int_R (4 + x^2 + y^2)\,dA\\ &= \rho\int_0^{2\pi}\int_0^2 (4 + r^2)r\,dr\,d\theta \qquad \text{Polar coordinates}\\ &= \rho\int_0^{2\pi} 12\,d\theta\\ &= 24\pi\rho.\end{aligned}$$

For an oriented surface S given by the vector-valued function

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad \text{Parametric surface}$$

defined over a region D in the uv-plane, you can define the flux integral of $\mathbf{F}$ across S as

$$\begin{aligned}\int\!\!\int_S \mathbf{F}\cdot\mathbf{N}\,dS &= \int\!\!\int_D \mathbf{F}\cdot\left(\frac{\mathbf{r}_u\times\mathbf{r}_v}{\|\mathbf{r}_u\times\mathbf{r}_v\|}\right)\|\mathbf{r}_u\times\mathbf{r}_v\|\,dA\\ &= \int\!\!\int_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA.\end{aligned}$$

Note the similarity of this integral to the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds.$$

A summary of formulas for line and surface integrals is presented on page 1121.

EXAMPLE 6 Finding the Flux of an Inverse Square Field

Find the flux over the sphere S given by

$$x^2 + y^2 + z^2 = a^2 \qquad \text{Sphere } S$$

where $\mathbf{F}$ is an inverse square field given by

$$\mathbf{F}(x, y, z) = \frac{kq}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{kq\mathbf{r}}{\|\mathbf{r}\|^3} \qquad \text{Inverse square field } \mathbf{F}$$

and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Assume S is oriented outward, as shown in Figure 15.53.

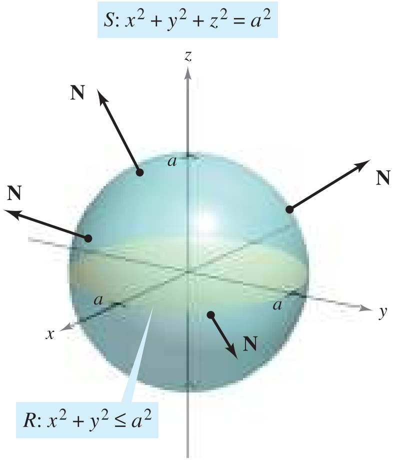


Figure 15.53

Solution The sphere is given by

$$\begin{aligned}\mathbf{r}(u, v) &= x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}\\ &= a \sin u \cos v\mathbf{i} + a \sin u \sin v\mathbf{j} + a \cos u\mathbf{k}\end{aligned}$$

where $0 \le u \le \pi$ and $0 \le v \le 2\pi$. The partial derivatives of $\mathbf{r}$ are

$$\mathbf{r}_u(u, v) = a \cos u \cos v\,\mathbf{i} + a \cos u \sin v\mathbf{j} - a \sin u\mathbf{k}$$

and

$$\mathbf{r}_v(u, v) = -a \sin u \sin v\mathbf{i} + a \sin u \cos v\mathbf{j}$$

which implies that the normal vector $\mathbf{r}_u \times \mathbf{r}_v$ is

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ a \cos u \cos v & a \cos u \sin v & -a \sin u\\ -a \sin u \sin v & a \sin u \cos v & 0\end{vmatrix}\\ &= a^2(\sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \sin u \cos u\mathbf{k}).\end{aligned}$$

Now, using

$$\begin{aligned}\mathbf{F}(x, y, z) &= \frac{kq\mathbf{r}}{\|\mathbf{r}\|^3}\\ &= kq\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\|^3}\\ &= \frac{kq}{a^3}(a \sin u \cos v\mathbf{i} + a \sin u \sin v\mathbf{j} + a \cos u\mathbf{k})\end{aligned}$$

it follows that

$$\begin{aligned}\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) &= \frac{kq}{a^3}[(a \sin u \cos v\mathbf{i} + a \sin u \sin v\mathbf{j} + a \cos u\mathbf{k}) \cdot\\ &\qquad a^2(\sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \sin u \cos u\mathbf{k})]\\ &= kq(\sin^3 u \cos^2 v + \sin^3 u \sin^2 v + \sin u \cos^2 u)\\ &= kq \sin u.\end{aligned}$$

Finally, the flux over the sphere S is given by

$$\begin{aligned}\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS &= \int_D\!\!\int (kq \sin u)\, dA\\ &= \int_0^{2\pi}\!\!\int_0^{\pi} kq \sin u\, du\, dv\\ &= 4\pi kq.\end{aligned}$$

■

The result in Example 6 shows that the flux across a sphere S in an inverse square field is independent of the radius of S. In particular, if $\mathbf{E}$ is an electric field, the result in Example 6, along with Coulomb's Law, yields one of the basic laws of electrostatics, known as **Gauss's Law:**

$$\int_S\!\!\int \mathbf{E}\cdot\mathbf{N}\,dS = 4\pi kq \qquad \text{Gauss's Law}$$

where q is a point charge located at the center of the sphere and k is the Coulomb constant. Gauss's Law is valid for more general closed surfaces that enclose the origin, and relates the flux out of the surface to the total charge q inside the surface.

This section concludes with a summary of different forms of line integrals and surface integrals.

SUMMARY OF LINE AND SURFACE INTEGRALS

Line Integrals

$$\begin{aligned} ds &= \|\mathbf{r}'(t)\|\,dt \\ &= \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt \end{aligned}$$

$$\int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t))\,ds \qquad \text{Scalar form}$$

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \mathbf{F}\cdot\mathbf{T}\,ds \\ &= \int_a^b \mathbf{F}(x(t), y(t), z(t))\cdot\mathbf{r}'(t)\,dt \end{aligned} \qquad \text{Vector form}$$

Surface Integrals $[z = g(x, y)]$

$$dS = \sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\,dA$$

$$\int_S\!\!\int f(x, y, z)\,dS = \int_R\!\!\int f(x, y, g(x, y))\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\,dA \qquad \text{Scalar form}$$

$$\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS = \int_R\!\!\int \mathbf{F}\cdot[-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}]\,dA \qquad \text{Vector form (upward normal)}$$

Surface Integrals (parametric form)

$$dS = \|\mathbf{r}_u(u, v)\times\mathbf{r}_v(u, v)\|\,dA$$

$$\int_S\!\!\int f(x, y, z)\,dS = \int_D\!\!\int f(x(u, v), y(u, v), z(u, v))\,dS \qquad \text{Scalar form}$$

$$\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS = \int_D\!\!\int \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA \qquad \text{Vector form}$$

15.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, evaluate $\displaystyle\int_S\!\!\int (x - 2y + z)\,dS$.

1. S: $z = 4 - x, \quad 0 \le x \le 4, \quad 0 \le y \le 3$
2. S: $z = 15 - 2x + 3y, \quad 0 \le x \le 2, \quad 0 \le y \le 4$
3. S: $z = 2, \quad x^2 + y^2 \le 1$
4. S: $z = \frac{2}{3}x^{3/2}, \quad 0 \le x \le 1, \quad 0 \le y \le x$

In Exercises 5 and 6, evaluate $\displaystyle\int_S\!\!\int xy\,dS$.

5. S: $z = 3 - x - y$, first octant
6. S: $z = h, \quad 0 \le x \le 2, \quad 0 \le y \le \sqrt{4 - x^2}$

CAS **In Exercises 7 and 8, use a computer algebra system to evaluate**

$$\int_S\!\!\int xy\,dS.$$

7. S: $z = 9 - x^2, \quad 0 \le x \le 2, \quad 0 \le y \le x$
8. S: $z = \frac{1}{2}xy, \quad 0 \le x \le 4, \quad 0 \le y \le 4$

CAS **In Exercises 9 and 10, use a computer algebra system to evaluate**

$$\int_S\!\!\int (x^2 - 2xy)\,dS.$$

9. S: $z = 10 - x^2 - y^2, \quad 0 \le x \le 2, \quad 0 \le y \le 2$
10. S: $z = \cos x, \quad 0 \le x \le \dfrac{\pi}{2}, \quad 0 \le y \le \dfrac{1}{2}x$

Mass **In Exercises 11 and 12, find the mass of the surface lamina S of density ρ.**

11. S: $2x + 3y + 6z = 12$, first octant, $\rho(x, y, z) = x^2 + y^2$
12. S: $z = \sqrt{a^2 - x^2 - y^2}, \quad \rho(x, y, z) = kz$

In Exercises 13–16, evaluate $\displaystyle\int_S\!\!\int f(x, y)\,dS$.

13. $f(x, y) = y + 5$
    S: $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + 2v\mathbf{k}, \quad 0 \le u \le 1, \quad 0 \le v \le 2$
14. $f(x, y) = xy$
    S: $\mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j} + v\mathbf{k}$
    $0 \le u \le \dfrac{\pi}{2}, \quad 0 \le v \le 1$
15. $f(x, y) = x + y$
    S: $\mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j} + v\mathbf{k}$
    $0 \le u \le \dfrac{\pi}{2}, \quad 0 \le v \le 1$
16. $f(x, y) = x + y$
    S: $\mathbf{r}(u, v) = 4u\cos v\,\mathbf{i} + 4u\sin v\,\mathbf{j} + 3u\mathbf{k}$
    $0 \le u \le 4, \quad 0 \le v \le \pi$

In Exercises 17–22, evaluate $\displaystyle\int_S\!\!\int f(x, y, z)\,dS$.

17. $f(x, y, z) = x^2 + y^2 + z^2$
    S: $z = x + y, \quad x^2 + y^2 \le 1$
18. $f(x, y, z) = \dfrac{xy}{z}$
    S: $z = x^2 + y^2, \quad 4 \le x^2 + y^2 \le 16$
19. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$
    S: $z = \sqrt{x^2 + y^2}, \quad x^2 + y^2 \le 4$
20. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$
    S: $z = \sqrt{x^2 + y^2}, \quad (x - 1)^2 + y^2 \le 1$
21. $f(x, y, z) = x^2 + y^2 + z^2$
    S: $x^2 + y^2 = 9, \quad 0 \le x \le 3, \quad 0 \le y \le 3, \quad 0 \le z \le 9$
22. $f(x, y, z) = x^2 + y^2 + z^2$
    S: $x^2 + y^2 = 9, \quad 0 \le x \le 3, \quad 0 \le z \le x$

In Exercises 23–28, find the flux of F through S,

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\,dS$$

where N is the upward unit normal vector to S.

23. $\mathbf{F}(x, y, z) = 3z\mathbf{i} - 4\mathbf{j} + y\mathbf{k}$
    S: $z = 1 - x - y$, first octant
24. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$
    S: $z = 6 - 3x - 2y$, first octant
25. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
    S: $z = 1 - x^2 - y^2, \quad z \ge 0$
26. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
    S: $x^2 + y^2 + z^2 = 36$, first octant
27. $\mathbf{F}(x, y, z) = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$
    S: $z = x^2 + y^2, \quad x^2 + y^2 \le 4$
28. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$
    S: $z = \sqrt{a^2 - x^2 - y^2}$

In Exercises 29 and 30, find the flux of F over the closed surface. (Let N be the outward unit normal vector of the surface.)

29. $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
    S: $z = 16 - x^2 - y^2, \quad z = 0$
30. $\mathbf{F}(x, y, z) = 4xy\mathbf{i} + z^2\mathbf{j} + yz\mathbf{k}$
    S: unit cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$
31. ***Electrical Charge*** Let $\mathbf{E} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ be an electrostatic field. Use Gauss's Law to find the total charge enclosed by the closed surface consisting of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and its circular base in the xy-plane.

32. ***Electrical Charge*** Let $\mathbf{E} = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$ be an electrostatic field. Use Gauss's Law to find the total charge enclosed by the closed surface consisting of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and its circular base in the xy-plane.

Moment of Inertia **In Exercises 33 and 34, use the following formulas for the moments of inertia about the coordinate axes of a surface lamina of density ρ.**

$$I_x = \int_S\!\!\int (y^2 + z^2)\rho(x, y, z)\, dS$$

$$I_y = \int_S\!\!\int (x^2 + z^2)\rho(x, y, z)\, dS$$

$$I_z = \int_S\!\!\int (x^2 + y^2)\rho(x, y, z)\, dS$$

33. Verify that the moment of inertia of a conical shell of uniform density about its axis is $\frac{1}{2}ma^2$, where m is the mass and a is the radius and height.

34. Verify that the moment of inertia of a spherical shell of uniform density about its diameter is $\frac{2}{3}ma^2$, where m is the mass and a is the radius.

Moment of Inertia **In Exercises 35 and 36, find I_z for the given lamina with uniform density of 1. Use a computer algebra system to verify your results.**

35. $x^2 + y^2 = a^2, \quad 0 \le z \le h$

36. $z = x^2 + y^2, \quad 0 \le z \le h$

CAS ***Flow Rate*** **In Exercises 37 and 38, use a computer algebra system to find the rate of mass flow of a fluid of density ρ through the surface S oriented upward if the velocity field is given by $\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$.**

37. $S\colon\ z = 16 - x^2 - y^2, \quad z \ge 0$

38. $S\colon\ z = \sqrt{16 - x^2 - y^2}$

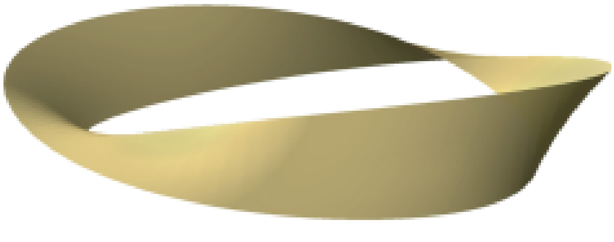
WRITING ABOUT CONCEPTS

39. Define a surface integral of the scalar function f over a surface $z = g(x, y)$. Explain how to evaluate the surface integral.

40. Describe an orientable surface.

41. Define a flux integral and explain how it is evaluated.

42. Is the surface shown in the figure orientable? Explain.



Double twist

CAS **43.** ***Investigation***

(a) Use a computer algebra system to graph the vector-valued function

$$\mathbf{r}(u, v) = (4 - v\sin u)\cos(2u)\mathbf{i} + (4 - v\sin u)\sin(2u)\mathbf{j} + v\cos u\mathbf{k}, \quad 0 \le u \le \pi, \quad -1 \le v \le 1.$$

This surface is called a Möbius strip.

(b) Explain why this surface is not orientable.

(c) Use a computer algebra system to graph the space curve represented by $\mathbf{r}(u, 0)$. Identify the curve.

(d) Construct a Möbius strip by cutting a strip of paper, making a single twist, and pasting the ends together.

(e) Cut the Möbius strip along the space curve graphed in part (c), and describe the result.

CAPSTONE

44. Consider the vector field

$$\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$$

and the orientable surface S given in parametric form by

$$\mathbf{r}(u, v) = (u + v^2)\mathbf{i} + (u - v)\mathbf{j} + u^2\mathbf{k},$$
$$0 \le u \le 2, \ -1 \le v \le 1.$$

(a) Find and interpret $\mathbf{r}_u \times \mathbf{r}_v$.

(b) Find $\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)$ as a function of u and v.

(c) Find u and v at the point $P(3, 1, 4)$.

(d) Explain how to find the normal component of $\mathbf{F}$ to the surface at P. Then find this value.

(e) Evaluate the flux integral $\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS$.

SECTION PROJECT

Hyperboloid of One Sheet

Consider the parametric surface given by the function

$$\mathbf{r}(u, v) = a\cosh u\cos v\mathbf{i} + a\cosh u\sin v\mathbf{j} + b\sinh u\mathbf{k}.$$

(a) Use a graphing utility to graph $\mathbf{r}$ for various values of the constants a and b. Describe the effect of the constants on the shape of the surface.

(b) Show that the surface is a hyperboloid of one sheet given by

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 1.$$

(c) For fixed values $u = u_0$, describe the curves given by

$$\mathbf{r}(u_0, v) = a\cosh u_0\cos v\mathbf{i} + a\cosh u_0\sin v\mathbf{j} + b\sinh u_0\mathbf{k}.$$

(d) For fixed values $v = v_0$, describe the curves given by

$$\mathbf{r}(u, v_0) = a\cosh u\cos v_0\mathbf{i} + a\cosh u\sin v_0\mathbf{j} + b\sinh u\mathbf{k}.$$

(e) Find a normal vector to the surface at $(u, v) = (0, 0)$.

15.7 Divergence Theorem

- **Understand and use the Divergence Theorem.**
- **Use the Divergence Theorem to calculate flux.**

Divergence Theorem

Recall from Section 15.4 that an alternative form of Green's Theorem is

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \int_R\!\!\int \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$
$$= \int_R\!\!\int \operatorname{div} \mathbf{F}\, dA.$$

In an analogous way, the **Divergence Theorem** gives the relationship between a triple integral over a solid region Q and a surface integral over the surface of Q. In the statement of the theorem, the surface S is **closed** in the sense that it forms the complete boundary of the solid Q. Regions bounded by spheres, ellipsoids, cubes, tetrahedrons, or combinations of these surfaces are typical examples of closed surfaces. Assume that Q is a solid region on which a triple integral can be evaluated, and that the closed surface S is oriented by *outward* unit normal vectors, as shown in Figure 15.54. With these restrictions on S and Q, the Divergence Theorem is as follows.

Mary Evans Picture Collection

CARL FRIEDRICH GAUSS (1777–1855)

The **Divergence Theorem** is also called **Gauss's Theorem,** after the famous German mathematician Carl Friedrich Gauss. Gauss is recognized, with Newton and Archimedes, as one of the three greatest mathematicians in history. One of his many contributions to mathematics was made at the age of 22, when, as part of his doctoral dissertation, he proved the *Fundamental Theorem of Algebra.*

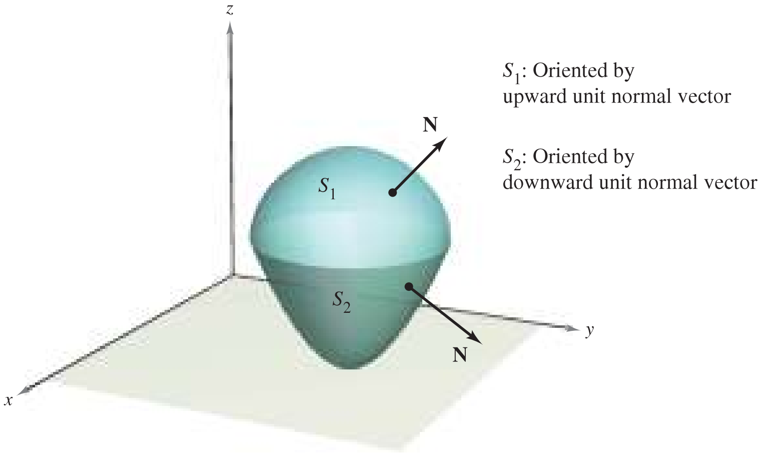

z
S_1: Oriented by upward unit normal vector
N
S_2: Oriented by downward unit normal vector
S_1
S_2
y
N
x

Figure 15.54

THEOREM 15.12 THE DIVERGENCE THEOREM

Let Q be a solid region bounded by a closed surface S oriented by a unit normal vector directed outward from Q. If $\mathbf{F}$ is a vector field whose component functions have continuous first partial derivatives in Q, then

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_Q \operatorname{div} \mathbf{F}\, dV.$$

NOTE As noted at the left above, the Divergence Theorem is sometimes called Gauss's Theorem. It is also sometimes called Ostrogradsky's Theorem, after the Russian mathematician Michel Ostrogradsky (1801–1861). ■

NOTE This proof is restricted to a *simple* solid region. The general proof is best left to a course in advanced calculus.

PROOF If you let $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, the theorem takes the form

$$\int\!\!\int_S \mathbf{F} \cdot \mathbf{N}\, dS = \int\!\!\int_S (M\mathbf{i} \cdot \mathbf{N} + N\mathbf{j} \cdot \mathbf{N} + P\mathbf{k} \cdot \mathbf{N})\, dS$$

$$= \int\!\!\int_Q\!\!\int \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dV.$$

You can prove this by verifying that the following three equations are valid.

$$\int\!\!\int_S M\mathbf{i} \cdot \mathbf{N}\, dS = \int\!\!\int_Q\!\!\int \frac{\partial M}{\partial x}\, dV$$

$$\int\!\!\int_S N\mathbf{j} \cdot \mathbf{N}\, dS = \int\!\!\int_Q\!\!\int \frac{\partial N}{\partial y}\, dV$$

$$\int\!\!\int_S P\mathbf{k} \cdot \mathbf{N}\, dS = \int\!\!\int_Q\!\!\int \frac{\partial P}{\partial z}\, dV$$

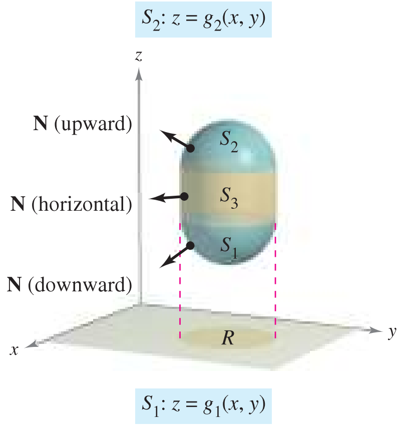


Figure 15.55

Because the verifications of the three equations are similar, only the third is discussed. Restrict the proof to a **simple solid** region with upper surface

$z = g_2(x, y)$ Upper surface

and lower surface

$z = g_1(x, y)$ Lower surface

whose projections onto the xy-plane coincide and form region R. If Q has a lateral surface like S_3 in Figure 15.55, then a normal vector is horizontal, which implies that $P\mathbf{k} \cdot \mathbf{N} = 0$. Consequently, you have

$$\int\!\!\int_S P\mathbf{k} \cdot \mathbf{N}\, dS = \int\!\!\int_{S_1} P\mathbf{k} \cdot \mathbf{N}\, dS + \int\!\!\int_{S_2} P\mathbf{k} \cdot \mathbf{N}\, dS + 0.$$

On the upper surface S_2, the outward normal vector is upward, whereas on the lower surface S_1, the outward normal vector is downward. So, by Theorem 15.11, you have the following.

$$\int\!\!\int_{S_1} P\mathbf{k} \cdot \mathbf{N}\, dS = \int\!\!\int_R P(x, y, g_1(x, y))\mathbf{k} \cdot \left(\frac{\partial g_1}{\partial x}\mathbf{i} + \frac{\partial g_1}{\partial y}\mathbf{j} - \mathbf{k}\right) dA$$

$$= -\int\!\!\int_R P(x, y, g_1(x, y))\, dA$$

$$\int\!\!\int_{S_2} P\mathbf{k} \cdot \mathbf{N}\, dS = \int\!\!\int_R P(x, y, g_2(x, y))\mathbf{k} \cdot \left(-\frac{\partial g_2}{\partial x}\mathbf{i} - \frac{\partial g_2}{\partial y}\mathbf{j} + \mathbf{k}\right) dA$$

$$= \int\!\!\int_R P(x, y, g_2(x, y))\, dA$$

Adding these results, you obtain

$$\int\!\!\int_S P\mathbf{k} \cdot \mathbf{N}\, dS = \int\!\!\int_R [P(x, y, g_2(x, y)) - P(x, y, g_1(x, y))]\, dA$$

$$= \int\!\!\int_R \left[\int_{g_1(x, y)}^{g_2(x, y)} \frac{\partial P}{\partial z}\, dz\right] dA$$

$$= \int\!\!\int_Q\!\!\int \frac{\partial P}{\partial z}\, dV.$$

■

EXAMPLE 1 Using the Divergence Theorem

Let Q be the solid region bounded by the coordinate planes and the plane $2x + 2y + z = 6$, and let $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$. Find

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS$$

where S is the surface of Q.

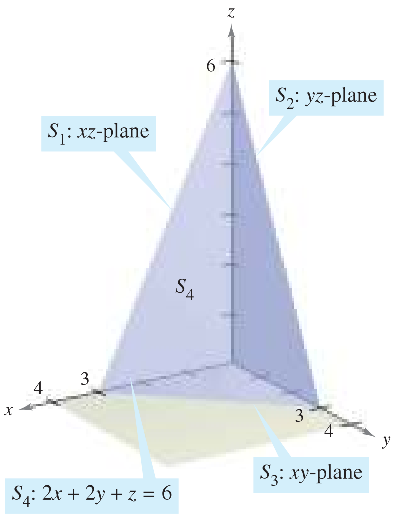


Figure 15.56

Solution From Figure 15.56, you can see that Q is bounded by four subsurfaces. So, you would need four *surface integrals* to evaluate

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS.$$

However, by the Divergence Theorem, you need only one triple integral. Because

$$\begin{aligned}\text{div } \mathbf{F} &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\\ &= 1 + 2y + 1\\ &= 2 + 2y\end{aligned}$$

you have

$$\begin{aligned}\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS &= \iiint_Q \text{div } \mathbf{F}\, dV\\ &= \int_0^3\int_0^{3-y}\int_0^{6-2x-2y} (2 + 2y)\, dz\, dx\, dy\\ &= \int_0^3\int_0^{3-y} (2z + 2yz)\Big]_0^{6-2x-2y}\, dx\, dy\\ &= \int_0^3\int_0^{3-y} (12 - 4x + 8y - 4xy - 4y^2)\, dx\, dy\\ &= \int_0^3 \Big[12x - 2x^2 + 8xy - 2x^2y - 4xy^2\Big]_0^{3-y}\, dy\\ &= \int_0^3 (18 + 6y - 10y^2 + 2y^3)\, dy\\ &= \left[18y + 3y^2 - \frac{10y^3}{3} + \frac{y^4}{2}\right]_0^3\\ &= \frac{63}{2}.\end{aligned}$$

■

TECHNOLOGY If you have access to a computer algebra system that can evaluate triple-iterated integrals, use it to verify the result in Example 1. When you are using such a utility, note that the first step is to convert the triple integral to an iterated integral—this step must be done by hand. To give yourself some practice with this important step, find the limits of integration for the following iterated integrals. Then use a computer to verify that the value is the same as that obtained in Example 1.

$$\int_?^?\int_?^?\int_?^? (2 + 2y)\, dy\, dz\, dx, \qquad \int_?^?\int_?^?\int_?^? (2 + 2y)\, dx\, dy\, dz$$

EXAMPLE 2 Verifying the Divergence Theorem

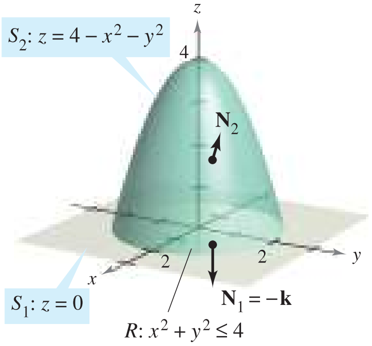


Figure 15.57

Let Q be the solid region between the paraboloid

$$z = 4 - x^2 - y^2$$

and the xy-plane. Verify the Divergence Theorem for

$$\mathbf{F}(x, y, z) = 2z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}.$$

Solution From Figure 15.57 you can see that the outward normal vector for the surface S_1 is $\mathbf{N}_1 = -\mathbf{k}$, whereas the outward normal vector for the surface S_2 is

$$\mathbf{N}_2 = \frac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}.$$

So, by Theorem 15.11, you have

$$\begin{aligned}
\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS
&= \int_{S_1}\!\!\int \mathbf{F} \cdot \mathbf{N}_1\, dS + \int_{S_2}\!\!\int \mathbf{F} \cdot \mathbf{N}_2\, dS \\
&= \int_{S_1}\!\!\int \mathbf{F} \cdot (-\mathbf{k})\, dS + \int_{S_2}\!\!\int \mathbf{F} \cdot \frac{(2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})}{\sqrt{4x^2 + 4y^2 + 1}}\, dS \\
&= \int_R\!\!\int -y^2\, dA + \int_R\!\!\int (4xz + 2xy + y^2)\, dA \\
&= -\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} y^2\, dx\, dy + \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4xz + 2xy + y^2)\, dx\, dy \\
&= \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4xz + 2xy)\, dx\, dy \\
&= \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} [4x(4 - x^2 - y^2) + 2xy]\, dx\, dy \\
&= \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (16x - 4x^3 - 4xy^2 + 2xy)\, dx\, dy \\
&= \int_{-2}^{2} \Big[8x^2 - x^4 - 2x^2y^2 + x^2y\Big]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\, dy \\
&= \int_{-2}^{2} 0\, dy \\
&= 0.
\end{aligned}$$

On the other hand, because

$$\text{div}\, \mathbf{F} = \frac{\partial}{\partial x}[2z] + \frac{\partial}{\partial y}[x] + \frac{\partial}{\partial z}[y^2] = 0 + 0 + 0 = 0$$

you can apply the Divergence Theorem to obtain the equivalent result

$$\begin{aligned}
\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS &= \iiint_Q \text{div}\, \mathbf{F}\, dV \\
&= \iiint_Q 0\, dV = 0.
\end{aligned}$$

■

EXAMPLE 3 Using the Divergence Theorem

Let Q be the solid bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 6$, and the xy-plane, as shown in Figure 15.58. Find

$$\int\!\!\int_S \mathbf{F} \cdot \mathbf{N}\, dS$$

where S is the surface of Q and

$$\mathbf{F}(x, y, z) = (x^2 + \sin z)\mathbf{i} + (xy + \cos z)\mathbf{j} + e^y\mathbf{k}.$$

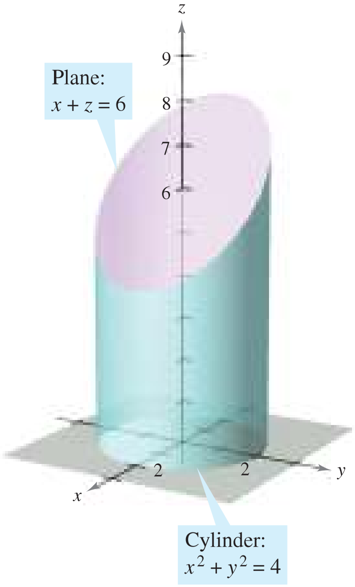


Figure 15.58

Solution Direct evaluation of this surface integral would be difficult. However, by the Divergence Theorem, you can evaluate the integral as follows.

$$\begin{aligned}
\int\!\!\int_S \mathbf{F} \cdot \mathbf{N}\, dS &= \int\!\!\int_Q\!\!\int \operatorname{div} \mathbf{F}\, dV \\
&= \int\!\!\int_Q\!\!\int (2x + x + 0)\, dV \\
&= \int\!\!\int_Q\!\!\int 3x\, dV \\
&= \int_0^{2\pi}\int_0^2\int_0^{6 - r\cos\theta} (3r\cos\theta) r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^2 (18r^2\cos\theta - 3r^3\cos^2\theta)\, dr\, d\theta \\
&= \int_0^{2\pi} (48\cos\theta - 12\cos^2\theta) d\theta \\
&= \left[48\sin\theta - 6\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} \\
&= -12\pi
\end{aligned}$$

Notice that cylindrical coordinates with $x = r\cos\theta$ and $dV = r\, dz\, dr\, d\theta$ were used to evaluate the triple integral. ■

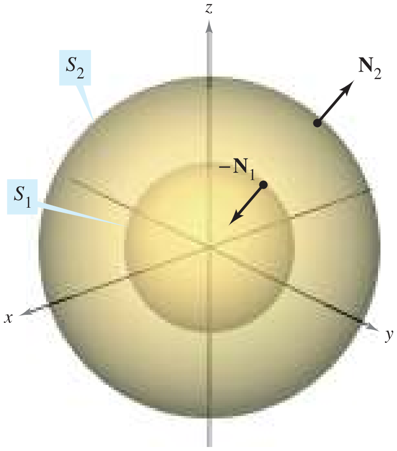


Figure 15.59

Even though the Divergence Theorem was stated for a simple solid region Q bounded by a closed surface, the theorem is also valid for regions that are the finite unions of simple solid regions. For example, let Q be the solid bounded by the closed surfaces S_1 and S_2, as shown in Figure 15.59. To apply the Divergence Theorem to this solid, let $S = S_1 \cup S_2$. The normal vector $\mathbf{N}$ to S is given by $-\mathbf{N}_1$ on S_1 and by $\mathbf{N}_2$ on S_2. So, you can write

$$\begin{aligned}
\int\!\!\int_Q\!\!\int \operatorname{div} \mathbf{F}\, dV &= \int\!\!\int_S \mathbf{F} \cdot \mathbf{N}\, dS \\
&= \int\!\!\int_{S_1} \mathbf{F} \cdot (-\mathbf{N}_1)\, dS + \int\!\!\int_{S_2} \mathbf{F} \cdot \mathbf{N}_2\, dS \\
&= -\int\!\!\int_{S_1} \mathbf{F} \cdot \mathbf{N}_1\, dS + \int\!\!\int_{S_2} \mathbf{F} \cdot \mathbf{N}_2\, dS.
\end{aligned}$$

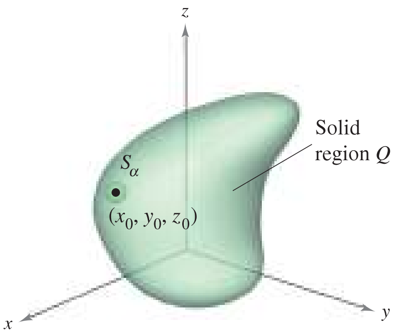


Figure 15.60

Flux and the Divergence Theorem

To help understand the Divergence Theorem, consider the two sides of the equation

$$\int\!\!\int_S \mathbf{F}\cdot\mathbf{N}\,dS = \int\!\!\int_Q\!\!\int \operatorname{div}\mathbf{F}\,dV.$$

You know from Section 15.6 that the flux integral on the left determines the total fluid flow across the surface S per unit of time. This can be approximated by summing the fluid flow across small patches of the surface. The triple integral on the right measures this same fluid flow across S, but from a very different perspective—namely, by calculating the flow of fluid into (or out of) small *cubes* of volume ΔV_i. The flux of the ith cube is approximately

$$\text{Flux of } i\text{th cube} \approx \operatorname{div}\mathbf{F}(x_i, y_i, z_i)\,\Delta V_i$$

for some point (x_i, y_i, z_i) in the ith cube. Note that for a cube in the interior of Q, the gain (or loss) of fluid through any one of its six sides is offset by a corresponding loss (or gain) through one of the sides of an adjacent cube. After summing over all the cubes in Q, the only fluid flow that is not canceled by adjoining cubes is that on the outside edges of the cubes on the boundary. So, the sum

$$\sum_{i=1}^{n} \operatorname{div}\mathbf{F}(x_i, y_i, z_i)\,\Delta V_i$$

approximates the total flux into (or out of) Q, and therefore through the surface S.

To see what is meant by the divergence of $\mathbf{F}$ at a point, consider ΔV_α to be the volume of a small sphere S_α of radius α and center (x_0, y_0, z_0), contained in region Q, as shown in Figure 15.60. Applying the Divergence Theorem to S_α produces

$$\begin{aligned}\text{Flux of } \mathbf{F} \text{ across } S_\alpha &= \int\!\!\int_{Q_\alpha}\!\!\int \operatorname{div}\mathbf{F}\,dV\\ &\approx \operatorname{div}\mathbf{F}(x_0, y_0, z_0,)\,\Delta V_\alpha\end{aligned}$$

where Q_α is the interior of S_α. Consequently, you have

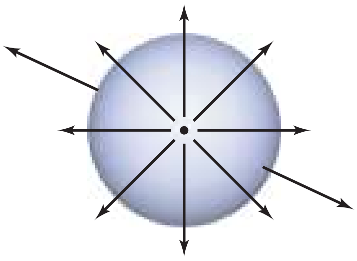
$$\operatorname{div}\mathbf{F}(x_0, y_0, z_0) \approx \frac{\text{flux of } \mathbf{F} \text{ across } S_\alpha}{\Delta V_\alpha}$$

and, by taking the limit as $\alpha \to 0$, you obtain the divergence of $\mathbf{F}$ at the point (x_0, y_0, z_0).

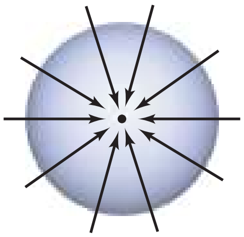
$$\begin{aligned}\operatorname{div}\mathbf{F}(x_0, y_0, z_0) &= \lim_{\alpha\to 0}\frac{\text{flux of } \mathbf{F} \text{ across } S_\alpha}{\Delta V_\alpha}\\ &= \text{flux per unit volume at } (x_0, y_0, z_0)\end{aligned}$$

The point (x_0, y_0, z_0) in a vector field is classified as a source, a sink, or incompressible, as follows.

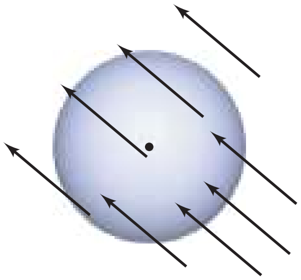
1. **Source,** if $\operatorname{div}\mathbf{F} > 0$ See Figure 15.61(a).
2. **Sink,** if $\operatorname{div}\mathbf{F} < 0$ See Figure 15.61(b).
3. **Incompressible,** if $\operatorname{div}\mathbf{F} = 0$ See Figure 15.61(c).



(a) Source: $\operatorname{div}\mathbf{F} > 0$



(b) Sink: $\operatorname{div}\mathbf{F} < 0$



(c) Incompressible: $\operatorname{div}\mathbf{F} = 0$

Figure 15.61

NOTE In hydrodynamics, a *source* is a point at which additional fluid is considered as being introduced to the region occupied by the fluid. A *sink* is a point at which fluid is considered as being removed. ■

EXAMPLE 4 Calculating Flux by the Divergence Theorem

Let Q be the region bounded by the sphere $x^2 + y^2 + z^2 = 4$. Find the outward flux of the vector field $\mathbf{F}(x, y, z) = 2x^3\mathbf{i} + 2y^3\mathbf{j} + 2z^3\mathbf{k}$ through the sphere.

Solution By the Divergence Theorem, you have

$$\begin{aligned}
\text{Flux across } S &= \int\!\!\int_S \mathbf{F}\cdot\mathbf{N}\,dS = \int\!\!\int\!\!\int_Q \operatorname{div}\mathbf{F}\,dV \\
&= \int\!\!\int\!\!\int_Q 6(x^2 + y^2 + z^2)\,dV \\
&= 6\int_0^2\int_0^{\pi}\int_0^{2\pi} \rho^4 \sin\phi\,d\theta\,d\phi\,d\rho \qquad \text{Spherical coordinates} \\
&= 6\int_0^2\int_0^{\pi} 2\pi\rho^4 \sin\phi\,d\phi\,d\rho \\
&= 12\pi\int_0^2 2\rho^4\,d\rho \\
&= 24\pi\left(\frac{32}{5}\right) \\
&= \frac{768\pi}{5}.
\end{aligned}$$

■

15.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, verify the Divergence Theorem by evaluating

$$\int\!\!\int_S \mathbf{F}\cdot\mathbf{N}\,dS$$

as a surface integral and as a triple integral.

1. $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + z^2\mathbf{k}$

S: cube bounded by the planes $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$

2. $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + z^2\mathbf{k}$

S: cylinder $x^2 + y^2 = 4$, $0 \le z \le h$

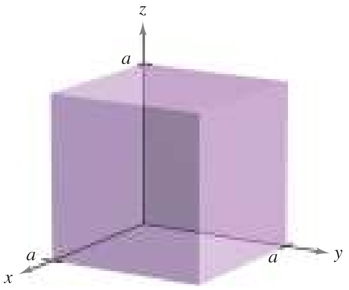


Figure for 1

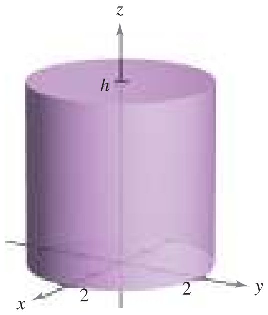


Figure for 2

3. $\mathbf{F}(x, y, z) = (2x - y)\mathbf{i} - (2y - z)\mathbf{j} + z\mathbf{k}$

S: surface bounded by the plane $2x + 4y + 2z = 12$ and the coordinate planes

4. $\mathbf{F}(x, y, z) = xy\mathbf{i} + z\mathbf{j} + (x + y)\mathbf{k}$

S: surface bounded by the planes $y = 4$ and $z = 4 - x$ and the coordinate planes

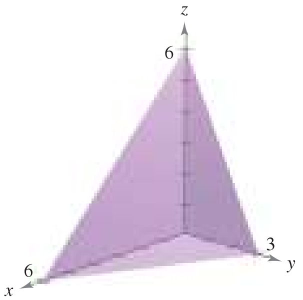


Figure for 3

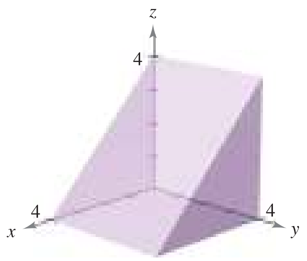


Figure for 4

5. $\mathbf{F}(x, y, z) = xz\mathbf{i} + zy\mathbf{j} + 2z^2\mathbf{k}$

S: surface bounded by $z = 1 - x^2 - y^2$ and $z = 0$

6. $\mathbf{F}(x, y, z) = xy^2\mathbf{i} + yx^2\mathbf{j} + e\mathbf{k}$

S: surface bounded by $z = \sqrt{x^2 + y^2}$ and $z = 4$

In Exercises 7–18, use the Divergence Theorem to evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$$

and find the outward flux of F through the surface of the solid bounded by the graphs of the equations. Use a computer algebra system to verify your results.

7. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

S: $x = 0, x = a, y = 0, y = a, z = 0, z = a$

8. $\mathbf{F}(x, y, z) = x^2z^2\mathbf{i} - 2y\mathbf{j} + 3xyz\mathbf{k}$

S: $x = 0, x = a, y = 0, y = a, z = 0, z = a$

9. $\mathbf{F}(x, y, z) = x^2\mathbf{i} - 2xy\mathbf{j} + xyz^2\mathbf{k}$

S: $z = \sqrt{a^2 - x^2 - y^2}, z = 0$

10. $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} - yz\mathbf{k}$

S: $z = \sqrt{a^2 - x^2 - y^2}, z = 0$

11. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

S: $x^2 + y^2 + z^2 = 9$

12. $\mathbf{F}(x, y, z) = xyz\mathbf{j}$

S: $x^2 + y^2 = 4, z = 0, z = 5$

13. $\mathbf{F}(x, y, z) = x\mathbf{i} + y^2\mathbf{j} - z\mathbf{k}$

S: $x^2 + y^2 = 25, z = 0, z = 7$

14. $\mathbf{F}(x, y, z) = (xy^2 + \cos z)\mathbf{i} + (x^2y + \sin z)\mathbf{j} + e^z\mathbf{k}$

S: $z = \frac{1}{2}\sqrt{x^2 + y^2}, z = 8$

15. $\mathbf{F}(x, y, z) = x^3\mathbf{i} + x^2y\mathbf{j} + x^2e^y\mathbf{k}$

S: $z = 4 - y, z = 0, x = 0, x = 6, y = 0$

16. $\mathbf{F}(x, y, z) = xe^z\mathbf{i} + ye^z\mathbf{j} + e^z\mathbf{k}$

S: $z = 4 - y, z = 0, x = 0, x = 6, y = 0$

17. $\mathbf{F}(x, y, z) = xy\mathbf{i} + 4y\mathbf{j} + xz\mathbf{k}$

S: $x^2 + y^2 + z^2 = 16$

18. $\mathbf{F}(x, y, z) = 2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

S: $z = \sqrt{4 - x^2 - y^2}, z = 0$

In Exercises 19 and 20, evaluate

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{N}\, dS$$

where S is the closed surface of the solid bounded by the graphs of $x = 4$ and $z = 9 - y^2$, and the coordinate planes.

19. $\mathbf{F}(x, y, z) = (4xy + z^2)\mathbf{i} + (2x^2 + 6yz)\mathbf{j} + 2xz\mathbf{k}$

20. $\mathbf{F}(x, y, z) = xy\cos z\,\mathbf{i} + yz\sin x\,\mathbf{j} + xyz\mathbf{k}$

WRITING ABOUT CONCEPTS

21. State the Divergence Theorem.

22. How do you determine if a point (x_0, y_0, z_0) in a vector field is a source, a sink, or incompressible?

23. (a) Use the Divergence Theorem to verify that the volume of the solid bounded by a surface S is

$$\iint_S x\, dy\, dz = \iint_S y\, dz\, dx = \iint_S z\, dx\, dy.$$

(b) Verify the result of part (a) for the cube bounded by $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, and $z = a$.

CAPSTONE

24. Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let S be the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$. Verify the Divergence Theorem by evaluating

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$$

as a surface integral and as a triple integral.

25. Verify that

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{N}\, dS = 0$$

for any closed surface S.

26. For the constant vector field $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, verify that

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = 0$$

where V is the volume of the solid bounded by the closed surface S.

27. Given the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, verify that

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = 3V$$

where V is the volume of the solid bounded by the closed surface S.

28. Given the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, verify that

$$\frac{1}{\|\mathbf{F}\|}\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \frac{3}{\|\mathbf{F}\|}\iiint_Q dV.$$

In Exercises 29 and 30, prove the identity, assuming that Q, S, and N meet the conditions of the Divergence Theorem and that the required partial derivatives of the scalar functions f and g are continuous. The expressions $D_{\mathbf{N}}f$ and $D_{\mathbf{N}}g$ are the derivatives in the direction of the vector N and are defined by

$$D_{\mathbf{N}}f = \nabla f \cdot \mathbf{N}, \quad D_{\mathbf{N}}g = \nabla g \cdot \mathbf{N}.$$

29. $$\iiint_Q (f\nabla^2 g + \nabla f \cdot \nabla g)\,dV = \iint_S f D_{\mathbf{N}}g\, dS$$

[*Hint:* Use $\operatorname{div}(f\mathbf{G}) = f \operatorname{div}\mathbf{G} + \nabla f \cdot \mathbf{G}$.]

30. $$\iiint_Q (f\nabla^2 g - g\nabla^2 f)\, dV = \iint_S (f D_{\mathbf{N}}g - g D_{\mathbf{N}}f)\, dS$$

(*Hint:* Use Exercise 29 twice.)

15.8 Stokes's Theorem

- **Understand and use Stokes's Theorem.**
- **Use curl to analyze the motion of a rotating liquid.**

Bettmann/Corbis

GEORGE GABRIEL STOKES (1819–1903)

Stokes became a Lucasian professor of mathematics at Cambridge in 1849. Five years later, he published the theorem that bears his name as a prize examination question there.

Stokes's Theorem

A second higher-dimension analog of Green's Theorem is called **Stokes's Theorem,** after the English mathematical physicist George Gabriel Stokes. Stokes was part of a group of English mathematical physicists referred to as the Cambridge School, which included William Thomson (Lord Kelvin) and James Clerk Maxwell. In addition to making contributions to physics, Stokes worked with infinite series and differential equations, as well as with the integration results presented in this section.

Stokes's Theorem gives the relationship between a surface integral over an oriented surface S and a line integral along a closed space curve C forming the boundary of S, as shown in Figure 15.62. The positive direction along C is counterclockwise relative to the normal vector $\mathbf{N}$. That is, if you imagine grasping the normal vector $\mathbf{N}$ with your right hand, with your thumb pointing in the direction of $\mathbf{N}$, your fingers will point in the positive direction C, as shown in Figure 15.63.

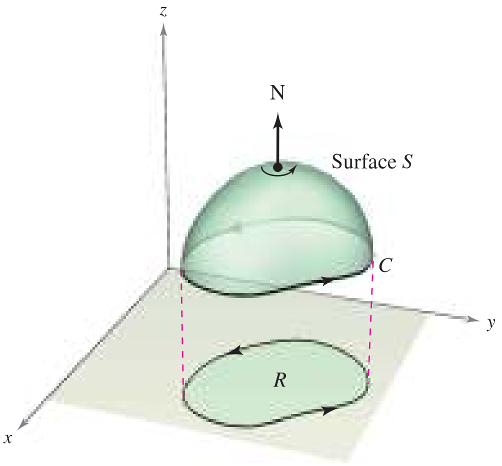
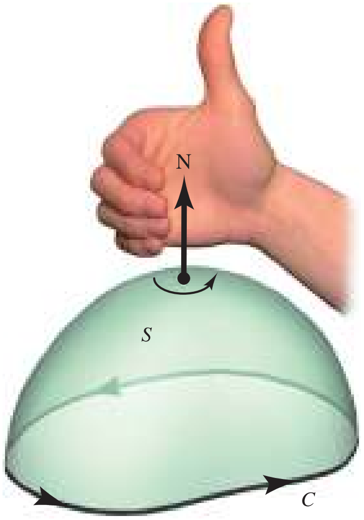


Figure 15.62



Direction along C is counterclockwise relative to $\mathbf{N}$.
Figure 15.63

THEOREM 15.13 STOKES'S THEOREM

Let S be an oriented surface with unit normal vector $\mathbf{N}$, bounded by a piecewise smooth simple closed curve C with a positive orientation. If $\mathbf{F}$ is a vector field whose component functions have continuous first partial derivatives on an open region containing S and C, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_S\!\!\int (\text{curl } \mathbf{F})\cdot \mathbf{N}\, dS.$$

NOTE The line integral may be written in the differential form $\int_C M\,dx + N\,dy + P\,dz$ or in the vector form $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$. ■

EXAMPLE 1 Using Stokes's Theorem

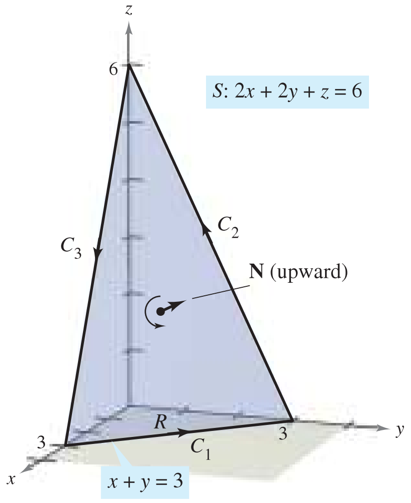


Figure 15.64

Let C be the oriented triangle lying in the plane $2x + 2y + z = 6$, as shown in Figure 15.64. Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F}(x, y, z) = -y^2\mathbf{i} + z\mathbf{j} + x\mathbf{k}$.

Solution Using Stokes's Theorem, begin by finding the curl of **F**.

$$\text{curl }\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & z & x \end{vmatrix} = -\mathbf{i} - \mathbf{j} + 2y\mathbf{k}$$

Considering $z = 6 - 2x - 2y = g(x, y)$, you can use Theorem 15.11 for an upward normal vector to obtain

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_S\int (\text{curl }\mathbf{F}) \cdot \mathbf{N}\, dS \\
&= \int_R\int (-\mathbf{i} - \mathbf{j} + 2y\mathbf{k}) \cdot [-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}]\, dA \\
&= \int_R\int (-\mathbf{i} - \mathbf{j} + 2y\mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j} + \mathbf{k})\, dA \\
&= \int_0^3 \int_0^{3-y} (2y - 4)\, dx\, dy \\
&= \int_0^3 (-2y^2 + 10y - 12)\, dy \\
&= \left[-\frac{2y^3}{3} + 5y^2 - 12y\right]_0^3 \\
&= -9.
\end{aligned}$$

■

Try evaluating the line integral in Example 1 directly, *without* using Stokes's Theorem. One way to do this would be to consider C as the union of C_1, C_2, and C_3, as follows.

C_1: $\mathbf{r}_1(t) = (3 - t)\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 3$
C_2: $\mathbf{r}_2(t) = (6 - t)\mathbf{j} + (2t - 6)\mathbf{k}, \quad 3 \le t \le 6$
C_3: $\mathbf{r}_3(t) = (t - 6)\mathbf{i} + (18 - 2t)\mathbf{k}, \quad 6 \le t \le 9$

The value of the line integral is

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot \mathbf{r}_1'(t)\, dt + \int_{C_2} \mathbf{F} \cdot \mathbf{r}_2'(t)\, dt + \int_{C_3} \mathbf{F} \cdot \mathbf{r}_3'(t)\, dt \\
&= \int_0^3 t^2\, dt + \int_3^6 (-2t + 6)\, dt + \int_6^9 (-2t + 12)\, dt \\
&= 9 - 9 - 9 \\
&= -9.
\end{aligned}$$

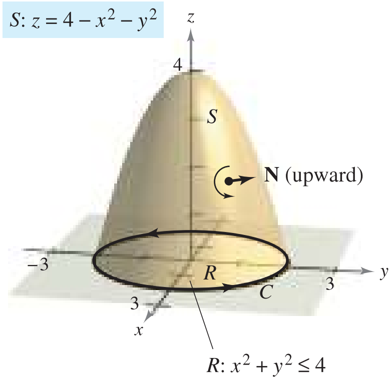


Figure 15.65

EXAMPLE 2 Verifying Stokes's Theorem

Let S be the portion of the paraboloid $z = 4 - x^2 - y^2$ lying above the xy-plane, oriented upward (see Figure 15.65). Let C be its boundary curve in the xy-plane, oriented counterclockwise. Verify Stokes's Theorem for

$$\mathbf{F}(x, y, z) = 2z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}$$

by evaluating the surface integral and the equivalent line integral.

Solution As a *surface integral*, you have $z = g(x, y) = 4 - x^2 - y^2$, $g_x = -2x$, $g_y = -2y$, and

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2z & x & y^2 \end{vmatrix} = 2y\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

By Theorem 15.11, you obtain

$$\begin{aligned}
\int_S\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS &= \int_R\!\!\int (2y\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\, dA \\
&= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4xy + 4y + 1)\, dy\, dx \\
&= \int_{-2}^{2} \Big[2xy^2 + 2y^2 + y \Big]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\
&= \int_{-2}^{2} 2\sqrt{4 - x^2}\, dx \\
&= \text{Area of circle of radius } 2 = 4\pi.
\end{aligned}$$

As a *line integral*, you can parametrize C as

$$\mathbf{r}(t) = 2 \cos t\,\mathbf{i} + 2 \sin t\,\mathbf{j} + 0\mathbf{k}, \quad 0 \le t \le 2\pi.$$

For $\mathbf{F}(x, y, z) = 2z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}$, you obtain

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C M\, dx + N\, dy + P\, dz \\
&= \int_C 2z\, dx + x\, dy + y^2\, dz \\
&= \int_0^{2\pi} [0 + 2 \cos t(2 \cos t) + 0]\, dt \\
&= \int_0^{2\pi} 4 \cos^2 t\, dt \\
&= 2 \int_0^{2\pi} (1 + \cos 2t)\, dt \\
&= 2\Big[t + \frac{1}{2} \sin 2t \Big]_0^{2\pi} \\
&= 4\pi.
\end{aligned}$$

■

Physical Interpretation of Curl

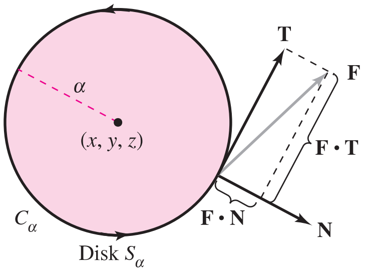


Figure 15.66

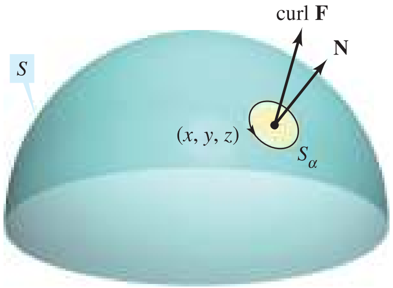


Figure 15.67

Stokes's Theorem provides insight into a physical interpretation of curl. In a vector field $\mathbf{F}$, let S_α be a *small* circular disk of radius α, centered at (x, y, z) and with boundary C_α, as shown in Figure 15.66. At each point on the circle C_α, $\mathbf{F}$ has a normal component $\mathbf{F} \cdot \mathbf{N}$ and a tangential component $\mathbf{F} \cdot \mathbf{T}$. The more closely $\mathbf{F}$ and $\mathbf{T}$ are aligned, the greater the value of $\mathbf{F} \cdot \mathbf{T}$. So, a fluid tends to move along the circle rather than across it. Consequently, you say that the line integral around C_α measures the **circulation of F around C_α.** That is,

$$\int_{C_\alpha} \mathbf{F} \cdot \mathbf{T}\, ds = \text{circulation of } \mathbf{F} \text{ around } C_\alpha.$$

Now consider a small disk S_α to be centered at some point (x, y, z) on the surface S, as shown in Figure 15.67. On such a small disk, curl $\mathbf{F}$ is nearly constant, because it varies little from its value at (x, y, z). Moreover, curl $\mathbf{F} \cdot \mathbf{N}$ is also nearly constant on S_α, because all unit normals to S_α are about the same. Consequently, Stokes's Theorem yields

$$\begin{aligned}\int_{C_\alpha} \mathbf{F} \cdot \mathbf{T}\, ds &= \int_{S_\alpha}\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS \\ &\approx (\text{curl } \mathbf{F}) \cdot \mathbf{N} \int_{S_\alpha}\!\!\int dS \\ &\approx (\text{curl } \mathbf{F}) \cdot \mathbf{N}(\pi\alpha^2).\end{aligned}$$

So,

$$\begin{aligned}(\text{curl } \mathbf{F}) \cdot \mathbf{N} &\approx \frac{\int_{C_\alpha} \mathbf{F} \cdot \mathbf{T}\, ds}{\pi\alpha^2} \\ &= \frac{\text{circulation of } \mathbf{F} \text{ around } C_\alpha}{\text{area of disk } S_\alpha} \\ &= \text{rate of circulation.}\end{aligned}$$

Assuming conditions are such that the approximation improves for smaller and smaller disks ($\alpha \to 0$), it follows that

$$(\text{curl } \mathbf{F}) \cdot \mathbf{N} = \lim_{\alpha \to 0} \frac{1}{\pi\alpha^2} \int_{C_\alpha} \mathbf{F} \cdot \mathbf{T}\, ds$$

which is referred to as the **rotation of F about N.** That is,

$$\text{curl } \mathbf{F}(x, y, z) \cdot \mathbf{N} = \text{rotation of } \mathbf{F} \text{ about } \mathbf{N} \text{ at } (x, y, z).$$

In this case, the rotation of $\mathbf{F}$ is maximum when curl $\mathbf{F}$ and $\mathbf{N}$ have the same direction. Normally, this tendency to rotate will vary from point to point on the surface S, and Stokes's Theorem

$$\underbrace{\int_S\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS}_{\text{Surface integral}} = \underbrace{\int_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Line integral}}$$

says that the collective measure of this *rotational* tendency taken over the entire surface S (surface integral) is equal to the tendency of a fluid to *circulate* around the boundary C (line integral).

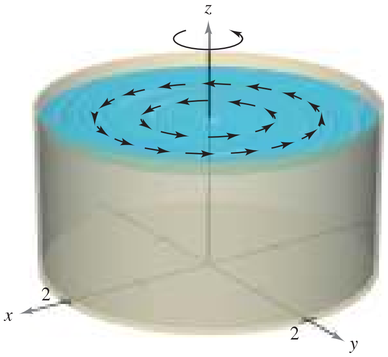


Figure 15.68

EXAMPLE 3 An Application of Curl

A liquid is swirling around in a cylindrical container of radius 2, so that its motion is described by the velocity field

$$\mathbf{F}(x, y, z) = -y\sqrt{x^2 + y^2}\,\mathbf{i} + x\sqrt{x^2 + y^2}\,\mathbf{j}$$

as shown in Figure 15.68. Find

$$\int_S\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS$$

where S is the upper surface of the cylindrical container.

Solution The curl of $\mathbf{F}$ is given by

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y\sqrt{x^2 + y^2} & x\sqrt{x^2 + y^2} & 0 \end{vmatrix} = 3\sqrt{x^2 + y^2}\,\mathbf{k}.$$

Letting $\mathbf{N} = \mathbf{k}$, you have

$$\begin{aligned} \int_S\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS &= \int_R\!\!\int 3\sqrt{x^2 + y^2}\, dA \\ &= \int_0^{2\pi}\!\!\int_0^2 (3r)r\, dr\, d\theta \\ &= \int_0^{2\pi} r^3 \Big]_0^2\, d\theta \\ &= \int_0^{2\pi} 8\, d\theta \\ &= 16\pi. \end{aligned}$$

NOTE If curl $\mathbf{F} = \mathbf{0}$ throughout region Q, the rotation of $\mathbf{F}$ about each unit normal $\mathbf{N}$ is 0. That is, $\mathbf{F}$ is irrotational. From earlier work, you know that this is a characteristic of conservative vector fields.

SUMMARY OF INTEGRATION FORMULAS

Fundamental Theorem of Calculus:

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

Fundamental Theorem of Line Integrals:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a))$$

Green's Theorem:

$$\int_C M\, dx + N\, dy = \int_R\!\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_R\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dA$$

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \int_R\!\!\int \text{div } \mathbf{F}\, dA$$

Divergence Theorem:

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int\!\!\int_Q\!\!\int \text{div } \mathbf{F}\, dV$$

Stokes's Theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS$$

15.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the curl of the vector field F.

1. $\mathbf{F}(x, y, z) = (2y - z)\mathbf{i} + e^z\mathbf{j} + xyz\mathbf{k}$
2. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + x^2\mathbf{k}$
3. $\mathbf{F}(x, y, z) = 2z\mathbf{i} - 4x^2\mathbf{j} + \arctan x\mathbf{k}$
4. $\mathbf{F}(x, y, z) = x \sin y\mathbf{i} - y \cos x\mathbf{j} + yz^2\mathbf{k}$
5. $\mathbf{F}(x, y, z) = e^{x^2+y^2}\mathbf{i} + e^{y^2+z^2}\mathbf{j} + xyz\mathbf{k}$
6. $\mathbf{F}(x, y, z) = \arcsin y\mathbf{i} + \sqrt{1 - x^2}\,\mathbf{j} + y^2\mathbf{k}$

In Exercises 7–10, verify Stokes's Theorem by evaluating $\int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_C \mathbf{F} \cdot d\mathbf{r}$ as a line integral and as a double integral.

7. $\mathbf{F}(x, y, z) = (-y + z)\mathbf{i} + (x - z)\mathbf{j} + (x - y)\mathbf{k}$
   $S: z = 9 - x^2 - y^2, \quad z \geq 0$
8. $\mathbf{F}(x, y, z) = (-y + z)\mathbf{i} + (x - z)\mathbf{j} + (x - y)\mathbf{k}$
   $S: z = \sqrt{1 - x^2 - y^2}$
9. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
   $S: 6x + 6y + z = 12$, first octant
10. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$
    $S: z = y^2, \quad 0 \leq x \leq a, \quad 0 \leq y \leq a$

In Exercises 11–20, use Stokes's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. Use a computer algebra system to verify your results. In each case, C is oriented counterclockwise as viewed from above.

11. $\mathbf{F}(x, y, z) = 2y\mathbf{i} + 3z\mathbf{j} + x\mathbf{k}$
    C: triangle with vertices $(2, 0, 0)$, $(0, 2, 0)$, $(0, 0, 2)$
12. $\mathbf{F}(x, y, z) = \arctan \dfrac{x}{y}\mathbf{i} + \ln\sqrt{x^2 + y^2}\,\mathbf{j} + \mathbf{k}$
    C: triangle with vertices $(0, 0, 0)$, $(1, 1, 1)$, $(0, 0, 2)$
13. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + 2x\mathbf{j} + y^2\mathbf{k}$
    $S: z = 1 - x^2 - y^2, \quad z \geq 0$
14. $\mathbf{F}(x, y, z) = 4xz\mathbf{i} + y\mathbf{j} + 4xy\mathbf{k}$
    $S: z = 9 - x^2 - y^2, \quad z \geq 0$
15. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
    $S: z = \sqrt{4 - x^2 - y^2}$
16. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + z^2\mathbf{j} - xyz\mathbf{k}$
    $S: z = \sqrt{4 - x^2 - y^2}$
17. $\mathbf{F}(x, y, z) = -\ln\sqrt{x^2 + y^2}\,\mathbf{i} + \arctan \dfrac{x}{y}\mathbf{j} + \mathbf{k}$
    S: $z = 9 - 2x - 3y$ over $r = 2 \sin 2\theta$ in the first octant
18. $\mathbf{F}(x, y, z) = yz\mathbf{i} + (2 - 3y)\mathbf{j} + (x^2 + y^2)\mathbf{k}, \quad x^2 + y^2 \leq 16$
    S: the first-octant portion of $x^2 + z^2 = 16$ over $x^2 + y^2 = 16$
19. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
    $S: z = x^2, \quad 0 \leq x \leq a, \quad 0 \leq y \leq a$
    $\mathbf{N}$ is the downward unit normal to the surface.
20. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad x^2 + y^2 \leq a^2$
    S: the first-octant portion of $z = x^2$ over $x^2 + y^2 = a^2$

Motion of a Liquid **In Exercises 21 and 22, the motion of a liquid in a cylindrical container of radius 1 is described by the velocity field $\mathbf{F}(x, y, z)$. Find $\int_S\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\,dS$, where S is the upper surface of the cylindrical container.**

21. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$
22. $\mathbf{F}(x, y, z) = -z\mathbf{i} + y\mathbf{k}$

WRITING ABOUT CONCEPTS

23. State Stokes's Theorem.
24. Give a physical interpretation of curl.

25. Let f and g be scalar functions with continuous partial derivatives, and let C and S satisfy the conditions of Stokes's Theorem. Verify each identity.
    (a) $\int_C (f \nabla g) \cdot d\mathbf{r} = \int_S\int (\nabla f \times \nabla g) \cdot \mathbf{N}\,dS$
    (b) $\int_C (f \nabla f) \cdot d\mathbf{r} = 0$
    (c) $\int_C (f \nabla g + g \nabla f) \cdot d\mathbf{r} = 0$
26. Demonstrate the results of Exercise 25 for the functions $f(x, y, z) = xyz$ and $g(x, y, z) = z$. Let S be the hemisphere $z = \sqrt{4 - x^2 - y^2}$.
27. Let $\mathbf{C}$ be a constant vector. Let S be an oriented surface with a unit normal vector $\mathbf{N}$, bounded by a smooth curve C. Prove that

$$\int_S\int \mathbf{C} \cdot \mathbf{N}\,dS = \frac{1}{2}\int_C (\mathbf{C} \times \mathbf{r}) \cdot d\mathbf{r}.$$

CAPSTONE

28. Verify Stokes's Theorem for each given vector field and upward oriented surface. Is the line integral or the double integral easier to set up? to evaluate? Explain.
    (a) $\mathbf{F}(x, y, z) = e^{y+z}\mathbf{i}$
    C: square with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$
    (b) $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$
    S: the portion of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 4$

PUTNAM EXAM CHALLENGE

29. Let $\mathbf{G}(x, y) = \left(\dfrac{-y}{x^2 + 4y^2}, \dfrac{x}{x^2 + 4y^2}, 0\right)$.

    Prove or disprove that there is a vector-valued function $\mathbf{F}(x, y, z) = (M(x, y, z),\ N(x, y, z),\ P(x, y, z))$ with the following properties.
    (i) M, N, P have continuous partial derivatives for all $(x, y, z) \neq (0, 0, 0)$;
    (ii) Curl $\mathbf{F} = \mathbf{0}$ for all $(x, y, z) \neq (0, 0, 0)$;
    (iii) $\mathbf{F}(x, y, 0) = \mathbf{G}(x, y)$.

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15 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, compute $\|\mathbf{F}\|$ and sketch several representative vectors in the vector field. Use a computer algebra system to verify your results.

1. $\mathbf{F}(x, y, z) = x\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ **2.** $\mathbf{F}(x, y) = \mathbf{i} - 2y\mathbf{j}$

In Exercises 3 and 4, find the gradient vector field for the scalar function.

3. $f(x, y, z) = 2x^2 + xy + z^2$ **4.** $f(x, y, z) = x^2e^{yz}$

In Exercises 5–12, determine whether the vector field is conservative. If it is, find a potential function for the vector field.

5. $\mathbf{F}(x, y) = -\dfrac{y}{x^2}\mathbf{i} + \dfrac{1}{x}\mathbf{j}$

6. $\mathbf{F}(x, y) = \dfrac{1}{y}\mathbf{i} - \dfrac{y}{x^2}\mathbf{j}$

7. $\mathbf{F}(x, y) = (xy^2 - x^2)\mathbf{i} + (x^2y + y^2)\mathbf{j}$

8. $\mathbf{F}(x, y) = (-2y^3 \sin 2x)\mathbf{i} + 3y^2(1 + \cos 2x)\mathbf{j}$

9. $\mathbf{F}(x, y, z) = 4xy^2\mathbf{i} + 2x^2\mathbf{j} + 2z\mathbf{k}$

10. $\mathbf{F}(x, y, z) = (4xy + z^2)\mathbf{i} + (2x^2 + 6yz)\mathbf{j} + 2xz\mathbf{k}$

11. $\mathbf{F}(x, y, z) = \dfrac{yz\mathbf{i} - xz\mathbf{j} - xy\mathbf{k}}{y^2z^2}$

12. $\mathbf{F}(x, y, z) = \sin z(y\mathbf{i} + x\mathbf{j} + \mathbf{k})$

In Exercises 13–20, find (a) the divergence of the vector field F and (b) the curl of the vector field F.

13. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy^2\mathbf{j} + x^2z\mathbf{k}$

14. $\mathbf{F}(x, y, z) = y^2\mathbf{j} - z^2\mathbf{k}$

15. $\mathbf{F}(x, y, z) = (\cos y + y\cos x)\mathbf{i} + (\sin x - x\sin y)\mathbf{j} + xyz\mathbf{k}$

16. $\mathbf{F}(x, y, z) = (3x - y)\mathbf{i} + (y - 2z)\mathbf{j} + (z - 3x)\mathbf{k}$

17. $\mathbf{F}(x, y, z) = \arcsin x\,\mathbf{i} + xy^2\mathbf{j} + yz^2\mathbf{k}$

18. $\mathbf{F}(x, y, z) = (x^2 - y)\mathbf{i} - (x + \sin^2 y)\mathbf{j}$

19. $\mathbf{F}(x, y, z) = \ln(x^2 + y^2)\mathbf{i} + \ln(x^2 + y^2)\mathbf{j} + z\mathbf{k}$

20. $\mathbf{F}(x, y, z) = \dfrac{z}{x}\mathbf{i} + \dfrac{z}{y}\mathbf{j} + z^2\mathbf{k}$

In Exercises 21–26, evaluate the line integral along the given path(s).

21. $\displaystyle\int_C (x^2 + y^2)\,ds$

(a) C: line segment from $(0, 0)$ to $(3, 4)$

(b) C: $x^2 + y^2 = 1$, one revolution counterclockwise, starting at $(1, 0)$

22. $\displaystyle\int_C xy\,ds$

(a) C: line segment from $(0, 0)$ to $(5, 4)$

(b) C: counterclockwise around the triangle with vertices $(0, 0)$, $(4, 0)$, $(0, 2)$

23. $\displaystyle\int_C (x^2 + y^2)\,ds$

C: $\mathbf{r}(t) = (1 - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$, $0 \le t \le 2\pi$

24. $\displaystyle\int_C (x^2 + y^2)\,ds$

C: $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}$, $0 \le t \le 2\pi$

25. $\displaystyle\int_C (2x - y)\,dx + (x + 2y)\,dy$

(a) C: line segment from $(0, 0)$ to $(3, -3)$

(b) C: one revolution counterclockwise around the circle $x = 3\cos t$, $y = 3\sin t$

26. $\displaystyle\int_C (2x - y)\,dx + (x + 3y)\,dy$

C: $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\sin t)\mathbf{j}$, $0 \le t \le \pi/2$

CAS **In Exercises 27 and 28, use a computer algebra system to evaluate the line integral over the given path.**

27. $\displaystyle\int_C (2x + y)\,ds$

$\mathbf{r}(t) = a\cos^3 t\,\mathbf{i} + a\sin^3 t\,\mathbf{j}$,

$0 \le t \le \pi/2$

28. $\displaystyle\int_C (x^2 + y^2 + z^2)\,ds$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^{3/2}\mathbf{k}$,

$0 \le t \le 4$

Lateral Surface Area **In Exercises 29 and 30, find the lateral surface area over the curve C in the xy-plane and under the surface $z = f(x, y)$.**

29. $f(x, y) = 3 + \sin(x + y)$

C: $y = 2x$ from $(0, 0)$ to $(2, 4)$

30. $f(x, y) = 12 - x - y$

C: $y = x^2$ from $(0, 0)$ to $(2, 4)$

In Exercises 31–36, evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$.

31. $\mathbf{F}(x, y) = xy\mathbf{i} + 2xy\mathbf{j}$

C: $\mathbf{r}(t) = t^2\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1$

32. $\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x + y)\mathbf{j}$

C: $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$

33. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

C: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi$

34. $\mathbf{F}(x, y, z) = (2y - z)\mathbf{i} + (z - x)\mathbf{j} + (x - y)\mathbf{k}$

C: curve of intersection of $x^2 + z^2 = 4$ and $y^2 + z^2 = 4$ from $(2, 2, 0)$ to $(0, 0, 2)$

35. $\mathbf{F}(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$

C: curve of intersection of $z = x^2 + y^2$ and $y = x$ from $(0, 0, 0)$ to $(2, 2, 8)$

36. $\mathbf{F}(x, y, z) = (x^2 - z)\mathbf{i} + (y^2 + z)\mathbf{j} + x\mathbf{k}$

C: curve of intersection of $z = x^2$ and $x^2 + y^2 = 4$ from $(0, -2, 0)$ to $(0, 2, 0)$

CAS **In Exercises 37 and 38, use a computer algebra system to evaluate the line integral.**

37. $\displaystyle\int_C xy\,dx + (x^2 + y^2)\,dy$

C: $y = x^2$ from $(0, 0)$ to $(2, 4)$ and $y = 2x$ from $(2, 4)$ to $(0, 0)$

38. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$

$\mathbf{F}(x, y) = (2x - y)\mathbf{i} + (2y - x)\mathbf{j}$

C: $\mathbf{r}(t) = (2\cos t + 2t\sin t)\mathbf{i} + (2\sin t - 2t\cos t)\mathbf{j}$, $0 \le t \le \pi$

39. ***Work*** Find the work done by the force field $\mathbf{F} = x\mathbf{i} - \sqrt{y}\,\mathbf{j}$ along the path $y = x^{3/2}$ from $(0, 0)$ to $(4, 8)$.

40. ***Work*** A 20-ton aircraft climbs 2000 feet while making a 90° turn in a circular arc of radius 10 miles. Find the work done by the engines.

In Exercises 41 and 42, evaluate the integral using the Fundamental Theorem of Line Integrals.

41. $\displaystyle\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz$

C: smooth curve from $(0, 0, 0)$ to $(1, 3, 2)$

42. $\displaystyle\int_C y\,dx + x\,dy + \frac{1}{z}\,dz$

C: smooth curve from $(0, 0, 1)$ to $(4, 4, 4)$

43. Evaluate the line integral $\displaystyle\int_C y^2\,dx + 2xy\,dy$.

(a) C: $\mathbf{r}(t) = (1 + 3t)\mathbf{i} + (1 + t)\mathbf{j}$, $0 \le t \le 1$

(b) C: $\mathbf{r}(t) = t\mathbf{i} + \sqrt{t}\,\mathbf{j}$, $1 \le t \le 4$

(c) Use the Fundamental Theorem of Line Integrals, where C is a smooth curve from $(1, 1)$ to $(4, 2)$.

44. ***Area and Centroid*** Consider the region bounded by the x-axis and one arch of the cycloid with parametric equations $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$. Use line integrals to find (a) the area of the region and (b) the centroid of the region.

In Exercises 45–50, use Green's Theorem to evaluate the line integral.

45. $\displaystyle\int_C y\,dx + 2x\,dy$

C: boundary of the square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$

46. $\displaystyle\int_C xy\,dx + (x^2 + y^2)\,dy$

C: boundary of the square with vertices $(0, 0)$, $(0, 2)$, $(2, 0)$, $(2, 2)$

47. $\displaystyle\int_C xy^2\,dx + x^2y\,dy$

C: $x = 4\cos t$, $y = 4\sin t$

48. $\displaystyle\int_C (x^2 - y^2)\,dx + 2xy\,dy$

C: $x^2 + y^2 = a^2$

49. $\displaystyle\int_C xy\,dx + x^2\,dy$

C: boundary of the region between the graphs of $y = x^2$ and $y = 1$

50. $\displaystyle\int_C y^2\,dx + x^{4/3}\,dy$

C: $x^{2/3} + y^{2/3} = 1$

CAS **In Exercises 51 and 52, use a computer algebra system to graph the surface represented by the vector-valued function.**

51. $\mathbf{r}(u, v) = \sec u\cos v\,\mathbf{i} + (1 + 2\tan u)\sin v\,\mathbf{j} + 2u\,\mathbf{k}$

$0 \le u \le \dfrac{\pi}{3}$, $0 \le v \le 2\pi$

52. $\mathbf{r}(u, v) = e^{-u/4}\cos v\,\mathbf{i} + e^{-u/4}\sin v\,\mathbf{j} + \dfrac{u}{6}\mathbf{k}$

$0 \le u \le 4$, $0 \le v \le 2\pi$

CAS **53.** ***Investigation*** Consider the surface represented by the vector-valued function

$\mathbf{r}(u, v) = 3\cos v\cos u\,\mathbf{i} + 3\cos v\sin u\,\mathbf{j} + \sin v\,\mathbf{k}$.

Use a computer algebra system to do the following.

(a) Graph the surface for $0 \le u \le 2\pi$ and $-\dfrac{\pi}{2} \le v \le \dfrac{\pi}{2}$.

(b) Graph the surface for $0 \le u \le 2\pi$ and $\dfrac{\pi}{4} \le v \le \dfrac{\pi}{2}$.

(c) Graph the surface for $0 \le u \le \dfrac{\pi}{4}$ and $0 \le v \le \dfrac{\pi}{2}$.

(d) Graph and identify the space curve for $0 \le u \le 2\pi$ and $v = \dfrac{\pi}{4}$.

(e) Approximate the area of the surface graphed in part (b).

(f) Approximate the area of the surface graphed in part (c).

54. Evaluate the surface integral $\displaystyle\int_S\!\!\int z\,dS$ over the surface S:

$\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + \sin v\,\mathbf{k}$

where $0 \le u \le 2$ and $0 \le v \le \pi$.

CAS **55.** Use a computer algebra system to graph the surface S and approximate the surface integral

$$\int_S\!\!\int (x + y)\,dS$$

where S is the surface

S: $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + (u - 1)(2 - u)\mathbf{k}$

over $0 \le u \le 2$ and $0 \le v \le 2\pi$.

56. ***Mass*** A cone-shaped surface lamina S is given by

$$z = a\left(a - \sqrt{x^2 + y^2}\right), \quad 0 \le z \le a^2.$$

At each point on S, the density is proportional to the distance between the point and the z-axis.

(a) Sketch the cone-shaped surface.

(b) Find the mass m of the lamina.

In Exercises 57 and 58, verify the Divergence Theorem by evaluating

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS$$

as a surface integral and as a triple integral.

57. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$

Q: solid region bounded by the coordinate planes and the plane $2x + 3y + 4z = 12$

58. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Q: solid region bounded by the coordinate planes and the plane $2x + 3y + 4z = 12$

In Exercises 59 and 60, verify Stokes's Theorem by evaluating

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

as a line integral and as a double integral.

59. $\mathbf{F}(x, y, z) = (\cos y + y\cos x)\mathbf{i} + (\sin x - x\sin y)\mathbf{j} + xyz\mathbf{k}$

S: portion of $z = y^2$ over the square in the xy-plane with vertices $(0, 0)$, $(a, 0)$, (a, a), $(0, a)$

$\mathbf{N}$ is the upward unit normal vector to the surface.

60. $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - z)\mathbf{j} + x^2\mathbf{k}$

S: first-octant portion of the plane $3x + y + 2z = 12$

61. Prove that it is not possible for a vector field with twice-differentiable components to have a curl of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

SECTION PROJECT

The Planimeter

You have learned many calculus techniques for finding the area of a planar region. Engineers use a mechanical device called a *planimeter* for measuring planar areas, which is based on the area formula given in Theorem 15.9 (page 1096). As you can see in the figure, the planimeter is fixed at point O (but free to pivot) and has a hinge at A. The end of the tracer arm AB moves counterclockwise around the region R. A small wheel at B is perpendicular to $\overline{AB}$ and is marked with a scale to measure how much it rolls as B traces out the boundary of region R. In this project you will show that the area of R is given by the length L of the tracer arm $\overline{AB}$ multiplied by the distance D that the wheel rolls.

Assume that point B traces out the boundary of R for $a \le t \le b$. Point A will move back and forth along a circular arc around the origin O. Let $\theta(t)$ denote the angle in the figure and let $(x(t), y(t))$ denote the coordinates of A.

(a) Show that the vector $\overrightarrow{OB}$ is given by the vector-valued function

$$\mathbf{r}(t) = [x(t) + L\cos\theta(t)]\mathbf{i} + [y(t) + L\sin\theta(t)]\mathbf{j}.$$

(b) Show that the following two integrals are equal to zero.

$$I_1 = \int_a^b \frac{1}{2}L^2\frac{d\theta}{dt}\,dt$$

$$I_2 = \int_a^b \frac{1}{2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

(c) Use the integral $\displaystyle\int_a^b [x(t)\sin\theta(t) - y(t)\cos\theta(t)]'\,dt$ to show that the following two integrals are equal.

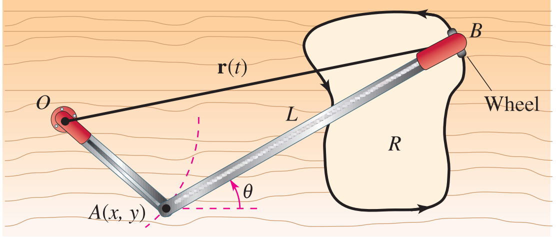
$$I_3 = \int_a^b \frac{1}{2}L\left(y\sin\theta\frac{d\theta}{dt} + x\cos\theta\frac{d\theta}{dt}\right)dt$$

$$I_4 = \int_a^b \frac{1}{2}L\left(-\sin\theta\frac{dx}{dt} + \cos\theta\frac{dy}{dt}\right)dt$$

(d) Let $\mathbf{N} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$. Explain why the distance D that the wheel rolls is given by

$$D = \int_C \mathbf{N} \cdot \mathbf{T}\,ds.$$

(e) Show that the area of region R is given by $I_1 + I_2 + I_3 + I_4 = DL$.



FOR FURTHER INFORMATION For more information about using calculus to find irregular areas, see "The Amateur Scientist" by C. L. Strong in the August 1958 issue of *Scientific American*.

P.S. PROBLEM SOLVING

1. Heat flows from areas of higher temperature to areas of lower temperature in the direction of greatest change. As a result, measuring heat flux involves the gradient of the temperature. The flux depends on the area of the surface. It is the normal direction to the surface that is important, because heat that flows in directions tangential to the surface will produce no heat loss. So, assume that the heat flux across a portion of the surface of area ΔS is given by $\Delta H \approx -k\nabla T \cdot \mathbf{N}\,dS$, where T is the temperature, $\mathbf{N}$ is the unit normal vector to the surface in the direction of the heat flow, and k is the thermal diffusivity of the material. The heat flux across the surface S is given by

$$H = \iint_S -k\nabla T \cdot \mathbf{N}\,dS.$$

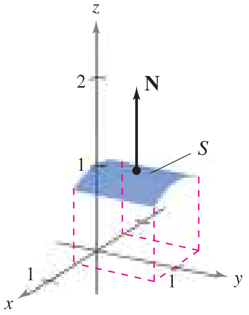
Consider a single heat source located at the origin with temperature

$$T(x, y, z) = \frac{25}{\sqrt{x^2 + y^2 + z^2}}.$$

(a) Calculate the heat flux across the surface

$$S = \left\{(x, y, z): z = \sqrt{1 - x^2}, -\frac{1}{2} \le x \le \frac{1}{2}, 0 \le y \le 1\right\}$$

as shown in the figure.



(b) Repeat the calculation in part (a) using the parametrization

$$x = \cos u, \quad y = v, \quad z = \sin u, \quad \frac{\pi}{3} \le u \le \frac{2\pi}{3}, \quad 0 \le v \le 1.$$

2. Consider a single heat source located at the origin with temperature

$$T(x, y, z) = \frac{25}{\sqrt{x^2 + y^2 + z^2}}.$$

(a) Calculate the heat flux across the surface

$$S = \left\{(x, y, z): z = \sqrt{1 - x^2 - y^2}, x^2 + y^2 \le 1\right\}$$

as shown in the figure.

(b) Repeat the calculation in part (a) using the parametrization

$$x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = \cos u, \quad 0 \le u \le \frac{\pi}{2}, \quad 0 \le v \le 2\pi.$$

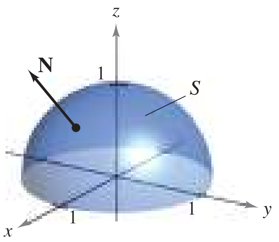


Figure for 2

3. Consider a wire of density $\rho(x, y, z)$ given by the space curve

$$C: \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b.$$

The **moments of inertia** about the x-, y-, and z-axes are given by

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)\,ds$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)\,ds$$

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)\,ds.$$

Find the moments of inertia for a wire of uniform density $\rho = 1$ in the shape of the helix

$\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 2t\mathbf{k}, \quad 0 \le t \le 2\pi$ (see figure).

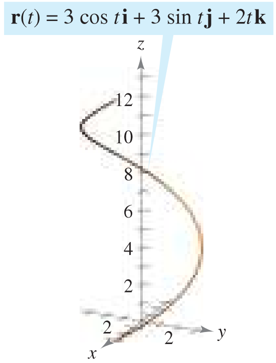


Figure for 3

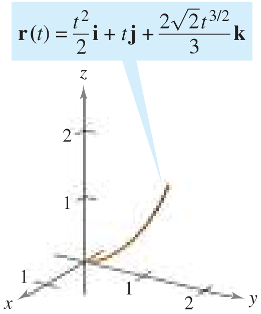


Figure for 4

4. Find the moments of inertia for the wire of density $\rho = \dfrac{1}{1 + t}$ given by the curve

$C: \mathbf{r}(t) = \dfrac{t^2}{2}\mathbf{i} + t\mathbf{j} + \dfrac{2\sqrt{2}t^{3/2}}{3}\mathbf{k}, \quad 0 \le t \le 1$ (see figure).

5. The **Laplacian** is the differential operator

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and **Laplace's equation** is

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0.$$

Any function that satisfies this equation is called **harmonic.** Show that the function $w = 1/f$ is harmonic.

CAS **6.** Consider the line integral

$$\int_C y^n\,dx + x^n\,dy$$

where C is the boundary of the region lying between the graphs of $y = \sqrt{a^2 - x^2}$ $(a > 0)$ and $y = 0$.

(a) Use a computer algebra system to verify Green's Theorem for n, an odd integer from 1 through 7.

(b) Use a computer algebra system to verify Green's Theorem for n, an even integer from 2 through 8.

(c) For n an odd integer, make a conjecture about the value of the integral.

7. Use a line integral to find the area bounded by one arch of the cycloid $x(\theta) = a(\theta - \sin\theta)$, $y(\theta) = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$, as shown in the figure.

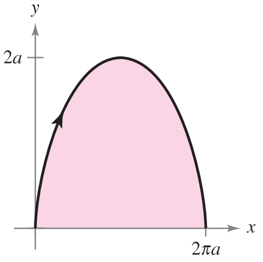


Figure for 7

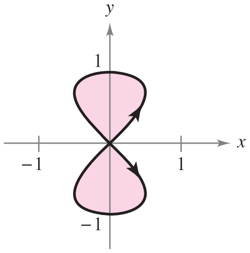


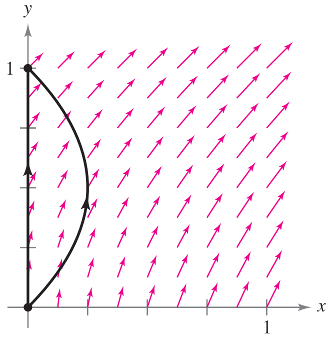
Figure for 8

8. Use a line integral to find the area bounded by the two loops of the eight curve

$$x(t) = \frac{1}{2}\sin 2t, \quad y(t) = \sin t, \quad 0 \le t \le 2\pi$$

as shown in the figure.

9. The force field $\mathbf{F}(x, y) = (x + y)\mathbf{i} + (x^2 + 1)\mathbf{j}$ acts on an object moving from the point $(0, 0)$ to the point $(0, 1)$, as shown in the figure.

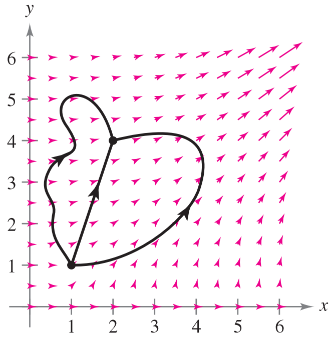


(a) Find the work done if the object moves along the path $x = 0$, $0 \le y \le 1$.

(b) Find the work done if the object moves along the path $x = y - y^2$, $0 \le y \le 1$.

(c) Suppose the object moves along the path $x = c(y - y^2)$, $0 \le y \le 1$, $c > 0$. Find the value of the constant c that minimizes the work.

10. The force field $\mathbf{F}(x, y) = (3x^2y^2)\mathbf{i} + (2x^3y)\mathbf{j}$ is shown in the figure below. Three particles move from the point $(1, 1)$ to the point $(2, 4)$ along different paths. Explain why the work done is the same for each particle, and find the value of the work.



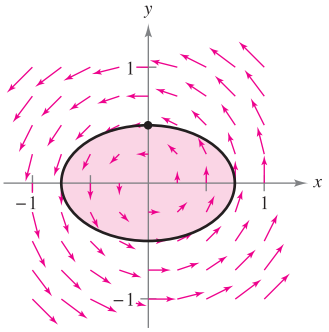
11. Let S be a smooth oriented surface with normal vector $\mathbf{N}$, bounded by a smooth simple closed curve C. Let $\mathbf{v}$ be a constant vector, and prove that

$$\int_S\!\!\int (2\mathbf{v} \cdot \mathbf{N})\,dS = \int_C (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{r}.$$

12. How does the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ compare with the magnitude of the work done by the force field

$$\mathbf{F}(x, y) = -\frac{1}{2}y\mathbf{i} + \frac{1}{2}x\mathbf{j}$$

on a particle that moves once around the ellipse (see figure)?



13. A cross section of Earth's magnetic field can be represented as a vector field in which the center of Earth is located at the origin and the positive y-axis points in the direction of the magnetic north pole. The equation for this field is

$$\begin{aligned}\mathbf{F}(x, y) &= M(x, y)\mathbf{i} + N(x, y)\mathbf{j}\\ &= \frac{m}{(x^2 + y^2)^{5/2}}[3xy\mathbf{i} + (2y^2 - x^2)\mathbf{j}]\end{aligned}$$

where m is the magnetic moment of Earth. Show that this vector field is conservative.

Appendices

Appendix A Proofs of Selected Theorems A2
Appendix B Integration Tables A21
Appendix C Precalculus Review (Online)
    C.1 Real Numbers and the Real Number Line
    C.2 The Cartesian Plane
    C.3 Review of Trigonometric Functions
Appendix D Rotation and the General Second-Degree Equation (Online)
Appendix E Complex Numbers (Online)
Appendix F Business and Economic Applications (Online)

A Proofs of Selected Theorems

THEOREM 1.2 PROPERTIES OF LIMITS (PROPERTIES 2, 3, 4, AND 5) (PAGE 59)

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \qquad \text{and} \qquad \lim_{x\to c} g(x) = K$$

2. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$

3. Product: $\lim_{x\to c} [f(x)g(x)] = LK$

4. Quotient: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$

5. Power: $\lim_{x\to c} [f(x)]^n = L^n$

PROOF To prove Property 2, choose $\varepsilon > 0$. Because $\varepsilon/2 > 0$, you know that there exists $\delta_1 > 0$ such that $0 < |x - c| < \delta_1$ implies $|f(x) - L| < \varepsilon/2$. You also know that there exists $\delta_2 > 0$ such that $0 < |x - c| < \delta_2$ implies $|g(x) - K| < \varepsilon/2$. Let δ be the smaller of δ_1 and δ_2; then $0 < |x - c| < \delta$ implies that

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(x) - K| < \frac{\varepsilon}{2}.$$

So, you can apply the triangle inequality to conclude that

$$|[f(x) + g(x)] - (L + K)| \leq |f(x) - L| + |g(x) - K| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which implies that

$$\lim_{x\to c} [f(x) + g(x)] = L + K = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

The proof that

$$\lim_{x\to c} [f(x) - g(x)] = L - K$$

is similar.

To prove Property 3, given that

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

you can write

$$f(x)g(x) = [f(x) - L][g(x) - K] + [Lg(x) + Kf(x)] - LK.$$

Because the limit of $f(x)$ is L, and the limit of $g(x)$ is K, you have

$$\lim_{x\to c} [f(x) - L] = 0 \quad \text{and} \quad \lim_{x\to c} [g(x) - K] = 0.$$

Let $0 < \varepsilon < 1$. Then there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then

$$|f(x) - L - 0| < \varepsilon \quad \text{and} \quad |g(x) - K - 0| < \varepsilon$$

which implies that

$$|[f(x) - L][g(x) - K] - 0| = |f(x) - L|\,|g(x) - K| < \varepsilon\varepsilon < \varepsilon.$$

So,

$$\lim_{x\to c} [f(x) - L][g(x) - K] = 0.$$

Furthermore, by Property 1, you have

$$\lim_{x\to c} Lg(x) = LK \quad \text{and} \quad \lim_{x\to c} Kf(x) = KL.$$

Finally, by Property 2, you obtain

$$\begin{aligned}\lim_{x\to c} f(x)g(x) &= \lim_{x\to c}[f(x) - L][g(x) - K] + \lim_{x\to c} Lg(x) + \lim_{x\to c} Kf(x) - \lim_{x\to c} LK\\ &= 0 + LK + KL - LK\\ &= LK.\end{aligned}$$

To prove Property 4, note that it is sufficient to prove that

$$\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{K}.$$

Then you can use Property 3 to write

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} f(x)\frac{1}{g(x)} = \lim_{x\to c} f(x) \cdot \lim_{x\to c} \frac{1}{g(x)} = \frac{L}{K}.$$

Let $\varepsilon > 0$. Because $\lim_{x\to c} g(x) = K$, there exists $\delta_1 > 0$ such that if

$$0 < |x - c| < \delta_1, \text{ then } |g(x) - K| < \frac{|K|}{2}$$

which implies that

$$|K| = |g(x) + [|K| - g(x)]| \leq |g(x)| + ||K| - g(x)| < |g(x)| + \frac{|K|}{2}.$$

That is, for $0 < |x - c| < \delta_1$,

$$\frac{|K|}{2} < |g(x)| \quad \text{or} \quad \frac{1}{|g(x)|} < \frac{2}{|K|}.$$

Similarly, there exists a $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$, then

$$|g(x) - K| < \frac{|K|^2}{2}\varepsilon.$$

Let δ be the smaller of δ_1 and δ_2. For $0 < |x - c| < \delta$, you have

$$\left|\frac{1}{g(x)} - \frac{1}{K}\right| = \left|\frac{K - g(x)}{g(x)K}\right| = \frac{1}{|K|} \cdot \frac{1}{|g(x)|}|K - g(x)| < \frac{1}{|K|} \cdot \frac{2}{|K|} \frac{|K|^2}{2}\varepsilon = \varepsilon.$$

So, $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{K}$.

Finally, the proof of Property 5 can be obtained by a straightforward application of mathematical induction coupled with Property 3. ■

THEOREM 1.4 THE LIMIT OF A FUNCTION INVOLVING A RADICAL (PAGE 60)

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}.$$

PROOF Consider the case for which $c > 0$ and n is any positive integer. For a given $\varepsilon > 0$, you need to find $\delta > 0$ such that

$$\left|\sqrt[n]{x} - \sqrt[n]{c}\right| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta$$

which is the same as saying

$$-\varepsilon < \sqrt[n]{x} - \sqrt[n]{c} < \varepsilon \quad \text{whenever} \quad -\delta < x - c < \delta.$$

Assume $\varepsilon < \sqrt[n]{c}$, which implies that $0 < \sqrt[n]{c} - \varepsilon < \sqrt[n]{c}$. Now, let δ be the smaller of the two numbers.

$$c - \left(\sqrt[n]{c} - \varepsilon\right)^n \quad \text{and} \quad \left(\sqrt[n]{c} + \varepsilon\right)^n - c$$

Then you have

$$\begin{aligned}
-\delta < x - c &< \delta \\
-\left[c - \left(\sqrt[n]{c} - \varepsilon\right)^n\right] < x - c &< \left(\sqrt[n]{c} + \varepsilon\right)^n - c \\
\left(\sqrt[n]{c} - \varepsilon\right)^n - c < x - c &< \left(\sqrt[n]{c} + \varepsilon\right)^n - c \\
\left(\sqrt[n]{c} - \varepsilon\right)^n < x &< \left(\sqrt[n]{c} + \varepsilon\right)^n \\
\sqrt[n]{c} - \varepsilon < \sqrt[n]{x} &< \sqrt[n]{c} + \varepsilon \\
-\varepsilon < \sqrt[n]{x} - \sqrt[n]{c} &< \varepsilon.
\end{aligned}$$

■

THEOREM 1.5 THE LIMIT OF A COMPOSITE FUNCTION (PAGE 61)

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

PROOF For a given $\varepsilon > 0$, you must find $\delta > 0$ such that

$$|f(g(x)) - f(L)| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Because the limit of $f(x)$ as $x \to L$ is $f(L)$, you know there exists $\delta_1 > 0$ such that

$$|f(u) - f(L)| < \varepsilon \quad \text{whenever} \quad |u - L| < \delta_1.$$

Moreover, because the limit of $g(x)$ as $x \to c$ is L, you know there exists $\delta > 0$ such that

$$|g(x) - L| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Finally, letting $u = g(x)$, you have

$$|f(g(x)) - f(L)| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

■

THEOREM 1.7 FUNCTIONS THAT AGREE AT ALL BUT ONE POINT (PAGE 62)

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

PROOF Let L be the limit of $g(x)$ as $x \to c$. Then, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $f(x) = g(x)$ in the open intervals $(c - \delta, c)$ and $(c, c + \delta)$, and

$$|g(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Because $f(x) = g(x)$ for all x in the open interval other than $x = c$, it follows that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

So, the limit of $f(x)$ as $x \to c$ is also L. ■

THEOREM 1.8 THE SQUEEZE THEOREM (PAGE 65)

If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ exists and is equal to L.

PROOF For $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|h(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta_1$$

and

$$|g(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta_2.$$

Because $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, there exists $\delta_3 > 0$ such that $h(x) \leq f(x) \leq g(x)$ for $0 < |x - c| < \delta_3$. Let δ be the smallest of δ_1, δ_2, and δ_3. Then, if $0 < |x - c| < \delta$, it follows that $|h(x) - L| < \varepsilon$ and $|g(x) - L| < \varepsilon$, which implies that

$$\begin{aligned} -\varepsilon < h(x) - L < \varepsilon \quad &\text{and} \quad -\varepsilon < g(x) - L < \varepsilon \\ L - \varepsilon < h(x) \quad &\text{and} \quad g(x) < L + \varepsilon. \end{aligned}$$

Now, because $h(x) \leq f(x) \leq g(x)$, it follows that $L - \varepsilon < f(x) < L + \varepsilon$, which implies that $|f(x) - L| < \varepsilon$. Therefore,

$$\lim_{x \to c} f(x) = L.$$

■

THEOREM 1.11 PROPERTIES OF CONTINUITY (PAGE 75)

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\dfrac{f}{g}$, if $g(c) \neq 0$

PROOF Because f and g are continuous at $x = c$, you can write

$$\lim_{x\to c} f(x) = f(c) \quad \text{and} \quad \lim_{x\to c} g(x) = g(c).$$

For Property 1, when b is a real number, it follows from Theorem 1.2 that

$$\lim_{x\to c}[(bf)(x)] = \lim_{x\to c}[bf(x)] = b\lim_{x\to c}[f(x)] = b\,f(c) = (bf)(c).$$

Thus, bf is continuous at $x = c$.

For Property 2, it follows from Theorem 1.2 that

$$\begin{aligned}\lim_{x\to c}(f \pm g)(x) &= \lim_{x\to c}[f(x) \pm g(x)]\\ &= \lim_{x\to c}[f(x)] \pm \lim_{x\to c}[g(x)]\\ &= f(c) \pm g(c)\\ &= (f \pm g)(c).\end{aligned}$$

Thus, $f \pm g$ is continuous at $x = c$.

For Property 3, it follows from Theorem 1.2 that

$$\begin{aligned}\lim_{x\to c}(fg)(x) &= \lim_{x\to c}[f(x)g(x)]\\ &= \lim_{x\to c}[f(x)]\lim_{x\to c}[g(x)]\\ &= f(c)g(c)\\ &= (fg)(c).\end{aligned}$$

Thus, fg is continuous at $x = c$.

For Property 4, when $g(c) \neq 0$, it follows from Theorem 1.2 that

$$\begin{aligned}\lim_{x\to c}\frac{f}{g}(x) &= \lim_{x\to c}\frac{f(x)}{g(x)}\\ &= \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}\\ &= \frac{f(c)}{g(c)}\\ &= \frac{f}{g}(\mathrm{c}).\end{aligned}$$

Thus, $\dfrac{f}{g}$ is continuous at $x = c$. ■

THEOREM 1.14 VERTICAL ASYMPTOTES (PAGE 85)

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.

PROOF Consider the case for which $f(c) > 0$, and there exists $b > c$ such that $c < x < b$ implies $g(x) > 0$. Then for $M > 0$, choose δ_1 such that

$$0 < x - c < \delta_1 \quad \text{implies that} \quad \frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}$$

and δ_2 such that

$$0 < x - c < \delta_2 \quad \text{implies that} \quad 0 < g(x) < \frac{f(c)}{2M}.$$

Now let δ be the smaller of δ_1 and δ_2. Then it follows that

$$0 < x - c < \delta \quad \text{implies that} \quad \frac{f(x)}{g(x)} > \frac{f(c)}{2}\left[\frac{2M}{f(c)}\right] = M.$$

So, it follows that

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \infty$$

and the line $x = c$ is a vertical asymptote of the graph of h. ■

ALTERNATIVE FORM OF THE DERIVATIVE (PAGE 101)

The derivative of f at c is given by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided this limit exists.

PROOF The derivative of f at c is given by

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

Let $x = c + \Delta x$. Then $x \to c$ as $\Delta x \to 0$. So, replacing $c + \Delta x$ by x, you have

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$ ■

THEOREM 2.10 THE CHAIN RULE (PAGE 131)

If $y = f(u)$ is a differentiable function of u, and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

PROOF In Section 2.4, you let $h(x) = f(g(x))$ and used the alternative form of the derivative to show that $h'(c) = f'(g(c))g'(c)$, provided $g(x) \neq g(c)$ for values of x other than c. Now consider a more general proof. Begin by considering the derivative of f.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

For a fixed value of x, define a function η such that

$$\eta(\Delta x) = \begin{cases} 0, & \Delta x = 0 \\ \dfrac{\Delta y}{\Delta x} - f'(x), & \Delta x \neq 0. \end{cases}$$

Because the limit of $\eta(\Delta x)$ as $\Delta x \to 0$ doesn't depend on the value of $\eta(0)$, you have

$$\lim_{\Delta x \to 0} \eta(\Delta x) = \lim_{\Delta x \to 0} \left[\frac{\Delta y}{\Delta x} - f'(x)\right] = 0$$

and you can conclude that η is continuous at 0. Moreover, because $\Delta y = 0$ when $\Delta x = 0$, the equation

$$\Delta y = \Delta x \eta(\Delta x) + \Delta x f'(x)$$

is valid whether Δx is zero or not. Now, by letting $\Delta u = g(x + \Delta x) - g(x)$, you can use the continuity of g to conclude that

$$\lim_{\Delta x \to 0} \Delta u = \lim_{\Delta x \to 0} [g(x + \Delta x) - g(x)] = 0$$

which implies that

$$\lim_{\Delta x \to 0} \eta(\Delta u) = 0.$$

Finally,

$$\Delta y = \Delta u \eta(\Delta u) + \Delta u f'(u) \to \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} \eta(\Delta u) + \frac{\Delta u}{\Delta x} f'(u), \quad \Delta x \neq 0$$

and taking the limit as $\Delta x \to 0$, you have

$$\begin{aligned} \frac{dy}{dx} = \frac{du}{dx}\left[\lim_{\Delta x \to 0} \eta(\Delta u)\right] + \frac{du}{dx} f'(u) &= \frac{dy}{dx}(0) + \frac{du}{dx} f'(u) \\ &= \frac{du}{dx} f'(u) \\ &= \frac{du}{dx} \cdot \frac{dy}{du}. \end{aligned}$$

■

CONCAVITY INTERPRETATION (PAGE 190)

1. Let f be differentiable on an open interval I. If the graph of f is concave *upward* on I, then the graph of f lies *above* all of its tangent lines on I.
2. Let f be differentiable on an open interval I. If the graph of f is concave *downward* on I, then the graph of f lies *below* all of its tangent lines on I.

PROOF Assume that f is concave upward on $I = (a, b)$. Then, f' is increasing on (a, b). Let c be a point in the interval $I = (a, b)$. The equation of the tangent line to the graph of f at c is given by

$$g(x) = f(c) + f'(c)(x - c).$$

If x is in the open interval (c, b), then the directed distance from point $(x, f(x))$ (on the graph of f) to the point $(x, g(x))$ (on the tangent line) is given by

$$\begin{aligned} d &= f(x) - [f(c) + f'(c)(x - c)] \\ &= f(x) - f(c) - f'(c)(x - c). \end{aligned}$$

Moreover, by the Mean Value Theorem there exists a number z in (c, x) such that

$$f'(z) = \frac{f(x) - f(c)}{x - c}.$$

So, you have

$$\begin{aligned} d &= f(x) - f(c) - f'(c)(x - c) \\ &= f'(z)(x - c) - f'(c)(x - c) \\ &= [f'(z) - f'(c)](x - c). \end{aligned}$$

The second factor $(x - c)$ is positive because $c < x$. Moreover, because f' is increasing, it follows that the first factor $[f'(z) - f'(c)]$ is also positive. Therefore, $d > 0$ and you can conclude that the graph of f lies above the tangent line at x. If x is in the open interval (a, c), a similar argument can be given. This proves the first statement. The proof of the second statement is similar. ■

THEOREM 3.7 TEST FOR CONCAVITY (PAGE 191)

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward in I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward in I.

PROOF For Property 1, assume $f''(x) > 0$ for all x in (a, b). Then, by Theorem 3.5, f' is increasing on $[a, b]$. Thus, by the definition of concavity, the graph of f is concave upward on (a, b).

For Property 2, assume $f''(x) < 0$ for all x in (a, b). Then, by Theorem 3.5, f' is decreasing on $[a, b]$. Thus, by the definition of concavity, the graph of f is concave downward on (a, b). ■

THEOREM 3.10 LIMITS AT INFINITY (PAGE 199)

If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then $\lim\limits_{x\to-\infty} \frac{c}{x^r} = 0.$

PROOF Begin by proving that

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

For $\varepsilon > 0$, let $M = 1/\varepsilon$. Then, for $x > M$, you have

$$x > M = \frac{1}{\varepsilon} \quad \Rightarrow \quad \frac{1}{x} < \varepsilon \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| < \varepsilon.$$

So, by the definition of a limit at infinity, you can conclude that the limit of $1/x$ as $x \to \infty$ is 0. Now, using this result, and letting $r = m/n$, you can write the following.

$$\begin{aligned}
\lim_{x\to\infty} \frac{c}{x^r} &= \lim_{x\to\infty} \frac{c}{x^{m/n}} \\
&= c\left[\lim_{x\to\infty}\left(\frac{1}{\sqrt[n]{x}}\right)^m\right] \\
&= c\left(\lim_{x\to\infty}\sqrt[n]{\frac{1}{x}}\right)^m \\
&= c\left(\sqrt[n]{\lim_{x\to\infty}\frac{1}{x}}\right)^m \\
&= c\left(\sqrt[n]{0}\right)^m \\
&= 0
\end{aligned}$$

The proof of the second part of the theorem is similar. ■

THEOREM 4.2 SUMMATION FORMULAS (PAGE 260)

1. $\displaystyle\sum_{i=1}^{n} c = cn$
2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

PROOF The proof of Property 1 is straightforward. By adding c to itself n times, you obtain a sum of cn.

To prove Property 2, write the sum in increasing and decreasing order and add corresponding terms, as follows.

$$\begin{array}{rccccccccc}
\sum_{i=1}^{n} i = & 1 & + & 2 & + & 3 & + \cdots + & (n-1) & + & n \\
& \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\sum_{i=1}^{n} i = & n & + & (n-1) & + & (n-2) & + \cdots + & 2 & + & 1 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
2\sum_{i=1}^{n} i = & \multicolumn{9}{c}{\underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)}_{n \text{ terms}}}
\end{array}$$

So,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

To prove Property 3, use mathematical induction. First, if $n = 1$, the result is true because

$$\sum_{i=1}^{1} i^2 = 1^2 = 1 = \frac{1(1+1)(2+1)}{6}.$$

Now, assuming the result is true for $n = k$, you can show that it is true for $n = k + 1$, as follows.

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k+1}{6}(2k^2 + k + 6k + 6) \\
&= \frac{k+1}{6}[(2k+3)(k+2)] \\
&= \frac{(k+1)(k+2)[2(k+1)+1]}{6}
\end{aligned}$$

Property 4 can be proved using a similar argument with mathematical induction. ■

THEOREM 4.8 PRESERVATION OF INEQUALITY (PAGE 278)

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \le \int_a^b f(x)\,dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \le g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

PROOF To prove Property 1, suppose, on the contrary, that

$$\int_a^b f(x)\,dx = I < 0.$$

Then, let $a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_n = b$ be a partition of $[a, b]$, and let

$$R = \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

be a Riemann sum. Because $f(x) \geq 0$, it follows that $R \geq 0$. Now, for $\|\Delta\|$ sufficiently small, you have $|R - I| < -I/2$, which implies that

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = R < I - \frac{I}{2} < 0$$

which is not possible. From this contradiction, you can conclude that

$$0 \leq \int_a^b f(x)\,dx.$$

To prove Property 2 of the theorem, note that $f(x) \leq g(x)$ implies that $g(x) - f(x) \geq 0$. So, you can apply the result of Property 1 to conclude that

$$\begin{aligned} 0 &\leq \int_a^b [g(x) - f(x)]\,dx \\ 0 &\leq \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \\ \int_a^b f(x)\,dx &\leq \int_a^b g(x)\,dx. \end{aligned}$$

■

PROPERTIES OF THE NATURAL LOGARITHMIC FUNCTION (PAGE 325)

The natural logarithmic function is one-to-one.

$$\lim_{x \to 0^+} \ln x = -\infty \qquad \text{and} \qquad \lim_{x \to \infty} \ln x = \infty$$

PROOF Recall from Section P.3 that a function f is one-to-one if for x_1 and x_2 in its domain

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

Let $f(x) = \ln x$. Then $f'(x) = \dfrac{1}{x} > 0$ for $x > 0$. So f is increasing on its entire domain $(0, \infty)$ and therefore is strictly monotonic (see Section 3.3). Choose x_1 and x_2 in the domain of f such that $x_1 \neq x_2$. Because f is strictly monotonic, it follows that either

$$f(x_1) < f(x_2) \quad \text{or} \quad f(x_1) > f(x_2).$$

In either case, $f(x_1) \neq f(x_2)$. So, $f(x) = \ln x$ is one-to-one. To verify the limits, begin by showing that $\ln 2 \geq \frac{1}{2}$. From the Mean Value Theorem for Integrals, you can write

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx = \frac{1}{c}(2 - 1) = \frac{1}{c}$$

where c is in $[1, 2]$.

This implies that

$$\begin{aligned} 1 \le &\ c \le 2 \\ 1 \ge &\ \frac{1}{c} \ge \frac{1}{2} \\ 1 \ge &\ \ln 2 \ge \frac{1}{2}. \end{aligned}$$

Now, let N be any positive (large) number. Because $\ln x$ is increasing, it follows that if $x > 2^{2N}$, then

$$\ln x > \ln 2^{2N} = 2N \ln 2.$$

However, because $\ln 2 \ge \frac{1}{2}$, it follows that

$$\ln x > 2N \ln 2 \ge 2N\left(\frac{1}{2}\right) = N.$$

This verifies the second limit. To verify the first limit, let $z = 1/x$. Then, $z \to \infty$ as $x \to 0^+$, and you can write

$$\begin{aligned} \lim_{x\to 0^+} \ln x &= \lim_{x\to 0^+}\left(-\ln \frac{1}{x}\right) \\ &= \lim_{z\to\infty} (-\ln z) \\ &= -\lim_{z\to\infty} \ln z \\ &= -\infty. \end{aligned}$$

■

THEOREM 5.8 CONTINUITY AND DIFFERENTIABILITY OF INVERSE FUNCTIONS (PAGE 347)

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable on an interval containing c and $f'(c) \ne 0$, then f^{-1} is differentiable at $f(c)$.

PROOF To prove Property 1, first show that if f is continuous on I and has an inverse function, then f is strictly monotonic on I. Suppose that f were not strictly monotonic. Then there would exist numbers x_1, x_2, x_3 in I such that $x_1 < x_2 < x_3$, but $f(x_2)$ is not between $f(x_1)$ and $f(x_3)$. Without loss of generality, assume $f(x_1) < f(x_3) < f(x_2)$. By the Intermediate Value Theorem, there exists a number x_0 between x_1 and x_2 such that $f(x_0) = f(x_3)$. So, f is not one-to-one and cannot have an inverse function. So, f must be strictly monotonic.

Because f is continuous, the Intermediate Value Theorem implies that the set of values of f

$$\{f(x): x \in I\}$$

forms an interval J. Assume that a is an interior point of J. From the previous argument, $f^{-1}(a)$ is an interior point of I. Let $\varepsilon > 0$. There exists $0 < \varepsilon_1 < \varepsilon$ such that

$$I_1 = (f^{-1}(a) - \varepsilon_1, f^{-1}(a) + \varepsilon_1) \subseteq I.$$

Because f is strictly monotonic on I_1, the set of values $\{f(x): x \in I_1\}$ forms an interval $J_1 \subseteq J$. Let $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq J_1$. Finally, if

$$|y - a| < \delta, \text{ then } |f^{-1}(y) - f^{-1}(a)| < \varepsilon_1 < \varepsilon.$$

So, f^{-1} is continuous at a. A similar proof can be given if a is an endpoint.

To prove Property 2, let y_1 and y_2 be in the domain of f^{-1}, with $y_1 < y_2$. Then, there exist x_1 and x_2 in the domain of f such that

$$f(x_1) = y_1 < y_2 = f(x_2).$$

Because f is increasing, $f(x_1) < f(x_2)$ holds precisely when $x_1 < x_2$. Therefore,

$$f^{-1}(y_1) = x_1 < x_2 = f^{-1}(y_2)$$

which implies that f^{-1} is increasing. (Property 3 can be proved in a similar way.)

Finally, to prove Property 4, consider the limit

$$(f^{-1})'(a) = \lim_{y \to a} \frac{f^{-1}(y) - f^{-1}(a)}{y - a}$$

where a is in the domain of f^{-1} and $f^{-1}(a) = c$. Because f is differentiable on an interval containing c, f is continuous on that interval, and so is f^{-1} at a. So, $y \to a$ implies that $x \to c$, and you have

$$\begin{aligned}(f^{-1})'(a) &= \lim_{x \to c} \frac{x - c}{f(x) - f(c)}\\ &= \lim_{x \to c} \frac{1}{\left(\dfrac{f(x) - f(c)}{x - c}\right)}\\ &= \frac{1}{\displaystyle\lim_{x \to c} \frac{f(x) - f(c)}{x - c}}\\ &= \frac{1}{f'(c)}.\end{aligned}$$

So, $(f^{-1})'(a)$ exists, and f^{-1} is differentiable at $f(c)$. ■

THEOREM 5.9 THE DERIVATIVE OF AN INVERSE FUNCTION (PAGE 347)

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

PROOF From the proof of Theorem 5.8, letting $a = x$, you know that g is differentiable. Using the Chain Rule, differentiate both sides of the equation $x = f(g(x))$ to obtain

$$1 = f'(g(x)) \frac{d}{dx}[g(x)].$$

Because $f'(g(x)) \neq 0$, you can divide by this quantity to obtain

$$\frac{d}{dx}[g(x)] = \frac{1}{f'(g(x))}.$$

■

THEOREM 5.10 OPERATIONS WITH EXPONENTIAL FUNCTIONS (PROPERTY 2) (PAGE 353)

2. $\dfrac{e^a}{e^b} = e^{a-b}$ (Let a and b be any real numbers.)

PROOF To prove Property 2, you can write

$$\ln\left(\frac{e^a}{e^b}\right) = \ln e^a - \ln e^b = a - b = \ln(e^{a-b})$$

Because the natural logarithmic function is one-to-one, you can conclude that

$$\frac{e^a}{e^b} = e^{a-b}.$$ ■

THEOREM 5.15 A LIMIT INVOLVING *e* (PAGE 366)

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$$

PROOF Let $y = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x$. Taking the natural logarithm of each side, you have

$$\ln y = \ln\left[\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x\right].$$

Because the natural logarithmic function is continuous, you can write

$$\ln y = \lim_{x\to\infty}\left[x \ln\left(1 + \frac{1}{x}\right)\right] = \lim_{x\to\infty}\left\{\frac{\ln[1 + (1/x)]}{1/x}\right\}.$$

Letting $x = \dfrac{1}{t}$, you have

$$\begin{aligned}\ln y &= \lim_{t\to 0^+}\frac{\ln(1+t)}{t} = \lim_{t\to 0^+}\frac{\ln(1+t) - \ln 1}{t}\\ &= \frac{d}{dx}\ln x \text{ at } x = 1\\ &= \frac{1}{x} \text{ at } x = 1\\ &= 1.\end{aligned}$$

Finally, because $\ln y = 1$, you know that $y = e$, and you can conclude that

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.$$ ■

THEOREM 5.16 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS (arcsin *u* and arccos *u*) (PAGE 376)

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

PROOF

Method 1: Apply Theorem 5.9.

Let $f(x) = \sin x$ and $g(x) = \arcsin x$. Because f is differentiable on $-\pi/2 \le y \le \pi/2$, you can apply Theorem 5.9.

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} = \frac{1}{\sqrt{1 - x^2}}$$

If u is a differentiable function of x, then you can use the Chain Rule to write

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1 - u^2}}, \quad \text{where} \quad u' = \frac{du}{dx}.$$

Method 2: Use implicit differentiation.

Let $y = \arccos x$, $0 \le y \le \pi$. So, $\cos y = x$, and you can use implicit differentiation as follows.

$$\cos y = x$$

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1 - \cos^2 y}} = \frac{-1}{\sqrt{1 - x^2}}$$

If u is a differentiable function of x, then you can use the Chain Rule to write

$$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1 - u^2}}, \quad \text{where} \quad u' = \frac{du}{dx}.$$

■

THEOREM 8.3 THE EXTENDED MEAN VALUE THEOREM (PAGE 570)

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \ne 0$ for any x in (a, b), then there exists a point c in (a, b) such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$.

PROOF You can assume that $g(a) \ne g(b)$, because otherwise, by Rolle's Theorem, it would follow that $g'(x) = 0$ for some x in (a, b). Now, define $h(x)$ as

$$h(x) = f(x) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g(x).$$

Then

$$h(a) = f(a) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g(a) = \frac{f(a)g(b) - f(b)g(a)}{g(b) - g(a)}$$

and

$$h(b) = f(b) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g(b) = \frac{f(a)g(b) - f(b)g(a)}{g(b) - g(a)}$$

and by Rolle's Theorem there exists a point c in (a, b) such that

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) = 0$$

which implies that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$.

■

THEOREM 8.4 L'HÔPITAL'S RULE (PAGE 570)

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

You can use the Extended Mean Value Theorem to prove L'Hôpital's Rule. Of the several different cases of this rule, the proof of only one case is illustrated. The remaining cases where $x \to c^-$ and $x \to c$ are left for you to prove.

PROOF Consider the case for which $\lim_{x \to c^+} f(x) = 0$ and $\lim_{x \to c^+} g(x) = 0$. Define the following new functions:

$$F(x) = \begin{cases} f(x), & x \neq c \\ 0, & x = c \end{cases} \quad \text{and} \quad G(x) = \begin{cases} g(x), & x \neq c \\ 0, & x = c \end{cases}.$$

For any x, $c < x < b$, F and G are differentiable on $(c, x]$ and continuous on $[c, x]$. You can apply the Extended Mean Value Theorem to conclude that there exists a number z in (c, x) such that

$$\frac{F'(z)}{G'(z)} = \frac{F(x) - F(c)}{G(x) - G(c)} = \frac{F(x)}{G(x)} = \frac{f'(z)}{g'(z)} = \frac{f(x)}{g(x)}.$$

Finally, by letting x approach c from the right, $x \to c^+$, you have $z \to c^+$ because $c < z < x$, and

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \lim_{x \to c^+} \frac{f'(z)}{g'(z)} = \lim_{z \to c^+} \frac{f'(z)}{g'(z)} = \lim_{x \to c^+} \frac{f'(x)}{g'(x)}.$$ ■

THEOREM 9.19 TAYLOR'S THEOREM (PAGE 656)

If a function f is differentiable through order $n + 1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}$.

PROOF To find $R_n(x)$, fix x in I $(x \neq c)$ and write $R_n(x) = f(x) - P_n(x)$ where $P_n(x)$ is the nth Taylor polynomial for $f(x)$. Then let g be a function of t defined by

$$g(t) = f(x) - f(t) - f'(t)(x - t) - \cdots - \frac{f^{(n)}(t)}{n!}(x - t)^n - R_n(x)\frac{(x - t)^{n+1}}{(x - c)^{n+1}}.$$

The reason for defining g in this way is that differentiation with respect to t has a telescoping effect. For example, you have

$$\frac{d}{dt}[-f(t) - f'(t)(x - t)] = -f'(t) + f'(t) - f''(t)(x - t) = -f''(t)(x - t).$$

The result is that the derivative $g'(t)$ simplifies to

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x - t)^n + (n + 1)R_n(x)\frac{(x - t)^n}{(x - c)^{n+1}}$$

for all t between c and x. Moreover, for a fixed x,

$$g(c) = f(x) - [P_n(x) + R_n(x)] = f(x) - f(x) = 0$$

and

$$g(x) = f(x) - f(x) - 0 - \cdot \cdot \cdot - 0 = f(x) - f(x) = 0.$$

Therefore, g satisfies the conditions of Rolle's Theorem, and it follows that there is a number z between c and x such that $g'(z) = 0$. Substituting z for t in the equation for $g'(t)$ and then solving for $R_n(x)$, you obtain

$$g'(z) = -\frac{f^{(n+1)}(z)}{n!}(x - z)^n + (n + 1)R_n(x)\frac{(x - z)^n}{(x - c)^{n+1}} = 0$$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}.$$

Finally, because $g(c) = 0$, you have

$$0 = f(x) - f(c) - f'(c)(x - c) - \cdot \cdot \cdot - \frac{f^{(n)}(c)}{n!}(x - c)^n - R_n(x)$$

$$f(x) = f(c) + f'(c)(x - c) + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x).$$ ■

THEOREM 9.20 CONVERGENCE OF A POWER SERIES (PAGE 662)

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$.
3. The series converges absolutely for all x.

The number R is the **radius of convergence** of the power series. If the series converges only at c, the radius of convergence is $R = 0$, and if the series converges for all x, the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

PROOF In order to simplify the notation, the theorem for the power series $\Sigma\, a_n x^n$ centered at $x = 0$ will be proved. The proof for a power series centered at $x = c$ follows easily. A key step in this proof uses the completeness property of the set of real numbers: If a nonempty set S of real numbers has an upper bound, then it must have a least upper bound (see page 603).

It must be shown that if a power series $\Sigma\, a_n x^n$ converges at $x = d$, $d \neq 0$, then it converges for all b satisfying $|b| < |d|$. Because $\Sigma\, a_n x^n$ converges, $\lim\limits_{x \to \infty} a_n d^n = 0$.

So, there exists $N > 0$ such that $a_n d^n < 1$ for all $n \ge N$. Then for $n \ge N$,

$$|a_n b^n| = \left|a_n b^n \frac{d^n}{d^n}\right| = |a_n d^n| \left|\frac{b^n}{d^n}\right| < \left|\frac{b^n}{d^n}\right|.$$

So, for $|b| < |d|$, $\left|\dfrac{b}{d}\right| < 1$, which implies that

$$\sum \left|\frac{b^n}{d^n}\right|$$

is a convergent geometric series. By the Comparison Test, the series $\sum a_n b^n$ converges.

Similarly, if the power series $\sum a_n x^n$ diverges at $x = b$, where $b \neq 0$, then it diverges for all d satisfying $|d| > |b|$. If $\sum a_n d^n$ converged, then the argument above would imply that $\sum a_n b^n$ converged as well.

Finally, to prove the theorem, suppose that neither Case 1 nor Case 3 is true. Then there exist points b and d such that $\sum a_n x^n$ converges at b and diverges at d. Let $S = \{x : \sum a_n x^n \text{ converges}\}$. S is nonempty because $b \in S$. If $x \in S$, then $|x| \le |d|$, which shows that $|d|$ is an upper bound for the nonempty set S. By the completeness property, S has a least upper bound, R.

Now, if $|x| > R$, then $x \notin S$, so $\sum a_n x^n$ diverges. And if $|x| < R$, then $|x|$ is not an upper bound for S, so there exists b in S satisfying $|b| > |x|$. Since $b \in S$, $\sum a_n b^n$ converges, which implies that $\sum a_n x^n$ converges. ■

THEOREM 10.16 CLASSIFICATION OF CONICS BY ECCENTRICITY (PAGE 750)

Let F be a fixed point (*focus*) and let D be a fixed line (*directrix*) in the plane. Let P be another point in the plane and let e (*eccentricity*) be the ratio of the distance between P and F to the distance between P and D. The collection of all points P with a given eccentricity is a conic.

1. The conic is an ellipse if $0 < e < 1$.
2. The conic is a parabola if $e = 1$.
3. The conic is a hyperbola if $e > 1$.

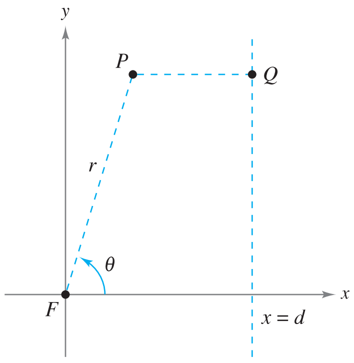


Figure A.1

PROOF If $e = 1$, then, by definition, the conic must be a parabola. If $e \neq 1$, then you can consider the focus F to lie at the origin and the directrix $x = d$ to lie to the right of the origin, as shown in Figure A.1. For the point $P = (r, \theta) = (x, y)$, you have $|PF| = r$ and $|PQ| = d - r\cos\theta$. Given that $e = |PF|/|PQ|$, it follows that

$$|PF| = |PQ|e \quad \Longrightarrow \quad r = e(d - r\cos\theta).$$

By converting to rectangular coordinates and squaring each side, you obtain

$$x^2 + y^2 = e^2(d - x)^2 = e^2(d^2 - 2dx + x^2).$$

Completing the square produces

$$\left(x + \frac{e^2 d}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2 d^2}{(1 - e^2)^2}.$$

If $e < 1$, this equation represents an ellipse. If $e > 1$, then $1 - e^2 < 0$, and the equation represents a hyperbola. ■

THEOREM 13.4 SUFFICIENT CONDITION FOR DIFFERENTIABILITY (PAGE 919)

If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.

PROOF Let S be the surface defined by $z = f(x, y)$, where f, f_x, and f_y are continuous at (x, y). Let A, B, and C be points on surface S, as shown in Figure A.2. From this figure, you can see that the change in f from point A to point C is given by

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= [f(x + \Delta x, y) - f(x, y)] + [f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)]\\ &= \Delta z_1 + \Delta z_2.\end{aligned}$$

Between A and B, y is fixed and x changes. So, by the Mean Value Theorem, there is a value x_1 between x and $x + \Delta x$ such that

$$\Delta z_1 = f(x + \Delta x, y) - f(x, y) = f_x(x_1, y)\,\Delta x.$$

Similarly, between B and C, x is fixed and y changes, and there is a value y_1 between y and $y + \Delta y$ such that

$$\Delta z_2 = f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) = f_y(x + \Delta x, y_1)\,\Delta y.$$

By combining these two results, you can write

$$\Delta z = \Delta z_1 + \Delta z_2 = f_x(x_1, y)\Delta x + f_y(x + \Delta x, y_1)\,\Delta y.$$

If you define ε_1 and ε_2 as $\varepsilon_1 = f_x(x_1, y) - f_x(x, y)$ and $\varepsilon_2 = f_y(x + \Delta x, y_1) - f_y(x, y)$, it follows that

$$\begin{aligned}\Delta z = \Delta z_1 + \Delta z_2 &= [\varepsilon_1 + f_x(x, y)]\,\Delta x + [\varepsilon_2 + f_y(x, y)]\,\Delta y\\ &= [f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y] + \varepsilon_1\Delta x + \varepsilon_2\Delta y.\end{aligned}$$

By the continuity of f_x and f_y and the fact that $x \le x_1 \le x + \Delta x$ and $y \le y_1 \le y + \Delta y$, it follows that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$. Therefore, by definition, f is differentiable. ■

THEOREM 13.6 CHAIN RULE: ONE INDEPENDENT VARIABLE (PAGE 925)

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

PROOF Because g and h are differentiable functions of t, you know that both Δx and Δy approach zero as Δt approaches zero. Moreover, because f is a differentiable function of x and y, you know that $\Delta w = (\partial w/\partial x)\,\Delta x + (\partial w/\partial y)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. So, for $\Delta t \neq 0$

$$\frac{\Delta w}{\Delta t} = \frac{\partial w}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}$$

from which it follows that

$$\frac{dw}{dt} = \lim_{\Delta t \to 0}\frac{\Delta w}{\Delta t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + 0\left(\frac{dx}{dt}\right) + 0\left(\frac{dy}{dt}\right) = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$ ■

B Integration Tables

Forms Involving u^n

1. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$

2. $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

Forms Involving $a + bu$

3. $\displaystyle\int \frac{u}{a+bu}\,du = \frac{1}{b^2}\bigl(bu - a\ln|a+bu|\bigr) + C$

4. $\displaystyle\int \frac{u}{(a+bu)^2}\,du = \frac{1}{b^2}\left(\frac{a}{a+bu} + \ln|a+bu|\right) + C$

5. $\displaystyle\int \frac{u}{(a+bu)^n}\,du = \frac{1}{b^2}\left[\frac{-1}{(n-2)(a+bu)^{n-2}} + \frac{a}{(n-1)(a+bu)^{n-1}}\right] + C, \quad n \neq 1, 2$

6. $\displaystyle\int \frac{u^2}{a+bu}\,du = \frac{1}{b^3}\left[-\frac{bu}{2}(2a - bu) + a^2\ln|a+bu|\right] + C$

7. $\displaystyle\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

8. $\displaystyle\int \frac{u^2}{(a+bu)^3}\,du = \frac{1}{b^3}\left[\frac{2a}{a+bu} - \frac{a^2}{2(a+bu)^2} + \ln|a+bu|\right] + C$

9. $\displaystyle\int \frac{u^2}{(a+bu)^n}\,du = \frac{1}{b^3}\left[\frac{-1}{(n-3)(a+bu)^{n-3}} + \frac{2a}{(n-2)(a+bu)^{n-2}} - \frac{a^2}{(n-1)(a+bu)^{n-1}}\right] + C, \quad n \neq 1, 2, 3$

10. $\displaystyle\int \frac{1}{u(a+bu)}\,du = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

11. $\displaystyle\int \frac{1}{u(a+bu)^2}\,du = \frac{1}{a}\left(\frac{1}{a+bu} + \frac{1}{a}\ln\left|\frac{u}{a+bu}\right|\right) + C$

12. $\displaystyle\int \frac{1}{u^2(a+bu)}\,du = -\frac{1}{a}\left(\frac{1}{u} + \frac{b}{a}\ln\left|\frac{u}{a+bu}\right|\right) + C$

13. $\displaystyle\int \frac{1}{u^2(a+bu)^2}\,du = -\frac{1}{a^2}\left[\frac{a+2bu}{u(a+bu)} + \frac{2b}{a}\ln\left|\frac{u}{a+bu}\right|\right] + C$

Forms Involving $a + bu + cu^2,\ b^2 \neq 4ac$

14. $$\int \frac{1}{a+bu+cu^2}\,du = \begin{cases} \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2cu+b}{\sqrt{4ac-b^2}} + C, & b^2 < 4ac \\ \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2cu+b-\sqrt{b^2-4ac}}{2cu+b+\sqrt{b^2-4ac}}\right| + C, & b^2 > 4ac \end{cases}$$

15. $\displaystyle\int \frac{u}{a+bu+cu^2}\,du = \frac{1}{2c}\left(\ln|a+bu+cu^2| - b\int \frac{1}{a+bu+cu^2}\,du\right)$

Forms Involving $\sqrt{a+bu}$

16. $\displaystyle\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

17. $$\int \frac{1}{u\sqrt{a+bu}}\,du = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|\dfrac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, & a > 0 \\ \dfrac{2}{\sqrt{-a}}\arctan\sqrt{\dfrac{a+bu}{-a}} + C, & a < 0 \end{cases}$$

18. $$\int \frac{1}{u^n\sqrt{a+bu}}\,du = \frac{-1}{a(n-1)}\left[\frac{\sqrt{a+bu}}{u^{n-1}} + \frac{(2n-3)b}{2}\int \frac{1}{u^{n-1}\sqrt{a+bu}}\,du\right],\ n \neq 1$$

19. $$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{1}{u\sqrt{a+bu}}\,du$$

20. $$\int \frac{\sqrt{a+bu}}{u^n}\,du = \frac{-1}{a(n-1)}\left[\frac{(a+bu)^{3/2}}{u^{n-1}} + \frac{(2n-5)b}{2}\int \frac{\sqrt{a+bu}}{u^{n-1}}\,du\right],\ n \neq 1$$

21. $$\int \frac{u}{\sqrt{a+bu}}\,du = \frac{-2(2a-bu)}{3b^2}\sqrt{a+bu} + C$$

22. $$\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2}{(2n+1)b}\left(u^n\sqrt{a+bu} - na\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du\right)$$

Forms Involving $a^2 \pm u^2$, $a > 0$

23. $$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C$$

24. $$\int \frac{1}{u^2-a^2}\,du = -\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$$

25. $$\int \frac{1}{(a^2 \pm u^2)^n}\,du = \frac{1}{2a^2(n-1)}\left[\frac{u}{(a^2 \pm u^2)^{n-1}} + (2n-3)\int \frac{1}{(a^2 \pm u^2)^{n-1}}\,du\right],\ n \neq 1$$

Forms Involving $\sqrt{u^2 \pm a^2}$, $a > 0$

26. $$\int \sqrt{u^2 \pm a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2 \pm a^2} \pm a^2 \ln\left|u + \sqrt{u^2 \pm a^2}\right|\right) + C$$

27. $$\int u^2\sqrt{u^2 \pm a^2}\,du = \frac{1}{8}\left[u(2u^2 \pm a^2)\sqrt{u^2 \pm a^2} - a^4 \ln\left|u + \sqrt{u^2 \pm a^2}\right|\right] + C$$

28. $$\int \frac{\sqrt{u^2+a^2}}{u}\,du = \sqrt{u^2+a^2} - a\ln\left|\frac{a + \sqrt{u^2+a^2}}{u}\right| + C$$

29. $$\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\,\mathrm{arcsec}\,\frac{|u|}{a} + C$$

30. $$\int \frac{\sqrt{u^2 \pm a^2}}{u^2}\,du = \frac{-\sqrt{u^2 \pm a^2}}{u} + \ln\left|u + \sqrt{u^2 \pm a^2}\right| + C$$

31. $$\int \frac{1}{\sqrt{u^2 \pm a^2}}\,du = \ln\left|u + \sqrt{u^2 \pm a^2}\right| + C$$

32. $$\int \frac{1}{u\sqrt{u^2+a^2}}\,du = \frac{-1}{a}\ln\left|\frac{a + \sqrt{u^2+a^2}}{u}\right| + C$$

33. $$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\,\mathrm{arcsec}\,\frac{|u|}{a} + C$$

34. $$\int \frac{u^2}{\sqrt{u^2 \pm a^2}}\,du = \frac{1}{2}\left(u\sqrt{u^2 \pm a^2} \mp a^2 \ln\left|u + \sqrt{u^2 \pm a^2}\right|\right) + C$$

35. $$\int \frac{1}{u^2\sqrt{u^2 \pm a^2}}\,du = \mp\frac{\sqrt{u^2 \pm a^2}}{a^2 u} + C$$

36. $$\int \frac{1}{(u^2 \pm a^2)^{3/2}}\,du = \frac{\pm u}{a^2\sqrt{u^2 \pm a^2}} + C$$

Forms Involving $\sqrt{a^2 - u^2}$, $a > 0$

37. $$\int \sqrt{a^2-u^2}\,du = \frac{1}{2}\left(u\sqrt{a^2-u^2} + a^2 \arcsin\frac{u}{a}\right) + C$$

38. $$\int u^2\sqrt{a^2-u^2}\,du = \frac{1}{8}\left[u(2u^2-a^2)\sqrt{a^2-u^2} + a^4 \arcsin\frac{u}{a}\right] + C$$

39. $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

40. $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \frac{-\sqrt{a^2-u^2}}{u} - \arcsin\frac{u}{a} + C$

41. $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\frac{u}{a} + C$

42. $\displaystyle\int \frac{1}{u\sqrt{a^2-u^2}}\,du = \frac{-1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

43. $\displaystyle\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = \frac{1}{2}\left(-u\sqrt{a^2-u^2} + a^2\arcsin\frac{u}{a}\right) + C$

44. $\displaystyle\int \frac{1}{u^2\sqrt{a^2-u^2}}\,du = \frac{-\sqrt{a^2-u^2}}{a^2u} + C$

45. $\displaystyle\int \frac{1}{(a^2-u^2)^{3/2}}\,du = \frac{u}{a^2\sqrt{a^2-u^2}} + C$

Forms Involving sin *u* or cos *u*

46. $\displaystyle\int \sin u\,du = -\cos u + C$

47. $\displaystyle\int \cos u\,du = \sin u + C$

48. $\displaystyle\int \sin^2 u\,du = \frac{1}{2}(u - \sin u\cos u) + C$

49. $\displaystyle\int \cos^2 u\,du = \frac{1}{2}(u + \sin u\cos u) + C$

50. $\displaystyle\int \sin^n u\,du = -\frac{\sin^{n-1}u\cos u}{n} + \frac{n-1}{n}\int \sin^{n-2}u\,du$

51. $\displaystyle\int \cos^n u\,du = \frac{\cos^{n-1}u\sin u}{n} + \frac{n-1}{n}\int \cos^{n-2}u\,du$

52. $\displaystyle\int u\sin u\,du = \sin u - u\cos u + C$

53. $\displaystyle\int u\cos u\,du = \cos u + u\sin u + C$

54. $\displaystyle\int u^n\sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$

55. $\displaystyle\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

56. $\displaystyle\int \frac{1}{1\pm\sin u}\,du = \tan u \mp \sec u + C$

57. $\displaystyle\int \frac{1}{1\pm\cos u}\,du = -\cot u \pm \csc u + C$

58. $\displaystyle\int \frac{1}{\sin u\cos u}\,du = \ln|\tan u| + C$

Forms Involving tan *u*, cot *u*, sec *u*, csc *u*

59. $\displaystyle\int \tan u\,du = -\ln|\cos u| + C$

60. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$

61. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$

62. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + C$ or $\displaystyle\int \csc u\,du = -\ln|\csc u + \cot u| + C$

63. $\displaystyle\int \tan^2 u\,du = -u + \tan u + C$

64. $\displaystyle\int \cot^2 u\,du = -u - \cot u + C$

65. $\displaystyle\int \sec^2 u\,du = \tan u + C$

66. $\displaystyle\int \csc^2 u\,du = -\cot u + C$

67. $\displaystyle\int \tan^n u\,du = \frac{\tan^{n-1}u}{n-1} - \int \tan^{n-2}u\,du,\ n \neq 1$

68. $\displaystyle\int \cot^n u\,du = -\frac{\cot^{n-1}u}{n-1} - \int (\cot^{n-2}u)\,du,\ n \neq 1$

69. $\displaystyle\int \sec^n u\,du = \frac{\sec^{n-2}u\tan u}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}u\,du,\ n \neq 1$

70. $\displaystyle\int \csc^n u\,du = -\frac{\csc^{n-2}u\cot u}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2}u\,du,\ n \neq 1$

71. $\displaystyle\int \frac{1}{1 \pm \tan u}\,du = \frac{1}{2}\left(u \pm \ln|\cos u \pm \sin u|\right) + C$

72. $\displaystyle\int \frac{1}{1 \pm \cot u}\,du = \frac{1}{2}\left(u \mp \ln|\sin u \pm \cos u|\right) + C$

73. $\displaystyle\int \frac{1}{1 \pm \sec u}\,du = u + \cot u \mp \csc u + C$

74. $\displaystyle\int \frac{1}{1 \pm \csc u}\,du = u - \tan u \pm \sec u + C$

Forms Involving Inverse Trigonometric Functions

75. $\displaystyle\int \arcsin u\,du = u \arcsin u + \sqrt{1 - u^2} + C$

76. $\displaystyle\int \arccos u\,du = u \arccos u - \sqrt{1 - u^2} + C$

77. $\displaystyle\int \arctan u\,du = u \arctan u - \ln\sqrt{1 + u^2} + C$

78. $\displaystyle\int \operatorname{arccot} u\,du = u \operatorname{arccot} u + \ln\sqrt{1 + u^2} + C$

79. $\displaystyle\int \operatorname{arcsec} u\,du = u \operatorname{arcsec} u - \ln\left|u + \sqrt{u^2 - 1}\right| + C$

80. $\displaystyle\int \operatorname{arccsc} u\,du = u \operatorname{arccsc} u + \ln\left|u + \sqrt{u^2 - 1}\right| + C$

Forms Involving e^u

81. $\displaystyle\int e^u\,du = e^u + C$

82. $\displaystyle\int ue^u\,du = (u - 1)e^u + C$

83. $\displaystyle\int u^n e^u\,du = u^n e^u - n\int u^{n-1} e^u\,du$

84. $\displaystyle\int \frac{1}{1 + e^u}\,du = u - \ln(1 + e^u) + C$

85. $\displaystyle\int e^{au} \sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a \sin bu - b \cos bu) + C$

86. $\displaystyle\int e^{au} \cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a \cos bu + b \sin bu) + C$

Forms Involving $\ln u$

87. $\displaystyle\int \ln u\,du = u(-1 + \ln u) + C$

88. $\displaystyle\int u \ln u\,du = \frac{u^2}{4}(-1 + 2 \ln u) + C$

89. $\displaystyle\int u^n \ln u\,du = \frac{u^{n+1}}{(n + 1)^2}\left[-1 + (n + 1) \ln u\right] + C,\ n \neq -1$

90. $\displaystyle\int (\ln u)^2\,du = u\left[2 - 2 \ln u + (\ln u)^2\right] + C$

91. $\displaystyle\int (\ln u)^n\,du = u(\ln u)^n - n\int (\ln u)^{n-1}\,du$

Forms Involving Hyperbolic Functions

92. $\displaystyle\int \cosh u\,du = \sinh u + C$

93. $\displaystyle\int \sinh u\,du = \cosh u + C$

94. $\displaystyle\int \operatorname{sech}^2 u\,du = \tanh u + C$

95. $\displaystyle\int \operatorname{csch}^2 u\,du = -\coth u + C$

96. $\displaystyle\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$

97. $\displaystyle\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

Forms Involving Inverse Hyperbolic Functions (in logarithmic form)

98. $\displaystyle\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$

99. $\displaystyle\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln\left|\frac{a + u}{a - u}\right| + C$

100. $\displaystyle\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a} \ln \frac{a + \sqrt{a^2 \pm u^2}}{|u|} + C$

Answers to Odd-Numbered Exercises

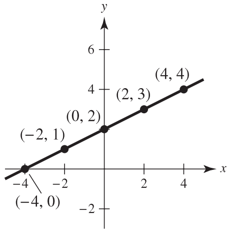
Chapter P

Section P.1 (page 8)

1. b **2.** d **3.** a **4.** c

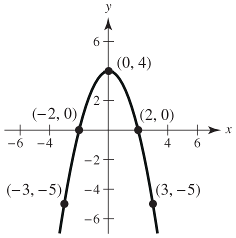
5. Answers will vary.

x	-4	-2	0	2	4
y	0	1	2	3	4



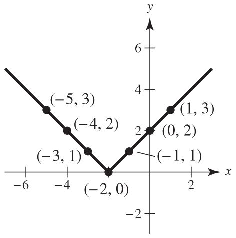
7. Answers will vary.

x	-3	-2	0	2	3
y	-5	0	4	0	-5



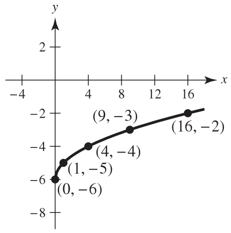
9. Answers will vary.

x	-5	-4	-3	-2	-1	0	1
y	3	2	1	0	1	2	3



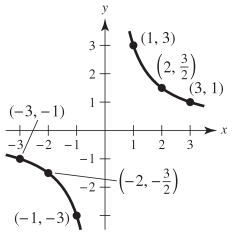
11. Answers will vary.

x	0	1	4	9	16
y	-6	-5	-4	-3	-2



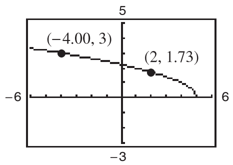
13. Answers will vary.

x	-3	-2	-1	0	1	2	3
y	-1	$-\frac{3}{2}$	-3	Undef.	3	$\frac{3}{2}$	1



15.
Xmin = -5
Xmax = 4
Xscl = 1
Ymin = -5
Ymax = 8
Yscl = 1

17. $y = \sqrt{5 - x}$



(a) $y \approx 1.73$ (b) $x = -4$

19. $(0, -5), \left(\frac{5}{2}, 0\right)$ **21.** $(0, -2), (-2, 0), (1, 0)$

23. $(0, 0), (4, 0), (-4, 0)$ **25.** $(4, 0)$ **27.** $(0, 0)$

29. Symmetric with respect to the y-axis

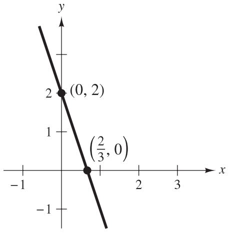
31. Symmetric with respect to the x-axis

33. Symmetric with respect to the origin **35.** No symmetry

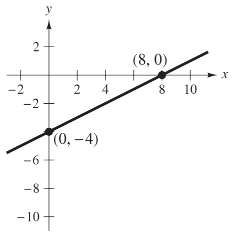
37. Symmetric with respect to the origin

39. Symmetric with respect to the y-axis

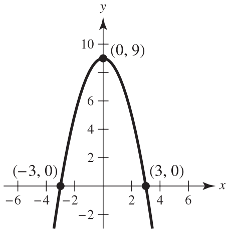
41. $y = 2 - 3x$
Symmetry: none



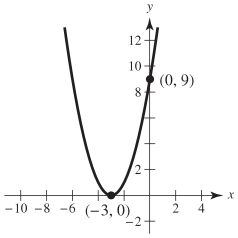
43. $y = \frac{1}{2}x - 4$
Symmetry: none



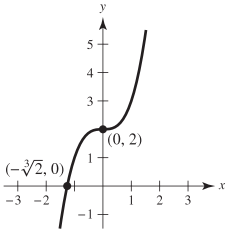
45. $y = 9 - x^2$
Symmetry: y-axis



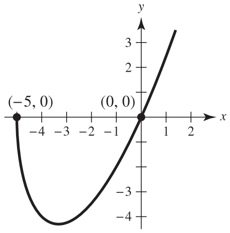
47. $y = (x + 3)^2$
Symmetry: none



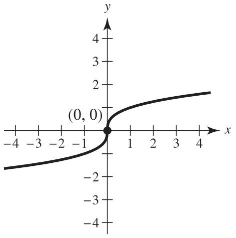
49. $y = x^3 + 2$
Symmetry: none



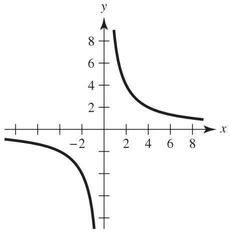
51. $y = x\sqrt{x + 5}$
Symmetry: none



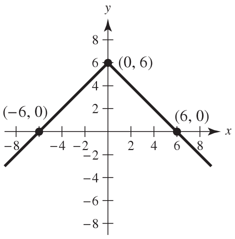
53. $x = y^3$
Symmetry: origin



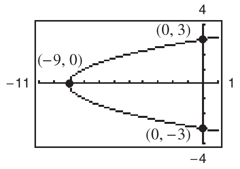
55. $y = 8/x$
Symmetry: origin



57. $y = 6 - |x|$
Symmetry: y-axis



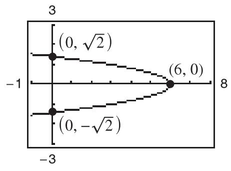
59. $y_1 = \sqrt{x + 9}$
$y_2 = -\sqrt{x + 9}$
Symmetry: x-axis



61. $y_1 = \sqrt{\dfrac{6 - x}{3}}$

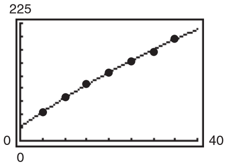
$y_2 = -\sqrt{\dfrac{6 - x}{3}}$

Symmetry: x-axis



63. $(3, 5)$ **65.** $(-1, 5), (2, 2)$ **67.** $(-1, -2), (2, 1)$
69. $(-1, -1), (0, 0), (1, 1)$
71. $(-1, -5), (0, -1), (2, 1)$ **73.** $(-2, 2), \left(-3, \sqrt{3}\right)$

75. (a) $y = -0.027t^2 + 5.73t + 26.9$
(b)



The model is a good fit for the data.

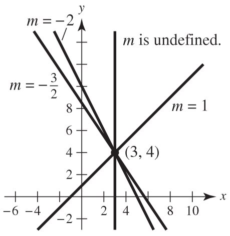
(c) 212.9

77. $x \approx 3133$ units **79.** $y = (x + 4)(x - 3)(x - 8)$
81. (a) Proof (b) Proof
83. False. $(4, -5)$ is not a point on the graph of $x = y^2 - 29$.
85. True **87.** $x^2 + (y - 4)^2 = 4$

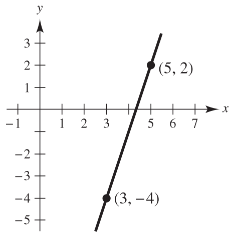
Section P.2 (page 16)

1. $m = 1$ **3.** $m = 0$ **5.** $m = -12$

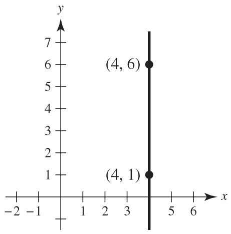
7.



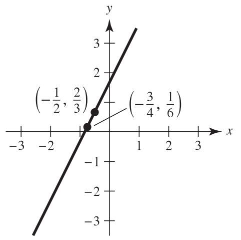
9. $m = 3$



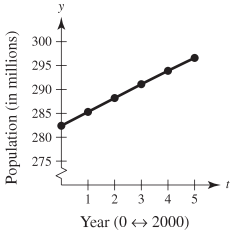
11. m is undefined.



13. $m = 2$

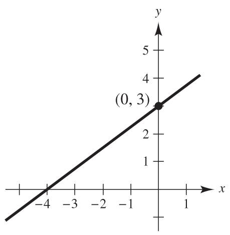


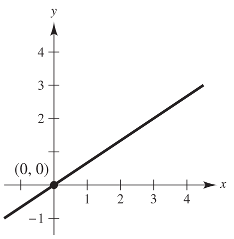
15. $(0, 2), (1, 2), (5, 2)$ **17.** $(0, 10), (2, 4), (3, 1)$
19. (a) $\frac{1}{3}$ (b) $10\sqrt{10}$ ft
21. (a)



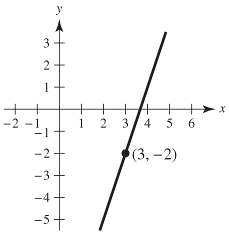
(b) Population increased least rapidly from 2004 to 2005.

23. $m = 4, (0, -3)$
25. $m = -\frac{1}{5}, (0, 4)$ **27.** m is undefined, no y-intercept
29. $3x - 4y + 12 = 0$ **31.** $2x - 3y = 0$

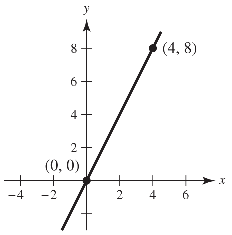




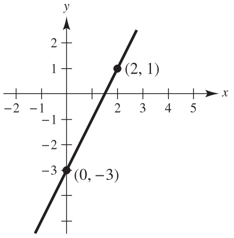
33. $3x - y - 11 = 0$



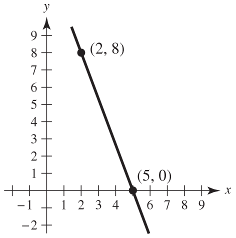
35. $2x - y = 0$



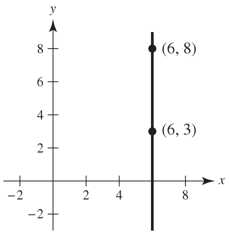
37. $2x - y - 3 = 0$



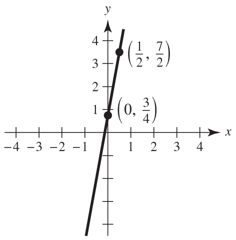
39. $8x + 3y - 40 = 0$



41. $x - 6 = 0$

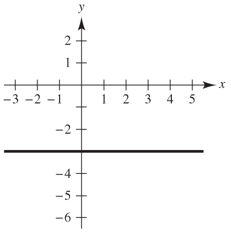


43. $22x - 4y + 3 = 0$

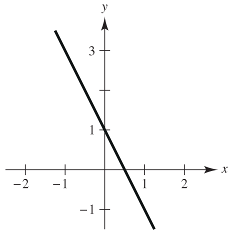


45. $x - 3 = 0$ **47.** $3x + 2y - 6 = 0$ **49.** $x + y - 3 = 0$

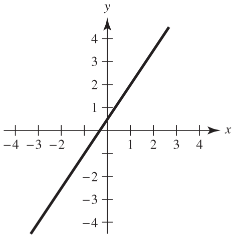
51.



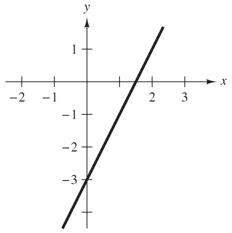
53.



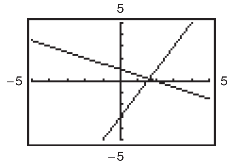
55.



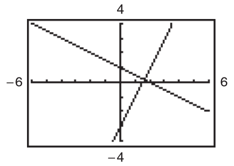
57.



59. (a)



(b)



The lines in (a) do not appear perpendicular, but they do in (b) because a square setting is used. The lines are perpendicular.

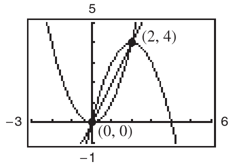
61. (a) $x + 7 = 0$ (b) $y + 2 = 0$

63. (a) $2x - y - 3 = 0$ (b) $x + 2y - 4 = 0$

65. (a) $40x - 24y - 9 = 0$ (b) $24x + 40y - 53 = 0$

67. $V = 250t - 150$ **69.** $V = -1600t + 30{,}000$

71. $y = 2x$ **73.** Not collinear, because $m_1 \neq m_2$

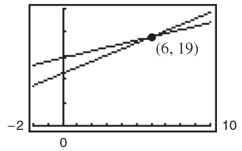


75. $\left(0, \dfrac{-a^2 + b^2 + c^2}{2c}\right)$ **77.** $\left(b, \dfrac{a^2 - b^2}{c}\right)$

79. $5F - 9C - 160 = 0$; $72°\text{F} \approx 22.2°\text{C}$

81. (a) $W_1 = 14.50 + 0.75x$, $W_2 = 11.20 + 1.30x$

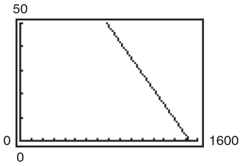
(b)



(c) When six units are produced, the wage for both options is \$19.00 per hour. Choose option 1 if you think you will produce less than six units and choose option 2 if you think you will produce more than six units.

83. (a) $x = (1530 - p)/15$

(b)



45 units

(c) 49 units

85. $12y + 5x - 169 = 0$ **87.** 2 **89.** $(5\sqrt{2})/2$ **91.** $2\sqrt{2}$

93. Proof **95.** Proof **97.** Proof **99.** True

Section P.3 (page 27)

1. (a) Domain of f: $[-4, 4]$; Range of f: $[-3, 5]$
Domain of g: $[-3, 3]$; Range of g: $[-4, 4]$
(b) $f(-2) = -1$; $g(3) = -4$
(c) $x = -1$ (d) $x = 1$ (e) $x = -1, x = 1$, and $x = 2$

3. (a) -4 (b) -25 (c) $7b - 4$ (d) $7x - 11$

5. (a) 5 (b) 0 (c) 1 (d) $4 + 2t - t^2$

7. (a) 1 (b) 0 (c) $-\frac{1}{2}$ **9.** $3x^2 + 3x\,\Delta x + (\Delta x)^2$, $\Delta x \neq 0$

11. $\left(\sqrt{x-1} - x + 1\right)/[(x - 2)(x - 1)]$
$= -1/\left[\sqrt{x-1}\left(1 + \sqrt{x-1}\right)\right], x \neq 2$

13. Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

15. Domain: $[0, \infty)$; Range: $[0, \infty)$

17. Domain: All real numbers t such that $t \neq 4n + 2$, where n is an integer; Range: $(-\infty, -1] \cup [1, \infty)$

19. Domain: $(-\infty, 0) \cup (0, \infty)$; Range: $(-\infty, 0) \cup (0, \infty)$

21. Domain: $[0, 1]$

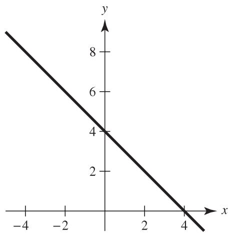
23. Domain: All real numbers x such that $x \neq 2n\pi$, where n is an integer

25. Domain: $(-\infty, -3) \cup (-3, \infty)$

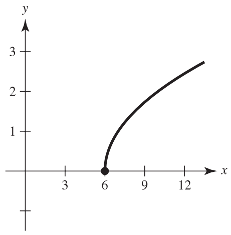
27. (a) -1 (b) 2 (c) 6 (d) $2t^2 + 4$
Domain: $(-\infty, \infty)$; Range: $(-\infty, 1) \cup [2, \infty)$

29. (a) 4 (b) 0 (c) -2 (d) $-b^2$
Domain: $(-\infty, \infty)$; Range: $(-\infty, 0] \cup [1, \infty)$

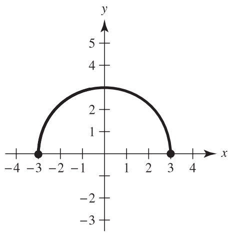
31. $f(x) = 4 - x$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



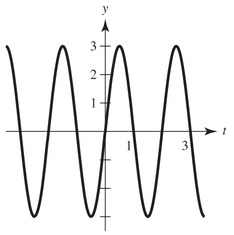
33. $h(x) = \sqrt{x - 6}$
Domain: $[6, \infty)$
Range: $[0, \infty)$



35. $f(x) = \sqrt{9 - x^2}$
Domain: $[-3, 3]$
Range: $[0, 3]$



37. $g(t) = 3 \sin \pi t$
Domain: $(-\infty, \infty)$
Range: $[-3, 3]$



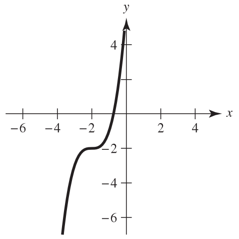
39. The student travels $\frac{1}{2}$ mile/minute during the first 4 minutes, is stationary for the next 2 minutes, and travels 1 mile/minute during the final 4 minutes.

41. y is not a function of x. **43.** y is a function of x.

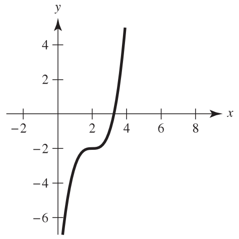
45. y is not a function of x. **47.** y is not a function of x.

49. *d* **50.** *b* **51.** *c* **52.** *a* **53.** *e* **54.** *g*

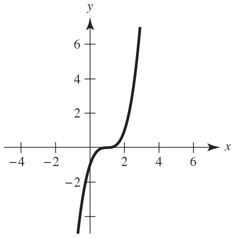
55. (a)



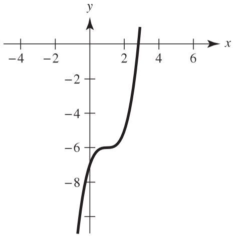
(b)



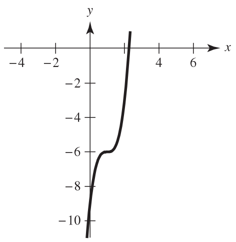
(c)



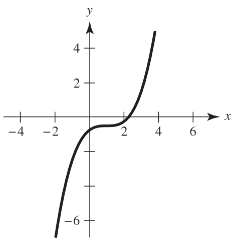
(d)



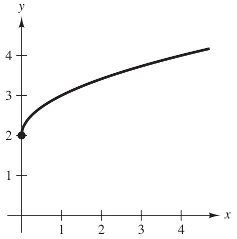
(e)



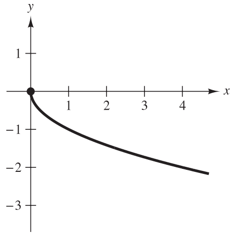
(f)



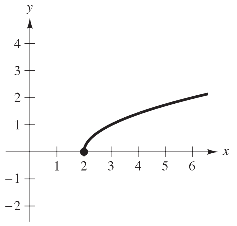
57. (a) Vertical translation



(b) Reflection about the x-axis



(c) Horizontal translation



59. (a) 0 (b) 0 (c) -1 (d) $\sqrt{15}$
(e) $\sqrt{x^2 - 1}$ (f) $x - 1$ $(x \geq 0)$

61. $(f \circ g)(x) = x$; Domain: $[0, \infty)$
$(g \circ f)(x) = |x|$; Domain: $(-\infty, \infty)$
No, their domains are different.

63. $(f \circ g)(x) = 3/(x^2 - 1)$; Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
$(g \circ f)(x) = (9/x^2) - 1$; Domain: $(-\infty, 0) \cup (0, \infty)$
No

65. (a) 4 (b) -2
(c) Undefined. The graph of g does not exist at $x = -5$.
(d) 3 (e) 2
(f) Undefined. The graph of f does not exist at $x = -4$.

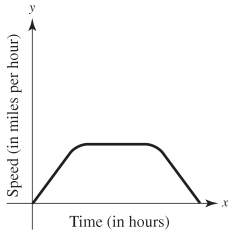
67. Answers will vary.
Example: $f(x) = \sqrt{x}$; $g(x) = x - 2$; $h(x) = 2x$

69. Even **71.** Odd **73.** (a) $\left(\frac{3}{2}, 4\right)$ (b) $\left(\frac{3}{2}, -4\right)$

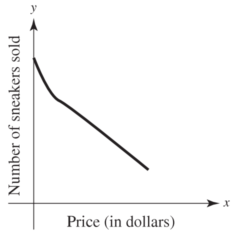
75. f is even. g is neither even nor odd. h is odd.

77. $f(x) = -5x - 6, -2 \leq x \leq 0$ **79.** $y = -\sqrt{-x}$

81. Answers will vary.
Sample answer:



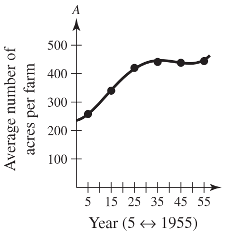
83. Answers will vary.
Sample answer:



85. $c = 25$

87. (a) $T(4) = 16°\text{C}, T(15) \approx 23°\text{C}$
(b) The changes in temperature occur 1 hour later.
(c) The temperatures are 1° lower.

89. (a)



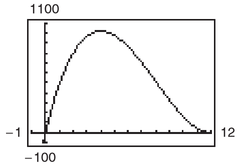
(b) $A(20) \approx 384$ acres/farm

91. $f(x) = |x| + |x-2| = \begin{cases} 2x-2, & \text{if } x \geq 2 \\ 2, & \text{if } 0 < x < 2 \\ -2x+2, & \text{if } x \leq 0 \end{cases}$

93. Proof **95.** Proof

97. (a) $V(x) = x(24-2x)^2,\ 0 < x < 12$

(b)



$4 \times 16 \times 16$ cm

(c)

Height, x	*Length and Width*	*Volume, V*
1	$24-2(1)$	$1[24-2(1)]^2 = 484$
2	$24-2(2)$	$2[24-2(2)]^2 = 800$
3	$24-2(3)$	$3[24-2(3)]^2 = 972$
4	$24-2(4)$	$4[24-2(4)]^2 = 1024$
5	$24-2(5)$	$5[24-2(5)]^2 = 980$
6	$24-2(6)$	$6[24-2(6)]^2 = 864$

The dimensions of the box that yield a maximum volume are $4 \times 16 \times 16$ cm.

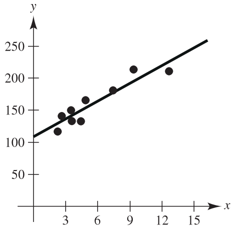
99. False. For example, if $f(x) = x^2$, then $f(-1) = f(1)$.

101. True **103.** Putnam Problem A1, 1988

Section P.4 (page 34)

1. Trigonometric **3.** No relationship

5. (a) and (b)

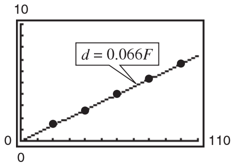


Approximately linear

(c) 136

7. (a) $d = 0.066F$

(b)

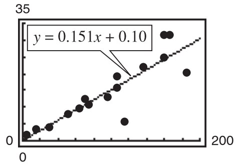


The model fits well.

(c) 3.63 cm

9. (a) $y = 0.151x + 0.10;\ r \approx 0.880$

(b)



(c) Greater per capita energy consumption by a country tends to correspond to greater per capita gross national product of the country. The four countries that differ most from the linear model are Venezuela, South Korea, Hong Kong, and the United Kingdom.

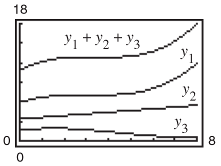
(d) $y = 0.155x + 0.22;\ r \approx 0.984$

11. (a) $y_1 = 0.04040t^3 - 0.3695t^2 + 1.123t + 5.88$

$y_2 = 0.264t + 3.35$

$y_3 = 0.01439t^3 - 0.1886t^2 + 0.476t + 1.59$

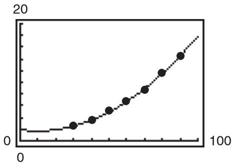
(b) $y_1 + y_2 + y_3 = 0.05479t^3 - 0.5581t^2 + 1.863t + 10.82$



About 47.5 cents/mi

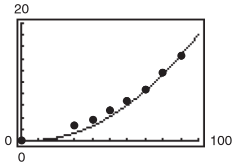
13. (a) $t = 0.002s^2 - 0.04s + 1.9$

(b)



(c) According to the model, the times required to attain speeds of less than 20 miles per hour are all about the same.

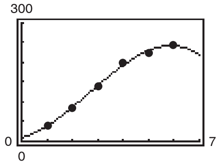
(d) $t = 0.002s^2 + 0.02s + 0.1$



(e) No. From the graph in part (b), you can see that the model from part (a) follows the data more closely than the model from part (d).

15. (a) $y = -1.806x^3 + 14.58x^2 + 16.4x + 10$

(b)

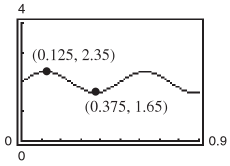


(c) 214 hp

17. (a) Yes. At time t there is one and only one displacement y.

(b) Amplitude: 0.35; Period: 0.5 (c) $y = 0.35 \sin(4\pi t) + 2$

(d)



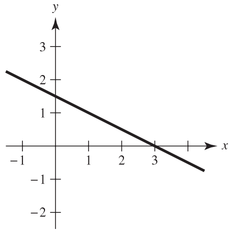
The model appears to fit the data well.

19. Answers will vary. **21.** Putnam Problem A2, 2004

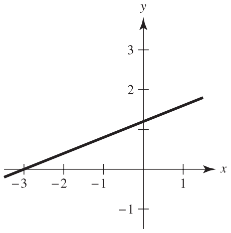
Review Exercises for Chapter P (page 37)

1. $\left(\frac{8}{5}, 0\right), (0, -8)$ **3.** $(3, 0), \left(0, \frac{3}{4}\right)$ **5.** y-axis symmetry

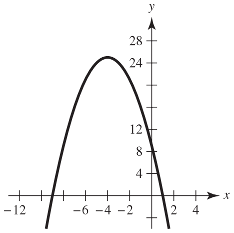
7.



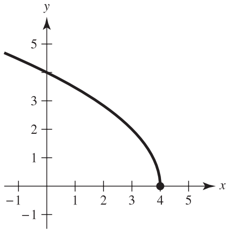
9.



11.



13.

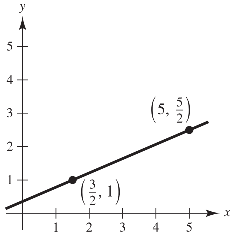


15.

Xmin = -5
Xmax = 5
Xscl = 1
Ymin = -30
Ymax = 10
Yscl = 5

17. $(-2, 3)$ **19.** $y = x^3 - 16x$

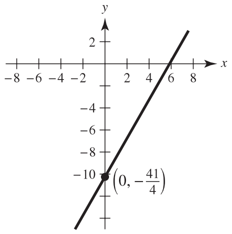
21.



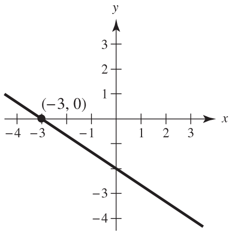
$m = \frac{3}{7}$

23. $t = \frac{1}{5}$

25. $y = \frac{7}{4}x - \frac{41}{4}$ or
$7x - 4y - 41 = 0$



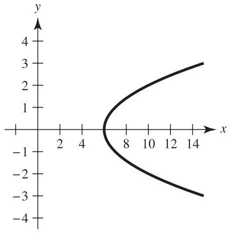
27. $y = -\frac{2}{3}x - 2$ or
$2x + 3y + 6 = 0$



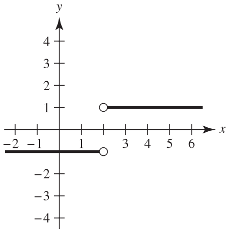
29. (a) $7x - 16y + 101 = 0$ (b) $5x - 3y + 30 = 0$
(c) $5x + 3y = 0$ (d) $x + 3 = 0$

31. $V = 12{,}500 - 850t$; $9950

33. Not a function



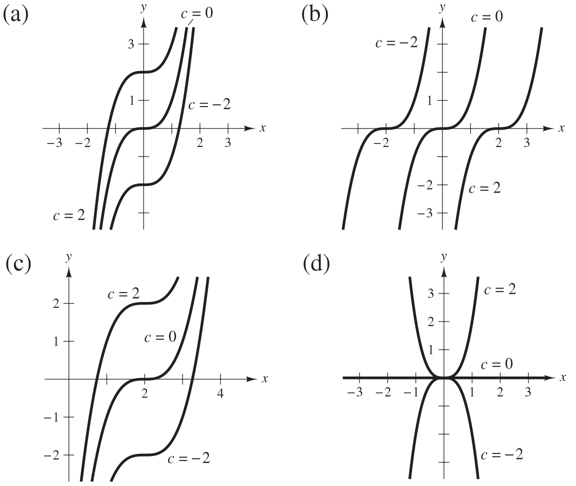
35. Function



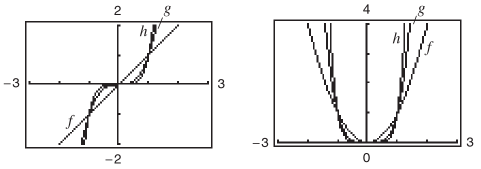
37. (a) Undefined (b) $-1/(1 + \Delta x), \Delta x \neq 0, -1$

39. (a) Domain: $[-6, 6]$; Range: $[0, 6]$
(b) Domain: $(-\infty, 5) \cup (5, \infty)$; Range: $(-\infty, 0) \cup (0, \infty)$
(c) Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

41.



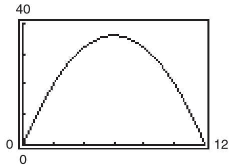
43. (a)



All the graphs pass through the origin. The graphs of the odd powers of x are symmetric with respect to the origin and the graphs of the even powers are symmetric with respect to the y-axis. As the powers increase, the graphs become flatter in the interval $-1 < x < 1$. Graphs of these equations with odd powers pass through Quadrants I and III. Graphs of these equations with even powers pass through Quadrants I and II.

(b) The graph of $y = x^7$ should pass through the origin and Quadrants I and III. It should be symmetric with respect to the origin and be fairly flat in the interval $(-1, 1)$. The graph of $y = x^8$ should pass through the origin and Quadrants I and II. It should be symmetric with respect to the y-axis and be fairly flat in the interval $(-1, 1)$.

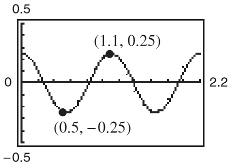
45. (a) $A = x(12 - x)$
(b) Domain: $(0, 12)$



(c) Maximum area:
36 in.^{2}; 6 × 6 in.

47. (a) Minimum degree: 3; Leading coefficient: negative
(b) Minimum degree: 4; Leading coefficient: positive
(c) Minimum degree: 2; Leading coefficient: negative
(d) Minimum degree: 5; Leading coefficient: positive

49. (a) Yes. For each time t there corresponds one and only one displacement y.
(b) Amplitude: 0.25; Period: 1.1 (c) $y \approx \frac{1}{4}\cos(5.7t)$
(d)

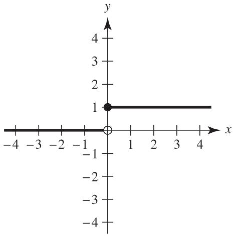


The model appears to fit the data.

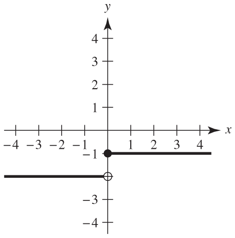
P.S. Problem Solving (page 39)

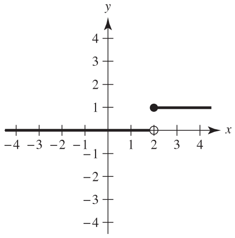
1. (a) Center: (3, 4); Radius: 5
(b) $y = -\frac{3}{4}x$ (c) $y = \frac{3}{4}x - \frac{9}{2}$ (d) $\left(3, -\frac{9}{4}\right)$

3.

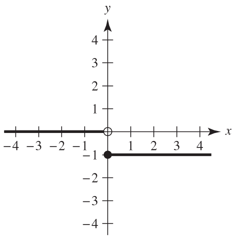


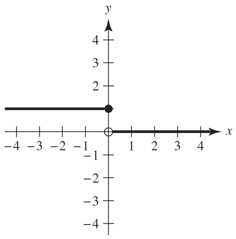
(a) $H(x) - 2 = \begin{cases} -1, & x \ge 0 \\ -2, & x < 0 \end{cases}$ (b) $H(x - 2) = \begin{cases} 1, & x \ge 2 \\ 0, & x < 2 \end{cases}$



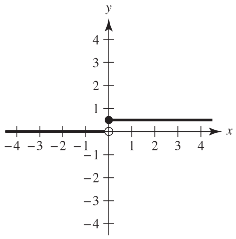


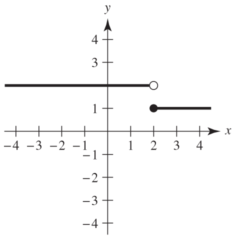
(c) $-H(x) = \begin{cases} -1, & x \ge 0 \\ 0, & x < 0 \end{cases}$ (d) $H(-x) = \begin{cases} 1, & x \le 0 \\ 0, & x > 0 \end{cases}$



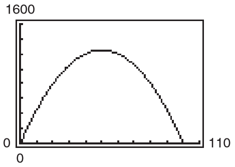


(e) $\frac{1}{2}H(x) = \begin{cases} \frac{1}{2}, & x \ge 0 \\ 0, & x < 0 \end{cases}$ (f) $-H(x - 2) + 2 = \begin{cases} 1, & x \ge 2 \\ 2, & x < 2 \end{cases}$





5. (a) $A(x) = x[(100 - x)/2]$; Domain: (0, 100)
(b)



Dimensions 50 m × 25 m yield maximum area of 1250 m^2.

(c) 50 m × 25 m; Area = 1250 m^2

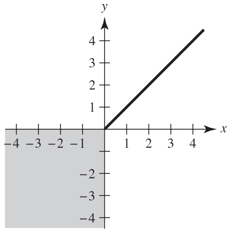
7. $T(x) = \left[2\sqrt{4 + x^2} + \sqrt{(3 - x)^2 + 1}\right]/4$

9. (a) 5, less (b) 3, greater (c) 4.1, less
(d) $4 + h$ (e) 4; Answers will vary.

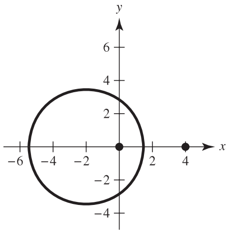
11. Using the definition of absolute value, you can rewrite the equation as

$$\begin{cases} 2y, & y > 0 \\ 0, & y \le 0 \end{cases} = \begin{cases} 2x, & x > 0 \\ 0, & x \le 0 \end{cases}.$$

For $x > 0$ and $y > 0$, you have $2y = 2x \rightarrow y = x$. For any $x \le 0$, y is any $y \le 0$. So, the graph of $y + |y| = x + |x|$ is as follows.

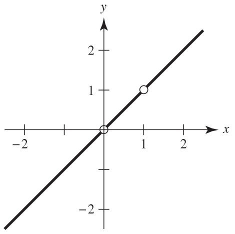


13. (a) $\left(x + \dfrac{4}{k - 1}\right)^2 + y^2 = \dfrac{16k}{(k - 1)^2}$
(b)



(c) As k becomes very large, $\dfrac{4}{k - 1} \rightarrow 0$ and $\dfrac{16k}{(k - 1)^2} \rightarrow 0$.
The center of the circle gets closer to (0, 0), and its radius approaches 0.

15. (a) Domain: $(-\infty, 1) \cup (1, \infty)$; Range: $(-\infty, 0) \cup (0, \infty)$
(b) $f(f(x)) = \dfrac{x - 1}{x}$
Domain: $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$
(c) $f(f(f(x))) = x$
Domain: $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$
(d)



The graph is not a line because there are holes at $x = 0$ and $x = 1$.

Chapter 1

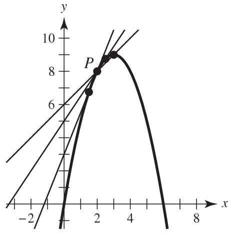
Section 1.1 (page 47)

1. Precalculus: 300 ft

3. Calculus: Slope of the tangent line at $x = 2$ is 0.16.

5. (a) Precalculus: 10 square units (b) Calculus: 5 square units

7. (a)



(b) $1; \frac{3}{2}; \frac{5}{2}$

(c) 2. Use points closer to P.

9. (a) Area ≈ 10.417; Area ≈ 9.145 (b) Use more rectangles.

11. (a) 5.66 (b) 6.11 (c) Increase the number of line segments.

Section 1.2 (page 54)

1.

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	0.2041	0.2004	0.2000	0.2000	0.1996	0.1961

$$\lim_{x\to 4} \frac{x-4}{x^2-3x-4} \approx 0.2000 \left(\text{Actual limit is } \frac{1}{5}.\right)$$

3.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	0.2050	0.2042	0.2041	0.2041	0.2040	0.2033

$$\lim_{x\to 0} \frac{\sqrt{x+6}-\sqrt{6}}{x} \approx 0.2041 \left(\text{Actual limit is } \frac{1}{2\sqrt{6}}.\right)$$

5.

x	2.9	2.99	2.999
$f(x)$	−0.0641	−0.0627	−0.0625

x	3.001	3.01	3.1
$f(x)$	−0.0625	−0.0623	−0.0610

$$\lim_{x\to 3} \frac{[1/(x+1)]-(1/4)}{x-3} \approx -0.0625 \left(\text{Actual limit is } -\frac{1}{16}.\right)$$

7.

x	−0.1	−0.01	−0.001
$f(x)$	0.9983	0.99998	1.0000

x	0.001	0.01	0.1
$f(x)$	1.0000	0.99998	0.9983

$$\lim_{x\to 0} \frac{\sin x}{x} \approx 1.0000 \text{ (Actual limit is 1.)}$$

9.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$$\lim_{x\to 1} \frac{x-2}{x^2+x-6} \approx 0.2500 \left(\text{Actual limit is } \frac{1}{4}.\right)$$

11.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.7340	0.6733	0.6673	0.6660	0.6600	0.6015

$$\lim_{x\to 1} \frac{x^4-1}{x^6-1} \approx 0.6666 \left(\text{Actual limit is } \frac{2}{3}.\right)$$

13.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	1.9867	1.9999	2.0000	2.0000	1.9999	1.9867

$$\lim_{x\to 0} \frac{\sin 2x}{x} \approx 2.0000 \text{ (Actual limit is 2.)}$$

15. 1 **17.** 2

19. Limit does not exist. The function approaches 1 from the right side of 2 but it approaches −1 from the left side of 2.

21. 0

23. Limit does not exist. The function oscillates between 1 and −1 as x approaches 0.

25. (a) 2

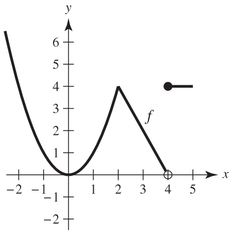
(b) Limit does not exist. The function approaches 1 from the right side of 1 but it approaches 3.5 from the left side of 1.

(c) Value does not exist. The function is undefined at $x = 4$.

(d) 2

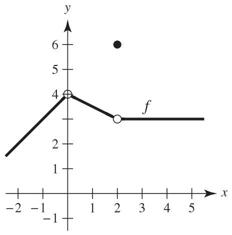
27. $\lim_{x\to c} f(x)$ exists for all points on the graph except where $c = -3$.

29.

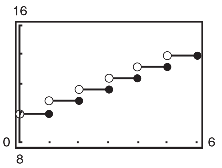


$\lim_{x\to c} f(x)$ exists for all points on the graph except where $c = 4$.

31.



33. (a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	11.57	12.36	12.36	12.36	12.36	12.36	12.36

$\lim_{t\to 3.5} C(t) = 12.36$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	10.78	11.57	11.57	11.57	12.36	12.36	12.36

The limit does not exist because the limits from the right and left are not equal.

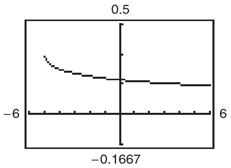
35. $\delta = 0.4$ **37.** $\delta = \frac{1}{11} \approx 0.091$

39. $L = 8$. Let $\delta = 0.01/3 \approx 0.0033$.

41. $L = 1$. Let $\delta = 0.01/5 = 0.002$.

43. 6 **45.** -3 **47.** 3 **49.** 0 **51.** 10 **53.** 2 **55.** 4

57.

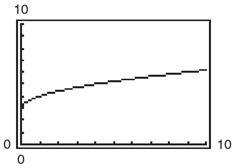


$\lim_{x\to 4} f(x) = \frac{1}{6}$

Domain: $[-5, 4) \cup (4, \infty)$

The graph has a hole at $x = 4$.

59.



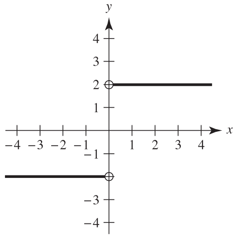
$\lim_{x\to 9} f(x) = 6$

Domain: $[0, 9) \cup (9, \infty)$

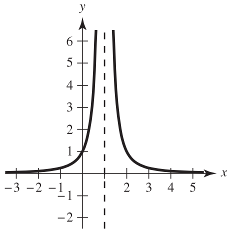
The graph has a hole at $x = 9$.

61. Answers will vary. Sample answer: As x approaches 8 from either side, $f(x)$ becomes arbitrarily close to 25.

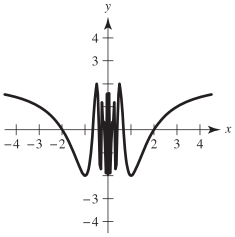
63. (i) The values of f approach different numbers as x approaches c from different sides of c.



(ii) The values of f increase or decrease without bound as x approaches c.



(iii) The values of f oscillate between two fixed numbers as x approaches c.



65. (a) $r = \dfrac{3}{\pi} \approx 0.9549$ cm

(b) $\dfrac{5.5}{2\pi} \le r \le \dfrac{6.5}{2\pi}$, or approximately $0.8754 < r < 1.0345$

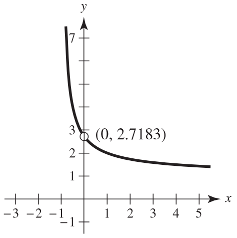
(c) $\lim_{r\to 3/\pi} 2\pi r = 6$; $\varepsilon = 0.5$; $\delta \approx 0.0796$

67.

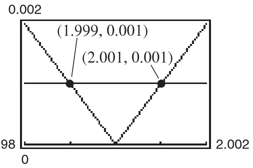
x	-0.001	-0.0001	-0.00001
$f(x)$	2.7196	2.7184	2.7183

x	0.00001	0.0001	0.001
$f(x)$	2.7183	2.7181	2.7169

$\lim_{x\to 0} f(x) \approx 2.7183$



69.



$\delta = 0.001$

$(1.999, 2.001)$

71. False. The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as $x \to c$.

73. False. See Exercise 17.

75. Yes. As x approaches 0.25 from either side, $\sqrt{x}$ becomes arbitrarily close to 0.5.

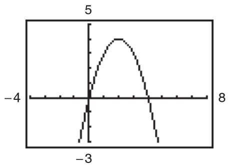
77. $\lim_{x\to 0} \dfrac{\sin nx}{x} = n$

79–81. Proofs **83.** Answers will vary.

85. Putnam Problem B1, 1986

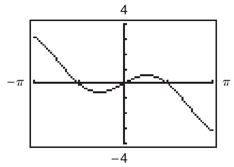
Section 1.3 (page 67)

1.



(a) 0 (b) -5

3.



(a) 0 (b) About 0.52 or $\pi/6$

5. 8 **7.** -1 **9.** 0 **11.** 7 **13.** 2 **15.** 1

17. $1/2$ **19.** $1/5$ **21.** 7 **23.** (a) 4 (b) 64 (c) 64

25. (a) 3 (b) 2 (c) 2 **27.** 1 **29.** $1/2$ **31.** 1

33. $1/2$ **35.** -1 **37.** (a) 10 (b) 5 (c) 6 (d) $3/2$

39. (a) 64 (b) 2 (c) 12 (d) 8

41. (a) -1 (b) -2

$g(x) = \dfrac{x^2 - x}{x}$ and $f(x) = x - 1$ agree except at $x = 0$.

43. (a) 2 (b) 0

$g(x) = \dfrac{x^3 - x}{x - 1}$ and $f(x) = x^2 + x$ agree except at $x = 1$.

45. -2

$f(x) = \dfrac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.

47. 12

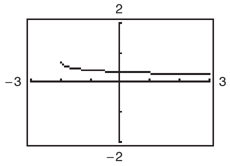
$f(x) = \dfrac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

49. -1 **51.** $1/8$ **53.** $5/6$ **55.** $1/6$ **57.** $\sqrt{5}/10$

59. $-1/9$ **61.** 2 **63.** $2x - 2$

65. $1/5$ **67.** 0 **69.** 0 **71.** 0 **73.** 1 **75.** $3/2$

77.



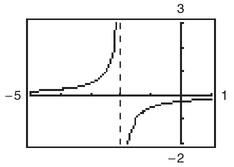
The graph has a hole at $x = 0$.

Answers will vary. Example:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	0.354	0.353	0.349

$$\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \approx 0.354 \left(\text{Actual limit is } \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.\right)$$

79.



The graph has a hole at $x = 0$.

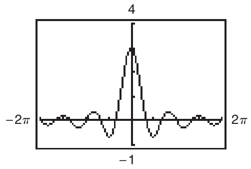
Answers will vary. Example:

x	-0.1	-0.01	-0.001
$f(x)$	-0.263	-0.251	-0.250

x	0.001	0.01	0.1
$f(x)$	-0.250	-0.249	-0.238

$$\lim_{x \to 0} \frac{[1/(2+x)] - (1/2)}{x} \approx -0.250 \left(\text{Actual limit is } -\frac{1}{4}.\right)$$

81.



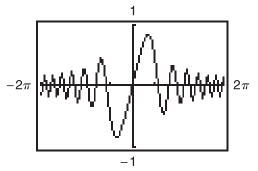
The graph has a hole at $t = 0$.

Answers will vary. Example:

t	-0.1	-0.01	0	0.01	0.1
$f(t)$	2.96	2.9996	?	2.9996	2.96

$$\lim_{t \to 0} \frac{\sin 3t}{t} = 3$$

83.



The graph has a hole at $x = 0$.

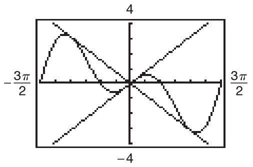
Answers will vary. Example:

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.1	-0.01	-0.001	?	0.001	0.01	0.1

$$\lim_{x \to 0} \frac{\sin x^2}{x} = 0$$

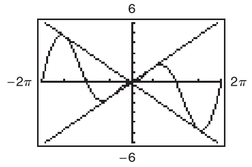
85. 3 **87.** $-1/(x + 3)^2$ **89.** 4

91.



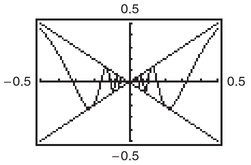
0

93.



0

95.



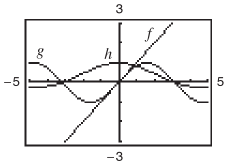
0

The graph has a hole at $x = 0$.

97. f and g agree at all but one point if c is a real number such that $f(x) = g(x)$ for all $x \neq c$.

99. An indeterminate form is obtained when evaluating a limit using direct substitution produces a meaningless fractional form, such as $\frac{0}{0}$.

101.



The magnitudes of $f(x)$ and $g(x)$ are approximately equal when x is close to 0. Therefore, their ratio is approximately 1.

103. -64 ft/sec (speed $= 64$ ft/sec) **105.** -29.4 m/sec

107. Let $f(x) = 1/x$ and $g(x) = -1/x$.

$\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 0} g(x)$ do not exist. However,

$$\lim_{x \to 0} [f(x) + g(x)] = \lim_{x \to 0} \left[\frac{1}{x} + \left(-\frac{1}{x}\right)\right] = \lim_{x \to 0} 0 = 0$$

and therefore does exist.

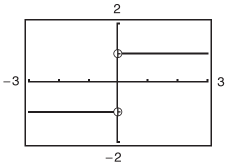
109–113. Proofs

115. Let $f(x) = \begin{cases} 4, & \text{if } x \geq 0 \\ -4, & \text{if } x < 0 \end{cases}$

$\lim\limits_{x \to 0} |f(x)| = \lim\limits_{x \to 0} 4 = 4$

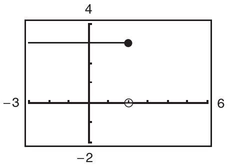
$\lim\limits_{x \to 0} f(x)$ does not exist because for $x < 0$, $f(x) = -4$ and for $x \geq 0$, $f(x) = 4$.

117. False. The limit does not exist because the function approaches 1 from the right side of 0 and approaches -1 from the left side of 0. (See graph below.)



119. True.

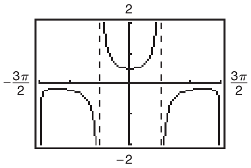
121. False. The limit does not exist because $f(x)$ approaches 3 from the left side of 2 and approaches 0 from the right side of 2. (See graph below.)



123. Proof

125. (a) All $x \neq 0, \dfrac{\pi}{2} + n\pi$

(b)



The domain is not obvious. The hole at $x = 0$ is not apparent from the graph.

(c) $\dfrac{1}{2}$ (d) $\dfrac{1}{2}$

127. The graphing utility was not set in *radian* mode.

Section 1.4 (page 78)

1. (a) 3 (b) 3 (c) 3; $f(x)$ is continuous on $(-\infty, \infty)$.

3. (a) 0 (b) 0 (c) 0; Discontinuity at $x = 3$

5. (a) -3 (b) 3 (c) Limit does not exist.
Discontinuity at $x = 2$

7. $\frac{1}{16}$ **9.** $\frac{1}{10}$

11. Limit does not exist. The function decreases without bound as x approaches -3 from the left.

13. -1 **15.** $-1/x^2$ **17.** $5/2$ **19.** 2

21. Limit does not exist. The function decreases without bound as x approaches π from the left and increases without bound as x approaches π from the right.

23. 8

25. Limit does not exist. The function approaches 5 from the left side of 3 but approaches 6 from the right side of 3.

27. Discontinuous at $x = -2$ and $x = 2$

29. Discontinuous at every integer

31. Continuous on $[-7, 7]$ **33.** Continuous on $[-1, 4]$

35. Nonremovable discontinuity at $x = 0$

37. Continuous for all real x

39. Nonremovable discontinuities at $x = -2$ and $x = 2$

41. Continuous for all real x

43. Nonremovable discontinuity at $x = 1$
Removable discontinuity at $x = 0$

45. Continuous for all real x

47. Removable discontinuity at $x = -2$
Nonremovable discontinuity at $x = 5$

49. Nonremovable discontinuity at $x = -7$

51. Continuous for all real x

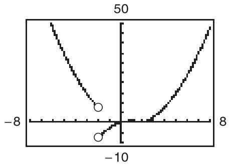
53. Nonremovable discontinuity at $x = 2$

55. Continuous for all real x

57. Nonremovable discontinuities at integer multiples of $\pi/2$

59. Nonremovable discontinuities at each integer

61.



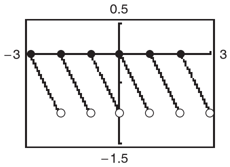
$\lim_{x\to 0^+} f(x) = 0$
$\lim_{x\to 0^-} f(x) = 0$
Discontinuity at $x = -2$

63. $a = 7$ **65.** $a = 2$ **67.** $a = -1,\ b = 1$

69. Continuous for all real x

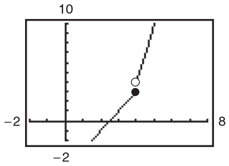
71. Nonremovable discontinuities at $x = 1$ and $x = -1$

73.



Nonremovable discontinuity at each integer

75.

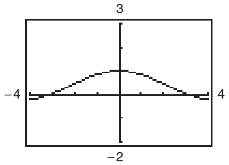


Nonremovable discontinuity at $x = 4$

77. Continuous on $(-\infty, \infty)$

79. Continuous on the open intervals . . . $(-6, -2)$, $(-2, 2)$, $(2, 6)$, ...

81.



The graph has a hole at $x = 0$. The graph appears to be continuous, but the function is not continuous on $[-4, 4]$. It is not obvious from the graph that the function has a discontinuity at $x = 0$.

83. Because $f(x)$ is continuous on the interval $[1, 2]$ and $f(1) = 37/12$ and $f(2) = -8/3$, by the Intermediate Value Theorem there exists a real number c in $[1, 2]$ such that $f(c) = 0$.

85. Because $f(x)$ is continuous on the interval $[0, \pi]$ and $f(0) = -3$ and $f(\pi) \approx 8.87$, by the Intermediate Value Theorem there exists a real number c in $[0, \pi]$ such that $f(c) = 0$.

87. 0.68, 0.6823 **89.** 0.56, 0.5636

91. $f(3) = 11$ **93.** $f(2) = 4$

95. (a) The limit does not exist at $x = c$.
(b) The function is not defined at $x = c$.
(c) The limit exists, but it is not equal to the value of the function at $x = c$.
(d) The limit does not exist at $x = c$.

97. If f and g are continuous for all real x, then so is $f + g$ (Theorem 1.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x, but f/g is not continuous at $x = \pm 1$.

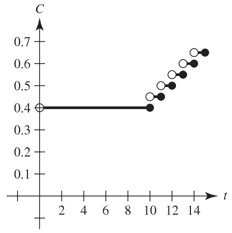
99. True

101. False. A rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n, respectively. It can have, at most, n discontinuities.

103. $\lim_{t\to 4^-} f(t) \approx 28$; $\lim_{t\to 4^+} f(t) \approx 56$

At the end of day 3, the amount of chlorine in the pool is about 28 oz. At the beginning of day 4, the amount of chlorine in the pool is about 56 oz.

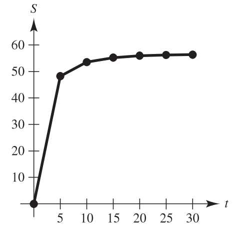
105. $C = \begin{cases} 0.40, & 0 < t \le 10 \\ 0.40 + 0.05[\![t - 9]\!], & t > 10, t \text{ is not an integer} \\ 0.40 + 0.05(t - 10), & t > 10, t \text{ is an integer} \end{cases}$



There is a nonremovable discontinuity at each integer greater than or equal to 10.

107–109. Proofs **111.** Answers will vary.

113. (a)

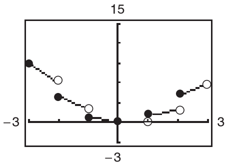


(b) There appears to be a limiting speed, and a possible cause is air resistance.

115. $c = \left(-1 \pm \sqrt{5}\right)/2$

117. Domain: $[-c^2, 0) \cup (0, \infty)$; Let $f(0) = 1/(2c)$

119. $h(x)$ has a nonremovable discontinuity at every integer except 0.



121. Putnam Problem B2, 1988

Section 1.5 (page 88)

1. $\lim_{x\to 4^+} \dfrac{1}{x-4} = \infty, \quad \lim_{x\to 4^-} \dfrac{1}{x-4} = -\infty$

3. $\lim_{x\to 4^+} \dfrac{1}{(x-4)^2} = \infty, \quad \lim_{x\to 4^-} \dfrac{1}{(x-4)^2} = \infty$

5. $\lim_{x\to -2^+} 2\left|\dfrac{x}{x^2-4}\right| = \infty, \quad \lim_{x\to -2^-} 2\left|\dfrac{x}{x^2-4}\right| = \infty$

7. $\lim_{x\to -2^+} \tan(\pi x/4) = -\infty, \quad \lim_{x\to -2^-} \tan(\pi x/4) = \infty$

9.

x	-3.5	-3.1	-3.01	-3.001
$f(x)$	0.31	1.64	16.6	167

x	-2.999	-2.99	-2.9	-2.5
$f(x)$	-167	-16.7	-1.69	-0.36

$\lim_{x\to -3^+} f(x) = -\infty \quad \lim_{x\to -3^-} f(x) = \infty$

11.

x	-3.5	-3.1	-3.01	-3.001
$f(x)$	3.8	16	151	1501

x	-2.999	-2.99	-2.9	-2.5
$f(x)$	-1499	-149	-14	-2.3

$\lim_{x\to -3^+} f(x) = -\infty \quad \lim_{x\to -3^-} f(x) = \infty$

13. $x = 0$ **15.** $x = \pm 2$ **17.** No vertical asymptote

19. $x = 2, \quad x = -1$ **21.** $t = 0$ **23.** $x = -2, x = 1$

25. No vertical asymptote **27.** No vertical asymptote

29. $x = \frac{1}{2} + n$, n is an integer.

31. $t = n\pi$, n is a nonzero integer.

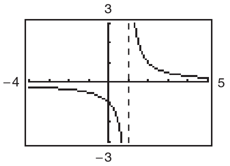
33. Removable discontinuity at $x = -1$

35. Vertical asymptote at $x = -1$ **37.** ∞ **39.** ∞

41. ∞ **43.** $-\frac{1}{5}$ **45.** $\frac{1}{2}$ **47.** $-\infty$ **49.** ∞ **51.** 0

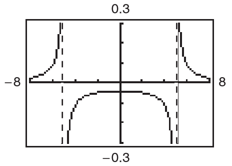
53. Limit does not exist.

55.



$\lim_{x\to 1^+} f(x) = \infty$

57.

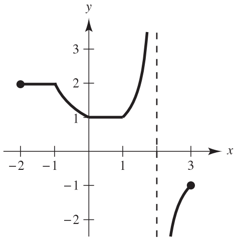


$\lim_{x\to 5^-} f(x) = -\infty$

59. Answers will vary.

61. Answers will vary. Example: $f(x) = \dfrac{x-3}{x^2 - 4x - 12}$

63.



65. ∞

67. (a) $\frac{1}{3}(200\pi)$ ft/sec

(b) 200π ft/sec

(c) $\lim_{\theta\to(\pi/2)^-} [50\pi \sec^2 \theta] = \infty$

69. (a) Domain: $x > 25$

(b)

x	30	40	50	60
y	150	66.667	50	42.857

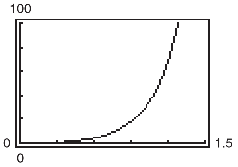
(c) $\lim_{x\to 25^+} \dfrac{25x}{x-25} = \infty$

As x gets closer and closer to 25 mi/h, y becomes larger and larger.

71. (a) $A = 50\tan\theta - 50\theta$; Domain: $(0, \pi/2)$

(b)

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$	0.47	4.21	18.0	68.6	630.1



(c) $\lim_{\theta \to \pi/2^-} A = \infty$

73. False; let $f(x) = (x^2 - 1)/(x - 1)$

75. False; let $f(x) = \tan x$

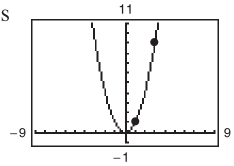
77. Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x^4}$, and let $c = 0$. $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$ and $\lim_{x\to 0} \dfrac{1}{x^4} = \infty$, but $\lim_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{x^4}\right) = \lim_{x\to 0} \left(\dfrac{x^2 - 1}{x^4}\right) = -\infty \neq 0$.

79. Given $\lim_{x\to c} f(x) = \infty$, let $g(x) = 1$. Then $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$ by Theorem 1.15.

81. Answers will vary.

Review Exercises for Chapter 1 (page 91)

1. Calculus



Estimate: 8.3

3.

x	−0.1	−0.01	−0.001
$f(x)$	−1.0526	−1.0050	−1.0005

x	0.001	0.01	0.1
$f(x)$	−0.9995	−0.9950	−0.9524

The estimate of the limit of $f(x)$, as x approaches zero, is -1.00.

5. 5; Proof **7.** -3; Proof **9.** (a) 4 (b) 5 **11.** 16

13. $\sqrt{6} \approx 2.45$ **15.** $-\frac{1}{4}$ **17.** $\frac{1}{2}$ **19.** -1 **21.** 75

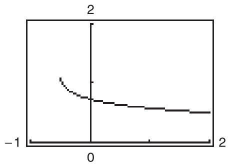
23. 0 **25.** $\sqrt{3}/2$ **27.** $-\frac{1}{2}$ **29.** $\frac{7}{12}$

31. (a)

x	1.1	1.01	1.001	1.0001
$f(x)$	0.5680	0.5764	0.5773	0.5773

$\lim_{x\to 1^+} f(x) \approx 0.5773$

(b)



The graph has a hole at $x = 1$.
$\lim_{x\to 1^+} f(x) \approx 0.5774$

(c) $\sqrt{3}/3$

33. -39.2 m/sec **35.** -1 **37.** 0

39. Limit does not exist. The limit as t approaches 1 from the left is 2 whereas the limit as t approaches 1 from the right is 1.

41. Continuous for all real x

43. Nonremovable discontinuity at each integer
Continuous on $(k, k + 1)$ for all integers k

45. Removable discontinuity at $x = 1$
Continuous on $(-\infty, 1) \cup (1, \infty)$

47. Nonremovable discontinuity at $x = 2$
Continuous on $(-\infty, 2) \cup (2, \infty)$

49. Nonremovable discontinuity at $x = -1$
Continuous on $(-\infty, -1) \cup (-1, \infty)$

51. Nonremovable discontinuity at each even integer
Continuous on $(2k, 2k + 2)$ for all integers k

53. $c = -\frac{1}{2}$ **55.** Proof

57. (a) -4 (b) 4 (c) Limit does not exist.

59. $x = 0$ **61.** $x = 10$ **63.** $-\infty$ **65.** $\frac{1}{3}$

67. $-\infty$ **69.** $-\infty$ **71.** $\frac{4}{5}$ **73.** ∞

75. (a) \$14,117.65 (b) \$80,000.00 (c) \$720,000.00 (d) ∞

P.S. Problem Solving (page 93)

1. (a) Perimeter $\triangle PAO = 1 + \sqrt{(x^2 - 1)^2 + x^2} + \sqrt{x^4 + x^2}$
Perimeter $\triangle PBO = 1 + \sqrt{x^4 + (x - 1)^2} + \sqrt{x^4 + x^2}$

(b)

x	4	2	1
Perimeter $\triangle PAO$	33.0166	9.0777	3.4142
Perimeter $\triangle PBO$	33.7712	9.5952	3.4142
$r(x)$	0.9777	0.9461	1.0000

x	0.1	0.01
Perimeter $\triangle PAO$	2.0955	2.0100
Perimeter $\triangle PBO$	2.0006	2.0000
$r(x)$	1.0475	1.0050

(c) 1

3. (a) Area (hexagon) $= (3\sqrt{3})/2 \approx 2.5981$
Area (circle) $= \pi \approx 3.1416$
Area (circle) − Area (hexagon) ≈ 0.5435

(b) $A_n = (n/2)\sin(2\pi/n)$

(c)

n	6	12	24	48	96
A_n	2.5981	3.0000	3.1058	3.1326	3.1394

(d) 3.1416 or π

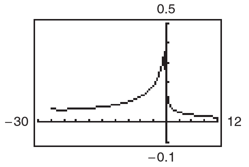
5. (a) $m = -\frac{12}{5}$ (b) $y = \frac{5}{12}x - \frac{169}{12}$

(c) $m_x = \dfrac{-\sqrt{169 - x^2} + 12}{x - 5}$

(d) $\frac{5}{12}$; It is the same as the slope of the tangent line found in (b).

7. (a) Domain: $[-27, 1) \cup (1, \infty)$

(b)

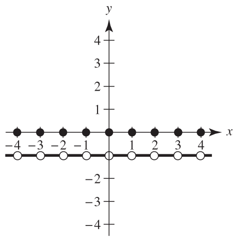


The graph has a hole at $x = 1$.

(c) $\frac{1}{14}$ (d) $\frac{1}{12}$

9. (a) g_1, g_4 (b) g_1 (c) g_1, g_3, g_4

11.



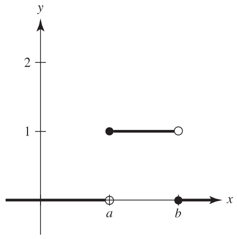
The graph jumps at every integer.

(a) $f(1) = 0,\ f(0) = 0,\ f\left(\frac{1}{2}\right) = -1,\ f(-2.7) = -1$

(b) $\lim\limits_{x\to 1^-} f(x) = -1,\ \lim\limits_{x\to 1^+} f(x) = -1,\ \lim\limits_{x\to 1/2} f(x) = -1$

(c) There is a discontinuity at each integer.

13. (a)



(b) (i) $\lim\limits_{x\to a^+} P_{a,b}(x) = 1$

(ii) $\lim\limits_{x\to a^-} P_{a,b}(x) = 0$

(iii) $\lim\limits_{x\to b^+} P_{a,b}(x) = 0$

(iv) $\lim\limits_{x\to b^-} P_{a,b}(x) = 1$

(c) Continuous for all positive real numbers except a and b

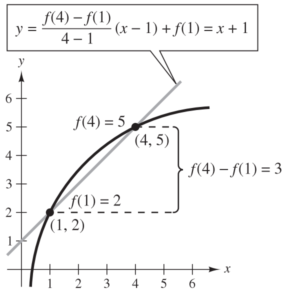
(d) The area under the graph of U and above the x-axis is 1.

Chapter 2

Section 2.1 (page 103)

1. (a) $m_1 = 0, m_2 = 5/2$ (b) $m_1 = -5/2, m_2 = 2$

3.



5. $m = -5$ **7.** $m = 4$

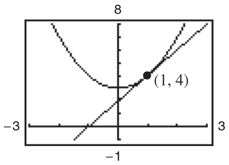
9. $m = 3$ **11.** $f'(x) = 0$ **13.** $f'(x) = -10$ **15.** $h'(s) = \frac{2}{3}$

17. $f'(x) = 2x + 1$ **19.** $f'(x) = 3x^2 - 12$

21. $f'(x) = \dfrac{-1}{(x - 1)^2}$ **23.** $f'(x) = \dfrac{1}{2\sqrt{x + 4}}$

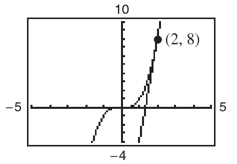
25. (a) Tangent line:
$y = 2x + 2$

(b)



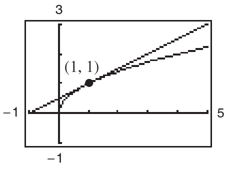
27. (a) Tangent line:
$y = 12x - 16$

(b)



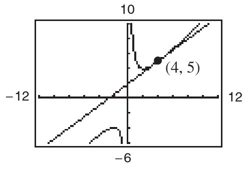
29. (a) Tangent line:
$y = \frac{1}{2}x + \frac{1}{2}$

(b)



31. (a) Tangent line:
$y = \frac{3}{4}x + 2$

(b)

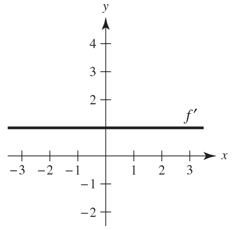


33. $y = 2x - 1$ **35.** $y = 3x - 2;\ y = 3x + 2$

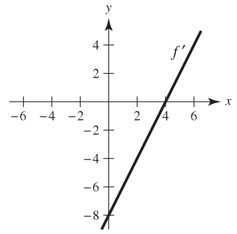
37. $y = -\frac{1}{2}x + \frac{3}{2}$ **39.** b **40.** d **41.** a **42.** c

43. $g(4) = 5;\ g'(4) = -\frac{5}{3}$

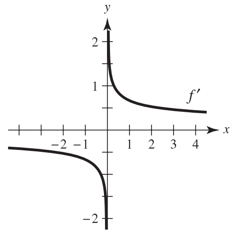
45.



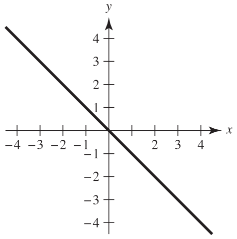
47.



49.



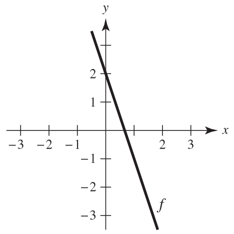
51. Answers will vary.
Sample answer: $y = -x$



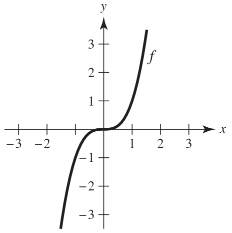
53. $f(x) = 5 - 3x$
$c = 1$

55. $f(x) = -x^2$
$c = 6$

57. $f(x) = -3x + 2$

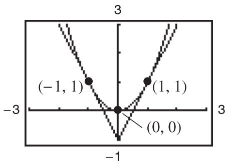


59. Answers will vary.
Sample answer: $f(x) = x^3$



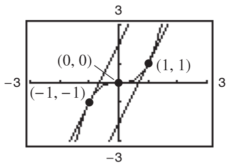
61. $y = 2x + 1;\ y = -2x + 9$

63. (a)



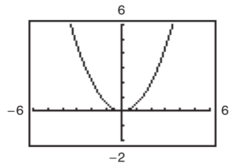
For this function, the slopes of the tangent lines are always distinct for different values of x.

(b)



For this function, the slopes of the tangent lines are sometimes the same.

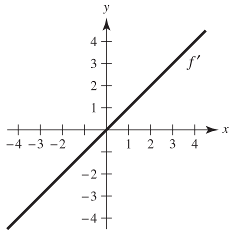
65. (a)



$f'(0)=0, f'\left(\frac{1}{2}\right)=\frac{1}{2}, f'(1)=1, f'(2)=2$

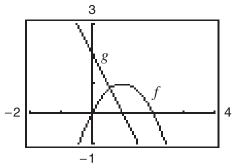
(b) $f'\left(-\frac{1}{2}\right)=-\frac{1}{2}, f'(-1)=-1, f'(-2)=-2$

(c)



(d) $f'(x)=x$

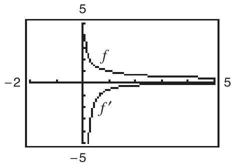
67.



$g(x)\approx f'(x)$

69. $f(2)=4; f(2.1)=3.99; f'(2)\approx -0.1$

71.



As x approaches infinity, the graph of f approaches a line of slope 0. Thus $f'(x)$ approaches 0.

73. 6 **75.** 4 **77.** $g(x)$ is not differentiable at $x=0$.

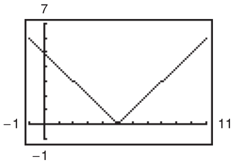
79. $f(x)$ is not differentiable at $x=6$.

81. $h(x)$ is not differentiable at $x=-7$.

83. $(-\infty, 3)\cup(3,\infty)$ **85.** $(-\infty,-4)\cup(-4,\infty)$

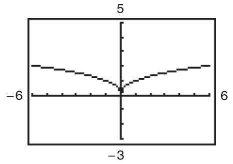
87. $(1,\infty)$

89.



$(-\infty,5)\cup(5,\infty)$

91.



$(-\infty,0)\cup(0,\infty)$

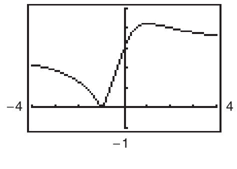
93. The derivative from the left is -1 and the derivative from the right is 1, so f is not differentiable at $x=1$.

95. The derivatives from both the right and the left are 0, so $f'(1)=0$.

97. f is differentiable at $x=2$.

99. (a) $d=\left(3|m+1|\right)/\sqrt{m^2+1}$

(b)



Not differentiable at $m=-1$

101. False. The slope is $\lim_{\Delta x\to 0}\dfrac{f(2+\Delta x)-f(2)}{\Delta x}$.

103. False. For example: $f(x)=|x|$. The derivative from the left and the derivative from the right both exist but are not equal.

105. Proof

Section 2.2 (page 115)

1. (a) $\frac{1}{2}$ (b) 3 **3.** 0 **5.** $7x^6$ **7.** $-5/x^6$ **9.** $1/(5x^{4/5})$

11. 1 **13.** $-4t+3$ **15.** $2x+12x^2$ **17.** $3t^2+10t-3$

19. $\dfrac{\pi}{2}\cos\theta+\sin\theta$ **21.** $2x+\dfrac{1}{2}\sin x$ **23.** $-\dfrac{1}{x^2}-3\cos x$

	Function	*Rewrite*	*Derivative*	*Simplify*
25.	$y=\dfrac{5}{2x^2}$	$y=\dfrac{5}{2}x^{-2}$	$y'=-5x^{-3}$	$y'=-\dfrac{5}{x^3}$
27.	$y=\dfrac{6}{(5x)^3}$	$y=\dfrac{6}{125}x^{-3}$	$y'=-\dfrac{18}{125}x^{-4}$	$y'=-\dfrac{18}{125x^4}$
29.	$y=\dfrac{\sqrt{x}}{x}$	$y=x^{-1/2}$	$y'=-\dfrac{1}{2}x^{-3/2}$	$y'=-\dfrac{1}{2x^{3/2}}$

31. -2 **33.** 0 **35.** 8 **37.** 3 **39.** $2x+6/x^3$

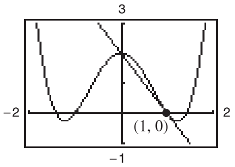
41. $2t+12/t^4$ **43.** $8x+3$ **45.** $(x^3-8)/x^3$

47. $3x^2+1$ **49.** $\dfrac{1}{2\sqrt{x}}-\dfrac{2}{x^{2/3}}$ **51.** $\dfrac{4}{5s^{1/5}}-\dfrac{2}{3s^{1/3}}$

53. $\dfrac{3}{\sqrt{x}}-5\sin x$

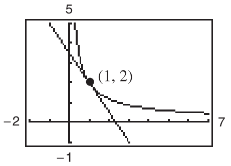
55. (a) $2x+y-2=0$

(b)



57. (a) $3x+2y-7=0$

(b)

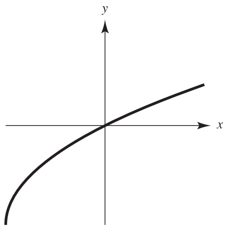


59. $(-1,2), (0,3), (1,2)$ **61.** No horizontal tangents

63. (π,π) **65.** $k=-1,\ k=-9$

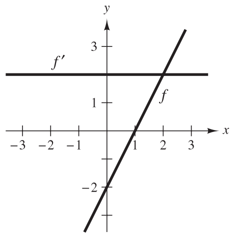
67. $k=3$ **69.** $k=4/27$

71.



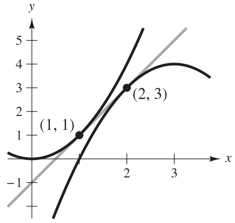
73. $g'(x)=f'(x)$

75.

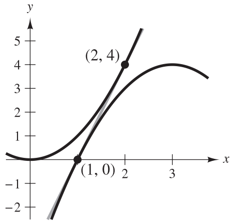


The rate of change of f is constant and therefore f' is a constant function.

77. $y = 2x - 1$

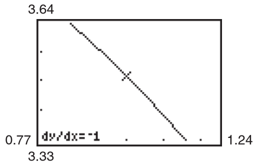


$y = 4x - 4$



79. $f'(x) = 3 + \cos x \neq 0$ for all x. **81.** $x - 4y + 4 = 0$

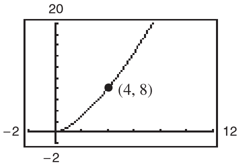
83.



$f'(1)$ appears to be close to -1.

$f'(1) = -1$

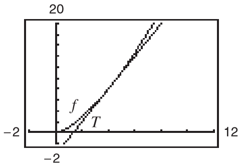
85. (a)



$(3.9, 7.7019)$,
$S(x) = 2.981x - 3.924$

(b) $T(x) = 3(x - 4) + 8 = 3x - 4$
The slope (and equation) of the secant line approaches that of the tangent line at $(4, 8)$ as you choose points closer and closer to $(4, 8)$.

(c)



The approximation becomes less accurate.

(d)

Δx	-3	-2	-1	-0.5	-0.1	0
$f(4 + \Delta x)$	1	2.828	5.196	6.548	7.702	8
$T(4 + \Delta x)$	-1	2	5	6.5	7.7	8

Δx	0.1	0.5	1	2	3
$f(4 + \Delta x)$	8.302	9.546	11.180	14.697	18.520
$T(4 + \Delta x)$	8.3	9.5	11	14	17

87. False. Let $f(x) = x$ and $g(x) = x + 1$.

89. False. $dy/dx = 0$ **91.** True

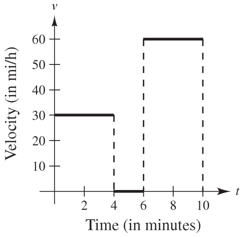
93. Average rate: 4
Instantaneous rates:
$f'(1) = 4; f'(2) = 4$

95. Average rate: $\frac{1}{2}$
Instantaneous rates:
$f'(1) = 1; f'(2) = \frac{1}{4}$

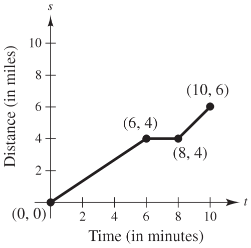
97. (a) $s(t) = -16t^2 + 1362$; $v(t) = -32t$ (b) -48 ft/sec
(c) $s'(1) = -32$ ft/sec; $s'(2) = -64$ ft/sec
(d) $t = \dfrac{\sqrt{1362}}{4} \approx 9.226$ sec (e) -295.242 ft/sec

99. $v(5) = 71$ m/sec; $v(10) = 22$ m/sec

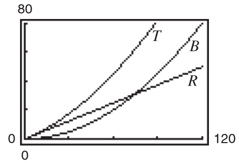
101.



103.



105. (a) $R(v) = 0.417v - 0.02$
(b) $B(v) = 0.0056v^2 + 0.001v + 0.04$
(c) $T(v) = 0.0056v^2 + 0.418v + 0.02$
(d)



(e) $T'(v) = 0.0112v + 0.418$
$T'(40) = 0.866$
$T'(80) = 1.314$
$T'(100) = 1.538$

(f) Stopping distance increases at an increasing rate.

107. $V'(6) = 108$ cm^3/cm **109.** Proof

111. (a) The rate of change of the number of gallons of gasoline sold when the price is \$2.979
(b) In general, the rate of change when $p = 2.979$ should be negative. As prices go up, sales go down.

113. $y = 2x^2 - 3x + 1$ **115.** $9x + y = 0, 9x + 4y + 27 = 0$

117. $a = \frac{1}{3}, b = -\frac{4}{3}$

119. $f_1(x) = |\sin x|$ is differentiable for all $x \neq n\pi$, n an integer.
$f_2(x) = \sin|x|$ is differentiable for all $x \neq 0$.

Section 2.3 (page 126)

1. $2(2x^3 - 6x^2 + 3x - 6)$ **3.** $(1 - 5t^2)/(2\sqrt{t})$

5. $x^2(3\cos x - x\sin x)$ **7.** $(1 - x^2)/(x^2 + 1)^2$

9. $(1 - 5x^3)/\left[2\sqrt{x}(x^3 + 1)^2\right]$ **11.** $(x\cos x - 2\sin x)/x^3$

13. $f'(x) = (x^3 + 4x)(6x + 2) + (3x^2 + 2x - 5)(3x^2 + 4)$
$= 15x^4 + 8x^3 + 21x^2 + 16x - 20$
$f'(0) = -20$

15. $f'(x) = \dfrac{x^2 - 6x + 4}{(x - 3)^2}$
$f'(1) = -\dfrac{1}{4}$

17. $f'(x) = \cos x - x\sin x$
$f'\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{8}(4 - \pi)$

	Function	*Rewrite*	*Differentiate*	*Simplify*
19.	$y = \dfrac{x^2 + 3x}{7}$	$y = \dfrac{1}{7}x^2 + \dfrac{3}{7}x$	$y' = \dfrac{2}{7}x + \dfrac{3}{7}$	$y' = \dfrac{2x + 3}{7}$
21.	$y = \dfrac{6}{7x^2}$	$y = \dfrac{6}{7}x^{-2}$	$y' = -\dfrac{12}{7}x^{-3}$	$y' = -\dfrac{12}{7x^3}$
23.	$y = \dfrac{4x^{3/2}}{x}$	$y = 4x^{1/2}$, $x > 0$	$y' = 2x^{-1/2}$	$y' = \dfrac{2}{\sqrt{x}}$, $x > 0$

25. $\dfrac{(x^2 - 1)(-3 - 2x) - (4 - 3x - x^2)(2x)}{(x^2 - 1)^2} = \dfrac{3}{(x + 1)^2}, x \neq 1$

27. $1 - 12/(x + 3)^2 = (x^2 + 6x - 3)/(x + 3)^2$

29. $\dfrac{3}{2}x^{-1/2} + \dfrac{1}{2}x^{-3/2} = (3x + 1)/2x^{3/2}$

31. $6s^2(s^3 - 2)$ **33.** $-(2x^2 - 2x + 3)/[x^2(x - 3)^2]$

35. $(6x^2+5)(x-3)(x+2)+(2x^3+5x)(1)(x+2)$
$+(2x^3+5x)(x-3)(1)$
$=10x^4-8x^3-21x^2-10x-30$

37. $\dfrac{(x^2-c^2)(2x)-(x^2+c^2)(2x)}{(x^2-c^2)^2}=-\dfrac{4xc^2}{(x^2-c^2)^2}$

39. $t(t\cos t+2\sin t)$ **41.** $-(t\sin t+\cos t)/t^2$

43. $-1+\sec^2 x=\tan^2 x$ **45.** $\dfrac{1}{4t^{3/4}}-6\csc t\cot t$

47. $\dfrac{-6\cos^2 x+6\sin x-6\sin^2 x}{4\cos^2 x}=\dfrac{3}{2}(-1+\tan x\sec x-\tan^2 x)$
$=\dfrac{3}{2}\sec x(\tan x-\sec x)$

49. $\csc x\cot x-\cos x=\cos x\cot^2 x$ **51.** $x(x\sec^2 x+2\tan x)$

53. $2x\cos x+2\sin x-x^2\sin x+2x\cos x$
$=4x\cos x+(2-x^2)\sin x$

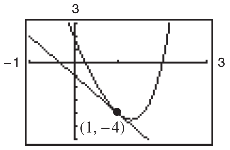
55. $\left(\dfrac{x+1}{x+2}\right)(2)+(2x-5)\left[\dfrac{(x+2)(1)-(x+1)(1)}{(x+2)^2}\right]$
$=\dfrac{2x^2+8x-1}{(x+2)^2}$

57. $\dfrac{1-\sin\theta+\theta\cos\theta}{(1-\sin\theta)^2}$ **59.** $y'=\dfrac{-2\csc x\cot x}{(1-\csc x)^2},\quad -4\sqrt{3}$

61. $h'(t)=\sec t(t\tan t-1)/t^2,\quad 1/\pi^2$

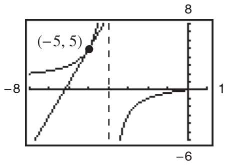
63. (a) $y=-3x-1$

(b)



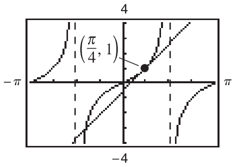
65. (a) $y=4x+25$

(b)



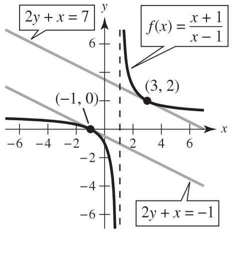
67. (a) $4x-2y-\pi+2=0$ **69.** $2y+x-4=0$

(b)



71. $25y-12x+16=0$ **73.** $(1, 1)$ **75.** $(0, 0), (2, 4)$

77. Tangent lines: $2y+x=7$; $2y+x=-1$



79. $f(x)+2=g(x)$ **81.** (a) $p'(1)=1$ (b) $q'(4)=-1/3$

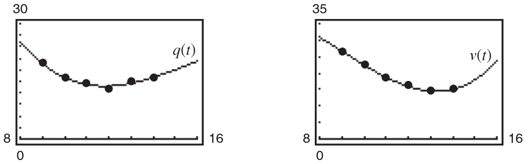
83. $(18t+5)/(2\sqrt{t})$ cm²/sec

85. (a) −\$38.13 thousand/100 components
(b) −\$10.37 thousand/100 components
(c) −\$3.80 thousand/100 components
The cost decreases with increasing order size.

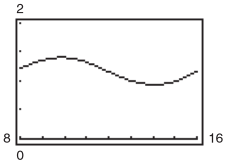
87. 31.55 bacteria/h **89.** Proof

91. (a) $q(t)=-0.0546t^3+2.529t^2-36.89t+186.6$
$v(t)=0.0796t^3-2.162t^2+15.32t+5.9$

(b)



(c) $A=\dfrac{0.0796t^3-2.162t^2+15.32t+5.9}{-0.0546t^3+2.529t^2-36.89t+186.6}$



A represents the average value (in billions of dollars) per one million personal computers.

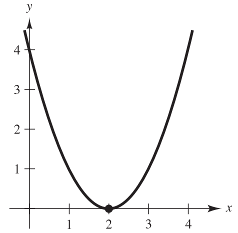
(d) $A'(t)$ represents the rate of change of the average value per one million personal computers for the given year.

93. $12x^2+12x-6$ **95.** $3/\sqrt{x}$ **97.** $2/(x-1)^3$

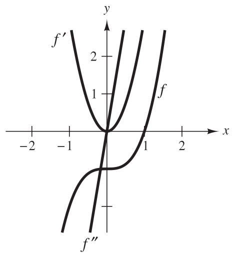
99. $2\cos x-x\sin x$ **101.** $2x$ **103.** $1/\sqrt{x}$

105. 0 **107.** −10

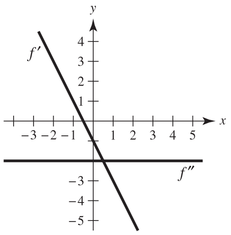
109. Answers will vary. For example: $f(x)=(x-2)^2$



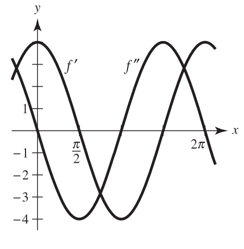
111.



113.



115.



117. $v(3)=27$ m/sec
$a(3)=-6$ m/sec²
The speed of the object is decreasing.

119.

t	0	1	2	3	4
$s(t)$	0	57.75	99	123.75	132
$v(t)$	66	49.5	33	16.5	0
$a(t)$	−16.5	−16.5	−16.5	−16.5	−16.5

The average velocity on [0, 1] is 57.75, on [1, 2] is 41.25, on [2, 3] is 24.75, and on [3, 4] is 8.25.

121. $f^{(n)}(x) = n(n-1)(n-2)\cdots(2)(1) = n!$

123. (a) $f''(x) = g(x)h''(x) + 2g'(x)h'(x) + g''(x)h(x)$

$f'''(x) = g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x)$

$f^{(4)}(x) = g(x)h^{(4)}(x) + 4g'(x)h'''(x) + 6g''(x)h''(x) + 4g'''(x)h'(x) + g^{(4)}(x)h(x)$

(b) $f^{(n)}(x) = g(x)h^{(n)}(x) + \dfrac{n!}{1!(n-1)!}g'(x)h^{(n-1)}(x) + \dfrac{n!}{2!(n-2)!}g''(x)h^{(n-2)}(x) + \cdots + \dfrac{n!}{(n-1)!1!}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$

125. $n = 1$: $f'(x) = x\cos x + \sin x$

$n = 2$: $f'(x) = x^2\cos x + 2x\sin x$

$n = 3$: $f'(x) = x^3\cos x + 3x^2\sin x$

$n = 4$: $f'(x) = x^4\cos x + 4x^3\sin x$

General rule: $f'(x) = x^n\cos x + nx^{(n-1)}\sin x$

127. $y' = -1/x^2$, $y'' = 2/x^3$,

$x^3y'' + 2x^2y' = x^3(2/x^3) + 2x^2(-1/x^2)$
$= 2 - 2 = 0$

129. $y' = 2\cos x$, $y'' = -2\sin x$,

$y'' + y = -2\sin x + 2\sin x + 3 = 3$

131. False. $dy/dx = f(x)g'(x) + g(x)f'(x)$ **133.** True

135. True **137.** $f(x) = 3x^2 - 2x - 1$

139. $f'(x) = 2|x|$; $f''(0)$ does not exist. **141.** Proof

Section 2.4 (page 137)

	$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1.	$y = (5x-8)^4$	$u = 5x - 8$	$y = u^4$
3.	$y = \sqrt{x^3-7}$	$u = x^3 - 7$	$y = \sqrt{u}$
5.	$y = \csc^3 x$	$u = \csc x$	$y = u^3$

7. $12(4x-1)^2$ **9.** $-108(4-9x)^3$

11. $\frac{1}{2}(5-t)^{-1/2}(-1) = -1/\left(2\sqrt{5-t}\right)$

13. $\frac{1}{3}(6x^2+1)^{-2/3}(12x) = 4x/\sqrt[3]{(6x^2+1)^2}$

15. $\frac{1}{2}(9-x^2)^{-3/4}(-2x) = -x/\sqrt[4]{(9-x^2)^3}$ **17.** $-1/(x-2)^2$

19. $-2(t-3)^{-3}(1) = -2/(t-3)^3$ **21.** $-1/\left[2\sqrt{(x+2)^3}\right]$

23. $x^2[4(x-2)^3(1)] + (x-2)^4(2x) = 2x(x-2)^3(3x-2)$

25. $x\left(\dfrac{1}{2}\right)(1-x^2)^{-1/2}(-2x) + (1-x^2)^{1/2}(1) = \dfrac{1-2x^2}{\sqrt{1-x^2}}$

27. $\dfrac{(x^2+1)^{1/2}(1) - x(1/2)(x^2+1)^{-1/2}(2x)}{x^2+1} = \dfrac{1}{\sqrt{(x^2+1)^3}}$

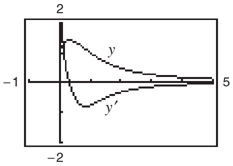
29. $\dfrac{-2(x+5)(x^2+10x-2)}{(x^2+2)^3}$ **31.** $\dfrac{-9(1-2v)^2}{(v+1)^4}$

33. $2((x^2+3)^5 + x)(5(x^2+3)^4(2x) + 1)$
$= 20x(x^2+3)^9 + 2(x^2+3)^5 + 20x^2(x^2+3)^4 + 2x$

35. $\dfrac{1}{2}(2 + (2 + x^{1/2})^{1/2})^{-1/2}\left(\dfrac{1}{2}(2 + x^{1/2})^{-1/2}\right)\left(\dfrac{1}{2}x^{-1/2}\right)$

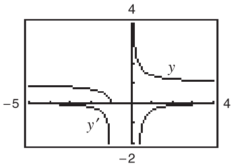
$= \dfrac{1}{8\sqrt{x}\left(\sqrt{2+\sqrt{x}}\right)\left(\sqrt{2+\sqrt{2+\sqrt{x}}}\right)}$

37. $(1 - 3x^2 - 4x^{3/2})/[2\sqrt{x}(x^2+1)^2]$



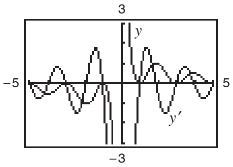
The zero of y' corresponds to the point on the graph of the function where the tangent line is horizontal.

39. $-\dfrac{\sqrt{\dfrac{x+1}{x}}}{2x(x+1)}$



y' has no zeros.

41. $-[\pi x\sin(\pi x) + \cos(\pi x) + 1]/x^2$



The zeros of y' correspond to the points on the graph of the function where the tangent lines are horizontal.

43. (a) 1 (b) 2; The slope of $\sin ax$ at the origin is a.

45. $-4\sin 4x$ **47.** $15\sec^2 3x$ **49.** $2\pi^2 x\cos(\pi x)^2$

51. $2\cos 4x$ **53.** $(-1 - \cos^2 x)/\sin^3 x$

55. $8\sec^2 x\tan x$ **57.** $10\tan 5\theta\sec^2 5\theta$

59. $\sin 2\theta\cos 2\theta = \dfrac{1}{2}\sin 4\theta$

61. $\dfrac{6\pi\sin(\pi t - 1)}{\cos^3(\pi t - 1)}$ **63.** $\dfrac{1}{2\sqrt{x}} + 2x\cos(2x)^2$

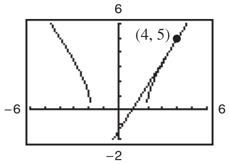
65. $2\sec^2 2x\cos(\tan 2x)$

67. $s'(t) = \dfrac{t+3}{\sqrt{t^2+6t-2}}, \dfrac{6}{5}$ **69.** $f'(x) = \dfrac{-15x^2}{(x^3-2)^2}, -\dfrac{3}{5}$

71. $f'(t) = \dfrac{-5}{(t-1)^2}, -5$ **73.** $y' = -12\sec^3 4x\tan 4x, 0$

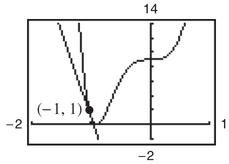
75. (a) $8x - 5y - 7 = 0$

(b)



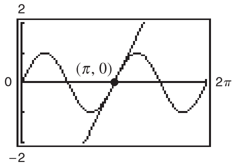
77. (a) $24x + y + 23 = 0$

(b)



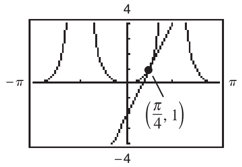
79. (a) $2x - y - 2\pi = 0$

(b)



81. (a) $4x - y + (1 - \pi) = 0$

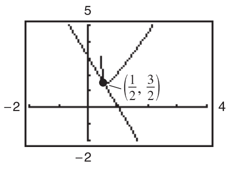
(b)



83. (a) $g'(1/2) = -3$

(b) $3x + y - 3 = 0$

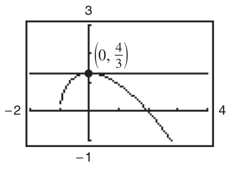
(c)



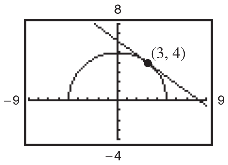
85. (a) $s'(0) = 0$

(b) $y = \frac{4}{3}$

(c)



87. $3x + 4y - 25 = 0$



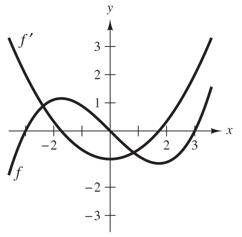
89. $\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right), \left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2}\right), \left(\frac{3\pi}{2}, 0\right)$ **91.** $2940(2 - 7x)^2$

93. $\dfrac{2}{(x - 6)^3}$

95. $2(\cos x^2 - 2x^2 \sin x^2)$ **97.** $h''(x) = 18x + 6, 24$

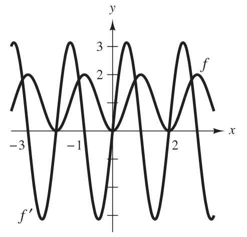
99. $f''(x) = -4x^2 \cos(x^2) - 2 \sin(x^2), 0$

101.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

103.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

105. The rate of change of g is three times as fast as the rate of change of f.

107. (a) $g'(x) = f'(x)$ (b) $h'(x) = 2f'(x)$
(c) $r'(x) = -3f'(-3x)$ (d) $s'(x) = f'(x + 2)$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$h'(x)$	8	$\frac{4}{3}$	$-\frac{2}{3}$	-2	-4	-8
$r'(x)$		12	1			
$s'(x)$	$-\frac{1}{3}$	-1	-2	-4		

109. (a) $\frac{1}{2}$
(b) $s'(5)$ does not exist because g is not differentiable at 6.

111. (a) 1.461 (b) −1.016

113. 0.2 rad, 1.45 rad/sec **115.** 0.04224 cm/sec

117. (a) $x = -1.637t^3 + 19.31t^2 - 0.5t - 1$
(b) $\dfrac{dC}{dt} = -294.66t^2 + 2317.2t - 30$
(c) Because x, the number of units produced in t hours, is not a linear function, and therefore the cost with respect to time t is not linear.

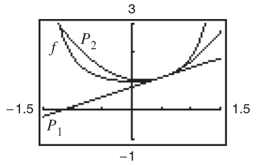
119. (a) Yes, if $f(x + p) = f(x)$ for all x, then $f'(x + p) = f'(x)$, which shows that f' is periodic as well.
(b) Yes, if $g(x) = f(2x)$, then $g'(x) = 2f'(2x)$. Because f' is periodic, so is g'.

121. (a) 0
(b) $f'(x) = 2 \sec x \cdot \sec x \tan x = 2 \sec^2 x \tan x$
$g'(x) = 2 \tan x \sec^2 x = 2 \sec^2 x \tan x$
$f'(x) = g'(x)$

123. Proof **125.** $f'(x) = 2x\left(\dfrac{x^2 - 9}{|x^2 - 9|}\right), x \neq \pm 3$

127. $f'(x) = \cos x \sin x/|\sin x|, x \neq k\pi$

129. (a) $P_1(x) = 2/3(x - \pi/6) + 2/\sqrt{3}$
$P_2(x) = 5/(3\sqrt{3})(x - \pi/6)^2 + 2/3(x - \pi/6) + 2/\sqrt{3}$
(b)



(c) P_2
(d) The accuracy worsens as you move away from $x = \pi/6$.

131. False. If $f(x) = \sin^2 2x$, then $f'(x) = 2(\sin 2x)(2 \cos 2x)$.

133. Putnam Problem A1, 1967

Section 2.5 (page 146)

1. $-x/y$ **3.** $-\sqrt{y/x}$ **5.** $(y - 3x^2)/(2y - x)$

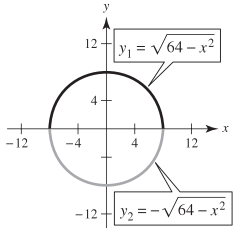
7. $(1 - 3x^2y^3)/(3x^3y^2 - 1)$

9. $(6xy - 3x^2 - 2y^2)/(4xy - 3x^2)$

11. $\cos x/[4 \sin(2y)]$ **13.** $(\cos x - \tan y - 1)/(x \sec^2 y)$

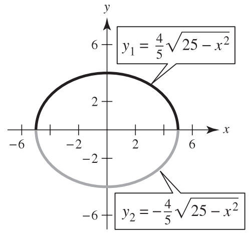
15. $[y \cos(xy)]/[1 - x \cos(xy)]$

17. (a) $y_1 = \sqrt{64 - x^2}$; $y_2 = -\sqrt{64 - x^2}$
(b)



(c) $y' = \mp \dfrac{x}{\sqrt{64 - x^2}} = -\dfrac{x}{y}$
(d) $y' = -\dfrac{x}{y}$

19. (a) $y_1 = \frac{4}{5}\sqrt{25 - x^2}$; $y_2 = -\frac{4}{5}\sqrt{25 - x^2}$
(b)



(c) $y' = \mp \dfrac{4x}{5\sqrt{25 - x^2}} = -\dfrac{16x}{25y}$
(d) $y' = -\dfrac{16x}{25y}$

21. $-\dfrac{y}{x}, -\dfrac{1}{6}$ **23.** $\dfrac{98x}{y(x^2 + 49)^2}$, Undefined **25.** $-\sqrt[3]{\dfrac{y}{x}}, -\dfrac{1}{2}$

27. $-\sin^2(x + y)$ or $-\dfrac{x^2}{x^2 + 1}, 0$ **29.** $-\dfrac{1}{2}$ **31.** 0

33. $y = -x + 7$ **35.** $y = -x + 2$

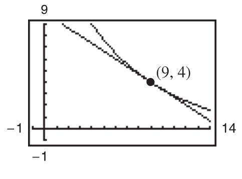
37. $y = \sqrt{3}x/6 + 8\sqrt{3}/3$ **39.** $y = -\frac{2}{11}x + \frac{30}{11}$

41. (a) $y = -2x + 4$ (b) Answers will vary.

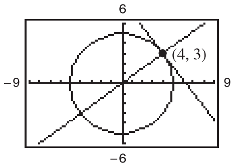
43. $\cos^2 y, -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}, \dfrac{1}{1 + x^2}$ **45.** $-4/y^3$

47. $-36/y^3$ **49.** $(3x)/(4y)$

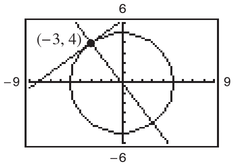
51. $2x + 3y - 30 = 0$



53. At $(4, 3)$:
Tangent line: $4x + 3y - 25 = 0$
Normal line: $3x - 4y = 0$



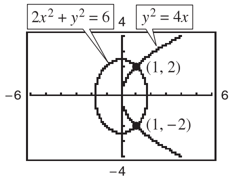
At $(-3, 4)$:
Tangent line: $3x - 4y + 25 = 0$
Normal line: $4x + 3y = 0$



55. $x^2 + y^2 = r^2 \Rightarrow y' = -x/y \Rightarrow y/x =$ slope of normal line. Then for (x_0, y_0) on the circle, $x_0 \neq 0$, an equation of the normal line is $y = (y_0/x_0)x$, which passes through the origin. If $x_0 = 0$, the normal line is vertical and passes through the origin.

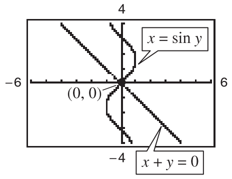
57. Horizontal tangents: $(-4, 0), (-4, 10)$
Vertical tangents: $(0, 5), (-8, 5)$

59.



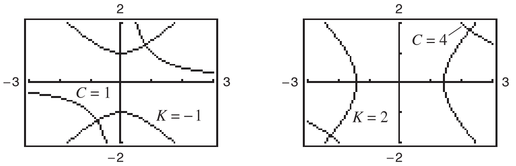
At $(1, 2)$:
Slope of ellipse: -1
Slope of parabola: 1
At $(1, -2)$:
Slope of ellipse: 1
Slope of parabola: -1

61.



At $(0, 0)$:
Slope of line: -1
Slope of sine curve: 1

63. Derivatives: $\frac{dy}{dx} = -\frac{y}{x}, \frac{dy}{dx} = \frac{x}{y}$

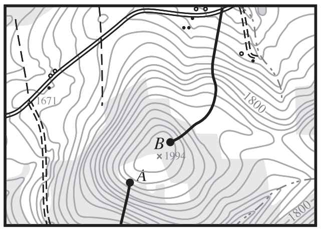


65. (a) $\frac{dy}{dx} = \frac{3x^3}{y}$ (b) $y\frac{dy}{dt} = 3x^3\frac{dx}{dt}$

67. (a) $\frac{dy}{dx} = \frac{-3\cos \pi x}{\sin \pi y}$ (b) $-\sin \pi y\left(\frac{dy}{dt}\right) = 3\cos \pi x\left(\frac{dx}{dt}\right)$

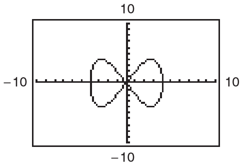
69. Answers will vary. In the explicit form of a function, the variable is explicitly written as a function of x. In an implicit equation, the function is only implied by an equation. An example of an implicit function is $x^2 + xy = 5$. In explicit form it would be $y = (5 - x^2)/x$.

71.

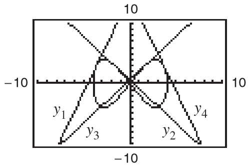


Use starting point B.

73. (a)



(b)



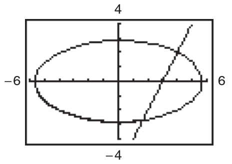
(c) $\left(\frac{8\sqrt{7}}{7}, 5\right)$

$$y_1 = \tfrac{1}{3}\left[\left(\sqrt{7} + 7\right)x + \left(8\sqrt{7} + 23\right)\right]$$
$$y_2 = -\tfrac{1}{3}\left[\left(-\sqrt{7} + 7\right)x - \left(23 - 8\sqrt{7}\right)\right]$$
$$y_3 = -\tfrac{1}{3}\left[\left(\sqrt{7} - 7\right)x - \left(23 - 8\sqrt{7}\right)\right]$$
$$y_4 = -\tfrac{1}{3}\left[\left(\sqrt{7} + 7\right)x - \left(8\sqrt{7} + 23\right)\right]$$

75. Proof **77.** $(6, -8), (-6, 8)$

79. $y = -\frac{\sqrt{3}}{2}x + 2\sqrt{3},\ y = \frac{\sqrt{3}}{2}x - 2\sqrt{3}$

81. (a) $y = 2x - 6$
(b)



(c) $\left(\frac{28}{17}, -\frac{46}{17}\right)$

Section 2.6 (page 154)

1. (a) $\frac{3}{4}$ (b) 20 **3.** (a) $-\frac{5}{8}$ (b) $\frac{3}{2}$

5. (a) -8 cm/sec (b) 0 cm/sec (c) 8 cm/sec

7. (a) 8 cm/sec (b) 4 cm/sec (c) 2 cm/sec

9. In a linear function, if x changes at a constant rate, so does y. However, unless $a = 1$, y does not change at the same rate as x.

11. $(4x^3 + 6x)/\sqrt{x^4 + 3x^2 + 1}$

13. (a) $64\pi\ \text{cm}^2/\text{min}$ (b) $256\pi\ \text{cm}^2/\text{min}$

15. (a) Proof
(b) When $\theta = \frac{\pi}{6}, \frac{dA}{dt} = \frac{\sqrt{3}}{8}s^2$. When $\theta = \frac{\pi}{3}, \frac{dA}{dt} = \frac{1}{8}s^2$.
(c) If s and $d\theta/dt$ are constant, dA/dt is proportional to $\cos\theta$.

17. (a) $2/(9\pi)$ cm/min (b) $1/(18\pi)$ cm/min

19. (a) $144\ \text{cm}^2/\text{sec}$ (b) $720\ \text{cm}^2/\text{sec}$ **21.** $8/(405\pi)$ ft/min

23. (a) 12.5% (b) $\frac{1}{144}$ m/min

25. (a) $-\frac{7}{12}$ ft/sec; $-\frac{3}{2}$ ft/sec; $-\frac{48}{7}$ ft/sec
(b) $\frac{527}{24}\ \text{ft}^2/\text{sec}$ (c) $\frac{1}{12}$ rad/sec

27. Rate of vertical change: $\frac{1}{5}$ m/sec
Rate of horizontal change: $-\sqrt{3}/15$ m/sec

29. (a) -750 mi/h (b) 30 min

31. $-50/\sqrt{85} \approx -5.42$ ft/sec **33.** (a) $\frac{25}{3}$ ft/sec (b) $\frac{10}{3}$ ft/sec

35. (a) 12 sec (b) $\frac{1}{2}\sqrt{3}$ m (c) $\left(\sqrt{5}\pi\right)/120$ m/sec

37. Evaporation rate proportional to $S \Rightarrow \dfrac{dV}{dt} = k(4\pi r^2)$

$V = \left(\dfrac{4}{3}\right)\pi r^3 \Rightarrow \dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$. So $k = \dfrac{dr}{dt}$.

39. 0.6 ohm/sec **41.** $\dfrac{dv}{dt} = \dfrac{16r}{v}\sec^2\theta\dfrac{d\theta}{dt}, \dfrac{d\theta}{dt} = \dfrac{v}{16r}\cos^2\theta\dfrac{dv}{dt}$

43. $\dfrac{2\sqrt{21}}{525} \approx 0.017$ rad/sec

45. (a) $\dfrac{200\pi}{3}$ ft/sec (b) 200π ft/sec (c) About 427.43π ft/sec

47. About 84.9797 mi/h

49. (a) $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$ means that y changes three times as fast as x changes.
(b) y changes slowly when $x \approx 0$ or $x \approx L$. y changes more rapidly when x is near the middle of the interval.

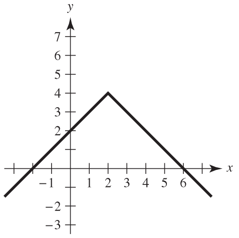
51. -18.432 ft/sec^2 **53.** About -97.96 m/sec

Review Exercises for Chapter 2 (page 158)

1. $f'(x) = 2x - 4$ **3.** $f'(x) = -2/(x-1)^2$

5. f is differentiable at all $x \neq 3$.

7.

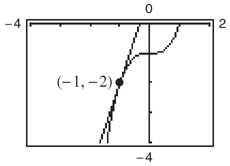


(a) Yes
(b) No, because the derivatives from the left and right are not equal.

9. $-\frac{3}{2}$

11. (a) $y = 3x + 1$ **13.** 8

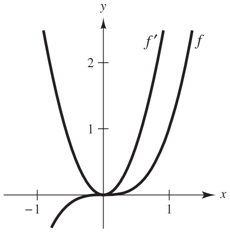
(b)



15. 0 **17.** $8x^7$ **19.** $52t^3$ **21.** $3x^2 - 22x$ **23.** $\dfrac{3}{\sqrt{x}} + \dfrac{1}{\sqrt[3]{x^2}}$

25. $-4/(3t^3)$ **27.** $4 - 5\cos\theta$ **29.** $-3\sin\theta - (\cos\theta)/4$

31.

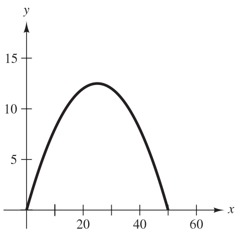


$f' > 0$ where the slopes of tangent lines to the graph of f are positive.

33. (a) 50 vibrations/sec/lb
(b) 33.33 vibrations/sec/lb

35. 1354.24 ft or 412.77 m

37. (a)



(b) 50
(c) $x = 25$
(d) $y' = 1 - 0.04x$

x	0	10	25	30	50
y'	1	0.6	0	-0.2	-1

(e) $y'(25) = 0$

39. (a) $x'(t) = 2t - 3$ (b) $(-\infty, 1.5)$ (c) $x = -\frac{1}{4}$ (d) 1

41. $4(5x^3 - 15x^2 - 11x - 8)$ **43.** $\sqrt{x}\cos x + \sin x/(2\sqrt{x})$

45. $-(x^2+1)/(x^2-1)^2$ **47.** $(8x)/(9-4x^2)^2$

49. $\dfrac{4x^3\cos x + x^4\sin x}{\cos^2 x}$ **51.** $3x^2\sec x\tan x + 6x\sec x$

53. $-x\sin x$ **55.** $y = 4x - 3$ **57.** $y = 0$

59. $v(4) = 20$ m/sec; $a(4) = -8$ m/sec^2

61. $-48t$ **63.** $\frac{225}{4}\sqrt{x}$ **65.** $6\sec^2\theta\tan\theta$

67. $y'' + y = -(2\sin x + 3\cos x) + (2\sin x + 3\cos x) = 0$

69. $\dfrac{2(x+5)(-x^2-10x+3)}{(x^2+3)^3}$

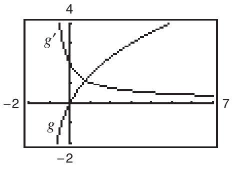
71. $s(s^2-1)^{3/2}(8s^3 - 3s + 25)$

73. $-45\sin(9x+1)$ **75.** $\frac{1}{2}(1 - \cos 2x) = \sin^2 x$

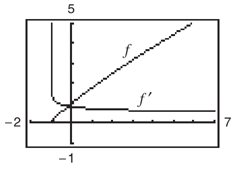
77. $\sin^{1/2}x\cos x - \sin^{5/2}x\cos x = \cos^3 x\sqrt{\sin x}$

79. $\dfrac{(x+2)(\pi\cos\pi x) - \sin\pi x}{(x+2)^2}$ **81.** -2 **83.** 0

85. $(x+2)/(x+1)^{3/2}$ **87.** $5/[6(t+1)^{1/6}]$

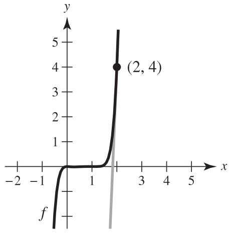


g' is not equal to zero for any x.

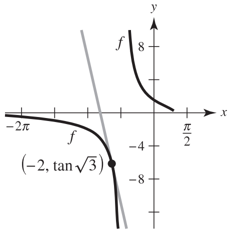


f' has no zeros.

89. (a) $f'(2) = 24$ (b) $y = 24t - 44$
(c)



91. (a) $f'(-2) = -\dfrac{1}{2\sqrt{3}\cos^2\sqrt{3}} \approx -11.1983$
(b) $y = -\dfrac{\sqrt{3}(x+2)}{6\cos^2\sqrt{3}} + \tan\sqrt{3}$
(c)



93. $14 - 4\cos 2x$ **95.** $2\csc^2 x \cot x$

97. $[8(2t+1)]/(1-t)^4$

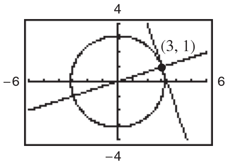
99. $18\sec^2 3\theta \tan 3\theta + \sin(\theta - 1)$

101. (a) $-18.667°/\text{h}$ (b) $-7.284°/\text{h}$
(c) $-3.240°/\text{h}$ (d) $-0.747°/\text{h}$

103. $-\dfrac{2x+3y}{3(x+y^2)}$ **105.** $\dfrac{\sqrt{y}(2\sqrt{x}-\sqrt{y})}{\sqrt{x}(\sqrt{x}+8\sqrt{y})} = \dfrac{2x-9y}{9x-32y}$

107. $\dfrac{y\sin x + \sin y}{\cos x - x\cos y}$

109. Tangent line: $3x + y - 10 = 0$
Normal line: $x - 3y = 0$



111. (a) $2\sqrt{2}$ units/sec (b) 4 units/sec (c) 8 units/sec

113. $\frac{2}{25}$ m/min **115.** -38.34 m/sec

P.S. Problem Solving (page 161)

1. (a) $r = \frac{1}{2};\ x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$
(b) Center: $\left(0, \frac{5}{4}\right);\ x^2 + \left(y - \frac{5}{4}\right)^2 = 1$

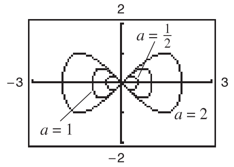
3. (a) $P_1(x) = 1$ (b) $P_2(x) = 1 - \frac{1}{2}x^2$
(c)

x	-1.0	-0.1	-0.001	0	0.001	0.1	1.0
$\cos x$	0.5403	0.9950	1.000	1	1.000	0.9950	0.5403
$P_2(x)$	0.5	0.995	1.000	1	1.000	0.995	0.5

$P_2(x)$ is a good approximation of $f(x) = \cos x$ when x is very close to 0.
(d) $P_3(x) = x - \frac{1}{6}x^3$

5. $p(x) = 2x^3 + 4x^2 - 5$

7. (a) Graph $\begin{cases} y_1 = \dfrac{1}{a}\sqrt{x^2(a^2-x^2)} \\ y_2 = -\dfrac{1}{a}\sqrt{x^2(a^2-x^2)} \end{cases}$ as separate equations.
(b) Answers will vary. Sample answer:



The intercepts will always be $(0, 0)$, $(a, 0)$, and $(-a, 0)$, and the maximum and minimum y-values appear to be $\pm\frac{1}{2}a$.
(c) $\left(\dfrac{a\sqrt{2}}{2}, \dfrac{a}{2}\right), \left(\dfrac{a\sqrt{2}}{2}, -\dfrac{a}{2}\right), \left(-\dfrac{a\sqrt{2}}{2}, \dfrac{a}{2}\right), \left(-\dfrac{a\sqrt{2}}{2}, -\dfrac{a}{2}\right)$

9. (a) When the man is 90 ft from the light, the tip of his shadow is $112\frac{1}{2}$ ft from the light. The tip of the child's shadow is $111\frac{1}{9}$ ft from the light, so the man's shadow extends $1\frac{7}{18}$ ft beyond the child's shadow.
(b) When the man is 60 ft from the light, the tip of his shadow is 75 ft from the light. The tip of the child's shadow is $77\frac{7}{9}$ ft from the light, so the child's shadow extends $2\frac{7}{9}$ ft beyond the man's shadow.
(c) $d = 80$ ft
(d) Let x be the distance of the man from the light and let s be the distance from the light to the tip of the shadow.
If $0 < x < 80$, $ds/dt = -50/9$.
If $x > 80$, $ds/dt = -25/4$.
There is a discontinuity at $x = 80$.

11. Proof. The graph of L is a line passing through the origin $(0, 0)$.

13. (a)

$z°$	0.1	0.01	0.0001
$\dfrac{\sin z}{z}$	0.0174532837	0.0174532924	0.0174532925

(b) $\pi/180$ (c) $(\pi/180)\cos z$
(d) $S(90) = 1$, $C(180) = -1$; $(\pi/180)C(z)$
(e) Answers will vary.

15. (a) j would be the rate of change of acceleration.
(b) $j = 0$. Acceleration is constant, so there is no change in acceleration.
(c) a: position function, d: velocity function,
b: acceleration function, c: jerk function

Chapter 3

Section 3.1 (page 169)

1. $f'(0) = 0$ **3.** $f'(2) = 0$ **5.** $f'(-2)$ is undefined.

7. 2, absolute maximum (and relative maximum)

9. 1, absolute maximum (and relative maximum);
2, absolute minimum (and relative minimum);
3, absolute maximum (and relative maximum)

11. $x = 0, x = 2$ **13.** $t = 8/3$ **15.** $x = \pi/3, \pi, 5\pi/3$

17. Minimum: $(2, 1)$
Maximum: $(-1, 4)$

19. Minimum: $(1, -1)$
Maximum: $(4, 8)$

21. Minimum: $\left(-1, -\frac{5}{2}\right)$
Maximum: $(2, 2)$

23. Minimum: $(0, 0)$
Maximum: $(-1, 5)$

25. Minimum: $(0, 0)$
Maxima: $\left(-1, \frac{1}{4}\right)$ and $\left(1, \frac{1}{4}\right)$

27. Minimum: $(1, -1)$
Maximum: $\left(0, -\frac{1}{2}\right)$

29. Minimum: $(-1, -1)$
Maximum: $(3, 3)$

31. Minimum value is -2 for $-2 \le x < -1$.
Maximum: $(2, 2)$

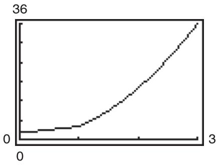
33. Minimum: $\left(1/6, \sqrt{3}/2\right)$
Maximum: $(0, 1)$

35. Minimum: $(\pi, -3)$
Maxima: $(0, 3)$ and $(2\pi, 3)$

37. (a) Minimum: $(0, -3)$;
Maximum: $(2, 1)$
(b) Minimum: $(0, -3)$
(c) Maximum: $(2, 1)$
(d) No extrema

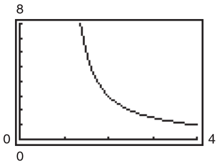
39. (a) Minimum: $(1, -1)$;
Maximum: $(-1, 3)$
(b) Maximum: $(3, 3)$
(c) Minimum: $(1, -1)$
(d) Minimum: $(1, -1)$

41.



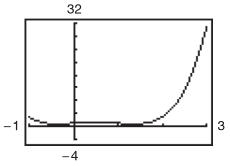
Minimum: $(0, 2)$
Maximum: $(3, 36)$

43.



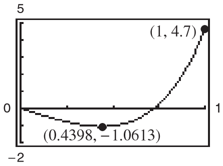
Minimum: $(4, 1)$

45.



Minima: $\left(\frac{-\sqrt{3}+1}{2}, \frac{3}{4}\right)$ and $\left(\frac{\sqrt{3}+1}{2}, \frac{3}{4}\right)$
Maximum: $(3, 31)$

47. (a)

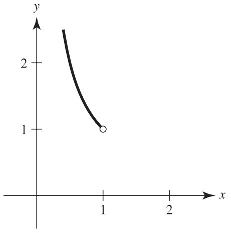


(b) Minimum: $(0.4398, -1.0613)$

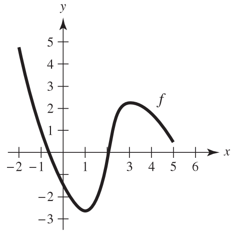
49. Maximum: $\left|f''\left(\sqrt[3]{-10+\sqrt{108}}\right)\right| = f''\left(\sqrt{3}-1\right) \approx 1.47$

51. Maximum: $\left|f^{(4)}(0)\right| = \frac{56}{81}$

53. Answers will vary. Let $f(x) = 1/x$. f is continuous on $(0, 1)$ but does not have a maximum or minimum.



55. Answers will vary. Example:



57. (a) Yes (b) No **59.** (a) No (b) Yes

61. Maximum: $P(12) = 72$; No. P is decreasing for $I > 12$.

63. $\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553$ rad

65. True **67.** True **69.** Proof **71.** Putnam Problem B3, 2004

Section 3.2 (page 176)

1. $f(-1) = f(1) = 1$; f is not continuous on $[-1, 1]$.

3. $f(0) = f(2) = 0$; f is not differentiable on $(0, 2)$.

5. $(2, 0), (-1, 0); f'\left(\frac{1}{2}\right) = 0$ **7.** $(0, 0), (-4, 0); f'\left(-\frac{8}{3}\right) = 0$

9. $f'(-1) = 0$ **11.** $f'\left(\frac{3}{2}\right) = 0$

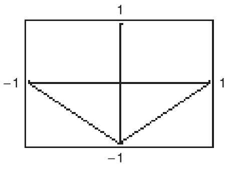
13. $f'\left(\frac{6-\sqrt{3}}{3}\right) = 0$; $f'\left(\frac{6+\sqrt{3}}{3}\right) = 0$

15. Not differentiable at $x = 0$ **17.** $f'\left(-2+\sqrt{5}\right) = 0$

19. $f'(\pi/2) = 0$; $f'(3\pi/2) = 0$ **21.** $f'(0.249) \approx 0$

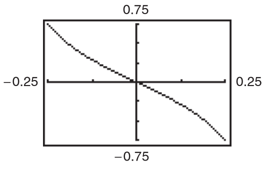
23. Not continuous on $[0, \pi]$

25.



Rolle's Theorem does not apply.

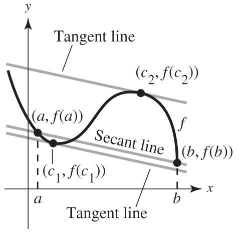
27.



Rolle's Theorem does not apply.

29. (a) $f(1) = f(2) = 38$
(b) Velocity $= 0$ for some t in $(1, 2)$; $t = \frac{3}{2}$ sec

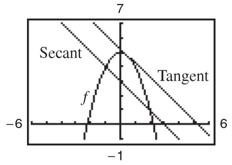
31.



33. The function is not continuous on $[0, 6]$.

35. The function is not continuous on $[0, 6]$.

37. (a) Secant line: $x + y - 3 = 0$ (b) $c = \frac{1}{2}$
(c) Tangent line: $4x + 4y - 21 = 0$
(d)

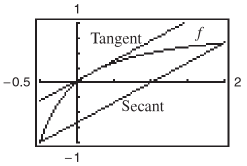


39. $f'(-1/2) = -1$ **41.** $f'\left(1/\sqrt{3}\right) = 3, f'\left(-1/\sqrt{3}\right) = 3$

43. $f'(8/27) = 1$ **45.** f is not differentiable at $x = -\frac{1}{2}$.

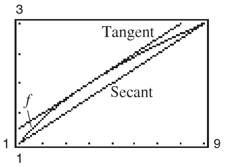
47. $f'(\pi/2) = 0$

49. (a)–(c)



(b) $y = \frac{2}{3}(x-1)$
(c) $y = \frac{1}{3}\left(2x + 5 - 2\sqrt{6}\right)$

51. (a)–(c)



(b) $y = \frac{1}{4}x + \frac{3}{4}$
(c) $y = \frac{1}{4}x + 1$

53. (a) -14.7 m/sec (b) 1.5 sec

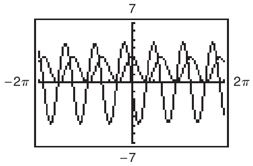
55. No. Let $f(x) = x^2$ on $[-1, 2]$.

57. No. $f(x)$ is not continuous on $[0, 1]$. So it does not satisfy the hypothesis of Rolle's Theorem.

59. By the Mean Value Theorem, there is a time when the speed of the plane must equal the average speed of 454.5 miles/hour. The speed was 400 miles/hour when the plane was accelerating to 454.5 miles/hour and decelerating from 454.5 miles/hour.

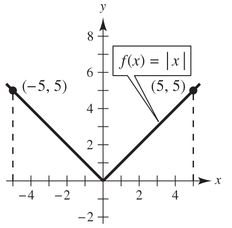
61. Proof

63. (a)



(b) Yes; yes
(c) Because $f(-1) = f(1) = 0$, Rolle's Theorem applies on $[-1, 1]$. Because $f(1) = 0$ and $f(2) = 3$, Rolle's Theorem does not apply on $[1, 2]$.
(d) $\lim_{x\to 3^-} f'(x) = 0$; $\lim_{x\to 3^+} f'(x) = 0$

65.



67. Proof **69.** Proof

71. $a = 6, b = 1, c = 2$ **73.** $f(x) = 5$ **75.** $f(x) = x^2 - 1$

77. False. f is not continuous on $[-1, 1]$. **79.** True

81–89. Proofs

Section 3.3 (page 186)

1. (a) $(0, 6)$ (b) $(6, 8)$

3. Increasing on $(3, \infty)$; Decreasing on $(-\infty, 3)$

5. Increasing on $(-\infty, -2)$ and $(2, \infty)$; Decreasing on $(-2, 2)$

7. Increasing on $(-\infty, -1)$; Decreasing on $(-1, \infty)$

9. Increasing on $(1, \infty)$; Decreasing on $(-\infty, 1)$

11. Increasing on $(-2\sqrt{2}, 2\sqrt{2})$
Decreasing on $(-4, -2\sqrt{2})$ and $(2\sqrt{2}, 4)$

13. Increasing on $(0, \pi/2)$ and $(3\pi/2, 2\pi)$;
Decreasing on $(\pi/2, 3\pi/2)$

15. Increasing on $(0, 7\pi/6)$ and $(11\pi/6, 2\pi)$;
Decreasing on $(7\pi/6, 11\pi/6)$

17. (a) Critical number: $x = 2$
(b) Increasing on $(2, \infty)$; Decreasing on $(-\infty, 2)$
(c) Relative minimum: $(2, -4)$

19. (a) Critical number: $x = 1$
(b) Increasing on $(-\infty, 1)$; Decreasing on $(1, \infty)$
(c) Relative maximum: $(1, 5)$

21. (a) Critical numbers: $x = -2, 1$
(b) Increasing on $(-\infty, -2)$ and $(1, \infty)$; Decreasing on $(-2, 1)$
(c) Relative maximum: $(-2, 20)$; Relative minimum: $(1, -7)$

23. (a) Critical numbers: $x = -\frac{5}{3}, 1$
(b) Increasing on $\left(-\infty, -\frac{5}{3}\right)$, $(1, \infty)$
Decreasing on $\left(-\frac{5}{3}, 1\right)$
(c) Relative maximum: $\left(-\frac{5}{3}, \frac{256}{27}\right)$
Relative minimum: $(1, 0)$

25. (a) Critical numbers: $x = \pm 1$
(b) Increasing on $(-\infty, -1)$ and $(1, \infty)$; Decreasing on $(-1, 1)$
(c) Relative maximum: $\left(-1, \frac{4}{5}\right)$; Relative minimum: $\left(1, -\frac{4}{5}\right)$

27. (a) Critical number: $x = 0$
(b) Increasing on $(-\infty, \infty)$
(c) No relative extrema

29. (a) Critical number: $x = -2$
(b) Increasing on $(-2, \infty)$; Decreasing on $(-\infty, -2)$
(c) Relative minimum: $(-2, 0)$

31. (a) Critical number: $x = 5$
(b) Increasing on $(-\infty, 5)$; Decreasing on $(5, \infty)$
(c) Relative maximum: $(5, 5)$

33. (a) Critical numbers: $x = \pm\sqrt{2}/2$; Discontinuity: $x = 0$
(b) Increasing on $(-\infty, -\sqrt{2}/2)$ and $(\sqrt{2}/2, \infty)$
Decreasing on $(-\sqrt{2}/2, 0)$ and $(0, \sqrt{2}/2)$
(c) Relative maximum: $(-\sqrt{2}/2, -2\sqrt{2})$
Relative minimum: $(\sqrt{2}/2, 2\sqrt{2})$

35. (a) Critical number: $x = 0$; Discontinuities: $x = \pm 3$
(b) Increasing on $(-\infty, -3)$ and $(-3, 0)$
Decreasing on $(0, 3)$ and $(3, \infty)$
(c) Relative maximum: $(0, 0)$

37. (a) Critical numbers: $x = -3, 1$; Discontinuity: $x = -1$
(b) Increasing on $(-\infty, -3)$ and $(1, \infty)$
Decreasing on $(-3, -1)$ and $(-1, 1)$
(c) Relative maximum: $(-3, -8)$; Relative minimum: $(1, 0)$

39. (a) Critical number: $x = 0$
(b) Increasing on $(-\infty, 0)$; Decreasing on $(0, \infty)$
(c) No relative extrema

41. (a) Critical number: $x = 1$
(b) Increasing on $(-\infty, 1)$; Decreasing on $(1, \infty)$
(c) Relative maximum: $(1, 4)$

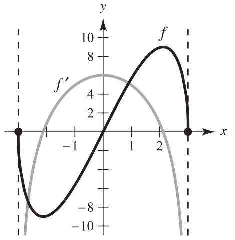
43. (a) Critical numbers: $x = \pi/6, 5\pi/6$
Increasing on $(0, \pi/6)$, $(5\pi/6, 2\pi)$
Decreasing on $(\pi/6, 5\pi/6)$
(b) Relative maximum: $(\pi/6, (\pi + 6\sqrt{3})/12)$
Relative minimum: $(5\pi/6, (5\pi - 6\sqrt{3})/12)$

45. (a) Critical numbers: $x = \pi/4, 5\pi/4$
Increasing on $(0, \pi/4)$, $(5\pi/4, 2\pi)$
Decreasing on $(\pi/4, 5\pi/4)$
(b) Relative maximum: $(\pi/4, \sqrt{2})$
Relative minimum: $(5\pi/4, -\sqrt{2})$

47. (a) Critical numbers:
$x = \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4$
Increasing on $(\pi/4, \pi/2)$, $(3\pi/4, \pi)$, $(5\pi/4, 3\pi/2)$, $(7\pi/4, 2\pi)$
Decreasing on $(0, \pi/4)$, $(\pi/2, 3\pi/4)$, $(\pi, 5\pi/4)$, $(3\pi/2, 7\pi/4)$
(b) Relative maxima: $(\pi/2, 1)$, $(\pi, 1)$, $(3\pi/2, 1)$
Relative minima: $(\pi/4, 0)$, $(3\pi/4, 0)$, $(5\pi/4, 0)$, $(7\pi/4, 0)$

49. (a) Critical numbers: $\pi/2, 7\pi/6, 3\pi/2, 11\pi/6$
Increasing on $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{7\pi}{6}, \dfrac{3\pi}{2}\right), \left(\dfrac{11\pi}{6}, 2\pi\right)$
Decreasing on $\left(\dfrac{\pi}{2}, \dfrac{7\pi}{6}\right), \left(\dfrac{3\pi}{2}, \dfrac{11\pi}{6}\right)$
(b) Relative maxima: $\left(\dfrac{\pi}{2}, 2\right), \left(\dfrac{3\pi}{2}, 0\right)$
Relative minima: $\left(\dfrac{7\pi}{6}, -\dfrac{1}{4}\right), \left(\dfrac{11\pi}{6}, -\dfrac{1}{4}\right)$

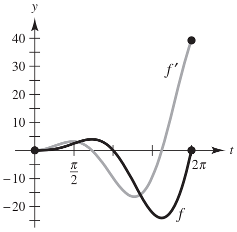
51. (a) $f'(x) = 2(9 - 2x^2)/\sqrt{9 - x^2}$
(b)



(c) Critical numbers:
$x = \pm 3\sqrt{2}/2$
(d) $f' > 0$ on $(-3\sqrt{2}/2, 3\sqrt{2}/2)$
$f' < 0$ on $(-3, -3\sqrt{2}/2)$, $(3\sqrt{2}/2, 3)$

53. (a) $f'(t) = t(t \cos t + 2 \sin t)$

(b)

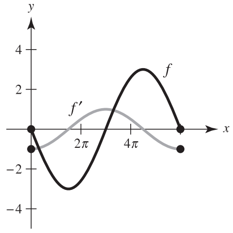


(c) Critical numbers:
$t = 2.2889, 5.0870$

(d) $f' > 0$ on $(0, 2.2889)$, $(5.0870, 2\pi)$
$f' < 0$ on $(2.2889, 5.0870)$

55. (a) $f'(x) = -\cos(x/3)$

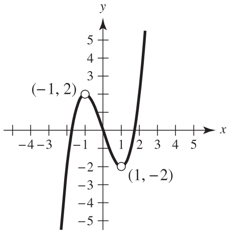
(b)



(c) Critical numbers:
$x = 3\pi/2, 9\pi/2$

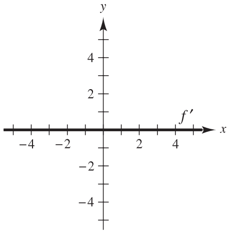
(d) $f' > 0$ on $\left(\dfrac{3\pi}{2}, \dfrac{9\pi}{2}\right)$
$f' < 0$ on $\left(0, \dfrac{3\pi}{2}\right), \left(\dfrac{9\pi}{2}, 6\pi\right)$

57. $f(x)$ is symmetric with respect to the origin.
Zeros: $(0, 0), (\pm\sqrt{3}, 0)$

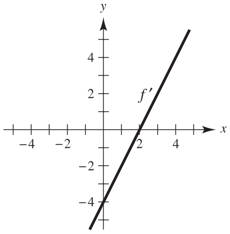


$g(x)$ is continuous on $(-\infty, \infty)$ and $f(x)$ has holes at $x = 1$ and $x = -1$.

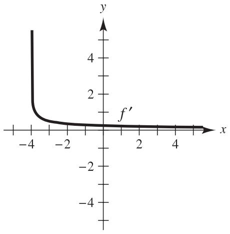
59.



61.



63.



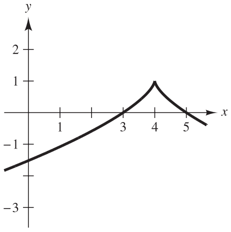
65. (a) Increasing on $(2, \infty)$; Decreasing on $(-\infty, 2)$
(b) Relative minimum: $x = 2$

67. (a) Increasing on $(-\infty, -1)$ and $(0, 1)$;
Decreasing on $(-1, 0)$ and $(1, \infty)$
(b) Relative maxima: $x = -1$ and $x = 1$
Relative minimum: $x = 0$

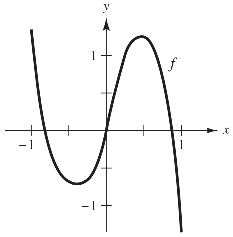
69. (a) Critical numbers: $x = -1, x = 1, x = 2$
(b) Relative maximum at $x = 1$, relative minimum at $x = 2$, and neither at $x = -1$

71. $g'(0) < 0$ **73.** $g'(-6) < 0$ **75.** $g'(0) > 0$

77. Answers will vary. Sample answer:



79. (a)



(b) Critical numbers: $x \approx -0.40$ and $x \approx 0.48$
(c) Relative maximum: $(0.48, 1.25)$
Relative minimum: $(-0.40, 0.75)$

81. (a) $s'(t) = 9.8(\sin \theta)t$; speed $= |9.8(\sin \theta)t|$
(b)

θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$s'(t)$	0	$4.9\sqrt{2}t$	$4.9\sqrt{3}t$	$9.8t$	$4.9\sqrt{3}t$	$4.9\sqrt{2}t$	0

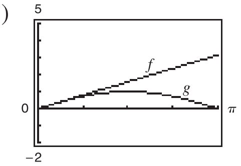
The speed is maximum at $\theta = \pi/2$.

83. (a)

x	0.5	1	1.5	2	2.5	3
f(x)	0.5	1	1.5	2	2.5	3
g(x)	0.48	0.84	1.00	0.91	0.60	0.14

$f(x) > g(x)$

(b)



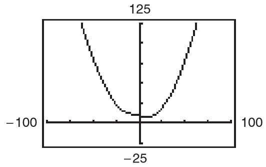
(c) Proof

$f(x) > g(x)$

85. $r = 2R/3$

87. (a) $\dfrac{dR}{dT} = \dfrac{0.004T^3 - 4}{2\sqrt{0.001T^4 - 4T + 100}}$
Critical number: $T = 10$
Minimum resistance: About 8.3666 ohms

(b)



Minimum resistance: About 8.3666 ohms

89. (a) $v(t) = 6 - 2t$ (b) $(0, 3)$ (c) $(3, \infty)$ (d) $t = 3$

91. (a) $v(t) = 3t^2 - 10t + 4$
(b) $\left(0, \left(5 - \sqrt{13}\right)/3\right)$ and $\left(\left(5 + \sqrt{13}\right)/3, \infty\right)$
(c) $\left(\dfrac{5 - \sqrt{13}}{3}, \dfrac{5 + \sqrt{13}}{3}\right)$ (d) $t = \dfrac{5 \pm \sqrt{13}}{3}$

93. Answers will vary.

95. (a) Minimum degree: 3
(b) $a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0$
$a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 = 2$
$3a_3(0)^2 + 2a_2(0) + a_1 = 0$
$3a_3(2)^2 + 2a_2(2) + a_1 = 0$
(c) $f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2$

97. (a) Minimum degree: 4
(b) $a_4(0)^4 + a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0$
$a_4(2)^4 + a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 = 4$
$a_4(4)^4 + a_3(4)^3 + a_2(4)^2 + a_1(4) + a_0 = 0$
$4a_4(0)^3 + 3a_3(0)^2 + 2a_2(0) + a_1 = 0$
$4a_4(2)^3 + 3a_3(2)^2 + 2a_2(2) + a_1 = 0$
$4a_4(4)^3 + 3a_3(4)^2 + 2a_2(4) + a_1 = 0$
(c) $f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$

99. True **101.** False. Let $f(x) = x^3$.

103. False. Let $f(x) = x^3$. There is a critical number at $x = 0$, but not a relative extremum.

105–107. Proofs

Section 3.4 (page 195)

1. $f' > 0, f'' > 0$ **3.** $f' < 0, f'' < 0$

5. Concave upward: $(-\infty, \infty)$

7. Concave upward: $(-\infty, 1)$; Concave downward: $(1, \infty)$

9. Concave upward: $(-\infty, 2)$; Concave downward: $(2, \infty)$

11. Concave upward: $(-\infty, -2), (2, \infty)$
Concave downward: $(-2, 2)$

13. Concave upward: $(-\infty, -1), (1, \infty)$
Concave downward: $(-1, 1)$

15. Concave upward: $(-2, 2)$
Concave downward: $(-\infty, -2), (2, \infty)$

17. Concave upward: $(-\pi/2, 0)$; Concave downward: $(0, \pi/2)$

19. Points of inflection: $(-2, -8), (0, 0)$
Concave upward: $(-\infty, -2), (0, \infty)$
Concave downward: $(-2, 0)$

21. Point of inflection: $(2, 8)$; Concave downward: $(-\infty, 2)$
Concave upward: $(2, \infty)$

23. Points of inflection: $\left(\pm 2\sqrt{3}/3, -20/9\right)$
Concave upward: $\left(-\infty, -2\sqrt{3}/3\right), \left(2\sqrt{3}/3, \infty\right)$
Concave downward: $\left(-2\sqrt{3}/3, 2\sqrt{3}/3\right)$

25. Points of inflection: $(2, -16), (4, 0)$
Concave upward: $(-\infty, 2), (4, \infty)$; Concave downward: $(2, 4)$

27. Concave upward: $(-3, \infty)$

29. Points of inflection: $\left(-\sqrt{3}/3, 3\right), \left(\sqrt{3}/3, 3\right)$
Concave upward: $\left(-\infty, -\sqrt{3}/3\right), \left(\sqrt{3}/3, \infty\right)$
Concave downward: $\left(-\sqrt{3}/3, \sqrt{3}/3\right)$

31. Point of inflection: $(2\pi, 0)$
Concave upward: $(2\pi, 4\pi)$; Concave downward: $(0, 2\pi)$

33. Concave upward: $(0, \pi), (2\pi, 3\pi)$
Concave downward: $(\pi, 2\pi), (3\pi, 4\pi)$

35. Points of inflection: $(\pi, 0), (1.823, 1.452), (4.46, -1.452)$
Concave upward: $(1.823, \pi), (4.46, 2\pi)$
Concave downward: $(0, 1.823), (\pi, 4.46)$

37. Relative minimum: $(5, 0)$ **39.** Relative maximum: $(3, 9)$

41. Relative maximum: $(0, 3)$; Relative minimum: $(2, -1)$

43. Relative minimum: $(3, -25)$

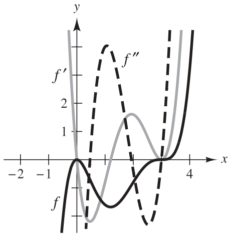
45. Relative maximum: $(2.4, 268.74)$; Relative minimum: $(0, 0)$

47. Relative minimum: $(0, -3)$

49. Relative maximum: $(-2, -4)$; Relative minimum: $(2, 4)$

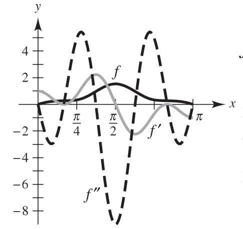
51. No relative extrema, because f is nonincreasing.

53. (a) $f'(x) = 0.2x(x - 3)^2(5x - 6)$
$f''(x) = 0.4(x - 3)(10x^2 - 24x + 9)$
(b) Relative maximum: $(0, 0)$
Relative minimum: $(1.2, -1.6796)$
Points of inflection: $(0.4652, -0.7048)$, $(1.9348, -0.9048)$, $(3, 0)$
(c)



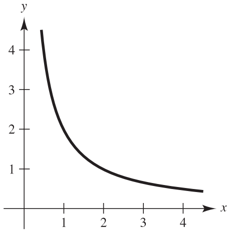
f is increasing when f' is positive, and decreasing when f' is negative. f is concave upward when f'' is positive, and concave downward when f'' is negative.

55. (a) $f'(x) = \cos x - \cos 3x + \cos 5x$
$f''(x) = -\sin x + 3\sin 3x - 5\sin 5x$
(b) Relative maximum: $(\pi/2, 1.53333)$
Points of inflection: $(\pi/6, 0.2667), (1.1731, 0.9637),$
$(1.9685, 0.9637), (5\pi/6, 0.2667)$
(c)

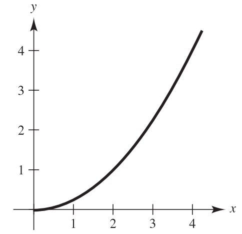


f is increasing when f' is positive, and decreasing when f' is negative. f is concave upward when f'' is positive, and concave downward when f'' is negative.

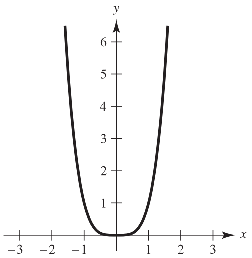
57. (a)



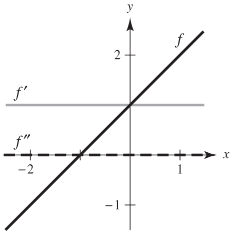
(b)



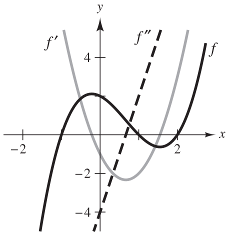
59. Answers will vary. Example:
$f(x) = x^4$; $f''(0) = 0$, but $(0, 0)$ is not a point of inflection.



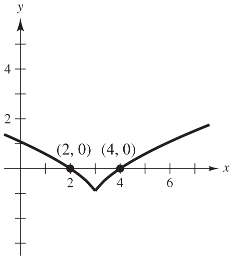
61.



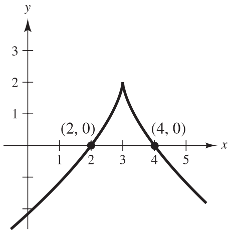
63.



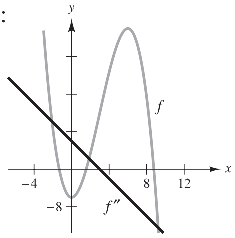
65.



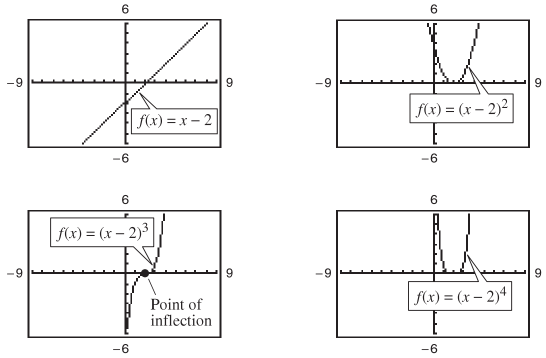
67.



69. Example:



71. (a) $f(x) = (x - 2)^n$ has a point of inflection at $(2, 0)$ if n is odd and $n \geq 3$.



(b) Proof

73. $f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$

75. (a) $f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$ (b) Two miles from touchdown

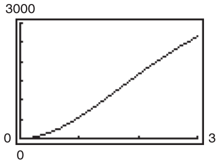
77. $x = \left(\dfrac{15 - \sqrt{33}}{16}\right)L \approx 0.578L$ **79.** $x = 100$ units

81. (a)

t	0.5	1	1.5	2	2.5	3
S	151.5	555.6	1097.6	1666.7	2193.0	2647.1

$1.5 < t < 2$

(b)



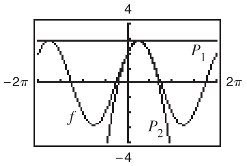
$t \approx 1.5$

(c) About 1.633 yr

83. $P_1(x) = 2\sqrt{2}$

$P_2(x) = 2\sqrt{2} - \sqrt{2}(x - \pi/4)^2$

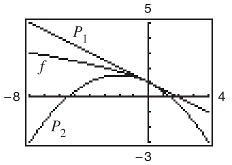
The values of f, P_1, and P_2 and their first derivatives are equal when $x = \pi/4$. The approximations worsen as you move away from $x = \pi/4$.



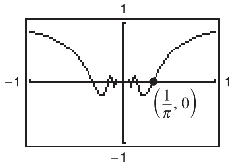
85. $P_1(x) = 1 - x/2$

$P_2(x) = 1 - x/2 - x^2/8$

The values of f, P_1, and P_2 and their first derivatives are equal when $x = 0$. The approximations worsen as you move away from $x = 0$.



87.



89. Proof **91.** True

93. False. f is concave upward at $x = c$ if $f''(c) > 0$. **95.** Proof

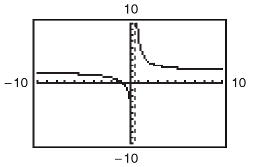
Section 3.5 (page 205)

1. f **2.** c **3.** d **4.** a **5.** b **6.** e

7.

x	10^0	10^1	10^2	10^3
$f(x)$	7	2.2632	2.0251	2.0025

x	10^4	10^5	10^6
$f(x)$	2.0003	2.0000	2.0000

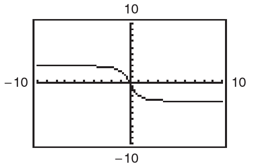


$$\lim_{x\to\infty} \frac{4x + 3}{2x - 1} = 2$$

9.

x	10^0	10^1	10^2	10^3
$f(x)$	-2	-2.9814	-2.9998	-3.0000

x	10^4	10^5	10^6
$f(x)$	-3.0000	-3.0000	-3.0000

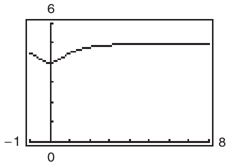


$$\lim_{x\to\infty} \frac{-6x}{\sqrt{4x^2 + 5}} = -3$$

11.

x	10^0	10^1	10^2	10^3
$f(x)$	4.5000	4.9901	4.9999	5.0000

x	10^4	10^5	10^6
$f(x)$	5.0000	5.0000	5.0000



$$\lim_{x\to\infty} \left(5 - \frac{1}{x^2 + 1}\right) = 5$$

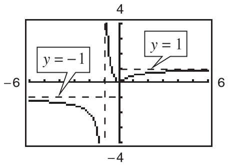
13. (a) ∞ (b) 5 (c) 0 **15.** (a) 0 (b) 1 (c) ∞

17. (a) 0 (b) $-\frac{2}{3}$ (c) $-\infty$ **19.** 4 **21.** $\frac{2}{3}$ **23.** 0

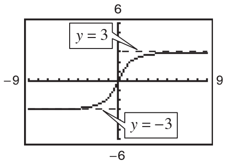
25. $-\infty$ **27.** -1 **29.** -2 **31.** $\frac{1}{2}$ **33.** ∞

35. 0 **37.** 0

39.



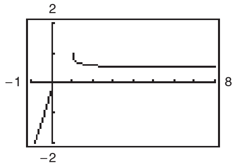
41.



43. 1 **45.** 0 **47.** $\frac{1}{6}$

49.

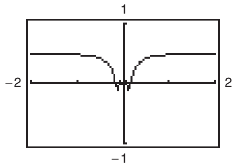
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1.000	0.513	0.501	0.500	0.500	0.500	0.500



$$\lim_{x\to\infty}\left[x - \sqrt{x(x-1)}\right] = \tfrac{1}{2}$$

51.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.479	0.500	0.500	0.500	0.500	0.500	0.500

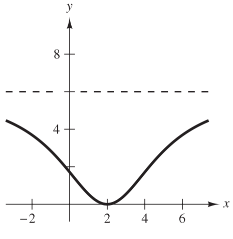


The graph has a hole at $x = 0$.

$$\lim_{x\to\infty} x \sin\frac{1}{2x} = \frac{1}{2}$$

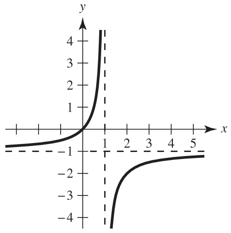
53. As x becomes large, $f(x)$ approaches 4.

55. Answers will vary. Example: let $f(x) = \dfrac{-6}{0.1(x-2)^2 + 1} + 6$.

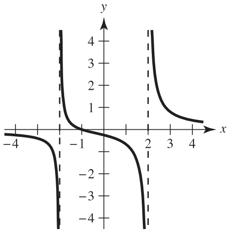


57. (a) 5 (b) -5

59.



61.



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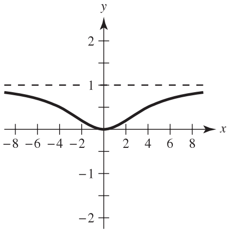

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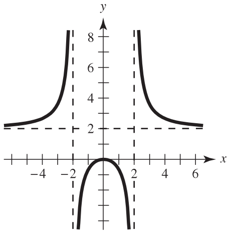

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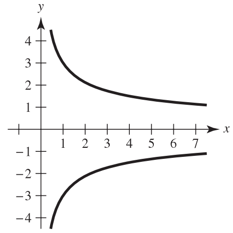

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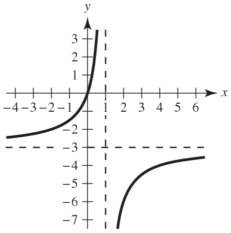

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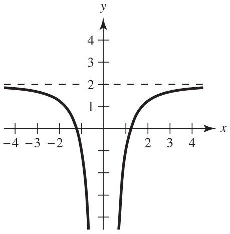
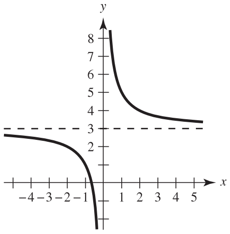
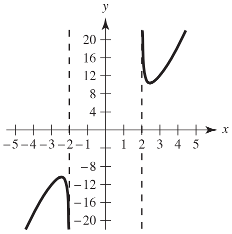


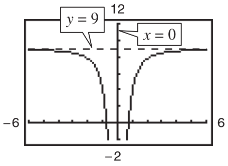
73.



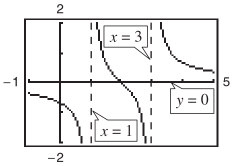
75.



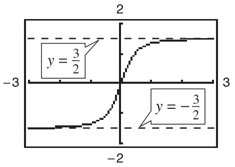
77.



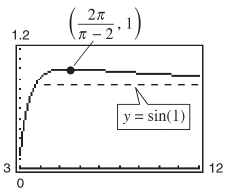
79.



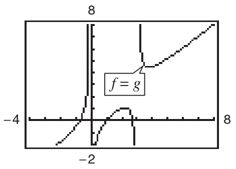
81.



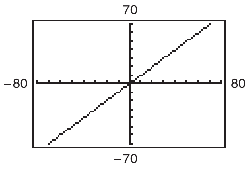
83.



85. (a)



(c)

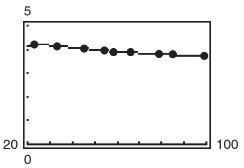


(b) Proof

The slant asymptote $y = x$

87. 100% **89.** $\lim_{t\to\infty} N(t) = +\infty$; $\lim_{t\to\infty} E(t) = c$

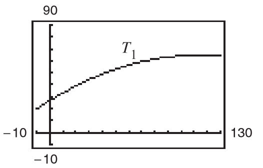
91. (a)



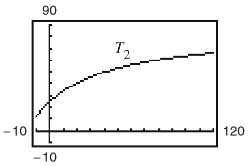
(b) Yes. $\lim_{t \to \infty} y = 3.351$

93. (a) $T_1 = -0.003t^2 + 0.68t + 26.6$

(b)



(c)

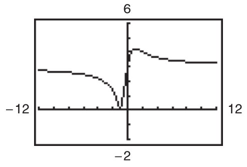


(d) $T_1(0) \approx 26.6°$, $T_2(0) \approx 25.0°$ (e) 86

(f) The limiting temperature is 86°.
No. T_1 has no horizontal asymptote.

95. (a) $d(m) = \dfrac{|3m + 3|}{\sqrt{m^2 + 1}}$

(b)



(c) $\lim_{m \to \infty} d(m) = 3$

$\lim_{m \to -\infty} d(m) = 3$

As m approaches $\pm\infty$, the distance approaches 3.

97. (a) $\lim_{x \to \infty} f(x) = 2$ (b) $x_1 = \sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}}$, $x_2 = -\sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}}$

(c) $\sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}}$ (d) $-\sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}}$

99. (a) Answers will vary. $M = \dfrac{5\sqrt{33}}{11}$ **101–105.** Proofs

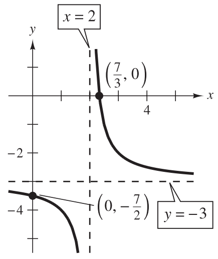
(b) Answers will vary. $M = \dfrac{29\sqrt{177}}{59}$

107. False. Let $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$. $f'(x) > 0$ for all real numbers.

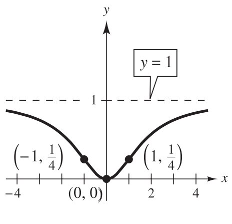
Section 3.6 (page 215)

1. d **2.** c **3.** a **4.** b

5.



7.



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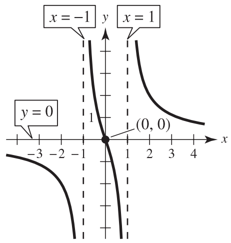

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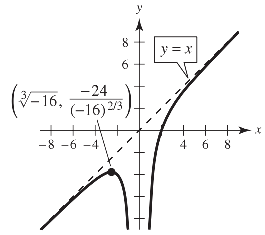
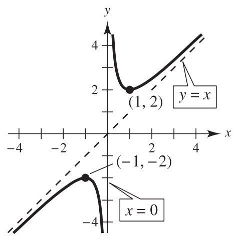


13.



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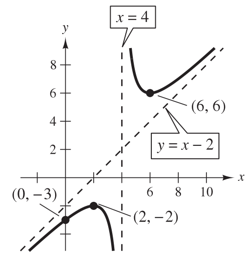
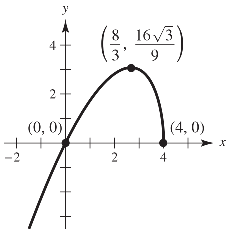


17.



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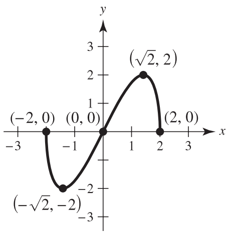

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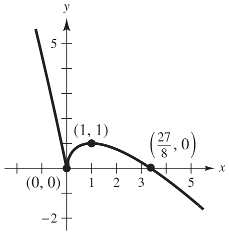
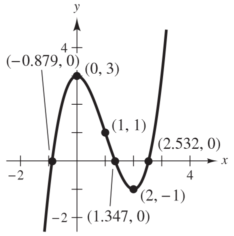
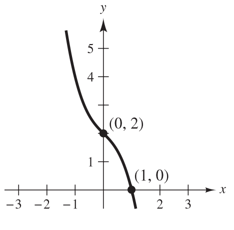


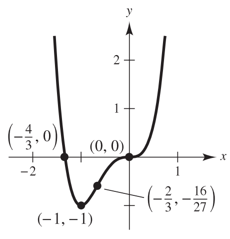
23.



25.



27.



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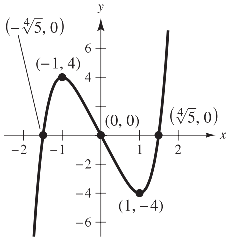
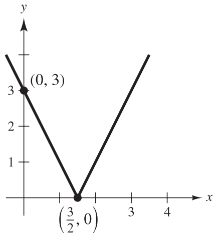
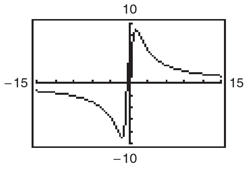


31.

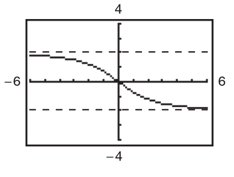


33.



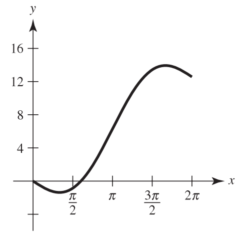
Minimum: $(-1.10, -9.05)$
Maximum: $(1.10, 9.05)$
Points of inflection:
$(-1.84, -7.86)$, $(1.84, 7.86)$
Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 0$

35.

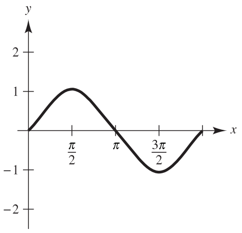


Point of inflection: $(0, 0)$
Horizontal asymptotes: $y = \pm 2$

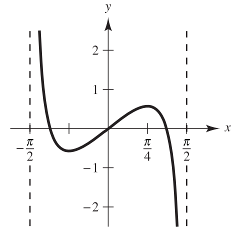
37.



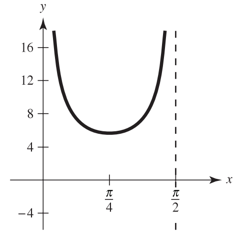
39.



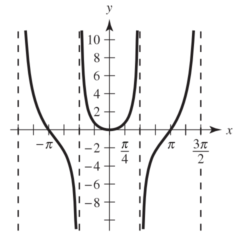
41.



43.

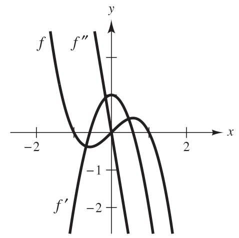


45.



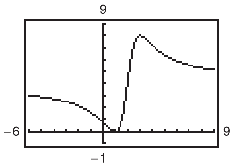
47. f is decreasing on $(2, 8)$ and therefore $f(3) > f(5)$.

49.



The zeros of f' correspond to the points where the graph of f has horizontal tangents. The zero of f'' corresponds to the point where the graph of f' has a horizontal tangent.

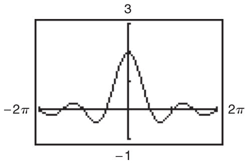
51.



The graph crosses the horizontal asymptote $y = 4$.

The graph of a function f does not cross its vertical asymptote $x = c$ because $f(c)$ does not exist.

53.

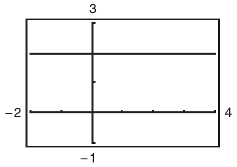


The graph has a hole at $x = 0$.

The graph crosses the horizontal asymptote $y = 0$.

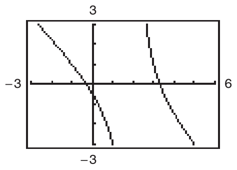
The graph of a function f does not cross its vertical asymptote $x = c$ because $f(c)$ does not exist.

55.



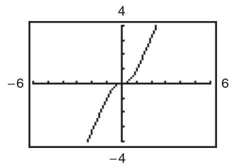
The graph has a hole at $x = 3$. The rational function is not reduced to lowest terms.

57.



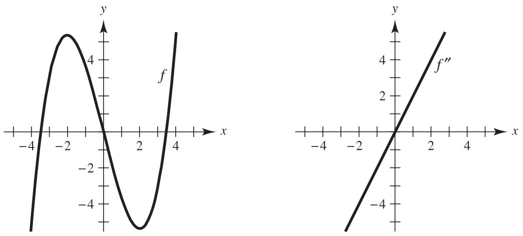
The graph appears to approach the line $y = -x + 1$, which is the slant asymptote.

59.

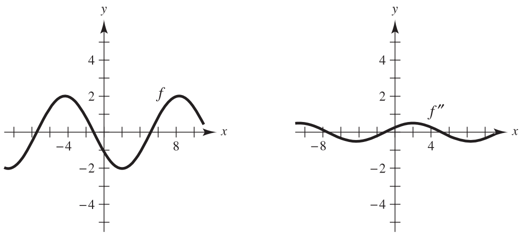


The graph appears to approach the line $y = 2x$, which is the slant asymptote.

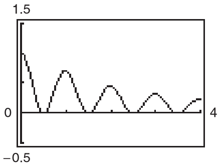
61.



63.



65. (a)



The graph has holes at $x = 0$ and at $x = 4$.

Visually approximated critical numbers: $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$

(b) $f'(x) = \dfrac{-x\cos^2(\pi x)}{(x^2+1)^{3/2}} - \dfrac{2\pi\sin(\pi x)\cos(\pi x)}{\sqrt{x^2+1}}$

Approximate critical numbers: $\frac{1}{2}, 0.97, \frac{3}{2}, 1.98, \frac{5}{2}, 2.98, \frac{7}{2}$

The critical numbers where maxima occur appear to be integers in part (a), but by approximating them using f' you can see that they are not integers.

67. Answers will vary. Example: $y = 1/(x - 3)$

69. Answers will vary. Example: $y = (3x^2 - 7x - 5)/(x - 3)$

71. (a) $f'(x) = 0$ for $x = \pm 2$; $f'(x) > 0$ for $(-\infty, -2), (2, \infty)$

$f'(x) < 0$ for $(-2, 2)$

(b) $f''(x) = 0$ for $x = 0$; $f''(x) > 0$ for $(0, \infty)$

$f''(x) < 0$ for $(-\infty, 0)$

(c) $(0, \infty)$

(d) f' is minimum for $x = 0$.

f is decreasing at the greatest rate at $x = 0$.

73. Answers will vary. Sample answer: The graph has a vertical asymptote at $x = b$. If a and b are both positive, or both negative, then the graph of f approaches ∞ as x approaches b, and the graph has a minimum at $x = -b$. If a and b have opposite signs, then the graph of f approaches $-\infty$ as x approaches b, and the graph has a maximum at $x = -b$.

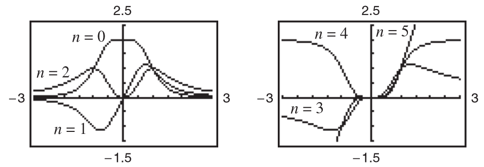
75. (a) If n is even, f is symmetric with respect to the y-axis.

If n is odd, f is symmetric with respect to the origin.

(b) $n = 0, 1, 2, 3$ (c) $n = 4$

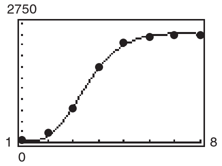
(d) When $n = 5$, the slant asymptote is $y = 2x$.

(e)



n	0	1	2	3	4	5
M	1	2	3	2	1	0
N	2	3	4	5	2	3

77. (a)



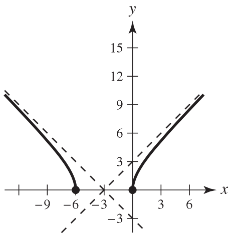
(b) 2434

(c) The number of bacteria reaches its maximum early on the seventh day.

(d) The rate of increase in the number of bacteria is greatest in the early part of the third day.

(e) 13,250/7

79. $y = x + 3$, $y = -x - 3$



Section 3.7 (page 223)

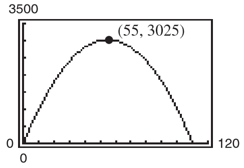
1. (a) and (b)

First Number x	Second Number	Product P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$
70	$110 - 70$	$70(110 - 70) = 2800$
80	$110 - 80$	$80(110 - 80) = 2400$
90	$110 - 90$	$90(110 - 90) = 1800$
100	$110 - 100$	$100(110 - 100) = 1000$

The maximum is attained near $x = 50$ and 60.

(c) $P = x(110 - x)$

(d)



(e) 55 and 55

3. $S/2$ and $S/2$ **5.** 21 and 7 **7.** 54 and 27

9. $l = w = 20$ m **11.** $l = w = 4\sqrt{2}$ ft **13.** $(1, 1)$

15. $\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$

17. Dimensions of page: $(2 + \sqrt{30})$ in. × $(2 + \sqrt{30})$ in.

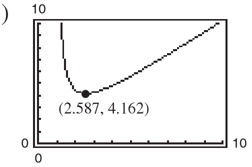
19. $x = Q_0/2$ **21.** 700×350 m

23. (a) Proof (b) $V_1 = 99\text{ in.}^3$, $V_2 = 125\text{ in.}^3$, $V_3 = 117\text{ in.}^3$
(c) $5 \times 5 \times 5$ in.

25. Rectangular portion: $16/(\pi + 4) \times 32/(\pi + 4)$ ft

27. (a) $L = \sqrt{x^2 + 4 + \dfrac{8}{x - 1} + \dfrac{4}{(x - 1)^2}},\quad x > 1$

(b)

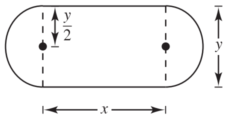


Minimum when $x \approx 2.587$

(c) (0, 0), (2, 0), (0, 4)

29. Width: $5\sqrt{2}/2$; Length: $5\sqrt{2}$

31. (a)



(b)

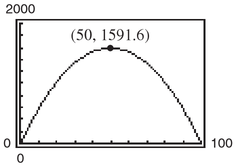
Length x	Width y	Area xy
10	$2/\pi(100 - 10)$	$(10)(2/\pi)(100 - 10) \approx 573$
20	$2/\pi(100 - 20)$	$(20)(2/\pi)(100 - 20) \approx 1019$
30	$2/\pi(100 - 30)$	$(30)(2/\pi)(100 - 30) \approx 1337$
40	$2/\pi(100 - 40)$	$(40)(2/\pi)(100 - 40) \approx 1528$
50	$2/\pi(100 - 50)$	$(50)(2/\pi)(100 - 50) \approx 1592$
60	$2/\pi(100 - 60)$	$(60)(2/\pi)(100 - 60) \approx 1528$

The maximum area of the rectangle is approximately 1592 m^2.

(c) $A = 2/\pi(100x - x^2),\quad 0 < x < 100$

(d) $\dfrac{dA}{dx} = \dfrac{2}{\pi}(100 - 2x)$
$= 0$ when $x = 50$
The maximum value is approximately 1592 when $x = 50$.

(e)



33. $18 \times 18 \times 36$ in. **35.** $32\pi r^3/81$

37. No. The volume changes because the shape of the container changes when squeezed.

39. $r = \sqrt[3]{21/(2\pi)} \approx 1.50$ ($h = 0$, so the solid is a sphere.)

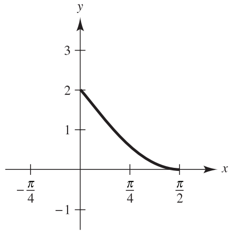
41. Side of square: $\dfrac{10\sqrt{3}}{9 + 4\sqrt{3}}$; Side of triangle: $\dfrac{30}{9 + 4\sqrt{3}}$

43. $w = (20\sqrt{3})/3$ in., $h = (20\sqrt{6})/3$ in. **45.** $\theta = \pi/4$

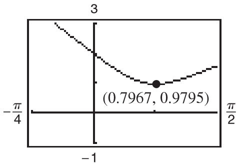
47. $h = \sqrt{2}$ ft **49.** One mile from the nearest point on the coast

51. Proof

53.



(a) Origin to y-intercept: 2
Origin to x-intercept: $\pi/2$
(b) $d = \sqrt{x^2 + (2 - 2\sin x)^2}$



(c) Minimum distance is 0.9795 when $x \approx 0.7967$.

55. $F = kW/\sqrt{k^2 + 1};\ \theta = \arctan k$

57. (a)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 10°$	$8\sin 10°$	≈ 22.1
8	$8 + 16\cos 20°$	$8\sin 20°$	≈ 42.5
8	$8 + 16\cos 30°$	$8\sin 30°$	≈ 59.7
8	$8 + 16\cos 40°$	$8\sin 40°$	≈ 72.7
8	$8 + 16\cos 50°$	$8\sin 50°$	≈ 80.5
8	$8 + 16\cos 60°$	$8\sin 60°$	≈ 83.1

(b)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 70°$	$8\sin 70°$	≈ 80.7
8	$8 + 16\cos 80°$	$8\sin 80°$	≈ 74.0
8	$8 + 16\cos 90°$	$8\sin 90°$	≈ 64.0

The maximum cross-sectional area is approximately 83.1 ft^2.

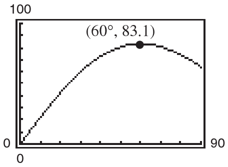
(c) $A = 64(1 + \cos\theta)\sin\theta,\ 0° < \theta < 90°$

(d) $\frac{dA}{d\theta} = 64(2\cos\theta - 1)(\cos\theta + 1)$

$= 0$ when $\theta = 60°, 180°, 300°$

The maximum area occurs when $\theta = 60°$.

(e)



59. 4045 units **61.** $y = \frac{64}{141}x;\ S_1 \approx 6.1$ mi

63. $y = \frac{3}{10}x;\ S_3 \approx 4.50$ mi **65.** Putnam Problem A1, 1986

Section 3.8 (page 233)

In the answers for Exercises 1 and 3, the values in the tables have been rounded for convenience. Because a calculator or a computer program calculates internally using more digits than they display, you may produce slightly different values than those shown in the tables.

1.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	2.2000	−0.1600	4.4000	−0.0364	2.2364
2	2.2364	0.0015	4.4728	0.0003	2.2361

3.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.6	−0.0292	−0.9996	0.0292	1.5708
2	1.5708	0	−1	0	1.5708

5. −1.587 **7.** 0.682 **9.** 1.250, 5.000

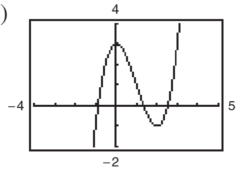
11. 0.900, 1.100, 1.900 **13.** 1.935 **15.** 0.569

17. 4.493 **19.** (a) Proof (b) $\sqrt{5} \approx 2.236;\ \sqrt{7} \approx 2.646$

21. $f'(x_1) = 0$ **23.** $2 = x_1 = x_3 = \ldots;\ 1 = x_2 = x_4 = \ldots$

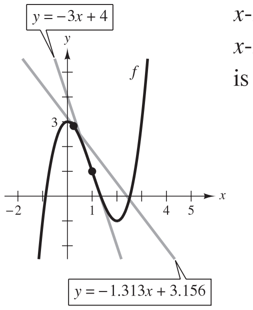
25. 0.74 **27.** Proof

29. (a)



(b) 1.347 (c) 2.532

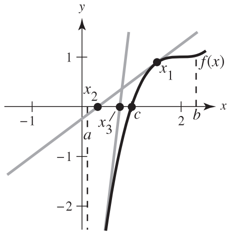
(d)



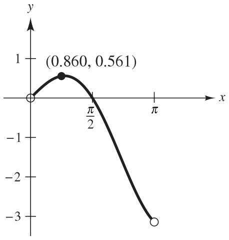
x-intercept of $y = -3x + 4$ is $\frac{4}{3}$.
x-intercept of $y = -1.313x + 3.156$ is approximately 2.404.

(e) If the initial estimate $x = x_1$ is not sufficiently close to the desired zero of a function, the x-intercept of the corresponding tangent line to the function may approximate a second zero of the function.

31. Answers will vary. Sample answer: If f is a function continuous on $[a, b]$ and differentiable on (a, b), where $c \in [a, b]$ and $f(c) = 0$, Newton's Method uses tangent lines to approximate c. First, estimate an initial x_1 close to c. (See graph.) Then determine x_2 using $x_2 = x_1 - f(x_1)/f'(x_1)$. Calculate a third estimate x_3 using $x_3 = x_2 - f(x_2)/f'(x_2)$. Continue this process until $|x_n - x_{n+1}|$ is within the desired accuracy and let x_{n+1} be the final approximation of c.



33. 0.860 **35.** (1.939, 0.240)



37. $x \approx 1.563$ mi **39.** 15.1, 26.8 **41.** False: let $f(x) = \dfrac{x^2 - 1}{x - 1}$.

43. True **45.** 0.217

Section 3.9 (page 240)

1. $T(x) = 4x - 4$

x	1.9	1.99	2	2.01	2.1
$f(x)$	3.610	3.960	4	4.040	4.410
$T(x)$	3.600	3.960	4	4.040	4.400

3. $T(x) = 80x - 128$

x	1.9	1.99	2	2.01	2.1
$f(x)$	24.761	31.208	32	32.808	40.841
$T(x)$	24.000	31.200	32	32.800	40.000

5. $T(x) = (\cos 2)(x - 2) + \sin 2$

x	1.9	1.99	2	2.01	2.1
$f(x)$	0.946	0.913	0.909	0.905	0.863
$T(x)$	0.951	0.913	0.909	0.905	0.868

7. $\Delta y = 0.331$; $dy = 0.3$ **9.** $\Delta y = -0.039$; $dy = -0.040$

11. $6x\,dx$ **13.** $-\dfrac{3}{(2x - 1)^2}\,dx$ **15.** $\dfrac{1 - 2x^2}{\sqrt{1 - x^2}}\,dx$

17. $(3 - \sin 2x)\,dx$ **19.** $-\pi \sin\left(\dfrac{6\pi x - 1}{2}\right) dx$

21. (a) 0.9 (b) 1.04 **23.** (a) 1.05 (b) 0.98

25. (a) 8.035 (b) 7.95 **27.** $\pm\frac{5}{8}$ in.^{2} **29.** $\pm 8\pi$ in.^{2}

31. (a) $\frac{5}{6}$% (b) 1.25%

33. (a) $\pm 5.12\pi$ in.^{3} (b) $\pm 1.28\pi$ in.^{2} (c) 0.75%, 0.5%

35. 80π cm^3 **37.** (a) $\frac{1}{4}$% (b) 216 sec = 3.6 min

39. (a) 0.87% (b) 2.16% **41.** 6407 ft

43. $f(x) = \sqrt{x}$, $dy = \dfrac{1}{2\sqrt{x}}\,dx$

$$f(99.4) \approx \sqrt{100} + \frac{1}{2\sqrt{100}}(-0.6) = 9.97$$

Calculator: 9.97

45. $f(x) = \sqrt[4]{x}$, $dy = \dfrac{1}{4x^{3/4}}\,dx$

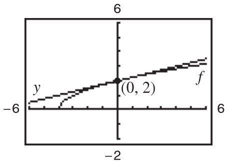
$$f(624) \approx \sqrt[4]{625} + \frac{1}{4(625)^{3/4}}(-1) = 4.998$$

Calculator: 4.998

47. $y - f(0) = f'(0)(x - 0)$

$y - 2 = \frac{1}{4}x$

$y = 2 + x/4$



49. The value of dy becomes closer to the value of Δy as Δx decreases.

51. (a) $f(x) = \sqrt{x}$; $dy = \dfrac{1}{2\sqrt{x}}\,dx$

$$f(4.02) \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.02) = 2 + \frac{1}{4}(0.02)$$

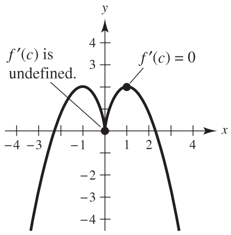
(b) $f(x) = \tan x$; $dy = \sec^2 x\,dx$

$$f(0.05) \approx \tan 0 + \sec^2(0)\,(0.05) = 0 + 1(0.05)$$

53. True **55.** True

Review Exercises for Chapter 3 (page 242)

1. Let f be defined at c. If $f'(c) = 0$ or if f' is undefined at c, then c is a critical number of f.

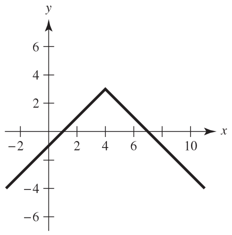


3. Maximum: (0, 0) **5.** Maximum: $(2\pi, 17.57)$

Minimum: $\left(-\frac{5}{2}, -\frac{25}{4}\right)$ Minimum: (2.73, 0.88)

7. $f(0) \neq f(4)$ **9.** Not continuous on $[-2, 2]$

11. (a) (b) f is not differentiable at $x = 4$.



13. $f'\left(\dfrac{2744}{729}\right) = \dfrac{3}{7}$ **15.** f is not differentiable at $x = 5$.

17. $f'(0) = 1$ **19.** $c = \dfrac{x_1 + x_2}{2}$

21. Critical number: $x = -\frac{3}{2}$

Increasing on $\left(-\frac{3}{2}, \infty\right)$; Decreasing on $\left(-\infty, -\frac{3}{2}\right)$

23. Critical numbers: $x = 1, \frac{7}{3}$

Increasing on $(-\infty, 1)$, $\left(\frac{7}{3}, \infty\right)$; Decreasing on $\left(1, \frac{7}{3}\right)$

25. Critical number: $x = 1$

Increasing on $(1, \infty)$; Decreasing on $(0, 1)$

27. Relative maximum: $\left(-\dfrac{\sqrt{15}}{6}, \dfrac{5\sqrt{15}}{9}\right)$

Relative minimum: $\left(\dfrac{\sqrt{15}}{6}, -\dfrac{5\sqrt{15}}{9}\right)$

29. Relative minimum: $(2, -12)$

31. (a) $y = \frac{1}{4}$ in.; $v = 4$ in./sec (b) Proof

(c) Period: $\pi/6$; Frequency: $6/\pi$

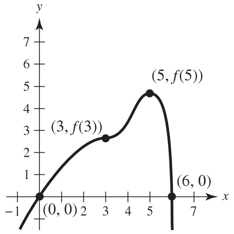
33. $(3, -54)$; Concave upward: $(3, \infty)$;
Concave downward: $(-\infty, 3)$

35. $(\pi/2, \pi/2), (3\pi/2, 3\pi/2)$; Concave upward: $(\pi/2, 3\pi/2)$
Concave downward: $(0, \pi/2), (3\pi/2, 2\pi)$

37. Relative minimum: $(-9, 0)$

39. Relative maxima: $(\sqrt{2}/2, 1/2), (-\sqrt{2}/2, 1/2)$
Relative minimum: $(0, 0)$

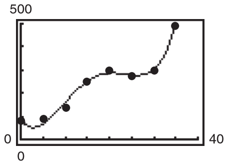
41.



43. Increasing and concave down

45. (a) $D = 0.00430t^4 - 0.2856t^3 + 5.833t^2 - 26.85t + 87.1$

(b)



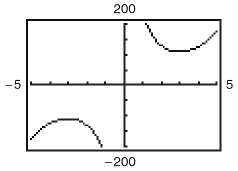
(c) Maximum in 2005; Minimum in 1972 (d) 2005

47. 8 **49.** $\frac{2}{3}$ **51.** $-\infty$ **53.** 0 **55.** 6

57. Vertical asymptote: $x = 0$; Horizontal asymptote: $y = -2$

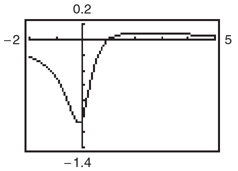
59. Vertical asymptote: $x = 4$; Horizontal asymptote: $y = 2$

61.



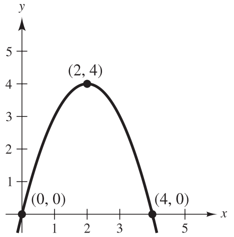
Vertical asymptote: $x = 0$
Relative minimum: $(3, 108)$
Relative maximum: $(-3, -108)$

63.

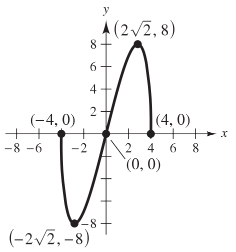


Horizontal asymptote: $y = 0$
Relative minimum:
$(-0.155, -1.077)$
Relative maximum:
$(2.155, 0.077)$

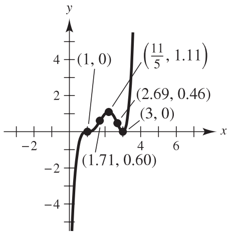
65.



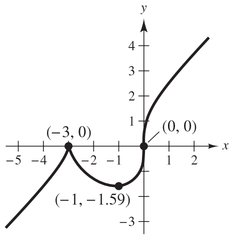
67.



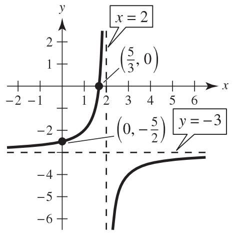
69.



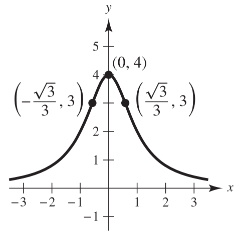
71.



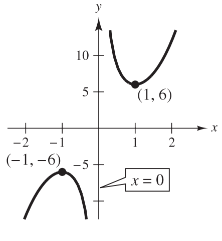
73.



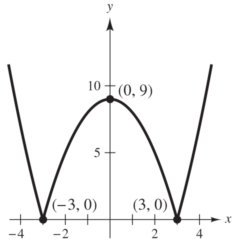
75.



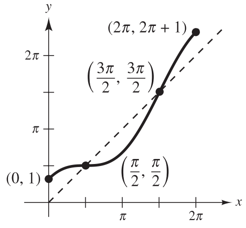
77.



79.



81.



83. (a) and (b) Maximum: $(1, 3)$
Minimum: $(1, 1)$

85. $t \approx 4.92 \approx$ 4:55 P.M.; $d \approx 64$ km

87. $(0, 0), (5, 0), (0, 10)$ **89.** Proof **91.** 14.05 ft

93. $3(3^{2/3} + 2^{2/3})^{3/2} \approx 21.07$ ft **95.** $v \approx 54.77$ mi/h

97. $-1.532, -0.347, 1.879$ **99.** $-1.164, 1.453$

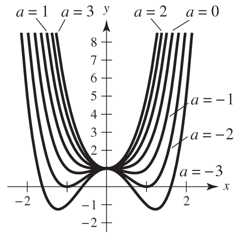
101. $dy = (1 - \cos x + x \sin x)\,dx$

103. $dS = \pm 1.8\pi \text{ cm}^2, \dfrac{dS}{S} \times 100 \approx \pm 0.56\%$

$dV = \pm 8.1\pi \text{ cm}^3, \dfrac{dV}{V} \times 100 \approx \pm 0.83\%$

P.S. Problem Solving (page 245)

1. Choices of a may vary.



(a) One relative minimum at $(0, 1)$ for $a \ge 0$
(b) One relative maximum at $(0, 1)$ for $a < 0$
(c) Two relative minima for $a < 0$ when $x = \pm\sqrt{-a/2}$
(d) If $a < 0$, there are three critical points; if $a \ge 0$, there is only one critical point.

3. All c where c is a real number **5–7.** Proofs

9. About 9.19 ft

11. Minimum: $(\sqrt{2} - 1)d$; There is no maximum.

13. (a)–(c) Proofs

15. (a)

x	0	0.5	1	2
$\sqrt{1 + x}$	1	1.2247	1.4142	1.7321
$\frac{1}{2}x + 1$	1	1.25	1.5	2

(b) Proof

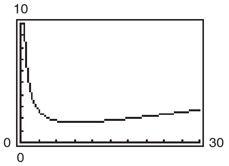
17. (a)

v	20	40	60	80	100
s	5.56	11.11	16.67	22.22	27.78
d	5.1	13.7	27.2	44.2	66.4

$d(s) = 0.071s^2 + 0.389s + 0.727$

(b) The distance between the back of the first vehicle and the front of the second vehicle is $d(s)$, the safe stopping distance. The first vehicle passes the given point in $5.5/s$ seconds, and the second vehicle takes $d(s)/s$ additional seconds. So, $T = d(s)/s + 5.5/s$.

(c)

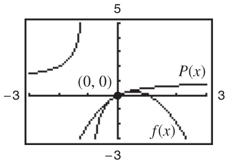


$s \approx 9.365$ m/sec

(d) $s \approx 9.365$ m/sec; 1.719 sec; 33.714 km/h (e) 10.597 m

19. (a) $P(x) = x - x^2$

(b)



Chapter 4

Section 4.1 (page 255)

1–3. Proofs **5.** $y = 3t^3 + C$ **7.** $y = \frac{2}{5}x^{5/2} + C$

	Original Integral	Rewrite	Integrate	Simplify
9.	$\int \sqrt[3]{x}\,dx$	$\int x^{1/3}\,dx$	$\frac{x^{4/3}}{4/3} + C$	$\frac{3}{4}x^{4/3} + C$
11.	$\int \frac{1}{x\sqrt{x}}\,dx$	$\int x^{-3/2}\,dx$	$\frac{x^{-1/2}}{-1/2} + C$	$-\frac{2}{\sqrt{x}} + C$
13.	$\int \frac{1}{2x^3}\,dx$	$\frac{1}{2}\int x^{-3}\,dx$	$\frac{1}{2}\left(\frac{x^{-2}}{-2}\right) + C$	$-\frac{1}{4x^2} + C$

15. $\frac{1}{2}x^2 + 7x + C$ **17.** $x^2 - x^3 + C$ **19.** $\frac{1}{6}x^6 + x + C$

21. $\frac{2}{5}x^{5/2} + x^2 + x + C$ **23.** $\frac{3}{5}x^{5/3} + C$ **25.** $-1/(4x^4) + C$

27. $\frac{2}{3}x^{3/2} + 12x^{1/2} + C = \frac{2}{3}x^{1/2}(x + 18) + C$

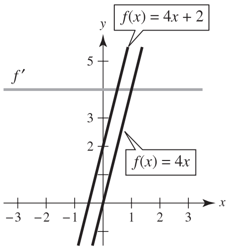
29. $x^3 + \frac{1}{2}x^2 - 2x + C$

31. $\frac{2}{7}y^{7/2} + C$ **33.** $x + C$ **35.** $5\sin x - 4\cos x + C$

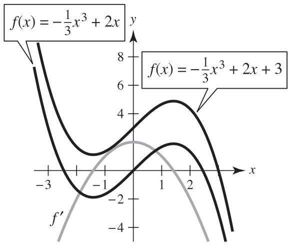
37. $t + \csc t + C$ **39.** $\tan\theta + \cos\theta + C$ **41.** $\tan y + C$

43. $-\csc x + C$

45. Answers will vary.
Example:

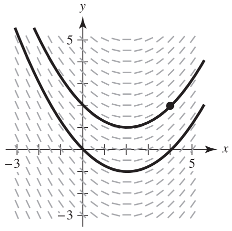


47. Answers will vary.
Example:

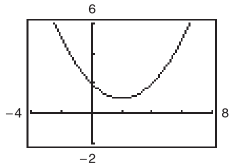


49. $y = x^2 - x + 1$

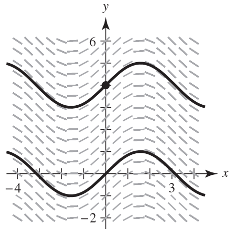
51. (a) Answers will vary.
Example:



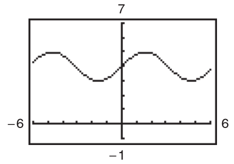
(b) $y = \frac{1}{4}x^2 - x + 2$



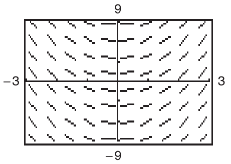
53. (a) Answers will vary.
Example:



(b) $y = \sin x + 4$

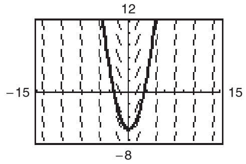


55. (a)



(b) $y = x^2 - 6$

(c)



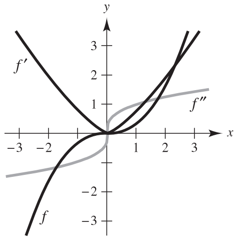
57. $f(x) = 3x^2 + 8$ **59.** $h(t) = 2t^4 + 5t - 11$

61. $f(x) = x^2 + x + 4$ **63.** $f(x) = -4\sqrt{x} + 3x$

65. (a) $h(t) = \frac{3}{4}t^2 + 5t + 12$ (b) 69 cm

67. When you evaluate the integral $\int f(x)\,dx$, you are finding a function $F(x)$ that is an antiderivative of $f(x)$. So, there is no difference.

69.



71. 62.25 ft **73.** $v_0 \approx 187.617$ ft/sec

75. $v(t) = -9.8t + C_1 = -9.8t + v_0$
$f(t) = -4.9t^2 + v_0t + C_2 = -4.9t^2 + v_0t + s_0$

77. 7.1 m **79.** 320 m; -32 m/sec

81. (a) $v(t) = 3t^2 - 12t + 9$; $a(t) = 6t - 12$
(b) (0, 1), (3, 5) (c) -3

83. $a(t) = -1/(2t^{3/2})$; $x(t) = 2\sqrt{t} + 2$

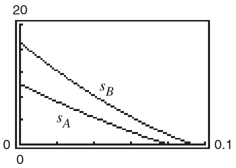
85. (a) 1.18 m/sec² (b) 190 m

87. (a) 300 ft (b) 60 ft/sec ≈ 41 mi/h

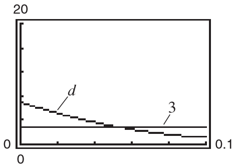
89. (a) Airplane A: $s_A = \frac{625}{2}t^2 - 150t + 10$

Airplane B: $s_B = \frac{49,275}{68}t^2 - 250t + 17$

(b)



(c) $d = \dfrac{28{,}025}{68}t^2 - 100t + 7$

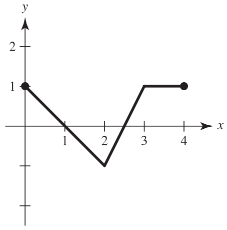


Yes, $d < 3$ for $t > 0.0505$ h

91. True **93.** True

95. False. f has an infinite number of antiderivatives, each differing by a constant.

97.



99. Proof

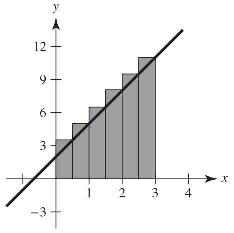
Section 4.2 (page 267)

1. 75 **3.** $\dfrac{158}{85}$ **5.** $4c$ **7.** $\displaystyle\sum_{i=1}^{11} \frac{1}{5i}$ **9.** $\displaystyle\sum_{j=1}^{6}\left[7\left(\frac{j}{6}\right) + 5\right]$

11. $\displaystyle\frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right)\right]$ **13.** $\displaystyle\frac{3}{n}\sum_{i=1}^{n}\left[2\left(1 + \frac{3i}{n}\right)^2\right]$ **15.** 84

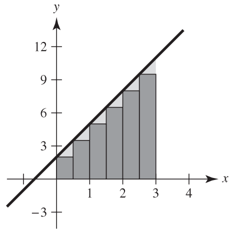
17. 1200 **19.** 2470 **21.** 12,040 **23.** 2930

25. (a)



Area ≈ 21.75

(b)



Area ≈ 17.25

27. $13 < (\text{Area of region}) < 15$

29. $55 < (\text{Area of region}) < 74.5$

31. $0.7908 < (\text{Area of region}) < 1.1835$

33. The area of the shaded region falls between 12.5 square units and 16.5 square units.

35. The area of the shaded region falls between 7 square units and 11 square units.

37. $\frac{81}{4}$ **39.** 9 **41.** $A \approx S \approx 0.768$
$A \approx s \approx 0.518$
43. $A \approx S \approx 0.746$
$A \approx s \approx 0.646$

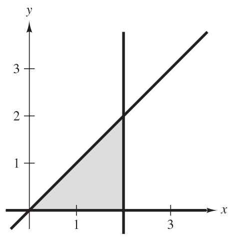
45. $(n + 2)/n$
$n = 10$: $S = 1.2$
$n = 100$: $S = 1.02$
$n = 1000$: $S = 1.002$
$n = 10{,}000$: $S = 1.0002$

47. $[2(n + 1)(n - 1)]/n^2$
$n = 10$: $S = 1.98$
$n = 100$: $S = 1.9998$
$n = 1000$: $S = 1.999998$
$n = 10{,}000$: $S = 1.99999998$

49. $\displaystyle\lim_{n\to\infty}\left[\frac{12(n + 1)}{n}\right] = 12$ **51.** $\displaystyle\lim_{n\to\infty}\frac{1}{6}\left(\frac{2n^3 - 3n^2 + n}{n^3}\right) = \frac{1}{3}$

53. $\displaystyle\lim_{n\to\infty}[(3n + 1)/n] = 3$

55. (a)



(b) $\Delta x = (2 - 0)/n = 2/n$

(c) $\displaystyle s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x = \sum_{i=1}^{n}[(i - 1)(2/n)](2/n)$

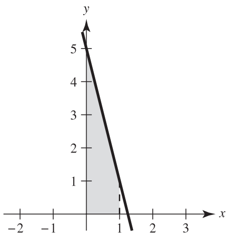
(d) $\displaystyle S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n}[i(2/n)](2/n)$

(e)

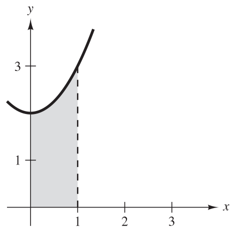
n	5	10	50	100
$s(n)$	1.6	1.8	1.96	1.98
$S(n)$	2.4	2.2	2.04	2.02

(f) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}[(i - 1)(2/n)](2/n) = 2;\ \lim_{n\to\infty}\sum_{i=1}^{n}[i(2/n)](2/n) = 2$

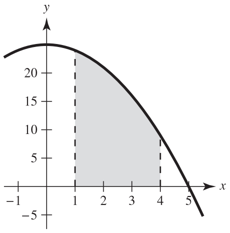
57. $A = 3$



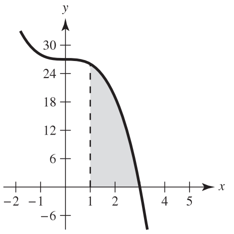
59. $A = \frac{7}{3}$



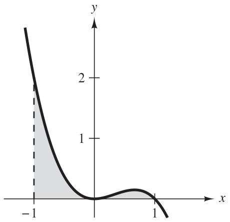
61. $A = 54$



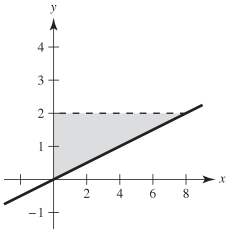
63. $A = 34$



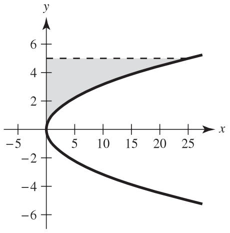
65. $A = \frac{2}{3}$



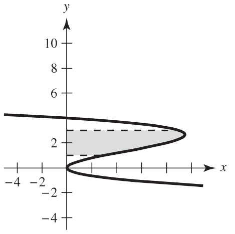
67. $A = 8$



69. $A = \frac{125}{3}$



71. $A = \frac{44}{3}$



73. $\frac{69}{8}$ **75.** 0.345

77.

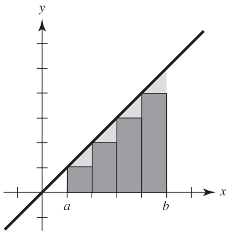
n	4	8	12	16	20
Approximate Area	5.3838	5.3523	5.3439	5.3403	5.3384

79.

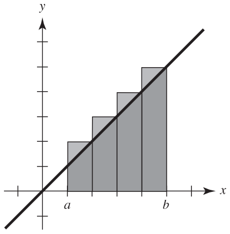
n	4	8	12	16	20
Approximate Area	2.2223	2.2387	2.2418	2.2430	2.2435

81. b

83. You can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of the inscribed rectangles in the figure below is the lower sum.

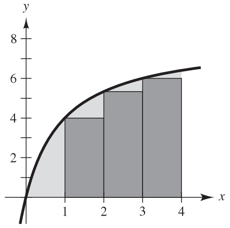


The sum of the areas of the circumscribed rectangles in the figure below is the upper sum.



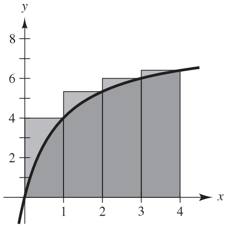
The rectangles in the first graph do not contain all of the area of the region, and the rectangles in the second graph cover more than the area of the region. The exact value of the area lies between these two sums.

85. (a)



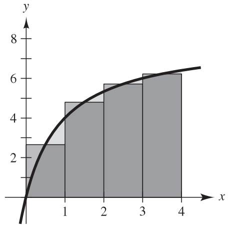
$s(4) = \frac{46}{3}$

(b)



$S(4) = \frac{326}{15}$

(c)



$M(4) = \frac{6112}{315}$

(d) Proof

(e)

n	4	8	20	100	200
$s(n)$	15.333	17.368	18.459	18.995	19.060
$S(n)$	21.733	20.568	19.739	19.251	19.188
$M(n)$	19.403	19.201	19.137	19.125	19.125

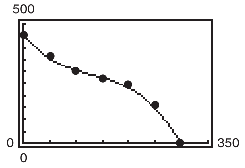
(f) Because f is an increasing function, $s(n)$ is always increasing and $S(n)$ is always decreasing.

87. True

89. Suppose there are n rows and $n + 1$ columns. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total. So, $2[1 + 2 + \cdots + n] = n(n + 1)$ and $1 + 2 + \cdots + n = [n(n + 1)]/2$.

91. (a) $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(b)



(c) 76,897.5 ft^2

93. Proof

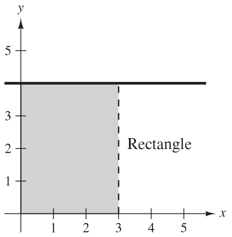
Section 4.3 (page 278)

1. $2\sqrt{3} \approx 3.464$ **3.** 32 **5.** 0 **7.** $\frac{10}{3}$ **9.** $\int_{-1}^{5} (3x + 10)\,dx$

11. $\int_{0}^{3} \sqrt{x^2 + 4}\,dx$ **13.** $\int_{0}^{4} 5\,dx$ **15.** $\int_{-4}^{4} \left(4 - |x|\right)dx$

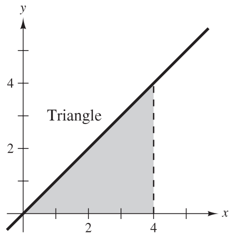
17. $\int_{-5}^{5} (25 - x^2)\,dx$ **19.** $\int_{0}^{\pi/2} \cos x\,dx$ **21.** $\int_{0}^{2} y^3\,dy$

23.



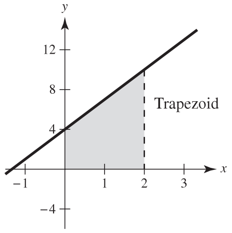
$A = 12$

25.



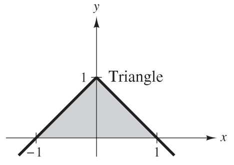
$A = 8$

27.



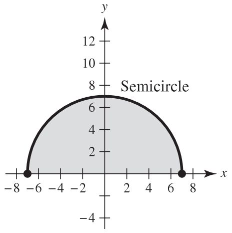
$A = 14$

29.



$A = 1$

31.



$A = 49\pi/2$

33. -6 **35.** 48 **37.** -12

39. 16 **41.** (a) 13 (b) -10 (c) 0 (d) 30

43. (a) 8 (b) -12 (c) -4 (d) 30 **45.** -48, 88

47. (a) $-\pi$ (b) 4 (c) $-(1 + 2\pi)$ (d) $3 - 2\pi$ (e) $5 + 2\pi$ (f) $23 - 2\pi$

49. (a) 14 (b) 4 (c) 8 (d) 0 **51.** 81

53. $\sum_{i=1}^{n} f(x_i)\,\Delta x > \int_{1}^{5} f(x)\,dx$

55. No. There is a discontinuity at $x = 4$. **57.** a **59.** d

61.

n	4	8	12	16	20
$L(n)$	3.6830	3.9956	4.0707	4.1016	4.1177
$M(n)$	4.3082	4.2076	4.1838	4.1740	4.1690
$R(n)$	3.6830	3.9956	4.0707	4.1016	4.1177

63.

n	4	8	12	16	20
$L(n)$	0.5890	0.6872	0.7199	0.7363	0.7461
$M(n)$	0.7854	0.7854	0.7854	0.7854	0.7854
$R(n)$	0.9817	0.8836	0.8508	0.8345	0.8247

65. True **67.** True

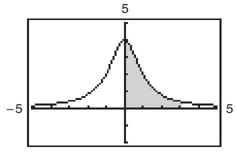
69. False: $\int_0^2 (-x)\,dx = -2$ **71.** 272 **73.** Proof

75. No. No matter how small the subintervals, the number of both rational and irrational numbers within each subinterval is infinite and $f(c_i) = 0$ or $f(c_i) = 1$.

77. $a = -1$ and $b = 1$ maximize the integral. **79.** $\frac{1}{3}$

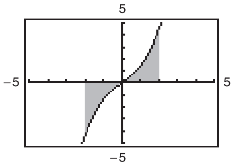
Section 4.4 (page 293)

1.



Positive

3.



Zero

5. 12 **7.** -2 **9.** $-\frac{10}{3}$ **11.** $\frac{1}{3}$ **13.** $\frac{1}{2}$ **15.** $\frac{2}{3}$ **17.** -4

19. $-\frac{1}{18}$ **21.** $-\frac{27}{20}$ **23.** $\frac{25}{2}$ **25.** $\frac{64}{3}$ **27.** $\pi + 2$

29. $\pi/4$ **31.** $2\sqrt{3}/3$ **33.** 0 **35.** $\frac{1}{6}$ **37.** 1 **39.** $\frac{52}{3}$

41. 20 **43.** $\frac{32}{3}$ **45.** $3\sqrt[3]{2}/2 \approx 1.8899$

47. $\frac{1444}{225} \approx 6.4178$ **49.** $\pm\arccos\sqrt{\pi}/2 \approx \pm 0.4817$

51. Average value $= 6$
$x = \pm\sqrt{3} \approx \pm 1.7321$

53. Average value $= \frac{1}{4}$
$x = \sqrt[3]{2}/2 \approx 0.6300$

55. Average value $= 2/\pi$
$x \approx 0.690, x \approx 2.451$

57. About 540 ft

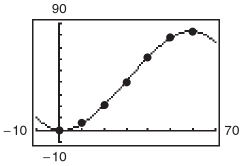
59. (a) 8 (b) $\frac{4}{3}$ (c) $\int_1^7 f(x)\,dx = 20$; Average value $= \frac{10}{3}$

61. (a) $F(x) = 500\sec^2 x$ (b) $1500\sqrt{3}/\pi \approx 827$ N

63. About 0.5318 L

65. (a) $v = -0.00086t^3 + 0.0782t^2 - 0.208t + 0.10$

(b)



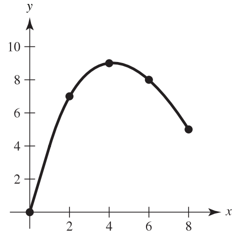
(c) 2475.6 m

67. $F(x) = 2x^2 - 7x$
$F(2) = -6$
$F(5) = 15$
$F(8) = 72$

69. $F(x) = -20/x + 20$
$F(2) = 10$
$F(5) = 16$
$F(8) = \frac{35}{2}$

71. $F(x) = \sin x - \sin 1$
$F(2) = \sin 2 - \sin 1 \approx 0.0678$
$F(5) = \sin 5 - \sin 1 \approx -1.8004$
$F(8) = \sin 8 - \sin 1 \approx 0.1479$

73. (a) $g(0) = 0, g(2) \approx 7, g(4) \approx 9, g(6) \approx 8, g(8) \approx 5$
(b) Increasing: $(0, 4)$; Decreasing: $(4, 8)$
(c) A maximum occurs at $x = 4$.
(d)

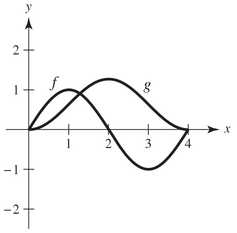


75. $\frac{1}{2}x^2 + 2x$ **77.** $\frac{3}{4}x^{4/3} - 12$ **79.** $\tan x - 1$

81. $x^2 - 2x$ **83.** $\sqrt{x^4 + 1}$ **85.** $x\cos x$ **87.** 8

89. $\cos x\sqrt{\sin x}$ **91.** $3x^2\sin x^6$

93.



An extremum of g occurs at $x = 2$.

95. (a) $C(x) = 1000(12x^{5/4} + 125)$
(b) $C(1) = \$137{,}000$
$C(5) \approx \$214{,}721$
$C(10) \approx \$338{,}394$

97. (a) $\frac{3}{2}$ ft to the right (b) $\frac{113}{10}$ ft **99.** (a) 0 ft (b) $\frac{63}{2}$ ft

101. (a) 2 ft to the right (b) 2 ft **103.** 28 units **105.** 8190 L

107. $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

109. $f(x) = \sec^2 x$ has a nonremovable discontinuity at $x = \pi/2$.

111. $2/\pi \approx 63.7\%$ **113.** True

115. $f'(x) = \dfrac{1}{(1/x)^2 + 1}\left(-\dfrac{1}{x^2}\right) + \dfrac{1}{x^2 + 1} = 0$

Because $f'(x) = 0$, $f(x)$ is constant.

117. (a) 0 (b) 0 (c) $xf(x) + \int_0^x f(t)\,dt$ (d) 0

Section 4.5 (page 306)

	$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
1.	$\int (8x^2 + 1)^2(16x)\,dx$	$8x^2 + 1$	$16x\,dx$
3.	$\int \dfrac{x}{\sqrt{x^2 + 1}}\,dx$	$x^2 + 1$	$2x\,dx$
5.	$\int \tan^2 x\sec^2 x\,dx$	$\tan x$	$\sec^2 x\,dx$

7. No **9.** Yes **11.** $\frac{1}{5}(1 + 6x)^5 + C$

13. $\frac{2}{3}(25 - x^2)^{3/2} + C$ **15.** $\frac{1}{12}(x^4 + 3)^3 + C$

17. $\frac{1}{15}(x^3 - 1)^5 + C$ **19.** $\frac{1}{3}(t^2 + 2)^{3/2} + C$

21. $-\frac{15}{8}(1 - x^2)^{4/3} + C$ **23.** $1/[4(1 - x^2)^2] + C$

25. $-1/[3(1 + x^3)] + C$ **27.** $-\sqrt{1 - x^2} + C$

29. $-\frac{1}{4}(1 + 1/t)^4 + C$ **31.** $\sqrt{2x} + C$

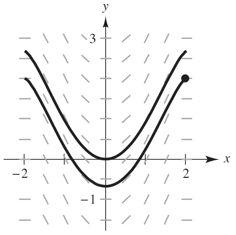
33. $\frac{2}{5}x^{5/2} + \frac{10}{3}x^{3/2} - 16x^{1/2} + C = \frac{1}{15}\sqrt{x}(6x^2 + 50x - 240) + C$

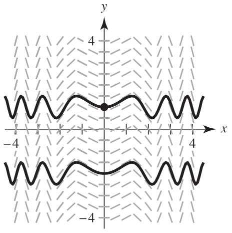
35. $\frac{1}{4}t^4 - 4t^2 + C$

37. $6y^{3/2} - \frac{2}{5}y^{5/2} + C = \frac{2}{5}y^{3/2}(15 - y) + C$

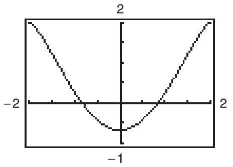
39. $2x^2 - 4\sqrt{16 - x^2} + C$ **41.** $-1/[2(x^2 + 2x - 3)] + C$

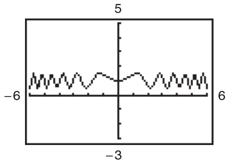
43. (a) Answers will vary. Example: **45.** (a) Answers will vary. Example:





(b) $y = -\frac{1}{3}(4 - x^2)^{3/2} + 2$ (b) $y = \frac{1}{2}\sin x^2 + 1$





47. $-\cos(\pi x) + C$ **49.** $-\frac{1}{4}\cos 4x + C$ **51.** $-\sin(1/\theta) + C$

53. $\frac{1}{4}\sin^2 2x + C$ or $-\frac{1}{4}\cos^2 2x + C_1$ or $-\frac{1}{8}\cos 4x + C_2$

55. $\frac{1}{5}\tan^5 x + C$ **57.** $\frac{1}{2}\tan^2 x + C$ or $\frac{1}{2}\sec^2 x + C_1$

59. $-\cot x - x + C$ **61.** $f(x) = 2\cos(x/2) + 4$

63. $f(x) = -\frac{1}{2}\cos 4x - 1$ **65.** $f(x) = \frac{1}{12}(4x^2 - 10)^3 - 8$

67. $\frac{2}{5}(x + 6)^{5/2} - 4(x + 6)^{3/2} + C = \frac{2}{5}(x + 6)^{3/2}(x - 4) + C$

69. $-\left[\frac{2}{3}(1 - x)^{3/2} - \frac{4}{5}(1 - x)^{5/2} + \frac{2}{7}(1 - x)^{7/2}\right] + C =$
$-\frac{2}{105}(1 - x)^{3/2}(15x^2 + 12x + 8) + C$

71. $\frac{1}{8}\left[\frac{2}{5}(2x - 1)^{5/2} + \frac{4}{3}(2x - 1)^{3/2} - 6(2x - 1)^{1/2}\right] + C =$
$\left(\sqrt{2x - 1}/15\right)(3x^2 + 2x - 13) + C$

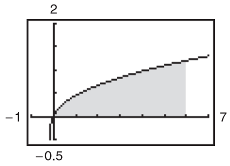
73. $-x - 1 - 2\sqrt{x + 1} + C$ or $-\left(x + 2\sqrt{x + 1}\right) + C_1$

75. 0 **77.** $12 - \frac{8}{9}\sqrt{2}$ **79.** 2 **81.** $\frac{1}{2}$ **83.** $\frac{4}{15}$ **85.** $3\sqrt{3}/4$

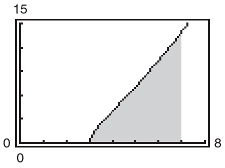
87. $f(x) = (2x^3 + 1)^3 + 3$ **89.** $f(x) = \sqrt{2x^2 - 1} - 3$

91. 1209/28 **93.** 4 **95.** $2\left(\sqrt{3} - 1\right)$

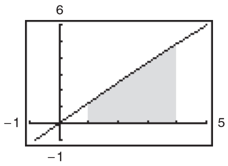
97. $\frac{14}{3}$



99. $\frac{144}{5}$



101. 9.21



103. $\frac{272}{15}$ **105.** $\frac{2}{3}$ **107.** (a) $\frac{64}{3}$ (b) $\frac{128}{3}$ (c) $-\frac{64}{3}$ (d) 64

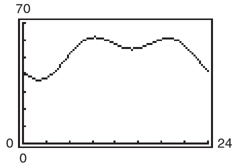
109. $2\int_0^3 (4x^2 - 6)\,dx = 36$

111. If $u = 5 - x^2$, then $du = -2x\,dx$ and
$\int x(5 - x^2)^3\,dx = -\frac{1}{2}\int(5 - x^2)^3(-2x)\,dx = -\frac{1}{2}\int u^3\,du.$

113. 16 **115.** $250,000

117. (a) Relative minimum: (6.7, 0.7) or July
Relative maximum: (1.3, 5.1) or February
(b) 36.68 in. (c) 3.99 in.

119. (a)



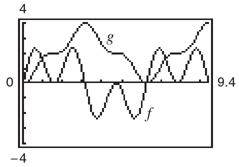
Maximum flow: $R \approx 61.713$ at $t = 9.36$.

(b) 1272 thousand gallons

121. (a) $P_{0.50,\,0.75} \approx 35.3\%$ (b) $b \approx 58.6\%$

123. (a) $9.17 (b) $3.14

125. (a)

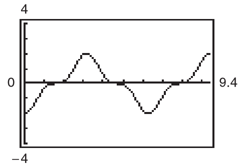


(b) g is nonnegative because the graph of f is positive at the beginning, and generally has more positive sections than negative ones.

(c) The points on g that correspond to the extrema of f are points of inflection of g.

(d) No, some zeros of f, such as $x = \pi/2$, do not correspond to extrema of g. The graph of g continues to increase after $x = \pi/2$ because f remains above the x-axis.

(e)



The graph of h is that of g shifted 2 units downward.

127. (a) Proof (b) Proof

129. False. $\int (2x + 1)^2\,dx = \frac{1}{6}(2x + 1)^3 + C$

131. True **133.** True **135–137.** Proofs

139. Putnam Problem A1, 1958

Section 4.6 (page 316)

	Trapezoidal	*Simpson's*	*Exact*
1.	2.7500	2.6667	2.6667
3.	4.2500	4.0000	4.0000
5.	20.2222	20.0000	20.0000
7.	12.6640	12.6667	12.6667
9.	0.3352	0.3334	0.3333

	Trapezoidal	*Simpson's*	*Graphing Utility*
11.	3.2833	3.2396	3.2413
13.	0.3415	0.3720	0.3927
15.	0.5495	0.5483	0.5493
17.	−0.0975	−0.0977	−0.0977
19.	0.1940	0.1860	0.1858

21. Trapezoidal: Linear (1st-degree) polynomials
Simpson's: Quadratic (2nd-degree) polynomials

23. (a) 1.500 (b) 0.000 **25.** (a) 0.01 (b) 0.0005

27. (a) 0.1615 (b) 0.0066 **29.** (a) $n = 366$ (b) $n = 26$

31. (a) $n = 77$ (b) $n = 8$ **33.** (a) $n = 287$ (b) $n = 16$

35. (a) $n = 130$ (b) $n = 12$ **37.** (a) $n = 643$ (b) $n = 48$

39. (a) 24.5 (b) 25.67 **41.** Answers will vary.

43.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.8739	0.7960	0.6239	0.7489	0.7709
8	0.8350	0.7892	0.7100	0.7725	0.7803
10	0.8261	0.7881	0.7261	0.7761	0.7818
12	0.8200	0.7875	0.7367	0.7783	0.7826
16	0.8121	0.7867	0.7496	0.7808	0.7836
20	0.8071	0.7864	0.7571	0.7821	0.7841

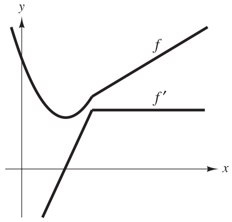
45. 0.701 **47.** 17.476

49. (a) Trapezoidal Rule: 12.518; Simpson's Rule: 12.592
(b) $y = -1.37266x^3 + 4.0092x^2 - 0.620x + 4.28$
$\int_0^2 y\,dx \approx 12.521$

51. 3.14159 **53.** 7435 m^2 **55.** 2.477

Review Exercises for Chapter 4 (page 318)

1.

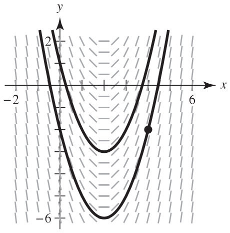


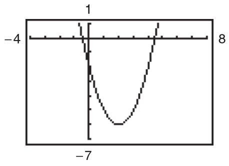
3. $\frac{4}{3}x^3 + \frac{1}{2}x^2 + 3x + C$

5. $x^2/2 - 4/x^2 + C$ **7.** $x^2 + 9\cos x + C$

9. $y = 1 - 3x^2$

11. (a) Answers will vary. (b) $y = x^2 - 4x - 2$
Example:





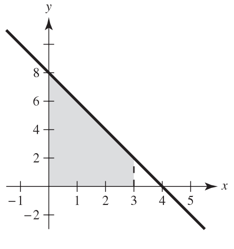
13. 240 ft/sec **15.** (a) 3 sec; 144 ft (b) $\frac{3}{2}$ sec (c) 108 ft

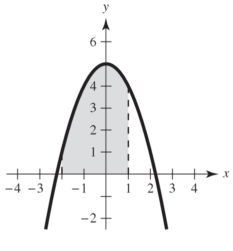
17. $\sum_{n=1}^{10} \frac{1}{3n}$ **19.** 420 **21.** 3310

23. (a) $\sum_{i=1}^{10} (2i - 1)$ (b) $\sum_{i=1}^{n} i^3$ (c) $\sum_{i=1}^{10} (4i + 2)$

25. $9.038 <$ (Area of region) < 13.038

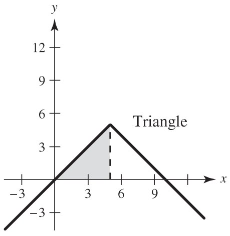
27. $A = 15$ **29.** $A = 12$





31. $\frac{27}{2}$ **33.** $\int_4^6 (2x - 3)\,dx$ **35.** $\int_{-4}^{0} (2x + 8)\,dx$

37.

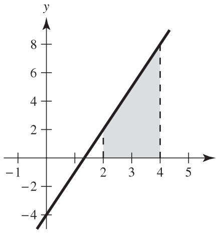


$A = \frac{25}{2}$

39. (a) 17 (b) 7 (c) 9 (d) 84

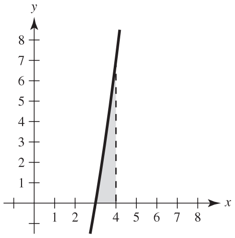
41. 56 **43.** 0 **45.** $\frac{422}{5}$ **47.** $(\sqrt{2} + 2)/2$

49.



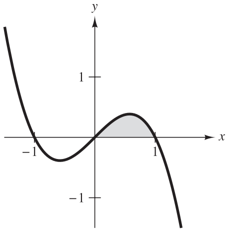
$A = 10$

51.



$A = \frac{10}{3}$

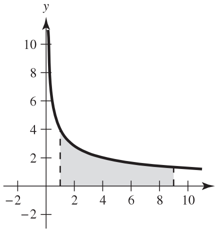
53.



$A = \frac{1}{4}$

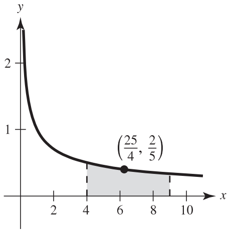
55. $-\cos 2 + 1 \approx 1.416$

57.



$A = 16$

59. Average value $= \frac{2}{5}$, $x = \frac{25}{4}$



61. $x^2\sqrt{1 + x^3}$ **63.** $x^2 + 3x + 2$

65. $-\frac{1}{7}x^7 + \frac{9}{5}x^5 - 9x^3 + 27x + C$ **67.** $\frac{2}{3}\sqrt{x^3 + 3} + C$

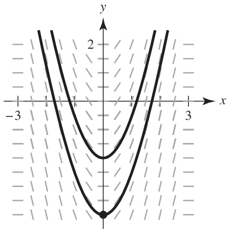
69. $-\frac{1}{30}(1 - 3x^2)^5 + C = \frac{1}{30}(3x^2 - 1)^5 + C$

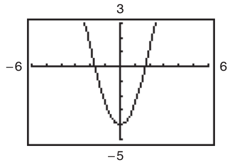
71. $\frac{1}{4}\sin^4 x + C$ **73.** $-2\sqrt{1 - \sin\theta} + C$

75. $\frac{1}{3\pi}(1 + \sec \pi x)^3 + C$

77. $21/4$ **79.** 2 **81.** $28\pi/15$ **83.** 2

85. (a) Answers will vary. (b) $y = -\frac{1}{3}(9 - x^2)^{3/2} + 5$
Example:





87. $\frac{468}{7}$

89. (a) $\int_0^{12} [2.880 + 2.125 \sin(0.578t + 0.745)]\, dt \approx 36.63$ in.
(b) 2.22 in.

91. Trapezoidal Rule: 0.285
Simpson's Rule: 0.284
Graphing Utility: 0.284

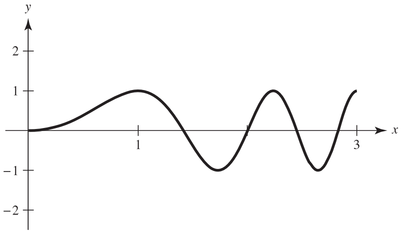
93. Trapezoidal Rule: 0.637
Simpson's Rule: 0.685
Graphing Utility: 0.704

P.S. Problem Solving (page 321)

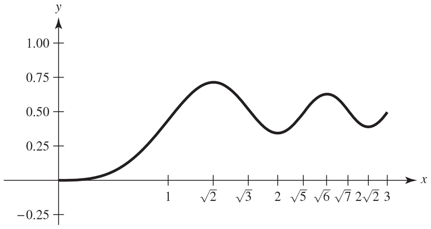
1. (a) $L(1) = 0$ (b) $L'(x) = 1/x$, $L'(1) = 1$
(c) $x \approx 2.718$ (d) Proof

3. (a) $\lim\limits_{n\to\infty} \left[\frac{32}{n^5}\sum_{i=1}^{n} i^4 - \frac{64}{n^4}\sum_{i=1}^{n} i^3 + \frac{32}{n^3}\sum_{i=1}^{n} i^2\right]$
(b) $(16n^4 - 16)/(15n^4)$ (c) $16/15$

5. (a)

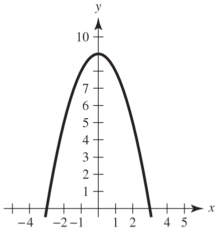


(b)



(c) Relative maxima at $x = \sqrt{2}, \sqrt{6}$
Relative minima at $x = 2, 2\sqrt{2}$
(d) Points of inflection at $x = 1, \sqrt{3}, \sqrt{5}, \sqrt{7}$

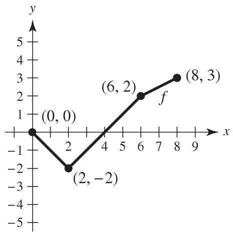
7. (a)



Area = 36

(b) Base = 6, height = 9
Area = $\frac{2}{3}bh = \frac{2}{3}(6)(9) = 36$
(c) Proof

9. (a)



(b)

x	0	1	2	3	4	5	6	7	8
$F(x)$	0	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	-4	$-\frac{7}{2}$	-2	$\frac{1}{4}$	3

(c) $x = 4, 8$ (d) $x = 2$

11. Proof **13.** $\frac{2}{3}$ **15.** $1 \le \int_0^1 \sqrt{1 + x^4}\, dx \le \sqrt{2}$

17. (a) Proof (b) Proof (c) Proof

19. (a) $R(n), I, T(n), L(n)$
(b) $S(4) = \frac{1}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)] \approx 5.42$

21. $a = -4, b = 4$

Chapter 5

Section 5.1 (page 331)

1.

x	0.5	1.5	2	2.5
$\int_1^x (1/t)\, dt$	-0.6932	0.4055	0.6932	0.9163

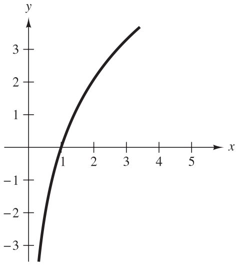
x	3	3.5	4
$\int_1^x (1/t)\, dt$	1.0987	1.2529	1.3865

3. (a) 3.8067 (b) $\ln 45 = \int_1^{45} \frac{1}{t}\, dt \approx 3.8067$

5. (a) -0.2231 (b) $\ln 0.8 = \int_1^{0.8} \frac{1}{t}\, dt \approx -0.2231$

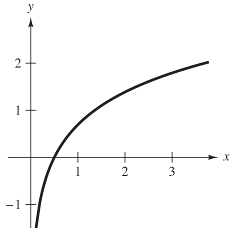
7. b **8.** d **9.** a **10.** c

11.



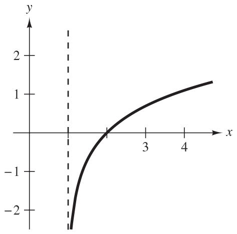
Domain: $x > 0$

13.



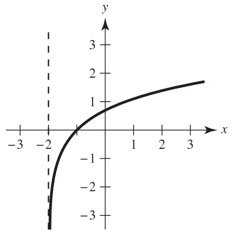
Domain: $x > 0$

15.



Domain: $x > 1$

17.



Domain: $x > -2$

19. (a) 1.7917 (b) -0.4055 (c) 4.3944 (d) 0.5493

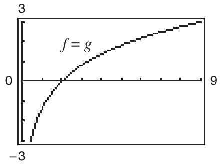
21. $\ln x - \ln 4$ **23.** $\ln x + \ln y - \ln z$

25. $\ln x + \frac{1}{2}\ln(x^2 + 5)$ **27.** $\frac{1}{2}[\ln(x - 1) - \ln x]$

29. $\ln z + 2\ln(z - 1)$

31. $\ln \frac{x - 2}{x + 2}$ **33.** $\ln \sqrt[3]{\frac{x(x + 3)^2}{x^2 - 1}}$ **35.** $\ln\left(9/\sqrt{x^2 + 1}\right)$

37. (a)



(b) $f(x) = \ln \frac{x^2}{4} = \ln x^2 - \ln 4$
$= 2\ln x - \ln 4$
$= g(x)$

39. $-\infty$ **41.** $\ln 4 \approx 1.3863$ **43.** $y = 3x - 3$

45. $y = 4x - 4$ **47.** $1/x$ **49.** $2/x$ **51.** $4(\ln x)^3/x$

53. $2/(t + 1)$ **55.** $\dfrac{2x^2 - 1}{x(x^2 - 1)}$ **57.** $\dfrac{1 - x^2}{x(x^2 + 1)}$

59. $\dfrac{1 - 2\ln t}{t^3}$ **61.** $\dfrac{2}{x\ln x^2} = \dfrac{1}{x\ln x}$ **63.** $\dfrac{1}{1 - x^2}$

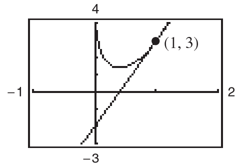
65. $\dfrac{-4}{x(x^2 + 4)}$ **67.** $\dfrac{\sqrt{x^2 + 1}}{x^2}$ **69.** $\cot x$

71. $-\tan x + \dfrac{\sin x}{\cos x - 1}$ **73.** $\dfrac{3\cos x}{(\sin x - 1)(\sin x + 2)}$

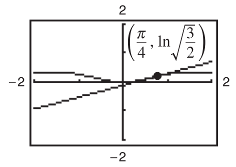
75. $[\ln(2x) + 1]/x$

77. (a) $5x - y - 2 = 0$ **79.** (a) $y = \frac{1}{3}x - \frac{1}{12}\pi + \frac{1}{2}\ln\left(\frac{3}{2}\right)$

(b)

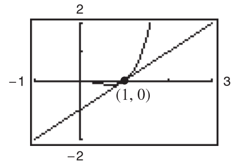


(b)



81. (a) $y = x - 1$ **83.** $2xy/(3 - 2y^2)$

(b)



85. $\dfrac{y(1 - 6x^2)}{1 + y}$ **87.** $y = x - 1$

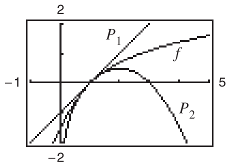
89. $xy'' + y' = x(-2/x^2) + (2/x) = 0$

91. Relative minimum: $\left(1, \frac{1}{2}\right)$

93. Relative minimum: $(e^{-1}, -e^{-1})$

95. Relative minimum: (e, e); Point of inflection: $(e^2, e^2/2)$

97. $P_1(x) = x - 1$; $P_2(x) = x - 1 - \frac{1}{2}(x - 1)^2$



The values of f, P_1, and P_2 and their first derivatives agree at $x = 1$.

99. $x \approx 0.567$ **101.** $(2x^2 + 1)/\sqrt{x^2 + 1}$

103. $\dfrac{3x^3 + 15x^2 - 8x}{2(x + 1)^3\sqrt{3x - 2}}$ **105.** $\dfrac{(2x^2 + 2x - 1)\sqrt{x - 1}}{(x + 1)^{3/2}}$

107. The domain of the natural logarithmic function is $(0, \infty)$ and the range is $(-\infty, \infty)$. The function is continuous, increasing, and one-to-one, and its graph is concave downward. In addition, if a and b are positive numbers and n is rational, then $\ln(1) = 0$, $\ln(a \cdot b) = \ln a + \ln b$, $\ln(a^n) = n\ln a$, and $\ln(a/b) = \ln a - \ln b$.

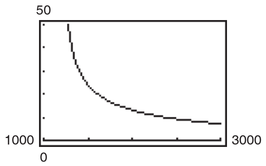
109. (a) Yes. If the graph of g is increasing , then $g'(x) > 0$. Since $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and thus $f'(x) > 0$. Therefore, the graph of f is increasing.

(b) No. Let $f(x) = x^2 + 1$ (positive and concave up) and let $g(x) = \ln(x^2 + 1)$ (not concave up).

111. False; $\ln x + \ln 25 = \ln 25x$.

113. False; π is a constant, so $\dfrac{d}{dx}[\ln \pi] = 0$.

115. (a)



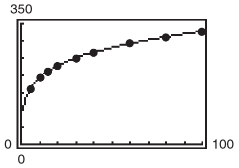
(b) 30 yr; $503,434.80

(c) 20 yr; $386,685.60

(d) When $x = 1398.43$, $dt/dx \approx -0.0805$.
When $x = 1611.19$, $dt/dx \approx -0.0287$.

(e) Two benefits of a higher monthly payment are a shorter term and the total amount paid is lower.

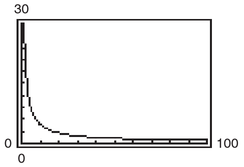
117. (a)



(b) $T'(10) \approx 4.75°/\text{lb/in.}^2$

$T'(70) \approx 0.97°/\text{lb/in.}^2$

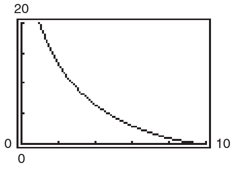
(c)



$\lim\limits_{p\to\infty} T'(p) = 0$

Answers will vary.

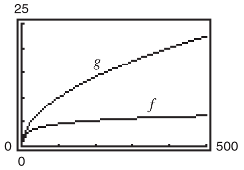
119. (a)



(b) When $x = 5$,
$dy/dx = -\sqrt{3}$.
When $x = 9$,
$dy/dx = -\sqrt{19}/9$.

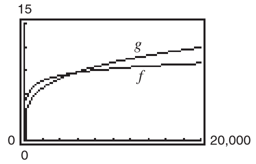
(c) $\lim\limits_{x\to 10^-} \dfrac{dy}{dx} = 0$

121. (a)



For $x > 4$, $g'(x) > f'(x)$.
g is increasing at a faster rate than f for large values of x.

(b)



For $x > 256$, $g'(x) > f'(x)$.
g is increasing at a faster rate than f for large values of x.

$f(x) = \ln x$ increases very slowly for large values of x.

Section 5.2 (page 340)

1. $5\ln|x| + C$ **3.** $\ln|x + 1| + C$ **5.** $\frac{1}{2}\ln|2x + 5| + C$

7. $\frac{1}{2}\ln|x^2 - 3| + C$ **9.** $\ln|x^4 + 3x| + C$

11. $x^2/2 - \ln(x^4) + C$ **13.** $\frac{1}{3}\ln|x^3 + 3x^2 + 9x| + C$

15. $\frac{1}{2}x^2 - 4x + 6\ln|x + 1| + C$ **17.** $\frac{1}{3}x^3 + 5\ln|x - 3| + C$

19. $\frac{1}{3}x^3 - 2x + \ln\sqrt{x^2 + 2} + C$ **21.** $\frac{1}{3}(\ln x)^3 + C$

23. $2\sqrt{x + 1} + C$ **25.** $2\ln|x - 1| - 2/(x - 1) + C$

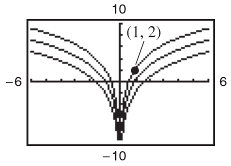
27. $\sqrt{2x} - \ln\left|1 + \sqrt{2x}\right| + C$

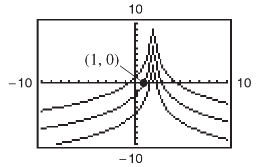
29. $x + 6\sqrt{x} + 18\ln\left|\sqrt{x} - 3\right| + C$ **31.** $3\ln\left|\sin\dfrac{\theta}{3}\right| + C$

33. $-\frac{1}{2}\ln|\csc 2x + \cot 2x| + C$ **35.** $\frac{1}{3}\sin 3\theta - \theta + C$

37. $\ln|1 + \sin t| + C$ **39.** $\ln|\sec x - 1| + C$

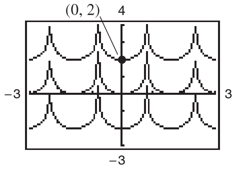
41. $y = 4\ln|x| + C$ **43.** $y = -3\ln|2 - x| + C$



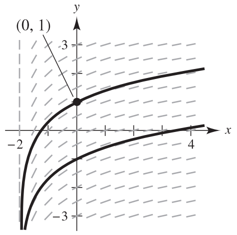


The graph has a hole at $x = 2$.

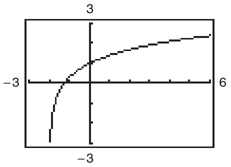
45. $s = -\frac{1}{2}\ln|\cos 2\theta| + C$ **47.** $f(x) = -2\ln x + 3x - 2$



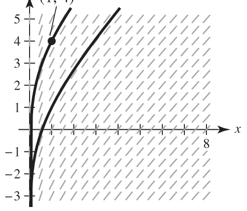
49. (a)



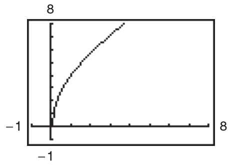
(b) $y = \ln\left(\frac{x+2}{2}\right) + 1$



51. (a)



(b) $y = \ln x + x + 3$



53. $\frac{5}{3}\ln 13 \approx 4.275$ **55.** $\frac{7}{3}$ **57.** $-\ln 3 \approx -1.099$

59. $\ln\left|\frac{2 - \sin 2}{1 - \sin 1}\right| \approx 1.929$ **61.** $2\left[\sqrt{x} - \ln\left(1 + \sqrt{x}\right)\right] + C$

63. $\ln\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + 2\sqrt{x} + C$ **65.** $\ln\left(\sqrt{2} + 1\right) - \frac{\sqrt{2}}{2} \approx 0.174$

67. $1/x$ **69.** $1/x$ **71.** d **73.** $6\ln 3$ **75.** $\frac{1}{2}\ln 2$

77. $\frac{15}{2} + 8\ln 2 \approx 13.045$

79. $(12/\pi)\ln\left(2 + \sqrt{3}\right) \approx 5.03$

81. Trapezoidal Rule: 20.2
Simpson's Rule: 19.4667

83. Trapezoidal Rule: 5.3368
Simpson's Rule: 5.3632

85. Power Rule **87.** Log Rule **89.** $x = 2$ **91.** Proof

93. $-\ln|\cos x| + C = \ln|1/\cos x| + C = \ln|\sec x| + C$

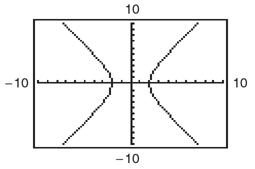
95. $\ln|\sec x + \tan x| + C = \ln\left|\frac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right| + C$
$= -\ln|\sec x - \tan x| + C$

97. 1 **99.** $1/(e - 1) \approx 0.582$

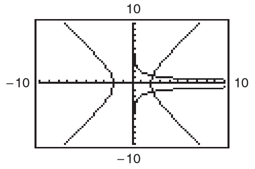
101. $P(t) = 1000(12\ln|1 + 0.25t| + 1)$; $P(3) \approx 7715$

103. $168.27 **105.** False. $\frac{1}{2}(\ln x) = \ln x^{1/2}$ **107.** True

109. (a)



(b) Answers will vary.
Example:
$y^2 = e^{-\ln x + \ln 4} = 4/x$



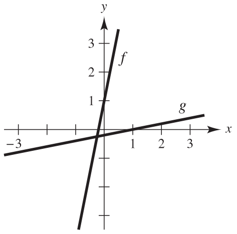
(c) Answers will vary.

111. Proof

Section 5.3 (page 349)

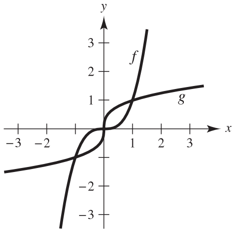
1. (a) $f(g(x)) = 5[(x - 1)/5] + 1 = x$
$g(f(x)) = [(5x + 1) - 1]/5 = x$

(b)



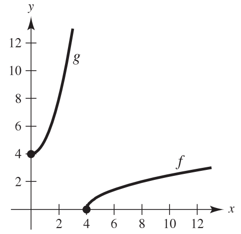
3. (a) $f(g(x)) = \left(\sqrt[3]{x}\right)^3 = x$; $g(f(x)) = \sqrt[3]{x^3} = x$

(b)



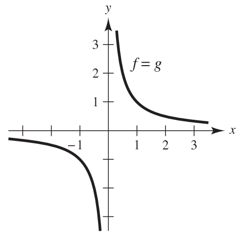
5. (a) $f(g(x)) = \sqrt{x^2 + 4 - 4} = x$;
$g(f(x)) = \left(\sqrt{x - 4}\right)^2 + 4 = x$

(b)



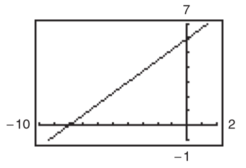
7. (a) $f(g(x)) = \frac{1}{1/x} = x$; $g(f(x)) = \frac{1}{1/x} = x$

(b)



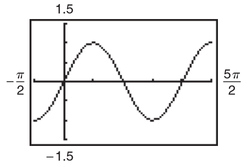
9. c **10.** b **11.** a **12.** d

13.



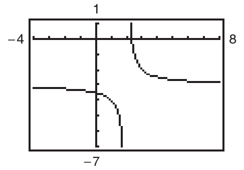
One-to-one, inverse exists.

15.



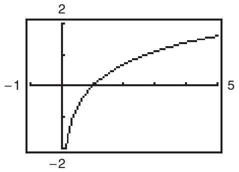
Not one-to-one, inverse does not exist.

17.



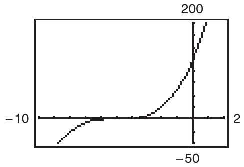
One-to-one, inverse exists.

19.



One-to-one, inverse exists.

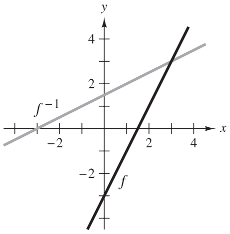
21.



One-to-one, inverse exists.

23. (a) $f^{-1}(x) = (x+3)/2$

(b)

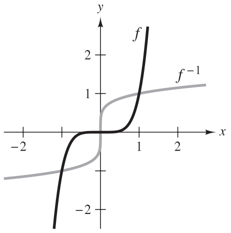


(c) f and f^{-1} are symmetric about $y = x$.

(d) Domain of f and f^{-1}: all real numbers
Range of f and f^{-1}: all real numbers

25. (a) $f^{-1}(x) = x^{1/5}$

(b)

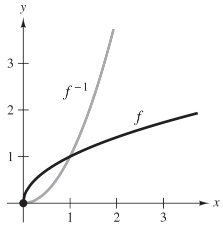


(c) f and f^{-1} are symmetric about $y = x$.

(d) Domain of f and f^{-1}: all real numbers
Range of f and f^{-1}: all real numbers

27. (a) $f^{-1}(x) = x^2, \ x \geq 0$

(b)

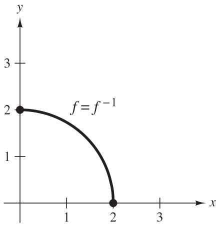


(c) f and f^{-1} are symmetric about $y = x$.

(d) Domain of f and f^{-1}: $x \geq 0$
Range of f and f^{-1}: $y \geq 0$

29. (a) $f^{-1}(x) = \sqrt{4 - x^2}$, $0 \leq x \leq 2$

(b)

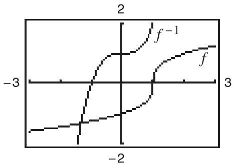


(c) f and f^{-1} are symmetric about $y = x$.

(d) Domain of f and f^{-1}: $0 \leq x \leq 2$
Range of f and f^{-1}: $0 \leq y \leq 2$

31. (a) $f^{-1}(x) = x^3 + 1$

(b)

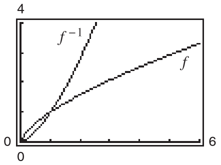


(c) f and f^{-1} are symmetric about $y = x$.

(d) Domain of f and f^{-1}: all real numbers
Range of f and f^{-1}: all real numbers

33. (a) $f^{-1}(x) = x^{3/2}, \ x \geq 0$

(b)

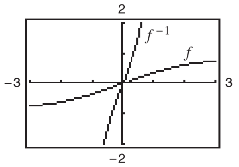


(c) f and f^{-1} are symmetric about $y = x$.

(d) Domain of f and f^{-1}: $x \geq 0$
Range of f and f^{-1}: $y \geq 0$

35. (a) $f^{-1}(x) = \sqrt{7}x/\sqrt{1 - x^2}, \ -1 < x < 1$

(b)



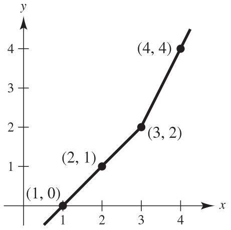
(c) f and f^{-1} are symmetric about $y = x$.

(d) Domain of f: all real numbers
Domain of f^{-1}: $-1 < x < 1$
Range of f: $-1 < y < 1$
Range of f^{-1}: all real numbers

37.

x	0	1	2	4
$f(x)$	1	2	3	4

x	1	2	3	4
$f^{-1}(x)$	0	1	2	4



39. (a) Proof
(b) $y = \frac{20}{7}(80 - x)$
x: total cost
y: number of pounds of the less expensive commodity
(c) $[62.5, 80]$ (d) 20 lb

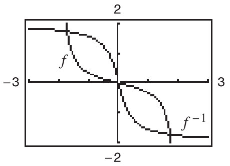
41. Inverse exists. **43.** Inverse does not exist.

45. Inverse exists. **47.** $f'(x) = 2(x - 4) > 0$ on $(4, \infty)$

49. $f'(x) = -8/x^3 < 0$ on $(0, \infty)$

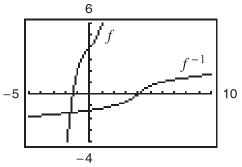
51. $f'(x) = -\sin x < 0$ on $(0, \pi)$

53. $f^{-1}(x) = \begin{cases} \left[1 - \sqrt{1 + 16x^2}\right]/(2x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$



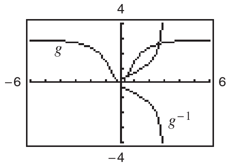
The graph of f^{-1} is a reflection of the graph of f in the line $y = -x$.

55. (a) and (b)



(c) f is one-to-one and has an inverse function.

57. (a) and (b)



(c) g is not one-to-one and does not have an inverse function.

59. One-to-one
$f^{-1}(x) = x^2 + 2, \ x \geq 0$

61. One-to-one
$f^{-1}(x) = 2 - x, \ x \geq 0$

63. $f^{-1}(x) = \sqrt{x} + 3, \ x \geq 0$ (Answer is not unique.)

65. $f^{-1}(x) = x - 3, \ x \geq 0$ (Answer is not unique.)

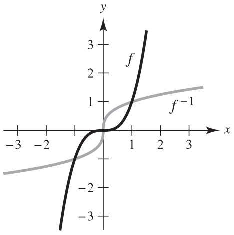
67. Inverse exists. Volume is an increasing function, and therefore is one-to-one. The inverse function gives the time t corresponding to the volume V.

69. Inverse does not exist. **71.** $1/27$ **73.** $1/5$

75. $2\sqrt{3}/3$ **77.** -2 **79.** $1/13$

81. (a) Domain of f: $(-\infty, \infty)$ (b) Range of f: $(-\infty, \infty)$
Domain of f^{-1}: $(-\infty, \infty)$ Range of f^{-1}: $(-\infty, \infty)$

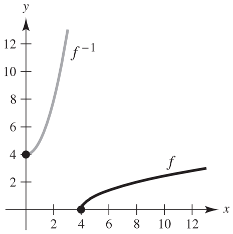
(c)



(d) $f'\left(\frac{1}{2}\right) = \frac{3}{4}$, $(f^{-1})'\left(\frac{1}{8}\right) = \frac{4}{3}$

83. (a) Domain of f: $[4, \infty)$ (b) Range of f: $[0, \infty)$
Domain of f^{-1}: $[0, \infty)$ Range of f^{-1}: $[4, \infty)$

(c)



(d) $f'(5) = \frac{1}{2}$, $(f^{-1})'(1) = 2$

85. $-\frac{1}{11}$ **87.** 32 **89.** 600

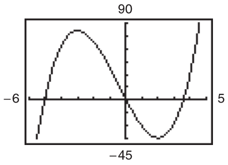
91. $(g^{-1} \circ f^{-1})(x) = (x + 1)/2$ **93.** $(f \circ g)^{-1}(x) = (x + 1)/2$

95. Let $y = f(x)$ be one-to-one. Solve for x as a function of y. Interchange x and y to get $y = f^{-1}(x)$. Let the domain of f^{-1} be the range of f. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.
Example: $f(x) = x^3$; $y = x^3$; $x = \sqrt[3]{y}$; $y = \sqrt[3]{x}$; $f^{-1}(x) = \sqrt[3]{x}$

97. Many x-values yield the same y-value. For example, $f(\pi) = 0 = f(0)$. The graph is not continuous at $[(2n - 1)\pi]/2$ where n is an integer.

99. $\frac{1}{4}$ **101.** False. Let $f(x) = x^2$. **103.** True

105. (a)



f does not pass the horizontal line test.

(b) $c = 2$

107–109. Proofs **111.** Proof; concave upward

113. Proof; $\sqrt{5}/5$

115. (a) Proof (b) $f^{-1}(x) = \dfrac{b - dx}{cx - a}$
(c) $a = -d$, or $b = c = 0$, $a = d$

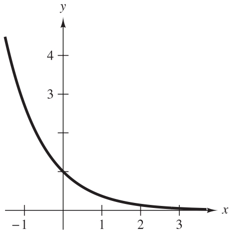
Section 5.4 (page 358)

1. $x = 4$ **3.** $x \approx 2.485$ **5.** $x = 0$ **7.** $x \approx 0.511$

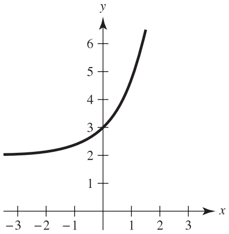
9. $x \approx 8.862$ **11.** $x \approx 7.389$ **13.** $x \approx 10.389$

15. $x \approx 5.389$

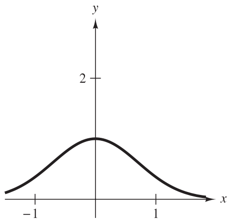
17.



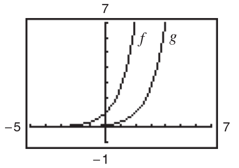
19.



21.

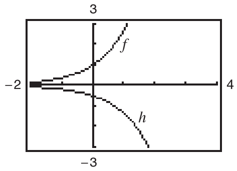


23. (a)



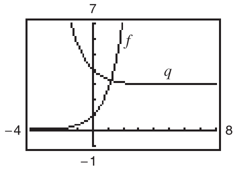
Translation two units to the right

(b)



Reflection in the x-axis and a vertical shrink

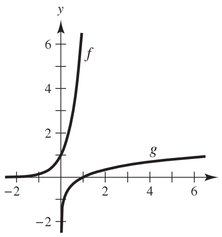
(c)



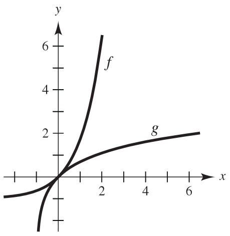
Reflection in the y-axis and a translation three units upward

25. c **26.** d **27.** a **28.** b

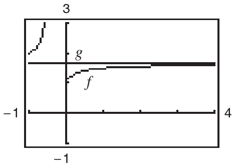
29.



31.



33.



$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = e^{0.5}$

35. $2.7182805 < e$ **37.** (a) $y = 3x + 1$ (b) $y = -3x + 1$

39. $2e^{2x}$ **41.** $e^{\sqrt{x}}/(2\sqrt{x})$ **43.** e^{x-4} **45.** $e^x\left(\dfrac{1}{x} + \ln x\right)$

47. $e^x(x^3 + 3x^2)$ **49.** $3(e^{-t} + e^t)^2(e^t - e^{-t})$

51. $2e^{2x}/(1 + e^{2x})$ **53.** $-2(e^x - e^{-x})/(e^x + e^{-x})^2$

55. $-2e^x/(e^x - 1)^2$ **57.** $2e^x \cos x$ **59.** $\cos(x)/x$

61. $y = -x + 2$ **63.** $y = -4(x + 1)$ **65.** $y = ex$

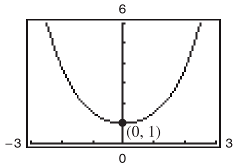
67. $y = (1/e)x - 1/e$ **69.** $\dfrac{10 - e^y}{xe^y + 3}$

71. $y = (-e - 1)x + 1$ **73.** $3(6x + 5)e^{-3x}$

75. $y'' - y = 0$
$4e^{-x} - 4e^{-x} = 0$

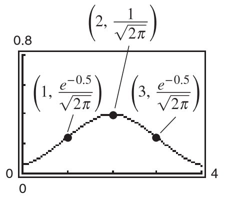
77. $y'' - 2y' + 3y = 0$
$e^x\left[-\cos\sqrt{2}x - \sin\sqrt{2}x - 2\sqrt{2}\sin\sqrt{2}x + 2\sqrt{2}\cos\sqrt{2}x\right] -$
$2e^x\left[-\sqrt{2}\sin\sqrt{2}x + \sqrt{2}\cos\sqrt{2}x + \cos\sqrt{2}x + \sin\sqrt{2}x\right] +$
$3e^x\left[\cos\sqrt{2}x + \sin\sqrt{2}x\right] = 0$
$0 = 0$

79. Relative minimum: $(0, 1)$



81. Relative maximum: $\left(2, 1/\sqrt{2\pi}\right)$

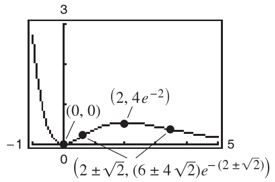
Points of inflection: $\left(1, \dfrac{e^{-0.5}}{\sqrt{2\pi}}\right), \left(3, \dfrac{e^{-0.5}}{\sqrt{2\pi}}\right)$



83. Relative minimum: $(0, 0)$

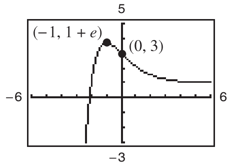
Relative maximum: $(2, 4e^{-2})$

Points of inflection: $\left(2 \pm \sqrt{2}, \left(6 \pm 4\sqrt{2}\right)e^{-(2 \pm \sqrt{2})}\right)$



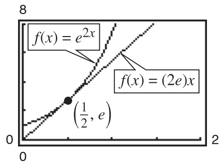
85. Relative maximum: $(-1, 1 + e)$

Point of inflection: $(0, 3)$

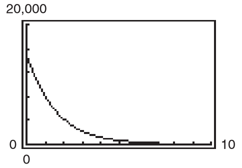


87. $A = \sqrt{2}e^{-1/2}$

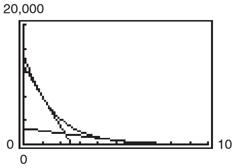
89. $\left(\frac{1}{2}, e\right)$



91. (a)



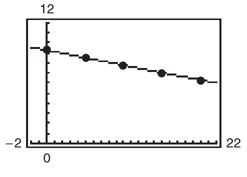
(c)



(b) When $t = 1$, $\dfrac{dV}{dt} \approx -5028.84$.

When $t = 5$, $\dfrac{dV}{dt} \approx -406.89$.

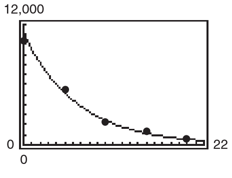
93. (a)



$\ln P = -0.1499h + 9.3018$

(b) $P = 10{,}957.7e^{-0.1499h}$

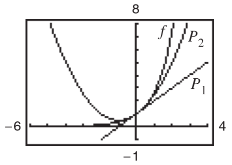
(c)



(d) $h = 5$: -776

$h = 18$: -111

95. $P_1 = 1 + x$; $P_2 = 1 + x + \frac{1}{2}x^2$



The values of f, P_1, and P_2 and their first derivatives agree at $x = 0$.

97. $12! = 479{,}001{,}600$

Stirling's Formula: $12! \approx 475{,}687{,}487$

99. $e^{5x} + C$ **101.** $\frac{1}{2}e^{2x-1} + C$ **103.** $\frac{1}{3}e^{x^3} + C$

105. $2e^{\sqrt{x}} + C$

107. $x - \ln(e^x + 1) + C_1$ or $-\ln(1 + e^{-x}) + C_2$

109. $-\frac{2}{3}(1 - e^x)^{3/2} + C$ **111.** $\ln|e^x - e^{-x}| + C$

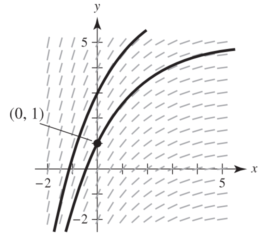
113. $-\frac{5}{2}e^{-2x} + e^{-x} + C$ **115.** $\ln|\cos e^{-x}| + C$

117. $(e^2 - 1)/(2e^2)$ **119.** $(e - 1)/(2e)$ **121.** $(e/3)(e^2 - 1)$

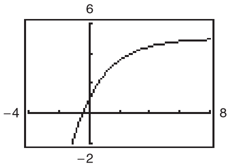
123. $\ln\left(\dfrac{1 + e^6}{2}\right)$ **125.** $(1/\pi)[e^{\sin(\pi^2/2)} - 1]$

127. $[1/(2a)]e^{ax^2} + C$ **129.** $f(x) = \frac{1}{2}(e^x + e^{-x})$

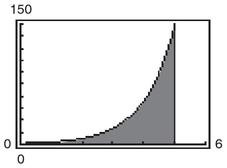
131. (a)

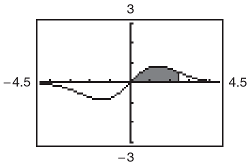


(b) $y = -4e^{-x/2} + 5$



133. $e^5 - 1 \approx 147.413$ **135.** $2(1 - e^{-3/2}) \approx 1.554$





137. Midpoint Rule: 92.190; Trapezoidal Rule: 93.837; Simpson's Rule: 92.7385

139. The probability that a given battery will last between 48 months and 60 months is approximately 47.72%.

141. (a) $t = \dfrac{1}{2k}\ln\dfrac{B}{A}$

(b) $x''(t) = k^2(Ae^{kt} + Be^{-kt})$, k^2 is the constant of proportionality.

143. $f(x) = e^x$

The domain of $f(x)$ is $(-\infty, \infty)$ and the range of $f(x)$ is $(0, \infty)$. $f(x)$ is continuous, increasing, one-to-one, and concave upward on its entire domain.

$\lim\limits_{x\to-\infty} e^x = 0$ and $\lim\limits_{x\to\infty} e^x = \infty$

145. (a) Log Rule (b) Substitution

147. $\displaystyle\int_0^x e^t\,dt \geq \int_0^x 1\,dt$; $e^x - 1 \geq x$; $e^x \geq x + 1$ for $x \geq 0$

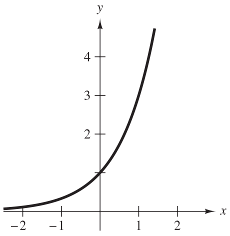
149. $x \approx 0.567$ **151.** Proof

Section 5.5 (page 368)

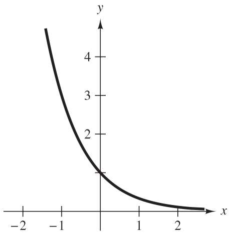
1. -3 **3.** 0 **5.** (a) $\log_2 8 = 3$ (b) $\log_3(1/3) = -1$

7. (a) $10^{-2} = 0.01$ (b) $\left(\frac{1}{2}\right)^{-3} = 8$

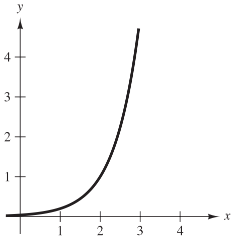
9.



11.



13.



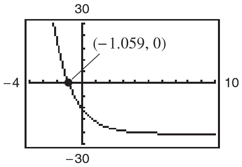
15. d **16.** c **17.** b **18.** a

19. (a) $x = 3$ (b) $x = -1$ **21.** (a) $x = \frac{1}{3}$ (b) $x = \frac{1}{16}$

23. (a) $x = -1, 2$ (b) $x = \frac{1}{3}$ **25.** 1.965 **27.** -6.288

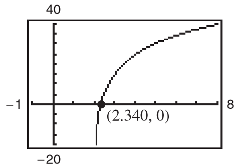
29. 12.253 **31.** 33.000 **33.** ± 11.845

35.



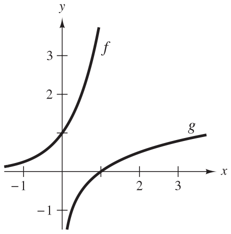
$(-1.059, 0)$

37.



$(2.340, 0)$

39.



41. $(\ln 4)4^x$ **43.** $(-4 \ln 5)5^{-4x}$

45. $9^x(x \ln 9 + 1)$ **47.** $t2^t(t \ln 2 + 2)$

49. $-2^{-\theta}[(\ln 2) \cos \pi\theta + \pi \sin \pi\theta]$

51. $5/[(\ln 4)(5x + 1)]$ **53.** $2/[(\ln 5)(t - 4)]$

55. $x/[(\ln 5)(x^2 - 1)]$ **57.** $(x - 2)/[(\ln 2)x(x - 1)]$

59. $(3x - 2)/[(2x \ln 3)(x - 1)]$

61. $5(1 - \ln t)/(t^2 \ln 2)$ **63.** $y = -2x \ln 2 - 2 \ln 2 + 2$

65. $y = [1/(27 \ln 3)]x + 3 - 1/\ln 3$ **67.** $2(1 - \ln x)x^{(2/x)-2}$

69. $(x - 2)^{x+1}[(x + 1)/(x - 2) + \ln(x - 2)]$

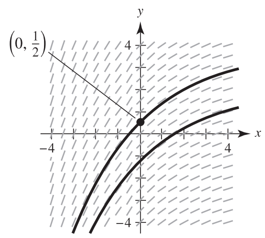
71. $y = x$ **73.** $y = \dfrac{\cos e}{e}x - \cos e + 1$ **75.** $3^x/\ln 3 + C$

77. $\dfrac{1}{3}x^3 - \dfrac{2^{-x}}{\ln 2} + C$ **79.** $[-1/(2 \ln 5)](5^{-x^2}) + C$

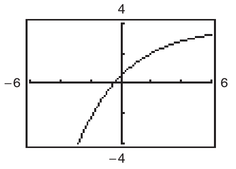
81. $\ln(3^{2x} + 1)/(2 \ln 3) + C$ **83.** $7/(2 \ln 2)$

85. $4/\ln 5 - 2/\ln 3$ **87.** $26/\ln 3$

89. (a)



(b) $y = \dfrac{3(1 - 0.4^{x/3})}{\ln 2.5} + \dfrac{1}{2}$



91. (a) $x > 0$ (b) 10^x (c) $3 \leq f(x) \leq 4$
(d) $0 < x < 1$ (e) 10 (f) 100^n

93. (a) ax^{a-1} (b) $(\ln a)a^x$ (c) $x^x(1 + \ln x)$ (d) 0

95. (a) $40.64 (b) $C'(1) \approx 0.051P$, $C'(8) \approx 0.072P$
(c) ln 1.05

97.

n	1	2	4	12
A	$1410.60	$1414.78	$1416.91	$1418.34

n	365	Continuous
A	$1419.04	$1419.07

99.

n	1	2	4	12
A	$4321.94	$4399.79	$4440.21	$4467.74

n	365	Continuous
A	$4481.23	$4481.69

101.

t	1	10	20	30
P	$95,122.94	$60,653.07	$36,787.94	$22,313.02

t	40	50
P	$13,533.53	$8208.50

103.

t	1	10	20	30
P	$95,132.82	$60,716.10	$36,864.45	$22,382.66

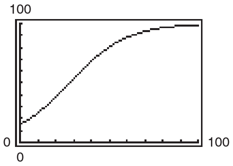
t	40	50
P	$13,589.88	$8251.24

105. c

107. (a) 6.7 million ft^3/acre

(b) $t = 20$: $\dfrac{dV}{dt} = 0.073$; $t = 60$: $\dfrac{dV}{dt} = 0.040$

109. (a)



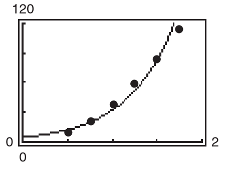
(b) 16.7%

(c) $x \approx 38.8$ or 38,800 egg masses

(d) $x \approx 27.75$ or 27,750 egg masses

111. (a) $B = 4.75(6.774)^d$

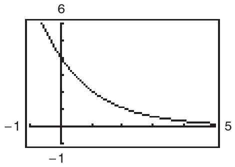
(b)



(c) When $d = 0.8$, the rate of growth is 41.99.
When $d = 1.5$, the rate of growth is 160.21.

113. (a) 5.67; 5.67; 5.67

(b)



(c) $f(t) = g(t) = h(t)$. No, because the definite integrals of two functions over a given interval may be equal even though the functions are not equal.

115. $y = 1200(0.6^t)$ **117.** e **119.** e^2

121. False: e is an irrational number. **123.** True **125.** True

127. (a) $(2^3)^2 = 2^6 = 64$

$2^{(3^2)} = 2^9 = 512$

(b) No. $f(x) = (x^x)^x = x^{(x^2)}$ and $g(x) = x^{(x^x)}$

(c) $f'(x) = x^{x^2}(x + 2x \ln x)$

$g'(x) = x^{x^x + x - 1}[x(\ln x)^2 + x \ln x + 1]$

129. Proof

131. (a) $\dfrac{dy}{dx} = \dfrac{y^2 - yx \ln y}{x^2 - xy \ln x}$

(b) (i) 1 when $c \neq 0, c \neq e$ (ii) -3.1774 (iii) -0.3147

(c) (e, e)

133. Putnam Problem A15, 1940

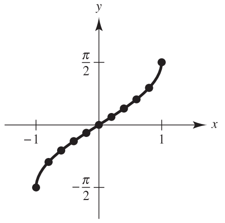
Section 5.6 (page 379)

1. (a)

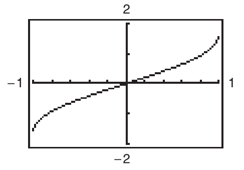
x	-1	-0.8	-0.6	-0.4	-0.2
y	-1.57	-0.93	-0.64	-0.41	-0.20

x	0	0.2	0.4	0.6	0.8	1
y	0	0.20	0.41	0.64	0.93	1.57

(b)



(c)



(d) Intercept: $(0, 0)$;
Symmetry: origin

3. $(-\sqrt{2}/2, 3\pi/4), (1/2, \pi/3), (\sqrt{3}/2, \pi/6)$

5. $\pi/6$ **7.** $\pi/3$ **9.** $\pi/6$ **11.** $-\pi/4$ **13.** 2.50

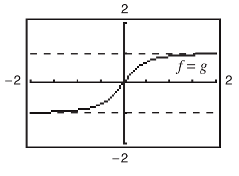
15. $\arccos(1/1.269) \approx 0.66$ **17.** (a) $3/5$ (b) $5/3$

19. (a) $-\sqrt{3}$ (b) $-\frac{13}{5}$ **21.** x **23.** $\sqrt{1 - x^2}/x$ **25.** $1/x$

27. $\sqrt{1 - 4x^2}$ **29.** $\sqrt{x^2 - 1}/|x|$

31. $\sqrt{x^2 - 9}/3$ **33.** $\sqrt{x^2 + 2}/x$

35. (a)



(b) Proof

(c) Horizontal asymptotes:
$y = -1, y = 1$

37. $x = \frac{1}{3}\left[\sin\left(\frac{1}{2}\right) + \pi\right] \approx 1.207$ **39.** $x = \frac{1}{3}$

41. (a) and (b) Proofs **43.** $2/\sqrt{2x - x^2}$

45. $-3/\sqrt{4 - x^2}$ **47.** $e^x/(1 + e^{2x})$

49. $\left(3x - \sqrt{1 - 9x^2} \arcsin 3x\right)/\left(x^2\sqrt{1 - 9x^2}\right)$

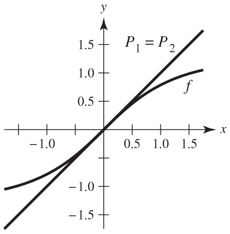
51. $-t/\sqrt{1 - t^2}$ **53.** $2 \arccos x$ **55.** $1/(1 - x^4)$

57. $\arcsin x$ **59.** $x^2/\sqrt{16 - x^2}$ **61.** $2/(1 + x^2)^2$

63. $y = \frac{1}{3}\left(4\sqrt{3}x - 2\sqrt{3} + \pi\right)$

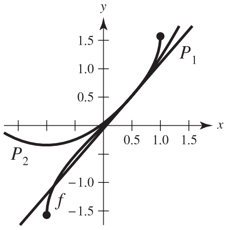
65. $y = \frac{1}{4}x + (\pi - 2)/4$ **67.** $y = (2\pi - 4)x + 4$

69. $P_1(x) = x$; $P_2(x) = x$



71. $P_1(x) = \dfrac{\pi}{6} + \dfrac{2\sqrt{3}}{3}\left(x - \dfrac{1}{2}\right)$

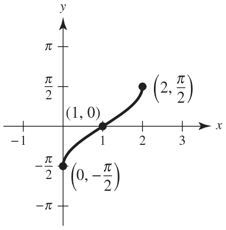
$P_2(x) = \dfrac{\pi}{6} + \dfrac{2\sqrt{3}}{3}\left(x - \dfrac{1}{2}\right) + \dfrac{2\sqrt{3}}{9}\left(x - \dfrac{1}{2}\right)^2$



73. Relative maximum: $(1.272, -0.606)$
Relative minimum: $(-1.272, 3.747)$

75. Relative maximum: $(2, 2.214)$

77.

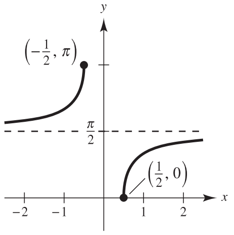


Maximum: $\left(2, \dfrac{\pi}{2}\right)$

Minimum: $\left(0, -\dfrac{\pi}{2}\right)$

Point of inflection: $(1, 0)$

79.



Maximum: $\left(-\dfrac{1}{2}, \pi\right)$

Minimum: $\left(\dfrac{1}{2}, 0\right)$

Asymptote: $y = \dfrac{\pi}{2}$

81. $y = -2\pi x/(\pi + 8) + 1 - \pi^2/(2\pi + 16)$

83. $y = -x + \sqrt{2}$

85. If the domains were not restricted, the trigonometric functions would have no inverses because they would not be one-to-one.

87. If $x > 0$, $y = \operatorname{arccot} x = \arctan \dfrac{1}{x}$; If $x < 0$, $y = \arctan \dfrac{1}{x} + \pi$.

89. (a) $\arcsin(\arcsin 0.5) \approx 0.551$

$\arcsin(\arcsin 1)$ does not exist.

(b) $\sin(-1) \leq x \leq \sin(1)$

91. False. The range of arccos is $[0, \pi]$. **93.** True **95.** True

97. (a) $\theta = \operatorname{arccot}(x/5)$

(b) $x = 10$: 16 rad/h; $x = 3$: 58.824 rad/h

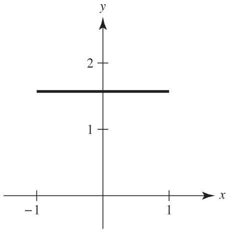
99. (a) $h(t) = -16t^2 + 256$; $t = 4$ sec

(b) $t = 1$: -0.0520 rad/sec; $t = 2$: -0.1116 rad/sec

101. $50\sqrt{2} \approx 70.71$ ft **103.** (a) and (b) Proofs

105. $k \le -1$ or $k \ge 1$

107. (a)

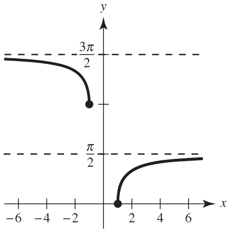


(b) The graph is a horizontal line at $\frac{\pi}{2}$.

(c) Proof

109. $c = 2$

111. (a)



(b) Proof

Section 5.7 (page 387)

1. $\arcsin \frac{x}{3} + C$ **3.** $\frac{7}{4} \arctan \frac{x}{4} + C$ **5.** $\operatorname{arcsec}|2x| + C$

7. $\arcsin(x + 1) + C$ **9.** $\frac{1}{2} \arcsin t^2 + C$

11. $\frac{1}{10} \arctan \frac{t^2}{5} + C$ **13.** $\frac{1}{4} \arctan(e^{2x}/2) + C$

15. $\arcsin\left(\frac{\tan x}{5}\right) + C$ **17.** $\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2 + 1) + C$

19. $2 \arcsin \sqrt{x} + C$ **21.** $\frac{1}{2}\ln(x^2 + 1) - 3 \arctan x + C$

23. $8 \arcsin[(x - 3)/3] - \sqrt{6x - x^2} + C$ **25.** $\pi/6$

27. $\pi/6$ **29.** $\frac{1}{2}(\sqrt{3} - 2) \approx -0.134$

31. $\frac{1}{5} \arctan \frac{3}{5} \approx 0.108$ **33.** $\arctan 5 - \frac{\pi}{4} \approx 0.588$

35. $\pi/4$ **37.** $\frac{1}{32}\pi^2 \approx 0.308$ **39.** $\pi/2$

41. $\ln|x^2 + 6x + 13| - 3 \arctan[(x + 3)/2] + C$

43. $\arcsin[(x + 2)/2] + C$ **45.** $-\sqrt{-x^2 - 4x} + C$

47. $4 - 2\sqrt{3} + \frac{1}{6}\pi \approx 1.059$ **49.** $\frac{1}{2}\arctan(x^2 + 1) + C$

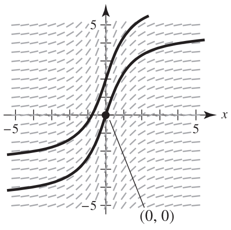
51. $2\sqrt{e^t - 3} - 2\sqrt{3} \arctan\left(\sqrt{e^t - 3}/\sqrt{3}\right) + C$ **53.** $\pi/6$

55. a and b **57.** a, b, and c

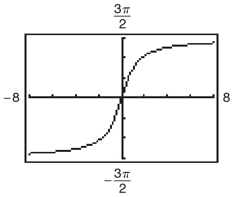
59. No. This integral does not correspond to any of the basic integration rules.

61. $y = \arcsin(x/2) + \pi$

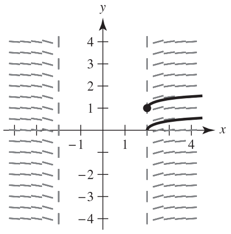
63. (a)



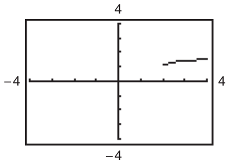
(b) $y = 3 \arctan x$



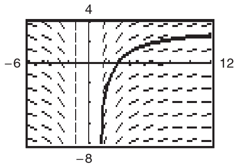
65. (a)



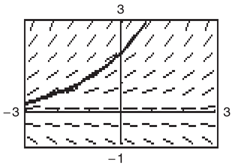
(b) $y = \frac{1}{2} \operatorname{arcsec}(x/2) + 1$, $x \ge 2$



67.



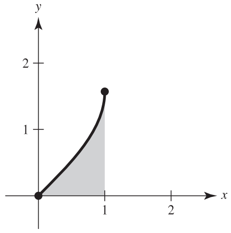
69.



71. $\pi/3$ **73.** $\pi/8$ **75.** $3\pi/2$

77. (a) Proof (b) $\ln\left(\sqrt{6}/2\right) + \left(9\pi - 4\pi\sqrt{3}\right)/36$

79. (a)



(b) 0.5708

(c) $(\pi - 2)/2$

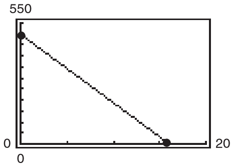
81. (a) $F(x)$ represents the average value of $f(x)$ over the interval $[x, x + 2]$. Maximum at $x = -1$.

(b) Maximum at $x = -1$.

83. False. $\displaystyle\int \frac{dx}{3x\sqrt{9x^2 - 16}} = \frac{1}{12} \operatorname{arcsec} \frac{|3x|}{4} + C$

85. True **87–89.** Proofs

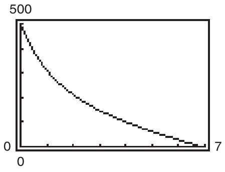
91. (a) $v(t) = -32t + 500$



(b) $s(t) = -16t^2 + 500t$; 3906.25 ft

(c) $v(t) = \sqrt{\frac{32}{k}} \tan\left[\arctan\left(500\sqrt{\frac{k}{32}}\right) - \sqrt{32k}\,t\right]$

(d)



(e) 1088 ft

(f) When air resistance is taken into account, the maximum height of the object is not as great.

$t_0 = 6.86$ sec

Section 5.8 (page 398)

1. (a) 10.018 (b) −0.964 **3.** (a) $\frac{4}{3}$ (b) $\frac{13}{12}$

5. (a) 1.317 (b) 0.962 **7–15.** Proofs

17. $\cosh x = \sqrt{13}/2$; $\tanh x = 3\sqrt{13}/13$; $\operatorname{csch} x = 2/3$; $\operatorname{sech} x = 2\sqrt{13}/13$; $\coth x = \sqrt{13}/3$

19. $3 \cosh 3x$ **21.** $-10x[\operatorname{sech}(5x^2)\tanh(5x^2)]$ **23.** $\coth x$

25. $\operatorname{csch} x$ **27.** $\sinh^2 x$ **29.** $\operatorname{sech} t$

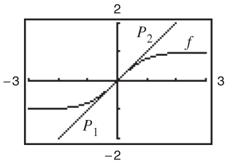
31. $y = -2x + 2$ **33.** $y = 1 - 2x$

35. Relative maxima: $(\pm\pi, \cosh \pi)$; Relative minimum: $(0, -1)$

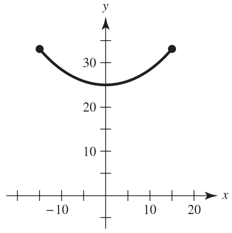
37. Relative maximum: $(1.20, 0.66)$
Relative minimum: $(-1.20, -0.66)$

39. $y = a \sinh x$; $y' = a \cosh x$; $y'' = a \sinh x$; $y''' = a \cosh x$; Therefore, $y''' - y' = 0$.

41. $P_1(x) = x$; $P_2(x) = x$



43. (a)



(b) 33.146 units; 25 units
(c) $m = \sinh(1) \approx 1.175$

45. $\frac{1}{2} \sinh 2x + C$ **47.** $-\frac{1}{2}\cosh(1 - 2x) + C$

49. $\frac{1}{3} \cosh^3(x - 1) + C$ **51.** $\ln|\sinh x| + C$

53. $-\coth(x^2/2) + C$ **55.** $\operatorname{csch}(1/x) + C$

57. $\frac{1}{2} \arctan x^2 + C$ **59.** $\ln(5/4)$ **61.** $\frac{1}{5} \ln 3$

63. $\pi/4$ **65.** $3/\sqrt{9x^2 - 1}$ **67.** $\dfrac{1}{2\sqrt{x}(1 - x)}$ **69.** $|\sec x|$

71. $\dfrac{-2 \operatorname{csch}^{-1} x}{|x|\sqrt{1 + x^2}}$ **73.** $2 \sinh^{-1}(2x)$ **75.** Answers will vary.

77. $\cosh x$, $\operatorname{sech} x$ **79.** ∞ **81.** 1 **83.** 0 **85.** 1

87. $\dfrac{\sqrt{3}}{18} \ln\left|\dfrac{1 + \sqrt{3}x}{1 - \sqrt{3}x}\right| + C$ **89.** $\ln\left(\sqrt{e^{2x} + 1} - 1\right) - x + C$

91. $2 \sinh^{-1}\sqrt{x} + C = 2 \ln\left(\sqrt{x} + \sqrt{1 + x}\right) + C$

93. $\dfrac{1}{4}\ln\left|\dfrac{x - 4}{x}\right| + C$ **95.** $\dfrac{1}{2\sqrt{6}} \ln\left|\dfrac{\sqrt{2}(x + 1) + \sqrt{3}}{\sqrt{2}(x + 1) - \sqrt{3}}\right| + C$

97. $\ln\left(\dfrac{3 + \sqrt{5}}{2}\right)$ **99.** $\dfrac{\ln 7}{12}$ **101.** $\dfrac{1}{4} \arcsin\left(\dfrac{4x - 1}{9}\right) + C$

103. $-\dfrac{x^2}{2} - 4x - \dfrac{10}{3} \ln\left|\dfrac{x - 5}{x + 1}\right| + C$

105. $8 \arctan(e^2) - 2\pi \approx 5.207$ **107.** $\frac{5}{2} \ln\left(\sqrt{17} + 4\right) \approx 5.237$

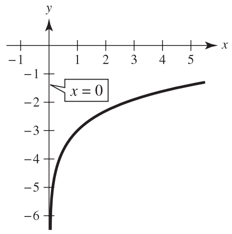
109. (a) $\ln\left(\sqrt{3} + 2\right)$ (b) $\sinh^{-1}\sqrt{3}$

111. $\frac{52}{31}$ kg **113.** $-\sqrt{a^2 - x^2}/x$ **115–123.** Proofs

125. Putnam Problem 8, 1939

Review Exercises for Chapter 5 (page 401)

1.



Vertical asymptote: $x = 0$

3. $\frac{1}{5}[\ln(2x + 1) + \ln(2x - 1) - \ln(4x^2 + 1)]$

5. $\ln\left(3\sqrt[3]{4 - x^2}/x\right)$ **7.** $e^4 - 1 \approx 53.598$

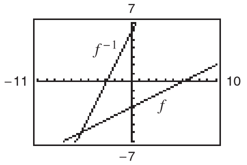
9. $1/(2x)$ **11.** $(1 + 2 \ln x)/\left(2\sqrt{\ln x}\right)$

13. $\dfrac{dy}{dx} = \dfrac{1}{b^2}\left(b - \dfrac{ab}{a + bx}\right) = \dfrac{x}{a + bx}$ **15.** $y = -x + 1$

17. $\frac{1}{7} \ln|7x - 2| + C$ **19.** $-\ln|1 + \cos x| + C$

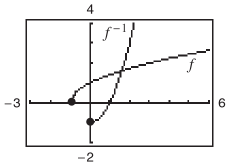
21. $3 + \ln 2$ **23.** $\ln\left(2 + \sqrt{3}\right)$

25. (a) $f^{-1}(x) = 2x + 6$
(b)



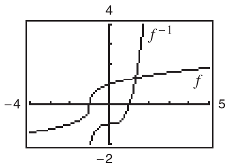
(c) Proof
(d) Domain of f and f^{-1}: all real numbers
Range of f and f^{-1}: all real numbers

27. (a) $f^{-1}(x) = x^2 - 1, \quad x \geq 0$
(b)



(c) Proof
(d) Domain of f: $x \geq -1$
Domain of f^{-1}: $x \geq 0$
Range of f: $y \geq 0$
Range of f^{-1}: $y \geq -1$

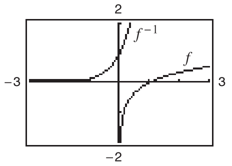
29. (a) $f^{-1}(x) = x^3 - 1$
(b)



(c) Proof
(d) Domain of f and f^{-1}: all real numbers
Range of f and f^{-1}: all real numbers

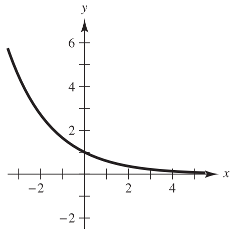
31. $1/\left[3\left(\sqrt[3]{-3}\right)^2\right] \approx 0.160$ **33.** $3/4$

35. (a) $f^{-1}(x) = e^{2x}$
(b)



(c) Proof
(d) Domain of f: $x > 0$
Domain of f^{-1}: all real numbers
Range of f: all real numbers
Range of f^{-1}: $y > 0$

37.



39. $te^t(t + 2)$

41. $(e^{2x} - e^{-2x})/\sqrt{e^{2x} + e^{-2x}}$ **43.** $x(2 - x)/e^x$

45. $y = -4x + 4$ **47.** $-y/[x(2y + \ln x)]$

49. $(1 - e^{-3})/6 \approx 0.158$ **51.** $(e^{4x} - 3e^{2x} - 3)/(3e^x) + C$

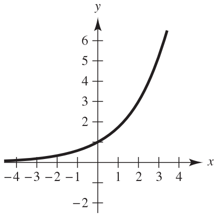
53. $-\frac{1}{2}e^{1-x^2} + C$ **55.** $\ln(e^2 + e + 1) \approx 2.408$

57. $y = e^x(a\cos 3x + b\sin 3x)$

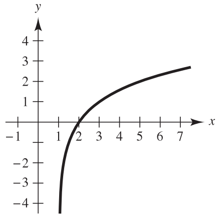
$$\begin{aligned} y' &= e^x[(-3a + b)\sin 3x + (a + 3b)\cos 3x] \\ y'' &= e^x[(-6a - 8b)\sin 3x + (-8a + 6b)\cos 3x] \\ y'' &- 2y' + 10y \\ &= e^x\{[(-6a - 8b) - 2(-3a + b) + 10b]\sin 3x + \\ &\quad [(-8a + 6b) - 2(a + 3b) + 10a]\cos 3x\} = 0 \end{aligned}$$

59. $-\frac{1}{2}(e^{-16} - 1) \approx 0.500$

61.



63.

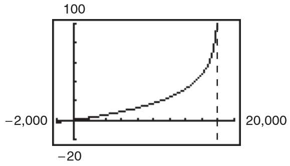


65. $3^{x-1}\ln 3$ **67.** $x^{2x+1}(2\ln x + 2 + 1/x)$

69. $-1/[\ln 3(2 - 2x)]$ **71.** $5^{(x+1)^2}/(2\ln 5) + C$

73. (a) Domain: $0 \le h < 18{,}000$

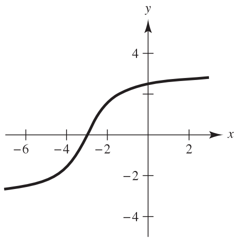
(b)



(c) $t = 0$

Vertical asymptote: $h = 18{,}000$

75.



77. (a) $1/2$ (b) $\sqrt{3}/2$

79. $(1 - x^2)^{-3/2}$ **81.** $\dfrac{x}{|x|\sqrt{x^2 - 1}} + \operatorname{arcsec} x$ **83.** $(\arcsin x)^2$

85. $\frac{1}{2}\arctan(e^{2x}) + C$ **87.** $\frac{1}{2}\arcsin x^2 + C$

89. $\frac{1}{4}[\arctan(x/2)]^2 + C$ **91.** $\frac{2}{3}\pi + \sqrt{3} - 2 \approx 1.826$

93. $y = A\sin\left(t\sqrt{k/m}\right)$

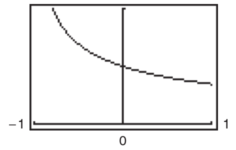
95. $y' = x\left(\dfrac{2}{1 - 4x^2}\right) + \tanh^{-1} 2x = \dfrac{2x}{1 - 4x^2} + \tanh^{-1} 2x$

97. $\frac{1}{3}\tanh x^3 + C$

P.S. Problem Solving (page 403)

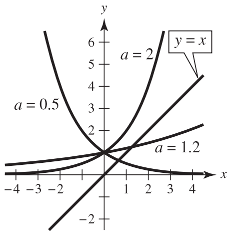
1. $a \approx 4.7648$; $\theta \approx 1.7263$ or 98.9°

3. (a)



(b) 1 (c) Proof

5.



$y = 0.5^x$ and $y = 1.2^x$ intersect the line $y = x$; $0 < a < e^{1/e}$

7. $e - 1$

9. (a) Area of region $A = \left(\sqrt{3} - \sqrt{2}\right)/2 \approx 0.1589$

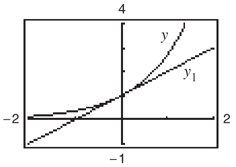
Area of region $B = \pi/12 \approx 0.2618$

(b) $\frac{1}{24}\left[3\pi\sqrt{2} - 12\left(\sqrt{3} - \sqrt{2}\right) - 2\pi\right] \approx 0.1346$

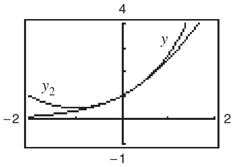
(c) 1.2958 (d) 0.6818

11. Proof **13.** $2\ln\frac{3}{2} \approx 0.8109$

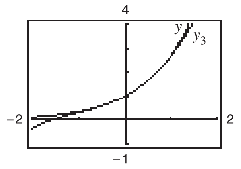
15. (a) (i)



(ii)

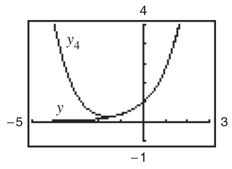


(iii)



(b) Pattern: $y_n = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots$

$$y_4 = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$



(c) The pattern implies that $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$.

Chapter 6

Section 6.1 (page 411)

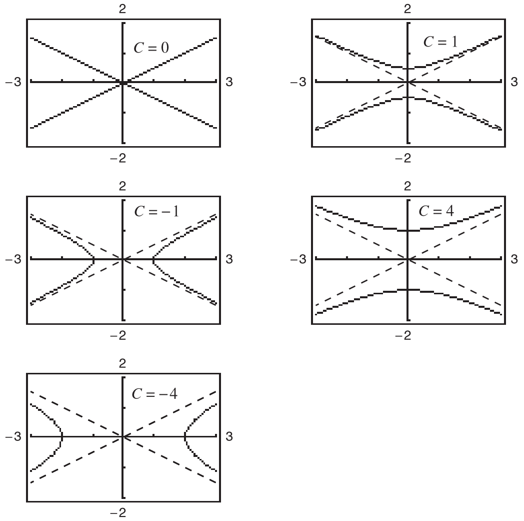
1–11. Proofs **13.** Not a solution **15.** Solution

17. Solution **19.** Solution **21.** Not a solution

23. Solution **25.** Not a solution **27.** Not a solution

29. $y = 3e^{-x/2}$ **31.** $4y^2 = x^3$

33.



35. $y = 3e^{-2x}$ **37.** $y = 2\sin 3x - \frac{1}{3}\cos 3x$

39. $y = -2x + \frac{1}{2}x^3$ **41.** $2x^3 + C$

43. $y = \frac{1}{2}\ln(1 + x^2) + C$ **45.** $y = x - \ln x^2 + C$

47. $y = -\frac{1}{2}\cos 2x + C$

49. $y = \frac{2}{5}(x - 6)^{5/2} + 4(x - 6)^{3/2} + C$

51. $y = \frac{1}{2}e^{x^2} + C$

53.

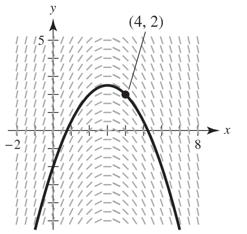
x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-4	Undef.	0	1	$\frac{4}{3}$	2

55.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	$-2\sqrt{2}$	-2	0	0	$-2\sqrt{2}$	-8

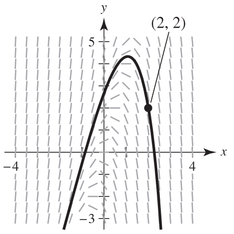
57. b **58.** c **59.** d **60.** a

61. (a) and (b)



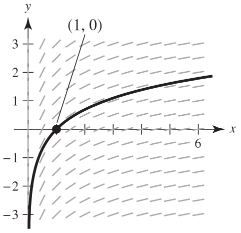
(c) As $x \to \infty$, $y \to -\infty$; as $x \to -\infty$, $y \to -\infty$

63. (a) and (b)



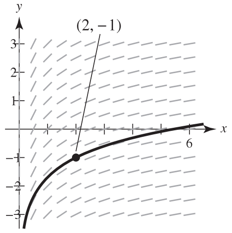
(c) As $x \to \infty$, $y \to -\infty$; as $x \to -\infty$, $y \to -\infty$

65. (a)



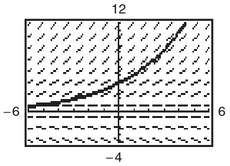
As $x \to \infty$, $y \to \infty$

(b)

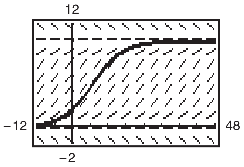


As $x \to \infty$, $y \to \infty$

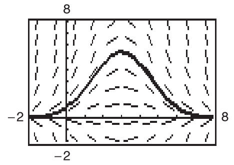
67. (a) and (b)



69. (a) and (b)



71. (a) and (b)



73.

n	0	1	2	3	4	5	6
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6
y_n	2	2.2	2.43	2.693	2.992	3.332	3.715

n	7	8	9	10
x_n	0.7	0.8	0.9	1.0
y_n	4.146	4.631	5.174	5.781

75.

n	0	1	2	3	4	5	6
x_n	0	0.05	0.1	0.15	0.2	0.25	0.3
y_n	3	2.7	2.438	2.209	2.010	1.839	1.693

n	7	8	9	10
x_n	0.35	0.4	0.45	0.5
y_n	1.569	1.464	1.378	1.308

77.

n	0	1	2	3	4	5	6
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6
y_n	1	1.1	1.212	1.339	1.488	1.670	1.900

n	7	8	9	10
x_n	0.7	0.8	0.9	1.0
y_n	2.213	2.684	3.540	5.958

79.

x	0	0.2	0.4	0.6	0.8	1
$y(x)$ (exact)	3.0000	3.6642	4.4755	5.4664	6.6766	8.1548
$y(x)$ ($h = 0.2$)	3.0000	3.6000	4.3200	5.1840	6.2208	7.4650
$y(x)$ ($h = 0.1$)	3.0000	3.6300	4.3923	5.3147	6.4308	7.7812

81.

x	0	0.2	0.4	0.6	0.8	1
$y(x)$ (exact)	0.0000	0.2200	0.4801	0.7807	1.1231	1.5097
$y(x)$ $(h = 0.2)$	0.0000	0.2000	0.4360	0.7074	1.0140	1.3561
$y(x)$ $(h = 0.1)$	0.0000	0.2095	0.4568	0.7418	1.0649	1.4273

83. (a) $y(1) = 112.7141°$; $y(2) = 96.3770°$; $y(3) = 86.5954°$
(b) $y(1) = 113.2441°$; $y(2) = 97.0158°$; $y(3) = 87.1729°$
(c) Euler's Method: $y(1) = 112.9828°$; $y(2) = 96.6998°$; $y(3) = 86.8863°$
Exact solution: $y(1) = 113.2441°$; $y(2) = 97.0158°$; $y(3) = 87.1729°$
The approximations are better using $h = 0.05$.

85. The general solution is a family of curves that satisfies the differential equation. A particular solution is one member of the family that satisfies given conditions.

87. Begin with a point (x_0, y_0) that satisfies the initial condition $y(x_0) = y_0$. Then, using a small step size h, calculate the point $(x_1, y_1) = (x_0 + h, y_0 + hF(x_0, y_0))$. Continue generating the sequence of points $(x_n + h, y_n + hF(x_n, y_n))$ or (x_{n+1}, y_{n+1}).

89. False: $y = x^3$ is a solution of $xy' - 3y = 0$, but $y = x^3 + 1$ is not a solution.

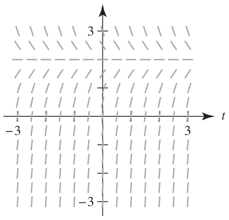
91. True

93. (a)

x	0	0.2	0.4	0.6	0.8	1
y	4	2.6813	1.7973	1.2048	0.8076	0.5413
y_1	4	2.56	1.6384	1.0486	0.6711	0.4295
y_2	4	2.4	1.44	0.864	0.5184	0.3110
e_1	0	0.1213	0.1589	0.1562	0.1365	0.1118
e_2	0	0.2813	0.3573	0.3408	0.2892	0.2303
r		0.4312	0.4447	0.4583	0.4720	0.4855

(b) If h is halved, then the error is approximately halved because r is approximately 0.5.
(c) The error will again be halved.

95. (a)



(b) $\lim_{t \to \infty} I(t) = 2$

97. $\omega = \pm 4$ **99.** Putnam Problem 3, Morning Session, 1954

Section 6.2 (page 420)

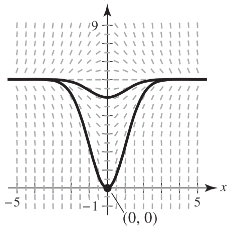
1. $y = \frac{1}{2}x^2 + 3x + C$ **3.** $y = Ce^x - 3$ **5.** $y^2 - 5x^2 = C$

7. $y = Ce^{(2x^{3/2})/3}$ **9.** $y = C(1 + x^2)$

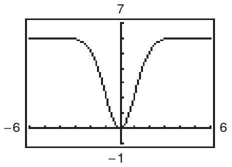
11. $dQ/dt = k/t^2$
$Q = -k/t + C$

13. $dN/ds = k(500 - s)$
$N = -(k/2)(500 - s)^2 + C$

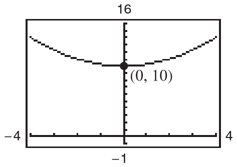
15. (a)



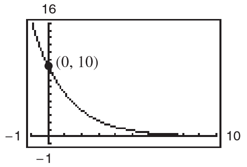
(b) $y = 6 - 6e^{-x^2/2}$



17. $y = \frac{1}{4}t^2 + 10$



19. $y = 10e^{-t/2}$



21. $dy/dx = ky$
$y = 6e^{(1/4)\ln(5/2)x} \approx 6e^{0.2291x}$
$y(8) \approx 37.5$

23. $dV/dt = kV$
$V = 20{,}000e^{(1/4)\ln(5/8)t} \approx 20{,}000e^{-0.1175t}$
$V(6) \approx 9882$

25. $y = (1/2)e^{[(\ln 10)/5]t} \approx (1/2)e^{0.4605t}$

27. $y = 5(5/2)^{1/4}e^{[\ln(2/5)/4]t} \approx 6.2872e^{-0.2291t}$

29. C is the initial value of y, and k is the proportionality constant.

31. Quadrants I and III; dy/dx is positive when both x and y are positive (Quadrant I) or when both x and y are negative (Quadrant III).

33. Amount after 1000 yr: 12.96 g;
Amount after 10,000 yr: 0.26 g

35. Initial quantity: 7.63 g;
Amount after 1000 yr: 4.95 g

37. Amount after 1000 yr: 4.43 g;
Amount after 10,000 yr: 1.49 g

39. Initial quantity: 2.16 g;
Amount after 10,000 yr: 1.62 g

41. 95.76%

43. Time to double: 11.55 yr; Amount after 10 yr: $7288.48

45. Annual rate: 8.94%; Amount after 10 yr: $1833.67

47. Annual rate: 9.50%; Time to double: 7.30 yr

49. $224,174.18 **51.** $61,377.75

53. (a) 10.24 yr (b) 9.93 yr (c) 9.90 yr (d) 9.90 yr

55. (a) 8.50 yr (b) 8.18 yr (c) 8.16 yr (d) 8.15 yr

57. (a) $P = 2.40e^{-0.006t}$ (b) 2.19 million
(c) Because $k < 0$, the population is decreasing.

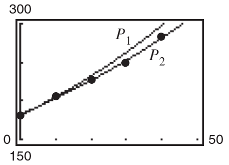
59. (a) $P = 5.66e^{0.024t}$ (b) 8.11 million
(c) Because $k > 0$, the population is increasing.

61. (a) $P = 23.55e^{0.036t}$ (b) 40.41 million
(c) Because $k > 0$, the population is increasing.

63. (a) $N = 100.1596(1.2455)^t$ (b) 6.3 h

65. (a) $N \approx 30(1 - e^{-0.0502t})$ (b) 36 days

67. (a) $P_1 \approx 181e^{0.01245t} \approx 181(1.01253)^t$
(b) $P_2 = 182.3248(1.01091)^t$
(c) (d) 2011



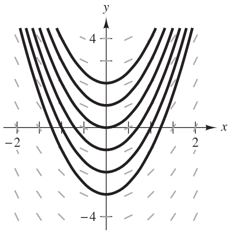
P_2 is a better approximation.

69. (a) 20 dB (b) 70 dB (c) 95 dB (d) 120 dB
71. 2024 ($t = 16$) **73.** 379.2°F
75. False. The rate of growth dy/dx is proportional to y.
77. True

Section 6.3 (page 431)

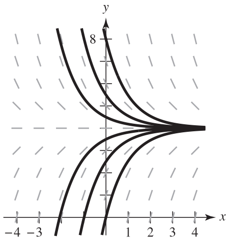
1. $y^2 - x^2 = C$ **3.** $15y^2 + 2x^3 = C$ **5.** $r = Ce^{0.75s}$
7. $y = C(x + 2)^3$ **9.** $y^2 = C - 8\cos x$
11. $y = -\frac{1}{4}\sqrt{1 - 4x^2} + C$ **13.** $y = Ce^{(\ln x)^2/2}$
15. $y^2 = 4e^x + 5$ **17.** $y = e^{-(x^2+2x)/2}$
19. $y^2 = 4x^2 + 3$ **21.** $u = e^{(1-\cos v^2)/2}$ **23.** $P = P_0e^{kt}$
25. $4y^2 - x^2 = 16$ **27.** $y = \frac{1}{3}\sqrt{x}$ **29.** $f(x) = Ce^{-x/2}$
31. Homogeneous of degree 3 **33.** Homogeneous of degree 3
35. Not homogeneous **37.** Homogeneous of degree 0
39. $|x| = C(x - y)^2$ **41.** $|y^2 + 2xy - x^2| = C$
43. $y = Ce^{-x^2/(2y^2)}$ **45.** $e^{y/x} = 1 + \ln x^2$ **47.** $x = e^{\sin(y/x)}$

49.



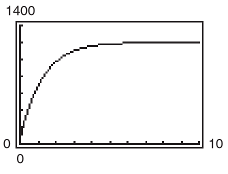
$y = \frac{1}{2}x^2 + C$

51.

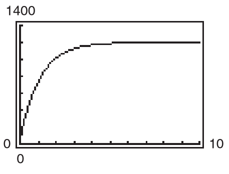


$y = 4 + Ce^{-x}$

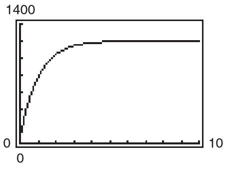
53. (a) $y \approx 0.1602$ (b) $y = 5e^{-3x^2}$ (c) $y \approx 0.2489$
55. (a) $y \approx 3.0318$ (b) $y^3 - 4y = x^2 + 12x - 13$ (c) $y = 3$
57. 97.9% of the original amount
59. (a) $dy/dx = k(y - 4)$ (b) a (c) Proof
60. (a) $dy/dx = k(x - 4)$ (b) b (c) Proof
61. (a) $dy/dx = ky(y - 4)$ (b) c (c) Proof
62. (a) $dy/dx = ky^2$ (b) d (c) Proof
63. (a) $w = 1200 - 1140e^{-0.8t}$



$w = 1200 - 1140e^{-0.9t}$

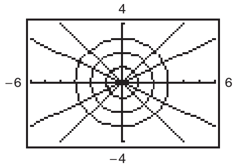


$w = 1200 - 1140e^{-t}$

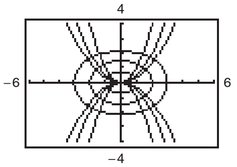


(b) 1.31 yr; 1.16 yr; 1.05 yr (c) 1200 lb

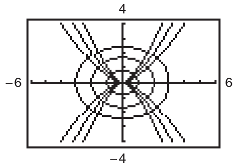
65. Circles: $x^2 + y^2 = C$
Lines: $y = Kx$
Graphs will vary.



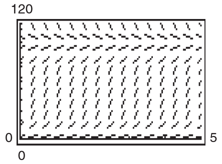
67. Parabolas: $x^2 = Cy$
Ellipses: $x^2 + 2y^2 = K$
Graphs will vary.



69. Curves: $y^2 = Cx^3$
Ellipses: $2x^2 + 3y^2 = K$
Graphs will vary.



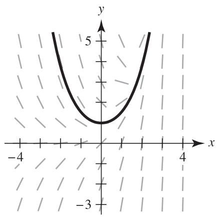
71. d **72.** a **73.** b **74.** c
75. (a) 0.75 (b) 2100 (c) 70 (d) 4.49 yr
(e) $dP/dt = 0.75P(1 - P/2100)$
77. (a) 3 (b) 100
(c) (d) 50

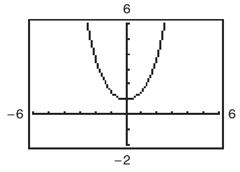


79. $y = 36/(1 + 8e^{-t})$ **81.** $y = 120/(1 + 14e^{-0.8t})$
83. (a) $P = \dfrac{200}{1 + 7e^{-0.2640t}}$ (b) 70 panthers (c) 7.37 yr
(d) $dP/dt = 0.2640P(1 - P/200)$; 65.6 (e) 100 yr
85. Answers will vary.
87. Two families of curves are mutually orthogonal if each curve in the first family intersects each curve in the second family at right angles.
89. Proof
91. False. $y' = x/y$ is separable, but $y = 0$ is not a solution.
93. False: $f(tx, ty) \neq t^nf(x, y)$. **95.** Putnam Problem A2, 1988

Section 6.4 (page 440)

1. Linear; can be written in the form $dy/dx + P(x)y = Q(x)$
3. Not linear; cannot be written in the form $dy/dx + P(x)y = Q(x)$
5. $y = 2x^2 + x + C/x$ **7.** $y = -16 + Ce^x$
9. $y = -1 + Ce^{\sin x}$ **11.** $y = (x^3 - 3x + C)/[3(x - 1)]$
13. $y = e^{x^3}(x + C)$
15. (a) Answers will vary. (c)





(b) $y = \frac{1}{2}(e^x + e^{-x})$
17. $y = 1 + 4/e^{\tan x}$ **19.** $y = \sin x + (x + 1)\cos x$
21. $xy = 4$ **23.** $y = -2 + x\ln|x| + 12x$

25. $P = -N/k + (N/k + P_0)e^{kt}$

27. (a) \$4,212,796.94 (b) \$31,424,909.75

29. (a) $\dfrac{dQ}{dt} = q - kQ$ (b) $Q = \dfrac{q}{k} + \left(Q_0 - \dfrac{q}{k}\right)e^{-kt}$ (c) $\dfrac{q}{k}$

31. Proof

33. (a) $Q = 25e^{-t/20}$ (b) $-20 \ln\left(\frac{3}{5}\right) \approx 10.2$ min (c) 0

35. (a) $t = 50$ min (b) $100 - \dfrac{25}{\sqrt{2}} \approx 82.32$ lb

(c) $t = 50$ min; $200 - \dfrac{50}{\sqrt{2}} \approx 164.64$ lb

37. $v(t) = -159.47(1 - e^{-0.2007t})$; -159.47 ft/sec

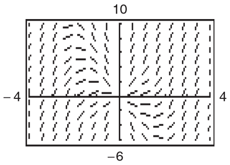
39. $I = \dfrac{E_0}{R} + Ce^{-Rt/L}$ **41.** $\dfrac{dy}{dx} + P(x)y = Q(x)$; $u(x) = e^{\int P(x)\,dx}$

43. c **44.** d **45.** a **46.** b

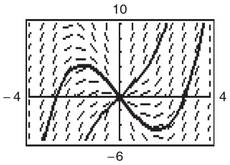
47. $1/y^2 = Ce^{2x^3} + \frac{1}{3}$ **49.** $y = 1/(Cx - x^2)$

51. $1/y^2 = 2x + Cx^2$ **53.** $y^{2/3} = 2e^x + Ce^{2x/3}$

55. (a)



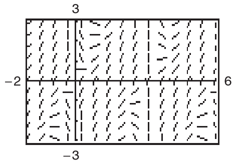
(c)



(b) $(-2, 4)$: $y = \frac{1}{2}x(x^2 - 8)$

$(2, 8)$: $y = \frac{1}{2}x(x^2 + 4)$

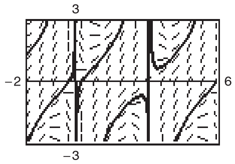
57. (a)



(b) $(1, 1)$: $y = (2 \cos 1 + \sin 1) \csc x - 2 \cot x$

$(3, -1)$: $y = (2 \cos 3 - \sin 3) \csc x - 2 \cot x$

(c)



59. $2e^x + e^{-2y} = C$ **61.** $y = Ce^{-\sin x} + 1$

63. $x^3y^2 + x^4y = C$ **65.** $y = [e^x(x - 1) + C]/x^2$

67. $x^4y^4 - 2x^2 = C$ **69.** $y = \frac{12}{5}x^2 + C/x^3$

71. False. $y' + xy = x^2$ is linear.

Review Exercises for Chapter 6 (page 443)

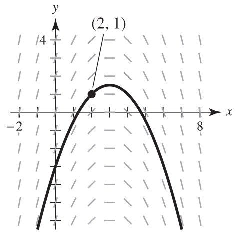
1. Yes **3.** $y = \frac{4}{3}x^3 + 7x + C$ **5.** $y = \frac{1}{2} \sin 2x + C$

7. $y = \frac{2}{5}(x - 5)^{5/2} + \frac{10}{3}(x - 5)^{3/2} + C$ **9.** $y = -e^{2-x} + C$

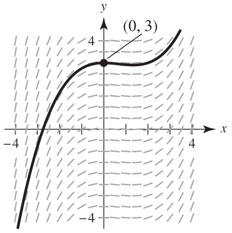
11.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-10	-4	-4	0	2	8

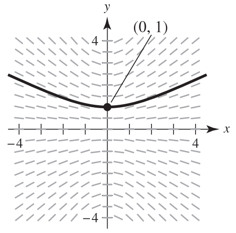
13. (a) and (b)



15. (a) and (b)



17. (a) and (b)



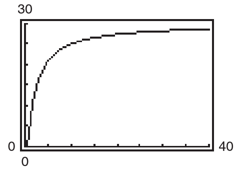
19. $y = 8x - \frac{1}{2}x^2 + C$

21. $y = -3 - 1/(x + C)$ **23.** $y = Ce^x/(2 + x)^2$

25. $y \approx \frac{3}{4}e^{0.379t}$ **27.** $y \approx 5e^{-0.680t}$ **29.** About 7.79 in.

31. (a) $S \approx 30e^{-1.7918/t}$ (b) 20,965 units

(c)

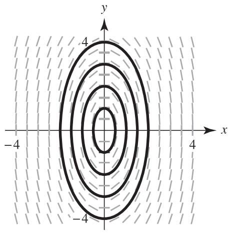


33. About 37.5 yr **35.** $y = \frac{1}{5}x^5 + 7 \ln|x| + C$

37. $y = Ce^{8x^2}$ **39.** $x/(x^2 - y^2) = C$

41. Proof; $y = -2x + \frac{1}{2}x^3$

43.



Graphs will vary.
$4x^2 + y^2 = C$

45. (a) 0.55 (b) 5250 (c) 150 (d) 6.41 yr

(e) $\dfrac{dP}{dt} = 0.55P\left(1 - \dfrac{P}{5250}\right)$

47. $y = \dfrac{80}{1 + 9e^{-t}}$

49. (a) $P(t) = \dfrac{20{,}400}{1 + 16e^{-0.553t}}$ (b) 17,118 trout (c) 4.94 yr

51. $y = -10 + Ce^x$ **53.** $y = e^{x/4}\left(\frac{1}{4}x + C\right)$

55. $y = (x + C)/(x - 2)$

57. $y = Ce^{3x} - \frac{1}{13}(2 \cos 2x + 3 \sin 2x)$

59. $y = e^{5x}/10 + Ce^{-5x}$ **61.** $y = 1/(1 + x + Ce^x)$

63. $y^{-2} = Cx^2 + 2/(3x)$

65. Answers will vary. Sample answer:
$(x^2 + 3y^2)\,dx - 2xy\,dy = 0$; $x^3 = C(x^2 + y^2)$

67. Answers will vary. Sample answer:
$x^3y' + 2x^2y = 1$; $x^2y = \ln|x| + C$

P.S. Problem Solving (page 445)

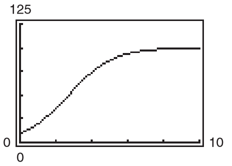
1. (a) $y = 1/(1-0.01t)^{100}$; $T = 100$

(b) $y = 1\Big/\left[\left(\dfrac{1}{y_0}\right)^{\varepsilon} - k\varepsilon t\right]^{1/\varepsilon}$; Explanations will vary.

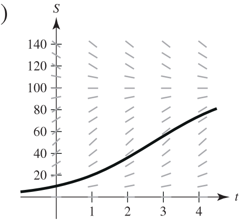
3. (a) $dS/dt = kS(L-S)$; $S = 100/(1+9e^{-0.8109t})$

(b) 2.7 mo

(c)



(d)



(e) Sales will decrease toward the line $S = L$.

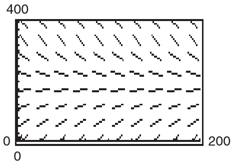
5. Proof; The graph of the logistics function is just a shift of the graph of the hyperbolic tangent.

7. 1481.45 sec ≈ 24 min, 41 sec

9. 2575.95 sec ≈ 42 min, 56 sec

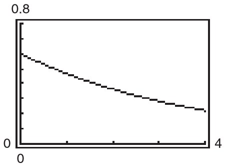
11. (a) $s = 184.21 - Ce^{-0.019t}$

(b)

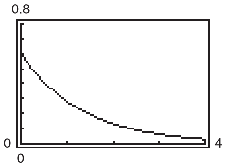


(c) As $t \to \infty$, $Ce^{-0.019t} \to 0$, and $s \to 184.21$.

13. (a) $C = 0.6e^{-0.25t}$



(b) $C = 0.6e^{-0.75t}$



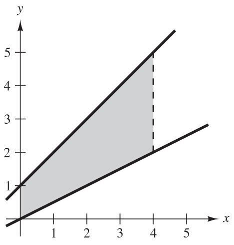
Chapter 7

Section 7.1 (page 454)

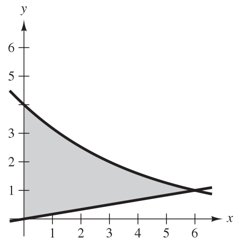
1. $-\int_0^6 (x^2 - 6x)\,dx$ **3.** $\int_0^3 (-2x^2 + 6x)\,dx$

5. $-6\int_0^1 (x^3 - x)\,dx$

7.



9.



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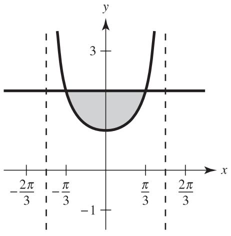

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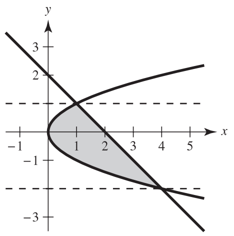
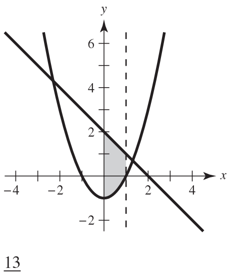


15. d

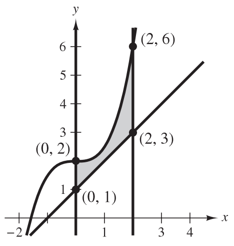
17. (a) $\frac{125}{6}$ (b) $\frac{125}{6}$ (c) Integrating with respect to y; Answers will vary.

19.



$\frac{13}{6}$

21.



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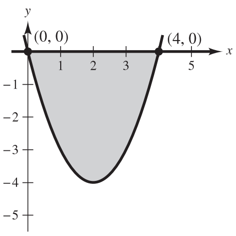
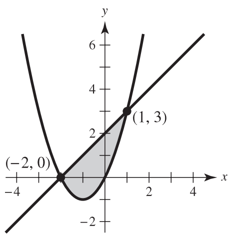
23.



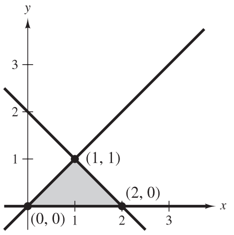
$\frac{32}{3}$

25.



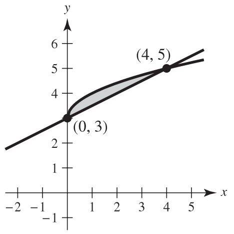
$\frac{9}{2}$

27.



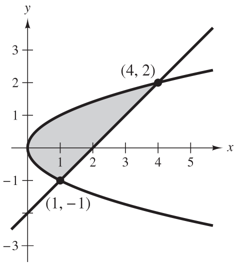
1

29.



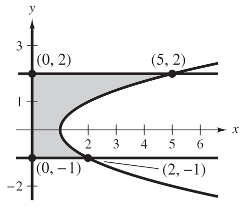
$\frac{4}{3}$

31.



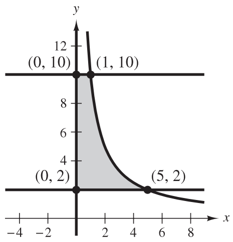
$\frac{9}{2}$

33.



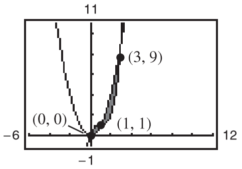
6

35.



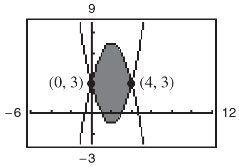
$10 \ln 5 \approx 16.094$

37. (a)



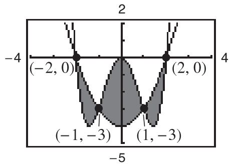
(b) $\frac{37}{12}$

39. (a)



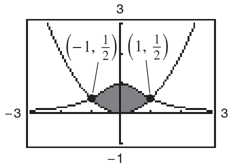
(b) $\frac{64}{3}$

41. (a)



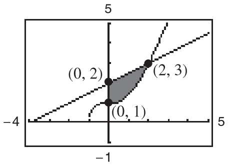
(b) 8

43. (a)



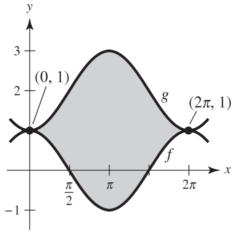
(b) $\pi/2 - 1/3 \approx 1.237$

45. (a)



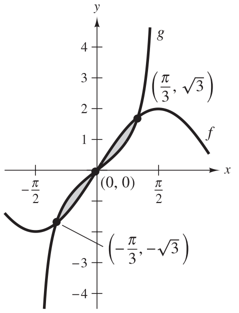
(b) ≈ 1.759

47.



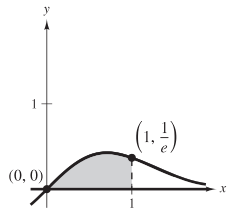
$4\pi \approx 12.566$

49.



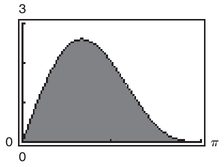
$2(1 - \ln 2) \approx 0.614$

51.



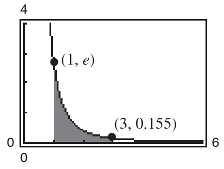
$(1/2)(1 - 1/e) \approx 0.316$

53. (a)



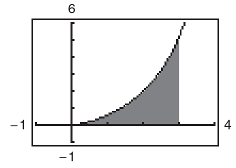
(b) 4

55. (a)



(b) About 1.323

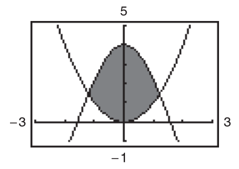
57. (a)



(b) The function is difficult to integrate.

(c) About 4.7721

59. (a)

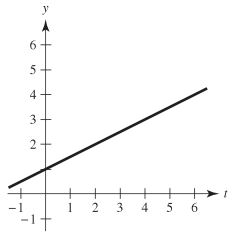


(b) The intersections are difficult to find.

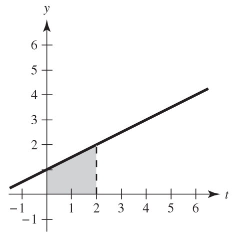
(c) About 6.3043

61. $F(x) = \frac{1}{4}x^2 + x$

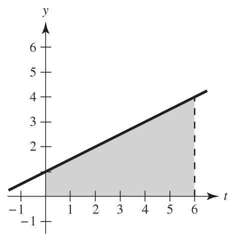
(a) $F(0) = 0$



(b) $F(2) = 3$

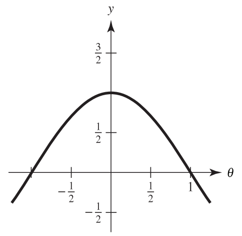


(c) $F(6) = 15$

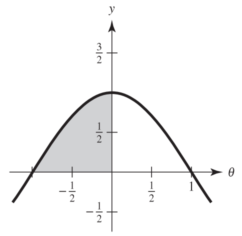


63. $F(\alpha) = (2/\pi)[\sin(\pi\alpha/2) + 1]$

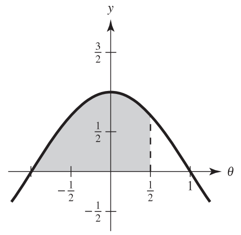
(a) $F(-1) = 0$



(b) $F(0) = 2/\pi \approx 0.6366$



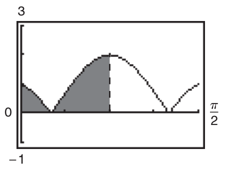
(c) $F(1/2) = \left(\sqrt{2} + 2\right)/\pi \approx 1.0868$



65. 14 **67.** 16

69. Answers will vary. Sample answers:
(a) About 966 ft^2 (b) About 1004 ft^2

71. $A = \frac{3\sqrt{3}}{4} - \frac{1}{2} \approx 0.7990$ **73.** $\int_{-2}^{1}[x^3 - (3x - 2)]\,dx = \frac{27}{4}$



75. $\displaystyle\int_0^1 \left[\frac{1}{x^2+1} - \left(-\frac{1}{2}x + 1\right)\right] dx \approx 0.0354$

77. Answers will vary. Example: $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$

$\displaystyle\int_{-1}^{1} [(1 - x^2) - (x^4 - 2x^2 + 1)]\, dx = \tfrac{4}{15}$

79. Offer 2 is better because the cumulative salary (area under the curve) is greater.

81. (a) The integral $\int_0^5 [v_1(t) - v_2(t)]\, dt = 10$ means that the first car traveled 10 more meters than the second car between 0 and 5 seconds.

The integral $\int_0^{10} [v_1(t) - v_2(t)]\, dt = 30$ means that the first car traveled 30 more meters than the second car between 0 and 10 seconds.

The integral $\int_{20}^{30} [v_1(t) - v_2(t)]\, dt = -5$ means that the second car traveled 5 more meters than the first car between 20 and 30 seconds.

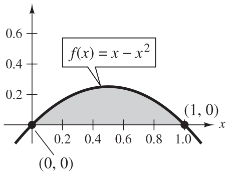
(b) No. You do not know when both cars started or the initial distance between the cars.

(c) The car with velocity v_1 is ahead by 30 meters.

(d) Car 1 is ahead by 8 meters.

83. $b = 9\left(1 - 1/\sqrt[3]{4}\right) \approx 3.330$ **85.** $a = 4 - 2\sqrt{2} \approx 1.172$

87. Answers will vary. Sample answer: $\frac{1}{6}$



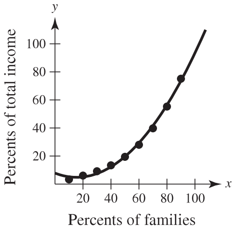
89. (a) $(-2, -11)$, $(0, 7)$ (b) $y = 9x + 7$

(c) 3.2, 6.4, 3.2; The area between the two inflection points is the sum of the areas between the other two regions.

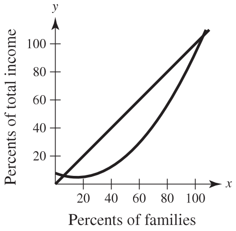
91. \$6.825 billion

93. (a) $y = 0.0124x^2 - 0.385x + 7.85$

(b)



(c)



(d) About 2006.7

95. $\frac{16}{3}\left(4\sqrt{2} - 5\right) \approx 3.503$

97. (a) About 6.031 m^2 (b) About 12.062 m^3 (c) 60,310 lb

99. True

101. False. Let $f(x) = x$ and $g(x) = 2x - x^2$. f and g intersect at $(1, 1)$, the midpoint of $[0, 2]$, but

$\displaystyle\int_a^b [f(x) - g(x)]\, dx = \int_0^2 [x - (2x - x^2)]\, dx = \tfrac{2}{3} \ne 0.$

103. $\sqrt{3}/2 + 7\pi/24 + 1 \approx 2.7823$

105. Putnam Problem A1, 1993

Section 7.2 (page 465)

1. $\displaystyle\pi\int_0^1 (-x + 1)^2\, dx = \frac{\pi}{3}$ **3.** $\displaystyle\pi\int_1^4 \left(\sqrt{x}\right)^2 dx = \frac{15\pi}{2}$

5. $\displaystyle\pi\int_0^1 [(x^2)^2 - (x^5)^2]\, dx = \frac{6\pi}{55}$ **7.** $\displaystyle\pi\int_0^4 \left(\sqrt{y}\right)^2 dy = 8\pi$

9. $\displaystyle\pi\int_0^1 (y^{3/2})^2\, dy = \frac{\pi}{4}$

11. (a) $9\pi/2$ (b) $\left(36\pi\sqrt{3}\right)/5$ (c) $\left(24\pi\sqrt{3}\right)/5$
(d) $\left(84\pi\sqrt{3}\right)/5$

13. (a) $32\pi/3$ (b) $64\pi/3$ **15.** 18π

17. $\pi\left(48 \ln 2 - \frac{27}{4}\right) \approx 83.318$ **19.** $124\pi/3$ **21.** $832\pi/15$

23. $\pi \ln 5$ **25.** $2\pi/3$ **27.** $(\pi/2)(1 - 1/e^2) \approx 1.358$

29. $277\pi/3$ **31.** 8π **33.** $\pi^2/2 \approx 4.935$

35. $(\pi/2)(e^2 - 1) \approx 10.036$ **37.** 1.969 **39.** 15.4115

41. $\pi/3$ **43.** $2\pi/15$ **45.** $\pi/2$ **47.** $\pi/6$

49. (a) The area appears to be close to 1 and therefore the volume (area squared × π) is near 3.

51. A sine curve on $[0, \pi/2]$ revolved about the x-axis

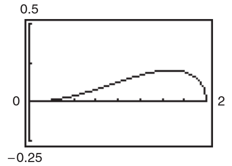
53. The parabola $y = 4x - x^2$ is a horizontal translation of the parabola $y = 4 - x^2$. Therefore, their volumes are equal.

55. (a) This statement is true. Explanations will vary.

(b) This statement is false. Explanations will vary.

57. 18π **59.** Proof **61.** $\pi r^2 h[1 - (h/H) + h^2/(3H^2)]$

63.

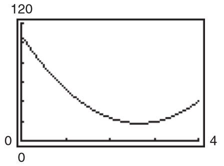


$\pi/30$

65. (a) 60π (b) 50π

67. (a) $V = \pi\left(4b^2 - \frac{64}{3}b + \frac{512}{15}\right)$

(b)



$b \approx 2.67$

(c) $b = \frac{8}{3} \approx 2.67$

69. (a) ii; right circular cylinder of radius r and height h

(b) iv; ellipsoid whose underlying ellipse has the equation $(x/b)^2 + (y/a)^2 = 1$

(c) iii; sphere of radius r

(d) i; right circular cone of radius r and height h

(e) v; torus of cross-sectional radius r and other radius R

71. (a) $\frac{81}{10}$ (b) $\frac{9}{2}$ **73.** $\frac{16}{3}r^3$ **75.** $V = \frac{4}{3}\pi(R^2 - r^2)^{3/2}$

77. 19.7443 **79.** (a) $\frac{2}{3}r^3$ (b) $\frac{2}{3}r^3 \tan\theta$; As $\theta \to 90°$, $V \to \infty$.

Section 7.3 (page 474)

1. $\displaystyle 2\pi\int_0^2 x^2\, dx = \frac{16\pi}{3}$ **3.** $\displaystyle 2\pi\int_0^4 x\sqrt{x}\, dx = \frac{128\pi}{5}$

5. $\displaystyle 2\pi\int_0^3 x^3\, dx = \frac{81}{2}\pi$ **7.** $\displaystyle 2\pi\int_0^2 x(4x - 2x^2)\, dx = \frac{16\pi}{3}$

9. $2\pi\int_0^2 x(x^2 - 4x + 4)\,dx = \dfrac{8\pi}{3}$

11. $2\pi\int_2^4 x\sqrt{x-2}\,dx = \dfrac{128\pi}{15}\sqrt{2}$

13. $2\pi\int_0^1 x\left(\dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)dx = \sqrt{2\pi}\left(1 - \dfrac{1}{\sqrt{e}}\right) \approx 0.986$

15. $2\pi\int_0^2 y(2 - y)\,dy = \dfrac{8\pi}{3}$

17. $2\pi\left[\int_0^{1/2} y\,dy + \int_{1/2}^1 y\left(\dfrac{1}{y} - 1\right)dy\right] = \dfrac{\pi}{2}$

19. $2\pi\left[\int_0^8 y^{4/3}\,dy\right] = \dfrac{768\pi}{7}$

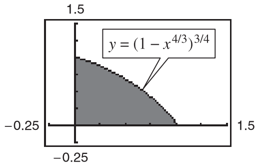
21. $2\pi\int_0^2 y(4 - 2y)\,dy = 16\pi/3$ **23.** 64π **25.** 16π

27. Shell method; it is much easier to put x in terms of y rather than vice versa.

29. (a) $128\pi/7$ (b) $64\pi/5$ (c) $96\pi/5$

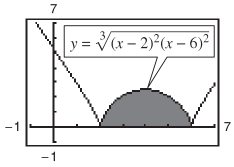
31. (a) $\pi a^3/15$ (b) $\pi a^3/15$ (c) $4\pi a^3/15$

33. (a)



(b) 1.506

35. (a)



(b) 187.25

37. d **39.** a, c, b

41. Both integrals yield the volume of the solid generated by revolving the region bounded by the graphs of $y = \sqrt{x-1}$, $y = 0$, and $x = 5$ about the x-axis.

43. (a) The rectangles would be vertical.
(b) The rectangles would be horizontal.

45. Diameter $= 2\sqrt{4 - 2\sqrt{3}} \approx 1.464$ **47.** $4\pi^2$

49. (a) Region bounded by $y = x^2$, $y = 0$, $x = 0$, $x = 2$
(b) Revolved about the y-axis

51. (a) Region bounded by $x = \sqrt{6 - y}$, $y = 0$, $x = 0$
(b) Revolved about $y = -2$

53. (a) Proof (b) (i) $V = 2\pi$ (ii) $V = 6\pi^2$

55. Proof

57. (a) $R_1(n) = n/(n + 1)$ (b) $\lim_{n\to\infty} R_1(n) = 1$
(c) $V = \pi ab^{n+2}[n/(n + 2)]$; $R_2(n) = n/(n + 2)$
(d) $\lim_{n\to\infty} R_2(n) = 1$
(e) As $n \to \infty$, the graph approaches the line $x = b$.

59. (a) and (b) About 121,475 ft^3 **61.** $c = 2$

63. (a) $64\pi/3$ (b) $2048\pi/35$ (c) $8192\pi/105$

Section 7.4 (page 485)

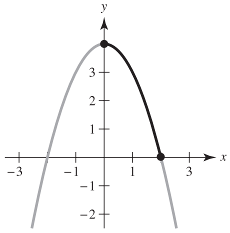
1. (a) and (b) 17 **3.** $\frac{5}{3}$ **5.** $\frac{2}{3}\left(2\sqrt{2} - 1\right) \approx 1.219$

7. $5\sqrt{5} - 2\sqrt{2} \approx 8.352$ **9.** 309.3195

11. $\ln\left[\left(\sqrt{2} + 1\right)/\left(\sqrt{2} - 1\right)\right] \approx 1.763$

13. $\frac{1}{2}(e^2 - 1/e^2) \approx 3.627$ **15.** $\frac{76}{3}$

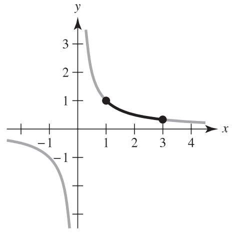
17. (a)



(b) $\int_0^2 \sqrt{1 + 4x^2}\,dx$

(c) About 4.647

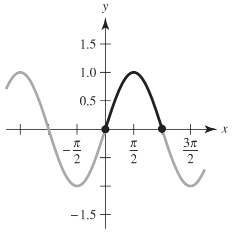
19. (a)



(b) $\int_1^3 \sqrt{1 + \dfrac{1}{x^4}}\,dx$

(c) About 2.147

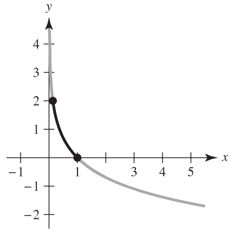
21. (a)



(b) $\int_0^{\pi} \sqrt{1 + \cos^2 x}\,dx$

(c) About 3.820

23. (a)

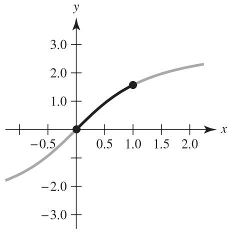


(b) $\int_0^2 \sqrt{1 + e^{-2y}}\,dy$

$= \int_{e^{-2}}^1 \sqrt{1 + \dfrac{1}{x^2}}\,dx$

(c) About 2.221

25. (a)



(b) $\int_0^1 \sqrt{1 + \left(\dfrac{2}{1 + x^2}\right)^2}\,dx$

(c) About 1.871

27. b **29.** (a) 64.125 (b) 64.525 (c) 64.666 (d) 64.672

31. $20[\sinh 1 - \sinh(-1)] \approx 47.0$ m **33.** About 1480

35. $3\arcsin\frac{2}{3} \approx 2.1892$

37. $2\pi\int_0^3 \dfrac{1}{3}x^3\sqrt{1 + x^4}\,dx = \dfrac{\pi}{9}\left(82\sqrt{82} - 1\right) \approx 258.85$

39. $2\pi\int_1^2 \left(\dfrac{x^3}{6} + \dfrac{1}{2x}\right)\left(\dfrac{x^2}{2} + \dfrac{1}{2x^2}\right)dx = \dfrac{47\pi}{16} \approx 9.23$

41. $2\pi\int_{-1}^1 2\,dx = 8\pi \approx 25.13$

43. $2\pi\int_1^8 x\sqrt{1 + \dfrac{1}{9x^{4/3}}}\,dx = \dfrac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right) \approx 199.48$

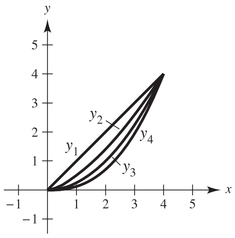
45. $2\pi\int_0^2 x\sqrt{1 + \dfrac{x^2}{4}}\,dx = \dfrac{\pi}{3}\left(16\sqrt{2} - 8\right) \approx 15.318$

47. 14.424

49. A rectifiable curve is a curve with a finite arc length.

51. The integral formula for the area of a surface of revolution is derived from the formula for the lateral surface area of the frustum of a right circular cone. The formula is $S = 2\pi rL$, where $r = \frac{1}{2}(r_1 + r_2)$, which is the average radius of the frustum, and L is the length of a line segment on the frustum. The representative element is $2\pi f(d_i)\sqrt{1 + (\Delta y_i/\Delta x_i)^2}\,\Delta x_i$.

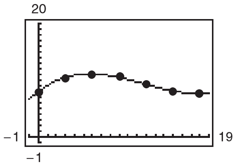
53. (a)



(b) y_1, y_2, y_3, y_4
(c) $s_1 \approx 5.657$; $s_2 \approx 5.759$; $s_3 \approx 5.916$; $s_4 \approx 6.063$

55. 20π **57.** $6\pi(3 - \sqrt{5}) \approx 14.40$

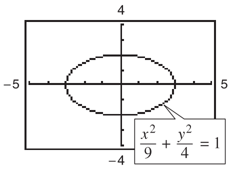
59. (a) Answers will vary. Sample answer: 5207.62 in.^{3}
(b) Answers will vary. Sample answer: 1168.64 in.^{2}
(c) $r = 0.0040y^3 - 0.142y^2 + 1.23y + 7.9$



(d) 5279.64 in.^{3}; 1179.5 in.^{2}

61. (a) $\pi(1 - 1/b)$ (b) $2\pi \int_1^b \sqrt{x^4 + 1}/x^3 \, dx$
(c) $\lim_{b\to\infty} V = \lim_{b\to\infty} \pi(1 - 1/b) = \pi$
(d) Because $\dfrac{\sqrt{x^4+1}}{x^3} > \dfrac{\sqrt{x^4}}{x^3} = \dfrac{1}{x} > 0$ on $[1, b]$,
you have $\displaystyle\int_1^b \frac{\sqrt{x^4+1}}{x^3}\,dx > \int_1^b \frac{1}{x}\,dx = \Big[\ln x\Big]_1^b = \ln b$
and $\lim_{b\to\infty} \ln b \to \infty$. So, $\displaystyle\lim_{b\to\infty} 2\pi \int_1^b \frac{\sqrt{x^4+1}}{x^3}\,dx = \infty$.

63. (a)



(b) $\displaystyle\int_0^3 \sqrt{1 + \frac{4x^2}{81 - 9x^2}}\,dx$

(c) You cannot evaluate this definite integral because the integrand is not defined at $x = 3$. Simpson's Rule will not work for the same reason.

65. Fleeing object: $\frac{2}{3}$ unit
Pursuer: $\displaystyle\frac{1}{2}\int_0^1 \frac{x+1}{\sqrt{x}}\,dx = \frac{4}{3} = 2\left(\frac{2}{3}\right)$

67. $384\pi/5$ **69.** Proof **71.** Proof

Section 7.5 (page 495)

1. 2000 ft-lb **3.** 896 N-m
5. 40.833 in.-lb ≈ 3.403 ft-lb **7.** 8750 N-cm = 87.5 N-m
9. 160 in.-lb ≈ 13.3 ft-lb **11.** 37.125 ft-lb
13. (a) 487.805 mile-tons ≈ 5.151×10^9 ft-lb
(b) 1395.349 mile-tons ≈ 1.473×10^{10} ft-lb
15. (a) 2.93×10^4 mile-tons ≈ 3.10×10^{11} ft-lb
(b) 3.38×10^4 mile-tons ≈ 3.57×10^{11} ft-lb
17. (a) 2496 ft-lb (b) 9984 ft-lb **19.** $470{,}400\pi$ N-m
21. 2995.2π ft-lb **23.** $20{,}217.6\pi$ ft-lb **25.** 2457π ft-lb
27. 600 ft-lb **29.** 450 ft-lb **31.** 168.75 ft-lb
33. If an object is moved a distance D in the direction of an applied constant force F, then the work W done by the force is defined as $W = FD$.
35. The situation in part (a) requires more work. There is no work required for part (b) because the distance is 0.
37. (a) 54 ft-lb (b) 160 ft-lb (c) 9 ft-lb (d) 18 ft-lb
39. $2000 \ln(3/2) \approx 810.93$ ft-lb **41.** $3k/4$
43. 3249.4 ft-lb **45.** 10,330.3 ft-lb

Section 7.6 (page 506)

1. $\bar{x} = -\frac{4}{3}$ **3.** $\bar{x} = 12$ **5.** (a) $\bar{x} = 16$ (b) $\bar{x} = -2$
7. $x = 6$ ft **9.** $(\bar{x}, \bar{y}) = \left(\frac{10}{9}, -\frac{1}{9}\right)$ **11.** $(\bar{x}, \bar{y}) = \left(2, \frac{48}{25}\right)$
13. $M_x = \rho/3$, $M_y = 4\rho/3$, $(\bar{x}, \bar{y}) = (4/3, 1/3)$
15. $M_x = 4\rho$, $M_y = 64\rho/5$, $(\bar{x}, \bar{y}) = (12/5, 3/4)$
17. $M_x = \rho/35$, $M_y = \rho/20$, $(\bar{x}, \bar{y}) = (3/5, 12/35)$
19. $M_x = 99\rho/5$, $M_y = 27\rho/4$, $(\bar{x}, \bar{y}) = (3/2, 22/5)$
21. $M_x = 192\rho/7$, $M_y = 96\rho$, $(\bar{x}, \bar{y}) = (5, 10/7)$
23. $M_x = 0$, $M_y = 256\rho/15$, $(\bar{x}, \bar{y}) = (8/5, 0)$
25. $M_x = 27\rho/4$, $M_y = -27\rho/10$, $(\bar{x}, \bar{y}) = (-3/5, 3/2)$

27. $A = \displaystyle\int_0^2 (2x - x^2)\,dx = \frac{4}{3}$

$M_x = \displaystyle\int_0^2 \left(\frac{2x + x^2}{2}\right)(2x - x^2)\,dx = \frac{32}{15}$

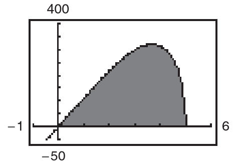
$M_y = \displaystyle\int_0^2 x(2x - x^2)\,dx = \frac{4}{3}$

29. $A = \displaystyle\int_0^3 (2x + 4)\,dx = 21$

$M_x = \displaystyle\int_0^3 \left(\frac{2x + 4}{2}\right)(2x + 4)\,dx = 78$

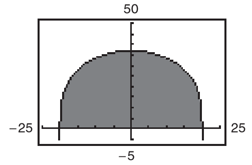
$M_y = \displaystyle\int_0^3 x(2x + 4)\,dx = 36$

31.



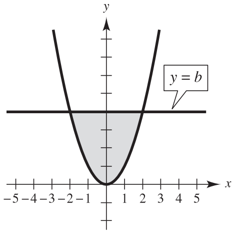
$(\bar{x}, \bar{y}) = (3.0, 126.0)$

33.



$(\bar{x}, \bar{y}) = (0, 16.2)$

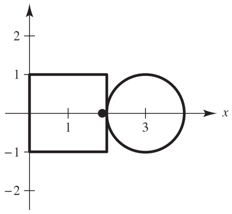
35. $(\bar{x}, \bar{y}) = \left(\dfrac{b}{3}, \dfrac{c}{3}\right)$ **37.** $(\bar{x}, \bar{y}) = \left(\dfrac{(a + 2b)c}{3(a + b)}, \dfrac{a^2 + ab + b^2}{3(a + b)}\right)$
39. $(\bar{x}, \bar{y}) = (0, 4b/(3\pi))$
41. (a)



(b) $\bar{x} = 0$ by symmetry
(c) $M_y = \displaystyle\int_{-\sqrt{b}}^{\sqrt{b}} x(b - x^2)\,dx = 0$ because $x(b - x^2)$ is an odd function.
(d) $\bar{y} > b/2$ because the area is greater for $y > b/2$.
(e) $\bar{y} = (3/5)b$

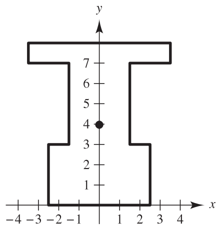
43. (a) $(\bar{x}, \bar{y}) = (0, 12.98)$
(b) $y = (-1.02 \times 10^{-5})x^4 - 0.0019x^2 + 29.28$
(c) $(\bar{x}, \bar{y}) = (0, 12.85)$

45.



$(\bar{x}, \bar{y}) = \left(\dfrac{4 + 3\pi}{4 + \pi}, 0\right)$

47.



$(\bar{x}, \bar{y}) = \left(0, \dfrac{135}{34}\right)$

49. $(\bar{x}, \bar{y}) = \left(\dfrac{2 + 3\pi}{2 + \pi}, 0\right)$ **51.** $160\pi^2 \approx 1579.14$

53. $128\pi/3 \approx 134.04$

55. The center of mass $(\bar{x}, \bar{y})$ is $\bar{x} = M_y/m$ and $\bar{y} = M_x/m$, where:
1. $m = m_1 + m_2 + \cdot \cdot \cdot + m_n$ is the total mass of the system.
2. $M_y = m_1 x_1 + m_2 x_2 + \cdot \cdot \cdot + m_n x_n$ is the moment about the y-axis.
3. $M_x = m_1 y_1 + m_2 y_2 + \cdot \cdot \cdot + m_n y_n$ is the moment about the x-axis.

57. See Theorem 7.1 on page 505. **59.** $(\bar{x}, \bar{y}) = (0, 2r/\pi)$

61. $(\bar{x}, \bar{y}) = \left(\dfrac{n + 1}{n + 2}, \dfrac{n + 1}{4n + 2}\right)$; As $n \to \infty$, the region shrinks toward the line segments $y = 0$ for $0 \le x \le 1$ and $x = 1$ for $0 \le y \le 1$; $(\bar{x}, \bar{y}) \to \left(1, \dfrac{1}{4}\right)$.

Section 7.7 (page 513)

1. 1497.6 lb **3.** 4992 lb **5.** 748.8 lb **7.** 1123.2 lb
9. 748.8 lb **11.** 1064.96 lb **13.** 117,600 N
15. 2,381,400 N **17.** 2814 lb **19.** 6753.6 lb **21.** 94.5 lb
23. Proof **25.** Proof **27.** 960 lb

29. Answers will vary. Sample answer (using Simpson's Rule): 3010.8 lb

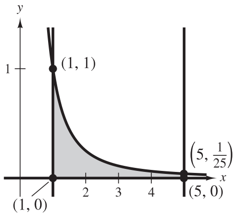
31. 8213.0 lb

33. $3\sqrt{2}/2 \approx 2.12$ ft; The pressure increases with increasing depth.

35. Because you are measuring total force against a region between two depths.

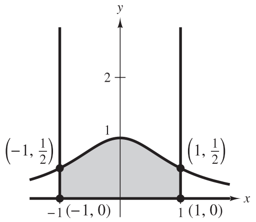
Review Exercises for Chapter 7 (page 515)

1.



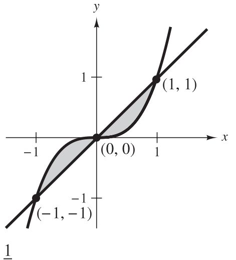
$4/5$

3.



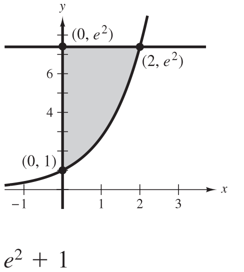
$\pi/2$

5.



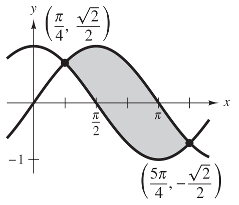
$\frac{1}{2}$

7.



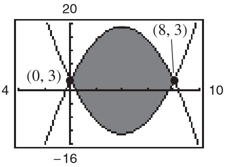
$e^2 + 1$

9.



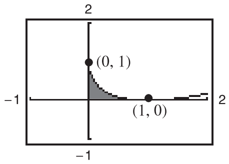
$2\sqrt{2}$

11.



$\frac{512}{3}$

13.



$\frac{1}{6}$

15. $\displaystyle\int_0^2 [0 - (y^2 - 2y)]\,dy = \int_{-1}^0 2\sqrt{x + 1}\,dx = \frac{4}{3}$

17. $\displaystyle\int_0^2 \left[1 - \left(1 - \frac{x}{2}\right)\right] dx + \int_2^3 [1 - (x - 2)]\,dx$
$\displaystyle= \int_0^1 [(y + 2) - (2 - 2y)]\,dy = \frac{3}{2}$

19. (a) 9920 ft^2 (b) $10{,}413\frac{1}{3}$ ft^2

21. (a) 9π (b) 18π (c) 9π (d) 36π

23. (a) 64π (b) 48π **25.** $\pi^2/4$

27. $(4\pi/3)(20 - 9 \ln 3) \approx 42.359$

29. (a) $\dfrac{4}{15}$ (b) $\dfrac{\pi}{12}$ (c) $\dfrac{32\pi}{105}$ **31.** 1.958 ft

33. $\frac{8}{15}\left(1 + 6\sqrt{3}\right) \approx 6.076$ **35.** 4018.2 ft

37. 15π **39.** 62.5 in.-lb $\approx$ 5.208 ft-lb

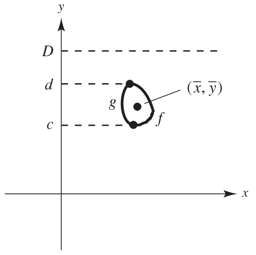
41. $122{,}980\pi$ ft-lb $\approx$ 193.2 foot-tons **43.** 200 ft-lb

45. $a = 15/4$ **47.** $(\bar{x}, \bar{y}) = (a/5, a/5)$ **49.** $(\bar{x}, \bar{y}) = (0, 2a^2/5)$

51. $(\bar{x}, \bar{y}) = \left(\dfrac{2(9\pi + 49)}{3(\pi + 9)}, 0\right)$

53. Let D = surface of liquid; ρ = weight per cubic volume.

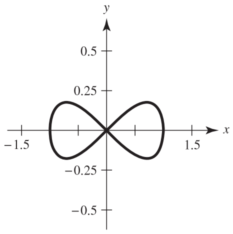
$$F = \rho\int_c^d (D - y)[f(y) - g(y)]\,dy$$
$$= \rho\left[\int_c^d D[f(y) - g(y)]\,dy - \int_c^d y[f(y) - g(y)]\,dy\right]$$
$$= \rho\left[\int_c^d [f(y) - g(y)]\,dy\right]\left[D - \frac{\displaystyle\int_c^d y[f(y) - g(y)]\,dy}{\displaystyle\int_c^d [f(y) - g(y)]\,dy}\right]$$
$$= \rho(\text{area})(D - \bar{y})$$
$$= \rho(\text{area})(\text{depth of centroid})$$



P.S. Problem Solving (page 517)

1. 3 **3.** $y = 0.2063x$

5.

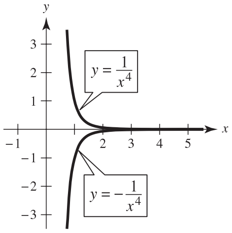


$\frac{5\sqrt{2}\pi}{3}$

7. $V = 2\pi\left[d + \frac{1}{2}\sqrt{w^2 + l^2}\right]lw$ **9.** $f(x) = 2e^{x/2} - 2$

11. 89.3%

13.



(a) $(\bar{x}, \bar{y}) = \left(\frac{63}{43}, 0\right)$

(b) $(\bar{x}, \bar{y}) = \left(\frac{3b(b+1)}{2(b^2 + b + 1)}, 0\right)$

(c) $\left(\frac{3}{2}, 0\right)$

15. Consumer surplus: 1600; Producer surplus: 400

17. Wall at shallow end: 9984 lb
Wall at deep end: 39,936 lb
Side wall: 19,968 + 26,624 = 46,592 lb

Chapter 8

Section 8.1 (page 524)

1. b **3.** c

5. $\int u^n\,du$ $u = 5x - 3, n = 4$ **7.** $\int \frac{du}{u}$ $u = 1 - 2\sqrt{x}$ **9.** $\int \frac{du}{\sqrt{a^2 - u^2}}$ $u = t, a = 1$

11. $\int \sin u\,du$ $u = t^2$ **13.** $\int e^u\,du$ $u = \sin x$ **15.** $2(x - 5)^7 + C$

17. $-7/[6(z - 10)^6] + C$ **19.** $\frac{1}{2}v^2 - 1/[6(3v - 1)^2] + C$

21. $-\frac{1}{3}\ln\left|-t^3 + 9t + 1\right| + C$ **23.** $\frac{1}{2}x^2 + x + \ln|x - 1| + C$

25. $\ln(1 + e^x) + C$ **27.** $\frac{x}{15}(48x^4 + 200x^2 + 375) + C$

29. $\sin(2\pi x^2)/(4\pi) + C$ **31.** $-(1/\pi)\csc \pi x + C$

33. $\frac{1}{11}e^{11x} + C$ **35.** $2\ln(1 + e^x) + C$ **37.** $(\ln x)^2 + C$

39. $-\ln(1 - \sin x) + C = \ln|\sec x(\sec x + \tan x)| + C$

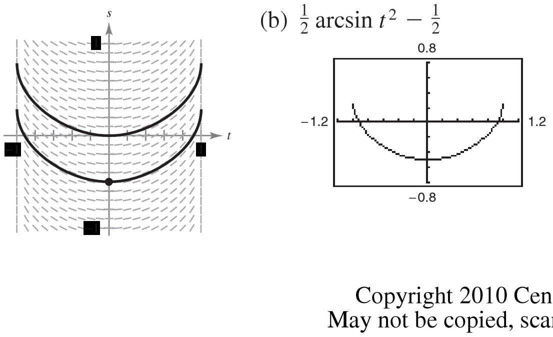
41. $\csc\theta + \cot\theta + C = (1 + \cos\theta)/\sin\theta + C$

43. $-\frac{1}{4}\arcsin(4t + 1) + C$ **45.** $\frac{1}{2}\ln|\cos(2/t)| + C$

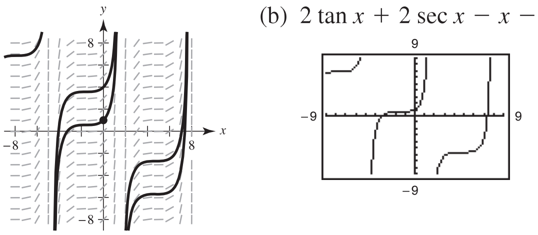
47. $6\arcsin[(x - 5)/5] + C$ **49.** $\frac{1}{4}\arctan[(2x + 1)/8] + C$

51. $\arcsin\left[(x + 2)/\sqrt{5}\right] + C$

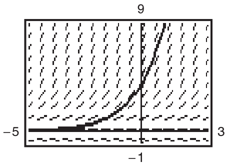
53. (a) (b) $\frac{1}{2}\arcsin t^2 - \frac{1}{2}$



55.
(a) (b) $2\tan x + 2\sec x - x - 1 + C$



57. $y = 4e^{0.8x}$



59. $y = \frac{1}{2}e^{2x} + 10e^x + 25x + C$

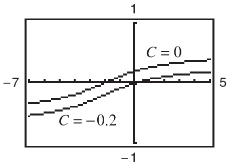
61. $r = 10\arcsin e^t + C$ **63.** $y = \frac{1}{2}\arctan(\tan x/2) + C$

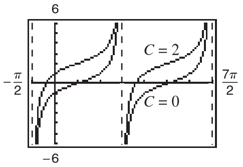
65. $\frac{1}{2}$ **67.** $\frac{1}{2}(1 - e^{-1}) \approx 0.316$ **69.** 8 **71.** $\pi/18$

73. $18\sqrt{6}/5 \approx 8.82$ **75.** $\frac{3}{2}\ln\left(\frac{34}{9}\right) + \frac{2}{3}\arctan\left(\frac{5}{3}\right) \approx 2.68$

77. $\frac{4}{3} \approx 1.333$

79. $\frac{1}{3}\arctan\left[\frac{1}{3}(x + 2)\right] + C$ **81.** $\tan\theta - \sec\theta + C$
Graphs will vary. Graphs will vary.
Example: Example:





One graph is a vertical translation of the other. One graph is a vertical translation of the other.

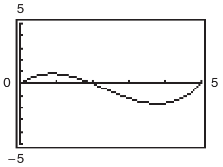
83. Power Rule: $\int u^n\,du = \frac{u^{n+1}}{n+1} + C;\ u = x^2 + 1, n = 3$

85. Log Rule: $\int \frac{du}{u} = \ln|u| + C;\ u = x^2 + 1$

87. $a = \sqrt{2}, b = \frac{\pi}{4};\ -\frac{1}{\sqrt{2}}\ln\left|\csc\left(x + \frac{\pi}{4}\right) + \cot\left(x + \frac{\pi}{4}\right)\right| + C$

89. $a = \frac{1}{2}$

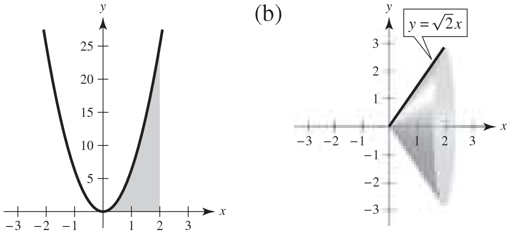
91.



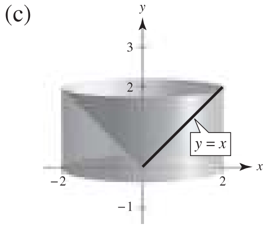
Negative; more area below the x-axis than above

93. a

95. (a) (b)



(c)



97. (a) $\pi(1-e^{-1})\approx 1.986$ (b) $b=\sqrt{\ln\left(\frac{3\pi}{3\pi-4}\right)}\approx 0.743$

99. $\ln(\sqrt{2}+1)\approx 0.8814$

101. $(8\pi/3)(10\sqrt{10}-1)\approx 256.545$

103. $\frac{1}{3}\arctan 3\approx 0.416$ **105.** About 1.0320

107. (a) $\frac{1}{3}\sin x(\cos^2 x+2)$
(b) $\frac{1}{15}\sin x(3\cos^4 x+4\cos^2 x+8)$
(c) $\frac{1}{35}\sin x(5\cos^6 x+6\cos^4 x+8\cos^2 x+16)$
(d) $\int\cos^{15}x\,dx=\int(1-\sin^2 x)^7\cos x\,dx$
You would expand $(1-\sin^2 x)^7$.

109. Proof

Section 8.2 (page 533)

1. $u=x,\ dv=e^{2x}\,dx$ **3.** $u=(\ln x)^2,\ dv=dx$

5. $u=x,\ dv=\sec^2 x\,dx$ **7.** $\frac{1}{16}x^4(4\ln x-1)+C$

9. $\frac{1}{9}\sin 3x-\frac{1}{3}x\cos 3x+C$ **11.** $-\frac{1}{16e^{4x}}(4x+1)+C$

13. $e^x(x^3-3x^2+6x-6)+C$

15. $\frac{1}{3}e^{x^3}+C$ **17.** $\frac{1}{4}[2(t^2-1)\ln|t+1|-t^2+2t]+C$

19. $\frac{1}{3}(\ln x)^3+C$ **21.** $e^{2x}/[4(2x+1)]+C$

23. $(x-1)^2e^x+C$ **25.** $\frac{2}{15}(x-5)^{3/2}(3x+10)+C$

27. $x\sin x+\cos x+C$

29. $(6x-x^3)\cos x+(3x^2-6)\sin x+C$

31. $-t\csc t-\ln|\csc t+\cot t|+C$

33. $x\arctan x-\frac{1}{2}\ln(1+x^2)+C$

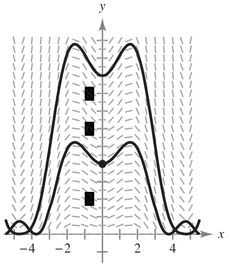
35. $\frac{1}{5}e^{2x}(2\sin x-\cos x)+C$

37. $\frac{1}{5}e^{-x}(2\sin 2x-\cos 2x)+C$ **39.** $y=\frac{1}{2}e^{x^2}+C$

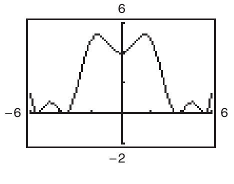
41. $y=\frac{2}{5}t^2\sqrt{3+5t}-\frac{8t}{75}(3+5t)^{3/2}+\frac{16}{1875}(3+5t)^{5/2}+C$
$=\frac{2}{625}\sqrt{3+5t}(25t^2-20t+24)+C$

43. $\sin y=x^2+C$

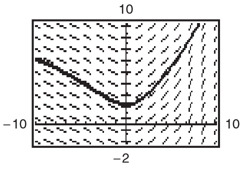
45. (a)



(b) $2\sqrt{y}-\cos x-x\sin x=3$



47.



49. $2e^{3/2}+4\approx 12.963$

51. $\frac{\pi}{8}-\frac{1}{4}\approx 0.143$

53. $(\pi-3\sqrt{3}+6)/6\approx 0.658$

55. $\frac{1}{2}[e(\sin 1-\cos 1)+1]\approx 0.909$

57. $\frac{4}{3}\sqrt{2}\ln 2-\frac{8}{9}\sqrt{2}+\frac{4}{9}\approx 0.494$

59. $8\ \mathrm{arcsec}\ 4+\sqrt{3}/2-\sqrt{15}/2-2\pi/3\approx 7.380$

61. $(e^{2x}/4)(2x^2-2x+1)+C$

63. $(3x^2-6)\sin x-(x^3-6x)\cos x+C$

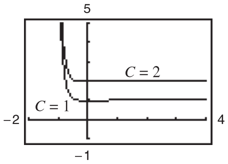
65. $x\tan x+\ln|\cos x|+C$ **67.** $2(\sin\sqrt{x}-\sqrt{x}\cos\sqrt{x})+C$

69. $\frac{128}{15}$ **71.** $\frac{1}{2}(x^4e^{x^2}-2x^2e^{x^2}+2e^{x^2})+C$

73. $\frac{1}{2}x[\cos(\ln x)+\sin(\ln x)]+C$ **75.** Product Rule

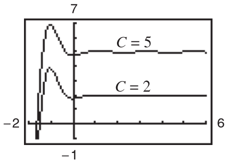
77. In order for the integration by parts technique to be efficient, you want dv to be the most complicated portion of the integrand and you want u to be the portion of the integrand whose derivative is a function simpler than u. If you let $u=\sin x$, then du is not a simpler function.

79. (a) $-(e^{-4t}/128)(32t^3+24t^2+12t+3)+C$
(b) Graphs will vary. Example:



(c) One graph is a vertical translation of the other.

81. (a) $\frac{1}{13}(2e^{-\pi}+3)\approx 0.2374$
(b) Graphs will vary. Example:



(c) One graph is a vertical translation of the other.

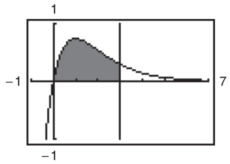
83. $\frac{2}{5}(2x-3)^{3/2}(x+1)+C$ **85.** $\frac{1}{3}\sqrt{4+x^2}(x^2-8)+C$

87. $n=0$: $x(\ln x-1)+C$
$n=1$: $\frac{1}{4}x^2(2\ln x-1)+C$
$n=2$: $\frac{1}{9}x^3(3\ln x-1)+C$
$n=3$: $\frac{1}{16}x^4(4\ln x-1)+C$
$n=4$: $\frac{1}{25}x^5(5\ln x-1)+C$
$\int x^n\ln x\,dx=\frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x-1]+C$

89–93. Proofs

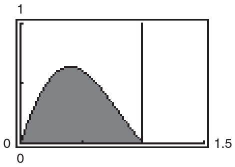
95. $\frac{1}{36}x^6(6\ln x-1)+C$ **97.** $\frac{1}{13}e^{2x}(2\cos 3x+3\sin 3x)+C$

99.



$2-\frac{8}{e^3}\approx 1.602$

101.



$\frac{\pi}{1+\pi^2}\left(\frac{1}{e}+1\right)\approx 0.395$

103. (a) 1 (b) $\pi(e-2)\approx 2.257$ (c) $\frac{1}{2}\pi(e^2+1)\approx 13.177$
(d) $\left(\frac{e^2+1}{4},\frac{e-2}{2}\right)\approx(2.097,0.359)$

105. In Example 6, we showed that the centroid of an equivalent region was $(1,\pi/8)$. By symmetry, the centroid of this region is $(\pi/8,1)$.

107. $[7/(10\pi)](1-e^{-4\pi})\approx 0.223$ **109.** $931,265

111. Proof **113.** $b_n = [8h/(n\pi)^2]\sin(n\pi/2)$

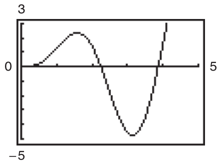
115. Shell: $V = \pi\left[b^2f(b) - a^2f(a) - \int_a^b x^2 f'(x)\,dx\right]$

Disk: $V = \pi\left[b^2f(b) - a^2f(a) - \int_{f(a)}^{f(b)} [f^{-1}(y)]^2\,dy\right]$

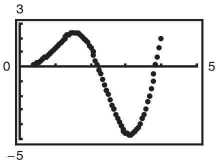
Both methods yield the same volume because $x = f^{-1}(y)$, $f'(x)\,dx = dy$, if $y = f(a)$ then $x = a$, and if $y = f(b)$ then $x = b$.

117. (a) $y = \frac{1}{4}(3\sin 2x - 6x\cos 2x)$

(b)

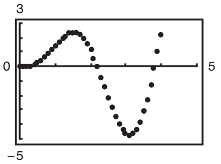


(c) You obtain the following points.



n	x_n	y_n
0	0	0
1	0.05	0
2	0.10	7.4875×10^{-4}
3	0.15	0.0037
4	0.20	0.0104
⋮	⋮	⋮
80	4.00	1.3181

(d) You obtain the following points.



n	x_n	y_n
0	0	0
1	0.1	0
2	0.2	0.0060
3	0.3	0.0293
4	0.4	0.0801
⋮	⋮	⋮
40	4.0	1.0210

119. The graph of $y = x\sin x$ is below the graph of $y = x$ on $[0, \pi/2]$.

Section 8.3 (page 542)

1. c **2.** a **3.** d **4.** b **5.** $-\frac{1}{6}\cos^6 x + C$

7. $\frac{1}{16}\sin^8 2x + C$ **9.** $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$

11. $-\frac{1}{3}(\cos 2\theta)^{3/2} + \frac{1}{7}(\cos 2\theta)^{7/2} + C$

13. $\frac{1}{12}(6x + \sin 6x) + C$

15. $\frac{3}{8}\alpha + \frac{1}{12}\sin 6\alpha + \frac{1}{96}\sin 12\alpha + C$

17. $\frac{1}{8}(2x^2 - 2x\sin 2x - \cos 2x) + C$ **19.** $\frac{16}{35}$

21. $63\pi/512$ **23.** $5\pi/32$ **25.** $\frac{1}{7}\ln|\sec 7x + \tan 7x| + C$

27. $\frac{1}{15}\tan 5x(3 + \tan^2 5x) + C$

29. $(\sec \pi x\tan \pi x + \ln|\sec \pi x + \tan \pi x|)/(2\pi) + C$

31. $\frac{1}{2}\tan^4(x/2) - \tan^2(x/2) - 2\ln|\cos(x/2)| + C$

33. $\frac{1}{2}\tan^2 x + C$ **35.** $\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$

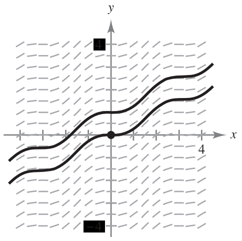
37. $\frac{1}{24}\sec^6 4x + C$ **39.** $\frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C$

41. $\ln|\sec x + \tan x| - \sin x + C$

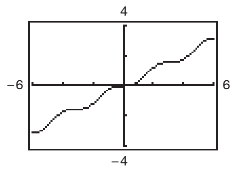
43. $(12\pi\theta - 8\sin 2\pi\theta + \sin 4\pi\theta)/(32\pi) + C$

45. $y = \frac{1}{9}\sec^3 3x - \frac{1}{3}\sec 3x + C$

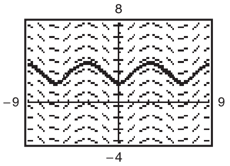
47. (a)



(b) $y = \frac{1}{2}x - \frac{1}{4}\sin 2x$



49.



51. $\frac{1}{16}(2\sin 4x + \sin 8x) + C$

53. $\frac{1}{12}(3\cos 2x - \cos 6x) + C$ **55.** $\frac{1}{8}(2\sin 2\theta - \sin 4\theta) + C$

57. $\frac{1}{4}(\ln|\csc^2 2x| - \cot^2 2x) + C$ **59.** $-\frac{1}{2}\cot 2x - \frac{1}{6}\cot^3 2x + C$

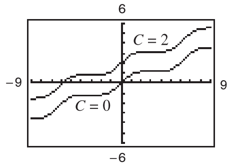
61. $\ln|\csc t - \cot t| + \cos t + C$

63. $\ln|\csc x - \cot x| + \cos x + C$ **65.** $t - 2\tan t + C$

67. π **69.** $3(1 - \ln 2)$ **71.** $\ln 2$ **73.** 4

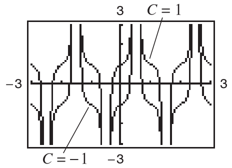
75. $\frac{1}{16}(6x + 8\sin x + \sin 2x) + C$

Graphs will vary. Example:



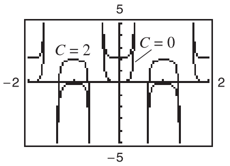
77. $[\sec^3 \pi x\tan \pi x +$
$\frac{3}{2}(\sec \pi x\tan \pi x + \ln|\sec \pi x + \tan \pi x|)]/(4\pi) + C$

Graphs will vary. Example:



79. $(\sec^5 \pi x)/(5\pi) + C$ **81.** $2\sqrt{2}/7$ **83.** $3\pi/16$

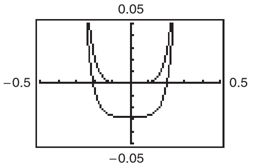
Graphs will vary. Example:



85. (a) Save one sine factor and convert the remaining factors to cosines. Then expand and integrate.
(b) Save one cosine factor and convert the remaining factors to sines. Then expand and integrate.
(c) Make repeated use of the power reducing formulas to convert the integrand to odd powers of the cosine. Then proceed as in part (b).

87. (a) $\frac{1}{2}\sin^2 x + C$ (b) $-\frac{1}{2}\cos^2 x + C$
(c) $\frac{1}{2}\sin^2 x + C$ (d) $-\frac{1}{4}\cos 2x + C$
The answers are all the same, they are just written in different forms. Using trigonometric identities, you can rewrite each answer in the same form.

89. (a) $\frac{1}{18}\tan^6 3x + \frac{1}{12}\tan^4 3x + C_1$, $\frac{1}{18}\sec^6 3x - \frac{1}{12}\sec^4 3x + C_2$
(b)



(c) Proof

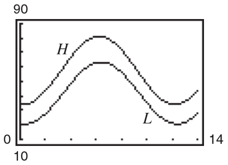
91. $\frac{1}{3}$ **93.** 1 **95.** $2\pi(1 - \pi/4) \approx 1.348$

97. (a) $\pi^2/2$ (b) $(\bar{x}, \bar{y}) = (\pi/2, \pi/8)$ **99–101.** Proofs

103. $-\frac{1}{15}\cos x(3\sin^4 x + 4\sin^2 x + 8) + C$

105. $\dfrac{5}{6\pi}\tan\dfrac{2\pi x}{5}\left(\sec^2\dfrac{2\pi x}{5} + 2\right) + C$

107. (a) $H(t) \approx 57.72 - 23.36\cos(\pi t/6) - 2.75\sin(\pi t/6)$
(b) $L(t) \approx 42.04 - 20.91\cos(\pi t/6) - 4.33\sin(\pi t/6)$
(c)



The maximum difference is at $t \approx 4.9$, or late spring.

109. Proof

Section 8.4 (page 551)

1. $x = 3\tan\theta$ **3.** $x = 4\sin\theta$ **5.** $x/\left(16\sqrt{16 - x^2}\right) + C$

7. $4\ln\left|\left(4 - \sqrt{16 - x^2}\right)/x\right| + \sqrt{16 - x^2} + C$

9. $\ln\left|x + \sqrt{x^2 - 25}\right| + C$

11. $\frac{1}{15}(x^2 - 25)^{3/2}(3x^2 + 50) + C$

13. $\frac{1}{3}(1 + x^2)^{3/2} + C$ **15.** $\frac{1}{2}[\arctan x + x/(1 + x^2)] + C$

17. $\frac{1}{2}x\sqrt{9 + 16x^2} + \frac{9}{8}\ln\left|4x + \sqrt{9 + 16x^2}\right| + C$

19. $\frac{25}{4}\arcsin(2x/5) + \frac{1}{2}x\sqrt{25 - 4x^2} + C$

21. $\sqrt{x^2 + 36} + C$ **23.** $\arcsin(x/4) + C$

25. $4\arcsin(x/2) + x\sqrt{4 - x^2} + C$ **27.** $\ln\left|x + \sqrt{x^2 - 4}\right| + C$

29. $-\dfrac{(1 - x^2)^{3/2}}{3x^3} + C$ **31.** $-\dfrac{1}{3}\ln\left|\dfrac{\sqrt{4x^2 + 9} + 3}{2x}\right| + C$

33. $3/\sqrt{x^2 + 3} + C$ **35.** $\frac{1}{3}(1 + e^{2x})^{3/2} + C$

37. $\frac{1}{2}\left(\arcsin e^x + e^x\sqrt{1 - e^{2x}}\right) + C$

39. $\frac{1}{4}\left[x/(x^2 + 2) + \left(1/\sqrt{2}\right)\arctan\left(x/\sqrt{2}\right)\right] + C$

41. $x\,\operatorname{arcsec} 2x - \frac{1}{2}\ln\left|2x + \sqrt{4x^2 - 1}\right| + C$

43. $\arcsin[(x - 2)/2] + C$

45. $\sqrt{x^2 + 6x + 12} - 3\ln\left|\sqrt{x^2 + 6x + 12} + (x + 3)\right| + C$

47. (a) and (b) $\sqrt{3} - \pi/3 \approx 0.685$

49. (a) and (b) $9\left(2 - \sqrt{2}\right) \approx 5.272$

51. (a) and (b) $-(9/2)\ln\left(2\sqrt{7}/3 - 4\sqrt{3}/3 - \sqrt{21}/3 + 8/3\right) + 9\sqrt{3} - 2\sqrt{7} \approx 12.644$

53. $\sqrt{x^2 - 9} - 3\arctan\left(\sqrt{x^2 - 9}/3\right) + 1$

55. $\frac{1}{2}(x - 15)\sqrt{x^2 + 10x + 9} + 33\ln\left|\sqrt{x^2 + 10x + 9} + (x + 5)\right| + C$

57. $\frac{1}{2}\left(x\sqrt{x^2 - 1} + \ln\left|x + \sqrt{x^2 - 1}\right|\right) + C$

59. (a) Let $u = a\sin\theta$, $\sqrt{a^2 - u^2} = a\cos\theta$, where $-\pi/2 \le \theta \le \pi/2$.
(b) Let $u = a\tan\theta$, $\sqrt{a^2 + u^2} = a\sec\theta$, where $-\pi/2 < \theta < \pi/2$.
(c) Let $u = a\sec\theta$, $\sqrt{u^2 - a^2} = \tan\theta$ if $u > a$ and $\sqrt{u^2 - a^2} = -\tan\theta$ if $u < -a$, where $0 \le \theta < \pi/2$ or $\pi/2 < \theta \le \pi$.

61. Trigonometric substitution: $x = \sec\theta$ **63.** True

65. False: $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{(1 + x^2)^{3/2}} = \int_0^{\pi/3} \cos\theta\, d\theta$

67. πab **69.** (a) $5\sqrt{2}$ (b) $25(1 - \pi/4)$ (c) $r^2(1 - \pi/4)$

71. $6\pi^2$ **73.** $\ln\left[\dfrac{5\left(\sqrt{2} + 1\right)}{\sqrt{26} + 1}\right] + \sqrt{26} - \sqrt{2} \approx 4.367$

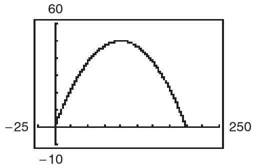
75. Length of one arch of sine curve: $y = \sin x$, $y' = \cos x$

$$L_1 = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$$

Length of one arch of cosine curve: $y = \cos x$, $y' = -\sin x$

$$\begin{aligned} L_2 &= \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x}\, dx \\ &= \int_{-\pi/2}^{\pi/2} \sqrt{1 + \cos^2(x - \pi/2)}\, dx, \quad u = x - \pi/2, du = dx \\ &= \int_{-\pi}^{0} \sqrt{1 + \cos^2 u}\, du = \int_0^{\pi} \sqrt{1 + \cos^2 u}\, du = L_1 \end{aligned}$$

77. (a)



(b) 200
(c) $100\sqrt{2} + 50\ln\left[\left(\sqrt{2} + 1\right)/\left(\sqrt{2} - 1\right)\right] \approx 229.559$

79. (0, 0.422) **81.** $(\pi/32)\left[102\sqrt{2} - \ln\left(3 + 2\sqrt{2}\right)\right] \approx 13.989$

83. (a) 187.2π lb (b) $62.4\pi d$ lb

85. Proof **87.** $12 + 9\pi/2 - 25\arcsin(3/5) \approx 10.050$

89. Putnum Problem A5, 2005

Section 8.5 (page 561)

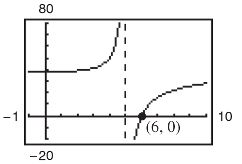
1. $\dfrac{A}{x} + \dfrac{B}{x - 8}$ **3.** $\dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 10}$ **5.** $\dfrac{A}{x} + \dfrac{B}{x - 6}$

7. $\frac{1}{6}\ln|(x - 3)/(x + 3)| + C$ **9.** $\ln|(x - 1)/(x + 4)| + C$

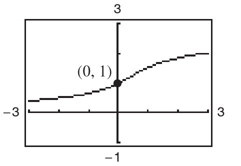
11. $\frac{3}{2}\ln|2x - 1| - 2\ln|x + 1| + C$

13. $5\ln|x - 2| - \ln|x + 2| - 3\ln|x| + C$

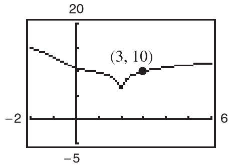
15. $x^2 + \frac{3}{2}\ln|x - 4| - \frac{1}{2}\ln|x + 2| + C$
17. $1/x + \ln|x^4 + x^3| + C$
19. $2\ln|x - 2| - \ln|x| - 3/(x - 2) + C$
21. $\ln|(x^2 + 1)/x| + C$
23. $\frac{1}{6}\left[\ln|(x - 2)/(x + 2)| + \sqrt{2}\arctan(x/\sqrt{2})\right] + C$
25. $\frac{1}{16}\ln|(4x^2 - 1)/(4x^2 + 1)| + C$
27. $\ln|x + 1| + \sqrt{2}\arctan[(x - 1)/\sqrt{2}] + C$
29. $\ln 3$ **31.** $\frac{1}{2}\ln(8/5) - \pi/4 + \arctan 2 \approx 0.557$
33. $y = 5\ln|x - 5| - 5x/(x - 5) + 30$



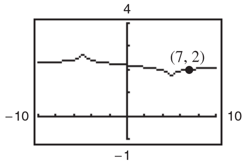
35. $y = (\sqrt{2}/2)\arctan(x/\sqrt{2}) - 1/[2(x^2 + 2)] + 5/4$



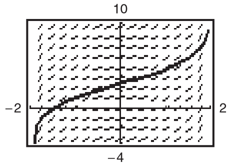
37. $y = \ln|x - 2| + \frac{1}{2}\ln|x^2 + x + 1|$
$- \sqrt{3}\arctan[(2x + 1)/\sqrt{3}] - \frac{1}{2}\ln 13$
$+ \sqrt{3}\arctan(7/\sqrt{3}) + 10$



39. $y = \frac{1}{10}\ln|(x - 5)/(x + 5)| + \frac{1}{10}\ln 6 + 2$



41. $\ln\left|\frac{\cos x}{\cos x - 1}\right| + C$ **43.** $\ln\left|\frac{\sin x}{1 + \sin x}\right| + C$
45. $\ln\left|\frac{\tan x + 2}{\tan x + 3}\right| + C$ **47.** $\frac{1}{5}\ln\left|\frac{e^x - 1}{e^x + 4}\right| + C$
49. $2\sqrt{x} + 2\ln\left|\frac{\sqrt{x} - 2}{\sqrt{x} + 2}\right| + C$ **51–53.** Proofs
55. $y = \frac{3}{2}\ln\left|\frac{2 + x}{2 - x}\right| + 3$ **57.** First divide x^3 by $(x - 5)$.

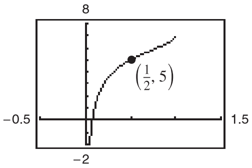


59. $12\ln\left(\frac{9}{8}\right) \approx 1.4134$ **61.** $6 - \frac{7}{4}\ln 7 \approx 2.5947$
63. 4.90 or $490,000
65. $V = 2\pi\left(\arctan 3 - \frac{3}{10}\right) \approx 5.963$; $(\bar{x}, \bar{y}) \approx (1.521, 0.412)$

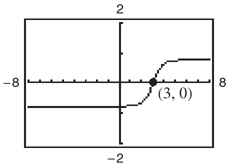
67. $x = n[e^{(n+1)kt} - 1]/[n + e^{(n+1)kt}]$ **69.** $\pi/8$

Section 8.6 (page 567)

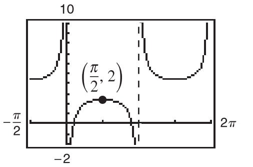
1. $-\frac{1}{2}x(10 - x) + 25\ln|5 + x| + C$
3. $\frac{1}{2}\left[e^x\sqrt{e^{2x} + 1} + \ln\left(e^x + \sqrt{e^{2x} + 1}\right)\right] + C$
5. $-\sqrt{1 - x^2}/x + C$
7. $\frac{1}{24}(3x + \sin 3x\cos 3x + 2\cos^3 3x\sin 3x) + C$
9. $-2(\cot\sqrt{x} + \csc\sqrt{x}) + C$ **11.** $x - \frac{1}{2}\ln(1 + e^{2x}) + C$
13. $\frac{1}{64}x^8(8\ln x - 1) + C$
15. (a) and (b) $\frac{1}{27}e^{3x}(9x^2 - 6x + 2) + C$
17. (a) and (b) $\ln|(x + 1)/x| - 1/x + C$
19. $\frac{1}{2}\left[(x^2 + 1)\operatorname{arccsc}(x^2 + 1) + \ln\left(x^2 + 1 + \sqrt{x^4 + 2x^2}\right)\right] + C$
21. $\sqrt{x^2 - 4}/(4x) + C$ **23.** $\frac{4}{25}[\ln|2 - 5x| + 2/(2 - 5x)] + C$
25. $e^x\arccos(e^x) - \sqrt{1 - e^{2x}} + C$
27. $\frac{1}{2}(x^2 + \cot x^2 + \csc x^2) + C$
29. $(\sqrt{2}/2)\arctan[(1 + \sin\theta)/\sqrt{2}] + C$
31. $-\sqrt{2 + 9x^2}/(2x) + C$
33. $\frac{1}{4}\left(2\ln|x| - 3\ln|3 + 2\ln|x||\right) + C$
35. $(3x - 10)/[2(x^2 - 6x + 10)] + \frac{3}{2}\arctan(x - 3) + C$
37. $\frac{1}{2}\ln\left|x^2 - 3 + \sqrt{x^4 - 6x^2 + 5}\right| + C$
39. $-\frac{1}{3}\sqrt{4 - x^2}(x^2 + 8) + C$
41. $2/(1 + e^x) - 1/[2(1 + e^x)^2] + \ln(1 + e^x) + C$
43. $\frac{1}{2}(e - 1) \approx 0.8591$ **45.** $\frac{32}{5}\ln 2 - \frac{31}{25} \approx 3.1961$
47. $\pi/2$ **49.** $\pi^3/8 - 3\pi + 6 \approx 0.4510$ **51–55.** Proofs
57. $y = -2\sqrt{1 - x}/\sqrt{x} + 7$



59. $y = \frac{1}{2}[(x - 3)/(x^2 - 6x + 10) + \arctan(x - 3)]$



61. $y = -\csc\theta + \sqrt{2} + 2$



63. $\frac{1}{\sqrt{5}}\ln\left|\frac{2\tan(\theta/2) - 3 - \sqrt{5}}{2\tan(\theta/2) - 3 + \sqrt{5}}\right| + C$ **65.** $\ln 2$
67. $\frac{1}{2}\ln(3 - 2\cos\theta) + C$ **69.** $-2\cos\sqrt{\theta} + C$ **71.** $4\sqrt{3}$
73. (a) $\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

$\int x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$

$\int x^3\ln x\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$

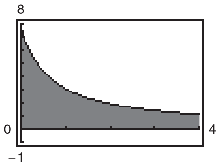
(b) $\int x^n \ln x\, dx = x^{n+1} \ln x/(n+1) - x^{n+1}/(n+1)^2 + C$

75. False. Substitutions may first have to be made to rewrite the integral in a form that appears in the table.

77. $32\pi^2$ **79.** 1919.145 ft-lb

81. (a) $V = 80 \ln\left(\sqrt{10} + 3\right) \approx 145.5 \text{ ft}^3$
$W = 11{,}840 \ln\left(\sqrt{10} + 3\right) \approx 21{,}530.4 \text{ lb}$
(b) (0, 1.19)

83. (a) $k = 30/\ln 7 \approx 15.42$ **85.** Putnam Problem A3, 1980
(b)



Section 8.7 (page 576)

1.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	1.3177	1.3332	1.3333	1.3333	1.3332	1.3177

$\frac{4}{3}$

3.

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	0.9900	90,483.7	3.7×10^9	4.5×10^{10}	0	0

0

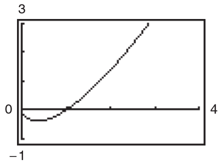
5. $\frac{3}{8}$ **7.** $\frac{1}{8}$ **9.** $\frac{5}{3}$ **11.** 4 **13.** 0 **15.** 2

17. ∞ **19.** $\frac{11}{4}$ **21.** $\frac{3}{5}$ **23.** 1 **25.** $\frac{5}{4}$ **27.** ∞

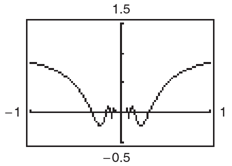
29. 0 **31.** 1 **33.** 0 **35.** 0 **37.** ∞

39. $\frac{5}{9}$ **41.** 1 **43.** ∞

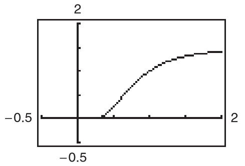
45. (a) Not indeterminate
(b) ∞
(c)



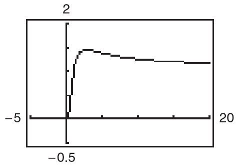
47. (a) $0 \cdot \infty$
(b) 1
(c)



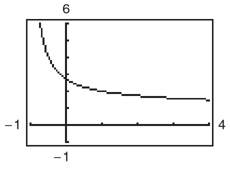
49. (a) Not indeterminate
(b) 0
(c)



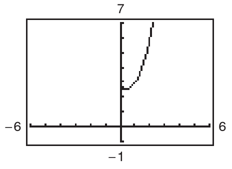
51. (a) ∞^0
(b) 1
(c)



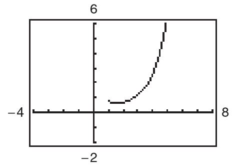
53. (a) 1^∞ (b) e
(c)



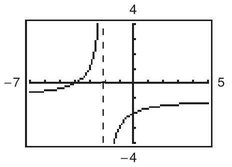
55. (a) 0^0 (b) 3
(c)



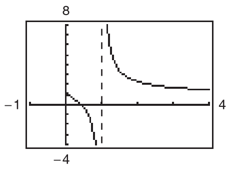
57. (a) 0^0 (b) 1
(c)



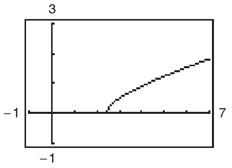
59. (a) $\infty - \infty$ (b) $-\frac{3}{2}$
(c)



61. (a) $\infty - \infty$ (b) ∞
(c)

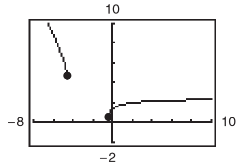


63. (a)



(b) $\frac{1}{2}$

65. (a)



(b) $\frac{5}{2}$

67. $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^\infty, 0^0, \infty - \infty$

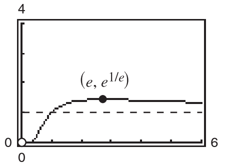
69. Answers will vary. Examples:
(a) $f(x) = x^2 - 25,\ g(x) = x - 5$
(b) $f(x) = (x - 5)^2,\ g(x) = x^2 - 25$
(c) $f(x) = x^2 - 25,\ g(x) = (x - 5)^3$

71.

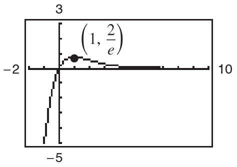
x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$	2.811	4.498	0.720	0.036	0.001	0.000

73. 0 **75.** 0 **77.** 0

79. Horizontal asymptote: $y = 1$
Relative maximum: $(e, e^{1/e})$



81. Horizontal asymptote: $y = 0$
Relative maximum: $(1, 2/e)$



83. Limit is not of the form $0/0$ or ∞/∞.

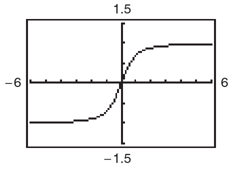
85. Limit is not of the form $0/0$ or ∞/∞.

87. Limit is not of the form $0/0$ or ∞/∞.

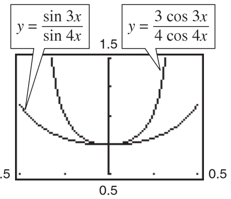
89. (a) $\lim\limits_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = \lim\limits_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$

Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails.

(b) 1
(c)



91.



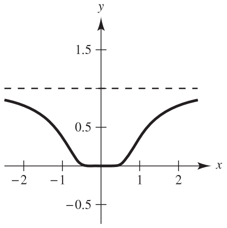
As $x \to 0$, the graphs get closer together (they both approach 0.75). By L'Hôpital's Rule, $\lim_{x\to 0} \frac{\sin 3x}{\sin 4x} = \lim_{x\to 0} \frac{3\cos 3x}{4\cos 4x} = \frac{3}{4}$.

93. $v = 32t + v_0$ **95.** Proof **97.** $c = \frac{2}{3}$ **99.** $c = \pi/4$

101. False: L'Hôpital's Rule does not apply, because $\lim_{x\to 0}(x^2 + x + 1) \neq 0$. **103.** True

105. $\frac{3}{4}$ **107.** $\frac{4}{3}$ **109.** $a = 1, b = \pm 2$ **111.** Proof

113.



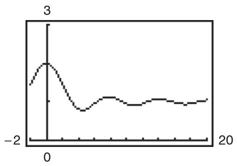
$g'(0) = 0$

115. (a) $0 \cdot \infty$ (b) 0

117. Proof

119. (a)–(c) 2

121. (a)



(b) $\lim_{x\to\infty} h(x) = 1$

(c) No

123. Putnam Problem A1, 1956

Section 8.8 (page 587)

1. Improper; $0 \le \frac{3}{5} \le 1$ **3.** Not improper; continuous on $[0, 1]$

5. Not improper; continuous on $[0, 2]$

7. Improper; infinite limits of integration

9. Infinite discontinuity at $x = 0$; 4

11. Infinite discontinuity at $x = 1$; diverges

13. Infinite limit of integration; $\frac{1}{4}$

15. Infinite discontinuity at $x = 0$; diverges

17. Infinite limit of integration; converges to 1 **19.** $\frac{1}{2}$

21. Diverges **23.** Diverges **25.** 2 **27.** $\frac{1}{2}$

29. $1/[2(\ln 4)^2]$ **31.** π **33.** $\pi/4$ **35.** Diverges

37. Diverges **39.** 6 **41.** $-\frac{1}{4}$ **43.** Diverges **45.** $\pi/3$

47. $\ln(2 + \sqrt{3})$ **49.** 0 **51.** $\pi/6$ **53.** $2\pi\sqrt{6}/3$

55. $p > 1$ **57.** Proof **59.** Diverges **61.** Converges

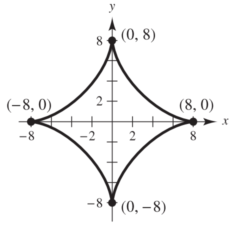
63. Converges **65.** Diverges **67.** Diverges **69.** Converges

71. An integral with infinite integration limits, an integral with an infinite discontinuity at or between the integration limits

73. The improper integral diverges. **75.** e **77.** π

79. (a) 1 (b) $\pi/2$ (c) 2π

81.



Perimeter = 48

83. $8\pi^2$ **85.** (a) $W = 20{,}000$ mile-tons (b) 4000 mi

87. (a) Proof (b) $P = 43.53\%$ (c) $E(x) = 7$

89. (a) \$757,992.41 (b) \$837,995.15 (c) \$1,066,666.67

91. $P = \left[2\pi NI\left(\sqrt{r^2 + c^2} - c\right)\right]/\left(kr\sqrt{r^2 + c^2}\right)$

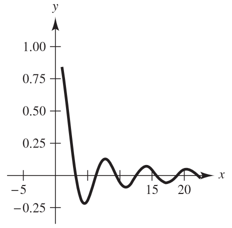
93. False. Let $f(x) = 1/(x + 1)$. **95.** True

97. (a) and (b) Proofs

(c) The definition of the improper integral $\int_{-\infty}^{\infty}$ is not $\lim_{a\to\infty} \int_{-a}^{a}$ but rather if you rewrite the integral that diverges, you can find that the integral converges.

99. (a) $\int_1^{\infty} \frac{1}{x^n}\,dx$ will converge if $n > 1$ and diverge if $n \le 1$.

(b)



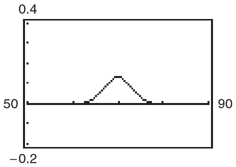
(c) Converges

101. (a) $\Gamma(1) = 1, \Gamma(2) = 1, \Gamma(3) = 2$ (b) Proof (c) $\Gamma(n) = (n - 1)!$

103. $1/s,\ s > 0$ **105.** $2/s^3,\ s > 0$ **107.** $s/(s^2 + a^2),\ s > 0$

109. $s/(s^2 - a^2),\ s > |a|$

111. (a)



(b) About 0.2525

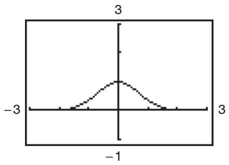
(c) 0.2525; same by symmetry

113. $c = 1$; $\ln(2)$

115. $8\pi[(\ln 2)^2/3 - (\ln 4)/9 + 2/27] \approx 2.01545$

117. $\int_0^1 2\sin(u^2)\,du$; 0.6278

119. (a)



(b) Proof

Review Exercises for Chapter 8 (page 591)

1. $\frac{1}{3}(x^2 - 36)^{3/2} + C$ **3.** $\frac{1}{2}\ln|x^2 - 49| + C$

5. $\ln(2) + \frac{1}{2} \approx 1.1931$ **7.** $100\arcsin(x/10) + C$

9. $\frac{1}{9}e^{3x}(3x - 1) + C$

11. $\frac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + C$

13. $\frac{2}{15}(x - 1)^{3/2}(3x + 2) + C$

15. $-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$

17. $\frac{1}{16}\left[(8x^2 - 1)\arcsin 2x + 2x\sqrt{1 - 4x^2}\right] + C$

19. $\sin(\pi x - 1)[\cos^2(\pi x - 1) + 2]/(3\pi) + C$

21. $\frac{2}{3}[\tan^3(x/2) + 3\tan(x/2)] + C$ **23.** $\tan\theta + \sec\theta + C$

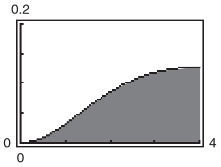
25. $3\pi/16 + \frac{1}{2} \approx 1.0890$ **27.** $3\sqrt{4 - x^2}/x + C$

29. $\frac{1}{3}(x^2 + 4)^{1/2}(x^2 - 8) + C$ **31.** π

33. (a), (b), and (c) $\frac{1}{3}\sqrt{4 + x^2}(x^2 - 8) + C$
35. $6 \ln|x + 3| - 5 \ln|x - 4| + C$
37. $\frac{1}{4}[6 \ln|x - 1| - \ln(x^2 + 1) + 6 \arctan x] + C$
39. $x - \frac{64}{11} \ln|x + 8| + \frac{9}{11} \ln|x - 3| + C$
41. $\frac{1}{25}[4/(4 + 5x) + \ln|4 + 5x|] + C$ **43.** $1 - \sqrt{2}/2$
45. $\frac{1}{2} \ln|x^2 + 4x + 8| - \arctan[(x + 2)/2] + C$
47. $\ln|\tan \pi x|/\pi + C$ **49.** Proof
51. $\frac{1}{8}(\sin 2\theta - 2\theta \cos 2\theta) + C$
53. $\frac{4}{3}[x^{3/4} - 3x^{1/4} + 3 \arctan(x^{1/4})] + C$
55. $2\sqrt{1 - \cos x} + C$ **57.** $\sin x \ln(\sin x) - \sin x + C$
59. $\frac{5}{2} \ln|(x - 5)/(x + 5)| + C$
61. $y = x \ln|x^2 + x| - 2x + \ln|x + 1| + C$ **63.** $\frac{1}{5}$
65. $\frac{1}{2}(\ln 4)^2 \approx 0.961$ **67.** π **69.** $\frac{128}{15}$
71. $(\bar{x}, \bar{y}) = (0, 4/(3\pi))$ **73.** 3.82 **75.** 0 **77.** ∞ **79.** 1
81. $1000e^{0.09} \approx 1094.17$ **83.** Converges; $\frac{32}{3}$ **85.** Diverges
87. Converges; 1 **89.** Converges; $\pi/4$
91. (a) \$6,321,205.59 (b) \$10,000,000
93. (a) 0.4581 (b) 0.0135

P.S. Problem Solving (page 593)

1. (a) $\frac{4}{3}, \frac{16}{15}$ (b) Proof **3.** ln 3 **5.** Proof
7. (a)



Area ≈ 0.2986

(b) $\ln 3 - \frac{4}{5}$
(c) $\ln 3 - \frac{4}{5}$

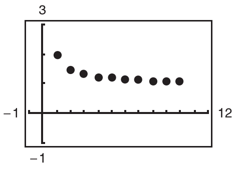
9. $\ln 3 - \frac{1}{2} \approx 0.5986$ **11.** Proof **13.** About 0.8670
15. (a) ∞ (b) 0 (c) $-\frac{2}{3}$
The form $0 \cdot \infty$ is indeterminant.
17. $\dfrac{1/12}{x} + \dfrac{1/42}{x - 3} + \dfrac{1/10}{x - 1} + \dfrac{111/140}{x + 4}$
19. Proof **21.** About 0.0158

Chapter 9

Section 9.1 (page 604)

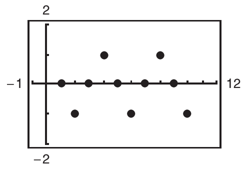
1. 3, 9, 27, 81, 243 **3.** $-\frac{1}{4}, \frac{1}{16}, -\frac{1}{64}, \frac{1}{256}, -\frac{1}{1024}$
5. 1, 0, −1, 0, 1 **7.** $-1, -\frac{1}{4}, \frac{1}{9}, \frac{1}{16}, -\frac{1}{25}$ **9.** $5, \frac{19}{4}, \frac{43}{9}, \frac{77}{16}, \frac{121}{25}$
11. 3, 4, 6, 10, 18 **13.** 32, 16, 8, 4, 2
15. c **16.** a **17.** d **18.** b **19.** b **20.** c
21. a **22.** d **23.** 14, 17; add 3 to preceding term
25. 80, 160; multiply preceding term by 2.
27. $\frac{3}{16}, -\frac{3}{32}$; multiply preceding term by $-\frac{1}{2}$
29. $11 \cdot 10 \cdot 9 = 990$ **31.** $n + 1$ **33.** $1/[(2n + 1)(2n)]$
35. 5 **37.** 2 **39.** 0

41.



Converges to 1

43.



Diverges

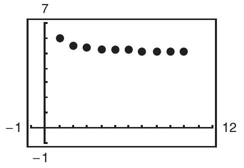
45. Converges to −1 **47.** Converges to 0
49. Diverges **51.** Converges to $\frac{3}{2}$ **53.** Converges to 0
55. Converges to 0 **57.** Converges to 0 **59.** Converges to 0
61. Diverges **63.** Converges to 0 **65.** Converges to 0
67. Converges to 1 **69.** Converges to e^k **71.** Converges to 0
73. Answers will vary. Sample answer: $3n - 2$
75. Answers will vary. Sample answer: $n^2 - 2$
77. Answers will vary. Sample answer: $(n + 1)/(n + 2)$
79. Answers will vary. Sample answer: $(n + 1)/n$
81. Answers will vary. Sample answer: $n/[(n + 1)(n + 2)]$
83. Answers will vary. Sample answer:

$$\frac{(-1)^{n-1}}{1 \cdot 3 \cdot 5 \cdots (2n - 1)} = \frac{(-1)^{n-1}2^n n!}{(2n)!}$$

85. Answers will vary. Sample answer: $(2n)!$
87. Monotonic, bounded **89.** Monotonic, bounded
91. Not monotonic, bounded **93.** Monotonic, bounded
95. Not monotonic, bounded **97.** Not monotonic, bounded

99. (a) $\left|5 + \dfrac{1}{n}\right| \le 6 \Rightarrow$ bounded
$a_n > a_{n+1} \Rightarrow$ monotonic
So, $\{a_n\}$ converges.

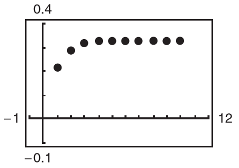
(b)



Limit = 5

101. (a) $\left|\dfrac{1}{3}\left(1 - \dfrac{1}{3^n}\right)\right| < \dfrac{1}{3} \Rightarrow$ bounded
$a_n < a_{n+1} \Rightarrow$ monotonic
So, $\{a_n\}$ converges.

(b)



Limit $= \frac{1}{3}$

103. $\{a_n\}$ has a limit because it is bounded and monotonic; since $2 \le a_n \le 4$, $2 \le L \le 4$.
105. (a) No; $\lim_{n\to\infty} A_n$ does not exist.
(b)

n	1	2	3	4
A_n	\$10,045.83	\$10,091.88	\$10,138.13	\$10,184.60

n	5	6	7
A_n	\$10,231.28	\$10,278.17	\$10,325.28

n	8	9	10
A_n	\$10,372.60	\$10,420.14	\$10,467.90

107. No. A sequence is said to converge when its terms approach a real number.
109. The graph on the left represents a sequence with alternating signs because the terms alternate from being above the x-axis to being below the x-axis.

111. (a) $\$4{,}500{,}000{,}000(0.8)^n$

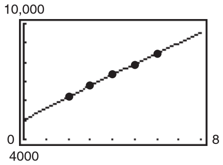
(b)

Year	1	2
Budget	\$3,600,000,000	\$2,880,000,000

Year	3	4
Budget	\$2,304,000,000	\$1,843,200,000

(c) Converges to 0

113. (a) $a_n = -5.364n^2 + 608.04n + 4998.3$



(b) \$11,522.4 billion

115. (a) $a_9 = a_{10} = 1{,}562{,}500/567$ (b) Decreasing
(c) Factorials increase more rapidly than exponentials.

117. 1, 1.4142, 1.4422, 1.4142, 1.3797, 1.3480; Converges to 1

119. True **121.** True **123.** True

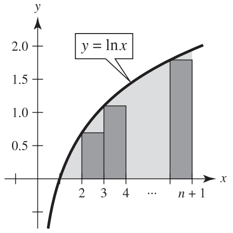
125. (a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144
(b) 1, 2, 1.5, 1.6667, 1.6, 1.6250, 1.6154, 1.6190, 1.6176, 1.6182
(c) Proof (d) $\rho = \left(1 + \sqrt{5}\right)/2 \approx 1.6180$

127. (a) 1.4142, 1.8478, 1.9616, 1.9904, 1.9976
(b) $a_n = \sqrt{2 + a_{n-1}}$ (c) $\lim\limits_{n\to\infty} a_n = 2$

129. (a) Proof (b) Proof (c) $\lim\limits_{n\to\infty} a_n = \left(1 + \sqrt{1 + 4k}\right)/2$

131. (a) Proof (b) Proof

133. (a) Proof
(b)



(c) Proof (d) Proof

(e) $\dfrac{\sqrt[20]{20!}}{20} \approx 0.4152$; $\dfrac{\sqrt[50]{50!}}{50} \approx 0.3897$; $\dfrac{\sqrt[100]{100!}}{100} \approx 0.3799$

135. Proof

137. Answers will vary. Sample answer: $a_n = (-1)^n$

139. Proof **141.** Putnam Problem A1, 1990

Section 9.2 (page 614)

1. 1, 1.25, 1.361, 1.424, 1.464

3. 3, −1.5, 5.25, −4.875, 10.3125

5. 3, 4.5, 5.25, 5.625, 5.8125

7. $\{a_n\}$ converges, $\Sigma\, a_n$ diverges

9. Geometric series: $r = \frac{7}{6} > 1$

11. Geometric series: $r = 1.055 > 1$ **13.** $\lim\limits_{n\to\infty} a_n = 1 \neq 0$

15. $\lim\limits_{n\to\infty} a_n = 1 \neq 0$ **17.** $\lim\limits_{n\to\infty} a_n = \frac{1}{2} \neq 0$ **19.** c; 3

20. b; 3 **21.** a; 3 **22.** d; 3 **23.** f; $\frac{34}{9}$ **24.** e; $\frac{5}{3}$

25. Geometric series: $r = \frac{5}{6} < 1$

27. Geometric series: $r = 0.9 < 1$

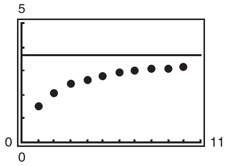
29. Telescoping series: $a_n = 1/n - 1/(n + 1)$; Converges to 1.

31. (a) $\frac{11}{3}$

(b)

n	5	10	20	50	100
S_n	2.7976	3.1643	3.3936	3.5513	3.6078

(c)



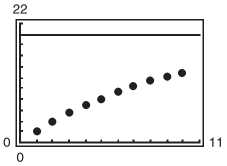
(d) The terms of the series decrease in magnitude relatively slowly, and the sequence of partial sums approaches the sum of the series relatively slowly.

33. (a) 20

(b)

n	5	10	20	50	100
S_n	8.1902	13.0264	17.5685	19.8969	19.9995

(c)



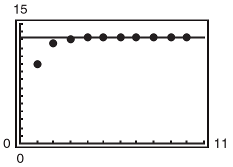
(d) The terms of the series decrease in magnitude relatively slowly, and the sequence of partial sums approaches the sum of the series relatively slowly.

35. (a) $\frac{40}{3}$

(b)

n	5	10	20	50	100
S_n	13.3203	13.3333	13.3333	13.3333	13.3333

(c)



(d) The terms of the series decrease in magnitude relatively rapidly, and the sequence of partial sums approaches the sum of the series relatively rapidly.

37. 2 **39.** $\frac{3}{4}$ **41.** $\frac{3}{4}$ **43.** 4 **45.** $\frac{10}{9}$ **47.** $\frac{9}{4}$ **49.** $\frac{1}{2}$

51. $\dfrac{\sin(1)}{1 - \sin(1)}$ **53.** (a) $\displaystyle\sum_{n=0}^{\infty} \frac{4}{10}(0.1)^n$ (b) $\dfrac{4}{9}$

55. (a) $\displaystyle\sum_{n=0}^{\infty} \frac{81}{100}(0.01)^n$ (b) $\dfrac{9}{11}$

57. (a) $\displaystyle\sum_{n=0}^{\infty} \frac{3}{40}(0.01)^n$ (b) $\dfrac{5}{66}$ **59.** Diverges **61.** Diverges

63. Converges **65.** Converges **67.** Diverges

69. Converges **71.** Diverges **73.** Diverges **75.** Diverges

77. See definitions on page 608.

79. The series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdot\cdot\cdot + ar^n + \cdot\cdot\cdot, a \neq 0$$

is a geometric series with ratio r. When $0 < |r| < 1$, the series converges to the sum $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}$.

81. The series in (a) and (b) are the same. The series in (c) is different unless $a_1 = a_2 = \cdot\cdot\cdot = a$ is constant.

83. $-2 < x < 2$; $x/(2 - x)$ **85.** $0 < x < 2$; $(x - 1)/(2 - x)$

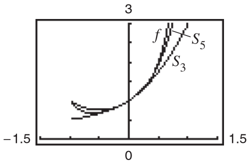
87. $-1 < x < 1$; $1/(1 + x)$

89. x: $(-\infty, -1) \cup (1, \infty)$; $x/(x - 1)$ **91.** $c = \left(\sqrt{3} - 1\right)/2$

93. Neither statement is true. The formula is valid for $-1 < x < 1$.

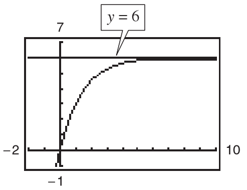
95. (a) x (b) $f(x) = 1/(1 - x)$, $|x| < 1$

(c)



Answers will vary.

97.



Horizontal asymptote: $y = 6$
The horizontal asymptote is the sum of the series.

99. The required terms for the two series are $n = 100$ and $n = 5$, respectively. The second series converges at a higher rate.

101. $160{,}000(1 - 0.95^n)$ units

103. $\sum_{i=0}^{\infty} 200(0.75)^i$; Sum = \$800 million

105. 152.42 feet **107.** $\frac{1}{8}$; $\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1/2}{1 - 1/2} = 1$

109. (a) $-1 + \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = -1 + \frac{a}{1 - r} = -1 + \frac{1}{1 - 1/2} = 1$
(b) No (c) 2

111. (a) 126 in.^{2} (b) 128 in.^{2}

113. The \$2,000,000 sweepstakes has a present value of \$1,146,992.12. After accruing interest over the 20-year period, it attains its full value.

115. (a) \$5,368,709.11 (b) \$10,737,418.23 (c) \$21,474,836.47

117. (a) \$14,773.59 (b) \$14,779.65

119. (a) \$91,373.09 (b) \$91,503.32 **121.** \$4,751,275.79

123. False. $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

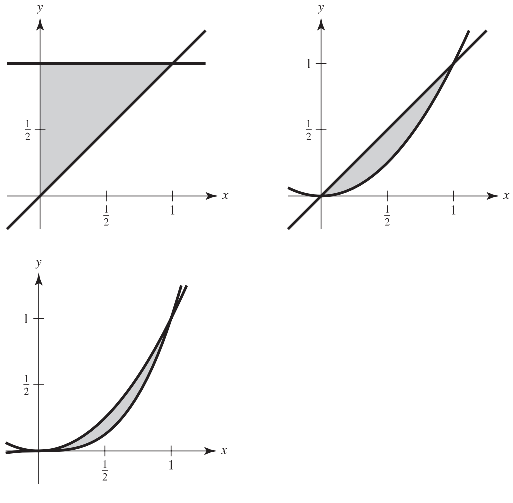
125. False. $\sum_{n=1}^{\infty} ar^n = \left(\frac{a}{1 - r}\right) - a$ The formula requires that the geometric series begins with $n = 0$.

127. True **129.** Proof

131. Answers will vary. Example: $\sum_{n=0}^{\infty} 1$, $\sum_{n=0}^{\infty} (-1)$

133–137. Proofs

139. (a)



(b) $\int_0^1 (1 - x)\,dx = \frac{1}{2}$

$\int_0^1 (x - x^2)\,dx = \frac{1}{6}$

$\int_0^1 (x^2 - x^3)\,dx = \frac{1}{12}$

(c) $a_n = \frac{1}{n} - \frac{1}{n + 1}$ and $\sum_{n=1}^{\infty} a_n = 1$; The sum of all the shaded regions is the area of the square, 1.

141. H = half-life of the drug
n = number of equal doses
P = number of units of the drug
t = equal time intervals

The total amount of the drug in the patient's system at the time the last dose is given is

$T_n = P + Pe^{kt} + Pe^{2kt} + \cdot\cdot\cdot + Pe^{(n-1)kt}$

where $k = -(\ln 2)/H$. One time interval after the last dose is administered is given by

$T_{n+1} = Pe^{kt} + Pe^{2kt} + Pe^{3kt} + \cdot\cdot\cdot + Pe^{nkt}$

and so on. Because $k < 0$, $T_{n+s} \to 0$ as $s \to \infty$.

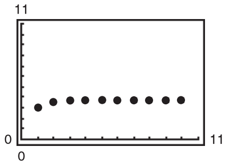
143. Putnam Problem A1, 1966

Section 9.3 (page 622)

1. Diverges **3.** Converges **5.** Converges **7.** Converges
9. Diverges **11.** Diverges **13.** Diverges **15.** Converges
17. Converges **19.** Converges **21.** Diverges
23. Diverges **25.** Diverges **27.** $f(x)$ is not positive for $x \geq 1$.
29. $f(x)$ is not always decreasing. **31.** Converges **33.** Diverges
35. Diverges **37.** Diverges **39.** Converges **41.** Converges
43. c; diverges **44.** f; diverges **45.** b; converges
46. a; diverges **47.** d; converges **48.** e; converges

49. (a)

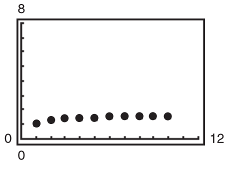
n	5	10	20	50	100
S_n	3.7488	3.75	3.75	3.75	3.75



The partial sums approach the sum 3.75 very quickly.

(b)

n	5	10	20	50	100
S_n	1.4636	1.5498	1.5962	1.6251	1.635

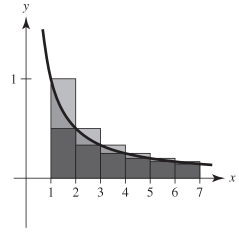


The partial sums approach the sum $\pi^2/6 \approx 1.6449$ more slowly than the series in part (a).

51. See Theorem 9.10 on page 619. Answers will vary. For example, convergence or divergence can be determined for the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$.

53. No. Because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=10{,}000}^{\infty} \frac{1}{n}$ also diverges. The convergence or divergence of a series is not determined by the first finite number of terms of the series.

55.



$$\sum_{n=1}^{6} a_n \geq \int_1^7 f(x)\,dx \geq \sum_{n=2}^{7} a_n$$

57. $p > 1$ **59.** $p > 1$ **61.** $p > 1$ **63.** Diverges

65. Converges **67.** Proof

69. $S_6 \approx 1.0811$ **71.** $S_{10} \approx 0.9818$ **73.** $S_4 \approx 0.4049$

$R_6 \approx 0.0015$ $R_{10} \approx 0.0997$ $R_4 \approx 5.6 \times 10^{-8}$

75. $N \geq 7$ **77.** $N \geq 2$ **79.** $N \geq 1000$

81. (a) $\sum_{n=2}^{\infty} \frac{1}{n^{1.1}}$ converges by the p-Series Test because $1.1 > 1$.

$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the Integral Test because $\int_2^{\infty} \frac{1}{x \ln x}\,dx$ diverges.

(b) $$\sum_{n=2}^{\infty} \frac{1}{n^{1.1}} = 0.4665 + 0.2987 + 0.2176 + 0.1703 + 0.1393 + \cdots$$

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} = 0.7213 + 0.3034 + 0.1803 + 0.1243 + 0.0930 + \cdots$$

(c) $n \geq 3.431 \times 10^{15}$

83. (a) Let $f(x) = 1/x$. f is positive, continuous, and decreasing on $[1, \infty)$.

$$S_n - 1 \leq \int_1^n \frac{1}{x}\,dx = \ln n$$

$$S_n \geq \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$$

So, $\ln(n+1) \leq S_n \leq 1 + \ln n$.

(b) $\ln(n+1) - \ln n \leq S_n - \ln n \leq 1$.

Also, $\ln(n+1) - \ln n > 0$ for $n \geq 1$. So, $0 \leq S_n - \ln n \leq 1$, and the sequence $\{a_n\}$ is bounded.

(c) $$a_n - a_{n+1} = [S_n - \ln n] - [S_{n+1} - \ln(n+1)] = \int_n^{n+1} \frac{1}{x}\,dx - \frac{1}{n+1} \geq 0$$

So, $a_n \geq a_{n+1}$.

(d) Because the sequence is bounded and monotonic, it converges to a limit, γ.

(e) 0.5822

85. (a) Diverges (b) Diverges

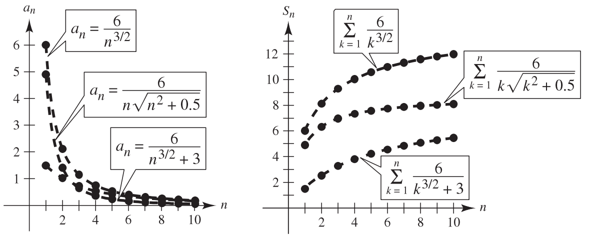
(c) $\sum_{n=2}^{\infty} x^{\ln n}$ converges for $x < 1/e$.

87. Diverges **89.** Converges **91.** Converges **93.** Diverges

95. Diverges **97.** Converges

Section 9.4 (page 630)

1. (a)



(b) $\sum_{n=1}^{\infty} \frac{6}{n^{3/2}}$; Converges

(c) The magnitudes of the terms are less than the magnitudes of the terms of the p-series. Therefore, the series converges.

(d) The smaller the magnitudes of the terms, the smaller the magnitudes of the terms of the sequence of partial sums.

3. Converges **5.** Diverges **7.** Converges **9.** Diverges

11. Converges **13.** Converges **15.** Diverges **17.** Diverges

19. Converges **21.** Converges **23.** Converges

25. Diverges **27.** Diverges **29.** Diverges; p-Series Test

31. Converges; Direct Comparison Test with $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$

33. Diverges; nth-Term Test **35.** Converges; Integral Test

37. $\lim_{n\to\infty} \frac{a_n}{1/n} = \lim_{n\to\infty} na_n$

$\lim_{n\to\infty} na_n \neq 0$, but is finite.

The series diverges by the Limit Comparison Test.

39. Diverges **41.** Converges

43. $\lim_{n\to\infty} n\left(\frac{n^3}{5n^4+3}\right) = \frac{1}{5} \neq 0$

So, $\sum_{n=1}^{\infty} \frac{n^3}{5n^4+3}$ diverges.

45. Diverges **47.** Converges

49. Convergence or divergence is dependent on the form of the general term for the series and not necessarily on the magnitudes of the terms.

51. See Theorem 9.13 on page 628. Answers will vary. For example, $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges because $\lim_{n\to\infty} \frac{1/\sqrt{n}-1}{1/\sqrt{n}} = 1$ and $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series).

53. (a) Proof

(b)

n	5	10	20	50	100
S_n	1.1839	1.2087	1.2212	1.2287	1.2312

(c) 0.1226 (d) 0.0277

55. False. Let $a_n = 1/n^3$ and $b_n = 1/n^2$.

57. True **59.** True **61.** Proof **63.** $\sum_{n=1}^{\infty} \frac{1}{n^2}, \sum_{n=1}^{\infty} \frac{1}{n^3}$

65–71. Proofs **73.** Putnam Problem B4, 1988

Section 9.5 (page 638)

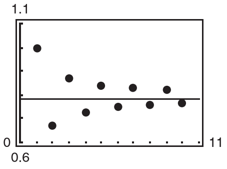
1. d **2.** f **3.** a **4.** b **5.** e **6.** c

7. (a)

n	1	2	3	4	5
S_n	1.0000	0.6667	0.8667	0.7238	0.8349

n	6	7	8	9	10
S_n	0.7440	0.8209	0.7543	0.8131	0.7605

(b)



(c) The points alternate sides of the horizontal line $y = \pi/4$ that represents the sum of the series. The distances between the successive points and the line decrease.

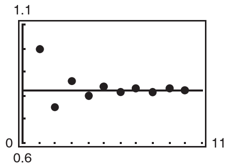
(d) The distance in part (c) is always less than the magnitude of the next term of the series.

9. (a)

n	1	2	3	4	5
S_n	1.0000	0.7500	0.8611	0.7986	0.8386

n	6	7	8	9	10
S_n	0.8108	0.8312	0.8156	0.8280	0.8180

(b)



(c) The points alternate sides of the horizontal line $y = \pi^2/12$ that represents the sum of the series. The distances between the successive points and the line decrease.

(d) The distance in part (c) is always less than the magnitude of the next term of the series.

11. Converges **13.** Converges **15.** Diverges
17. Converges **19.** Diverges **21.** Converges **23.** Diverges
25. Diverges **27.** Diverges **29.** Converges
31. Converges **33.** Converges **35.** Converges
37. $0.7305 \le S \le 0.7361$ **39.** $2.3713 \le S \le 2.4937$
41. (a) 7 terms (Note that the sum begins with $n = 0$.) (b) 0.368
43. (a) 3 terms (Note that the sum begins with $n = 0$.) (b) 0.842
45. (a) 1000 terms (b) 0.693 **47.** 10 **49.** 7
51. Converges absolutely **53.** Converges absolutely
55. Converges absolutely **57.** Converges conditionally
59. Diverges **61.** Converges conditionally
63. Converges absolutely **65.** Converges absolutely
67. Converges conditionally **69.** Converges absolutely
71. An alternating series is a series whose terms alternate in sign.
73. $|S - S_N| = |R_N| \le a_{N+1}$
75. Graph (b). The partial sums alternate above and below the horizontal line representing the sum.
77. True **79.** $p > 0$
81. Proof; The converse is false. For example: Let $a_n = 1/n$.
83. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, hence so does $\sum_{n=1}^{\infty} \frac{1}{n^4}$.
85. (a) No; $a_{n+1} \le a_n$ is not satisfied for all n. For example, $\frac{1}{9} < \frac{1}{8}$.
(b) Yes; 0.5
87. Converges; p-Series Test **89.** Diverges; nth-Term Test
91. Converges; Geometric Series Test
93. Converges; Integral Test
95. Converges; Alternating Series Test
97. The first term of the series is 0, not 1. You cannot regroup series terms arbitrarily.
99. Putnam Problem 2, afternoon session, 1954

Section 9.6 (page 647)

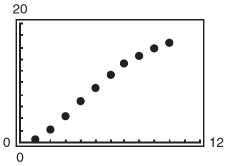
1–3. Proofs **5.** d **6.** c **7.** f **8.** b **9.** a **10.** e

11. (a) Proof

(b)

n	5	10	15	20	25
S_n	9.2104	16.7598	18.8016	19.1878	19.2491

(c)



(d) 19.26

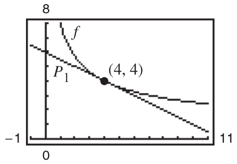
(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of partial sums approaches the sum of the series.

13. Converges **15.** Diverges **17.** Diverges
19. Converges **21.** Diverges **23.** Converges
25. Diverges **27.** Converges **29.** Converges
31. Diverges **33.** Converges **35.** Converges
37. Converges **39.** Diverges **41.** Converges
43. Diverges **45.** Converges **47.** Converges
49. Converges **51.** Converges; Alternating Series Test
53. Converges; p-Series Test **55.** Diverges; nth-Term Test
57. Diverges; Geometric Series Test
59. Converges; Limit Comparison Test with $b_n = 1/2^n$
61. Converges; Direct Comparison Test with $b_n = 1/3^n$
63. Converges; Ratio Test **65.** Converges; Ratio Test
67. Converges; Ratio Test **69.** a and c **71.** a and b
73. $\sum_{n=0}^{\infty} \frac{n+1}{7^{n+1}}$ **75.** (a) 9 (b) -0.7769
77. Diverges; $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$
79. Converges; $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$
81. Diverges; $\lim a_n \ne 0$ **83.** Converges **85.** Converges
87. $(-3, 3)$ **89.** $(-2, 0]$ **91.** $x = 0$
93. See Theorem 9.17 on page 641.
95. No; the series $\sum_{n=1}^{\infty} \frac{1}{n + 10{,}000}$ diverges.
97. Absolutely; by Theorem 9.17 **99–105.** Proofs
107. (a) Diverges (b) Converges (c) Converges
(d) Converges for all integers $x \ge 2$
109. Answers will vary.
111. Putnam Problem 7, morning session, 1951

Section 9.7 (page 658)

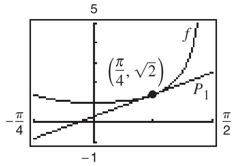
1. d **2.** c **3.** a **4.** b

5. $P_1 = -\frac{1}{2}x + 6$



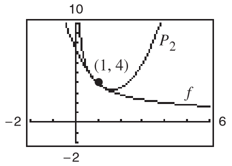
P_1 is the first-degree Taylor polynomial for f at 4.

7. $P_1 = \sqrt{2}x + \sqrt{2}(4 - \pi)/4$



P_1 is the first-degree Taylor polynomial for f at $\pi/4$.

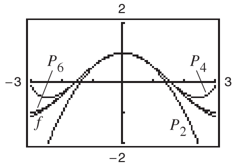
9.



x	0	0.8	0.9	1	1.1
$f(x)$	Error	4.4721	4.2164	4.0000	3.8139
$P_2(x)$	7.5000	4.4600	4.2150	4.0000	3.8150

x	1.2	2
$f(x)$	3.6515	2.8284
$P_2(x)$	3.6600	3.5000

11. (a)



(b) $f^{(2)}(0) = -1$ $\quad P_2^{(2)}(0) = -1$
$f^{(4)}(0) = 1$ $\quad P_4^{(4)}(0) = 1$
$f^{(6)}(0) = -1$ $\quad P_6^{(6)}(0) = -1$

(c) $f^{(n)}(0) = P_n^{(n)}(0)$

13. $1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$

15. $1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4$

17. $x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ **19.** $x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$

21. $1 - x + x^2 - x^3 + x^4 - x^5$ **23.** $1 + \frac{1}{2}x^2$

25. $2 - 2(x - 1) + 2(x - 1)^2 - 2(x - 1)^3$

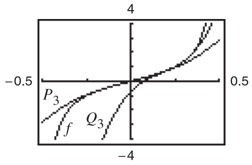
27. $2 + \frac{1}{4}(x - 4) - \frac{1}{64}(x - 4)^2 + \frac{1}{512}(x - 4)^3$

29. $\ln 2 + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + \frac{1}{24}(x - 2)^3 - \frac{1}{64}(x - 2)^4$

31. (a) $P_3(x) = \pi x + \dfrac{\pi^3}{3}x^3$

(b)

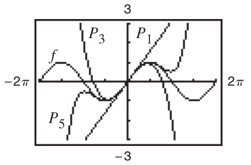
$$Q_3(x) = 1 + 2\pi\left(x - \frac{1}{4}\right) + 2\pi^2\left(x - \frac{1}{4}\right)^2 + \frac{8\pi^3}{3}\left(x - \frac{1}{4}\right)^3$$



33. (a)

x	0	0.25	0.50	0.75	1.00
$\sin x$	0	0.2474	0.4794	0.6816	0.8415
$P_1(x)$	0	0.25	0.50	0.75	1.00
$P_3(x)$	0	0.2474	0.4792	0.6797	0.8333
$P_5(x)$	0	0.2474	0.4794	0.6817	0.8417

(b)



(c) As the distance increases, the polynomial approximation becomes less accurate.

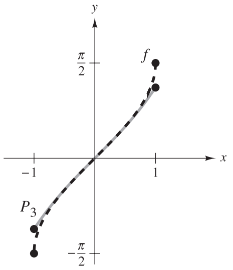
35. (a) $P_3(x) = x + \frac{1}{6}x^3$

(b)

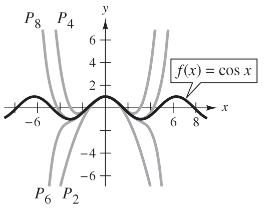
x	−0.75	−0.50	−0.25	0	0.25
$f(x)$	−0.848	−0.524	−0.253	0	0.253
$P_3(x)$	−0.820	−0.521	−0.253	0	0.253

x	0.50	0.75
$f(x)$	0.524	0.848
$P_3(x)$	0.521	0.820

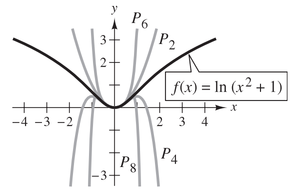
(c)



37.



39.



41. 4.3984 **43.** 0.7419 **45.** $R_4 \leq 2.03 \times 10^{-5}$; 0.000001

47. $R_3 \leq 7.82 \times 10^{-3}$; 0.00085 **49.** 3 **51.** 5

53. $n = 9$; $\ln(1.5) \approx 0.4055$ **55.** $n = 16$; $e^{-\pi(1.3)} \approx 0.01684$

57. $-0.3936 < x < 0$ **59.** $-0.9467 < x < 0.9467$

61. The graph of the approximating polynomial P and the elementary function f both pass through the point $(c, f(c))$, and the slope of the graph of P is the same as the slope of the graph of f at the point $(c, f(c))$. If P is of degree n, then the first n derivatives of f and P agree at c. This allows for the graph of P to resemble the graph of f near the point $(c, f(c))$.

63. See "Definitions of nth Taylor Polynomial and nth Maclaurin Polynomial" on page 652.

65. As the degree of the polynomial increases, the graph of the Taylor polynomial becomes a better and better approximation of the function within the interval of convergence. Therefore, the accuracy is increased.

67. (a) $f(x) \approx P_4(x) = 1 + x + (1/2)x^2 + (1/6)x^3 + (1/24)x^4$
$g(x) \approx Q_5(x) = x + x^2 + (1/2)x^3 + (1/6)x^4 + (1/24)x^5$
$Q_5(x) = xP_4(x)$
(b) $g(x) \approx P_6(x) = x^2 - x^4/3! + x^6/5!$
(c) $g(x) \approx P_4(x) = 1 - x^2/3! + x^4/5!$

69. (a) $Q_2(x) = -1 + (\pi^2/32)(x + 2)^2$
(b) $R_2(x) = -1 + (\pi^2/32)(x - 6)^2$
(c) No. Horizontal translations of the result in part (a) are possible only at $x = -2 + 8n$ (where n is an integer) because the period of f is 8.

71. Proof

73. As you move away from $x = c$, the Taylor polynomial becomes less and less accurate.

Section 9.8 (page 668)

1. 0 **3.** 2 **5.** $R = 1$ **7.** $R = \frac{1}{4}$ **9.** $R = \infty$
11. $(-4, 4)$ **13.** $(-1, 1]$ **15.** $(-\infty, \infty)$ **17.** $x = 0$
19. $(-4, 4)$ **21.** $(-5, 13]$ **23.** $(0, 2]$ **25.** $(0, 6)$
27. $\left(-\frac{1}{2}, \frac{1}{2}\right)$ **29.** $(-\infty, \infty)$ **31.** $(-1, 1)$ **33.** $x = 3$
35. $R = c$ **37.** $(-k, k)$ **39.** $(-1, 1)$
41. $\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$ **43.** $\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$
45. (a) $(-3, 3)$ (b) $(-3, 3)$ (c) $(-3, 3)$ (d) $[-3, 3)$
47. (a) $(0, 2]$ (b) $(0, 2)$ (c) $(0, 2)$ (d) $[0, 2]$
49. c; $S_1 = 1, S_2 = 1.33$ **50.** a; $S_1 = 1, S_2 = 1.67$
51. b; diverges **52.** d; alternating
53. b **54.** c **55.** d **56.** a

57. A series of the form

$$\sum_{n=0}^{\infty} a_n(x - c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots + a_n(x - c)^n + \cdots$$

is called a power series centered at c, where c is a constant.

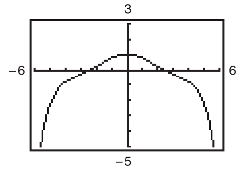
59. 1. A single point 2. An interval centered at c
3. The entire real line

61. Answers will vary.

63. (a) For $f(x)$: $(-\infty, \infty)$; For $g(x)$: $(-\infty, \infty)$
(b) Proof (c) Proof (d) $f(x) = \sin x$; $g(x) = \cos x$

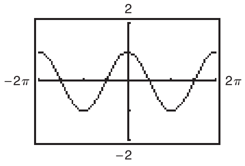
65–69. Proofs

71. (a) Proof (b) Proof
(c)



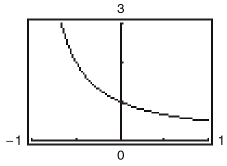
(d) 0.92

73.



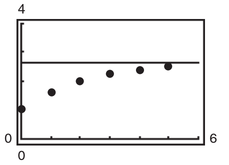
$f(x) = \cos x$

75.

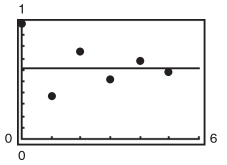


$f(x) = 1/(1 + x)$

77. (a) $\frac{8}{3}$



(b) $\frac{8}{13}$



(c) The alternating series converges more rapidly. The partial sums of the series of positive terms approach the sum from below. The partial sums of the alternating series alternate sides of the horizontal line representing the sum.

(d)

M	10	100	1000	10,000
N	5	14	24	35

79. False. Let $a_n = (-1)^n/(n2^n)$ **81.** True **83.** Proof
85. (a) $(-1, 1)$ (b) $f(x) = (c_0 + c_1x + c_2x^2)/(1 - x^3)$
87. Proof

Section 9.9 (page 676)

1. $\sum_{n=0}^{\infty} \frac{x^n}{4^{n+1}}$ **3.** $\sum_{n=0}^{\infty} \frac{3(-1)^n x^n}{4^{n+1}}$ **5.** $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^{n+1}}$ **7.** $\sum_{n=0}^{\infty} (3x)^n$
$(-1, 3)$ $\left(-\frac{1}{3}, \frac{1}{3}\right)$

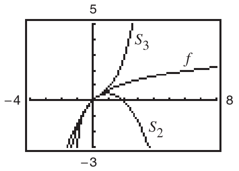
9. $-\frac{5}{9}\sum_{n=0}^{\infty}\left[\frac{2}{9}(x + 3)\right]^n$ **11.** $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^n}{3^{n+1}}$
$\left(-\frac{15}{2}, \frac{3}{2}\right)$ $\left(-\frac{3}{2}, \frac{3}{2}\right)$

13. $\sum_{n=0}^{\infty}\left[\frac{1}{(-3)^n} - 1\right]x^n$ **15.** $\sum_{n=0}^{\infty} x^n[1 + (-1)^n] = 2\sum_{n=0}^{\infty} x^{2n}$
$(-1, 1)$ $(-1, 1)$

17. $2\sum_{n=0}^{\infty} x^{2n}$ **19.** $\sum_{n=1}^{\infty} n(-1)^n x^{n-1}$ **21.** $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n + 1}$
$(-1, 1)$ $(-1, 1)$ $(-1, 1]$

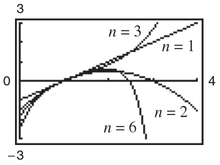
23. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ **25.** $\sum_{n=0}^{\infty} (-1)^n (2x)^{2n}$
$(-1, 1)$ $\left(-\frac{1}{2}, \frac{1}{2}\right)$

27.



x	0.0	0.2	0.4	0.6	0.8	1.0
S_2	0.000	0.180	0.320	0.420	0.480	0.500
$\ln(x + 1)$	0.000	0.182	0.336	0.470	0.588	0.693
S_3	0.000	0.183	0.341	0.492	0.651	0.833

29. (a)



(b) $\ln x, 0 < x \le 2, R = 1$
(c) -0.6931
(d) $\ln(0.5)$; The error is approximately 0.

31. c **32.** d **33.** a **34.** b **35.** 0.245 **37.** 0.125

39. $\sum_{n=1}^{\infty} nx^{n-1},\ -1 < x < 1$ **41.** $\sum_{n=0}^{\infty} (2n+1)x^n,\ -1 < x < 1$

43. $E(n) = 2$. Because the probability of obtaining a head on a single toss is $\frac{1}{2}$, it is expected that, on average, a head will be obtained in two tosses.

45. Because $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)}$, substitute $(-x)$ into the geometric series.

47. Because $\dfrac{5}{1+x} = 5\left(\dfrac{1}{1-(-x)}\right)$, substitute $(-x)$ into the geometric series and then multiply the series by 5.

49. Proof **51.** (a) Proof (b) 3.14

53. $\ln \frac{3}{2} \approx 0.4055$; See Exercise 21.

55. $\ln \frac{7}{5} \approx 0.3365$; See Exercise 53.

57. $\arctan \frac{1}{2} \approx 0.4636$; See Exercise 56.

59. $f(x) = \arctan x$ is an odd function (symmetric to the origin).

61. The series in Exercise 56 converges to its sum at a lower rate because its terms approach 0 at a much lower rate.

63. The series converges on the interval $(-5, 3)$ and perhaps also at one or both endpoints.

65. $\sqrt{3}\pi/6$ **67.** $S_1 = 0.3183098862,\ 1/\pi \approx 0.3183098862$

Section 9.10 (page 687)

1. $\sum_{n=0}^{\infty} \dfrac{(2x)^n}{n!}$ **3.** $\dfrac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \dfrac{(-1)^{n(n+1)/2}}{n!}\left(x - \dfrac{\pi}{4}\right)^n$

5. $\sum_{n=0}^{\infty} (-1)^n (x-1)^n$ **7.** $\sum_{n=0}^{\infty} \dfrac{(-1)^n (x-1)^{n+1}}{n+1}$

9. $\sum_{n=0}^{\infty} \dfrac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$ **11.** $1 + x^2/2! + 5x^4/4! + \cdot \cdot \cdot$

13–15. Proofs **17.** $\sum_{n=0}^{\infty} (-1)^n (n+1)x^n$

19. $1 + \sum_{n=1}^{\infty} \dfrac{1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2n-1)x^n}{2^n n!}$

21. $\dfrac{1}{2}\left[1 + \sum_{n=1}^{\infty} \dfrac{(-1)^n 1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2n-1)x^{2n}}{2^{3n} n!}\right]$

23. $1 + \dfrac{x}{2} + \sum_{n=2}^{\infty} \dfrac{(-1)^{n+1}\, 1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2n-3)x^n}{2^n n!}$

25. $1 + \dfrac{x^2}{2} + \sum_{n=2}^{\infty} \dfrac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2n-3)x^{2n}}{2^n n!}$

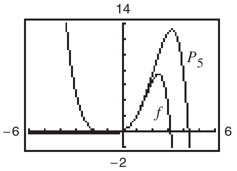
27. $\sum_{n=0}^{\infty} \dfrac{x^{2n}}{2^n n!}$ **29.** $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1} x^n}{n}$ **31.** $\sum_{n=0}^{\infty} \dfrac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$

33. $\sum_{n=0}^{\infty} \dfrac{(-1)^n 4^{2n} x^{2n}}{(2n)!}$ **35.** $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{3n}}{(2n)!}$ **37.** $\sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}$

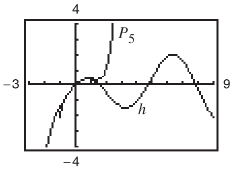
39. $\dfrac{1}{2}\left[1 + \sum_{n=0}^{\infty} \dfrac{(-1)^n (2x)^{2n}}{(2n)!}\right]$ **41.** $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+2}}{(2n+1)!}$

43. $\begin{cases} \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n+1)!}, & x \ne 0 \\ 1, & x = 0 \end{cases}$ **45.** Proof

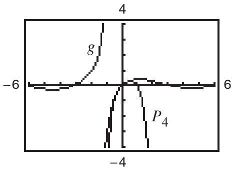
47. $P_5(x) = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5$



49. $P_5(x) = x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{3}{40}x^5$



51. $P_4(x) = x - x^2 + \frac{5}{6}x^3 - \frac{5}{6}x^4$



53. c; $f(x) = x \sin x$ **54.** d; $f(x) = x \cos x$ **55.** a; $f(x) = xe^x$

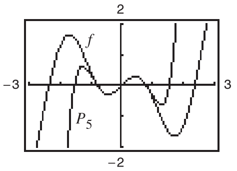
56. b; $f(x) = x^2\left(\dfrac{1}{1+x}\right)$ **57.** $\sum_{n=0}^{\infty} \dfrac{(-1)^{(n+1)} x^{2n+3}}{(2n+3)(n+1)!}$

59. 0.6931 **61.** 7.3891 **63.** 0 **65.** 1 **67.** 0.8075

69. 0.9461 **71.** 0.4872 **73.** 0.2010 **75.** 0.7040

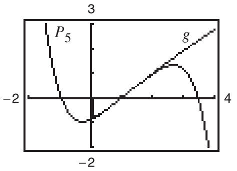
77. 0.3412

79. $P_5(x) = x - 2x^3 + \frac{2}{3}x^5$



$\left[-\frac{3}{4}, \frac{3}{4}\right]$

81. $P_5(x) = (x-1) - \frac{1}{24}(x-1)^3 + \frac{1}{24}(x-1)^4 - \frac{71}{1920}(x-1)^5$



$\left[\frac{1}{4}, 2\right]$

83. See "Guidelines for Finding a Taylor Series" on page 682.

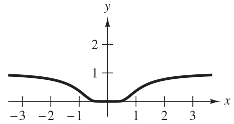
85. The binomial series is given by

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdot \cdot \cdot .$$

The radius of convergence is $R = 1$.

87. Proof

89. (a)



(b) Proof

(c) $\sum_{n=0}^{\infty} 0x^n = 0 \ne f(x)$

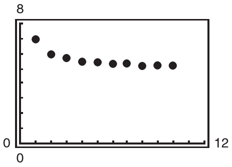
91. Proof **93.** 10 **95.** -0.0390625

97. $\sum_{n=0}^{\infty} \binom{k}{n} x^n$ **99.** Proof

Review Exercises for Chapter 9 (page 690)

1. $a_n = 1/(n! + 1)$ **3.** a **4.** c **5.** d **6.** b

7.



Converges to 5

9. Converges to 3 **11.** Diverges **13.** Converges to 0

15. Converges to 0 **17.** Converges to 0

19. (a)

n	1	2	3	4
A_n	\$8100.00	\$8201.25	\$8303.77	\$8407.56

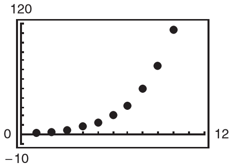
n	5	6	7	8
A_n	\$8512.66	\$8619.07	\$8726.80	\$8835.89

(b) \$13,148.96

21. (a)

n	5	10	15	20	25
S_n	13.2	113.3	873.8	6648.5	50,500.3

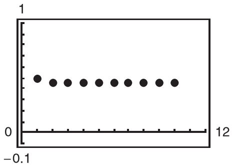
(b)



23. (a)

n	5	10	15	20	25
S_n	0.4597	0.4597	0.4597	0.4597	0.4597

(b)



25. 3 **27.** 5.5 **29.** (a) $\sum_{n=0}^{\infty} (0.09)(0.01)^n$ (b) $\frac{1}{11}$

31. Diverges **33.** Diverges **35.** $45\frac{1}{3}$ m **37.** \$7630.70

39. Converges **41.** Diverges **43.** Diverges

45. Converges **47.** Diverges **49.** Converges

51. Converges **53.** Diverges **55.** Diverges

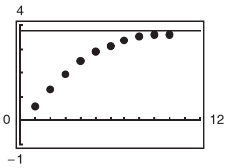
57. Converges **59.** Diverges

61. (a) Proof

(b)

n	5	10	15	20	25
S_n	2.8752	3.6366	3.7377	3.7488	3.7499

(c)



(d) 3.75

63. (a)

N	5	10	20	30	40
$\sum_{n=1}^{N} \frac{1}{n^2}$	1.4636	1.5498	1.5962	1.6122	1.6202
$\int_N^{\infty} \frac{1}{x^2}\,dx$	0.2000	0.1000	0.0500	0.0333	0.0250

(b)

N	5	10	20	30	40
$\sum_{n=1}^{N} \frac{1}{n^5}$	1.0367	1.0369	1.0369	1.0369	1.0369
$\int_N^{\infty} \frac{1}{x^5}\,dx$	0.0004	0.0000	0.0000	0.0000	0.0000

The series in part (b) converges more rapidly. This is evident from the integrals that give the remainders of the partial sums.

65. $P_3(x) = 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3$ **67.** 0.996 **69.** 0.559

71. (a) 4 (b) 6 (c) 5 (d) 10 **73.** $(-10, 10)$

75. $[1, 3]$ **77.** Converges only at $x = 2$ **79.** Proof

81. $\sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{x}{3}\right)^n$ **83.** $\sum_{n=0}^{\infty} \frac{2}{9}(n + 1)\left(\frac{x}{3}\right)^n$

85. $f(x) = \frac{3}{3 - 2x}, \left(-\frac{3}{2}, \frac{3}{2}\right)$ **87.** $\frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}}{n!}\left(x - \frac{3\pi}{4}\right)^n$

89. $\sum_{n=0}^{\infty} \frac{(x \ln 3)^n}{n!}$ **91.** $-\sum_{n=0}^{\infty} (x + 1)^n$

93. $1 + x/5 - 2x^2/25 + 6x^3/125 - 21x^4/625 + \cdots$

95. $\ln \frac{5}{4} \approx 0.2231$ **97.** $e^{1/2} \approx 1.6487$ **99.** $\cos \frac{2}{3} \approx 0.7859$

101. The series in Exercise 45 converges to its sum at a lower rate because its terms approach 0 at a lower rate.

103. (a)–(c) $1 + 2x + 2x^2 + \frac{4}{3}x^3$

105. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)(2n + 1)!}$ **107.** $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n + 1)^2}$ **109.** 0

P.S. Problem Solving (page 693)

1. (a) 1 (b) Answers will vary. Example: $0, \frac{1}{3}, \frac{2}{3}$ (c) 0

3. Proof **5.** (a) Proof (b) Yes (c) Any distance

7. For $a = b$, the series converges conditionally. For no values of a and b does the series converge absolutely.

9. 665,280 **11.** (a) Proof (b) Diverges

13. Proof **15.** (a) Proof (b) Proof

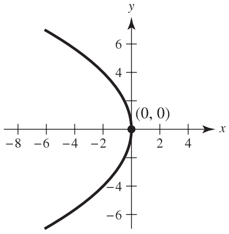
17. (a) The height is infinite. (b) The surface area is infinite. (c) Proof

Chapter 10

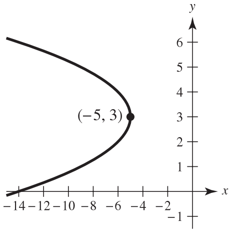
Section 10.1 (page 706)

1. h **2.** a **3.** e **4.** b **5.** f **6.** g **7.** c **8.** d

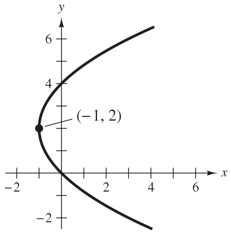
9. Vertex: $(0, 0)$
Focus: $(-2, 0)$
Directrix: $x = 2$



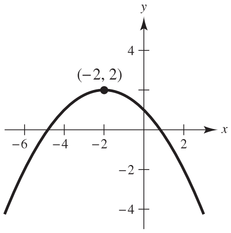
11. Vertex: $(-5, 3)$
Focus: $\left(-\frac{21}{4}, 3\right)$
Directrix: $x = -\frac{19}{4}$



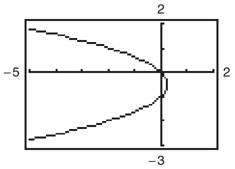
13. Vertex: $(-1, 2)$
Focus: $(0, 2)$
Directrix: $x = -2$



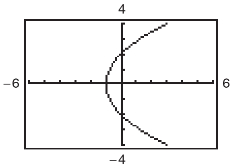
15. Vertex: $(-2, 2)$
Focus: $(-2, 1)$
Directrix: $y = 3$



17. Vertex: $\left(\frac{1}{4}, -\frac{1}{2}\right)$
Focus: $\left(0, -\frac{1}{2}\right)$
Directrix: $x = \frac{1}{2}$



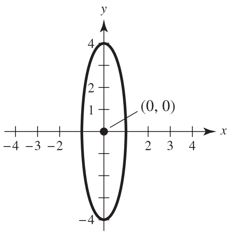
19. Vertex: $(-1, 0)$
Focus: $(0, 0)$
Directrix: $x = -2$



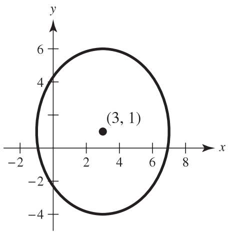
21. $y^2 - 8y + 8x - 24 = 0$ **23.** $x^2 - 32y + 160 = 0$

25. $x^2 + y - 4 = 0$ **27.** $5x^2 - 14x - 3y + 9 = 0$

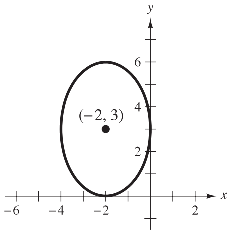
29. Center: $(0, 0)$
Foci: $\left(0, \pm\sqrt{15}\right)$
Vertices: $(0, \pm 4)$
$e = \sqrt{15}/4$



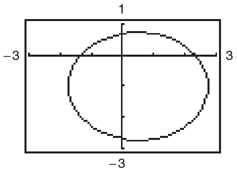
31. Center: $(3, 1)$
Foci: $(3, 4), (3, -2)$
Vertices: $(3, 6), (3, -4)$
$e = \frac{3}{5}$



33. Center: $(-2, 3)$
Foci: $\left(-2, 3 \pm \sqrt{5}\right)$
Vertices: $(-2, 6), (-2, 0)$
$e = \sqrt{5}/3$

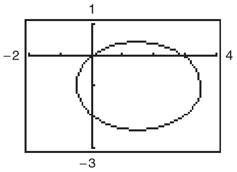


35. Center: $\left(\frac{1}{2}, -1\right)$
Foci: $\left(\frac{1}{2} \pm \sqrt{2}, -1\right)$
Vertices: $\left(\frac{1}{2} \pm \sqrt{5}, -1\right)$
To obtain the graph, solve
for y and get



$y_1 = -1 + \sqrt{(57 + 12x - 12x^2)/20}$ and
$y_2 = -1 - \sqrt{(57 + 12x - 12x^2)/20}$.
Graph these equations in the same viewing window.

37. Center: $\left(\frac{3}{2}, -1\right)$
Foci: $\left(\frac{3}{2} \pm \sqrt{2}, -1\right)$
Vertices: $\left(-\frac{1}{2}, -1\right), \left(\frac{7}{2}, -1\right)$
To obtain the graph, solve for y
and get

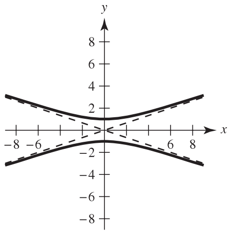


$y_1 = -1 + \sqrt{(7 + 12x - 4x^2)/8}$ and
$y_2 = -1 - \sqrt{(7 + 12x - 4x^2)/8}$.
Graph these equations in the same viewing window.

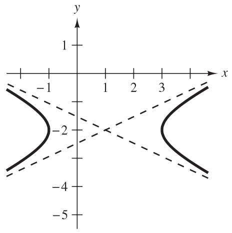
39. $x^2/36 + y^2/11 = 1$ **41.** $(x - 3)^2/9 + (y - 5)^2/16 = 1$

43. $x^2/16 + 7y^2/16 = 1$

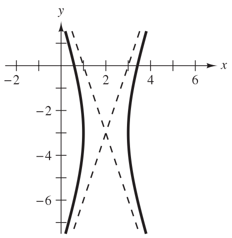
45. Center: $(0, 0)$
Foci: $\left(0, \pm\sqrt{10}\right)$
Vertices: $(0, \pm 1)$



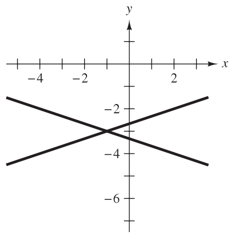
47. Center: $(1, -2)$
Foci: $\left(1 \pm \sqrt{5}, -2\right)$
Vertices: $(-1, -2), (3, -2)$



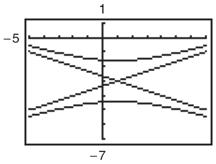
49. Center: $(2, -3)$
Foci: $\left(2 \pm \sqrt{10}, -3\right)$
Vertices: $(1, -3), (3, -3)$



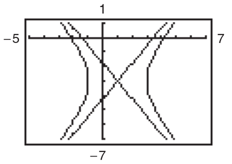
51. Degenerate hyperbola
Graph is two lines
$y = -3 \pm \frac{1}{3}(x + 1)$
intersecting at $(-1, -3)$.



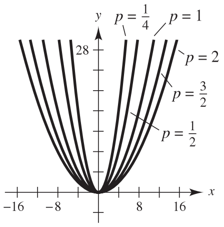
53. Center: $(1, -3)$
Foci: $\left(1, -3 \pm 2\sqrt{5}\right)$
Vertices: $\left(1, -3 \pm \sqrt{2}\right)$
Asymptotes:
$y = \frac{1}{3}x - \frac{1}{3} - 3;$
$y = -\frac{1}{3}x + \frac{1}{3} - 3$



55. Center: $(1, -3)$
Foci: $\left(1 \pm \sqrt{10}, -3\right)$
Vertices: $(-1, -3), (3, -3)$
Asymptotes:
$y = \sqrt{6}x/2 - \sqrt{6}/2 - 3;$
$y = -\sqrt{6}x/2 + \sqrt{6}/2 - 3$

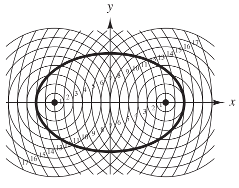


57. $x^2/1 - y^2/25 = 1$ **59.** $y^2/9 - (x-2)^2/(9/4) = 1$
61. $y^2/4 - x^2/12 = 1$ **63.** $(x-3)^2/9 - (y-2)^2/4 = 1$
65. (a) $\left(6, \sqrt{3}\right)$: $2x - 3\sqrt{3}y - 3 = 0$
$\left(6, -\sqrt{3}\right)$: $2x + 3\sqrt{3}y - 3 = 0$
(b) $\left(6, \sqrt{3}\right)$: $9x + 2\sqrt{3}y - 60 = 0$
$\left(6, -\sqrt{3}\right)$: $9x - 2\sqrt{3}y - 60 = 0$
67. Ellipse **69.** Parabola **71.** Circle
73. Circle **75.** Hyperbola
77. (a) A parabola is the set of all points (x, y) that are equidistant from a fixed line and a fixed point not on the line.
(b) For directrix $y = k - p$: $(x-h)^2 = 4p(y-k)$
For directrix $x = h - p$: $(y-k)^2 = 4p(x-h)$
(c) If P is a point on a parabola, then the tangent line to the parabola at P makes equal angles with the line passing through P and the focus, and with the line passing through P parallel to the axis of the parabola.
79. (a) A hyperbola is the set of all points (x, y) for which the absolute value of the difference between the distances from two distinct fixed points is constant.
(b) Transverse axis is horizontal: $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$
Transverse axis is vertical: $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$
(c) Transverse axis is horizontal:
$y = k + (b/a)(x-h)$ and $y = k - (b/a)(x-h)$
Transverse axis is vertical:
$y = k + (a/b)(x-h)$ and $y = k - (a/b)(x-h)$
81. $\frac{9}{4}$ m **83.** $y = 2ax_0x - ax_0^2$ **85.** (a) Proof (b) Proof
87. $x_0 = 2\sqrt{3}/3$; Distance from hill: $2\sqrt{3}/3 - 1$
89. $[16(4 + 3\sqrt{3} - 2\pi)]/3 \approx 15.536$ ft^2
91. (a) $y = (1/180)x^2$
(b) $10\left[2\sqrt{13} + 9\ln\left(\dfrac{2+\sqrt{13}}{3}\right)\right] \approx 128.4$ m
93.

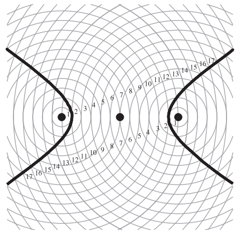


As p increases, the graph of $x^2 = 4py$ gets wider.

95. (a) $L = 2a$
(b) The thumbtacks are located at the foci, and the length of string is the constant sum of distances from the foci.
97.



99. Proof **101.** $e \approx 0.1776$
103. $e \approx 0.9671$ **105.** $\left(0, \frac{25}{3}\right)$
107. Minor-axis endpoints: $(-6, -2), (0, -2)$
Major-axis endpoints: $(-3, -6), (-3, 2)$
109. (a) Area $= 2\pi$
(b) Volume $= 8\pi/3$
Surface area $= [2\pi(9 + 4\sqrt{3}\pi)]/9 \approx 21.48$
(c) Volume $= 16\pi/3$
Surface area $= \dfrac{4\pi\left[6 + \sqrt{3}\ln\left(2 + \sqrt{3}\right)\right]}{3} \approx 34.69$
111. 37.96 **113.** 40 **115.** $(x-6)^2/9 - (y-2)^2/7 = 1$
117.



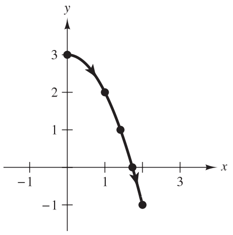
119. Proof
121. $x = \left(-90 + 96\sqrt{2}\right)/7 \approx 6.538$
$y = \left(160 - 96\sqrt{2}\right)/7 \approx 3.462$
123. There are four points of intersection.
At $\left(\dfrac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \dfrac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)$, the slopes of the tangent lines are $y'_e = -c/a$ and $y'_h = a/c$.
Because the slopes are negative reciprocals, the tangent lines are perpendicular. Similarly, the curves are perpendicular at the other three points of intersection.
125. False. See the definition of a parabola. **127.** True
129. True **131.** Putnam Problem B4, 1976

Section 10.2 (page 718)

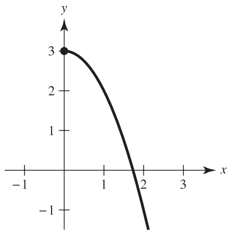
1. (a)

t	0	1	2	3	4
x	0	1	$\sqrt{2}$	$\sqrt{3}$	2
y	3	2	1	0	−1

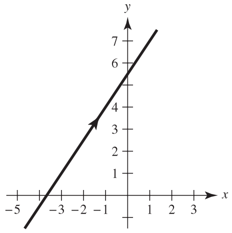
(b) and (c)



(d) $y = 3 - x^2, x \geq 0$

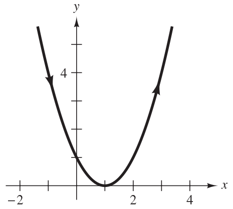


3.



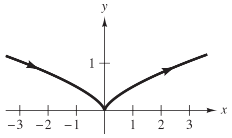
$3x - 2y + 11 = 0$

5.



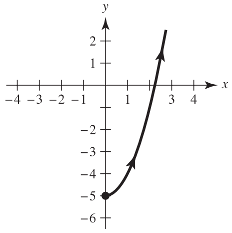
$y = (x - 1)^2$

7.



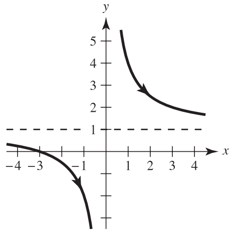
$y = \frac{1}{2}x^{2/3}$

9.



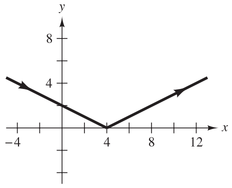
$y = x^2 - 5, \quad x \geq 0$

11.



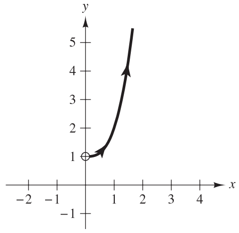
$y = (x + 3)/x$

13.



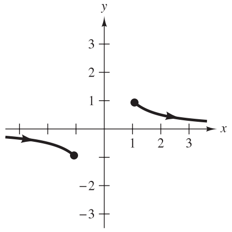
$y = |x - 4|/2$

15.



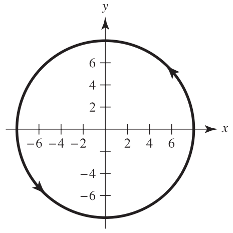
$y = x^3 + 1, \quad x > 0$

17.



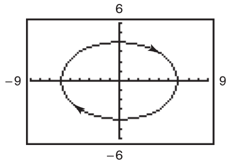
$y = 1/x, \quad |x| \geq 1$

19.



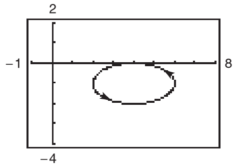
$x^2 + y^2 = 64$

21.



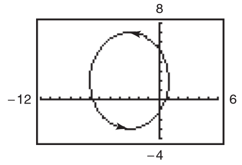
$$\frac{x^2}{36} + \frac{y^2}{16} = 1$$

23.



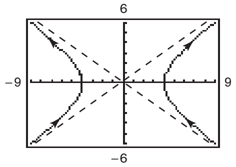
$$\frac{(x - 4)^2}{4} + \frac{(y + 1)^2}{1} = 1$$

25.



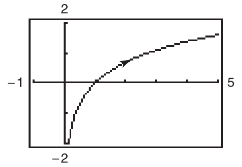
$$\frac{(x + 3)^2}{16} + \frac{(y - 2)^2}{25} = 1$$

27.



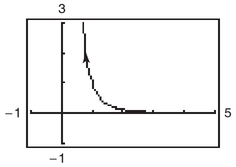
$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

29.



$y = \ln x$

31.



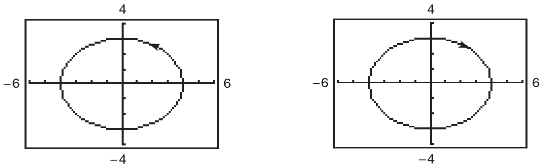
$y = 1/x^3, \quad x > 0$

33. Each curve represents a portion of the line $y = 2x + 1$.

	Domain	*Orientation*	*Smooth*
(a)	$-\infty < x < \infty$	Up	Yes
(b)	$-1 \leq x \leq 1$	Oscillates	No, $\frac{dx}{d\theta} = \frac{dy}{d\theta} = 0$ when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$
(c)	$0 < x < \infty$	Down	Yes
(d)	$0 < x < \infty$	Up	Yes

35. (a) and (b) represent the parabola $y = 2(1 - x^2)$ for $-1 \leq x \leq 1$. The curve is smooth. The orientation is from right to left in part (a) and in part (b).

37. (a)



(b) The orientation is reversed. (c) The orientation is reversed.

(d) Answers will vary. For example,

$x = 2 \sec t \qquad x = 2 \sec(-t)$

$y = 5 \sin t \qquad y = 5 \sin(-t)$

have the same graphs, but their orientations are reversed.

39. $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

41. $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$

43. $x = 4t$

$y = -7t$

(Solution is not unique.)

45. $x = 3 + 2 \cos \theta$

$y = 1 + 2 \sin \theta$

(Solution is not unique.)

47. $x = 10 \cos \theta$

$y = 6 \sin \theta$

(Solution is not unique.)

49. $x = 4 \sec \theta$

$y = 3 \tan \theta$

(Solution is not unique.)

51. $x = t$

$y = 6t - 5;$

$x = t + 1$

$y = 6t + 1$

(Solution is not unique.)

53. $x = t$

$y = t^3;$

$x = \tan t$

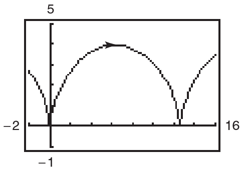
$y = \tan^3 t$

(Solution is not unique.)

55. $x = t + 3, y = 2t + 1$

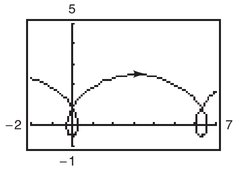
57. $x = t, y = t^2$

59.



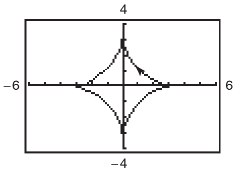
Not smooth at $\theta = 2n\pi$

61.



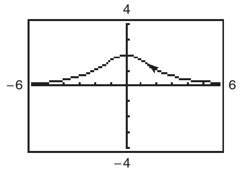
Smooth everywhere

63.



Not smooth at $\theta = \frac{1}{2}n\pi$

65.



Smooth everywhere

67. Each point (x, y) in the plane is determined by the plane curve $x = f(t)$, $y = g(t)$. For each t, plot (x, y). As t increases, the curve is traced out in a specific direction called the orientation of the curve.

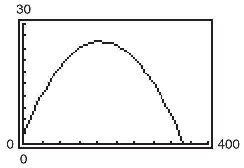
69. $x = a\theta - b \sin \theta$; $y = a - b \cos \theta$

71. False. The graph of the parametric equations is the portion of the line $y = x$ when $x \geq 0$.

73. True

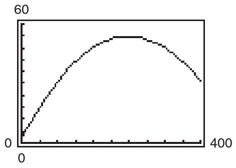
75. (a) $x = \left(\frac{440}{3} \cos \theta\right)t$; $y = 3 + \left(\frac{440}{3} \sin \theta\right)t - 16t^2$

(b)



Not a home run

(c)



Home run

(d) 19.4°

Section 10.3 (page 727)

1. $-3/t$ **3.** -1

5. $\frac{dy}{dx} = \frac{3}{4}, \frac{d^2y}{dx^2} = 0$; Neither concave upward nor concave downward

7. $dy/dx = 2t + 3$, $d^2y/dx^2 = 2$
At $t = -1$, $dy/dx = 1$, $d^2y/dx^2 = 2$; Concave upward

9. $dy/dx = -\cot \theta$, $d^2y/dx^2 = -(\csc \theta)^3/4$
At $\theta = \pi/4$, $dy/dx = -1$, $d^2y/dx^2 = -\sqrt{2}/2$;
Concave downward

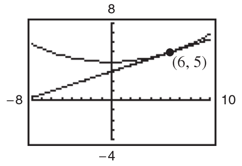
11. $dy/dx = 2 \csc \theta$, $d^2y/dx^2 = -2 \cot^3 \theta$
At $\theta = \pi/6$, $dy/dx = 4$, $d^2y/dx^2 = -6\sqrt{3}$;
Concave downward

13. $dy/dx = -\tan \theta$, $d^2y/dx^2 = \sec^4 \theta \csc \theta/3$
At $\theta = \pi/4$, $dy/dx = -1$, $d^2y/dx^2 = 4\sqrt{2}/3$;
Concave upward

15. $\left(-2/\sqrt{3}, 3/2\right)$: $3\sqrt{3}x - 8y + 18 = 0$
$(0, 2)$: $y - 2 = 0$
$\left(2\sqrt{3}, 1/2\right)$: $\sqrt{3}x + 8y - 10 = 0$

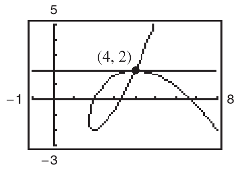
17. $(0, 0)$: $2y - x = 0$
$(-3, -1)$: $y + 1 = 0$
$(-3, 3)$: $2x - y + 9 = 0$

19. (a) and (d)



(b) At $t = 1$, $dx/dt = 6$, $dy/dt = 2$, and $dy/dx = 1/3$.
(c) $y = \frac{1}{3}x + 3$

21. (a) and (d)



(b) At $t = -1$, $dx/dt = -3$, $dy/dt = 0$, and $dy/dx = 0$.
(c) $y = 2$

23. $y = \pm\frac{3}{4}x$ **25.** $y = 3x - 5$ and $y = 1$

27. Horizontal: $(1, 0), (-1, \pi), (1, -2\pi)$
Vertical: $(\pi/2, 1), (-3\pi/2, -1), (5\pi/2, 1)$

29. Horizontal: $(4, 0)$
Vertical: None

31. Horizontal: $(5, -2), (3, 2)$
Vertical: None

33. Horizontal: $(0, 3), (0, -3)$
Vertical: $(3, 0), (-3, 0)$

35. Horizontal: $(5, -1), (5, -3)$
Vertical: $(8, -2), (2, -2)$

37. Horizontal: None
Vertical: $(1, 0), (-1, 0)$

39. Concave downward: $-\infty < t < 0$
Concave upward: $0 < t < \infty$

41. Concave upward: $t > 0$

43. Concave downward: $0 < t < \pi/2$
Concave upward: $\pi/2 < t < \pi$

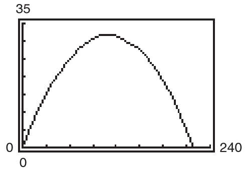
45. $\int_1^3 \sqrt{4t^2 - 3t + 9}\, dt$ **47.** $\int_{-2}^2 \sqrt{e^{2t} + 4}\, dt$

49. $4\sqrt{13} \approx 14.422$ **51.** $70\sqrt{5} \approx 156.525$

53. $\sqrt{2}(1 - e^{-\pi/2}) \approx 1.12$

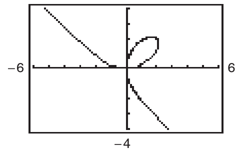
55. $\frac{1}{12}\left[\ln\left(\sqrt{37} + 6\right) + 6\sqrt{37}\right] \approx 3.249$ **57.** $6a$ **59.** $8a$

61. (a)



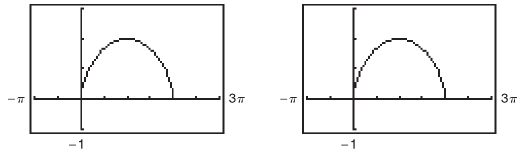
(b) 219.2 ft
(c) 230.8 ft

63. (a)



(b) $(0, 0), \left(4\sqrt[3]{2}/3, 4\sqrt[3]{4}/3\right)$
(c) About 6.557

65. (a)



(b) The average speed of the particle on the second path is twice the average speed of the particle on the first path.
(c) 4π

67. $S = 2\pi\int_0^4 \sqrt{10}(t+2)\,dt = 32\pi\sqrt{10} \approx 317.907$

69. $S = 2\pi\int_0^{\pi/2}\left(\sin\theta\cos\theta\sqrt{4\cos^2\theta+1}\right)d\theta = \dfrac{\left(5\sqrt{5}-1\right)\pi}{6}$
≈ 5.330

71. (a) $27\pi\sqrt{13}$ (b) $18\pi\sqrt{13}$ **73.** 50π **75.** $12\pi a^2/5$

77. See Theorem 10.7, Parametric Form of the Derivative, on page 721.

79. 6

81. (a) $S = 2\pi\int_a^b g(t)\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt$

(b) $S = 2\pi\int_a^b f(t)\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt$

83. Proof **85.** $3\pi/2$ **87.** d **88.** b **89.** f **90.** c

91. a **92.** e **93.** $\left(\frac{3}{4}, \frac{8}{5}\right)$ **95.** 288π

97. (a) $dy/dx = \sin\theta/(1-\cos\theta)$; $d^2y/dx^2 = -1/[a(\cos\theta - 1)^2]$

(b) $y = \left(2+\sqrt{3}\right)\left[x - a\left(\pi/6 - \frac{1}{2}\right)\right] + a\left(1 - \sqrt{3}/2\right)$

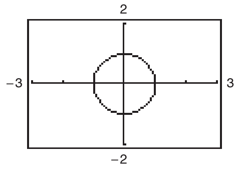
(c) $(a(2n+1)\pi, 2a)$

(d) Concave downward on $(0, 2\pi)$, $(2\pi, 4\pi)$, etc.

(e) $s = 8a$

99. Proof

101. (a)



(b) Circle of radius 1 and center at $(0, 0)$ except the point $(-1, 0)$

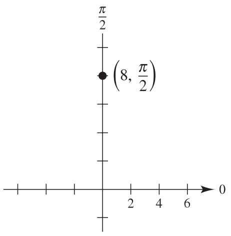
(c) As t increases from -20 to 0, the speed increases, and as t increases from 0 to 20, the speed decreases.

103. False: $\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left[\dfrac{g'(t)}{f'(t)}\right]}{f'(t)} = \dfrac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$.

105. About 982 ft

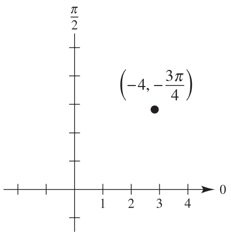
Section 10.4 (page 738)

1.



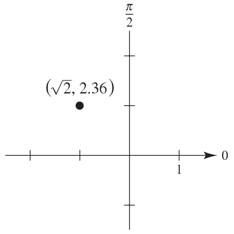
(0, 8)

3.



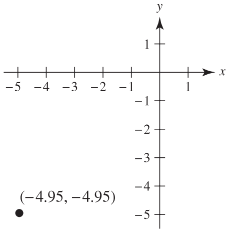
$\left(2\sqrt{2}, 2\sqrt{2}\right) \approx (2.828, 2.828)$

5.

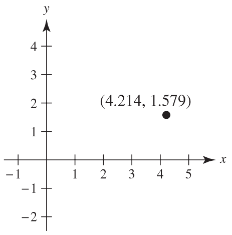


$(-1.004, 0.996)$

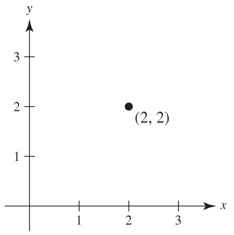
7.



9.

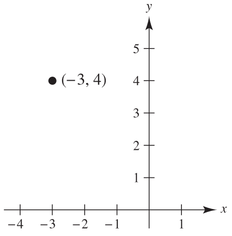


11.

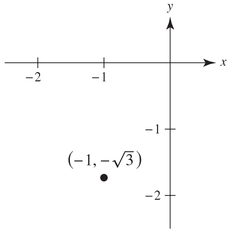


$\left(2\sqrt{2}, \pi/4\right), \left(-2\sqrt{2}, 5\pi/4\right)$

13.



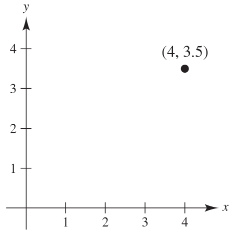
15.



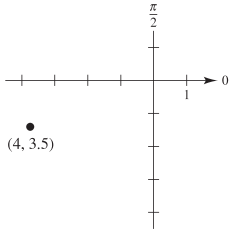
(5, 2.214), (−5, 5.356) (2, 4π/3), (−2, π/3)

17. (3.606, −0.588) **19.** (3.052, 0.960)

21. (a)

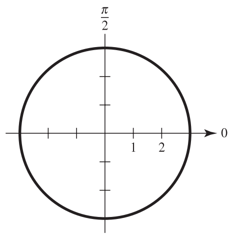


(b)

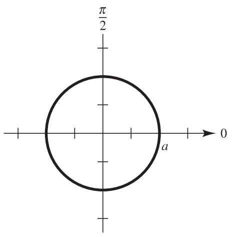


23. c **24.** b **25.** a **26.** d

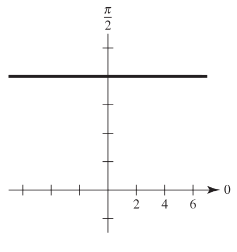
27. $r = 3$



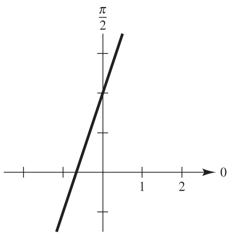
29. $r = a$



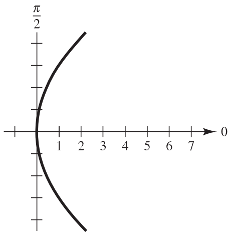
31. $r = 8\csc\theta$



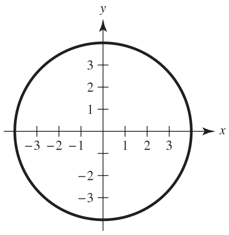
33. $r = \dfrac{-2}{3\cos\theta - \sin\theta}$



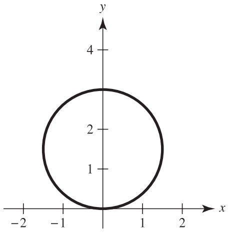
35. $r = 9\csc^2\theta\cos\theta$



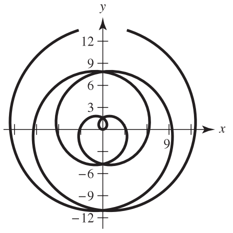
37. $x^2 + y^2 = 16$



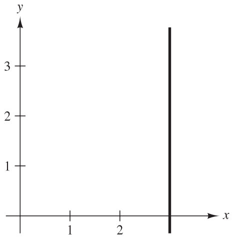
39. $x^2 + y^2 - 3y = 0$



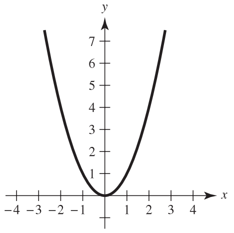
41. $\sqrt{x^2 + y^2} = \arctan(y/x)$



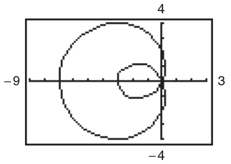
43. $x - 3 = 0$



45. $x^2 - y = 0$

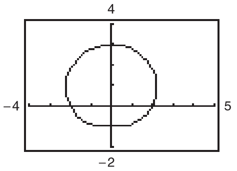


47.



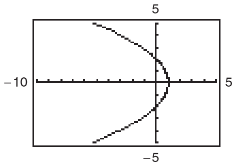
$0 \le \theta < 2\pi$

49.



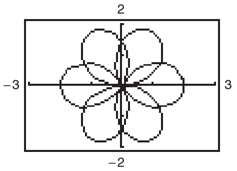
$0 \le \theta < 2\pi$

51.



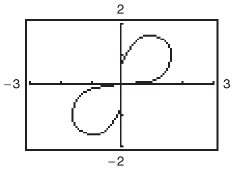
$-\pi < \theta < \pi$

53.



$0 \le \theta < 4\pi$

55.



$0 \le \theta < \pi/2$

57. $(x - h)^2 + (y - k)^2 = h^2 + k^2$

Radius: $\sqrt{h^2 + k^2}$

Center: (h, k)

59. $\sqrt{17}$ **61.** About 5.6

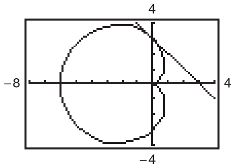
63. $\dfrac{dy}{dx} = \dfrac{2\cos\theta(3\sin\theta + 1)}{6\cos^2\theta - 2\sin\theta - 3}$

$(5, \pi/2)$: $dy/dx = 0$

$(2, \pi)$: $dy/dx = -2/3$

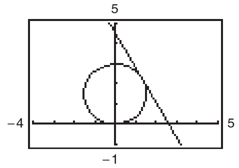
$(-1, 3\pi/2)$: $dy/dx = 0$

65. (a) and (b)



(c) $dy/dx = -1$

67. (a) and (b)



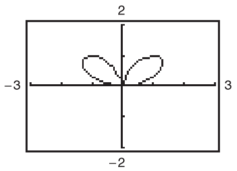
(c) $dy/dx = -\sqrt{3}$

69. Horizontal: $(2, 3\pi/2)$, $\left(\frac{1}{2}, \pi/6\right)$, $\left(\frac{1}{2}, 5\pi/6\right)$

Vertical: $\left(\frac{3}{2}, 7\pi/6\right)$, $\left(\frac{3}{2}, 11\pi/6\right)$

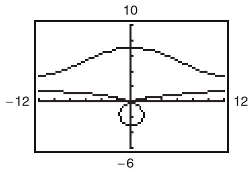
71. $(5, \pi/2)$, $(1, 3\pi/2)$

73.



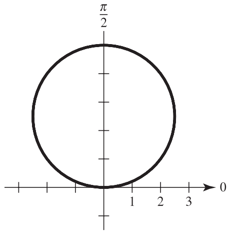
(0, 0), (1.4142, 0.7854), (1.4142, 2.3562)

75.



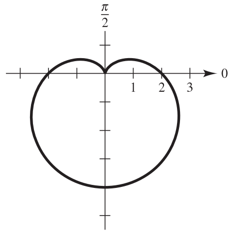
(7, 1.5708), (3, 4.7124)

77.



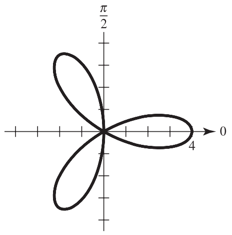
$\theta = 0$

79.



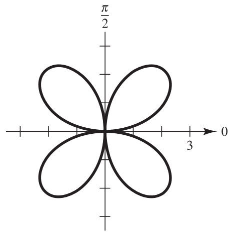
$\theta = \pi/2$

81.



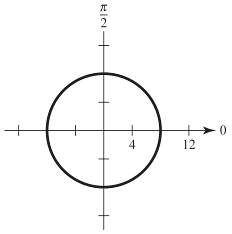
$\theta = \pi/6, \pi/2, 5\pi/6$

83.

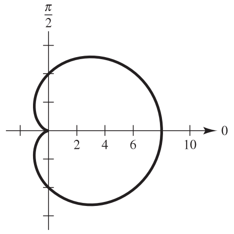


$\theta = 0, \pi/2$

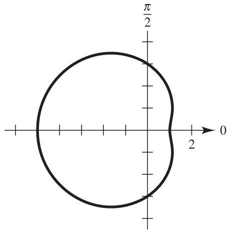
85.



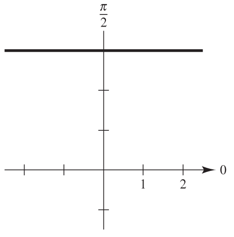
87.



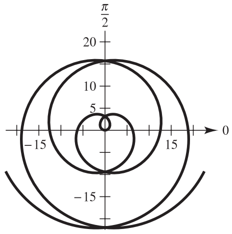
89.



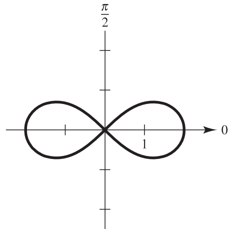
91.



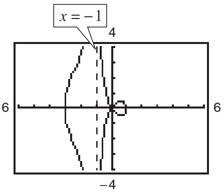
93.



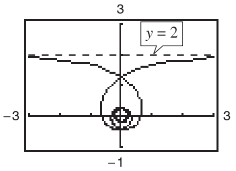
95.



97.



99.



101. The rectangular coordinate system is a collection of points of the form (x, y), where x is the directed distance from the y-axis to the point and y is the directed distance from the x-axis to the point. Every point has a unique representation.

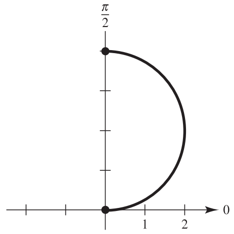
The polar coordinate system is a collection of points of the form (r, θ), where r is the directed distance from the origin O to a point P and θ is the directed angle, measured counterclockwise, from the polar axis to the segment $\overline{OP}$. Polar coordinates do not have unique representations.

103. Slope of tangent line to graph of $r = f(\theta)$ at (r, θ) is

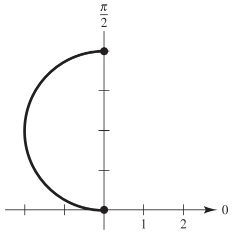
$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then $\theta = \alpha$ is tangent at the pole.

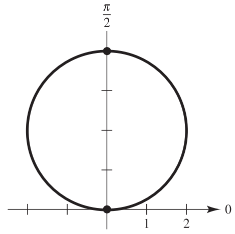
105. (a)



(b)



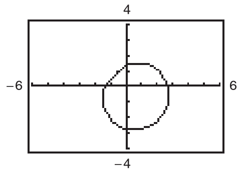
(c)



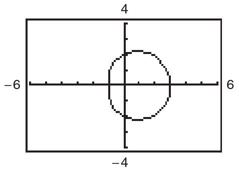
107. Proof

109. (a) $r = 2 - \sin(\theta - \pi/4)$

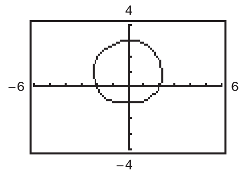
$$= 2 - \frac{\sqrt{2}(\sin\theta - \cos\theta)}{2}$$



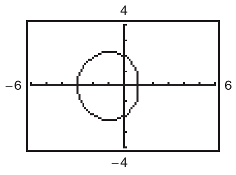
(b) $r = 2 + \cos\theta$



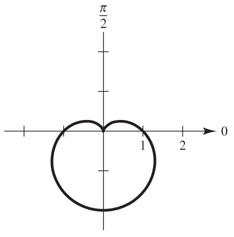
(c) $r = 2 + \sin\theta$



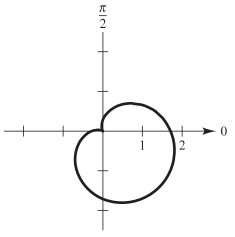
(d) $r = 2 - \cos\theta$



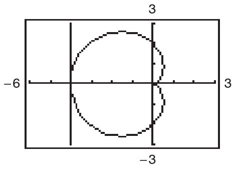
111. (a)



(b)

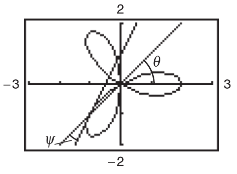


113.



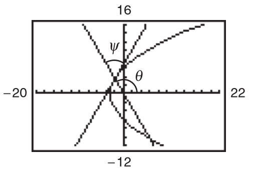
$\psi = \pi/2$

115.



$\psi = \arctan\frac{1}{3} \approx 18.4°$

117.



$\psi = \pi/3, 60°$

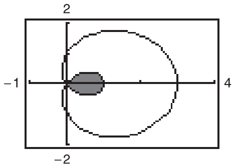
119. True **121.** True

Section 10.5 (page 747)

1. $8\int_0^{\pi/2} \sin^2\theta\, d\theta$ **3.** $\frac{1}{2}\int_{\pi/2}^{3\pi/2} (3 - 2\sin\theta)^2\, d\theta$ **5.** 9π

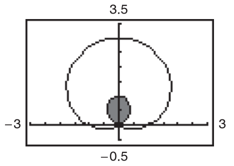
7. $\pi/3$ **9.** $\pi/8$ **11.** $3\pi/2$ **13.** 27π **15.** 4

17.



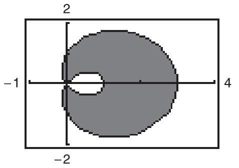
$(2\pi - 3\sqrt{3})/2$

19.



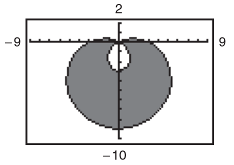
$(2\pi - 3\sqrt{3})/2$

21.



$\pi + 3\sqrt{3}$

23.



$9\pi + 27\sqrt{3}$

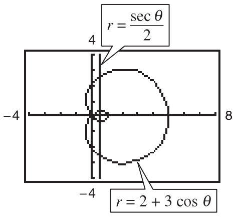
25. $(1, \pi/2), (1, 3\pi/2), (0, 0)$

27. $\left(\frac{2 - \sqrt{2}}{2}, \frac{3\pi}{4}\right), \left(\frac{2 + \sqrt{2}}{2}, \frac{7\pi}{4}\right), (0, 0)$

29. $\left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5\pi}{6}\right), (0, 0)$ **31.** $(2, 4), (-2, -4)$

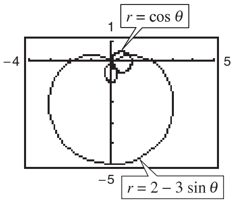
33. $(1, \pi/12), (1, 5\pi/12), (1, 7\pi/12), (1, 11\pi/12),$
$(1, 13\pi/12), (1, 17\pi/12), (1, 19\pi/12), (1, 23\pi/12)$

35.



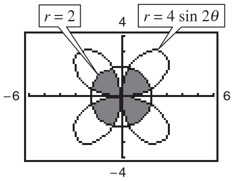
$(-0.581, \pm 2.607)$, $(2.581, \pm 1.376)$

37.



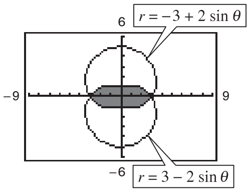
$(0, 0)$, $(0.935, 0.363)$, $(0.535, -1.006)$
The graphs reach the pole at different times (θ-values).

39.



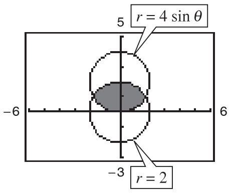
$\frac{4}{3}(4\pi - 3\sqrt{3})$

41.



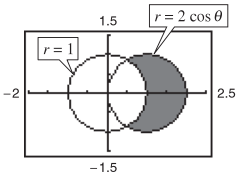
$11\pi - 24$

43.



$\frac{2}{3}(4\pi - 3\sqrt{3})$

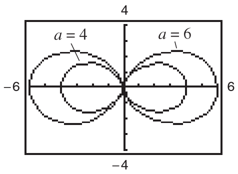
45.



$\pi/3 + \sqrt{3}/2$

47. $5\pi a^2/4$ **49.** $(a^2/2)(\pi - 2)$

51. (a) $(x^2 + y^2)^{3/2} = ax^2$
(b)

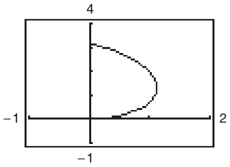


(c) $15\pi/2$

53. The area enclosed by the function is $\pi a^2/4$ if n is odd and is $\pi a^2/2$ if n is even.

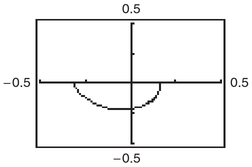
55. 16π **57.** 4π **59.** 8

61.



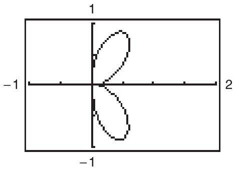
About 4.16

63.



About 0.71

65.



About 4.39

67. 36π

69. $\dfrac{2\pi\sqrt{1 + a^2}}{1 + 4a^2}(e^{\pi a} - 2a)$ **71.** 21.87

73. You will only find simultaneous points of intersection. There may be intersection points that do not occur with the same coordinates in the two graphs.

75. (a) $S = 2\pi\int_{\alpha}^{\beta} f(\theta)\sin\theta\sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$
(b) $S = 2\pi\int_{\alpha}^{\beta} f(\theta)\cos\theta\sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$

77. $40\pi^2$

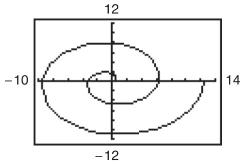
79. (a) 16π
(b)

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A	6.32	12.14	17.06	20.80	23.27	24.60	25.08

(c) and (d) For $\frac{1}{4}$ of area $(4\pi \approx 12.57)$: 0.42
For $\frac{1}{2}$ of area $(8\pi \approx 25.13)$: $1.57(\pi/2)$
For $\frac{3}{4}$ of area $(12\pi \approx 37.70)$: 2.73
(e) No. The results do not depend on the radius. Answers will vary.

81. Circle

83. (a)



The graph becomes larger and more spread out. The graph is reflected over the y-axis.

(b) $(an\pi, n\pi)$ where $n = 1, 2, 3, \ldots$
(c) About 21.26 (d) $4/3\pi^3$

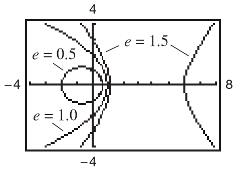
85. $r = \sqrt{2}\cos\theta$

87. False. The graphs of $f(\theta) = 1$ and $g(\theta) = -1$ coincide.

89. Proof

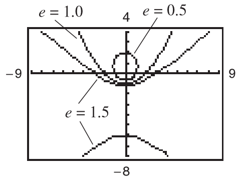
Section 10.6 (page 755)

1.



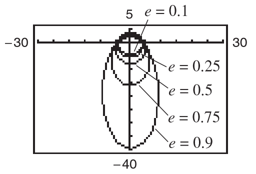
(a) Parabola (b) Ellipse
(c) Hyperbola

3.



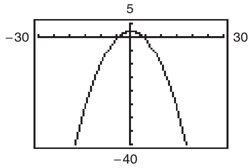
(a) Parabola (b) Ellipse
(c) Hyperbola

5. (a)



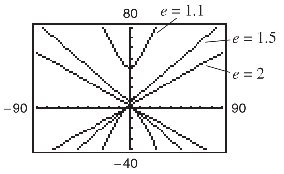
Ellipse
As $e \to 1^-$, the ellipse becomes more elliptical, and as $e \to 0^+$, it becomes more circular.

(b)



Parabola

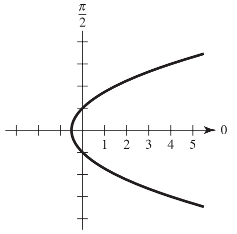
(c)



Hyperbola
As $e \to 1^+$, the hyperbola opens more slowly, and as $e \to \infty$, it opens more rapidly.

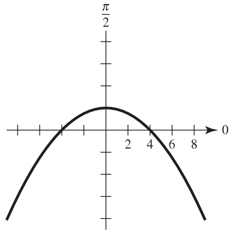
7. c **8.** f **9.** a **10.** e **11.** b **12.** d

13. $e = 1$
Distance $= 1$



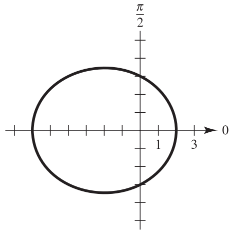
Parabola

15. $e = 1$
Distance $= 4$



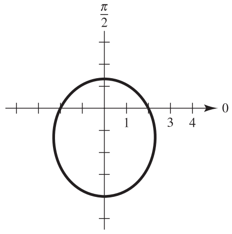
Parabola

17. $e = \frac{1}{2}$
Distance $= 6$



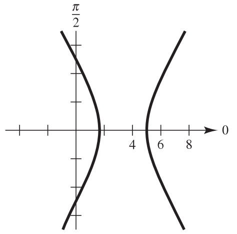
Ellipse

19. $e = \frac{1}{2}$
Distance $= 4$



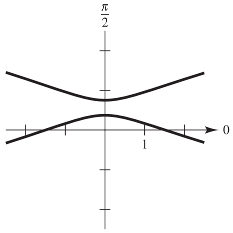
Ellipse

21. $e = 2$
Distance $= \frac{5}{2}$



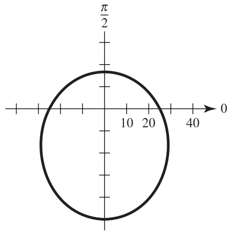
Hyperbola

23. $e = 3$
Distance $= \frac{1}{2}$



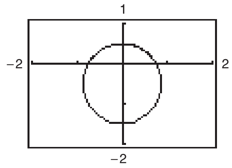
Hyperbola

25. $e = \frac{1}{2}$
Distance $= 50$



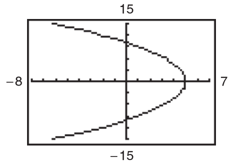
Ellipse

27.



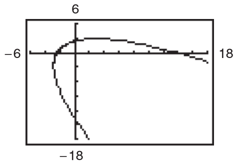
Ellipse
$e = \frac{1}{2}$

29.



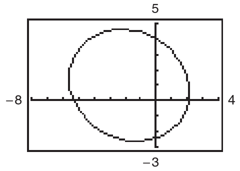
Parabola
$e = 1$

31.



Rotated $\pi/4$ radian counterclockwise.

33.



Rotated $\pi/6$ radian clockwise.

35. $r = \dfrac{8}{8 + 5\cos\left(\theta + \dfrac{\pi}{6}\right)}$

37. $r = 3/(1 - \cos\theta)$ **39.** $r = 1/(2 + \sin\theta)$

41. $r = 2/(1 + 2\cos\theta)$ **43.** $r = 2/(1 - \sin\theta)$

45. $r = 16/(5 + 3\cos\theta)$ **47.** $r = 9/(4 - 5\sin\theta)$

49. $r = 4/(2 + \cos\theta)$

51. If $0 < e < 1$, the conic is an ellipse.
If $e = 1$, the conic is a parabola.
If $e > 1$, the conic is a hyperbola.

53. If the foci are fixed and $e \to 0$, then $d \to \infty$. To see this, compare the ellipses

$$r = \frac{1/2}{1 + (1/2)\cos\theta},\ e = \frac{1}{2},\ d = 1 \text{ and}$$

$$r = \frac{5/16}{1 + (1/4)\cos\theta},\ e = \frac{1}{4},\ d = \frac{5}{4}.$$

55. Proof

57. $r^2 = \dfrac{9}{1 - (16/25)\cos^2\theta}$ **59.** $r^2 = \dfrac{-16}{1 - (25/9)\cos^2\theta}$

61. About 10.88 **63.** 3.37 **65.** $\dfrac{7979.21}{1 - 0.9372\cos\theta}$; 11,015 mi

67. $r = \dfrac{149{,}558{,}278.0560}{1 - 0.0167\cos\theta}$
Perihelion: 147,101,680 km
Aphelion: 152,098,320 km

69. $r = \dfrac{4{,}497{,}667{,}328}{1 - 0.0086\cos\theta}$
Perihelion: 4,459,317,200 km
Aphelion: 4,536,682,800 km

71. Answers will vary. Sample answers:
(a) 3.591×10^{18} km^2; 9.322 yr
(b) $\alpha \approx 0.361 + \pi$; Larger angle with the smaller ray to generate an equal area
(c) Part (a): 1.583×10^9 km; 1.698×10^8 km/yr
Part (b): 1.610×10^9 km; 1.727×10^8 km/yr

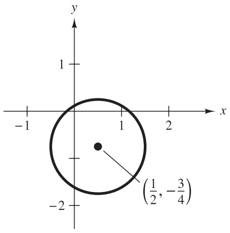
73. Proof

75. Let $r_1 = ed/(1 + \sin\theta)$ and $r_2 = ed/(1 - \sin\theta)$.
The points of intersection of r_1 and r_2 are $(ed, 0)$ and (ed, π). The slopes of the tangent lines to r_1 are -1 at $(ed, 0)$ and 1 at (ed, π). The slopes of the tangent lines to r_2 are 1 at $(ed, 0)$ and -1 at (ed, π). Therefore, at $(ed, 0)$, $m_1m_2 = -1$, and at (ed, π), $m_1m_2 = -1$, and the curves intersect at right angles.

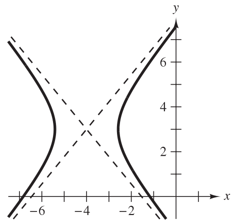
Review Exercises for Chapter 10 (page 758)

1. e **2.** c **3.** b **4.** d **5.** a **6.** f

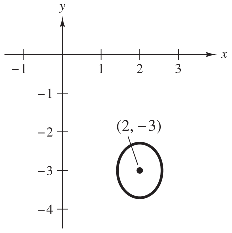
7. Circle
Center: $\left(\frac{1}{2}, -\frac{3}{4}\right)$
Radius: 1



9. Hyperbola
Center: $(-4, 3)$
Vertices: $\left(-4 \pm \sqrt{2}, 3\right)$



11. Ellipse
Center: $(2, -3)$
Vertices: $\left(2, -3 \pm \sqrt{2}/2\right)$

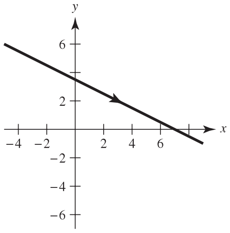


13. $y^2 - 4y - 12x + 4 = 0$ **15.** $(x - 1)^2/36 + y^2/20 = 1$

17. $x^2/49 - y^2/32 = 1$ **19.** About 15.87

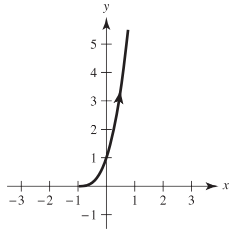
21. $4x + 4y - 7 = 0$ **23.** (a) $(0, 50)$ (b) About 38,294.49

25.



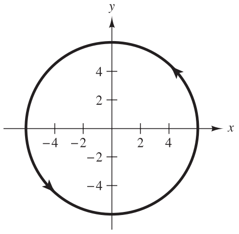
$x + 2y - 7 = 0$

27.



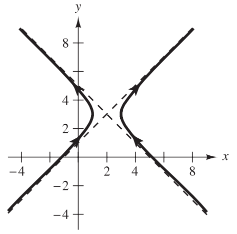
$y = (x + 1)^3, x > -1$

29.



$x^2 + y^2 = 36$

31.

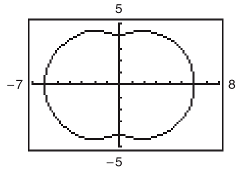


$(x - 2)^2 - (y - 3)^2 = 1$

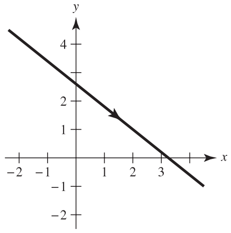
33. Answers will vary. Sample answer:
$x = 5t - 2$
$y = 6 - 4t$

35. $x = 4\cos\theta - 3$
$y = 4 + 3\sin\theta$

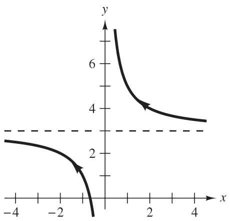
37.



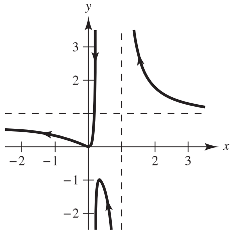
39. (a) $dy/dx = -\frac{4}{5}$;
Horizontal tangents: None
(b) $y = (-4x + 13)/5$
(c)



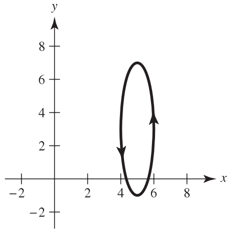
41. (a) $dy/dx = -2t^2$;
Horizontal tangents: None
(b) $y = 3 + 2/x$
(c)



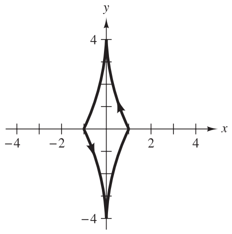
43. (a) $\dfrac{dy}{dx} = \dfrac{(t - 1)(2t + 1)^2}{t^2(t - 2)^2}$;
Horizontal tangent: $\left(\dfrac{1}{3}, -1\right)$
(b) $y = \dfrac{4x^2}{(5x - 1)(x - 1)}$
(c)



45. (a) $\dfrac{dy}{dx} = -4\cot\theta$;
Horizontal tangents: $(5, 7)$, $(5, -1)$
(b) $(x - 5)^2 + \dfrac{(y - 3)^2}{16} = 1$
(c)



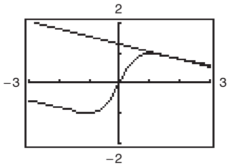
47. (a) $\dfrac{dy}{dx} = -4\tan\theta$;
Horizontal tangents: None
(b) $x^{2/3} + (y/4)^{2/3} = 1$
(c)



49. Horizontal: $(5, 0)$
Vertical: None

51. Horizontal: $(2, 2), (2, 0)$
Vertical: $(4, 1), (0, 1)$

53. (a) and (c)

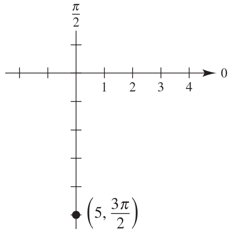


(b) $dx/d\theta = -4$, $dy/d\theta = 1$, $dy/dx = -\frac{1}{4}$

55. $\frac{1}{2}\pi^2 r$

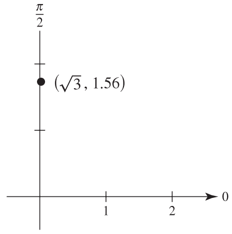
57. (a) $s = 12\pi\sqrt{10} \approx 119.215$ **59.** $A = 3\pi$
(b) $s = 4\pi\sqrt{10} \approx 39.738$

61.



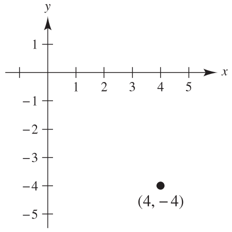
Rectangular: $(0, -5)$

63.



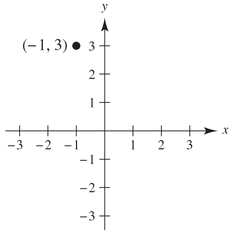
Rectangular: $(0.0187, 1.7320)$

65.



$\left(4\sqrt{2}, \frac{7\pi}{4}\right), \left(-4\sqrt{2}, \frac{3\pi}{4}\right)$

67.



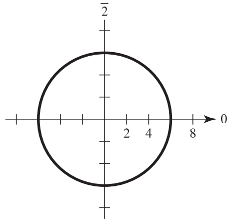
$(\sqrt{10}, 1.89), (-\sqrt{10}, 5.03)$

69. $x^2 + y^2 - 3x = 0$ **71.** $(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$

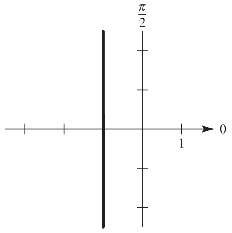
73. $(x^2 + y^2)^2 = x^2 - y^2$ **75.** $y^2 = x^2[(4 - x)/(4 + x)]$

77. $r = a\cos^2\theta\sin\theta$ **79.** $r^2 = a^2\theta^2$

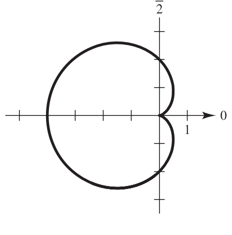
81. Circle



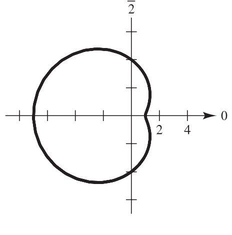
83. Line



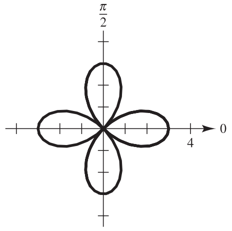
85. Cardioid



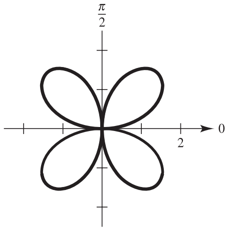
87. Limaçon



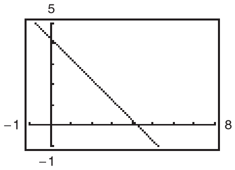
89. Rose curve



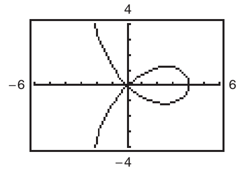
91. Rose curve



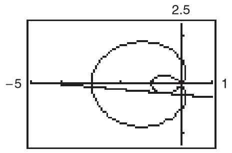
93.



95.



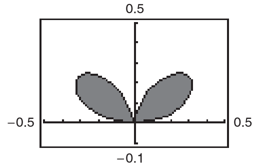
97. (a) $\theta = \pm\pi/3$
(b) Vertical: $(-1, 0), (3, \pi), \left(\frac{1}{2}, \pm 1.318\right)$
Horizontal: $(-0.686, \pm 0.568), (2.186, \pm 2.206)$
(c)



99. Proof **101.** $\dfrac{9\pi}{20}$ **103.** $\dfrac{9\pi}{2}$ **105.** 4

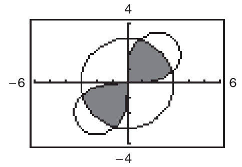
107. $\left(1 + \dfrac{\sqrt{2}}{2}, \dfrac{3\pi}{4}\right), \left(1 - \dfrac{\sqrt{2}}{2}, \dfrac{7\pi}{4}\right), (0, 0)$

109.



$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/2} \sin^2\theta\cos^4\theta\,d\theta \approx 0.10$$

111.

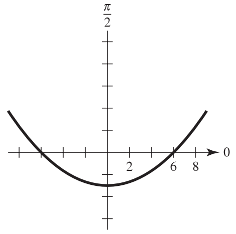


$$A = 2\left[\frac{1}{2}\int_0^{\pi/12} 18\sin 2\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} 9\,d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2} 18\sin 2\theta\,d\theta\right]$$
$$\approx 1.2058 + 9.4248 + 1.2058 = 11.8364$$

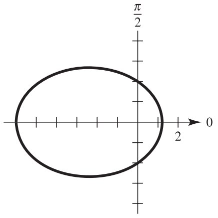
113. $4a$

115. $S = 2\pi\displaystyle\int_0^{\pi/2} (1 + 4\cos\theta)\sin\theta\sqrt{17 + 8\cos\theta}\,d\theta$
$= 34\pi\sqrt{17}/5 \approx 88.08$

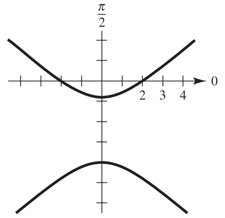
117. Parabola



119. Ellipse



121. Hyperbola

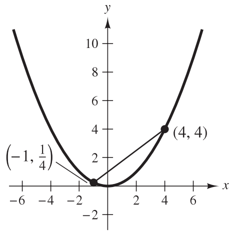


123. $r = 10 \sin \theta$

125. $r = 4/(1 - \cos \theta)$ **127.** $r = 5/(3 - 2 \cos \theta)$

P.S. Problem Solving (page 761)

1. (a)



(b) and (c) Proofs

3. Proof

5. (a) $r = 2a \tan \theta \sin \theta$

(b) $x = 2at^2/(1 + t^2)$

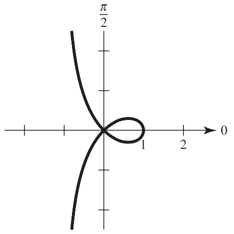
$y = 2at^3/(1 + t^2)$

(c) $y^2 = x^3/(2a - x)$

7. (a) $y^2 = x^2[(1 - x)/(1 + x)]$

(b) $r = \cos 2\theta \cdot \sec \theta$

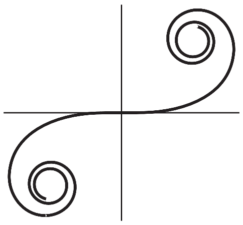
(c)



(d) $y = x, y = -x$

(e) $\left(\frac{\sqrt{5} - 1}{2}, \pm \frac{\sqrt{5} - 1}{2}\sqrt{-2 + \sqrt{5}}\right)$

9. (a)



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(b) Proof (c) $a, 2\pi$

11. $A = \frac{1}{2}ab$ **13.** $r^2 = 2 \cos 2\theta$

15. (a) First plane: $x_1 = \cos 70°(150 - 375t)$

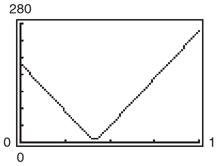
$y_1 = \sin 70°(150 - 375t)$

Second plane: $x_2 = \cos 45°(450t - 190)$

$y_2 = \sin 45°(190 - 450t)$

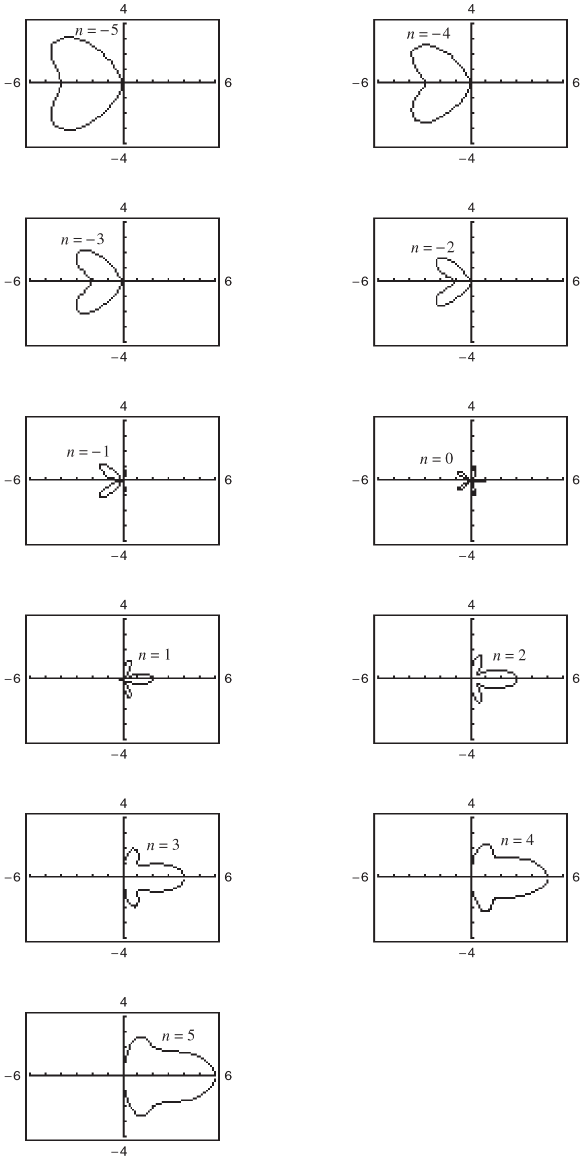
(b) $\{[\cos 45°(450t - 190) - \cos 70°(150 - 375t)]^2 + [\sin 45°(190 - 450t) - \sin 70°(150 - 375t)]^2\}^{1/2}$

(c)



0.4145 h; Yes

17.



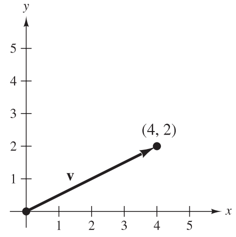
$n = 1, 2, 3, 4, 5$ produce "bells"; $n = -1, -2, -3, -4, -5$ produce "hearts."

Chapter 11

Section 11.1 (page 771)

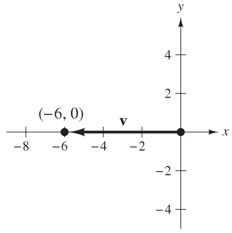
1. (a) $\langle 4, 2\rangle$

(b)



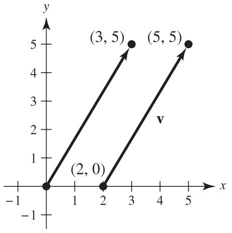
3. (a) $\langle -6, 0\rangle$

(b)



5. $\mathbf{u} = \mathbf{v} = \langle 2, 4\rangle$ **7.** $\mathbf{u} = \mathbf{v} = \langle 6, -5\rangle$

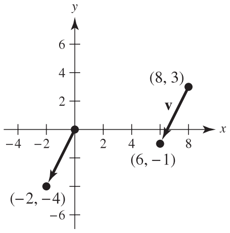
9. (a) and (d)



(b) $\langle 3, 5\rangle$

(c) $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j}$

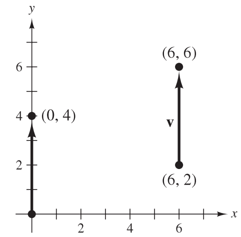
11. (a) and (d)



(b) $\langle -2, -4\rangle$

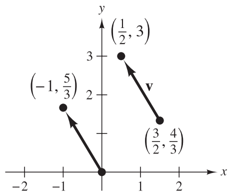
(c) $\mathbf{v} = -2\mathbf{i} - 4\mathbf{j}$

13. (a) and (d)



(b) $\langle 0, 4\rangle$ (c) $\mathbf{v} = 4\mathbf{j}$

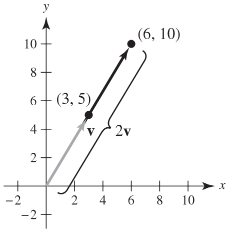
15. (a) and (d)



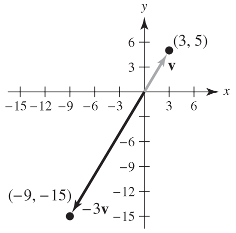
(b) $\left\langle -1, \frac{5}{3}\right\rangle$

(c) $\mathbf{v} = -\mathbf{i} + \frac{5}{3}\mathbf{j}$

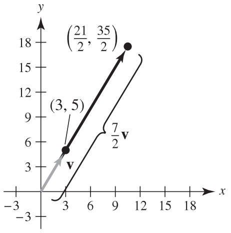
17. (a) $\langle 6, 10\rangle$



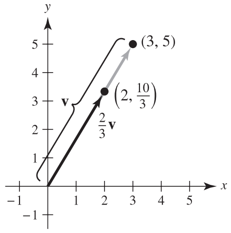
(b) $\langle -9, -15\rangle$



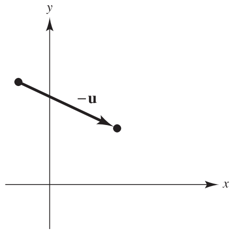
(c) $\left\langle \frac{21}{2}, \frac{35}{2}\right\rangle$



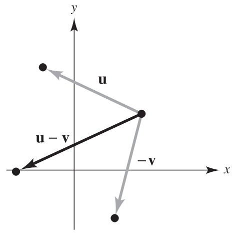
(d) $\left\langle 2, \frac{10}{3}\right\rangle$



19.

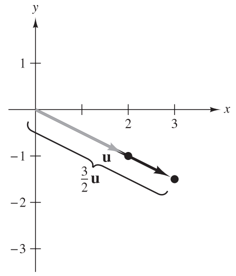


21.

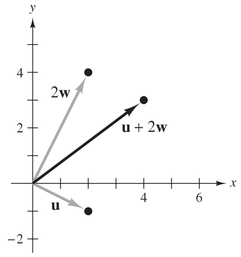


23. (a) $\left\langle \frac{8}{3}, 6\right\rangle$ (b) $\langle -2, -14\rangle$ (c) $\langle 18, -7\rangle$

25. $\left\langle 3, -\frac{3}{2}\right\rangle$



27. $\langle 4, 3\rangle$



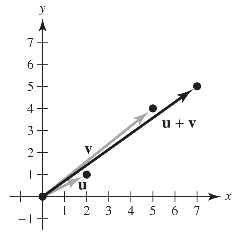
29. $(3, 5)$ **31.** 7 **33.** 5 **35.** $\sqrt{61}$

37. $\left\langle \sqrt{17}/17, 4\sqrt{17}/17\right\rangle$ **39.** $\left\langle 3\sqrt{34}/34, 5\sqrt{34}/34\right\rangle$

41. (a) $\sqrt{2}$ (b) $\sqrt{5}$ (c) 1 (d) 1 (e) 1 (f) 1

43. (a) $\sqrt{5}/2$ (b) $\sqrt{13}$ (c) $\sqrt{85}/2$ (d) 1 (e) 1 (f) 1

45.



$\|\mathbf{u}\| + \|\mathbf{v}\| = \sqrt{5} + \sqrt{41}$ and $\|\mathbf{u} + \mathbf{v}\| = \sqrt{74}$

$\sqrt{74} \le \sqrt{5} + \sqrt{41}$

47. $\langle 0, 6\rangle$ **49.** $\left\langle -\sqrt{5}, 2\sqrt{5}\right\rangle$ **51.** $\langle 3, 0\rangle$ **53.** $\left\langle -\sqrt{3}, 1\right\rangle$

55. $\left\langle \dfrac{2 + 3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}\right\rangle$ **57.** $\langle 2\cos 4 + \cos 2, 2\sin 4 + \sin 2\rangle$

59. Answers will vary. Example: A scalar is a single real number such as 2. A vector is a line segment having both direction and magnitude. The vector $\left\langle \sqrt{3}, 1\right\rangle$, given in component form, has a direction of $\pi/6$ and a magnitude of 2.

61. (a) Vector; has magnitude and direction

(b) Scalar; has only magnitude

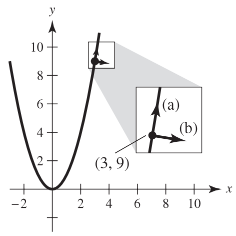
63. $a = 1, b = 1$ **65.** $a = 1, b = 2$ **67.** $a = \frac{2}{3}, b = \frac{1}{3}$

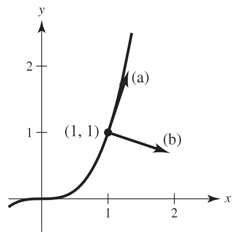
69. (a) $\pm\left(1/\sqrt{37}\right)\langle 1, 6\rangle$

(b) $\pm\left(1/\sqrt{37}\right)\langle 6, -1\rangle$

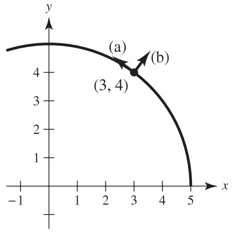
71. (a) $\pm\left(1/\sqrt{10}\right)\langle 1, 3\rangle$

(b) $\pm\left(1/\sqrt{10}\right)\langle 3, -1\rangle$





73. (a) $\pm\frac{1}{5}\langle -4, 3\rangle$ **75.** $\left\langle -\sqrt{2}/2, \sqrt{2}/2\right\rangle$
(b) $\pm\frac{1}{5}\langle 3, 4\rangle$



77. (a)–(c) Answers will vary.
(d) Magnitude ≈ 63.5, direction ≈ −8.26°

79. 1.33, 132.5° **81.** 10.7°, 584.6 lb **83.** 71.3°, 228.5 lb

85. (a) $\theta = 0°$ (b) $\theta = 180°$
(c) No, the resultant can only be less than or equal to the sum.

87. $(-4, -1)$, $(6, 5)$, $(10, 3)$

89. Tension in cable $AC \approx 2638.2$ lb
Tension in cable $BC \approx 1958.1$ lb

91. Horizontal: 1193.43 ft/sec **93.** 38.3° north of west
Vertical: 125.43 ft/sec 882.9 km/h

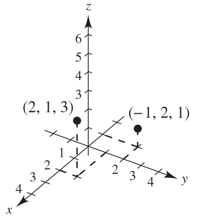
95. True **97.** True **99.** False. $\|a\mathbf{i} + b\mathbf{j}\| = \sqrt{2}|a|$

101–103. Proofs **105.** $x^2 + y^2 = 25$

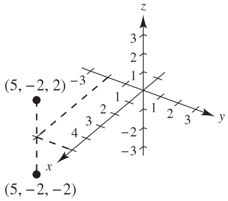
Section 11.2 (page 780)

1. $A(2, 3, 4)$, $B(-1, -2, 2)$

3.



5.



7. $(-3, 4, 5)$ **9.** $(12, 0, 0)$ **11.** 0

13. Six units above the xy-plane

15. Three units behind the yz-plane

17. To the left of the xz-plane

19. Within three units of the xz-plane

21. Three units below the xy-plane, and below either quadrant I or III

23. Above the xy-plane and above quadrants II or IV, *or* below the xy-plane and below quadrants I or III

25. $\sqrt{69}$ **27.** $\sqrt{61}$ **29.** 7, $7\sqrt{5}$, 14; Right triangle

31. $\sqrt{41}$, $\sqrt{41}$, $\sqrt{14}$; Isosceles triangle

33. $(0, 0, 9)$, $(2, 6, 12)$, $(6, 4, -3)$

35. $\left(\frac{3}{2}, -3, 5\right)$ **37.** $(x - 0)^2 + (y - 2)^2 + (z - 5)^2 = 4$

39. $(x - 1)^2 + (y - 3)^2 + (z - 0)^2 = 10$

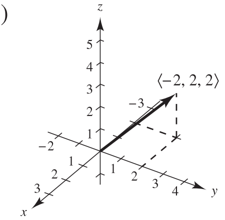
41. $(x - 1)^2 + (y + 3)^2 + (z + 4)^2 = 25$
Center: $(1, -3, -4)$
Radius: 5

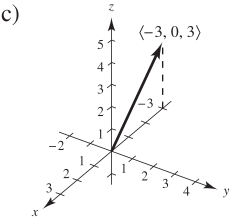
43. $\left(x - \frac{1}{3}\right)^2 + (y + 1)^2 + z^2 = 1$
Center: $\left(\frac{1}{3}, -1, 0\right)$
Radius: 1

45. A solid sphere with center $(0, 0, 0)$ and radius 6

47. Interior of sphere of radius 4 centered at $(2, -3, 4)$

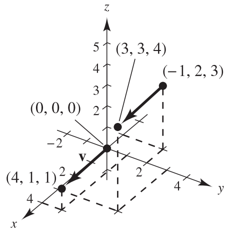
49. (a) $\langle -2, 2, 2\rangle$ **51.** (a) $\langle -3, 0, 3\rangle$
(b) $\mathbf{v} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ (b) $\mathbf{v} = -3\mathbf{i} + 3\mathbf{k}$
(c) (c)





53. $\mathbf{v} = \langle 1, -1, 6\rangle$ **55.** $\mathbf{v} = \langle -1, 0, -1\rangle$
$\|\mathbf{v}\| = \sqrt{38}$ $\|\mathbf{v}\| = \sqrt{2}$
$\mathbf{u} = \dfrac{1}{\sqrt{38}}\langle 1, -1, 6\rangle$ $\mathbf{u} = \dfrac{1}{\sqrt{2}}\langle -1, 0, -1\rangle$

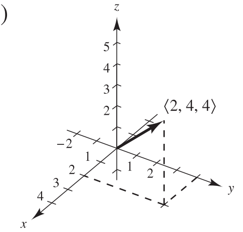
57. (a) and (d)

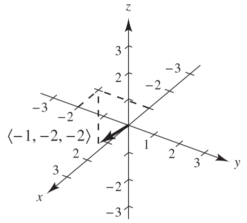


(b) $\langle 4, 1, 1\rangle$ (c) $\mathbf{v} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$

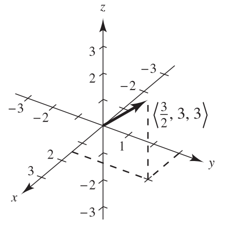
59. $(3, 1, 8)$

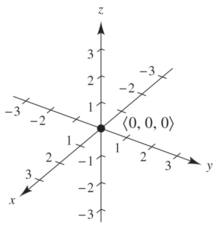
61. (a) (b)





(c) (d)





63. $\langle -1, 0, 4\rangle$ **65.** $\langle 6, 12, 6\rangle$ **67.** $\left\langle \frac{7}{2}, 3, \frac{5}{2}\right\rangle$

69. a and b **71.** a **73.** Collinear **75.** Not collinear

77. $\overrightarrow{AB} = \langle 1, 2, 3\rangle$
$\overrightarrow{CD} = \langle 1, 2, 3\rangle$
$\overrightarrow{BD} = \langle -2, 1, 1\rangle$
$\overrightarrow{AC} = \langle -2, 1, 1\rangle$
Because $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{BD} = \overrightarrow{AC}$, the given points form the vertices of a parallelogram.

79. 0 **81.** $\sqrt{34}$ **83.** $\sqrt{14}$

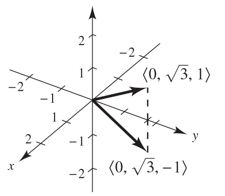
85. (a) $\frac{1}{3}\langle 2, -1, 2\rangle$ (b) $-\frac{1}{3}\langle 2, -1, 2\rangle$

87. (a) $\left(1/\sqrt{38}\right)\langle 3, 2, -5\rangle$ (b) $-\left(1/\sqrt{38}\right)\langle 3, 2, -5\rangle$

89. (a)–(d) Answers will vary.
(e) $\mathbf{u} + \mathbf{v} = \langle 4, 7.5, -2\rangle$
$\|\mathbf{u} + \mathbf{v}\| \approx 8.732$
$\|\mathbf{u}\| \approx 5.099$
$\|\mathbf{v}\| \approx 9.014$

91. $\pm\frac{7}{3}$ **93.** $\left\langle 0, 10/\sqrt{2}, 10/\sqrt{2}\right\rangle$ **95.** $\left\langle 1, -1, \frac{1}{2}\right\rangle$

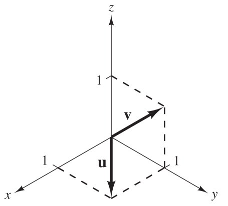
97.


z
2
1
−2
−1
−2
−1
1
2
−1
−2
$\langle 0, \sqrt{3}, 1\rangle$
$\langle 0, \sqrt{3}, -1\rangle$
x
y

$\left\langle 0, \sqrt{3}, \pm 1\right\rangle$

99. $(2, -1, 2)$

101. (a)



(b) $a = 0, a + b = 0, b = 0$
(c) $a = 1, a + b = 2, b = 1$
(d) Not possible

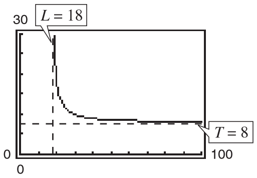
103. x_0 is directed distance to yz-plane.
y_0 is directed distance to xz-plane.
z_0 is directed distance to xy-plane.

105. $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$ **107.** **0**

109. (a) $T = 8L/\sqrt{L^2 - 18^2}, L > 18$
(b)

L	20	25	30	35	40	45	50
T	18.4	11.5	10	9.3	9.0	8.7	8.6

(c)



(d) Proof (e) 30 in.

111. $\left(\sqrt{3}/3\right)\langle 1, 1, 1\rangle$

113. Tension in cable AB: 202.919 N
Tension in cable AC: 157.909 N
Tension in cable AD: 226.521 N

115. $\left(x - \frac{4}{3}\right)^2 + (y - 3)^2 + \left(z + \frac{1}{3}\right)^2 = \frac{44}{9}$

Section 11.3 (page 789)

1. (a) 17 (b) 25 (c) 25 (d) $\langle -17, 85\rangle$ (e) 34

3. (a) -26 (b) 52 (c) 52 (d) $\langle 78, -52\rangle$ (e) -52

5. (a) 2 (b) 29 (c) 29 (d) $\langle 0, 12, 10\rangle$ (e) 4

7. (a) 1 (b) 6 (c) 6 (d) $\mathbf{i} - \mathbf{k}$ (e) 2

9. 20 **11.** $\pi/2$ **13.** $\arccos\left[-1/\left(5\sqrt{2}\right)\right] \approx 98.1°$

15. $\arccos\left(\sqrt{2}/3\right) \approx 61.9°$ **17.** $\arccos\left(-8\sqrt{13}/65\right) \approx 116.3°$

19. Neither **21.** Orthogonal **23.** Neither

25. Orthogonal **27.** Right triangle; answers will vary.

29. Acute triangle; answers will vary.

31. $\cos\alpha = \frac{1}{3}$
$\cos\beta = \frac{2}{3}$
$\cos\gamma = \frac{2}{3}$

33. $\cos\alpha = 0$
$\cos\beta = 3/\sqrt{13}$
$\cos\gamma = -2/\sqrt{13}$

35. $\alpha \approx 43.3°, \beta \approx 61.0°, \gamma \approx 119.0°$

37. $\alpha \approx 100.5°, \beta \approx 24.1°, \gamma \approx 68.6°$

39. Magnitude: 124.310 lb
$\alpha \approx 29.48°, \beta \approx 61.39°, \gamma \approx 96.53°$

41. $\alpha = 90°, \beta = 45°, \gamma = 45°$ **43.** (a) $\langle 2, 8\rangle$ (b) $\langle 4, -1\rangle$

45. (a) $\left\langle \frac{5}{2}, \frac{1}{2}\right\rangle$ (b) $\left\langle -\frac{1}{2}, \frac{5}{2}\right\rangle$ **47.** (a) $\langle -2, 2, 2\rangle$ (b) $\langle 2, 1, 1\rangle$

49. (a) $\left\langle 0, \frac{33}{25}, \frac{44}{25}\right\rangle$ (b) $\left\langle 2, -\frac{8}{25}, \frac{6}{25}\right\rangle$

51. See “Definition of Dot Product,” page 783.

53. (a) and (b) are defined. (c) and (d) are not defined because it is not possible to find the dot product of a scalar and a vector or to add a scalar to a vector.

55. See Figure 11.29 on page 787.

57. Yes.

$$\left\|\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\right\| = \left\|\frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u}\right\|$$
$$|\mathbf{u}\cdot\mathbf{v}|\frac{\|\mathbf{v}\|}{\|\mathbf{v}\|^2} = |\mathbf{v}\cdot\mathbf{u}|\frac{\|\mathbf{u}\|}{\|\mathbf{u}\|^2}$$
$$\frac{1}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{u}\|}$$
$$\|\mathbf{u}\| = \|\mathbf{v}\|$$

59. \$12,351.25; Total revenue **61.** (a)–(c) Answers will vary.

63. Answers will vary. **65.** $\mathbf{u}$

67. Answers will vary. Example: $\langle 12, 2\rangle$ and $\langle -12, -2\rangle$

69. Answers will vary. Example: $\langle 2, 0, 3\rangle$ and $\langle -2, 0, -3\rangle$

71. (a) 8335.1 lb (b) 47,270.8 lb

73. 425 ft-lb **75.** 2900.2 km-N

77. False. For example, $\langle 1, 1\rangle \cdot \langle 2, 3\rangle = 5$ and $\langle 1, 1\rangle \cdot \langle 1, 4\rangle = 5$, but $\langle 2, 3\rangle \neq \langle 1, 4\rangle$.

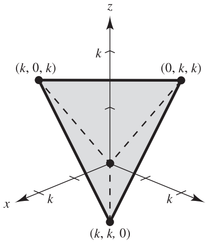
79. $\arccos\left(1/\sqrt{3}\right) \approx 54.7°$

81. (a) $(0, 0), (1, 1)$
(b) To $y = x^2$ at $(1, 1)$: $\left\langle \pm\sqrt{5}/5, \pm 2\sqrt{5}/5\right\rangle$
To $y = x^{1/3}$ at $(1, 1)$: $\left\langle \pm 3\sqrt{10}/10, \pm\sqrt{10}/10\right\rangle$
To $y = x^2$ at $(0, 0)$: $\langle \pm 1, 0\rangle$
To $y = x^{1/3}$ at $(0, 0)$: $\langle 0, \pm 1\rangle$
(c) At $(1, 1)$: $\theta = 45°$
At $(0, 0)$: $\theta = 90°$

83. (a) $(-1, 0), (1, 0)$
(b) To $y = 1 - x^2$ at $(1, 0)$: $\langle \pm\sqrt{5}/5, \mp 2\sqrt{5}/5 \rangle$
To $y = x^2 - 1$ at $(1, 0)$: $\langle \pm\sqrt{5}/5, \pm 2\sqrt{5}/5 \rangle$
To $y = 1 - x^2$ at $(-1, 0)$: $\langle \pm\sqrt{5}/5, \pm 2\sqrt{5}/5 \rangle$
To $y = x^2 - 1$ at $(-1, 0)$: $\langle \pm\sqrt{5}/5, \mp 2\sqrt{5}/5 \rangle$
(c) At $(1, 0)$: $\theta = 53.13°$
At $(-1, 0)$: $\theta = 53.13°$

85. Proof

87. (a)

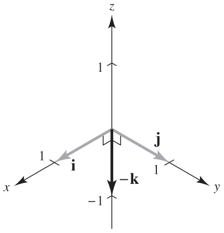


(b) $k\sqrt{2}$ (c) 60° (d) 109.5°

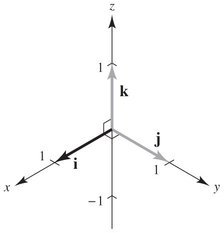
89–91. Proofs

Section 11.4 (page 798)

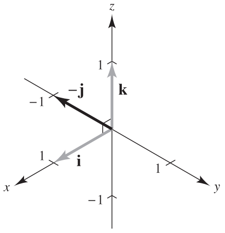
1. $-\mathbf{k}$



3. $\mathbf{i}$



5. $-\mathbf{j}$

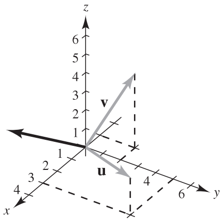


7. (a) $20\mathbf{i} + 10\mathbf{j} - 16\mathbf{k}$
(b) $-20\mathbf{i} - 10\mathbf{j} + 16\mathbf{k}$
(c) $\mathbf{0}$

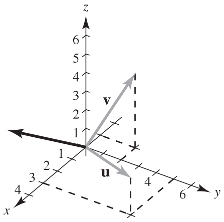
9. (a) $17\mathbf{i} - 33\mathbf{j} - 10\mathbf{k}$
(b) $-17\mathbf{i} + 33\mathbf{j} + 10\mathbf{k}$
(c) $\mathbf{0}$

11. $\langle 0, 0, 54 \rangle$ **13.** $\langle -1, -1, -1 \rangle$ **15.** $\langle -2, 3, -1 \rangle$

17.



19.



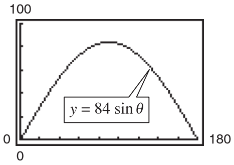
21. $\langle -73.5, 5.5, 44.75 \rangle, \left\langle -\dfrac{2.94}{\sqrt{11.8961}}, \dfrac{0.22}{\sqrt{11.8961}}, \dfrac{1.79}{\sqrt{11.8961}} \right\rangle$

23. $\langle -3.6, -1.4, 1.6 \rangle, \left\langle -\dfrac{1.8}{\sqrt{4.37}}, -\dfrac{0.7}{\sqrt{4.37}}, \dfrac{0.8}{\sqrt{4.37}} \right\rangle$

25. Answers will vary. **27.** 1 **29.** $6\sqrt{5}$ **31.** $9\sqrt{5}$

33. $\dfrac{11}{2}$ **35.** $\dfrac{\sqrt{16{,}742}}{2}$ **37.** $10 \cos 40° \approx 7.66$ ft-lb

39. (a) $84 \sin \theta$



(b) $42\sqrt{2} \approx 59.40$
(c) $\theta = 90°$; This is what should be expected. When $\theta = 90°$, the pipe wrench is horizontal.

41. 1 **43.** 6 **45.** 2 **47.** 75

49. At least one of the vectors is the zero vector.

51. See “Definition of Cross Product of Two Vectors in Space,” page 792.

53. The magnitude of the cross product will increase by a factor of 4.

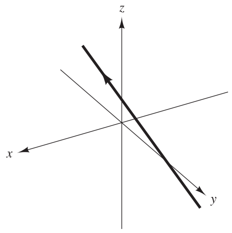
55. False. The cross product of two vectors is not defined in a two-dimensional coordinate system.

57. False. Let $\mathbf{u} = \langle 1, 0, 0 \rangle$, $\mathbf{v} = \langle 1, 0, 0 \rangle$, and $\mathbf{w} = \langle -1, 0, 0 \rangle$. Then $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w} = \mathbf{0}$, but $\mathbf{v} \neq \mathbf{w}$.

59–67. Proofs

Section 11.5 (page 807)

1. (a)



(b) $P = (1, 2, 2)$, $Q = (10, -1, 17)$, $\overrightarrow{PQ} = \langle 9, -3, 15 \rangle$
(There are many correct answers.) The components of the vector and the coefficients of t are proportional because the line is parallel to $\overrightarrow{PQ}$.
(c) $\left(-\frac{1}{5}, \frac{12}{5}, 0\right), (7, 0, 12), \left(0, \frac{7}{3}, \frac{1}{3}\right)$

3. (a) Yes (b) No

	Parametric Equations (*a*)	*Symmetric Equations* (*b*)	*Direction Numbers*
5.	$x = 3t$ $y = t$ $z = 5t$	$\dfrac{x}{3} = y = \dfrac{z}{5}$	3, 1, 5
7.	$x = -2 + 2t$ $y = 4t$ $z = 3 - 2t$	$\dfrac{x+2}{2} = \dfrac{y}{4} = \dfrac{z-3}{-2}$	2, 4, −2
9.	$x = 1 + 3t$ $y = -2t$ $z = 1 + t$	$\dfrac{x-1}{3} = \dfrac{y}{-2} = \dfrac{z-1}{1}$	3, −2, 1

	Parametric Equations (a)	Symmetric Equations (b)	Direction Numbers
11.	$x = 5 + 17t$ $y = -3 - 11t$ $z = -2 - 9t$	$\dfrac{x - 5}{17} = \dfrac{y + 3}{-11} = \dfrac{z + 2}{-9}$	17, −11, −9
13.	$x = 7 - 10t$ $y = -2 + 2t$ $z = 6$	Not possible	−10, 2, 0

15. $x = 2$, $y = 3$, $z = 4 + t$

17. $x = 2 + 3t$, $y = 3 + 2t$, $z = 4 - t$

19. $x = 5 + 2t$, $y = -3 - t$, $z = -4 + 3t$

21. $x = 2 - t$, $y = 1 + t$, $z = 2 + t$

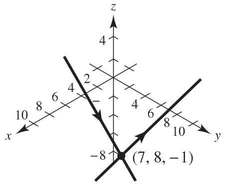
23. $P(3, -1, -2)$; $\mathbf{v} = \langle -1, 2, 0 \rangle$

25. $P(7, -6, -2)$; $\mathbf{v} = \langle 4, 2, 1 \rangle$

27. $L_1 = L_2$ and is parallel to L_3. **29.** L_1 and L_3 are identical.

31. $(2, 3, 1)$; $\cos\theta = 7\sqrt{17}/51$ **33.** Not intersecting

35.



$(7, 8, -1)$

37. (a) $P = (0, 0, -1)$, $Q = (0, -2, 0)$, $R = (3, 4, -1)$
$\overrightarrow{PQ} = \langle 0, -2, 1 \rangle$, $\overrightarrow{PR} = \langle 3, 4, 0 \rangle$
(There are many correct answers.)

(b) $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -4, 3, 6 \rangle$
The components of the cross product are proportional to the coefficients of the variables in the equation. The cross product is parallel to the normal vector.

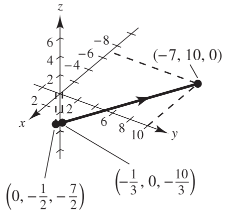
39. (a) Yes (b) Yes **41.** $y - 3 = 0$

43. $2x + 3y - z = 10$ **45.** $2x - y - 2z + 6 = 0$

47. $3x - 19y - 2z = 0$ **49.** $4x - 3y + 4z = 10$ **51.** $z = 3$

53. $x + y + z = 5$ **55.** $7x + y - 11z = 5$ **57.** $y - z = -1$

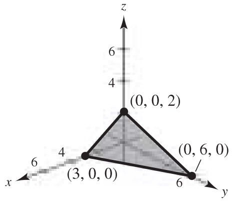
59.



61. $x - z = 0$ **63.** $9x - 3y + 2z - 21 = 0$

65. Orthogonal **67.** Neither; 83.5° **69.** Parallel

71.



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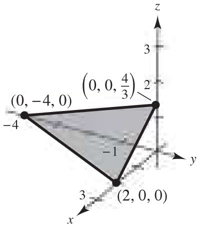

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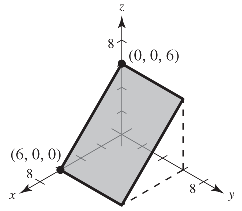

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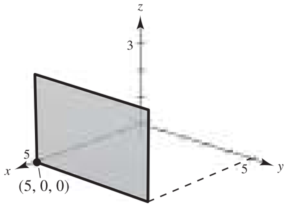

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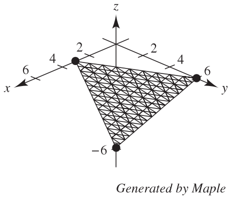

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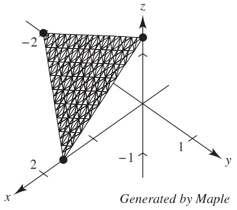
81.



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83. P_1 and P_2 are parallel. **85.** $P_1 = P_4$ and is parallel to P_2.

87. The planes have intercepts at $(c, 0, 0)$, $(0, c, 0)$, and $(0, 0, c)$ for each value of c.

89. If $c = 0$, $z = 0$ is the xy-plane; If $c \neq 0$, the plane is parallel to the x-axis and passes through $(0, 0, 0)$ and $(0, 1, -c)$.

91. (a) $\theta \approx 65.91°$

(b) $x = 2$
$y = 1 + t$
$z = 1 + 2t$

93. $(2, -3, 2)$; The line does not lie in the plane.

95. Not intersecting **97.** $6\sqrt{14}/7$ **99.** $11\sqrt{6}/6$

101. $2\sqrt{26}/13$ **103.** $27\sqrt{94}/188$ **105.** $\sqrt{2533}/17$

107. $7\sqrt{3}/3$ **109.** $\sqrt{66}/3$

111. Parametric equations: $x = x_1 + at$, $y = y_1 + bt$, and $z = z_1 + ct$

Symmetric equations: $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$

You need a vector $\mathbf{v} = \langle a, b, c \rangle$ parallel to the line and a point $P(x_1, y_1, z_1)$ on the line.

113. Simultaneously solve the two linear equations representing the planes and substitute the values back into one of the original equations. Then choose a value for t and form the corresponding parametric equations for the line of intersection.

115. (a) Parallel if vector $\langle a_1, b_1, c_1 \rangle$ is a scalar multiple of $\langle a_2, b_2, c_2 \rangle$; $\theta = 0$.

(b) Perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$; $\theta = \pi/2$.

117. $cbx + acy + abz = abc$

119. Sphere: $(x - 3)^2 + (y + 2)^2 + (z - 5)^2 = 16$

121. (a)

Year	1999	2000	2001	2002
z (approx.)	6.25	6.05	5.94	5.76

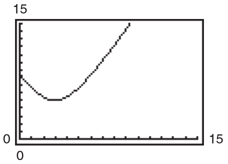
Year	2003	2004	2005
z (approx.)	5.66	5.56	5.56

The approximations are close to the actual values.

(b) Answers will vary.

123. (a) $\sqrt{70}$ in.

(b)



(c) The distance is never zero.

(d) 5 in.

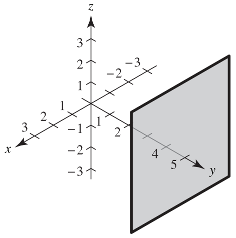
125. $\left(\frac{77}{13}, \frac{48}{13}, -\frac{23}{13}\right)$ **127.** $\left(-\frac{1}{2}, -\frac{9}{4}, \frac{1}{4}\right)$ **129.** True **131.** True

133. False. Plane $7x + y - 11z = 5$ and plane $5x + 2y - 4z = 1$ are perpendicular to plane $2x - 3y + z = 3$ but are not parallel.

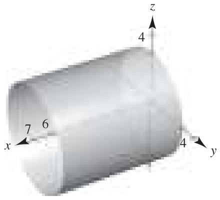
Section 11.6 (page 820)

1. c **2.** e **3.** f **4.** b **5.** d **6.** a

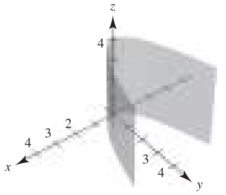
7. Plane


z
3
2
1
−2
−3
1
2
3
−1
−2
−3
1
2
4
5
x
y

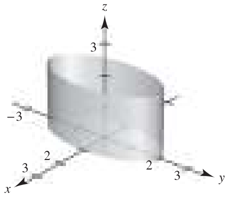
9. Right circular cylinder



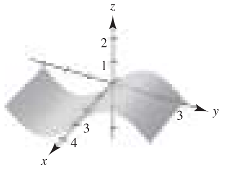
11. Parabolic cylinder



13. Elliptic cylinder

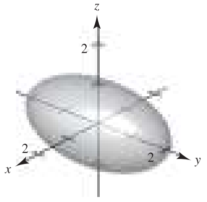


15. Cylinder

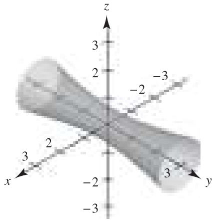


17. (a) (20, 0, 0)
(b) (10, 10, 20)
(c) (0, 0, 20)
(d) (0, 20, 0)

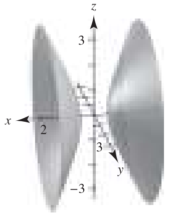
19. Ellipsoid



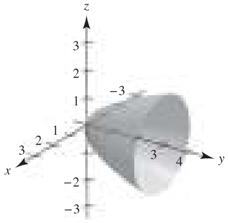
21. Hyperboloid of one sheet


z
3
2
−2
−3
3
2
3
−2
−3
x
y

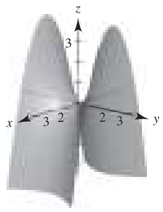
23. Hyperboloid of two sheets



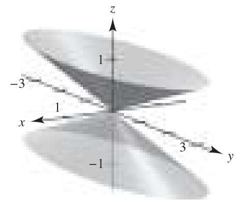
25. Elliptic paraboloid


z
3
2
1
−3
3
2
1
3
4
−2
−3
x
y

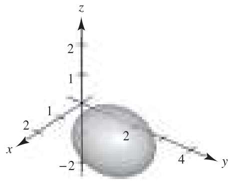
27. Hyperbolic paraboloid



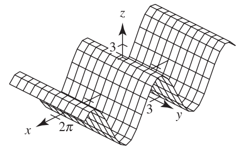
29. Elliptic cone



31. Ellipsoid


z
2
1
1
2
x
2
4
y
−2

33.



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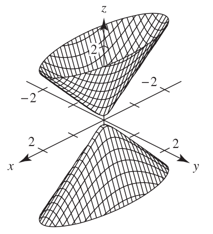

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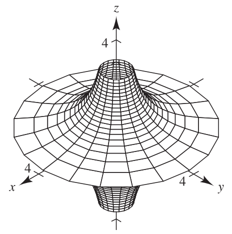

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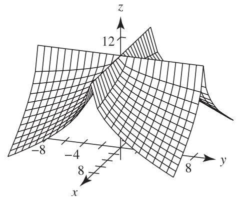

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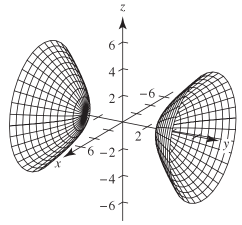
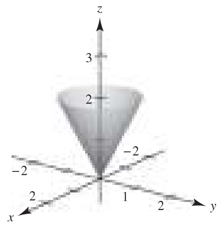

z
6
4
2
−6
2
6
−2
−4
−6
x
y

43.


z
3
2
−2
−2
2
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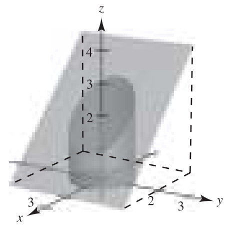
2
x
y

45.



47. $x^2 + z^2 = 4y$ **49.** $4x^2 + 4y^2 = z^2$ **51.** $y^2 + z^2 = 4/x^2$

53. $y = \sqrt{2z}$ $\left(\text{or } x = \sqrt{2z}\right)$

55. Let C be a curve in a plane and let L be a line not in a parallel plane. The set of all lines parallel to L and intersecting C is called a cylinder. C is called the generating curve of the cylinder, and the parallel lines are called rulings.

57. See pages 814 and 815. **59.** $128\pi/3$

61. (a) Major axis: $4\sqrt{2}$
Minor axis: 4
Foci: $(0, \pm 2, 2)$

(b) Major axis: $8\sqrt{2}$
Minor axis: 8
Foci: $(0, \pm 4, 8)$

63. $x^2 + z^2 = 8y$; Elliptic paraboloid

65. $x^2/3963^2 + y^2/3963^2 + z^2/3950^2 = 1$

67. $x = at, y = -bt, z = 0$;
$x = at, y = bt + ab^2, z = 2abt + a^2b^2$

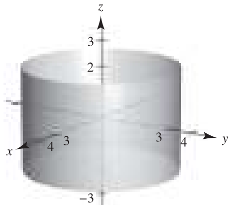
69. True

71. False. A trace of an ellipsoid can be a single point.

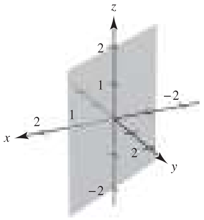
73. The Klein bottle does not have both an “inside” and an “outside.” It is formed by inserting the small open end through the side of the bottle and making it contiguous with the top of the bottle.

Section 11.7 (page 827)

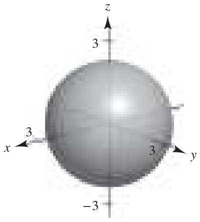
1. $(-7, 0, 5)$ **3.** $\left(3\sqrt{2}/2, 3\sqrt{2}/2, 1\right)$ **5.** $\left(-2\sqrt{3}, -2, 3\right)$
7. $(5, \pi/2, 1)$ **9.** $\left(2\sqrt{2}, -\pi/4, -4\right)$ **11.** $(2, \pi/3, 4)$
13. $z = 4$ **15.** $r^2 + z^2 = 17$ **17.** $r = \sec\theta\tan\theta$
19. $r^2\sin^2\theta = 10 - z^2$
21. $x^2 + y^2 = 9$



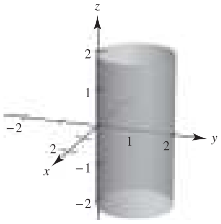
23. $x - \sqrt{3}y = 0$



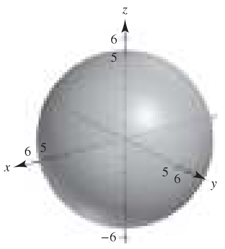
25. $x^2 + y^2 + z^2 = 5$



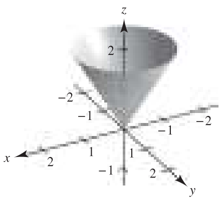
27. $x^2 + y^2 - 2y = 0$



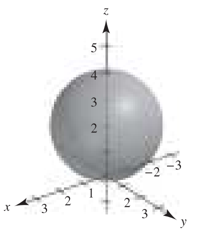
29. $(4, 0, \pi/2)$ **31.** $\left(4\sqrt{2}, 2\pi/3, \pi/4\right)$ **33.** $(4, \pi/6, \pi/6)$
35. $\left(\sqrt{6}, \sqrt{2}, 2\sqrt{2}\right)$ **37.** $(0, 0, 12)$ **39.** $\left(\frac{5}{2}, \frac{5}{2}, -5\sqrt{2}/2\right)$
41. $\rho = 2\csc\phi\csc\theta$ **43.** $\rho = 7$
45. $\rho = 4\csc\phi$ **47.** $\tan^2\phi = 2$
49. $x^2 + y^2 + z^2 = 25$



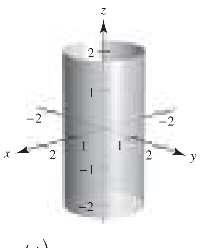
51. $3x^2 + 3y^2 - z^2 = 0$



53. $x^2 + y^2 + (z - 2)^2 = 4$



55. $x^2 + y^2 = 1$



57. $(4, \pi/4, \pi/2)$ **59.** $\left(4\sqrt{2}, \pi/2, \pi/4\right)$
61. $\left(2\sqrt{13}, -\pi/6, \arccos[3/\sqrt{13}]\right)$ **63.** $(13, \pi, \arccos[5/13])$
65. $(10, \pi/6, 0)$ **67.** $(36, \pi, 0)$
69. $\left(3\sqrt{3}, -\pi/6, 3\right)$ **71.** $\left(4, 7\pi/6, 4\sqrt{3}\right)$

	Rectangular	*Cylindrical*	*Spherical*
73.	$(4, 6, 3)$	$(7.211, 0.983, 3)$	$(7.810, 0.983, 1.177)$
75.	$(4.698, 1.710, 8)$	$(5, \pi/9, 8)$	$(9.434, 0.349, 0.559)$
77.	$(-7.071, 12.247, 14.142)$	$(14.142, 2.094, 14.142)$	$(20, 2\pi/3, \pi/4)$
79.	$(3, -2, 2)$	$(3.606, -0.588, 2)$	$(4.123, -0.588, 1.064)$
81.	$\left(\frac{5}{2}, \frac{4}{3}, -\frac{3}{2}\right)$	$(2.833, 0.490, -1.5)$	$(3.206, 0.490, 2.058)$
83.	$(-3.536, 3.536, -5)$	$(5, 3\pi/4, -5)$	$(7.071, 2.356, 2.356)$
85.	$(2.804, -2.095, 6)$	$(-3.5, 2.5, 6)$	$(6.946, 5.642, 0.528)$
87.	$(-1.837, 1.837, 1.5)$	$(2.598, 2.356, 1.5)$	$(3, 3\pi/4, \pi/3)$

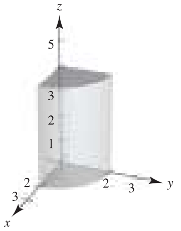
89. d **90.** e **91.** c **92.** a **93.** f **94.** b

95. Rectangular to cylindrical:
$r^2 = x^2 + y^2, \tan\theta = y/x, z = z$
Cylindrical to rectangular:
$x = r\cos\theta, y = r\sin\theta, z = z$

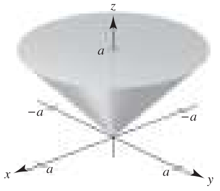
97. Rectangular to spherical:
$\rho^2 = x^2 + y^2 + z^2, \tan\theta = y/x, \phi = \arccos\left(z/\sqrt{x^2 + y^2 + z^2}\right)$
Spherical to rectangular:
$x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$

99. (a) $r^2 + z^2 = 25$ (b) $\rho = 5$
101. (a) $r^2 + (z - 1)^2 = 1$ (b) $\rho = 2\cos\phi$
103. (a) $r = 4\sin\theta$ (b) $\rho = 4\sin\theta/\sin\phi = 4\sin\theta\csc\phi$
105. (a) $r^2 = 9/(\cos^2\theta - \sin^2\theta)$
(b) $\rho^2 = 9\csc^2\phi/(\cos^2\theta - \sin^2\theta)$

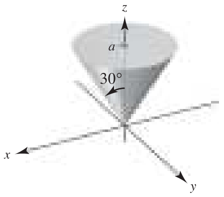
107.



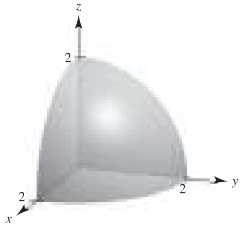
109.



111.



113.



115. Rectangular: $0 \le x \le 10$
$0 \le y \le 10$
$0 \le z \le 10$

117. Spherical: $4 \le \rho \le 6$

119. Cylindrical: $r^2 + z^2 \le 9, r \le 3\cos\theta, 0 \le \theta \le \pi$
121. False. $r = z$ represents a cone.
123. False. See page 823. **125.** Ellipse

Review Exercises for Chapter 11 (page 829)

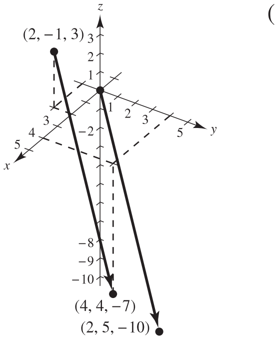
1. (a) $\mathbf{u} = \langle 3, -1\rangle$ (b) $\mathbf{u} = 3\mathbf{i} - \mathbf{j}$ (c) $2\sqrt{5}$ (d) $10\mathbf{i}$
$\mathbf{v} = \langle 4, 2\rangle$
3. $\mathbf{v} = \left\langle 4, 4\sqrt{3}\right\rangle$ **5.** $(-5, 4, 0)$

7. Above the xy-plane and to the right of the xz-plane *or* below the xy-plane and to the left of the xz-plane

9. $(x-3)^2 + (y+2)^2 + (z-6)^2 = \frac{225}{4}$

11. $(x-2)^2 + (y-3)^2 + z^2 = 9$
Center: $(2, 3, 0)$
Radius: 3

13. (a) and (d) (b) $\mathbf{u} = \langle 2, 5, -10 \rangle$
(c) $\mathbf{u} = 2\mathbf{i} + 5\mathbf{j} - 10\mathbf{k}$


z
(2, −1, 3)
3
2
1
3
4
5
x
1
2
3
5
y
−2
−8
−9
−10
(4, 4, −7)
(2, 5, −10)

15. Collinear **17.** $(1/\sqrt{38})\langle 2, 3, 5 \rangle$

19. (a) $\mathbf{u} = \langle -1, 4, 0 \rangle$, $\mathbf{v} = \langle -3, 0, 6 \rangle$ (b) 3 (c) 45

21. Orthogonal **23.** $\theta = \arccos\left(\dfrac{\sqrt{2}+\sqrt{6}}{4}\right) = 15°$ **25.** π

27. Answers will vary. Example: $\langle -6, 5, 0 \rangle$, $\langle 6, -5, 0 \rangle$

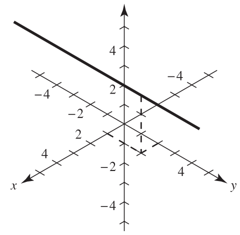
29. $\mathbf{u} \cdot \mathbf{u} = 14 = \|\mathbf{u}\|^2$ **31.** $\left\langle -\frac{15}{14}, \frac{5}{7}, -\frac{5}{14} \right\rangle$

33. $(1/\sqrt{5})(-2\mathbf{i} - \mathbf{j})$ or $(1/\sqrt{5})(2\mathbf{i} + \mathbf{j})$

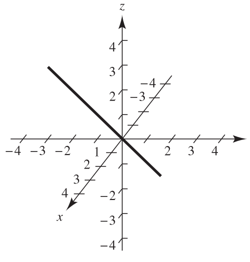
35. 4 **37.** $\sqrt{285}$ **39.** $100 \sec 20° \approx 106.4$ lb

41. (a) $x = 3 + 6t,\ y = 11t,\ z = 2 + 4t$
(b) $(x-3)/6 = y/11 = (z-2)/4$

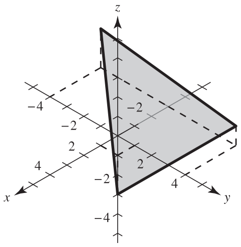
43. (a) $x = 1,\ y = 2 + t,\ z = 3$ (b) None
(c)

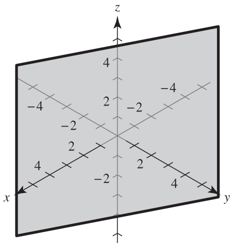

z
4
2
−4
−4
−2
2
4
x
−2
4
y
−4

45. (a) $x = t,\ y = -1 + t,\ z = 1$ (b) $x = y + 1,\ z = 1$
(c)


z
4
3
2
−4
−3
−4
−3
−2
1
2
3
4
2
3
4
x
−2
−3
−4

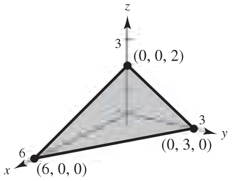
47. $27x + 4y + 32z + 33 = 0$ **49.** $x + 2y = 1$


z
−4
−2
−2
2
4
2
−2
4
x
y
−4

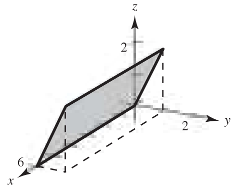

z
4
−4
−4
2
−2
−2
2
4
2
4
−2
x
y

51. $\frac{8}{7}$ **53.** $\sqrt{35}/7$

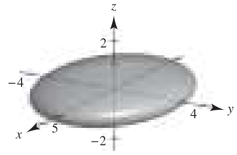
55. Plane


z
3
(0, 0, 2)
3
y
(0, 3, 0)
6
x
(6, 0, 0)

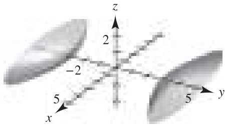
57. Plane



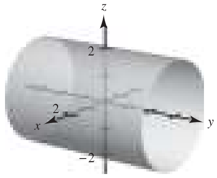
59. Ellipsoid



61. Hyperboloid of two sheets



63. Cylinder



65. Let $y = 2\sqrt{x}$ and revolve around the x-axis.

67. $x^2 + z^2 = 2y$

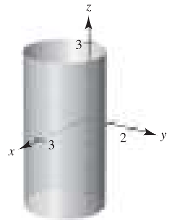
69. (a) $(4, 3\pi/4, 2)$ (b) $\left(2\sqrt{5}, 3\pi/4, \arccos\left[\sqrt{5}/5\right]\right)$

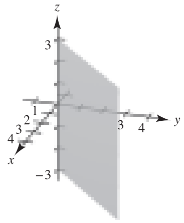
71. $\left(50\sqrt{5}, -\pi/6, \arccos\left[1/\sqrt{5}\right]\right)$

73. $\left(25\sqrt{2}/2, -\pi/4, -25\sqrt{2}/2\right)$

75. (a) $r^2 \cos 2\theta = 2z$ (b) $\rho = 2 \sec 2\theta \cos\phi \csc^2\phi$

77. $\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$ **79.** $x = y$



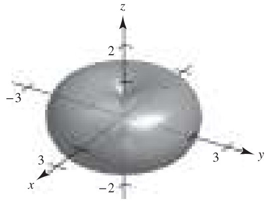

z
3
1
2
3
4
3
4
y
x
−3

P.S. Problem Solving (page 831)

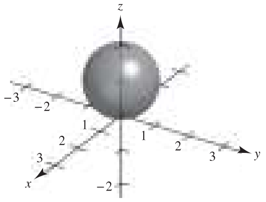
1–3. Proofs **5.** (a) $3\sqrt{2}/2 \approx 2.12$ (b) $\sqrt{5} \approx 2.24$

7. (a) $\pi/2$ (b) $\frac{1}{2}(\pi abk)k$
(c) $V = \frac{1}{2}(\pi ab)k^2$
$V = \frac{1}{2}(\text{area of base})\text{height}$

9. (a)



(b)


z
−3
−2
1
2
3
1
2
3
y
x
−2

11. Proof

13. (a) Tension: $2\sqrt{3}/3 \approx 1.1547$ lb

Magnitude of $\mathbf{u}$: $\sqrt{3}/3 \approx 0.5774$ lb

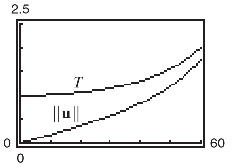
(b) $T = \sec\theta$; $\|\mathbf{u}\| = \tan\theta$; Domain: $0° \le \theta \le 90°$

(c)

θ	0°	10°	20°	30°
T	1	1.0154	1.0642	1.1547
$\|\mathbf{u}\|$	0	0.1763	0.3640	0.5774

θ	40°	50°	60°
T	1.3054	1.5557	2
$\|\mathbf{u}\|$	0.8391	1.1918	1.7321

(d)



(e) Both are increasing functions.

(f) $\displaystyle\lim_{\theta\to\pi/2^-} T = \infty$ and $\displaystyle\lim_{\theta\to\pi/2^-} \|\mathbf{u}\| = \infty$

Yes. As θ increases, both T and $\|\mathbf{u}\|$ increase.

15. $\langle 0, 0, \cos\alpha\sin\beta - \cos\beta\sin\alpha\rangle$; Proof

17. $D = \dfrac{|\overrightarrow{PQ}\cdot\mathbf{n}|}{\|\mathbf{n}\|}$

$= \dfrac{|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|}{\|\mathbf{u}\times\mathbf{v}\|} = \dfrac{|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|}{\|\mathbf{u}\times\mathbf{v}\|} = \dfrac{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{u}\times\mathbf{v}\|}$

19. Proof

Chapter 12

Section 12.1 (page 839)

1. $(-\infty, -1) \cup (-1, \infty)$ **3.** $(0, \infty)$

5. $[0, \infty)$ **7.** $(-\infty, \infty)$

9. (a) $\frac{1}{2}\mathbf{i}$ (b) $\mathbf{j}$ (c) $\frac{1}{2}(s+1)^2\mathbf{i} - s\mathbf{j}$ (d) $\frac{1}{2}\Delta t(\Delta t + 4)\mathbf{i} - \Delta t\mathbf{j}$

11. (a) $\ln 2\mathbf{i} + \dfrac{1}{2}\mathbf{j} + 6\mathbf{k}$ (b) Not possible

(c) $\ln(t-4)\mathbf{i} + \dfrac{1}{t-4}\mathbf{j} + 3(t-4)\mathbf{k}$

(d) $\ln(1+\Delta t)\mathbf{i} - \dfrac{\Delta t}{1+\Delta t}\mathbf{j} + 3\Delta t\mathbf{k}$

13. $\sqrt{t(1+25t)}$

15. $\mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}$

$x = 3t,\ y = t,\ z = 2t$

17. $\mathbf{r}(t) = (-2+t)\mathbf{i} + (5-t)\mathbf{j} + (-3+12t)\mathbf{k}$

$x = -2 + t,\ y = 5 - t,\ z = -3 + 12t$

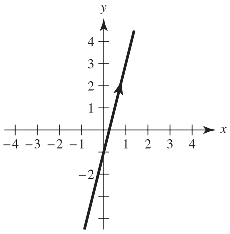
19. $t^2(5t-1)$; No, the dot product is a scalar.

21. b **22.** c **23.** d **24.** a

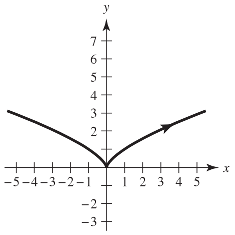
25. (a) $(-20, 0, 0)$ (b) $(10, 20, 10)$

(c) $(0, 0, 20)$ (d) $(20, 0, 0)$

27.



29.



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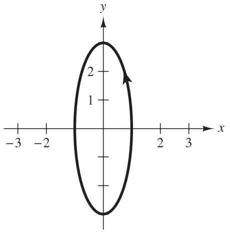

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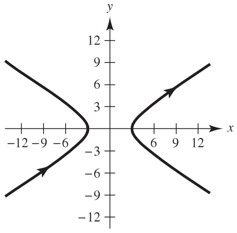

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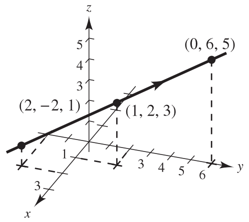

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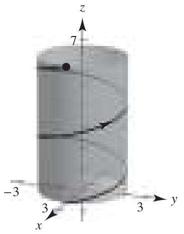
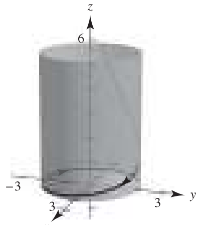
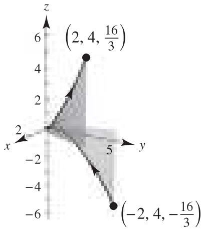


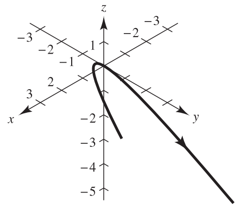
39.



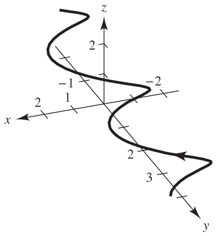
41.



43.

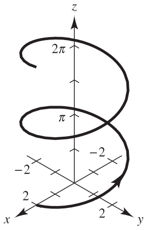


45.



Parabola Helix

47.



(a) The helix is translated two units back on the x-axis.

(b) The height of the helix increases at a greater rate.

(c) The orientation of the graph is reversed.

(d) The axis of the helix is the x-axis.

(e) The radius of the helix is increased from 2 to 6.

49–55. Answers will vary.

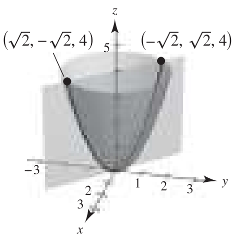
57. Answers will vary. Sample answer:

$\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j},\quad 0 \le t \le 2$

$\mathbf{r}_2(t) = (2-t)\mathbf{i} + 4\mathbf{j},\quad 0 \le t \le 2$

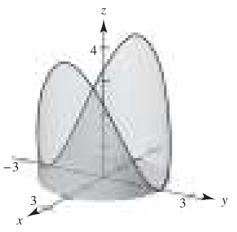
$\mathbf{r}_3(t) = (4-t)\mathbf{j},\quad 0 \le t \le 4$

59.



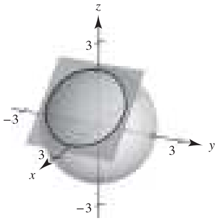
$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$

61.



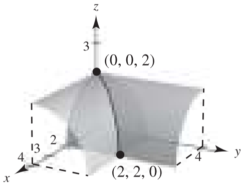
$\mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + 4\sin^2 t\mathbf{k}$

63.



$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + \sqrt{2}\cos t\mathbf{j} + (1 - \sin t)\mathbf{k}$ and
$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} - \sqrt{2}\cos t\mathbf{j} + (1 - \sin t)\mathbf{k}$

65.

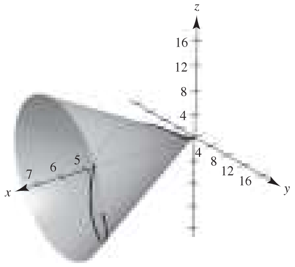


$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$

67. Let $x = t$, $y = 2t\cos t$, and $z = 2t\sin t$. Then

$$y^2 + z^2 = (2t\cos t)^2 + (2t\sin t)^2 = 4t^2\cos^2 t + 4t^2\sin^2 t = 4t^2(\cos^2 t + \sin^2 t) = 4t^2.$$

Because $x = t$, $y^2 + z^2 = 4x^2$.



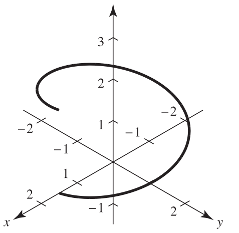
69. $\pi\mathbf{i} - \mathbf{j}$ **71.** $\mathbf{0}$ **73.** $\mathbf{i} + \mathbf{j} + \mathbf{k}$

75. $(-\infty, 0), (0, \infty)$ **77.** $[-1, 1]$

79. $(-\pi/2 + n\pi, \pi/2 + n\pi)$, n is an integer.

81. (a) $\mathbf{s}(t) = t^2\mathbf{i} + (t - 3)\mathbf{j} + (t + 3)\mathbf{k}$
(b) $\mathbf{s}(t) = (t^2 - 2)\mathbf{i} + (t - 3)\mathbf{j} + t\mathbf{k}$
(c) $\mathbf{s}(t) = t^2\mathbf{i} + (t + 2)\mathbf{j} + t\mathbf{k}$

83. Answers will vary. Sample answer:



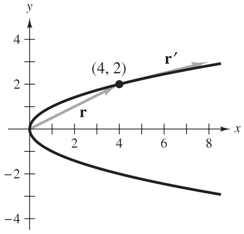
$\mathbf{r}(t) = 1.5\cos t\mathbf{i} + 1.5\sin t\mathbf{j} + \dfrac{1}{\pi}t\mathbf{k}$, $0 \le t \le 2\pi$

85–87. Proofs **89.** Yes; Yes **91.** Not necessarily

93. True **95.** True

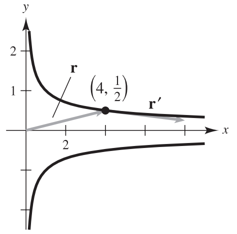
Section 12.2 (page 848)

1. $\mathbf{r}(2) = 4\mathbf{i} + 2\mathbf{j}$
$\mathbf{r}'(2) = 4\mathbf{i} + \mathbf{j}$



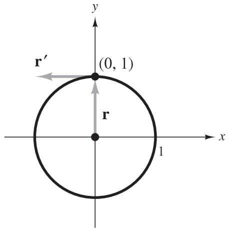
$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

3. $\mathbf{r}(2) = 4\mathbf{i} + \frac{1}{2}\mathbf{j}$
$\mathbf{r}'(2) = 4\mathbf{i} - \frac{1}{4}\mathbf{j}$



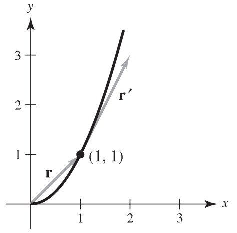
$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

5. $\mathbf{r}(\pi/2) = \mathbf{j}$
$\mathbf{r}'(\pi/2) = -\mathbf{i}$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

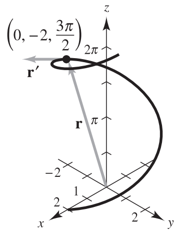
7. $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$
$\mathbf{r}'(0) = \mathbf{i} + 2\mathbf{j}$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

9. $\mathbf{r}\left(\dfrac{3\pi}{2}\right) = -2\mathbf{j} + \left(\dfrac{3\pi}{2}\right)\mathbf{k}$

$\mathbf{r}'\left(\dfrac{3\pi}{2}\right) = 2\mathbf{i} + \mathbf{k}$



11. $3t^2\mathbf{i} - 3\mathbf{j}$ **13.** $-2\sin t\mathbf{i} + 5\cos t\mathbf{j}$

15. $6\mathbf{i} - 14t\mathbf{j} + 3t^2\mathbf{k}$ **17.** $-3a\sin t\cos^2 t\mathbf{i} + 3a\sin^2 t\cos t\mathbf{j}$

19. $-e^{-t}\mathbf{i} + (5te^t + 5e^t)\mathbf{k}$

21. $\langle \sin t + t\cos t, \cos t - t\sin t, 1\rangle$

23. (a) $3t^2\mathbf{i} + t\mathbf{j}$ (b) $6t\mathbf{i} + \mathbf{j}$ (c) $18t^3 + t$

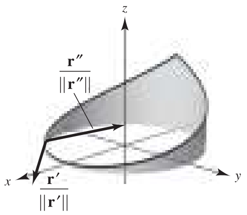
25. (a) $-4\sin t\mathbf{i} + 4\cos t\mathbf{j}$ (b) $-4\cos t\mathbf{i} - 4\sin t\mathbf{j}$ (c) 0

27. (a) $t\mathbf{i} - \mathbf{j} + \frac{1}{2}t^2\mathbf{k}$ (b) $\mathbf{i} + t\mathbf{k}$ (c) $t^3/2 + t$

29. (a) $\langle t\cos t, t\sin t, 1\rangle$
(b) $\langle \cos t - t\sin t, \sin t + t\cos t, 0\rangle$ (c) t

31. $$\frac{\mathbf{r}'(-1/4)}{\|\mathbf{r}'(-1/4)\|} = \frac{1}{\sqrt{4\pi^2 + 1}}\left(\sqrt{2}\pi\mathbf{i} + \sqrt{2}\pi\mathbf{j} - \mathbf{k}\right)$$

$$\frac{\mathbf{r}''(-1/4)}{\|\mathbf{r}''(-1/4)\|} = \frac{1}{2\sqrt{\pi^4 + 4}}\left(-\sqrt{2}\pi^2\mathbf{i} + \sqrt{2}\pi^2\mathbf{j} + 4\mathbf{k}\right)$$


z
$\frac{\mathbf{r}''}{\|\mathbf{r}''\|}$
x
$\frac{\mathbf{r}'}{\|\mathbf{r}'\|}$
y

33. $(-\infty, 0), (0, \infty)$ **35.** $(n\pi/2, (n+1)\pi/2)$

37. $(-\infty, \infty)$ **39.** $(-\infty, 0), (0, \infty)$

41. $(-\pi/2 + n\pi, \pi/2 + n\pi)$, n is an integer.

43. (a) $\mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$ (b) $2\mathbf{k}$ (c) $8t + 9t^2 + 5t^4$
(d) $-\mathbf{i} + (9 - 2t)\mathbf{j} + (6t - 3t^2)\mathbf{k}$
(e) $8t^3\mathbf{i} + (12t^2 - 4t^3)\mathbf{j} + (3t^2 - 24t)\mathbf{k}$
(f) $(10 + 2t^2)/\sqrt{10 + t^2}$

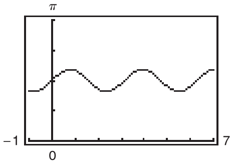
45. (a) $7t^6$ (b) $12t^5\mathbf{i} - 5t^4\mathbf{j}$

47. $\theta(t) = \arccos\left(\dfrac{-7 \sin t \cos t}{\sqrt{9\sin^2 t + 16\cos^2 t}\,\sqrt{9\cos^2 t + 16\sin^2 t}}\right)$

Maximum: $\theta\left(\dfrac{\pi}{4}\right) = \theta\left(\dfrac{5\pi}{4}\right) \approx 1.855$

Minimum: $\theta\left(\dfrac{3\pi}{4}\right) = \theta\left(\dfrac{7\pi}{4}\right) \approx 1.287$

Orthogonal: $\dfrac{n\pi}{2}$, n is an integer.



49. $\mathbf{r}'(t) = 3\mathbf{i} - 2t\mathbf{j}$ **51.** $\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{k}$

53. $t^2\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$ **55.** $\ln t\,\mathbf{i} + t\mathbf{j} - \frac{2}{5}t^{5/2}\mathbf{k} + \mathbf{C}$

57. $(t^2 - t)\mathbf{i} + t^4\mathbf{j} + 2t^{3/2}\mathbf{k} + \mathbf{C}$

59. $\tan t\,\mathbf{i} + \arctan t\,\mathbf{j} + \mathbf{C}$ **61.** $4\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$

63. $a\mathbf{i} + a\mathbf{j} + (\pi/2)\mathbf{k}$

65. $2\mathbf{i} + (e^2 - 1)\mathbf{j} - (e^2 + 1)\mathbf{k}$

67. $2e^{2t}\mathbf{i} + 3(e^t - 1)\mathbf{j}$ **69.** $600\sqrt{3}t\mathbf{i} + (-16t^2 + 600t)\mathbf{j}$

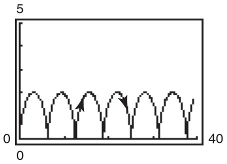
71. $((2 - e^{-t^2})/2)\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t + 1)\mathbf{k}$

73. See "Definition of the Derivative of a Vector-Valued Function" and Figure 12.8 on page 842.

75. The three components of $\mathbf{u}$ are increasing functions of t at $t = t_0$.

77–83. Proofs

85. (a)



The curve is a cycloid.

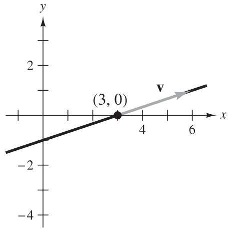
(b) The maximum of $\|\mathbf{r}'\|$ is 2; the minimum of $\|\mathbf{r}'\|$ is 0. The maximum and the minimum of $\|\mathbf{r}''\|$ is 1.

87. Proof **89.** True

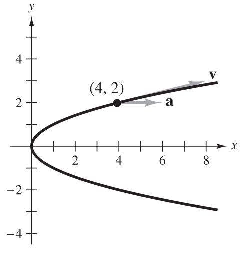
91. False: Let $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}$, then $d/dt[\|\mathbf{r}(t)\|] = 0$, but $\|\mathbf{r}'(t)\| = 1$.

Section 12.3 (page 856)

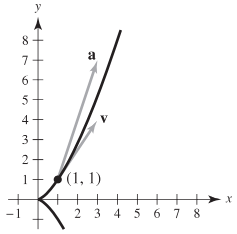
1. $\mathbf{v}(1) = 3\mathbf{i} + \mathbf{j}$
$\mathbf{a}(1) = \mathbf{0}$



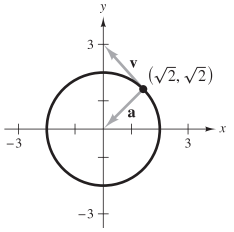
3. $\mathbf{v}(2) = 4\mathbf{i} + \mathbf{j}$
$\mathbf{a}(2) = 2\mathbf{i}$



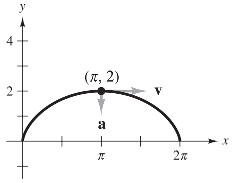
5. $\mathbf{v}(1) = 2\mathbf{i} + 3\mathbf{j}$
$\mathbf{a}(1) = 2\mathbf{i} + 6\mathbf{j}$



7. $\mathbf{v}(\pi/4) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$
$\mathbf{a}(\pi/4) = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$



9. $\mathbf{v}(\pi) = 2\mathbf{i}$
$\mathbf{a}(\pi) = -\mathbf{j}$



11. $\mathbf{v}(t) = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$
$\|\mathbf{v}(t)\| = \sqrt{35}$
$\mathbf{a}(t) = \mathbf{0}$

13. $\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$
$\|\mathbf{v}(t)\| = \sqrt{1 + 5t^2}$
$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

15. $\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \left(t/\sqrt{9 - t^2}\right)\mathbf{k}$
$\|\mathbf{v}(t)\| = \sqrt{(18 - t^2)/(9 - t^2)}$
$\mathbf{a}(t) = (-9/(9 - t^2)^{3/2})\mathbf{k}$

17. $\mathbf{v}(t) = 4\mathbf{i} - 3\sin t\,\mathbf{j} + 3\cos t\,\mathbf{k}$
$\|\mathbf{v}(t)\| = 5$
$\mathbf{a}(t) = -3\cos t\,\mathbf{j} - 3\sin t\,\mathbf{k}$

19. $\mathbf{v}(t) = (e^t\cos t - e^t\sin t)\mathbf{i} + (e^t\sin t + e^t\cos t)\mathbf{j} + e^t\mathbf{k}$
$\|\mathbf{v}(t)\| = e^t\sqrt{3}$
$\mathbf{a}(t) = -2e^t\sin t\,\mathbf{i} + 2e^t\cos t\,\mathbf{j} + e^t\mathbf{k}$

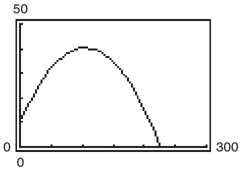
21. (a) $x = 1 + t$ (b) $(1.100, -1.200, 0.325)$
$y = -1 - 2t$
$z = \frac{1}{4} + \frac{3}{4}t$

23. $\mathbf{v}(t) = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$
$\mathbf{r}(t) = (t^2/2)(\mathbf{i} + \mathbf{j} + \mathbf{k})$
$\mathbf{r}(2) = 2(\mathbf{i} + \mathbf{j} + \mathbf{k})$

25. $\mathbf{v}(t) = \left(t^2/2 + \frac{9}{2}\right)\mathbf{j} + \left(t^2/2 - \frac{1}{2}\right)\mathbf{k}$
$\mathbf{r}(t) = \left(t^3/6 + \frac{9}{2}t - \frac{14}{3}\right)\mathbf{j} + \left(t^3/6 - \frac{1}{2}t + \frac{1}{3}\right)\mathbf{k}$
$\mathbf{r}(2) = \frac{17}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

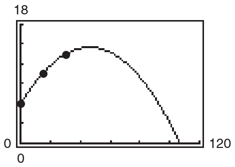
27. $\mathbf{v}(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$
$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$
$\mathbf{r}(2) = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j} + 2\mathbf{k}$

29. $\mathbf{r}(t) = 44\sqrt{3}t\mathbf{i} + (10 + 44t - 16t^2)\mathbf{j}$



31. $v_0 = 40\sqrt{6}$ ft/sec; 78 ft **33.** Proof

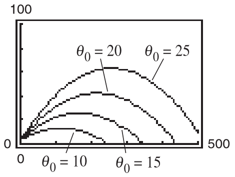
35. (a) $y = -0.004x^2 + 0.37x + 6$
$\mathbf{r}(t) = t\mathbf{i} + (-0.004t^2 + 0.37t + 6)\mathbf{j}$

(b)


(c) 14.56 ft

(d) Initial velocity: 67.4 ft/sec; $\theta \approx 20.14°$

37. (a) $\mathbf{r}(t) = \left(\frac{440}{3}\cos\theta_0\right)t\mathbf{i} + \left[3 + \left(\frac{440}{3}\sin\theta_0\right)t - 16t^2\right]\mathbf{j}$

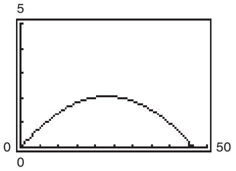
(b)


The minimum angle appears to be $\theta_0 = 20°$.

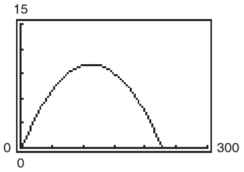
(c) $\theta_0 \approx 19.38°$

39. (a) $v_0 = 28.78$ ft/sec; $\theta = 58.28°$ (b) $v_0 \approx 32$ ft/sec

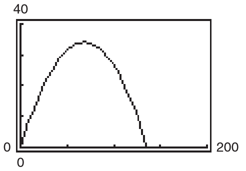
41. 1.91°

43. (a)


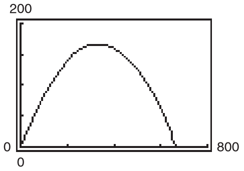
Maximum height: 2.1 ft
Range: 46.6 ft

(b)


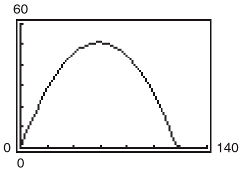
Maximum height: 10.0 ft
Range: 227.8 ft

(c)


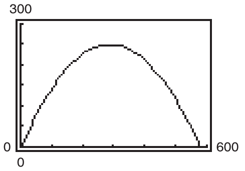
Maximum height: 34.0 ft
Range: 136.1 ft

(d)


Maximum height: 166.5 ft
Range: 666.1 ft

(e)


Maximum height: 51.0 ft
Range: 117.9 ft

(f)


Maximum height: 249.8 ft
Range: 576.9 ft

45. Maximum height: 129.1 m
Range: 886.3 m

47. $\mathbf{v}(t) = b\omega[(1 - \cos\omega t)\mathbf{i} + \sin\omega t\mathbf{j}]$
$\mathbf{a}(t) = b\omega^2(\sin\omega t\mathbf{i} + \cos\omega t\mathbf{j})$
(a) $\|\mathbf{v}(t)\| = 0$ when $\omega t = 0, 2\pi, 4\pi, \ldots$
(b) $\|\mathbf{v}(t)\|$ is maximum when $\omega t = \pi, 3\pi, \ldots$

49. $\mathbf{v}(t) = -b\omega\sin\omega t\mathbf{i} + b\omega\cos\omega t\mathbf{j}$
$\mathbf{v}(t) \cdot \mathbf{r}(t) = 0$

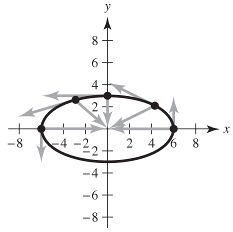
51. $\mathbf{a}(t) = -b\omega^2(\cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}) = -\omega^2\mathbf{r}(t)$; $\mathbf{a}(t)$ is a negative multiple of a unit vector from (0, 0) to $(\cos\omega t, \sin\omega t)$, so $\mathbf{a}(t)$ is directed toward the origin.

53. $8\sqrt{10}$ ft/sec **55–57.** Proofs

59. (a) $\mathbf{v}(t) = -6\sin t\mathbf{i} + 3\cos t\mathbf{j}$
$\|\mathbf{v}(t)\| = 3\sqrt{3\sin^2 t + 1}$
$\mathbf{a}(t) = -6\cos t\mathbf{i} - 3\sin t\mathbf{j}$

(b)

t	0	$\pi/4$	$\pi/2$	$2\pi/3$	π
Speed	3	$3\sqrt{10}/2$	6	$3\sqrt{13}/2$	3

(c)


(d) The speed is increasing when the angle between **v** and **a** is in the interval $[0, \pi/2)$, and decreasing when the angle is in the interval $(\pi/2, \pi]$.

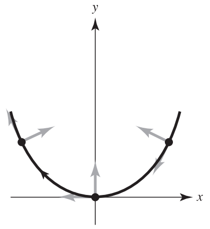
61. The velocity of an object involves both magnitude and direction of motion, whereas speed involves only magnitude.

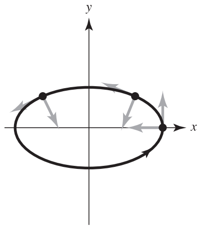
63. (a) Velocity: $\mathbf{r}_2'(t) = 2\mathbf{r}_1'(2t)$
Acceleration: $\mathbf{r}_2''(t) = 4\mathbf{r}_1''(2t)$
(b) In general, if $\mathbf{r}_3(t) = \mathbf{r}_1(\omega t)$, then:
Velocity: $\mathbf{r}_3'(t) = \omega\mathbf{r}_1'(\omega t)$
Acceleration: $\mathbf{r}_3''(t) = \omega^2\mathbf{r}_1''(\omega t)$

65. False; acceleration is the derivative of the velocity.

67. True

Section 12.4 (page 865)

1.


3.


5. $\mathbf{T}(1) = (\sqrt{2}/2)(\mathbf{i} + \mathbf{j})$ **7.** $\mathbf{T}(\pi/4) = (\sqrt{2}/2)(-\mathbf{i} + \mathbf{j})$

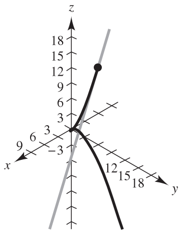
9. $\mathbf{T}(e) = (3e\mathbf{i} - \mathbf{j})/\sqrt{9e^2 + 1} \approx 0.9926\mathbf{i} - 0.1217\mathbf{j}$

11. $\mathbf{T}(0) = (\sqrt{2}/2)(\mathbf{i} + \mathbf{k})$
$x = t$
$y = 0$
$z = t$

13. $\mathbf{T}(0) = (\sqrt{10}/10)(3\mathbf{j} + \mathbf{k})$
$x = 3$
$y = 3t$
$z = t$

15. $\mathbf{T}(\pi/4) = \frac{1}{2}\langle -\sqrt{2}, \sqrt{2}, 0\rangle$
$x = \sqrt{2} - \sqrt{2}t$
$y = \sqrt{2} + \sqrt{2}t$
$z = 4$

17. $\mathbf{T}(3) = \frac{1}{19}\langle 1, 6, 18\rangle$
$x = 3 + t$
$y = 9 + 6t$
$z = 18 + 18t$



19. Tangent line: $x = 1 + t, y = t, z = 1 + \frac{1}{2}t$
$\mathbf{r}(1.1) \approx \langle 1.1, 0.1, 1.05\rangle$

21. 1.2° **23.** $\mathbf{N}(2) = (\sqrt{5}/5)(-2\mathbf{i} + \mathbf{j})$

25. $\mathbf{N}(2) = (-\sqrt{5}/5)(2\mathbf{i} - \mathbf{j})$

27. $\mathbf{N}(1) = (-\sqrt{14}/14)(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

29. $\mathbf{N}(3\pi/4) = (\sqrt{2}/2)(\mathbf{i} - \mathbf{j})$

31. $\mathbf{v}(t) = 4\mathbf{i}$
$\mathbf{a}(t) = \mathbf{0}$
$\mathbf{T}(t) = \mathbf{i}$
$\mathbf{N}(t)$ is undefined. The path is a line and the speed is constant.

33. $\mathbf{v}(t) = 8t\mathbf{i}$
$\mathbf{a}(t) = 8\mathbf{i}$
$\mathbf{T}(t) = \mathbf{i}$
$\mathbf{N}(t)$ is undefined. The path is a line and the speed is variable.

35. $\mathbf{T}(1) = (\sqrt{2}/2)(\mathbf{i} - \mathbf{j})$
$\mathbf{N}(1) = (\sqrt{2}/2)(\mathbf{i} + \mathbf{j})$
$a_{\mathbf{T}} = -\sqrt{2}$
$a_{\mathbf{N}} = \sqrt{2}$

37. $\mathbf{T}(1) = (-\sqrt{5}/5)(\mathbf{i} - 2\mathbf{j})$
$\mathbf{N}(1) = (-\sqrt{5}/5)(2\mathbf{i} + \mathbf{j})$
$a_{\mathbf{T}} = 14\sqrt{5}/5$
$a_{\mathbf{N}} = 8\sqrt{5}/5$

39. $\mathbf{T}(0) = (\sqrt{5}/5)(\mathbf{i} - 2\mathbf{j})$
$\mathbf{N}(0) = (\sqrt{5}/5)(2\mathbf{i} + \mathbf{j})$
$a_{\mathbf{T}} = -7\sqrt{5}/5$
$a_{\mathbf{N}} = 6\sqrt{5}/5$

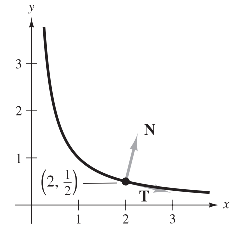
41. $\mathbf{T}(\pi/2) = (\sqrt{2}/2)(-\mathbf{i} + \mathbf{j})$
$\mathbf{N}(\pi/2) = (-\sqrt{2}/2)(\mathbf{i} + \mathbf{j})$
$a_{\mathbf{T}} = \sqrt{2}e^{\pi/2}$
$a_{\mathbf{N}} = \sqrt{2}e^{\pi/2}$

43. $\mathbf{T}(t_0) = (\cos \omega t_0)\mathbf{i} + (\sin \omega t_0)\mathbf{j}$
$\mathbf{N}(t_0) = (-\sin \omega t_0)\mathbf{i} + (\cos \omega t_0)\mathbf{j}$
$a_{\mathbf{T}} = \omega^2$
$a_{\mathbf{N}} = \omega^3 t_0$

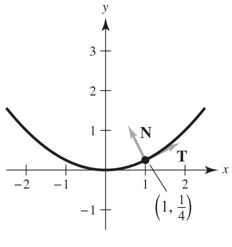
45. $\mathbf{T}(t) = -\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}$
$\mathbf{N}(t) = -\cos(\omega t)\mathbf{i} - \sin(\omega t)\mathbf{j}$
$a_{\mathbf{T}} = 0$
$a_{\mathbf{N}} = a\omega^2$

47. $\|\mathbf{v}(t)\| = a\omega$; The speed is constant because $a_{\mathbf{T}} = 0$.

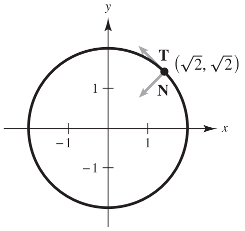
49. $\mathbf{r}(2) = 2\mathbf{i} + \frac{1}{2}\mathbf{j}$
$\mathbf{T}(2) = (\sqrt{17}/17)(4\mathbf{i} - \mathbf{j})$
$\mathbf{N}(2) = (\sqrt{17}/17)(\mathbf{i} + 4\mathbf{j})$



51. $\mathbf{r}(1/4) = \mathbf{i} + (1/4)\mathbf{j}$
$\mathbf{T}(1/4) = (\sqrt{5}/5)(2\mathbf{i} + \mathbf{j})$
$\mathbf{N}(1/4) = (2\sqrt{5}/5)[-(1/2)\mathbf{i} + \mathbf{j}]$



53. $\mathbf{r}(\pi/4) = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$
$\mathbf{T}(\pi/4) = (\sqrt{2}/2)(-\mathbf{i} + \mathbf{j})$
$\mathbf{N}(\pi/4) = (\sqrt{2}/2)(-\mathbf{i} - \mathbf{j})$



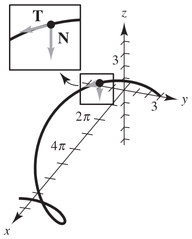
55. $\mathbf{T}(1) = (\sqrt{14}/14)(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$
$\mathbf{N}(1)$ is undefined.
$a_{\mathbf{T}}$ is undefined.
$a_{\mathbf{N}}$ is undefined.

57. $\mathbf{T}(\pi/3) = (\sqrt{5}/5)[-(\sqrt{3}/2)\mathbf{i} + (1/2)\mathbf{j} + 2\mathbf{k}]$
$\mathbf{N}(\pi/3) = -(1/2)\mathbf{i} - (\sqrt{3}/2)\mathbf{j}$
$a_{\mathbf{T}} = 0$
$a_{\mathbf{N}} = 1$

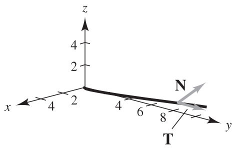
59. $\mathbf{T}(1) = (\sqrt{6}/6)(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
$\mathbf{N}(1) = (\sqrt{30}/30)(-5\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
$a_{\mathbf{T}} = 5\sqrt{6}/6$
$a_{\mathbf{N}} = \sqrt{30}/6$

61. $\mathbf{T}(0) = (\sqrt{3}/3)(\mathbf{i} + \mathbf{j} + \mathbf{k})$
$\mathbf{N}(0) = (\sqrt{2}/2)(\mathbf{i} - \mathbf{j})$
$a_{\mathbf{T}} = \sqrt{3}$
$a_{\mathbf{N}} = \sqrt{2}$

63. $\mathbf{T}(\pi/2) = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$
$\mathbf{N}(\pi/2) = -\mathbf{k}$
$a_{\mathbf{T}} = 0$
$a_{\mathbf{N}} = 3$



65. $\mathbf{T}(2) = (\sqrt{149}/149)(\mathbf{i} + 12\mathbf{j} + 2\mathbf{k})$
$\mathbf{N}(2) = (\sqrt{5513}/5513)(-74\mathbf{i} + 6\mathbf{j} + \mathbf{k})$
$a_{\mathbf{T}} = 74\sqrt{149}/149$
$a_{\mathbf{N}} = \sqrt{5513}/149$



67. Let C be a smooth curve represented by $\mathbf{r}$ on an open interval I. The unit tangent vector $\mathbf{T}(t)$ at t is defined as

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},\ \mathbf{r}'(t) \neq 0.$$

The principal unit normal vector $\mathbf{N}(t)$ at t is defined as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},\ \mathbf{T}'(t) \neq 0.$$

The tangential and normal components of acceleration are defined as $\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$.

69. (a) The particle's motion is in a straight line.
(b) The particle's speed is constant.

71. (a) $t = \frac{1}{2}$: $a_{\mathbf{T}} = \sqrt{2}\pi^2/2$, $a_{\mathbf{N}} = \sqrt{2}\pi^2/2$
$t = 1$: $a_{\mathbf{T}} = 0$, $a_{\mathbf{N}} = \pi^2$
$t = \frac{3}{2}$: $a_{\mathbf{T}} = -\sqrt{2}\pi^2/2$, $a_{\mathbf{N}} = \sqrt{2}\pi^2/2$
(b) $t = \frac{1}{2}$: Increasing because $a_{\mathbf{T}} > 0$.
$t = 1$: Maximum because $a_{\mathbf{T}} = 0$.
$t = \frac{3}{2}$: Decreasing because $a_{\mathbf{T}} < 0$.

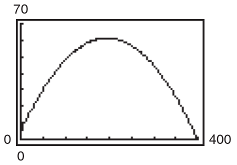
73. $\mathbf{T}(\pi/2) = (\sqrt{17}/17)(-4\mathbf{i} + \mathbf{k})$
$\mathbf{N}(\pi/2) = -\mathbf{j}$
$\mathbf{B}(\pi/2) = (\sqrt{17}/17)(\mathbf{i} + 4\mathbf{k})$

75. $\mathbf{T}(\pi/4) = (\sqrt{2}/2)(\mathbf{j} - \mathbf{k})$
$\mathbf{N}(\pi/4) = -(\sqrt{2}/2)(\mathbf{j} + \mathbf{k})$
$\mathbf{B}(\pi/4) = -\mathbf{i}$

77. $\mathbf{T}(\pi/3) = \left(\sqrt{5}/5\right)\left(\mathbf{i} - \sqrt{3}\mathbf{j} + \mathbf{k}\right)$
$\mathbf{N}(\pi/3) = -\frac{1}{2}\left(\sqrt{3}\mathbf{i} + \mathbf{j}\right)$
$\mathbf{B}(\pi/3) = \left(\sqrt{5}/10\right)\left(\mathbf{i} - \sqrt{3}\mathbf{j} - 4\mathbf{k}\right)$

79. $a_{\mathbf{T}} = \dfrac{-32(v_0 \sin\theta - 32t)}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$
$a_{\mathbf{N}} = \dfrac{32v_0\cos\theta}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$
At maximum height, $a_{\mathbf{T}} = 0$ and $a_{\mathbf{N}} = 32$.

81. (a) $\mathbf{r}(t) = 60\sqrt{3}t\mathbf{i} + (5 + 60t - 16t^2)\mathbf{j}$
(b)



Maximum height $\approx$ 61.245 ft
Range $\approx$ 398.186 ft

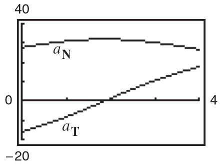
(c) $\mathbf{v}(t) = 60\sqrt{3}\mathbf{i} + (60 - 32t)\mathbf{j}$
$\|\mathbf{v}(t)\| = 8\sqrt{16t^2 - 60t + 225}$
$\mathbf{a}(t) = -32\mathbf{j}$

(d)

t	0.5	1.0	1.5
Speed	112.85	107.63	104.61

t	2.0	2.5	3.0
Speed	104	105.83	109.98

(e)



The speed is decreasing when $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ have opposite signs.

83. (a) $4\sqrt{625\pi^2 + 1} \approx 314$ mi/h
(b) $a_{\mathbf{T}} = 0, \quad a_{\mathbf{N}} = 1000\pi^2$
$a_{\mathbf{T}} = 0$ because the speed is constant.

85. (a) The centripetal component is quadrupled.
(b) The centripetal component is halved.

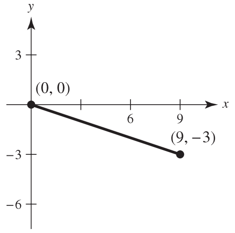
87. 4.82 mi/sec **89.** 4.67 mi/sec

91. False; centripetal acceleration may occur with constant speed.

93. (a) Proof (b) Proof **95–97.** Proofs

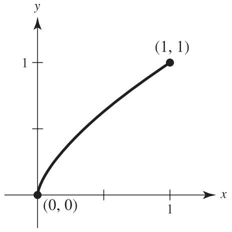
Section 12.5 (page 877)

1.



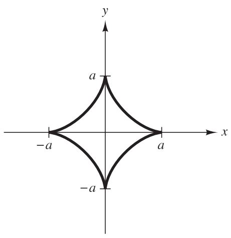
$3\sqrt{10}$

3.



$\left(13\sqrt{13} - 8\right)/27$

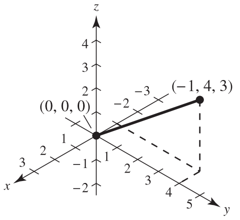
5.



$6a$

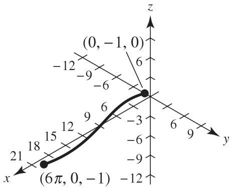
7. (a) $\mathbf{r}(t) = \left(50t\sqrt{2}\right)\mathbf{i} + \left(3 + 50t\sqrt{2} - 16t^2\right)\mathbf{j}$
(b) $\frac{649}{8} \approx 81$ ft (c) 315.5 ft (d) 362.9 ft

9.



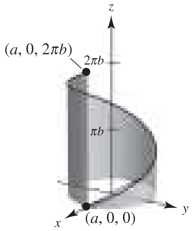
$\sqrt{26}$

11.



$3\sqrt{17}\pi/2$

13.



$2\pi\sqrt{a^2 + b^2}$

15. 8.37

17. (a) $2\sqrt{21} \approx 9.165$ (b) 9.529
(c) Increase the number of line segments. (d) 9.571

19. (a) $s = \sqrt{5}t$ (b) $\mathbf{r}(s) = 2\cos\dfrac{s}{\sqrt{5}}\mathbf{i} + 2\sin\dfrac{s}{\sqrt{5}}\mathbf{j} + \dfrac{s}{\sqrt{5}}\mathbf{k}$
(c) $s = \sqrt{5}$: (1.081, 1.683, 1.000)
$s = 4$: (−0.433, 1.953, 1.789)
(d) Proof

21. 0 **23.** $\frac{2}{5}$ **25.** 0 **27.** $\sqrt{2}/2$ **29.** 1

31. $\frac{1}{4}$ **33.** $1/a$ **35.** $\sqrt{2}/\left(4a\sqrt{1 - \cos\omega t}\right)$

37. $\sqrt{5}/(1 + 5t^2)^{3/2}$ **39.** $\frac{3}{25}$ **41.** $\frac{12}{125}$ **43.** $7\sqrt{26}/676$

45. $K = 0$, $1/K$ is undefined.

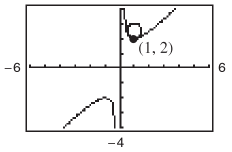
47. $K = 4/17^{3/2}, \quad 1/K = 17^{3/2}/4$ **49.** $K = 4, \quad 1/K = 1/4$

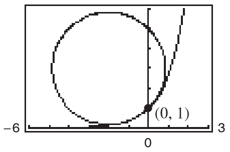
51. $K = 1/a, \quad 1/K = a$

53. $K = 12/145^{3/2}, \quad 1/K = 145^{3/2}/12$

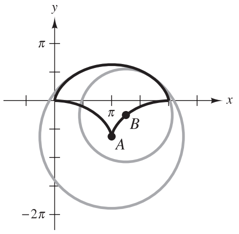
55. (a) $(x - \pi/2)^2 + y^2 = 1$
(b) Because the curvature is not as great, the radius of the curvature is greater.

57. $(x - 1)^2 + \left(y - \frac{5}{2}\right)^2 = \left(\frac{1}{2}\right)^2$ **59.** $(x + 2)^2 + (y - 3)^2 = 8$





61.



63. (a) $(1, 3)$ (b) 0

65. (a) $K \to \infty$ as $x \to 0$ (No maximum) (b) 0

67. (a) $\left(1/\sqrt{2}, -\ln 2/2\right)$ (b) 0

69. (a) $(\pm \operatorname{arcsinh}(1), 1)$ (b) 0

71. $(0, 1)$ **73.** $(\pi/2 + K\pi, 0)$

75. (a) $s = \displaystyle\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt$

(b) Plane: $K = \left\|\dfrac{d\mathbf{T}}{ds}\right\| = \|\mathbf{T}'(s)\|$

Space: $K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$

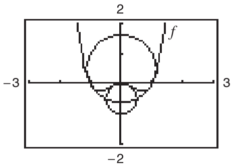
77. $K = |y''|$; Yes, for example, $y = x^4$ has a curvature of 0 at its relative minimum $(0, 0)$. The curvature is positive at any other point on the curve.

79. Proof

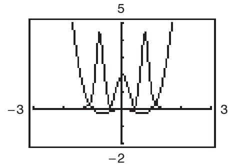
81. (a) $K = \dfrac{2|6x^2 - 1|}{(16x^6 - 16x^4 + 4x^2 + 1)^{3/2}}$

(b) $x = 0$: $x^2 + \left(y + \dfrac{1}{2}\right)^2 = \dfrac{1}{4}$

$x = 1$: $x^2 + \left(y - \dfrac{1}{2}\right)^2 = \dfrac{5}{4}$



(c)



The curvature tends to be greatest near the extrema of the function and decreases as $x \to \pm\infty$. However, f and K do not have the same critical numbers.

Critical numbers of f: $x = 0, \pm\sqrt{2}/2 \approx \pm 0.7071$

Critical numbers of K: $x = 0, \pm 0.7647, \pm 0.4082$

83. (a) 12.25 units (b) $\frac{1}{2}$ **85–87.** Proofs

89. (a) 0 (b) 0 **91.** $\frac{1}{4}$ **93.** Proof

95. $K = [1/(4a)]|\csc(\theta/2)|$ **97.** 3327.5 lb

Minimum: $K = 1/(4a)$

There is no maximum.

99. Proof

101. False. See Exploration on page 869. **103.** True

105–111. Proofs

Review Exercises for Chapter 12 (page 881)

1. (a) All reals except $(\pi/2) + n\pi$, n is an integer

(b) Continuous except at $t = (\pi/2) + n\pi$, n is an integer

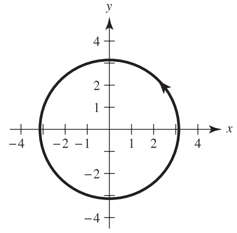
3. (a) $(0, \infty)$ (b) Continuous for all $t > 0$

5. (a) $\mathbf{i} - \sqrt{2}\mathbf{k}$ (b) $-3\mathbf{i} + 4\mathbf{j}$

(c) $(2c - 1)\mathbf{i} + (c - 1)^2\mathbf{j} - \sqrt{c + 1}\,\mathbf{k}$

(d) $2\Delta t\,\mathbf{i} + \Delta t(\Delta t + 2)\mathbf{j} - \left(\sqrt{\Delta t + 3} - \sqrt{3}\right)\mathbf{k}$

7.



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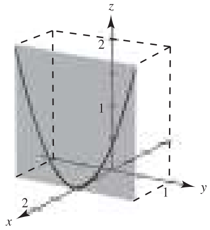

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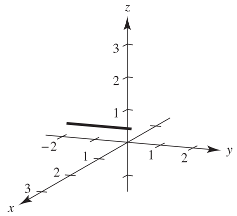

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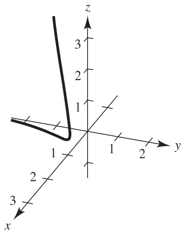


15. $\mathbf{r}_1(t) = 3t\mathbf{i} + 4t\mathbf{j}, \quad 0 \le t \le 1$

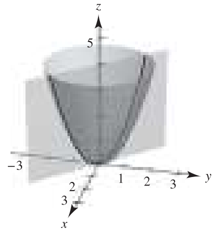
$\mathbf{r}_2(t) = 3\mathbf{i} + (4 - t)\mathbf{j}, \quad 0 \le t \le 4$

$\mathbf{r}_3(t) = (3 - t)\mathbf{i}, \quad 0 \le t \le 3$

17. $\mathbf{r}(t) = \langle -2 + 7t, -3 + 4t, 8 - 10t\rangle$

(Answer is not unique.)

19.



$x = t, y = -t, z = 2t^2$

21. $4\mathbf{i} + \mathbf{k}$

23. (a) $3\mathbf{i} + \mathbf{j}$ (b) $\mathbf{0}$ (c) $4t + 3t^2$

(d) $-5\mathbf{i} + (2t - 2)\mathbf{j} + 2t^2\mathbf{k}$

(e) $(10t - 1)/\sqrt{10t^2 - 2t + 1}$

(f) $\left(\frac{8}{3}t^3 - 2t^2\right)\mathbf{i} - 8t^3\mathbf{j} + (9t^2 - 2t + 1)\mathbf{k}$

25. $x(t)$ and $y(t)$ are increasing functions at $t = t_0$, and $z(t)$ is a decreasing function at $t = t_0$.

27. $\sin t\,\mathbf{i} + (t \sin t + \cos t)\mathbf{j} + \mathbf{C}$

29. $\frac{1}{2}\left(t\sqrt{1 + t^2} + \ln\left|t + \sqrt{1 + t^2}\right|\right) + \mathbf{C}$

31. $\frac{32}{3}\mathbf{j}$ **33.** $2(e - 1)\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$

35. $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (e^t + 2)\mathbf{j} - (e^{-t} + 4)\mathbf{k}$

37. $\mathbf{v}(t) = 4\mathbf{i} + 3t^2\mathbf{j} - \mathbf{k}$

$\|\mathbf{v}(t)\| = \sqrt{17 + 9t^4}$

$\mathbf{a}(t) = 6t\mathbf{j}$

39. $\mathbf{v}(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t, 3\rangle$

$\|\mathbf{v}(t)\| = 3\sqrt{\sin^2 t \cos^2 t + 1}$

$\mathbf{a}(t) = \langle 3\cos t(2\sin^2 t - \cos^2 t), 3\sin t(2\cos^2 t - \sin^2 t), 0\rangle$

41. $x(t) = t, y(t) = 16 + 8t, z(t) = 2 + \frac{1}{2}t$

$\mathbf{r}(4.1) \approx \langle 0.1, 16.8, 2.05\rangle$

43. About 191.0 ft **45.** About 38.1 m/sec

47. $\mathbf{v} = -\mathbf{i} + 3\mathbf{j}$
$\|\mathbf{v}\| = \sqrt{10}$
$\mathbf{a} = \mathbf{0}$
$\mathbf{a} \cdot \mathbf{T} = 0$
$\mathbf{a} \cdot \mathbf{N}$ does not exist.

49. $\mathbf{v} = \mathbf{i} + \left[1/\left(2\sqrt{t}\right)\right]\mathbf{j}$
$\|\mathbf{v}\| = \sqrt{4t+1}/\left(2\sqrt{t}\right)$
$\mathbf{a} = \left[-1/\left(4t\sqrt{t}\right)\right]\mathbf{j}$
$\mathbf{a} \cdot \mathbf{T} = -1/\left(4t\sqrt{t}\sqrt{4t+1}\right)$
$\mathbf{a} \cdot \mathbf{N} = 1/\left(2t\sqrt{4t+1}\right)$

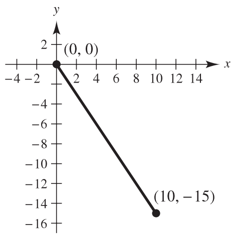
51. $\mathbf{v} = e^t\mathbf{i} - e^{-t}\mathbf{j}$
$\|\mathbf{v}\| = \sqrt{e^{2t} + e^{-2t}}$
$\mathbf{a} = e^t\mathbf{i} + e^{-t}\mathbf{j}$
$\mathbf{a} \cdot \mathbf{T} = \dfrac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$
$\mathbf{a} \cdot \mathbf{N} = \dfrac{2}{\sqrt{e^{2t} + e^{-2t}}}$

53. $\mathbf{v} = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$
$\|\mathbf{v}\| = \sqrt{1 + 5t^2}$
$\mathbf{a} = 2\mathbf{j} + \mathbf{k}$
$\mathbf{a} \cdot \mathbf{T} = \dfrac{5t}{\sqrt{1 + 5t^2}}$
$\mathbf{a} \cdot \mathbf{N} = \dfrac{\sqrt{5}}{\sqrt{1 + 5t^2}}$

55. $x = -\sqrt{3}t + 1$
$y = t + \sqrt{3}$
$z = t + (\pi/3)$

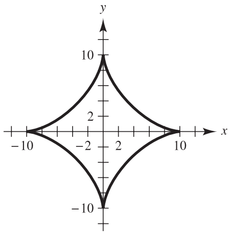
57. 4.58 mi/sec

59.



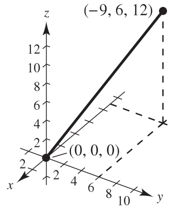
$5\sqrt{13}$

61.



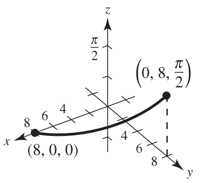
60

63.



$3\sqrt{29}$

65.



$\sqrt{65}\pi/2$

67. 0 **69.** $\left(2\sqrt{5}\right)/(4 + 5t^2)^{3/2}$ **71.** $\sqrt{2}/3$

73. $K = \sqrt{17}/289;\ r = 17\sqrt{17}$ **75.** $K = \sqrt{2}/4;\ r = 2\sqrt{2}$

77. The curvature changes abruptly from zero to a nonzero constant at the points B and C.

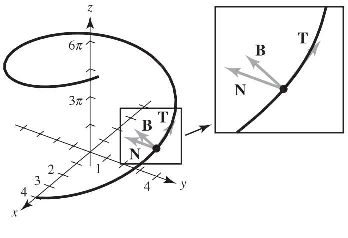
P.S. Problem Solving (page 883)

1. (a) a (b) πa (c) $K = \pi a$

3. Initial speed: 447.21 ft/sec; $\theta \approx 63.43°$

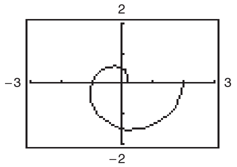
5–7. Proofs

9. Unit tangent: $\left\langle -\frac{4}{5}, 0, \frac{3}{5}\right\rangle$
Unit normal: $\langle 0, -1, 0\rangle$
Binormal: $\left\langle \frac{3}{5}, 0, \frac{4}{5}\right\rangle$



11. (a) Proof (b) Proof

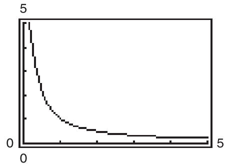
13. (a)



(b) 6.766

(c) $K = [\pi(\pi^2t^2 + 2)]/(\pi^2t^2 + 1)^{3/2}$
$K(0) = 2\pi$
$K(1) = [\pi(\pi^2 + 2)]/(\pi^2 + 1)^{3/2} \approx 1.04$
$K(2) \approx 0.51$

(d)



(e) $\lim_{t\to\infty} K = 0$

(f) As $t \to \infty$, the graph spirals outward and the curvature decreases.

Chapter 13

Section 13.1 (page 894)

1. Not a function because for some values of x and y (for example $x = y = 0$), there are two z-values.

3. z is a function of x and y. **5.** z is not a function of x and y.

7. (a) 6 (b) -4 (c) 150 (d) $5y$ (e) $2x$ (f) $5t$

9. (a) 5 (b) $3e^2$ (c) $2/e$ (d) $5e^y$ (e) xe^2 (f) te^t

11. (a) $\frac{2}{3}$ (b) 0 (c) $-\frac{3}{2}$ (d) $-\frac{10}{3}$

13. (a) $\sqrt{2}$ (b) $3 \sin 1$ (c) $-3\sqrt{3}/2$ (d) 4

15. (a) -4 (b) -6 (c) $-\frac{25}{4}$ (d) $\frac{9}{4}$

17. (a) $2,\ \Delta x \neq 0$ (b) $2y + \Delta y,\ \Delta y \neq 0$

19. Domain: $\{(x, y)\colon\ x$ is any real number, y is any real number$\}$
Range: $z \geq 0$

21. Domain: $\{(x, y)\colon\ y \geq 0\}$
Range: all real numbers

23. Domain: $\{(x, y)\colon\ x \neq 0, y \neq 0\}$
Range: all real numbers

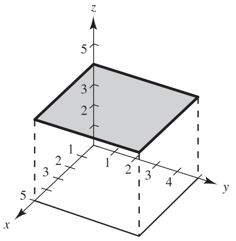
25. Domain: $\{(x, y)\colon\ x^2 + y^2 \leq 4\}$
Range: $0 \leq z \leq 2$

27. Domain: $\{(x, y)\colon\ -1 \leq x + y \leq 1\}$
Range: $0 \leq z \leq \pi$

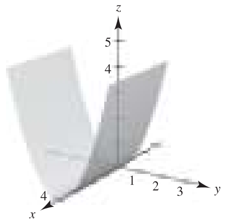
29. Domain: $\{(x, y)\colon\ y < -x + 4\}$
Range: all real numbers

31. (a) $(20, 0, 0)$ (b) $(-15, 10, 20)$
(c) $(20, 15, 25)$ (d) $(20, 20, 0)$

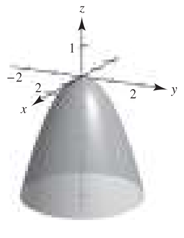
33.



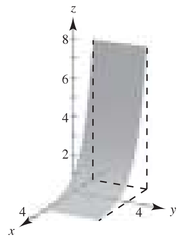
35.



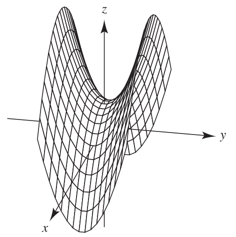
37.



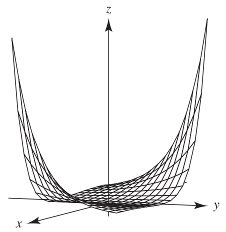
39.



41.

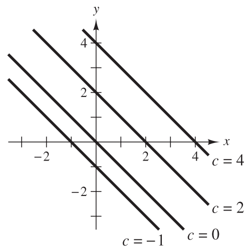


43.

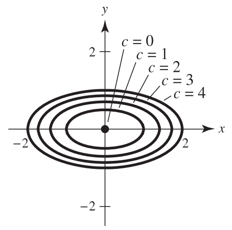


45. c **46.** d **47.** b **48.** a

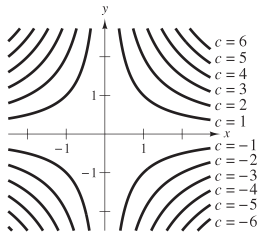
49. Lines: $x + y = c$


y
4
2
−2
2
4
x
−2
c = 4
c = 2
c = −1
c = 0

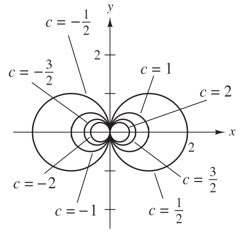
51. Ellipses: $x^2 + 4y^2 = c$ (except $x^2 + 4y^2 = 0$ is the point $(0, 0)$.)


y
2
c = 0
c = 1
c = 2
c = 3
c = 4
−2
2
x
−2

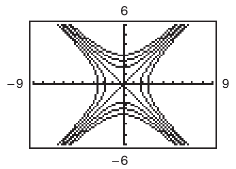
53. Hyperbolas: $xy = c$


y
c = 6
c = 5
c = 4
c = 3
c = 2
c = 1
1
−1
1
x
−1
c = −1
c = −2
c = −3
c = −4
c = −5
c = −6

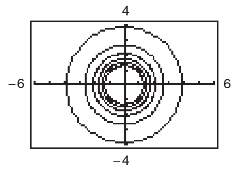
55. Circles passing through $(0, 0)$ Centered at $(1/(2c), 0)$


$c = -\frac{1}{2}$
y
2
$c = -\frac{3}{2}$
c = 1
c = 2
x
2
c = −2
$c = \frac{3}{2}$
c = −1
$c = \frac{1}{2}$

57.



59.



61. The graph of a function of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of f. The graph can be interpreted as a surface in space. Level curves are the scalar fields $f(x, y) = c$, where c is a constant.

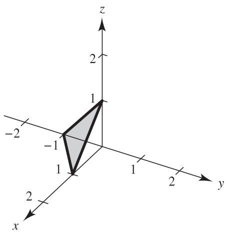
63. $f(x, y) = x/y$; the level curves are the lines $y = (1/c)x$.

65. The surface may be shaped like a saddle. For example, let $f(x, y) = xy$. The graph is not unique; any vertical translation will produce the same level curves.

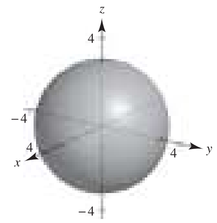
67.

	Inflation Rate		
Tax Rate	0	0.03	0.05
0	$1790.85	$1332.56	$1099.43
0.28	$1526.43	$1135.80	$937.09
0.35	$1466.07	$1090.90	$900.04

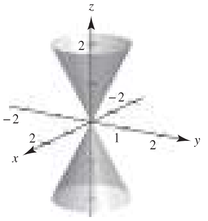
69.


z
2
1
−2
−1
1
1
2
y
2
x

71.

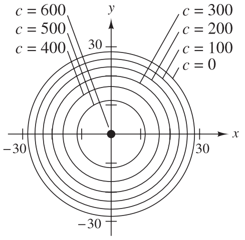


73.



75. (a) 243 board-ft
(b) 507 board-ft

77.


c = 600
c = 500
c = 400
y
30
c = 300
c = 200
c = 100
c = 0
−30
30
x
−30

79. Proof

81. $C = 1.20xy + 1.50(xz + yz)$

83. (a) $k = \frac{520}{3}$
(b) $P = 520T/(3V)$
The level curves are lines.

85. (a) C (b) A (c) B

87. (a) No; the level curves are uneven and sporadically spaced.
(b) Use more colors.

89. False: let $f(x, y) = 4$. **91.** True

Section 13.2 (page 904)

1–3. Proofs **5.** 1 **7.** 12 **9.** 9, continuous

11. e^2, continuous **13.** 0, continuous for $y \neq 0$

15. $\frac{1}{2}$, continuous except at $(0, 0)$ **17.** 0, continuous

19. 0, continuous for $xy \neq 1$, $|xy| \leq 1$

21. $2\sqrt{2}$, continuous for $x + y + z \geq 0$ **23.** 0

25. Limit does not exist. **27.** 4 **29.** Limit does not exist.

31. Limit does not exist. **33.** 0

35. Limit does not exist. **37.** Continuous, 1

39.

(x, y)	$(1, 0)$	$(0.5, 0)$	$(0.1, 0)$	$(0.01, 0)$	$(0.001, 0)$
$f(x, y)$	0	0	0	0	0

$y = 0$: 0

(x, y)	$(1, 1)$	$(0.5, 0.5)$	$(0.1, 0.1)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

(x, y)	$(0.01, 0.01)$	$(0.001, 0.001)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$

$y = x$: $\frac{1}{2}$
Limit does not exist.
Continuous except at $(0, 0)$

41.

(x, y)	$(1, 1)$	$(0.25, 0.5)$	$(0.01, 0.1)$
$f(x, y)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

(x, y)	$(0.0001, 0.01)$	$(0.000001, 0.001)$
$f(x, y)$	$-\frac{1}{2}$	$-\frac{1}{2}$

$x = y^2$: $-\frac{1}{2}$

(x, y)	$(-1, 1)$	$(-0.25, 0.5)$	$(-0.01, 0.1)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

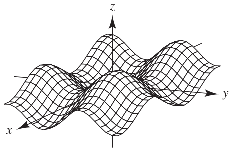
(x, y)	$(-0.0001, 0.01)$	$(-0.000001, 0.001)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$

$x = -y^2$: $\frac{1}{2}$
Limit does not exist.
Continuous except at $(0, 0)$

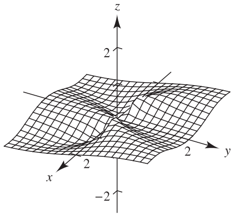
43. f is continuous. g is continuous except at $(0, 0)$. g has a removable discontinuity at $(0, 0)$.

45. f is continuous. g is continuous except at $(0, 0)$. g has a removable discontinuity at $(0, 0)$.

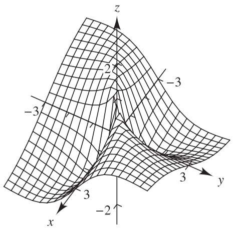
47. 0



49. Limit does not exist.



51. Limit does not exist.



53. 0 **55.** 0 **57.** 1 **59.** 1 **61.** 0

63. Continuous except at $(0, 0, 0)$ **65.** Continuous

67. Continuous **69.** Continuous

71. Continuous for $y \neq 2x/3$ **73.** (a) $2x$ (b) -4

75. (a) $1/y$ (b) $-x/y^2$ **77.** (a) $3 + y$ (b) $x - 2$

79. True **81.** False: let $f(x, y) = \begin{cases} \ln(x^2 + y^2), & x \neq 0, y \neq 0 \\ 0, & x = 0, y = 0 \end{cases}$.

83. (a) $(1 + a^2)/a, \quad a \neq 0$ (b) Limit does not exist.
(c) No, the limit does not exist. Different paths result in different limits.

85. 0 **87.** $\pi/2$ **89.** Proof

91. See "Definition of the Limit of a Function of Two Variables," on page 899; show that the value of $\displaystyle\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ is not the same for two different paths to (x_0, y_0).

93. (a) True. To find the first limit, you substitute $(2, 3)$ for (x, y). To find the second limit, you first substitute 3 for y to find a function of x. Then you substitute 2 for x.
(b) False. The convergence of one path does not imply the convergence of all paths.
(c) False. Let $f(x, y) = 4\left(\dfrac{(x - 2)^2 - (y - 3)^2}{(x - 2)^2 + (y - 3)^2}\right)^2$.
(d) True. When you multiply 0 by any real number, you always get 0.

Section 13.3 (page 914)

1. $f_x(4, 1) < 0$ **3.** $f_y(4, 1) > 0$

5. No. Because you are finding the partial derivative with respect to y, you consider x to be constant. So, the denominator is considered a constant and does not contain any variables.

7. Yes. Because you are finding the partial derivative with respect to x, you consider y to be constant. So, both the numerator and denominator contain variables.

9. $f_x(x, y) = 2$
$f_y(x, y) = -5$

11. $f_x(x, y) = 2xy^3$
$f_y(x, y) = 3x^2y^2$

13. $\partial z/\partial x = \sqrt{y}$
$\partial z/\partial y = x/\left(2\sqrt{y}\right)$

15. $\partial z/\partial x = 2x - 4y$
$\partial z/\partial y = -4x + 6y$

17. $\partial z/\partial x = ye^{xy}$
$\partial z/\partial y = xe^{xy}$

19. $\partial z/\partial x = 2xe^{2y}$
$\partial z/\partial y = 2x^2e^{2y}$

21. $\partial z/\partial x = 1/x$
$\partial z/\partial y = -1/y$

23. $\partial z/\partial x = 2x/(x^2 + y^2)$
$\partial z/\partial y = 2y/(x^2 + y^2)$

25. $\partial z/\partial x = (x^3 - 3y^3)/(x^2y)$
$\partial z/\partial y = (-x^3 + 12y^3)/(2xy^2)$

27. $h_x(x, y) = -2xe^{-(x^2+y^2)}$
$h_y(x, y) = -2ye^{-(x^2+y^2)}$

29. $f_x(x, y) = x/\sqrt{x^2 + y^2}$
$f_y(x, y) = y/\sqrt{x^2 + y^2}$

31. $\partial z/\partial x = -y \sin xy$
$\partial z/\partial y = -x \sin xy$

33. $\partial z/\partial x = 2 \sec^2(2x - y)$
$\partial z/\partial y = -\sec^2(2x - y)$

35. $\partial z/\partial x = ye^y \cos xy$
$\partial z/\partial y = e^y(x \cos xy + \sin xy)$

37. $\partial z/\partial x = 2\cosh(2x + 3y)$
$\partial z/\partial y = 3\cosh(2x + 3y)$

39. $f_x(x, y) = 1 - x^2$
$f_y(x, y) = y^2 - 1$

41. $f_x(x, y) = 3$
$f_y(x, y) = 2$

43. $f_x(x, y) = 1/\left(2\sqrt{x + y}\right)$
$f_y(x, y) = 1/\left(2\sqrt{x + y}\right)$

45. $\partial z/\partial x = -1$
$\partial z/\partial y = 0$

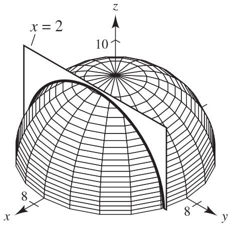
47. $\partial z/\partial x = -1$
$\partial z/\partial y = \frac{1}{2}$

49. $\partial z/\partial x = \frac{1}{4}$
$\partial z/\partial y = \frac{1}{4}$

51. $\partial z/\partial x = -\frac{1}{4}$
$\partial z/\partial y = \frac{1}{4}$

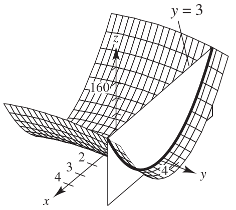
53. $g_x(1, 1) = -2$
$g_y(1, 1) = -2$

55.



$-\frac{1}{2}$

57.



18

59. $H_x(x, y, z) = \cos(x + 2y + 3z)$
$H_y(x, y, z) = 2\cos(x + 2y + 3z)$
$H_z(x, y, z) = 3\cos(x + 2y + 3z)$

61. $\dfrac{\partial w}{\partial x} = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}$
$\dfrac{\partial w}{\partial y} = \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}$
$\dfrac{\partial w}{\partial z} = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$

63. $F_x(x, y, z) = \dfrac{x}{x^2 + y^2 + z^2}$
$F_y(x, y, z) = \dfrac{y}{x^2 + y^2 + z^2}$
$F_z(x, y, z) = \dfrac{z}{x^2 + y^2 + z^2}$

65. $f_x = 3; f_y = 1; f_z = 2$

67. $f_x = 1; f_y = 1; f_z = 1$

69. $f_x = 0; f_y = 0; f_z = 1$

71. $\dfrac{\partial^2 z}{\partial x^2} = 0$
$\dfrac{\partial^2 z}{\partial y^2} = 6x$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = 6y$

73. $\dfrac{\partial^2 z}{\partial x^2} = 2$
$\dfrac{\partial^2 z}{\partial y^2} = 6$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = -2$

75. $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{y^2}{(x^2 + y^2)^{3/2}}$
$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{x^2}{(x^2 + y^2)^{3/2}}$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{-xy}{(x^2 + y^2)^{3/2}}$

77. $\dfrac{\partial^2 z}{\partial x^2} = e^x \tan y$
$\dfrac{\partial^2 z}{\partial y^2} = 2e^x \sec^2 y \tan y$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = e^x \sec^2 y$

79. $\dfrac{\partial^2 z}{\partial x^2} = -y^2 \cos xy$
$\dfrac{\partial^2 z}{\partial y^2} = -x^2 \cos xy$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = -xy \cos xy - \sin xy$

81. $x = 2, y = -2$ **83.** $x = -6, y = 4$

85. $x = 1, y = 1$ **87.** $x = 0, y = 0$

89. $\partial z/\partial x = \sec y$
$\partial z/\partial y = x \sec y \tan y$
$\partial^2 z/\partial x^2 = 0$
$\partial^2 z/\partial y^2 = x \sec y(\sec^2 y + \tan^2 y)$
$\partial^2 z/\partial y \partial x = \partial^2 z/\partial x \partial y = \sec y \tan y$
No values of x and y exist such that $f_x(x, y) = f_y(x, y) = 0$.

91. $\partial z/\partial x = (y^2 - x^2)/[x(x^2 + y^2)]$
$\partial z/\partial y = -2y/(x^2 + y^2)$
$\partial^2 z/\partial x^2 = (x^4 - 4x^2y^2 - y^4)/[x^2(x^2 + y^2)^2]$
$\partial^2 z/\partial y^2 = 2(y^2 - x^2)/(x^2 + y^2)^2$
$\partial^2 z/\partial y \partial x = \partial^2 z/\partial x \partial y = 4xy/(x^2 + y^2)^2$
No values of x and y exist such that $f_x(x, y) = f_y(x, y) = 0$.

93. $f_{xyy}(x, y, z) = f_{yxy}(x, y, z) = f_{yyx}(x, y, z) = 0$

95. $f_{xyy}(x, y, z) = f_{yxy}(x, y, z) = f_{yyx}(x, y, z) = z^2e^{-x} \sin yz$

97. $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0 + 0 = 0$

99. $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = e^x \sin y - e^x \sin y = 0$

101. $\partial^2 z/\partial t^2 = -c^2 \sin(x - ct) = c^2(\partial^2 z/\partial x^2)$

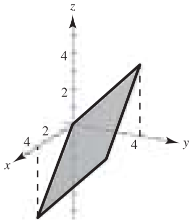
103. $\partial^2 z/\partial t^2 = -c^2/(x + ct)^2 = c^2(\partial^2 z/\partial x^2)$

105. $\partial z/\partial t = -e^{-t} \cos x/c = c^2(\partial^2 z/\partial x^2)$

107. Yes, $f(x, y) = \cos(3x - 2y)$. **109.** 0

111. If $z = f(x, y)$, then to find f_x you consider y constant and differentiate with respect to x. Similarly, to find f_y, you consider x constant and differentiate with respect to y.

113.



115. The mixed partial derivatives are equal. See Theorem 13.3.

117. (a) 72 (b) 72

119. $IQ_M = \dfrac{100}{C}, IQ_M(12, 10) = 10$

IQ increases at a rate of 10 points per year of mental age when the mental age is 12 and the chronological age is 10.

$IQ_C = -\dfrac{100M}{C^2}, IQ_C(12, 10) = -12$

IQ decreases at a rate of 12 points per year of chronological age when the mental age is 12 and the chronological age is 10.

121. An increase in either the charge for food and housing or the tuition will cause a decrease in the number of applicants.

123. $\partial T/\partial x = -2.4°/\text{m}, \partial T/\partial y = -9°/\text{m}$

125. $T = PV/(nR) \implies \partial T/\partial P = V/(nR)$
$P = nRT/V \implies \partial P/\partial V = -nRT/V^2$
$V = nRT/P \implies \partial V/\partial T = nR/P$
$\partial T/\partial P \cdot \partial P/\partial V \cdot \partial V/\partial T = -nRT/(VP) = -nRT/(nRT) = -1$

127. (a) $\partial z/\partial x = -0.92; \partial z/\partial y = 1.03$
(b) As the consumption of flavored milk (x) increases, the consumption of plain light and skim milks (z) decreases. As the consumption of plain reduced-fat milk (y) decreases, the consumption of plain light and skim milks also decreases.

129. False; Let $z = x + y + 1$. **131.** True

133. (a) $f_x(x, y) = \dfrac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

$f_y(x, y) = \dfrac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

(b) $f_x(0, 0) = 0,\ f_y(0, 0) = 0$

(c) $f_{xy}(0, 0) = -1,\ f_{yx}(0, 0) = 1$

(d) f_{xy} or f_{yx} or both are not continuous at $(0, 0)$.

135. (a) $f_x(0, 0) = 1, f_y(0, 0) = 1$

(b) $f_x(x, y)$ and $f_y(x, y)$ do not exist when $y = -x$.

Section 13.4 (page 923)

1. $dz = 4xy^3\,dx + 6x^2y^2\,dy$

3. $dz = 2(x\,dx + y\,dy)/(x^2 + y^2)^2$

5. $dz = (\cos y + y \sin x)\,dx - (x \sin y + \cos x)\,dy$

7. $dz = (e^x \sin y)\,dx + (e^x \cos y)\,dy$

9. $dw = 2z^3y \cos x\,dx + 2z^3 \sin x\,dy + 6z^2\,y \sin x\,dz$

11. (a) $f(2, 1) = 1, f(2.1, 1.05) = 1.05, \Delta z = 0.05$

(b) $dz = 0.05$

13. (a) $f(2, 1) = 11, f(2.1, 1.05) = 10.4875, \Delta z = -0.5125$

(b) $dz = -0.5$

15. (a) $f(2, 1) = e^2 \approx 7.3891, f(2.1, 1.05) = 1.05e^{2.1} \approx 8.5745,$ $\Delta z \approx 1.1854$

(b) $dz \approx 1.1084$

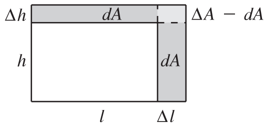
17. 0.44 **19.** -0.012

21. If $z = f(x, y)$ and Δx and Δy are increments of x and y, and x and y are independent variables, then the total differential of the dependent variable z is

$dz = (\partial z/\partial x)\,dx + (\partial z/\partial y)\,dy = f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y.$

23. The approximation of Δz by dz is called a linear approximation, where dz represents the change in height of a plane that is tangent to the surface at the point $P(x_0, y_0)$.

25. $dA = h\,dl + l\,dh$



$\Delta A - dA = dl\,dh$

27.

Δr	Δh	dV	ΔV	$\Delta V - dV$
0.1	0.1	8.3776	8.5462	0.1686
0.1	-0.1	5.0265	5.0255	-0.0010
0.001	0.002	0.1005	0.1006	0.0001
-0.0001	0.0002	-0.0034	-0.0034	0.0000

29. (a) $dz = -0.92\,dx + 1.03\,dy$

(b) $dz = \pm 0.4875;\ dz/z \approx 8.1\%$

31. 10% **33.** $dC = \pm 2.4418;\ dC/C = 19\%$

35. (a) $V = 18 \sin \theta$ ft^3; $\theta = \pi/2$

(b) 1.047 ft^3

37. 10% **39.** $L \approx 8.096 \times 10^{-4} \pm 6.6 \times 10^{-6}$ microhenrys

41. Answers will vary.
Example:
$\varepsilon_1 = \Delta x$
$\varepsilon_2 = 0$

43. Answers will vary.
Example:
$\varepsilon_1 = y\,\Delta x$
$\varepsilon_2 = 2x\,\Delta x + (\Delta x)^2$

45–47. Proofs

Section 13.5 (page 931)

1. $26t$ **3.** $e^t(\sin t + \cos t)$ **5.** (a) and (b) $-e^{-t}$

7. (a) and (b) $2e^{2t}$ **9.** (a) and (b) $3(2t^2 - 1)$

11. $-11\sqrt{29}/29 \approx -2.04$

13. $\dfrac{4}{(e^t + e^{-t})^2}$; 1

15. $\partial w/\partial s = 4s, 4$
$\partial w/\partial t = 4t, 0$

17. $\partial w/\partial s = 5 \cos(5s - t), 0$
$\partial w/\partial t = -\cos(5s - t), 0$

19. $\partial w/\partial r = 2r/\theta^2$
$\partial w/\partial \theta = -2r^2/\theta^3$

21. $\partial w/\partial r = 0$
$\partial w/\partial \theta = 1$

23. $\dfrac{\partial w}{\partial s} = t^2(3s^2 - t^2)$

$\dfrac{\partial w}{\partial t} = 2st(s^2 - 2t^2)$

25. $\dfrac{\partial w}{\partial s} = te^{s^2-t^2}(2s^2 + 1)$

$\dfrac{\partial w}{\partial t} = se^{s^2-t^2}(1 - 2t^2)$

27. $\dfrac{y - 2x + 1}{2y - x + 1}$

29. $-\dfrac{x^2 + y^2 + x}{x^2 + y^2 + y}$

31. $\dfrac{\partial z}{\partial x} = \dfrac{-x}{z}$

$\dfrac{\partial z}{\partial y} = \dfrac{-y}{z}$

33. $\dfrac{\partial z}{\partial x} = -\dfrac{x}{y + z}$

$\dfrac{\partial z}{\partial y} = -\dfrac{z}{y + z}$

35. $\dfrac{\partial z}{\partial x} = \dfrac{-\sec^2(x + y)}{\sec^2(y + z)}$

$\dfrac{\partial z}{\partial y} = -1 - \dfrac{\sec^2(x + y)}{\sec^2(y + z)}$

37. $\dfrac{\partial z}{\partial x} = -\dfrac{(ze^{xz} + y)}{xe^{xz}}$

$\dfrac{\partial z}{\partial y} = -e^{-xz}$

39. $\dfrac{\partial w}{\partial x} = -\dfrac{y + w}{x - z}$

$\dfrac{\partial w}{\partial y} = -\dfrac{x + z}{x - z}$

$\dfrac{\partial w}{\partial z} = \dfrac{w - y}{x - z}$

41. $\dfrac{\partial w}{\partial x} = \dfrac{y \sin xy}{z}$

$\dfrac{\partial w}{\partial y} = \dfrac{x \sin xy - z \cos yz}{z}$

$\dfrac{\partial w}{\partial z} = -\dfrac{y \cos yz + w}{z}$

43. (a) $f(tx, ty) = \dfrac{(tx)(ty)}{\sqrt{(tx)^2 + (ty)^2}} = t\left(\dfrac{xy}{\sqrt{x^2 + y^2}}\right) = tf(x, y);\ n = 1$

(b) $xf_x(x, y) + yf_y(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}} = 1f(x, y)$

45. (a) $f(tx, ty) = e^{tx/ty} = e^{x/y} = f(x, y);\ n = 0$

(b) $xf_x(x, y) + yf_y(x, y) = \dfrac{xe^{x/y}}{y} - \dfrac{xe^{x/y}}{y} = 0$

47. 47 **49.** $dw/dt = (\partial w/\partial x \cdot dx/dt) + (\partial w/\partial y \cdot dy/dt)$

51. $\dfrac{dy}{dx} = -\dfrac{f_x(x, y)}{f_y(x, y)}$

$\dfrac{\partial z}{\partial x} = -\dfrac{f_x(x, y, z)}{f_z(x, y, z)}$

$\dfrac{\partial z}{\partial y} = -\dfrac{f_y(x, y, z)}{f_z(x, y, z)}$

53. 4608π in.^{3}/min; 624π in.^{2}/min

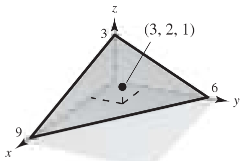
55. $\dfrac{dT}{dt} = \dfrac{1}{mR}\left[V\dfrac{dp}{dt} + p\dfrac{dV}{dt}\right]$ **57.** $28m$ cm^2/sec

59. Proof **61.** (a) Proof (b) Proof **63–65.** Proofs

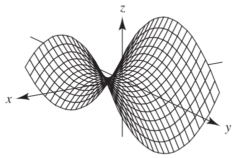
Section 13.6 (page 942)

1. 1 **3.** -1 **5.** $-e$ **7.** $-\frac{7}{25}$ **9.** $2\sqrt{3}/3$ **11.** $\frac{8}{3}$
13. $\sqrt{2}(x+y)$ **15.** $\left[(2+\sqrt{3})/2\right]\cos(2x+y)$ **17.** 6
19. $-8/\sqrt{5}$ **21.** $3\mathbf{i}+10\mathbf{j}$ **23.** $4\mathbf{i}-\mathbf{j}$ **25.** $6\mathbf{i}-10\mathbf{j}-8\mathbf{k}$
27. $3\sqrt{2}$ **29.** $2\sqrt{5}/5$ **31.** $2[(x+y)\mathbf{i}+x\mathbf{j}]$; $2\sqrt{2}$
33. $\tan y\mathbf{i}+x\sec^2 y\mathbf{j}$, $\sqrt{17}$ **35.** $e^{-x}(-y\mathbf{i}+\mathbf{j})$; $\sqrt{26}$
37. $\dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$, 1 **39.** $yz(yz\mathbf{i}+2xz\mathbf{j}+2xy\mathbf{k})$; $\sqrt{33}$

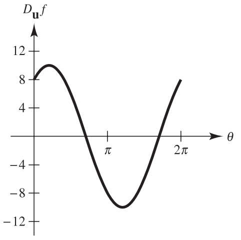
41.



43. (a) $-5\sqrt{2}/12$ (b) $\frac{3}{5}$ (c) $-\frac{1}{5}$ (d) $-11\sqrt{10}/60$
45. $\sqrt{13}/6$
47. (a) Answers will vary. Example: $-4\mathbf{i}+\mathbf{j}$
(b) $-\frac{2}{5}\mathbf{i}+\frac{1}{10}\mathbf{j}$ (c) $\frac{2}{5}\mathbf{i}-\frac{1}{10}\mathbf{j}$
The direction opposite that of the gradient
49. (a)



(b) $D_{\mathbf{u}}f(4,-3)=8\cos\theta+6\sin\theta$



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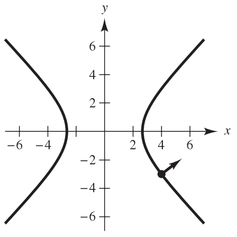
(c) $\theta\approx 2.21$, $\theta\approx 5.36$
Directions in which there is no change in f
(d) $\theta\approx 0.64$, $\theta\approx 3.79$
Directions of greatest rate of change in f
(e) 10; Magnitude of the greatest rate of change

(f)



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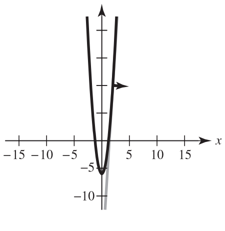
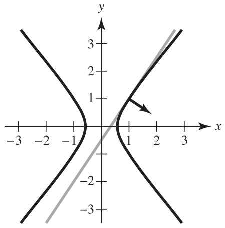
Orthogonal to the level curve

51. $-2\mathbf{i}-3\mathbf{j}$ **53.** $3\mathbf{i}-\mathbf{j}$
55. (a) $16\mathbf{i}-\mathbf{j}$ (b) $(\sqrt{257}/257)(16\mathbf{i}-\mathbf{j})$ (c) $y=16x-22$
(d)



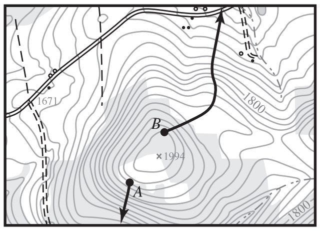
57. (a) $6\mathbf{i}-4\mathbf{j}$ (b) $(\sqrt{13}/13)(3\mathbf{i}-2\mathbf{j})$ (c) $y=\frac{3}{2}x-\frac{1}{2}$
(d)



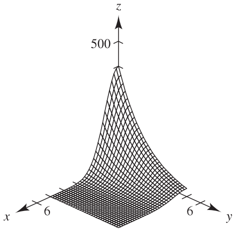
59. The directional derivative of $z=f(x,y)$ in the direction of $\mathbf{u}=\cos\theta\mathbf{i}+\sin\theta\mathbf{j}$ is

$$D_{\mathbf{u}}f(x,y)=\lim_{t\to 0}\frac{f(x+t\cos\theta,\,y+t\sin\theta)-f(x,y)}{t}$$

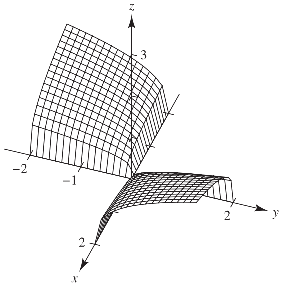
if the limit exists.
61. See the definition on page 936. See the properties on page 937.
63. The gradient vector is normal to the level curves.
65. $\frac{1}{625}(7\mathbf{i}-24\mathbf{j})$
67.



69. $y^2=10x$
71. (a)



(b) There is no change in heat in directions perpendicular to the gradient: $\pm(\mathbf{i}-6\mathbf{j})$.
(c) The greatest increase is in the direction of the gradient: $-3\mathbf{i}-\frac{1}{2}\mathbf{j}$.
73. True **75.** True **77.** $f(x,y,z)=e^x\cos y+\frac{1}{2}z^2+C$
79. (a) Proof (b) Proof
(c)



Section 13.7 (page 951)

1. The level surface can be written as $3x-5y+3z=15$, which is an equation of a plane in space.
3. The level surface can be written as $4x^2+9y^2-4z^2=0$, which is an elliptic cone that lies on the z-axis.

5. $\frac{1}{13}(3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k})$ **7.** $(\sqrt{6}/6)(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

9. $(\sqrt{145}/145)(12\mathbf{i} - \mathbf{k})$ **11.** $\frac{1}{13}(4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k})$

13. $(\sqrt{3}/3)(\mathbf{i} - \mathbf{j} + \mathbf{k})$ **15.** $(\sqrt{113}/113)(-\mathbf{i} - 6\sqrt{3}\mathbf{j} + 2\mathbf{k})$

17. $4x + 2y - z = 2$ **19.** $3x + 4y - 5z = 0$

21. $2x - 2y - z = 2$ **23.** $2x + 3y + 3z = 6$

25. $3x + 4y - 25z = 25(1 - \ln 5)$ **27.** $x - 4y + 2z = 18$

29. $6x - 3y - 2z = 11$ **31.** $x + y + z = 9$
$x - 3 = y - 3 = z - 3$

33. $2x + 4y + z = 14$
$$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-4}{1}$$

35. $6x - 4y - z = 5$
$$\frac{x-3}{6} = \frac{y-2}{-4} = \frac{z-5}{-1}$$

37. $10x + 5y + 2z = 30$
$$\frac{x-1}{10} = \frac{y-2}{5} = \frac{z-5}{2}$$

39. $x - y + 2z = \pi/2$
$$\frac{(x-1)}{1} = \frac{(y-1)}{-1} = \frac{z-(\pi/4)}{2}$$

41. (a) $\frac{x-1}{1} = \frac{y-1}{-1} = \frac{z-1}{1}$ (b) $\frac{1}{2}$, not orthogonal

43. (a) $\frac{x-3}{4} = \frac{y-3}{4} = \frac{z-4}{-3}$ (b) $\frac{16}{25}$, not orthogonal

45. (a) $\frac{x-3}{1} = \frac{y-1}{5} = \frac{z-2}{-4}$ (b) 0, orthogonal

47. 86.0° **49.** 77.4° **51.** $(0, 3, 12)$ **53.** $(2, 2, -4)$

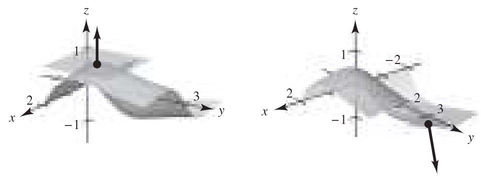
55. $(0, 0, 0)$ **57.** Proof **59.** (a) Proof (b) Proof

61. $(-2, 1, -1)$ or $(2, -1, 1)$

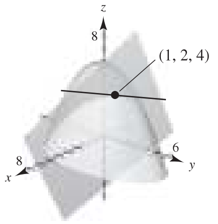
63. $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0)$
$+ F_z(x_0, y_0, z_0)(z - z_0) = 0$

65. Answers will vary.

67. (a) Line: $x = 1, y = 1, z = 1 - t$
Plane: $z = 1$
(b) Line: $x = -1, y = 2 + \frac{6}{25}t, z = -\frac{4}{5} - t$
Plane: $6y - 25z - 32 = 0$
(c)



69. (a) $x = 1 + t$
$y = 2 - 2t$
$z = 4$
$\theta \approx 48.2°$
(b)



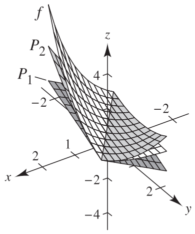
71. $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$
$F_x(x, y, z) = 2x/a^2$
$F_y(x, y, z) = 2y/b^2$
$F_z(x, y, z) = 2z/c^2$
Plane: $\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) + \frac{2z_0}{c^2}(z - z_0) = 0$
$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} = 1$$

73. $F(x, y, z) = a^2x^2 + b^2y^2 - z^2$
$F_x(x, y, z) = 2a^2x$
$F_y(x, y, z) = 2b^2y$
$F_z(x, y, z) = -2z$
Plane: $2a^2x_0(x - x_0) + 2b^2y_0(y - y_0) - 2z_0(z - z_0) = 0$
$a^2x_0x + b^2y_0y - z_0z = 0$
Therefore, the plane passes through the origin.

75. (a) $P_1(x, y) = 1 + x - y$
(b) $P_2(x, y) = 1 + x - y + \frac{1}{2}x^2 - xy + \frac{1}{2}y^2$
(c) If $x = 0$, $P_2(0, y) = 1 - y + \frac{1}{2}y^2$.
This is the second-degree Taylor polynomial for e^{-y}.
If $y = 0$, $P_2(x, 0) = 1 + x + \frac{1}{2}x^2$.
This is the second-degree Taylor polynomial for e^x.
(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1	1	1
0	0.1	0.9048	0.9000	0.9050
0.2	0.1	1.1052	1.1000	1.1050
0.2	0.5	0.7408	0.7000	0.7450
1	0.5	1.6487	1.5000	1.6250

(e)



77. Proof

Section 13.8 (page 960)

1. Relative minimum: $(1, 3, 0)$ **3.** Relative minimum: $(0, 0, 1)$

5. Relative minimum: $(-1, 3, -4)$

7. Relative minimum: $(1, 1, 11)$

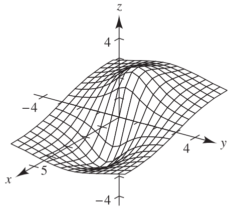
9. Relative maximum: $(5, -1, 2)$

11. Relative minimum: $(3, -4, -5)$

13. Relative minimum: $(0, 0, 0)$

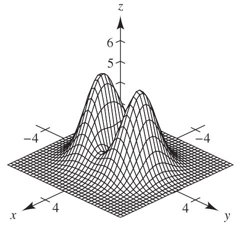
15. Relative maximum: $(0, 0, 4)$

17.



Relative maximum: $(-1, 0, 2)$
Relative minimum: $(1, 0, -2)$

19.



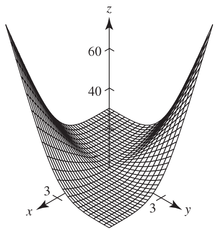
Relative minimum: $(0, 0, 0)$
Relative maxima: $(0, \pm 1, 4)$
Saddle points: $(\pm 1, 0, 1)$

21. Relative maximum: $(40, 40, 3200)$

23. Saddle point: $(0, 0, 0)$ **25.** Saddle point: $(1, -1, -1)$

27. There are no critical numbers.

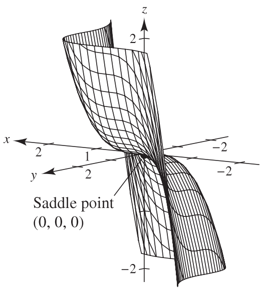
29. z is never negative. Minimum: $z = 0$ when $x = y \neq 0$.



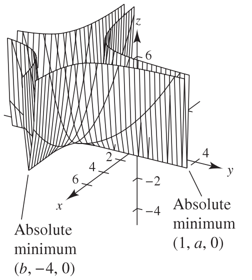
31. Insufficient information **33.** Saddle point

35. $-4 < f_{xy}(3, 7) < 4$

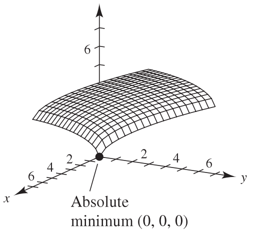
37. (a) $(0, 0)$ (b) Saddle point: $(0, 0, 0)$ (c) $(0, 0)$
(d)



39. (a) $(1, a), (b, -4)$ (b) Absolute minima: $(1, a, 0), (b, -4, 0)$
(c) $(1, a), (b, -4)$ (d)



41. (a) $(0, 0)$ (b) Absolute minimum: $(0, 0, 0)$ (c) $(0, 0)$
(d)



43. Relative minimum: $(0, 3, -1)$

45. Absolute maximum: $(4, 0, 21)$
Absolute minimum: $(4, 2, -11)$

47. Absolute maximum: $(0, 1, 10)$
Absolute minimum: $(1, 2, 5)$

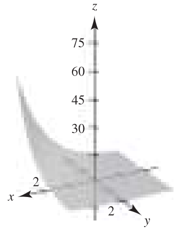
49. Absolute maxima: $(\pm 2, 4, 28)$
Absolute minimum: $(0, 1, -2)$

51. Absolute maxima: $(-2, -1, 9), (2, 1, 9)$
Absolute minima: $(x, -x, 0),\ |x| \leq 1$

53. Absolute maximum: $(1, 1, 1)$
Absolute minimum: $(0, 0, 0)$

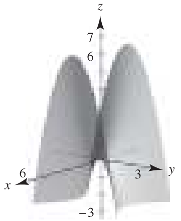
55. Point A is a saddle point.

57. Answers will vary.
Sample answer:



No extrema

59. Answers will vary.
Sample answer:



Saddle point

61. False. Let $f(x, y) = 1 - |x| - |y|$ at the point $(0, 0, 1)$.

63. False. Let $f(x, y) = x^2y^2$ (see Example 4 on page 958).

Section 13.9 (page 966)

1. $\sqrt{3}$ **3.** $\sqrt{7}$ **5.** $x = y = z = 3$ **7.** 10, 10, 10

9. 9 ft × 9 ft × 8.25 ft; \$26.73

11. Let $a + b + c = k$.

$$V = 4\pi abc/3 = \tfrac{4}{3}\pi ab(k - a - b) = \tfrac{4}{3}\pi(kab - a^2b - ab^2)$$

$$\left.\begin{aligned} V_a &= \tfrac{4}{3}\pi(kb - 2ab - b^2) = 0 \\ V_b &= \tfrac{4}{3}\pi(ka - a^2 - 2ab) = 0 \end{aligned}\right\} \begin{aligned} kb - 2ab - b^2 &= 0 \\ kb - a^2 - 2ab &= 0 \end{aligned}$$

So, $a = b$ and $b = k/3$. Thus, $a = b = c = k/3$.

13. Let x, y, and z be the length, width, and height, respectively, and let V_0 be the given volume. Then $V_0 = xyz$ and $z = V_0/xy$. The surface area is

$$S = 2xy + 2yz + 2xz = 2(xy + V_0/x + V_0/y).$$

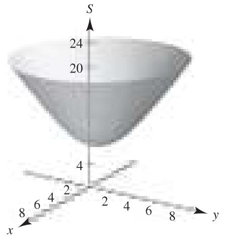
$$\left.\begin{aligned} S_x &= 2(y - V_0/x^2) = 0 \\ S_y &= 2(x - V_0/y^2) = 0 \end{aligned}\right\} \begin{aligned} x^2y - V_0 &= 0 \\ xy^2 - V_0 &= 0 \end{aligned}$$

So, $x = \sqrt[3]{V_0}$, $y = \sqrt[3]{V_0}$, and $z = \sqrt[3]{V_0}$.

15. $x_1 = 3; x_2 = 6$ **17.** Proof

19. $x = \sqrt{2}/2 \approx 0.707$ km
$y = (3\sqrt{2} + 2\sqrt{3})/6 \approx 1.284$ km

21. (a) $S = \sqrt{x^2 + y^2} + \sqrt{(x + 2)^2 + (y - 2)^2} + \sqrt{(x - 4)^2 + (y - 2)^2}$



The surface has a minimum.

(b) $S_x = \dfrac{x}{\sqrt{x^2 + y^2}} + \dfrac{x + 2}{\sqrt{(x + 2)^2 + (y - 2)^2}} + \dfrac{x - 4}{\sqrt{(x - 4)^2 + (y - 2)^2}}$

$S_y = \dfrac{y}{\sqrt{x^2 + y^2}} + \dfrac{y - 2}{\sqrt{(x + 2)^2 + (y - 2)^2}} + \dfrac{y - 2}{\sqrt{(x - 4)^2 + (y - 2)^2}}$

(c) $-\dfrac{1}{\sqrt{2}}\mathbf{i} - \left(\dfrac{1}{\sqrt{2}} - \dfrac{2}{\sqrt{10}}\right)\mathbf{j}$

$\theta \approx 186.0°$

(d) $t = 1.344$; $(x_2, y_2) \approx (0.05, 0.90)$

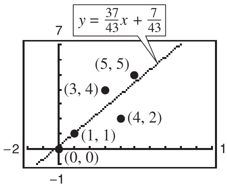
(e) $(x_4, y_4) \approx (0.06, 0.45)$; $S = 7.266$

(f) $-\nabla S(x, y)$ gives the direction of greatest rate of decrease of S. Use $\nabla S(x, y)$ when finding a maximum.

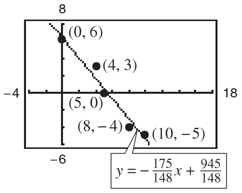
23. Write the equation to be maximized or minimized as a function of two variables. Take the partial derivatives and set them equal to zero or undefined to obtain the critical points. Use the Second Partials Test to test for relative extrema using the critical points. Check the boundary points.

25. (a) $y = \frac{3}{4}x + \frac{4}{3}$ (b) $\frac{1}{6}$ **27.** (a) $y = -2x + 4$ (b) 2

29. $y = \frac{37}{43}x + \frac{7}{43}$

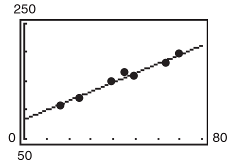


31. $y = -\frac{175}{148}x + \frac{945}{148}$



33. (a) $y = 1.6x + 84$

(b)



(c) 1.6

35. $y = 14x + 19$

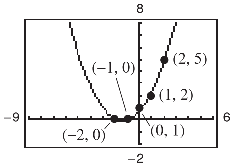
41.4 bushels per acre

37. $a\sum_{i=1}^{n} x_i^4 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 y_i$

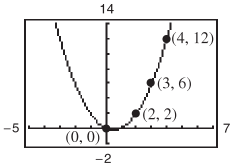
$a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$

$a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i + cn = \sum_{i=1}^{n} y_i$

39. $y = \frac{3}{7}x^2 + \frac{6}{5}x + \frac{26}{35}$

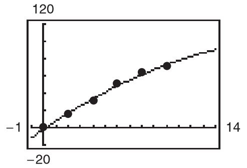


41. $y = x^2 - x$



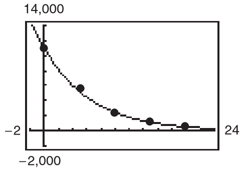
43. (a) $y = -0.22x^2 + 9.66x - 1.79$

(b)



45. (a) $\ln P = -0.1499h + 9.3018$ (b) $P = 10{,}957.7e^{-0.1499h}$

(c)

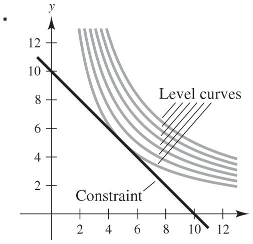


(d) Proof

47. Proof

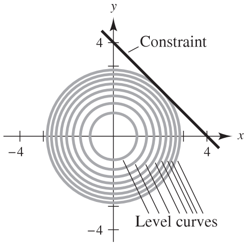
Section 13.10 (page 976)

1.



$f(5, 5) = 25$

3.



$f(2, 2) = 8$

5. $f(1, 2) = 5$ **7.** $f(25, 50) = 2600$

9. $f(1, 1) = 2$ **11.** $f(3, 3, 3) = 27$ **13.** $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$

15. Maxima: $f\left(\sqrt{2}/2, \sqrt{2}/2\right) = \frac{5}{2}$

$f\left(-\sqrt{2}/2, -\sqrt{2}/2\right) = \frac{5}{2}$

Minima: $f\left(-\sqrt{2}/2, \sqrt{2}/2\right) = -\frac{1}{2}$

$f\left(\sqrt{2}/2, -\sqrt{2}/2\right) = -\frac{1}{2}$

17. $f(8, 16, 8) = 1024$ **19.** $\sqrt{2}/2$ **21.** $3\sqrt{2}$ **23.** $\sqrt{11}/2$

25. 0.188 **27.** $\sqrt{3}$ **29.** $(-4, 0, 4)$

31. Optimization problems that have restrictions or constraints on the values that can be used to produce the optimal solutions are called constrained optimization problems.

33. $\sqrt{3}$ **35.** $x = y = z = 3$

37. 9 ft × 9 ft × 8.25 ft; \$26.73 **39.** $a = b = c = k/3$

41. Proof **43.** $2\sqrt{3}a/3 \times 2\sqrt{3}b/3 \times 2\sqrt{3}c/3$

45. $\sqrt[3]{360} \times \sqrt[3]{360} \times \frac{4}{3}\sqrt[3]{360}$ ft

47. $r = \sqrt[3]{\dfrac{v_0}{2\pi}}$ and $h = 2\sqrt[3]{\dfrac{v_0}{2\pi}}$ **49.** Proof

51. $P(15{,}625/18, 3125) \approx 226{,}869$

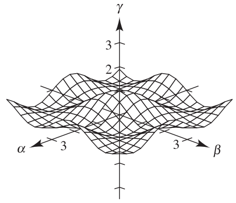
53. $x \approx 191.3$

$y \approx 688.7$

Cost ≈ \$55,095.60

55. (a) $g(\pi/3, \pi/3, \pi/3) = \frac{1}{8}$

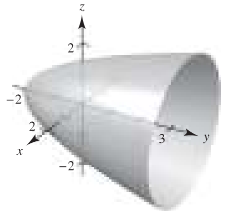
(b)



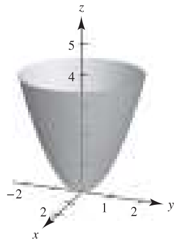
Maximum values occur when $\alpha = \beta$.

Review Exercises for Chapter 13 (page 978)

1.



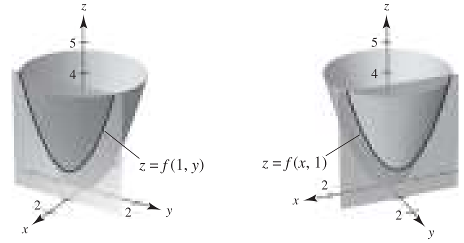
3. (a)



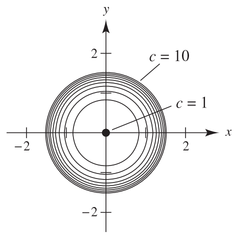
(b) g is a vertical translation of f two units upward.

(c) g is a horizontal translation of f two units to the right.

(d)



5.



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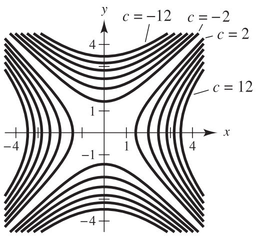
7.



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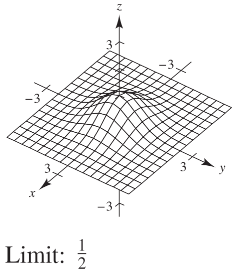
9.



11. Limit: $\frac{1}{2}$
Continuous except at $(0, 0)$

13. Limit: 0
Continuous

15. $f_x(x, y) = e^x \cos y$
$f_y(x, y) = -e^x \sin y$

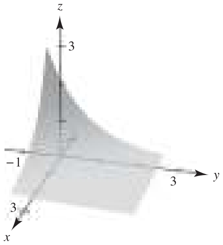
17. $\partial z/\partial x = -e^{-x}$
$\partial z/\partial y = -e^{-y}$

19. $g_x(x, y) = [y(y^2 - x^2)]/(x^2 + y^2)^2$
$g_y(x, y) = [x(x^2 - y^2)]/(x^2 + y^2)^2$

21. $f_x(x, y, z) = -yz/(x^2 + y^2)$
$f_y(x, y, z) = xz/(x^2 + y^2)$
$f_z(x, y, z) = \arctan y/x$

23. $u_x(x, t) = cne^{-n^2 t} \cos nx$
$u_t(x, t) = -cn^2 e^{-n^2 t} \sin nx$

25. Answers will vary. Example:



27. $f_{xx}(x, y) = 6$
$f_{yy}(x, y) = 12y$
$f_{xy}(x, y) = f_{yx}(x, y) = -1$

29. $h_{xx}(x, y) = -y \cos x$
$h_{yy}(x, y) = -x \sin y$
$h_{xy}(x, y) = h_{yx}(x, y) = \cos y - \sin x$

31. $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 2 + (-2) = 0$

33. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{6x^2y - 2y^3}{(x^2 + y^2)^3} + \dfrac{-6x^2y + 2y^3}{(x^2 + y^2)^3} = 0$

35. $(xy \cos xy + \sin xy)\, dx + (x^2 \cos xy)\, dy$

37. 0.6538 cm, 5.03% **39.** $\pm\pi$ in.^{3}

41. $dw/dt = (8t - 1)/(4t^2 - t + 4)$

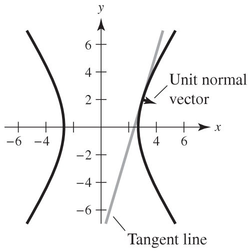
43. $\partial w/\partial r = (4r^2t - 4rt^2 - t^3)/(2r - t)^2$
$\partial w/\partial t = (4r^2t - rt^2 + 4r^3)/(2r - t)^2$

45. $\partial z/\partial x = (-2x - y)/(y + 2z)$
$\partial z/\partial y = (-x - 2y - z)/(y + 2z)$

47. -50 **49.** $\frac{2}{3}$ **51.** $\langle 4, 4\rangle, 4\sqrt{2}$ **53.** $\left\langle -\frac{1}{2}, 0\right\rangle, \frac{1}{2}$

55. (a) $54\mathbf{i} - 16\mathbf{j}$ (b) $\dfrac{27}{\sqrt{793}}\mathbf{i} - \dfrac{8}{\sqrt{793}}\mathbf{j}$ (c) $y = \dfrac{27}{8}x - \dfrac{65}{8}$

(d)

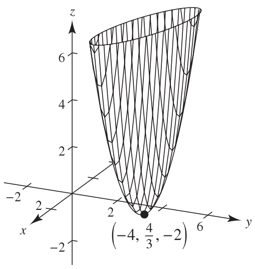


57. Tangent plane: $4x + 4y - z = 8$
Normal line: $x = 2 + 4t,\ y = 1 + 4t,\ z = 4 - t$

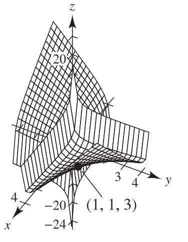
59. Tangent plane: $z = 4$
Normal line: $x = 2,\ y = -3,\ z = 4 + t$

61. $(x - 2)/1 = (y - 2)/1 = (z - 5)/(-4)$ **63.** $\theta \approx 36.7°$

65. Relative minimum:
$\left(-4, \frac{4}{3}, -2\right)$



67. Relative minimum: $(1, 1, 3)$



69. The level curves are hyperbolas. The critical point $(0, 0)$ may be a saddle point or an extremum.

71. $x_1 = 94, x_2 = 157$ **73.** $f(49.4, 253) = 13{,}201.8$

75. (a) $y = 0.004x^2 + 0.07x + 19.4$ (b) 50.6 kg

77. Maximum: $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$

79. $x = \sqrt{2}/2 \approx 0.707$ km; $y = \sqrt{3}/3 \approx 0.577$ km;
$z = \left(60 - 3\sqrt{2} - 2\sqrt{3}\right)/6 \approx 8.716$ km

P.S. Problem Solving (page 981)

1. (a) 12 square units (b) Proof (c) Proof

3. (a) $y_0z_0(x - x_0) + x_0z_0(y - y_0) + x_0y_0(z - z_0) = 0$

(b) $x_0y_0z_0 = 1 \implies z_0 = 1/x_0y_0$

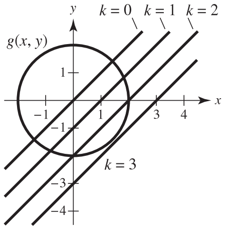
Then the tangent plane is

$$y_0\left(\frac{1}{x_0y_0}\right)(x - x_0) + x_0\left(\frac{1}{x_0y_0}\right)(y - y_0) + x_0y_0\left(z - \frac{1}{x_0y_0}\right) = 0.$$

Intercepts: $(3x_0, 0, 0), (0, 3y_0, 0), \left(0, 0, \dfrac{3}{x_0y_0}\right)$

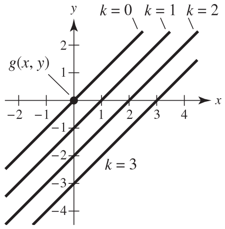
$V = \frac{1}{3}bh = \frac{9}{2}$

5. (a)



Maximum value: $2\sqrt{2}$

(b)



Maximum and minimum value: 0
The method of Lagrange multipliers does not work because $\nabla g(x_0, y_0) = \mathbf{0}$.

7. $2\sqrt[3]{150} \times 2\sqrt[3]{150} \times 5\sqrt[3]{150}/3$

9. (a)
$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= xCy^{1-a}ax^{a-1} + yCx^a(1-a)y^{1-a-1} \\ &= ax^aCy^{1-a} + (1-a)x^aC(y^{1-a}) \\ &= Cx^ay^{1-a}[a + (1-a)] \\ &= Cx^ay^{1-a} \\ &= f(x, y) \end{aligned}$$

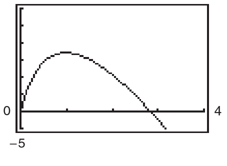
(b)
$$\begin{aligned} f(tx, ty) &= C(tx)^a(ty)^{1-a} \\ &= Ctx^ay^{1-a} \\ &= tCx^ay^{1-a} \\ &= tf(x, y) \end{aligned}$$

11. (a) $x = 32\sqrt{2}t$
$y = 32\sqrt{2}t - 16t^2$

(b) $\alpha = \arctan\left(\dfrac{y}{x + 50}\right) = \arctan\left(\dfrac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)$

(c) $\dfrac{d\alpha}{dt} = \dfrac{-16\left(8\sqrt{2}t^2 + 25t - 25\sqrt{2}\right)}{64t^4 - 256\sqrt{2}t^3 + 1024t^2 + 800\sqrt{2}t + 625}$

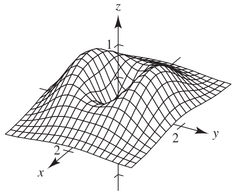
(d)



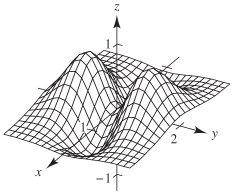
No; The rate of change of α is greatest when the projectile is closest to the camera.

(e) α is maximum when $t = 0.98$ second.
No; the projectile is at its maximum height when $t = \sqrt{2} \approx 1.41$ seconds.

13. (a)



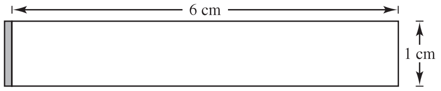
(b)



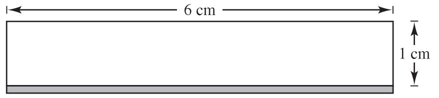
Minimum: $(0, 0, 0)$ Minima: $(\pm 1, 0, -e^{-1})$
Maxima: $(0, \pm 1, 2e^{-1})$ Maxima: $(0, \pm 1, 2e^{-1})$
Saddle points: $(\pm 1, 0, e^{-1})$ Saddle point: $(0, 0, 0)$

(c) $\alpha > 0$ $\alpha < 0$
Minimum: $(0, 0, 0)$ Minima: $(\pm 1, 0, \alpha e^{-1})$
Maxima: $(0, \pm 1, \beta e^{-1})$ Maxima: $(0, \pm 1, \beta e^{-1})$
Saddle points: $(\pm 1, 0, \alpha e^{-1})$ Saddle point: $(0, 0, 0)$

15. (a)



(b)



(c) Height

(d) $dl = 0.01, dh = 0$: $dA = 0.01$
$dl = 0, dh = 0.01$: $dA = 0.06$

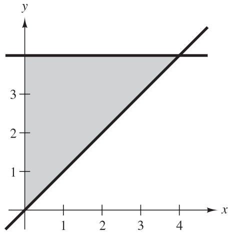
17–19. Proofs

Chapter 14

Section 14.1 (page 990)

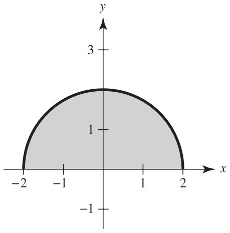
1. $2x^2$ **3.** $y\ln(2y)$ **5.** $(4x^2 - x^4)/2$
7. $(y/2)[(\ln y)^2 - y^2]$ **9.** $x^2(1 - e^{-x^2} - x^2e^{-x^2})$ **11.** 3
13. $\frac{8}{3}$ **15.** $\frac{1}{2}$ **17.** 2 **19.** $\frac{1}{3}$ **21.** 1629 **23.** $\frac{2}{3}$ **25.** 4
27. $\pi/2$ **29.** $\pi^2/32 + \frac{1}{8}$ **31.** $\frac{1}{2}$ **33.** Diverges **35.** 24
37. $\frac{16}{3}$ **39.** $\frac{8}{3}$ **41.** 5 **43.** πab **45.** $\frac{9}{2}$

47.



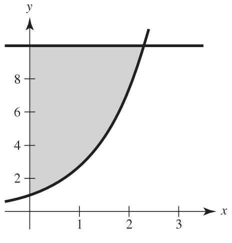
$\displaystyle\int_0^4\int_x^4 f(x, y)\,dy\,dx$

49.



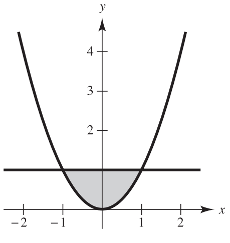
$\displaystyle\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x, y)\,dx\,dy$

51.



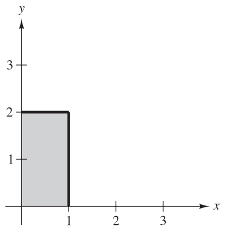
$\displaystyle\int_0^{\ln 10}\int_{e^x}^{10} f(x, y)\,dy\,dx$

53.



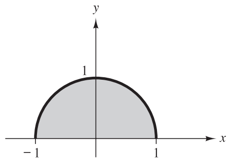
$\displaystyle\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\,dx\,dy$

55.



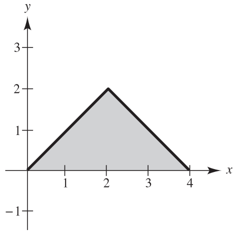
$$\int_0^1\int_0^2 dy\,dx = \int_0^2\int_0^1 dx\,dy = 2$$

57.



$$\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy = \int_{-1}^1\int_0^{\sqrt{1-x^2}} dy\,dx = \frac{\pi}{2}$$

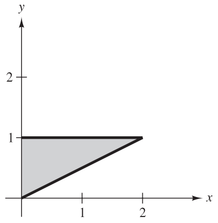
59.



y
x
3
2
1
−1
1
2
3
4

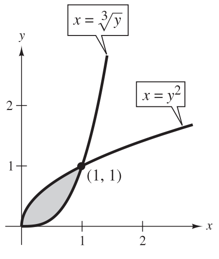
$$\int_0^2\int_0^x dy\,dx + \int_2^4\int_0^{4-x} dy\,dx = \int_0^2\int_y^{4-y} dx\,dy = 4$$

61.



$$\int_0^2\int_{x/2}^1 dy\,dx = \int_0^1\int_0^{2y} dx\,dy = 1$$

63.



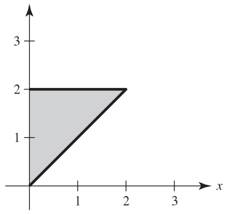
$x = \sqrt[3]{y}$
$x = y^2$
(1, 1)
y
x
2
1
1
2

$$\int_0^1\int_{y^2}^{\sqrt[3]{y}} dx\,dy = \int_0^1\int_{x^3}^{\sqrt{x}} dy\,dx = \frac{5}{12}$$

65. The first integral arises using vertical representative rectangles. The second two integrals arise using horizontal representative rectangles.

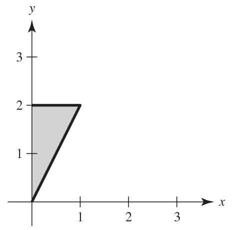
Value of the integrals: $15{,}625\pi/24$

67.



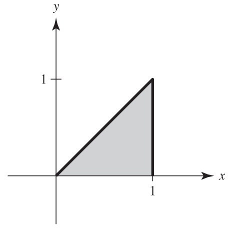
$$\int_0^2\int_x^2 x\sqrt{1+y^3}\,dy\,dx = \frac{26}{9}$$

69.



$$\int_0^1\int_{2x}^2 4e^{y^2}\,dy\,dx = e^4 - 1 \approx 53.598$$

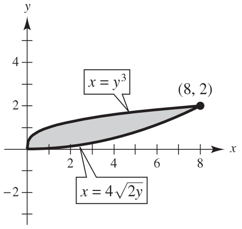
71.



$$\int_0^1\int_y^1 \sin x^2\,dx\,dy = \frac{1}{2}(1-\cos 1) \approx 0.230$$

73. $\frac{1664}{105}$ **75.** $(\ln 5)^2$

77. (a)



y
x
$x = y^3$
(8, 2)
$x = 4\sqrt{2y}$
4
2
−2
2
4
6
8

(b) $\displaystyle\int_0^8\int_{x^2/32}^{\sqrt[3]{x}} (x^2y - xy^2)\,dy\,dx$ (c) $67{,}520/693$

79. 20.5648 **81.** $15\pi/2$

83. An iterated integral is an integral of a function of several variables. Integrate with respect to one variable while holding the other variables constant.

85. If all four limits of integration are constant, the region of integration is rectangular.

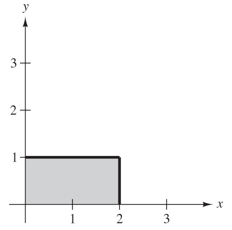
87. True

Section 14.2 (page 1000)

1. 24 (approximation is exact)

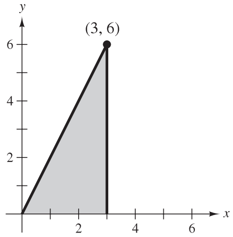
3. Approximation: 52; Exact: $\frac{160}{3}$ **5.** 400; 272

7.



8

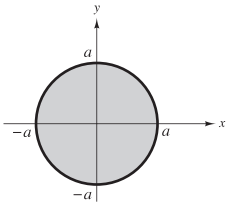
9.



y
x
(3, 6)
6
4
2
2
4
6

36

11.



0

13. $\int_0^3\int_0^5 xy\,dy\,dx = \frac{225}{4}$

$\int_0^5\int_0^3 xy\,dx\,dy = \frac{225}{4}$

15. $\int_1^2\int_x^{2x} \frac{y}{x^2+y^2}\,dy\,dx = \frac{1}{2}\ln\frac{5}{2}$

$\int_1^2\int_1^y \frac{y}{x^2+y^2}\,dx\,dy + \int_2^4\int_{y/2}^2 \frac{y}{x^2+y^2}\,dx\,dy = \frac{1}{2}\ln\frac{5}{2}$

17. $\int_0^1\int_{4-x}^{4-x^2} -2y\,dy\,dx = -\frac{6}{5}$

$\int_3^4\int_{4-y}^{\sqrt{4-y}} -2y\,dx\,dy = -\frac{6}{5}$

19. $\int_0^3\int_{4y/3}^{\sqrt{25-y^2}} x\,dx\,dy = 25$

$\int_0^4\int_0^{3x/4} x\,dy\,dx + \int_4^5\int_0^{\sqrt{25-x^2}} x\,dy\,dx = 25$

21. 4 **23.** 4 **25.** 12 **27.** $\frac{3}{8}$ **29.** 1 **31.** $32\sqrt{2}\,\pi/3$

33. $\int_0^1\int_0^x xy\,dy\,dx = \frac{1}{8}$ **35.** $\int_0^2\int_0^4 x^2\,dy\,dx = \frac{32}{3}$

37. $2\int_0^1\int_0^x \sqrt{1-x^2}\,dy\,dx = \frac{2}{3}$

39. $\int_0^2\int_0^{\sqrt{4-x^2}} (x+y)\,dy\,dx = \frac{16}{3}$

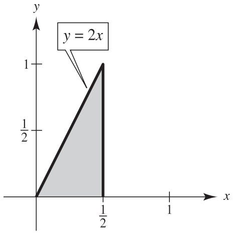
41. $2\int_0^2\int_0^{\sqrt{1-(x-1)^2}} (2x-x^2-y^2)\,dy\,dx$

43. $4\int_0^2\int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$

45. $\int_0^2\int_{-\sqrt{2-2(y-1)^2}}^{\sqrt{2-2(y-1)^2}} (4y-x^2-2y^2)\,dx\,dy$

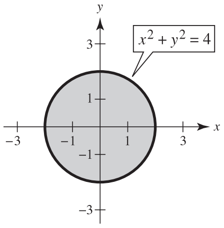
47. $81\pi/2$ **49.** 1.2315 **51.** Proof

53.



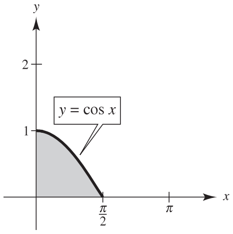
$\int_0^1\int_{y/2}^{1/2} e^{-x^2}\,dx\,dy = 1 - e^{-1/4} \approx 0.221$

55.



$\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-y^2}\,dy\,dx = \frac{64}{3}$

57.



$\int_0^1\int_0^{\arccos y} \sin x\sqrt{1+\sin^2 x}\,dx\,dy = \frac{1}{3}\left(2\sqrt{2}-1\right)$

59. 2 **61.** $\frac{8}{3}$ **63.** $(e-1)^2$ **65.** 25,645.24

67. See "Definition of Double Integral" on page 994. The double integral of a function $f(x, y) \geq 0$ over the region of integration yields the volume of that region.

69. (a) The total snowfall in county R
(b) The average snowfall in county R

71. No; 6π is the greatest possible value. **73.** Proof; $\frac{1}{5}$

75. Proof; $\frac{7}{27}$ **77.** 2500 m^3 **79.** (a) 1.784 (b) 1.788

81. (a) 11.057 (b) 11.041 **83.** d

85. False. $V = 8\int_0^1\int_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\,dx\,dy.$

87. $\frac{1}{2}(1-e)$ **89.** R: $x^2+y^2 \leq 9$ **91.** About 0.82736

93. Putnam Problem A2, 1989

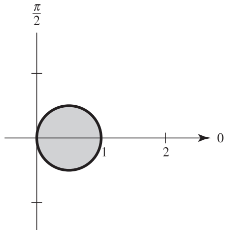
Section 14.3 (page 1009)

1. Rectangular **3.** Polar

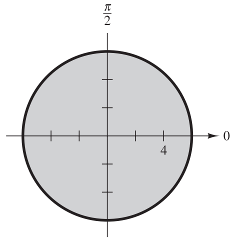
5. The region R is a half-circle of radius 8. It can be described in polar coordinates as
$R = \{(r, \theta): 0 \leq r \leq 8, 0 \leq \theta \leq \pi\}.$

7. The region R is a cardioid with $a = b = 3$. It can be described in polar coordinates as
$R = \{(r, \theta): 0 \leq r \leq 3 + 3\sin\theta, 0 \leq \theta \leq 2\pi\}.$

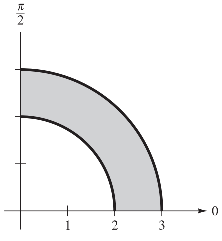
9. $\pi/4$



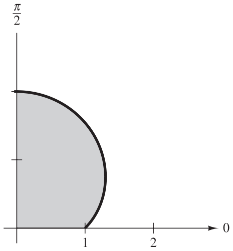
11. 0



13. $5\sqrt{5}\pi/6$



15. $\frac{9}{8} + 3\pi^2/32$



17. $a^3/3$ **19.** 4π **21.** $243\pi/10$ **23.** $\frac{2}{3}$ **25.** $(\pi/2)\sin 1$

27. $\int_0^{\pi/4}\int_0^{2\sqrt{2}} r^2\,dr\,d\theta = \frac{4\sqrt{2}\pi}{3}$

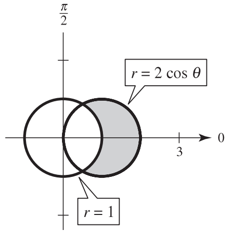
29. $\int_0^{\pi/2}\int_0^2 r^2(\cos\theta + \sin\theta)\,dr\,d\theta = \frac{16}{3}$

31. $\int_0^{\pi/4}\int_1^2 r\theta\,dr\,d\theta = \frac{3\pi^2}{64}$ **33.** $\frac{1}{8}$ **35.** $\frac{250\pi}{3}$

37. $\frac{64}{9}(3\pi - 4)$ **39.** $2\sqrt{4 - 2\sqrt[3]{2}}$ **41.** 1.2858

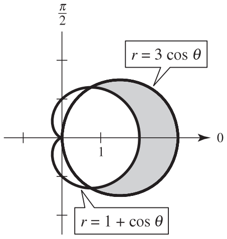
43. 9π **45.** $3\pi/2$ **47.** π

49.



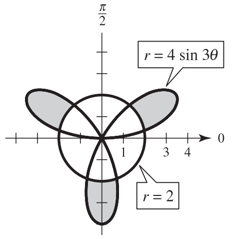
$\frac{\pi}{3} + \frac{\sqrt{3}}{2}$

51.



π

53.



$\frac{4\pi}{3} + 2\sqrt{3}$

55. Let R be a region bounded by the graphs of $r = g_1(\theta)$ and $r = g_2(\theta)$ and the lines $\theta = a$ and $\theta = b$. When using polar coordinates to evaluate a double integral over R, R can be partitioned into small polar sectors.

57. r-simple regions have fixed bounds for θ and variable bounds for r.

θ-simple regions have variable bounds for θ and fixed bounds for r.

59. (a) $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\,dy\,dx$

(b) $\int_0^{2\pi}\int_0^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

(c) Choose the integral in part (b) because the limits of integration are less complicated.

61. Insert a factor of r; Sector of a circle **63.** 56.051 **65.** c

67. False. Let $f(r, \theta) = r - 1$ and let R be a sector where $0 \le r \le 6$ and $0 \le \theta \le \pi$.

69. (a) 2π (b) $\sqrt{2\pi}$ **71.** 486,788

73. (a) $\int_2^4\int_{y/\sqrt{3}}^{y} f\,dx\,dy$

(b) $\int_{2/\sqrt{3}}^{2}\int_2^{\sqrt{3}x} f\,dy\,dx + \int_2^{4/\sqrt{3}}\int_x^{\sqrt{3}x} f\,dy\,dx + \int_{4/\sqrt{3}}^{4}\int_x^4 f\,dy\,dx$

(c) $\int_{\pi/4}^{\pi/3}\int_{2\csc\theta}^{4\csc\theta} f\,r\,dr\,d\theta$

75. $A = \frac{\Delta\theta r_2^2}{2} - \frac{\Delta\theta r_1^2}{2} = \Delta\theta\left(\frac{r_1 + r_2}{2}\right)(r_2 - r_1) = r\,\Delta r\,\Delta\theta$

Section 14.4 (page 1018)

1. $m = 4$ **3.** $m = \frac{1}{8}$

5. (a) $m = ka^2, (a/2, a/2)$ (b) $m = ka^3/2, (a/2, 2a/3)$
(c) $m = ka^3/2, (2a/3, a/2)$

7. (a) $m = ka^2/2, (a/3, 2a/3)$ (b) $m = ka^3/3, (3a/8, 3a/4)$
(c) $m = ka^3/6, (a/2, 3a/4)$

9. (a) $\left(\frac{a}{2} + 5, \frac{a}{2}\right)$ (b) $\left(\frac{a}{2} + 5, \frac{2a}{3}\right)$

(c) $\left(\frac{2(a^2 + 15a + 75)}{3(a + 10)}, \frac{a}{2}\right)$

11. $m = k/4, (2/3, 8/15)$ **13.** $m = 30k, (14/5, 4/5)$

15. (a) $m = k(e - 1), \left(\frac{1}{e - 1}, \frac{e + 1}{4}\right)$

(b) $m = \frac{k}{4}(e^2 - 1), \left(\frac{e^2 + 1}{2(e^2 - 1)}, \frac{4(e^3 - 1)}{9(e^2 - 1)}\right)$

17. $m = 256k/15, (0, 16/7)$ **19.** $m = \frac{2kL}{\pi}, \left(\frac{L}{2}, \frac{\pi}{8}\right)$

21. $m = \frac{k\pi a^2}{8}, \left(\frac{4\sqrt{2}a}{3\pi}, \frac{4a(2 - \sqrt{2})}{3\pi}\right)$

23. $m = \frac{k}{8}(1 - 5e^{-4}), \left(\frac{e^4 - 13}{e^4 - 5}, \frac{8}{27}\left[\frac{e^6 - 7}{e^6 - 5e^2}\right]\right)$

25. $m = k\pi/3, \left(81\sqrt{3}/(40\pi), 0\right)$

27. $\bar{\bar{x}} = \sqrt{3}b/3$ **29.** $\bar{\bar{x}} = a/2$ **31.** $\bar{\bar{x}} = a/2$
$\bar{\bar{y}} = \sqrt{3}h/3$ $\bar{\bar{y}} = a/2$ $\bar{\bar{y}} = a/2$

33. $I_x = kab^4/4$
$I_y = kb^2a^3/6$
$I_0 = (3kab^4 + 2ka^3b^2)/12$
$\bar{\bar{x}} = \sqrt{3}a/3$
$\bar{\bar{y}} = \sqrt{2}b/2$

35. $I_x = 32k/3$
$I_y = 16k/3$
$I_0 = 16k$
$\bar{\bar{x}} = 2\sqrt{3}/3$
$\bar{\bar{y}} = 2\sqrt{6}/3$

37. $I_x = 16k$
$I_y = 512k/5$
$I_0 = 592k/5$
$\bar{\bar{x}} = 4\sqrt{15}/5$
$\bar{\bar{y}} = \sqrt{6}/2$

39. $I_x = 3k/56$
$I_y = k/18$
$I_0 = 55k/504$
$\bar{\bar{x}} = \sqrt{30}/9$
$\bar{\bar{y}} = \sqrt{70}/14$

41. $2k\int_{-b}^{b}\int_0^{\sqrt{b^2-x^2}} (x - a)^2\,dy\,dx = \frac{k\pi b^2}{4}(b^2 + 4a^2)$

43. $\int_0^4\int_0^{\sqrt{x}} kx(x - 6)^2\,dy\,dx = \frac{42{,}752k}{315}$

45. $\int_0^a\int_0^{\sqrt{a^2-x^2}} k(a - y)(y - a)^2\,dy\,dx = ka^5\left(\frac{7\pi}{16} - \frac{17}{15}\right)$

47. See definitions on page 1014. **49.** Answers will vary.

51. $L/3$ **53.** $L/2$ **55.** Proof

Section 14.5 (page 1025)

1. 24 **3.** 12π **5.** $\frac{1}{2}\left[4\sqrt{17} + \ln\left(4 + \sqrt{17}\right)\right]$

7. $\frac{4}{27}\left(31\sqrt{31} - 8\right)$ **9.** $\sqrt{2} - 1$ **11.** $\sqrt{2}\pi$

13. $2\pi a\left(a - \sqrt{a^2 - b^2}\right)$ **15.** $48\sqrt{14}$ **17.** 20π

19. $\int_0^1\int_0^x \sqrt{5 + 4x^2}\,dy\,dx = \frac{27 - 5\sqrt{5}}{12} \approx 1.3183$

21. $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{1 + 4x^2 + 4y^2}\,dy\,dx$
$= \frac{\pi}{6}\left(37\sqrt{37} - 1\right) \approx 117.3187$

23. $\int_0^1\int_0^1 \sqrt{1 + 4x^2 + 4y^2}\,dy\,dx \approx 1.8616$ **25.** e

27. 2.0035 **29.** $\int_{-1}^{1}\int_{-1}^{1} \sqrt{1 + 9(x^2 - y)^2 + 9(y^2 - x)^2}\, dy\, dx$

31. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + e^{-2x}}\, dy\, dx$

33. $\int_{0}^{4}\int_{0}^{10} \sqrt{1 + e^{2xy}(x^2 + y^2)}\, dy\, dx$

35. If f and its first partial derivatives are continuous on the closed region R in the xy-plane, then the area of the surface S given by $z = f(x, y)$ over R is

$$\int_R\int \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA.$$

37. No. The size and shape of the graph stay the same, just the position is changed. So, the surface area does not increase.

39. 16 **41.** (a) $812\pi\sqrt{609}$ cm^3 (b) $100\pi\sqrt{609}$ cm^2

Section 14.6 (page 1035)

1. 18 **3.** $\frac{1}{10}$ **5.** $\frac{15}{2}(1 - 1/e)$ **7.** $-\frac{40}{3}$ **9.** $\frac{324}{5}$

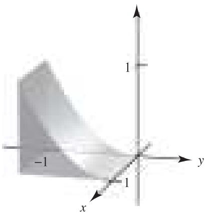
11. 2.44167 **13.** $V = \int_{0}^{5}\int_{0}^{5-x}\int_{0}^{5-x-y} dz\, dy\, dx$

15. $V = \int_{-\sqrt{6}}^{\sqrt{6}}\int_{-\sqrt{6-y^2}}^{\sqrt{6-y^2}}\int_{0}^{6-x^2-y^2} dz\, dx\, dy$

17. $V = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{(x^2+y^2)/2}^{\sqrt{80-x^2-y^2}} dz\, dy\, dx$

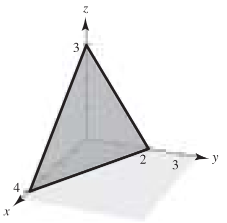
19. $\frac{256}{15}$ **21.** $4\pi a^3/3$ **23.** $\frac{256}{15}$ **25.** 10

27.



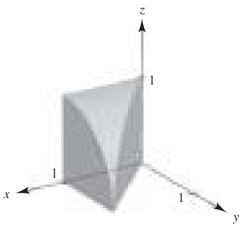
$\int_{0}^{1}\int_{0}^{1}\int_{-1}^{-\sqrt{z}} dy\, dz\, dx$

29.



$\int_{0}^{3}\int_{0}^{(12-4z)/3}\int_{0}^{(12-4z-3x)/6} dy\, dx\, dz$

31.



$\int_{0}^{1}\int_{0}^{x}\int_{0}^{\sqrt{1-y^2}} dz\, dy\, dx$

33. $\int_{0}^{1}\int_{0}^{x}\int_{0}^{3} xyz\, dz\, dy\, dx,\ \int_{0}^{1}\int_{y}^{1}\int_{0}^{3} xyz\, dz\, dx\, dy,$
$\int_{0}^{1}\int_{0}^{3}\int_{0}^{x} xyz\, dy\, dz\, dx,\ \int_{0}^{3}\int_{0}^{1}\int_{0}^{x} xyz\, dy\, dx\, dz,$
$\int_{0}^{3}\int_{0}^{1}\int_{y}^{1} xyz\, dx\, dy\, dz,\ \int_{0}^{1}\int_{0}^{3}\int_{y}^{1} xyz\, dx\, dz\, dy$

35. $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{4} xyz\, dz\, dy\, dx,\ \int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{0}^{4} xyz\, dz\, dx\, dy,$
$\int_{-3}^{3}\int_{0}^{4}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz\, dy\, dz\, dx,\ \int_{0}^{4}\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz\, dy\, dx\, dz,$
$\int_{0}^{4}\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz\, dx\, dy\, dz,\ \int_{-3}^{3}\int_{0}^{4}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz\, dx\, dz\, dy$

37. $\int_{0}^{1}\int_{0}^{1-z}\int_{0}^{1-y^2} dx\, dy\, dz,\ \int_{0}^{1}\int_{0}^{1-y}\int_{0}^{1-y^2} dx\, dz\, dy,$
$\int_{0}^{1}\int_{0}^{2z-z^2}\int_{0}^{1-z} 1\, dy\, dx\, dz + \int_{0}^{1}\int_{2z-z^2}^{1}\int_{0}^{\sqrt{1-x}} 1\, dy\, dx\, dz,$
$\int_{0}^{1}\int_{1-\sqrt{1-x}}^{1}\int_{0}^{1-z} 1\, dy\, dz\, dx + \int_{0}^{1}\int_{0}^{1-\sqrt{1-x}}\int_{0}^{\sqrt{1-x}} 1\, dy\, dz\, dx,$
$\int_{0}^{1}\int_{0}^{\sqrt{1-x}}\int_{0}^{1-y} dz\, dy\, dx$

39. $m = 8k$ **41.** $m = 128k/3$
$\bar{x} = \frac{3}{2}$ $\bar{z} = 1$

43. $m = k\int_{0}^{b}\int_{0}^{b}\int_{0}^{b} xy\, dz\, dy\, dx$

$M_{yz} = k\int_{0}^{b}\int_{0}^{b}\int_{0}^{b} x^2y\, dz\, dy\, dx$

$M_{xz} = k\int_{0}^{b}\int_{0}^{b}\int_{0}^{b} xy^2\, dz\, dy\, dx$

$M_{xy} = k\int_{0}^{b}\int_{0}^{b}\int_{0}^{b} xyz\, dz\, dy\, dx$

45. $\bar{x}$ will be greater than 2, and $\bar{y}$ and $\bar{z}$ will be unchanged.

47. $\bar{x}$ and $\bar{z}$ will be unchanged, and $\bar{y}$ will be greater than 0.

49. $(0, 0, 3h/4)$ **51.** $\left(0, 0, \frac{3}{2}\right)$ **53.** $\left(5, 6, \frac{5}{4}\right)$

55. (a) $I_x = 2ka^5/3$ (b) $I_x = ka^8/8$
$I_y = 2ka^5/3$ $I_y = ka^8/8$
$I_z = 2ka^5/3$ $I_z = ka^8/8$

57. (a) $I_x = 256k$ (b) $I_x = 2048k/3$
$I_y = 512k/3$ $I_y = 1024k/3$
$I_z = 256k$ $I_z = 2048k/3$

59. Proof **61.** $\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-x} (x^2 + y^2)\sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx$

63. (a) $m = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} kz\, dz\, dy\, dx$

(b) $\bar{x} = \bar{y} = 0$, by symmetry.

$\bar{z} = \frac{1}{m}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} kz^2\, dz\, dy\, dx$

(c) $I_z = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} kz(x^2 + y^2)\, dz\, dy\, dx$

65. See "Definition of Triple Integral" on page 1027 and Theorem 14.4, "Evaluation by Iterated Integrals," on page 1028.

67. (a) Solid B
- (b) Solid B has the greater moment of inertia because it is more dense.
- (c) Solid A will reach the bottom first. Because Solid B has a greater moment of inertia, it has a greater resistance to rotational motion.

69. $\frac{13}{3}$ **71.** $\frac{3}{2}$

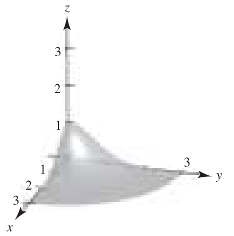
73. Q: $3z^2 + y^2 + 2x^2 \le 1$; $4\sqrt{6}\pi/45 \approx 0.684$

75. $a = 2, \frac{16}{3}$ **77.** Putnam Problem B1, 1965

Section 14.7 (page 1043)

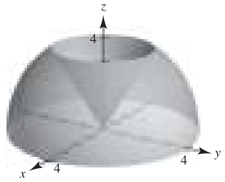
1. 27 **3.** $\frac{52}{45}$ **5.** $\pi/8$ **7.** $\pi(e^4 + 3)$

9.



$(1 - e^{-9})\pi/4$

11.



$64\sqrt{3}\pi/3$

13. Cylindrical: $\int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r^2\cos\theta\,dz\,dr\,d\theta = 0$

Spherical: $\int_0^{2\pi}\int_0^{\arctan(1/2)}\int_0^{4\sec\phi} \rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta$

$+ \int_0^{2\pi}\int_{\arctan(1/2)}^{\pi/2}\int_0^{\cot\phi\csc\phi} \rho^3\sin^2\phi\cos\phi\,d\rho\,d\phi\,d\theta = 0$

15. Cylindrical: $\int_0^{2\pi}\int_0^{a}\int_a^{a+\sqrt{a^2-r^2}} r^2\cos\theta\,dz\,dr\,d\theta = 0$

Spherical: $\int_0^{\pi/4}\int_0^{2\pi}\int_{a\sec\phi}^{2a\cos\phi} \rho^3\sin^2\phi\cos\theta\,d\rho\,d\theta\,d\phi = 0$

17. $(2a^3/9)(3\pi - 4)$ **19.** $\pi/16$ **21.** $(2a^3/9)(3\pi - 4)$

23. $48k\pi$ **25.** $\pi r_0^2h/3$ **27.** $(0, 0, h/5)$

29. $I_z = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^3\,dz\,dr\,d\theta = 3mr_0^2/10$

31. Proof **33.** $9\pi\sqrt{2}$ **35.** $16\pi^2$

37. $k\pi a^4$ **39.** $(0, 0, 3r/8)$ **41.** $k\pi/192$

43. Rectangular to cylindrical: $r^2 = x^2 + y^2$
$\tan\theta = y/x$
$z = z$
Cylindrical to rectangular: $x = r\cos\theta$
$y = r\sin\theta$
$z = z$

45. $\int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta,\, r\sin\theta)}^{h_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)r\,dz\,dr\,d\theta$

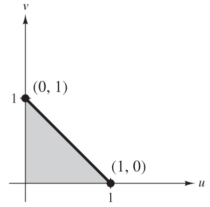
47. (a) r constant: right circular cylinder about z-axis
θ constant: plane parallel to z-axis
z constant: plane parallel to xy-plane
(b) ρ constant: sphere
θ constant: plane parallel to z-axis
ϕ constant: cone

49. $\frac{1}{2}\pi^2a^4$ **51.** Putnam Problem A1, 2006

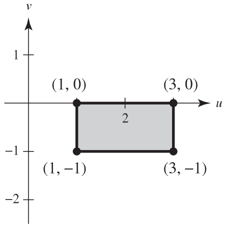
Section 14.8 (page 1050)

1. $-\frac{1}{2}$ **3.** $1 + 2v$ **5.** 1 **7.** $-e^{2u}$

9.



11.



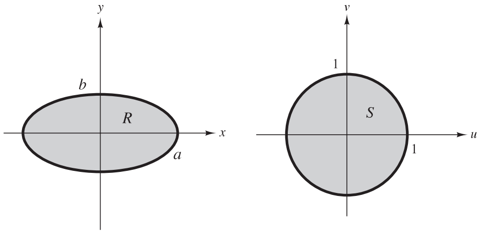
13. $\int\int_R 3xy\,dA = \int_{-2/3}^{2/3}\int_{1-x}^{(1/2)x+2} 3xy\,dy\,dx$

$+ \int_{2/3}^{4/3}\int_{(1/2)x}^{(1/2)x+2} 3xy\,dy\,dx + \int_{4/3}^{8/3}\int_{(1/2)x}^{4-x} 3xy\,dy\,dx = \frac{164}{9}$

15. $\frac{8}{3}$ **17.** 36 **19.** $(e^{-1/2} - e^{-2})\ln 8 \approx 0.9798$ **21.** 96

23. $12(e^4 - 1)$ **25.** $\frac{100}{9}$ **27.** $\frac{2}{5}a^{5/2}$ **29.** One

31. (a)



(b) ab (c) πab

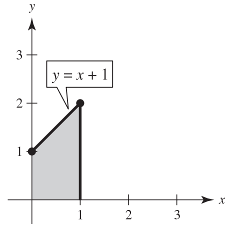
33. See "Definition of the Jacobian" on page 1045. **35.** u^2v

37. $-uv$ **39.** $-\rho^2\sin\phi$ **41.** Putnam Problem A2, 1994

Review Exercises for Chapter 14 (page 1052)

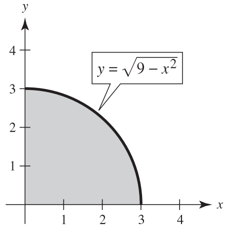
1. $x - x^3 + x^3\ln x^2$

3.



$\frac{29}{6}$

5.



36

7. $\int_0^3\int_0^{(3-x)/3} dy\,dx = \int_0^1\int_0^{3-3y} dx\,dy = \frac{3}{2}$

9. $\int_{-5}^{3}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} dy\,dx$

$= \int_{-5}^{-4}\int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx\,dy + \int_{-4}^{4}\int_{-\sqrt{25-y^2}}^{3} dx\,dy$

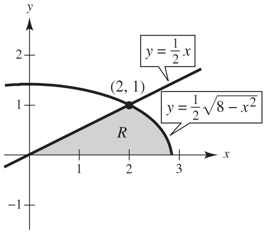
$+ \int_{4}^{5}\int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx\,dy$

$= 25\pi/2 + 12 + 25\arcsin\frac{3}{5} \approx 67.36$

11. $4\int_0^1\int_0^{x\sqrt{1-x^2}} dy\,dx = 4\int_0^{1/2}\int_{\sqrt{(1-\sqrt{1-4y^2})/2}}^{\sqrt{(1+\sqrt{1-4y^2})/2}} dx\,dy = \frac{4}{3}$

13. $\int_2^5\int_{x-3}^{\sqrt{x-1}} dy\,dx + 2\int_1^2\int_0^{\sqrt{x-1}} dy\,dx = \int_{-1}^2\int_{y^2+1}^{y+3} dx\,dy = \frac{9}{2}$

15. Both integrations are over the common region R, as shown in the figure. Both integrals yield $\frac{4}{3} + \frac{4}{3}\sqrt{2}$.



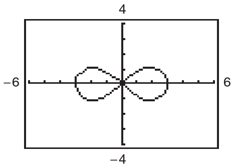
17. $\frac{3296}{15}$ **19.** $\frac{40}{3}$ **21.** 13.67°C **23.** $k = 1, 0.070$ **25.** c

27. True **29.** True **31.** $(h^3/6)\left[\ln\left(\sqrt{2} + 1\right) + \sqrt{2}\right]$

33. $9\pi/2$ **35.** $\pi h^3/3$

37. (a) $r = 3\sqrt{\cos 2\theta}$ (b) 9

(c) $3\left(3\pi - 16\sqrt{2} + 20\right) \approx 20.392$



(b) 9 (c) $3\left(3\pi - 16\sqrt{2} + 20\right) \approx 20.392$

39. (a) $m = k/4, \left(\frac{32}{45}, \frac{64}{55}\right)$ (b) $m = 17k/30, \left(\frac{936}{1309}, \frac{784}{663}\right)$

41. $I_x = ka^2b^3/6$

$I_y = ka^4b/4$

$I_0 = (2ka^2b^3 + 3ka^4b)/12$

$\bar{\bar{x}} = a/\sqrt{2}$

$\bar{\bar{y}} = b/\sqrt{3}$

43. $\dfrac{\left(101\sqrt{101} - 1\right)\pi}{6}$ **45.** $\frac{1}{6}\left(37\sqrt{37} - 1\right)$

47. (a) 30,415.74 ft^3 (b) 2081.53 ft^2 **49.** $324\pi/5$

51. $(abc/3)(a^2 + b^2 + c^2)$ **53.** $8\pi/15$ **55.** $\frac{32}{3}\left(\pi/2 - \frac{2}{3}\right)$

57. $\left(0, 0, \frac{1}{4}\right)$ **59.** $(3a/8, 3a/8, 3a/8)$ **61.** $833k\pi/3$

63. (a) $\dfrac{1}{3}\pi h^2(3a - h)$ (b) $\left(0, 0, \dfrac{3(2a - h)^2}{4(3a - h)}\right)$ (c) $\left(0, 0, \dfrac{3}{8}a\right)$

(d) a (e) $(\pi/30)h^3(20a^2 - 15ah + 3h^2)$ (f) $4\pi a^5/15$

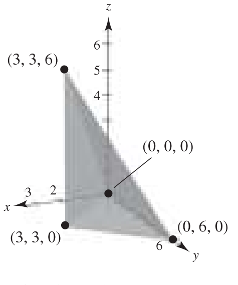
65. Volume of a torus formed by a circle of radius 3, centered at (0, 3, 0) and revolved about the z-axis

67. -9 **69.** $5 \ln 5 - 3 \ln 3 - 2 \approx 2.751$

P.S. Problem Solving (page 1055)

1. $8\left(2 - \sqrt{2}\right)$ **3.** (a)–(g) Proofs **5.** $\frac{1}{3}$

7.



$$\int_0^3\int_0^{2x}\int_x^{6-x} dy\,dz\,dx = 18$$

9. $\sqrt{\pi}/4$

11. If $a, k > 0$, then $1 = ka^2$ or $a = 1/\sqrt{k}$.

13. Answers will vary.

15. The greater the angle between the given plane and the xy-plane, the greater the surface area. So, $z_2 < z_1 < z_4 < z_3$.

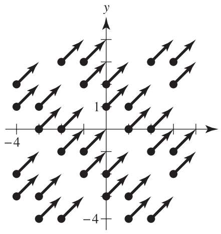
17. The results are not the same. Fubini's Theorem is not valid because f is not continuous on the region $0 \le x \le 1$, $0 \le y \le 1$.

Chapter 15

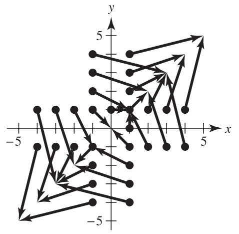
Section 15.1 (page 1067)

1. d **2.** c **3.** e **4.** b **5.** a **6.** f

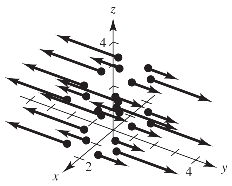
7. $\sqrt{2}$



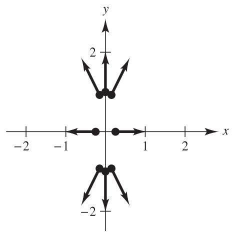
9. $\sqrt{x^2 + y^2}$



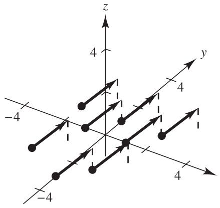
11. $3|y|$



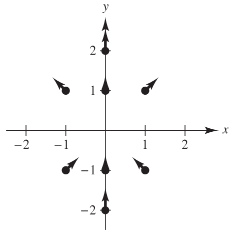
13. $\sqrt{16x^2 + y^2}$



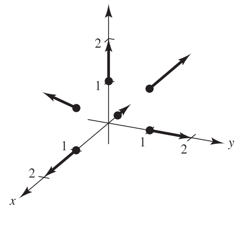
15. $\sqrt{3}$



17.



19.



21. $2x\mathbf{i} + 4y\mathbf{j}$

23. $(10x + 3y)\mathbf{i} + (3x + 2y)\mathbf{j}$ **25.** $6yz\mathbf{i} + 6xz\mathbf{j} + 6xy\mathbf{k}$

27. $2xye^{x^2}\mathbf{i} + e^{x^2}\mathbf{j} + \mathbf{k}$

29. $[xy/(x + y) + y\ln(x + y)]\mathbf{i} + [xy/(x + y) + x\ln(x + y)]\mathbf{j}$

31–33. Proofs **35.** Conservative because $\partial N/\partial x = \partial M/\partial y$.

37. Not conservative because $\partial N/\partial x \neq \partial M/\partial y$.

39. Conservative: $f(x, y) = xy + K$

41. Conservative: $f(x, y) = x^2y + K$

43. Not conservative **45.** Not conservative

47. Conservative: $f(x, y) = e^x \cos y + K$ **49.** $4\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

51. $-2\mathbf{k}$ **53.** $2x/(x^2 + y^2)\mathbf{k}$

55. $\cos(y - z)\mathbf{i} + \cos(z - x)\mathbf{j} + \cos(x - y)\mathbf{k}$

57. Conservative: $f(x, y, z) = \frac{1}{2}(x^2y^2z^2) + K$

59. Not conservative **61.** Conservative: $f(x, y, z) = xz/y + K$

63. $2x + 4y$ **65.** $\cos x - \sin y + 2z$ **67.** 4 **69.** 0

71. See "Definition of Vector Field" on page 1058. Some physical examples of vector fields include velocity fields, gravitational fields, and electric force fields.

73. See "Definition of Curl of a Vector Field" on page 1064.

75. $9x\mathbf{j} - 2y\mathbf{k}$ **77.** $z\mathbf{j} + y\mathbf{k}$ **79.** $3z + 2x$ **81.** 0

83–89. Proofs

91.
$$\begin{aligned} f(x, y, z) &= \|\mathbf{F}(x, y, z)\| = \sqrt{x^2 + y^2 + z^2} \\ \ln f &= \frac{1}{2}\ln(x^2 + y^2 + z^2) \\ \nabla \ln f &= \frac{x}{x^2 + y^2 + z^2}\mathbf{i} + \frac{y}{x^2 + y^2 + z^2}\mathbf{j} + \frac{z}{x^2 + y^2 + z^2}\mathbf{k} \\ &= \frac{\mathbf{F}}{f^2} \end{aligned}$$

93.
$$\begin{aligned} f^n &= \|\mathbf{F}(x, y, z)\|^n = \left(\sqrt{x^2 + y^2 + z^2}\right)^n \\ \nabla f^n &= n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1}\left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}\right) \\ &= nf^{n-2}\mathbf{F} \end{aligned}$$

95. True

97. False. Curl f is meaningful only for vector fields, when direction is involved.

Section 15.2 (page 1079)

1. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t\mathbf{j}, & 0 \le t \le 1 \\ (2 - t)\mathbf{i} + \sqrt{2 - t}\,\mathbf{j}, & 1 \le t \le 2 \end{cases}$

3. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 3 \\ 3\mathbf{i} + (t - 3)\mathbf{j}, & 3 \le t \le 6 \\ (9 - t)\mathbf{i} + 3\mathbf{j}, & 6 \le t \le 9 \\ (12 - t)\mathbf{j}, & 9 \le t \le 12 \end{cases}$

5. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}$, $0 \le t \le 2\pi$ **7.** 20 **9.** $5\pi/2$

11. (a) C: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$ (b) $2\sqrt{2}/3$

13. (a) C: $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \le t \le \pi/2$ (b) $\pi/2$

15. (a) C: $\mathbf{r}(t) = t\mathbf{i}$, $0 \le t \le 1$ (b) $1/2$

17. (a) C: $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 1 \\ (2 - t)\mathbf{i} + (t - 1)\mathbf{j}, & 1 \le t \le 2 \\ (3 - t)\mathbf{j}, & 2 \le t \le 3 \end{cases}$

(b) $\frac{19}{6}(1 + \sqrt{2})$

19. (a) C: $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 1 \\ \mathbf{i} + t\mathbf{k}, & 0 \le t \le 1 \\ \mathbf{i} + t\mathbf{j} + \mathbf{k}, & 0 \le t \le 1 \end{cases}$ (b) $\frac{23}{6}$

21. $8\sqrt{5}\pi(1 + 4\pi^2/3) \approx 795.7$ **23.** 2

25. $(k/12)(41\sqrt{41} - 27)$ **27.** 1 **29.** $\frac{1}{2}$ **31.** $\frac{9}{4}$

33. About 249.49 **35.** 66 **37.** 0 **39.** $-10\pi^2$

41. Positive **43.** Zero

45. (a) $\frac{236}{3}$; Orientation is from left to right, so the value is positive.

(b) $-\frac{236}{3}$; Orientation is from right to left, so the value is negative.

47.
$$\begin{aligned} &\mathbf{F}(t) = -2t\mathbf{i} - t\mathbf{j} \\ &\mathbf{r}'(t) = \mathbf{i} - 2\mathbf{j} \\ &\mathbf{F}(t) \cdot \mathbf{r}'(t) = -2t + 2t = 0 \\ &\int_C \mathbf{F} \cdot d\mathbf{r} = 0 \end{aligned}$$

49.
$$\begin{aligned} &\mathbf{F}(t) = (t^3 - 2t^2)\mathbf{i} + (t - t^2/2)\mathbf{j} \\ &\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} \\ &\mathbf{F}(t) \cdot \mathbf{r}'(t) = t^3 - 2t^2 + 2t^2 - t^3 = 0 \\ &\int_C \mathbf{F} \cdot d\mathbf{r} = 0 \end{aligned}$$

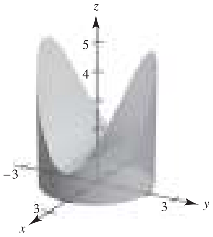
51. 1010 **53.** $\frac{190}{3}$ **55.** 25 **57.** $\frac{63}{2}$ **59.** $-\frac{11}{6}$ **61.** $\frac{316}{3}$

63. $5h$ **65.** $\frac{1}{2}$ **67.** $(h/4)\left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right]$

69. $\frac{1}{120}\left(25\sqrt{5} - 11\right)$

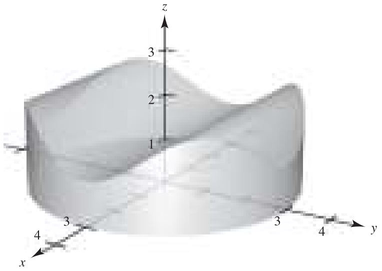
71. (a) $12\pi \approx 37.70$ cm² (b) $12\pi/5 \approx 7.54$ cm³

(c)



73. $I_x = I_y = a^3\pi$

75. (a)



(b) 9π cm² ≈ 28.274 cm²

(c) $$\text{Volume} = 2\int_0^3 2\sqrt{9 - y^2}\left[1 + 4\frac{y^2}{9}\left(1 - \frac{y^2}{9}\right)\right] dy$$
$$= 27\pi/2 \approx 42.412 \text{ cm}^3$$

77. 1750 ft-lb

79. See "Definition of Line Integral" on page 1070 and Theorem 15.4, "Evaluation of a Line Integral as a Definite Integral" on page 1071.

81. z_3, z_1, z_2, z_4; The greater the height of the surface over the curve $y = \sqrt{x}$, the greater the lateral surface area.

83. False: $\displaystyle\int_C xy\,ds = \sqrt{2}\int_0^1 t^2\,dt$.

85. False: the orientations are different. **87.** -12

Section 15.3 (page 1090)

1. (a) $\displaystyle\int_0^1 (t^2 + 2t^4)\,dt = \frac{11}{15}$

(b) $\displaystyle\int_0^{\pi/2} (\sin^2\theta\cos\theta + 2\sin^4\theta\cos\theta)\,d\theta = \frac{11}{15}$

3. (a) $\displaystyle\int_0^{\pi/3} (\sec\theta\tan^2\theta - \sec^3\theta)\,d\theta \approx -1.317$

(b) $\displaystyle\int_0^3 \left[\frac{\sqrt{t}}{2\sqrt{t + 1}} - \frac{\sqrt{t + 1}}{2\sqrt{t}}\right] dt \approx -1.317$

5. Conservative **7.** Not conservative **9.** Conservative
11. (a) 1 (b) 1 **13.** (a) 0 (b) $-\frac{1}{3}$ (c) $-\frac{1}{2}$
15. (a) 64 (b) 0 (c) 0 (d) 0 **17.** (a) $\frac{64}{3}$ (b) $\frac{64}{3}$
19. (a) 32 (b) 32 **21.** (a) $\frac{2}{3}$ (b) $\frac{17}{6}$ **23.** (a) 0 (b) 0
25. 72 **27.** -1 **29.** 0 **31.** (a) 2 (b) 2 (c) 2
33. 11 **35.** 30,366 **37.** 0
39. (a) $d\mathbf{r} = (\mathbf{i} - \mathbf{j})\,dt \Longrightarrow \int_0^{50} 175\,dt = 8750$ ft-lb
(b) $d\mathbf{r} = \left(\mathbf{i} - \frac{1}{25}(50 - t)\mathbf{j}\right)dt \Longrightarrow 7\int_0^{50} (50 - t)\,dt$
$= 8750$ ft-lb
41. See Theorem 15.5, “Fundamental Theorem of Line Integrals,” on page 1084.
43. (a) 2π (b) 2π (c) -2π (d) 0
45. Yes, because the work required to get from point to point is independent of the path taken.
47. False. It would be true if **F** were conservative.
49. True **51.** Proof
53. (a) Proof (b) $-\pi$ (c) π
(d) -2π; does not contradict Theorem 15.7 because **F** is not continuous at $(0, 0)$ in R enclosed by C.
(e) $\nabla\left(\arctan\frac{x}{y}\right) = \frac{1/y}{1 + (x/y)^2}\,\mathbf{i} + \frac{-x/y^2}{1 + (x/y)^2}\,\mathbf{j}$

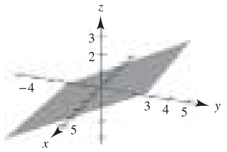
Section 15.4 (page 1099)

1. $\frac{1}{30}$ **3.** 0 **5.** About 19.99 **7.** $\frac{9}{2}$ **9.** 56 **11.** $\frac{4}{3}$ **13.** 0
15. 0 **17.** $\frac{1}{12}$ **19.** 32π **21.** π **23.** $\frac{225}{2}$ **25.** πa^2 **27.** $\frac{9}{2}$
29. See Theorem 15.8 on page 1093. **31.** Proof **33.** $\left(0, \frac{8}{5}\right)$
35. $\left(\frac{8}{15}, \frac{8}{21}\right)$ **37.** $3\pi a^2/2$ **39.** $\pi - 3\sqrt{3}/2$
41. (a) $51\pi/2$ (b) $243\pi/2$
43. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy = \int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = 0;$
$I = -2\pi$ when C is a circle that contains the origin.
45. $\frac{19}{2}$ **47–49.** Proofs

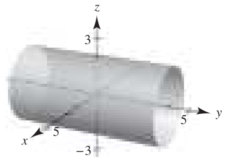
Section 15.5 (page 1109)

1. e **2.** f **3.** b **4.** a **5.** d **6.** c

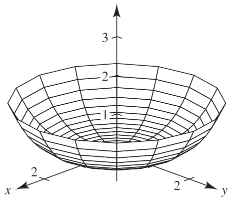
7. $y - 2z = 0$
Plane



9. $x^2 + z^2 = 4$
Cylinder



11.



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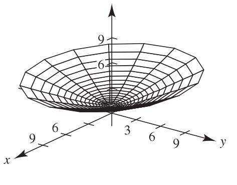

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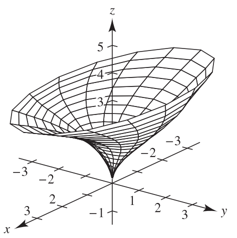
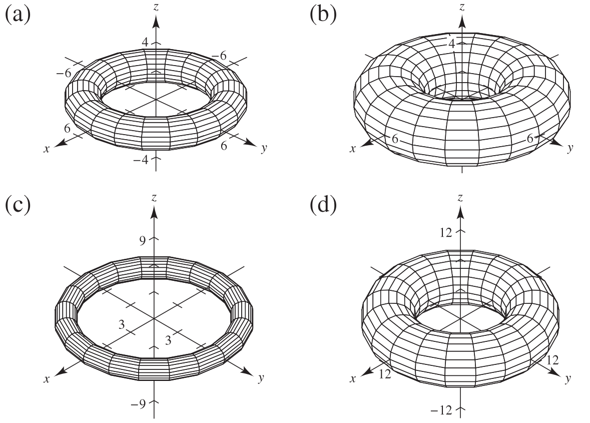
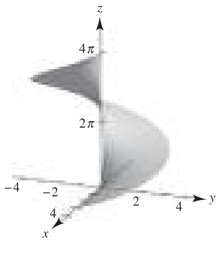


17. The paraboloid is reflected (inverted) through the xy-plane.
19. The height of the paraboloid is increased from 4 to 9.
21. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + v\mathbf{k}$
23. $\mathbf{r}(u, v) = \frac{1}{2}u\cos v\mathbf{i} + u\mathbf{j} + \frac{1}{3}u\sin v\mathbf{k},\ u \geq 0,\ 0 \leq v \leq 2\pi$ *or* $\mathbf{r}(x, y) = x\mathbf{i} + \sqrt{4x^2 + 9y^2}\mathbf{j} + z\mathbf{k}$
25. $\mathbf{r}(u, v) = 5\cos u\mathbf{i} + 5\sin u\mathbf{j} + v\mathbf{k}$
27. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + u^2\mathbf{k}$
29. $\mathbf{r}(u, v) = v\cos u\mathbf{i} + v\sin u\mathbf{j} + 4\mathbf{k},\quad 0 \leq v \leq 3$
31. $x = u,\ y = \frac{u}{2}\cos v,\ z = \frac{u}{2}\sin v,\quad 0 \leq u \leq 6,\ 0 \leq v \leq 2\pi$
33. $x = \sin u\cos v,\ y = \sin u\sin v,\ z = u$
$0 \leq u \leq \pi,\ 0 \leq v \leq 2\pi$
35. $x - y - 2z = 0$ **37.** $4y - 3z = 12$ **39.** $8\sqrt{2}$
41. $2\pi ab$ **43.** $\pi ab^2\sqrt{a^2 + 1}$
45. $(\pi/6)\left(17\sqrt{17} - 1\right) \approx 36.177$
47. See “Definition of Parametric Surface” on page 1102.
49–51. Proofs
53.



The radius of the generating circle that is revolved about the z-axis is b, and its center is a units from the axis of revolution.
55. 400π m²
57.

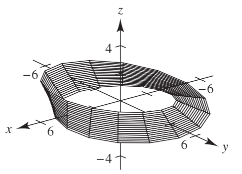


$2\pi\left[\frac{3}{2}\sqrt{13} + 2\ln\left(3 + \sqrt{13}\right) - 2\ln 2\right]$

59. Answers will vary. Sample answer: Let
$x = (2 - u)(5 + \cos v)\cos 3\pi u$
$y = (2 - u)(5 + \cos v)\sin 3\pi u$
$z = 5u + (2 - u)\sin v$
where $-\pi \le u \le \pi$ and $-\pi \le v \le \pi$.

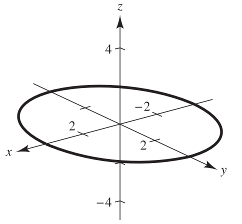
Section 15.6 (page 1122)

1. $12\sqrt{2}$ **3.** 2π **5.** $27\sqrt{3}/8$ **7.** $\left(391\sqrt{17} + 1\right)/240$
9. About -11.47 **11.** $\frac{364}{3}$ **13.** $12\sqrt{5}$ **15.** 8 **17.** $\sqrt{3}\pi$
19. $32\pi/3$ **21.** 486π **23.** $-\frac{4}{3}$ **25.** $3\pi/2$ **27.** 20π
29. 384π **31.** 0 **33.** Proof **35.** $2\pi a^3 h$ **37.** $64\pi\rho$
39. See Theorem 15.10, "Evaluating a Surface Integral," on page 1112.
41. See "Definition of Flux Integral," on page 1118; see Theorem 15.11, "Evaluating a Flux Integral," on page 1118.
43. (a)



(b) If a normal vector at a point P on the surface is moved around the Möbius strip once, it will point in the opposite direction.

(c)



Circle

(d) Construction
(e) A strip with a double twist that is twice as long as the Möbius strip.

Section 15.7 (page 1130)

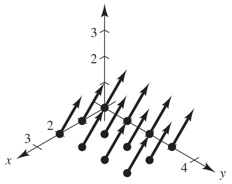
1. a^4 **3.** 18 **5.** π **7.** $3a^4$ **9.** 0 **11.** 108π
13. 0 **15.** 2304 **17.** $1024\pi/3$ **19.** 0
21. See Theorem 15.12, "The Divergence Theorem," on page 1124.
23–29. Proofs

Section 15.8 (page 1137)

1. $(xz - e^z)\mathbf{i} - (yz + 1)\mathbf{j} - 2\mathbf{k}$ **3.** $[2 - 1/(1 + x^2)]\mathbf{j} - 8x\mathbf{k}$
5. $z(x - 2e^{y^2+z^2})\mathbf{i} - yz\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k}$ **7.** 18π **9.** 0
11. -12 **13.** 2π **15.** 0 **17.** $\frac{8}{3}$ **19.** $a^5/4$ **21.** 0
23. See Theorem 15.13, "Stokes's Theorem," on page 1132.
25–27. Proofs **29.** Putnam Problem A5, 1987

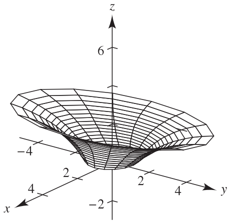
Review Exercises for Chapter 15 (page 1138)

1. $\sqrt{x^2 + 5}$

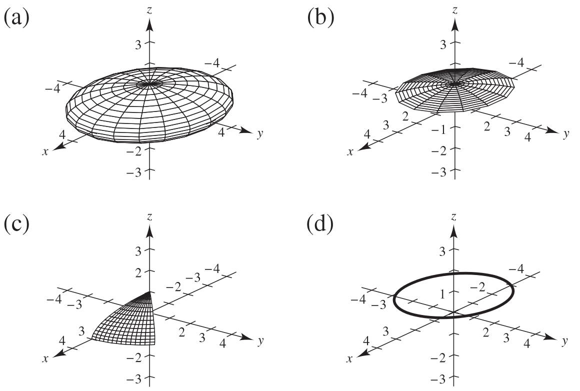


3. $(4x + y)\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$

5. Conservative: $f(x, y) = y/x + K$
7. Conservative: $f(x, y) = \frac{1}{2}x^2y^2 - \frac{1}{3}x^3 + \frac{1}{3}y^3 + K$
9. Not conservative **11.** Conservative: $f(x, y, z) = x/(yz) + K$
13. (a) $\operatorname{div}\mathbf{F} = 2x + 2xy + x^2$ (b) $\operatorname{curl}\mathbf{F} = -2xz\mathbf{j} + y^2\mathbf{k}$
15. (a) $\operatorname{div}\mathbf{F} = -y\sin x - x\cos y + xy$
(b) $\operatorname{curl}\mathbf{F} = xz\mathbf{i} - yz\mathbf{j}$
17. (a) $\operatorname{div}\mathbf{F} = 1/\sqrt{1 - x^2} + 2xy + 2yz$
(b) $\operatorname{curl}\mathbf{F} = z^2\mathbf{i} + y^2\mathbf{k}$
19. (a) $\operatorname{div}\mathbf{F} = \dfrac{2x + 2y}{x^2 + y^2} + 1$
(b) $\operatorname{curl}\mathbf{F} = \dfrac{2x - 2y}{x^2 + y^2}\mathbf{k}$
21. (a) $\frac{125}{3}$ (b) 2π **23.** 6π **25.** (a) 18 (b) 18π
27. $9a^2/5$ **29.** $\left(\sqrt{5}/3\right)(19 - \cos 6) \approx 13.446$ **31.** 1
33. $2\pi^2$ **35.** 36 **37.** $\frac{4}{3}$ **39.** $\frac{8}{3}\left(3 - 4\sqrt{2}\right) \approx -7.085$
41. 6 **43.** (a) 15 (b) 15 (c) 15
45. 1 **47.** 0 **49.** 0
51.

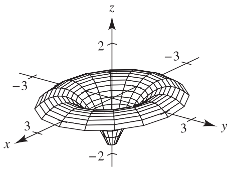


53.



Circle
(e) About 14.436 (f) About 4.269

55.



0

57. 66 **59.** $2a^6/5$ **61.** Proof

P.S. Problem Solving (page 1141)

1. (a) $\left(25\sqrt{2}/6\right)k\pi$ (b) $\left(25\sqrt{2}/6\right)k\pi$
3. $I_x = \left(\sqrt{13}\pi/3\right)(27 + 32\pi^2)$; $I_y = \left(\sqrt{13}\pi/3\right)(27 + 32\pi^2)$;
$I_z = 18\sqrt{13}\pi$
5. Proof **7.** $3a^2\pi$ **9.** (a) 1 (b) $\frac{13}{15}$ (c) $\frac{5}{2}$ **11.** Proof
13. $M = 3mxy(x^2 + y^2)^{-5/2}$
$\partial M/\partial y = 3mx(x^2 - 4y^2)/(x^2 + y^2)^{7/2}$
$N = m(2y^2 - x^2)(x^2 + y^2)^{-5/2}$
$\partial N/\partial x = 3mx(x^2 - 4y^2)/(x^2 + y^2)^{7/2}$
Therefore, $\partial N/\partial x = \partial M/\partial y$ and $\mathbf{F}$ is conservative.

Index

A

Abel, Niels Henrik (1802–1829), 232
Absolute convergence, 636
Absolute maximum of a function, 164
  of two variables, 954
Absolute minimum of a function, 164
  of two variables, 954
Absolute value, 50
  derivative involving, 330
  function, 22
Absolute Value Theorem, 600
Absolute zero, 74
Absolutely convergent series, 636
Acceleration, 125, 851, 875
  centripetal component of, 863
  tangential and normal components of, 863, 877
  vector, 862, 877
Accumulation function, 288
Addition of vectors, 766, 777
Additive Identity Property of Vectors, 767
Additive Interval Property, 276
Additive Inverse Property of Vectors, 767
Agnesi, Maria Gaetana (1718–1799), 201
d'Alembert, Jean Le Rond (1717–1783), 908
Algebraic function(s), 24, 25, 378
  derivatives of, 136
Algebraic properties of the cross product, 793
Alternating series, 633
  geometric, 633
  harmonic, 634, 636, 638
Alternating Series Remainder, 635
Alternating Series Test, 633
Alternative form
  of the derivative, 101, A7
  of the directional derivative, 936
  of Green's Theorem, 1098, 1099
  of Log Rule for Integration, 334
  of Mean Value Theorem, 175
Angle
  between two nonzero vectors, 784
  between two planes, 802
  of incidence, 698
  of inclination of a plane, 949
  of reflection, 698
Angular speed, 1017
Antiderivative, 248
  of f with respect to x, 249
  finding by integration by parts, 527
  general, 249
  notation for, 249
  representation of, 248
  of a vector-valued function, 846
Antidifferentiation, 249
  of a composite function, 297
Aphelion, 708, 757
Apogee, 708
Applied minimum and maximum problems, guidelines for solving, 219
Approximating zeros
  bisection method, 78
  Intermediate Value Theorem, 77
  Newton's Method, 229
Approximation,
  linear, 235, 920
  Padé, 333
  polynomial, 650
  Stirling's, 529
  tangent line, 235
  Two-point Gaussian Quadrature, 321
Arc length, 478, 479, 870
  derivative of, 870
  parameter, 870, 871
  in parametric form, 724
  of a polar curve, 745
  of a space curve, 869
  in the xy-plane, 1021
Arccosecant function, 373
Arccosine function, 373
Arccotangent function, 373
Archimedes (287–212 B.C.), 261
  Principle, 518
  spiral of, 725, 733, 749
Arcsecant function, 373
Arcsine function, 373
  series for, 684
Arctangent function, 373
  series for, 684
Area
  found by exhaustion method, 261
  line integral for, 1096
  of a parametric surface, 1106
  in polar coordinates, 741
  problem, 45, 46
  of a rectangle, 261
  of a region between two curves, 449
  of a region in the plane, 265
  of a surface of revolution, 483
    in parametric form, 726
    in polar coordinates, 746
  of the surface S, 1021
  in the xy-plane, 1021
Associative Property of Vector Addition, 767
Astroid, 146
Astroidal sphere, 1111
Asymptote(s)
  horizontal, 199
  of a hyperbola, 703
  slant, 211
  vertical, 84, 85, A7
Average rate of change, 12
Average value of a function
  on an interval, 286
  over a region R, 999
  over a solid region Q, 1037
Average velocity, 113
Axis
  conjugate, of a hyperbola, 703
  major, of an ellipse, 699
  minor, of an ellipse, 699
  of a parabola, 697
  polar, 731
  of revolution, 458
  transverse, of a hyperbola, 703

B

Barrow, Isaac (1630–1677), 145
Base(s), 327, 362
  of the natural exponential function, 362
  of a natural logarithm, 327
  other than e,
    derivatives for, 364
    exponential function, 362
    logarithmic function, 363
Basic differentiation rules for elementary functions, 378
Basic equation obtained in a partial fraction decomposition, 556
  guidelines for solving, 560
Basic integration rules, 250, 385, 522
  procedures for fitting integrands to, 523
Basic limits, 59
Basic types of transformations, 23
Bearing, 770
Bernoulli equation, 438
  general solution of, 439
Bernoulli, James (1654–1705), 717
Bernoulli, John (1667–1748), 554
Bessel function, 669, 670
Bifolium, 146
Binomial series, 683
Bisection method, 78
Bose-Einstein condensate, 74
Boundary point of a region R, 898
Bounded
  above, 603
  below, 603
  monotonic sequence, 603
  region R, 954
  sequence, 603

- Brachistochrone problem, 717
- Breteuil, Emilie de (1706–1749), 490
- Bullet-nose curve, 138

C

- Cantor set, 693
- Capillary action, 1026
- Cardioid, 736, 737
- Carrying capacity, 427, 429
- Catenary, 393
- Cauchy, Augustin-Louis (1789–1857), 75
- Cauchy-Riemann differential equations, 932
- Cauchy-Schwarz Inequality, 791
- Cavalieri's Theorem, 468
- Center
  - of curvature, 874
  - of an ellipse, 699
  - of gravity, 500, 501
    - of a one-dimensional system, 500
    - of a two-dimensional system, 501
  - of a hyperbola, 703
  - of mass, 499, 500, 501
    - of a one-dimensional system, 499, 500
    - of a planar lamina, 502
      - of variable density, 1014
    - of a solid region Q, 1032
    - of a two-dimensional system, 501
  - of a power series, 661
- Centered at c, 650
- Central force field, 1059
- Centripetal component of acceleration, 863
- Centripetal force, 868
- Centroid, 503
  - of a simple region, 1014
- Chain Rule, 130, 131, 136, A8
  - implicit differentiation, 930
  - one independent variable, 925, A20
  - three or more independent variables, 928
  - and trigonometric functions, 135
  - two independent variables, 925
- Change in x, 97
- Change in y, 97
- Change of variables, 300
  - for definite integrals, 303
  - for double integrals, 1047
  - guidelines for making, 301
  - for homogeneous equations, 426
  - to polar form, 1006
  - using a Jacobian, 1045
- Charles, Jacques (1746–1823), 74
- Charles's Law, 74
- Circle, 146, 696, 737
- Circle of curvature, 161, 874
- Circulation of **F** around C_α, 1135
- Circumscribed rectangle, 263
- Cissoid, 146
  - of Diocles, 761
- Classification of conics by eccentricity, 750, A19
- Closed
  - curve, 1088
  - disk, 898
  - region R, 898
  - surface, 1124
- Cobb-Douglas production function, 891
- Coefficient, 24
  - correlation, 31
  - leading, 24
- Collinear, 17
- Combinations of functions, 25
- Common logarithmic function, 363
- Common types of behavior associated with nonexistence of a limit, 51
- Commutative Property
  - of the dot product, 783
  - of vector addition, 767
- Comparison Test
  - Direct, 626
  - for improper integrals, 588
  - Limit, 628
- Completeness, 77, 603
- Completing the square, 383
- Component of acceleration
  - centripetal, 863
  - normal, 863, 877
  - tangential, 863, 877
- Component form of a vector in the plane, 765
- Component functions, 834
- Components of a vector, 787
  - along $\mathbf{v}$, 787
  - in the direction of $\mathbf{v}$, 788
  - orthogonal to $\mathbf{v}$, 787
  - in the plane, 765
- Composite function, 25
  - antidifferentiation of, 297
  - continuity of, 75
  - derivative of, 130
  - limit of, 61, A4
  - of two variables, 887
    - continuity of, 903
- Composition of functions, 25, 887
- Compound interest formulas, 366
- Compounding, continuous, 366
- Computer graphics, 892
- Concave downward, 190, A9
- Concave upward, 190, A9
- Concavity, 190, A9
  - test for, 191, A9
- Conditional convergence, 636
- Conditionally convergent series, 636
- Conic(s), 696
  - circle, 696
  - classification by eccentricity, 750, A19
  - degenerate, 696
  - directrix of, 750
  - eccentricity, 750
  - ellipse, 696, 699
  - focus of, 750
  - hyperbola, 696, 703
  - parabola, 696, 697
  - polar equations of, 751
- Conic section, 696
- Conjugate axis of a hyperbola, 703
- Connected region, 1086
- Conservative vector field, 1061, 1083
  - independence of path, 1086
  - test for, 1062, 1065
- Constant
  - Euler's, 625
  - force, 489
  - function, 24
  - of integration, 249
  - Multiple Rule, 110, 136
    - differential form, 238
  - Rule, 107, 136
  - spring, 34
  - term of a polynomial function, 24
- Constraint, 970
- Continued fraction expansion, 693
- Continuity
  - on a closed interval, 73
  - of a composite function, 75
    - of two variables, 903
  - differentiability implies, 103
  - and differentiability of inverse functions, 347, A13
  - implies integrability, 273
  - properties of, 75, A6
  - of a vector-valued function, 838
- Continuous, 70
  - at c, 59, 70
  - on the closed interval $[a, b]$, 73
  - compounding, 366
  - everywhere, 70
  - function of two variables, 900
  - on an interval, 838
  - from the left and from the right, 73
  - on an open interval (a, b), 70
  - in the open region R, 900, 904
  - at a point, 838, 902, 904
  - vector field, 1058
- Continuously differentiable, 478
- Contour lines, 889
- Converge, 231, 597, 608
- Convergence
  - absolute, 636
  - conditional, 636
  - endpoint, 664
  - of a geometric series, 610
  - of improper integral with infinite discontinuities, 583
  - integration limits, 580
  - interval of, 662, 666, A18

  of Newton's Method, 231, 232
  of a power series, 662, A18
  of *p*-series, 621
  radius of, 662, 666, A18
  of a sequence, 597
  of a series, 608
  of Taylor series, 680
  tests for series
    Alternating Series Test, 633
    Direct Comparison Test, 626
    geometric series, 610
    guidelines, 645
    Integral Test, 619
    Limit Comparison Test, 628
    p-series, 621
    Ratio Test, 641
    Root Test, 644
    summary, 646
Convergent power series, form of, 678
Convergent series, limit of *n*th term of, 612
Convex limaçon, 737
Coordinate conversion
  cylindrical to rectangular, 822
  cylindrical to spherical, 825
  polar to rectangular, 732
  rectangular to cylindrical, 822
  rectangular to polar, 732
  rectangular to spherical, 825
  spherical to cylindrical, 825
  spherical to rectangular, 825
Coordinate planes, 775
  xy-plane, 775
  xz-plane, 775
  yz-plane, 775
Coordinate system
  cylindrical, 822
  polar, 731
  spherical, 825
  three-dimensional, 775
Coordinates, polar, 731
  area in, 741
  area of a surface of revolution in, 746
  converting to rectangular, 732
  Distance Formula in, 739
Coordinates, rectangular, converting to polar, 732
Copernicus, Nicolaus (1473–1543), 699
Cornu spiral, 761, 883
Correlation coefficient, 31
Cosecant function
  derivative of, 123, 136
  integral of, 339
  inverse of, 373
    derivative of, 376
Cosine function, 22
  derivative of, 112, 136
  integral of, 339
  inverse of, 373
    derivative of, 376, A15
  series for, 684
Cotangent function
  derivative of, 123, 136
  integral of, 339
  inverse of, 373
    derivative of, 376
Coulomb's Law, 491, 1059
Critical number(s)
  of a function, 166
  relative extrema occur only at, 166
Critical point(s)
  of a function of two variables, 955
  relative extrema occur only at, 955
Cross product of two vectors in space, 792
  algebraic properties of, 793
  determinant form, 792
  geometric properties of, 794
  torque, 796
Cruciform, 146
Cubic function, 24
Cubing function, 22
Curl of a vector field, 1064
  and divergence, 1066
Curtate cycloid, 719
Curvature, 872
  center of, 874
  circle of, 161, 874
  formulas for, 873, 877
  radius of, 874
  in rectangular coordinates, 874, 877
  related to acceleration and speed, 875
Curve
  astroid, 146
  bifolium, 146
  bullet-nose, 138
  cissoid, 146
  closed, 1088
  cruciform, 146
  equipotential, 428
  folium of Descartes, 146, 749
  isothermal, 428
  kappa, 145, 147
  lateral surface area over, 1081
  lemniscate, 40, 144, 147, 737
  level, 889
  logistic, 429, 562
  natural equation for, 883
  orientation of, 1069
  piecewise smooth, 716, 1069
  plane, 711, 834
  pursuit, 395, 397
  rectifiable, 478
  rose, 734, 737
  simple, 1093
  smooth, 478, 716, 844, 859, 1069
    piecewise, 716, 1069
  space, 834
  tangent line to, 860
Curve sketching, summary of, 209
Cusps, 844
Cycloid, 716, 720
  curtate, 719
  prolate, 723
Cylinder, 812
  directrix of, 812
  equations of, 812
  generating curve of, 812
  right, 812
  rulings of, 812
Cylindrical coordinate system, 822
  pole of, 822
Cylindrical coordinates
  converting to rectangular, 822
  converting to spherical, 825
Cylindrical surface, 812

D

Darboux's Theorem, 245
Decay model, exponential, 416
Decomposition of $N(x)/D(x)$ into partial fractions, 555
Decreasing function, 179
  test for, 179
Definite integral(s), 273
  approximating
    Midpoint Rule, 269, 313
    Simpson's Rule, 314
    Trapezoidal Rule, 312
  as the area of a region, 274
  change of variables, 303
  evaluation of a line integral as a, 1071
  properties of, 277
  two special, 276
  of a vector-valued function, 846
Degenerate conic, 696
  line, 696
  point, 696
  two intersecting lines, 696
Degree of a polynomial function, 24
Delta, δ, δ-neighborhood, 898
Demand, 18
Density, 502
Density function ρ, 1012, 1032
Dependent variable, 19
  of a function of two variables, 886
Derivative(s)
  of algebraic functions, 136
  alternative form, 101, A7
  of arc length function, 870
  Chain Rule, 130, 131, 136, A8
    implicit differentiation, 930
    one independent variable, 925
    three or more independent variables, 928

    - two independent variables, 925
  - of a composite function, 130
  - Constant Multiple Rule, 110, 136
  - Constant Rule, 107, 136
  - of cosecant function, 123, 136
  - of cosine function, 112, 136
  - of cotangent function, 123, 136
  - Difference Rule, 111, 136
  - directional, 933, 934, 941
  - of an exponential function, base a, 364
  - of a function, 99
  - General Power Rule, 132, 136
  - higher-order, 125
  - of hyperbolic functions, 392
  - implicit, 142
  - of an inverse function, 347, A14
  - of inverse trigonometric functions, 376, A15
  - involving absolute value, 330
  - from the left and from the right, 101
  - of a logarithmic function, base a, 364
  - of the natural exponential function, 354
  - of the natural logarithmic function, 328
  - notation, 99
  - parametric form, 721
  - partial, 908
  - Power Rule, 108, 136
  - Product Rule, 119, 136
  - Quotient Rule, 121, 136
  - of secant function, 123, 136
  - second, 125
  - Simple Power Rule, 108, 136
  - simplifying, 134
  - of sine function, 112, 136
  - Sum Rule, 111, 136
  - of tangent function, 123, 136
  - third, 125
  - of trigonometric functions, 123, 136
  - of a vector-valued function, 842
    - higher-order, 843
    - properties of, 844
- Descartes, René (1596–1650), 2
- Determinant form of cross product, 792
- Difference quotient, 20, 97
- Difference Rule, 111, 136
  - differential form, 238
- Difference of two functions, 25
- Difference of two vectors, 766
- Differentiability
  - implies continuity, 103, 921
  - and continuity of inverse functions, 347, A13
  - sufficient condition for, 919, A19
- Differentiable at x, 99
- Differentiable, continuously, 478
- Differentiable function
  - on the closed interval $[a, b]$, 101
  - on an open interval (a, b), 99
  - in a region R, 919
  - of three variables, 920
  - of two variables, 919
  - vector-valued, 842
- Differential, 236
  - as an approximation, 920
  - function of three or more variables, 918
  - function of three variables, 920
  - function of two variables, 918
  - of x, 236
  - of y, 236
- Differential equation, 249, 406
  - Bernoulli equation, 438
  - Cauchy-Riemann, 932
  - doomsday, 445
  - Euler's Method, 410
  - first-order linear, 434, 440
  - general solution of, 249, 406
  - Gompertz, 445
  - homogeneous, 425
    - change of variables, 426
  - initial condition, 253, 407
  - integrating factor, 434
  - logistic, 245, 429
  - order of, 406
  - particular solution of, 253, 407
  - separable, 423
  - separation of variables, 415, 423
  - singular solution of, 406
  - solution of, 406
  - summary of first-order, 440
- Differential form, 236
  - of a line integral, 1077
- Differential formulas, 238
  - constant multiple, 238
  - product, 238
  - quotient, 238
  - sum or difference, 238
- Differential operator, 1064, 1066
  - Laplacian, 1141
- Differentiation, 99
  - basic rules for elementary functions, 378
  - implicit, 141
    - Chain Rule, 930
    - guidelines for, 142
  - involving inverse hyperbolic functions, 396
  - logarithmic, 329
  - numerical, 103
  - partial, 908
  - of power series, 666
  - of a vector-valued function, 843
- Differentiation rules
  - basic, 378
  - Chain, 130, 131, 136, A8
  - Constant, 107, 136
  - Constant Multiple, 110, 136
  - cosecant function, 123, 136
  - cosine function, 112, 136
  - cotangent function, 123, 136
  - Difference, 111, 136
  - general, 136
  - General Power, 132, 136
  - Power, 108, 136
    - for Real Exponents, 365
  - Product, 119, 136
  - Quotient, 121, 136
  - secant function, 123, 136
  - Simple Power, 108, 136
  - sine function, 112, 136
  - Sum, 111, 136
  - summary of, 136
  - tangent function, 123, 136
- Diminishing returns, point of, 227
- Dimpled limaçon, 737
- Direct Comparison Test, 626
- Direct substitution, 59, 60
- Directed distance, 501
- Directed line segment, 764
  - equivalent, 764
  - initial point of, 764
  - length of, 764
  - magnitude of, 764
  - terminal point of, 764
- Direction angles of a vector, 786
- Direction cosines of a vector, 786
- Direction field, 256, 325, 408
- Direction of motion, 850
- Direction numbers, 800
- Direction vector, 800
- Directional derivative, 933, 934
  - alternative form of, 936
  - of f in the direction of **u**, 934, 941
  - of a function in three variables, 941
- Directrix
  - of a conic, 750
  - of a cylinder, 812
  - of a parabola, 697
- Dirichlet, Peter Gustav (1805–1859), 51
- Dirichlet function, 51
- Discontinuity, 71
  - infinite, 580
  - nonremovable, 71
  - removable, 71
- Disk, 458, 898
  - closed, 898
  - method, 459
    - compared to shell, 471
  - open, 898
- Displacement of a particle, 291, 292
- Distance
  - between a point and a line in space, 806
  - between a point and a plane, 805
  - directed, 501
  - total, traveled on $[a, b]$, 292
- Distance Formula
  - in polar coordinates, 739

  - in space, 776
- Distributive Property
  - for the dot product, 783
  - for vectors, 767
- Diverge, 597, 608
- Divergence
  - of improper integral with infinite
    - discontinuities, 583
    - integration limits, 580
  - of a sequence, 597
  - of a series, 608
  - tests for series
    - Direct Comparison Test, 626
    - geometric series, 610
    - guidelines, 645
    - Integral Test, 619
    - Limit Comparison Test, 628
    - nth-Term Test, 612
    - p-series, 621
    - Ratio Test, 641
    - Root Test, 644
    - summary, 646
  - of a vector field, 1066
    - and curl, 1066
- Divergence Theorem, 1099, 1124
- Divergence-free vector field, 1066
- Divide out like factors, 63
- Domain
  - feasible, 218
  - of a function, 19
    - explicitly defined, 21
    - of two variables, 886
  - implied, 21
  - of a power series, 662
  - of a vector-valued function, 835
- Doomsday equation, 445
- Dot product
  - Commutative Property of, 783
  - Distributive Property for, 783
  - form of work, 789
  - projection using the, 788
  - properties of, 783
  - of two vectors, 783
- Double integral, 992, 993, 994
  - change of variables for, 1047
  - of f over R, 994
  - properties of, 994
- Doyle Log Rule, 896
- Dummy variable, 275
- Dyne, 489

E

- e, the number, 327
  - limit involving, 366, A15
- Eccentricity, 750, A19
  - classification of conics by, 750, A19
  - of an ellipse, 701
  - of a hyperbola, 704
- Eight curve, 161
- Electric force field, 1059
- Elementary function(s), 24, 378
  - basic differentiation rules for, 378
  - polynomial approximation of, 650
  - power series for, 684
- Eliminating the parameter, 713
- Ellipse, 696, 699
  - center of, 699
  - eccentricity of, 701
  - foci of, 699
  - major axis of, 699
  - minor axis of, 699
  - reflective property of, 701
  - rotated, 146
  - standard equation of, 699
  - vertices of, 699
- Ellipsoid, 813, 814
- Elliptic cone, 813, 815
- Elliptic integral, 317
- Elliptic paraboloid, 813, 815
- Endpoint convergence, 664
- Endpoint extrema, 164
- Energy
  - kinetic, 1089
  - potential, 1089
- Epicycloid, 719, 720, 724
- Epsilon-delta, ε-δ, definition of limit, 52
- Equal vectors, 765, 777
- Equality of mixed partial derivatives, 912
- Equation(s)
  - basic, 556
    - guidelines for solving, 560
  - Bernoulli, 438
  - of conics, polar, 751
  - of a cylinder, 812
  - doomsday, 445
  - of an ellipse, 699
  - general second-degree, 696
  - Gompertz, 445
  - graph of, 2
  - harmonic, 1141
  - of a hyperbola, 703
  - Laplace's, 1141
  - of a line
    - general form, 14
    - horizontal, 14
    - point-slope form, 11, 14
    - slope-intercept form, 13, 14
    - in space, parametric, 800
    - in space, symmetric, 800
    - summary, 14
    - vertical, 14
  - of a parabola, 697
  - parametric, 711, 1102
    - finding, 715
    - graph of, 711
  - of a plane in space
    - general form, 801
    - standard form, 801
  - primary, 218, 219
  - related-rate, 149
  - secondary, 219
  - separable, 423
  - solution point of, 2
  - of tangent plane, 946
- Equilibrium, 499
- Equipotential
  - curves, 428
  - lines, 889
- Equivalent
  - conditions, 1088
  - directed line segments, 764
- Error
  - in approximating a Taylor polynomial, 656
  - in measurement, 237
    - percent error, 237
    - propagated error, 237
    - relative error, 237
  - in Simpson's Rule, 315
  - in Trapezoidal Rule, 315
- Escape velocity, 94
- Euler, Leonhard (1707–1783), 24
- Euler's
  - constant, 625
  - Method, 410
- Evaluate a function, 19
- Evaluating
  - a flux integral, 1118
  - a surface integral, 1112
- Evaluation
  - by iterated integrals, 1028
  - of a line integral as a definite integral, 1071
- Even function, 26
  - integration of, 305
  - test for, 26
- Everywhere continuous, 70
- Evolute, 879
- Existence
  - of an inverse function, 345
  - of a limit, 73
  - theorem, 77, 164
- Expanded about c, approximating
  - polynomial, 650
- Explicit form of a function, 19, 141
- Explicitly defined domain, 21
- Exponential decay, 416
- Exponential function, 24
  - to base a, 362
    - derivative of, 364
  - integration rules, 356
  - natural, 352
    - derivative of, 354
    - properties of, 353
  - operations with, 353, A15

    series for, 684
Exponential growth and decay model, 416
    initial value, 416
    proportionality constant, 416
Exponentiate, 353
Extended Mean Value Theorem, 245, 570, A16
Extrema
    endpoint, 164
    of a function, 164, 954
    guidelines for finding, 167
    relative, 165
Extreme Value Theorem, 164, 954
Extreme values of a function, 164

F

Factorial, 599
Family of functions, 273
Famous curves
    astroid, 146
    bifolium, 146
    bullet-nose curve, 138
    circle, 146, 696, 737
    cissoid, 146
    cruciform, 146
    eight curve, 161
    folium of Descartes, 146, 749
    kappa curve, 145, 147
    lemniscate, 40, 144, 147, 737
    parabola, 2, 146, 696, 697
    pear-shaped quartic, 161
    rotated ellipse, 146
    rotated hyperbola, 146
    serpentine, 127
    top half of circle, 138
    witch of Agnesi, 127, 146, 201, 841
Faraday, Michael (1791–1867), 1085
Feasible domain, 218
Fermat, Pierre de (1601–1665), 166
Fibonacci sequence, 606, 617
Field
    central force, 1059
    direction, 256, 325, 408
    electric force, 1059
    force, 1058
    gravitational, 1059
    inverse square, 1059
    slope, 256, 306, 325, 408
    vector, 1058
        over a plane region R, 1058
        over a solid region Q, 1058
    velocity, 1058, 1059
Finite Fourier series, 544
First Derivative Test, 181
First moments, 1016, 1032
First partial derivatives, 908
    notation for, 909
First-order differential equations
    linear, 434, 440
        solution of, 435
    summary of, 440
Fitting integrands to basic rules, 523
Fixed plane, 880
Fixed point, 233
Fluid(s)
    force, 510
    pressure, 509
    weight-densities of, 509
Flux integral, 1118
    evaluating, 1118
Focal chord of a parabola, 697
Focus
    of a conic, 750
    of an ellipse, 699
    of a hyperbola, 703
    of a parabola, 697
Folium of Descartes, 146, 749
Force, 489
    constant, 489
    exerted by a fluid, 510
    of friction, 876
    resultant, 770
    variable, 490
Force field, 1058
    central, 1059
    electric, 1059
    work, 1074
Form of a convergent power series, 678
Fourier, Joseph (1768–1830), 671
Fourier series, finite, 544
Fourier Sine Series, 535
Fraction expansion, continued, 693
Fractions, partial, 554
    decomposition of $N(x)/D(x)$, into, 555
    method of, 554
Frenet-Serret formulas, 884
Fresnel function, 321
Friction, 876
Fubini's Theorem, 996
    for a triple integral, 1028
Function(s), 6, 19
    absolute maximum of, 164
    absolute minimum of, 164
    absolute value, 22
    acceleration, 125
    accumulation, 288
    addition of, 25
    algebraic, 24, 25, 378
    antiderivative of, 248
    arc length, 478, 479, 870
    arccosecant, 373
    arccosine, 373
    arccotangent, 373
    arcsecant, 373
    arcsine, 373
    arctangent, 373
    average value of, 286, 999
    Bessel, 669, 670
    Cobb-Douglas production, 891
    combinations of, 25
    common logarithmic, 363
    component, 834
    composite, 25, 887
    composition of, 25, 887
    concave downward, 190, A9
    concave upward, 190, A9
    constant, 24
    continuous, 70
    continuously differentiable, 478
    cosine, 22
    critical number of, 166
    cubic, 24
    cubing, 22
    decreasing, 179
        test for, 179
    defined by power series, properties of, 666
    density, 1012, 1032
    derivative of, 99
    difference of, 25
    differentiable, 99, 101
    Dirichlet, 51
    domain of, 19
    elementary, 24, 378
        algebraic, 24, 25
        exponential, 24
        logarithmic, 24
        trigonometric, 24
    evaluate, 19
    even, 26
    explicit form, 19, 141
    exponential to base a, 362
    extrema of, 164
    extreme values of, 164
    family of, 273
    feasible domain of, 218
    Fresnel, 321
    Gamma, 578, 590
    global maximum of, 164
    global minimum of, 164
    graph of, guidelines for analyzing, 209
    greatest integer, 72
    Gudermannian, 404
    Heaviside, 39
    homogeneous, 425, 931
    hyperbolic, 390
    identity, 22
    implicit form, 19
    implicitly defined, 141
    increasing, 179
        test for, 179
    inner product of two, 544
    integrable, 273
    inverse, 343

inverse hyperbolic, 394
inverse trigonometric, 373
involving a radical, limit of, 60, A4
jerk, 162
limit of, 48
linear, 24
local extrema of, 165
local maximum of, 165
local minimum of, 165
logarithmic, 324
    to base *a*, 363
logistic growth, 367
natural exponential, 352
natural logarithmic, 324
notation, 19
odd, 26
one-to-one, 21
onto, 21
orthogonal, 544
point of inflection, 192, 193
polynomial, 24, 60, 887
position, 32, 113, 855
potential, 1061
product of, 25
pulse, 94
quadratic, 24
quotient of, 25
radius, 818
range of, 19
rational, 22, 25, 887
real-valued, 19
relative extrema of, 165, 954
relative maximum of, 165, 954
relative minimum of, 165, 954
representation by power series, 671
Riemann zeta, 625
signum, 82
sine, 22
sine integral, 322
square root, 22
squaring, 22
standard normal probability density, 355
step, 72
strictly monotonic, 180, 345
sum of, 25
that agree at all but one point, 62, A5
of three variables
    continuity of, 904
    directional derivative of, 941
    gradient of, 941
transcendental, 25, 378
transformation of a graph of, 23
    horizontal shift, 23
    reflection about origin, 23
    reflection about *x*-axis, 23
    reflection about *y*-axis, 23
    reflection in the line $y = x$, 344
    vertical shift, 23
trigonometric, 24
of two variables, 886
    absolute maximum of, 954
    absolute minimum of, 954
    continuity of, 902
    critical point of, 955
    dependent variable, 886
    differentiability implies continuity, 921
    differentiable, 919
    differential of, 918
    domain of, 886
    gradient of, 936
    graph of, 888
    independent variables, 886
    limit of, 899
    maximum of, 954
    minimum of, 954
    nonremovable discontinuity of, 900
    partial derivative of, 908
    range of, 886
    relative extrema of, 954
    relative maximum of, 954, 957
    relative minimum of, 954, 957
    removable discontinuity of, 900
    total differential of, 918
unit pulse, 94
vector-valued, 834
Vertical Line Test, 22
of *x* and *y*, 886
zero of, 26
    approximating with Newton's Method, 229
Fundamental Theorem
    of Algebra, 1124
    of Calculus, 282
        guidelines for using, 283
        Second, 289
    of Line Integrals, 1083, 1084

G

Gabriel's Horn, 586, 1104
Galilei, Galileo (1564–1642), 378
Galois, Evariste (1811–1832), 232
Gamma Function, 578, 590
Gauss, Carl Friedrich (1777–1855), 260, 1124
Gaussian Quadrature Approximation, two-point, 321
Gauss's Law, 1121
Gauss's Theorem, 1124
General antiderivative, 249
General differentiation rules, 136
General form
    of the equation of a line, 14
    of the equation of a plane in space, 801
    of the equation of a quadric surface, 813
    of a second-degree equation, 694
General harmonic series, 621
General partition, 272
General Power Rule
    for differentiation, 132, 136
    for Integration, 302
General second-degree equation, 696
General solution
    of the Bernoulli equation, 439
    of a differential equation, 249, 406
Generating curve of a cylinder, 812
Geometric power series, 671
Geometric properties of the cross product, 794
Geometric property of triple scalar product, 797
Geometric series, 610
    alternating, 633
    convergence of, 610
    divergence of, 610
Gibbs, Josiah Willard (1839–1903), 1069
Global maximum of a function, 164
Global minimum of a function, 164
Golden ratio, 606
Gompertz equation, 445
Grad, 936
Gradient, 1058, 1061
    of a function of three variables, 941
    of a function of two variables, 936
    normal to level curves, 940
    normal to level surfaces, 950
    properties of, 937
    recovering a function from, 1065
Graph(s)
    of absolute value function, 22
    of cosine function, 22
    of cubing function, 22
    of an equation, 2
    of a function
        guidelines for analyzing, 209
        transformation of, 23
        of two variables, 888
    of hyperbolic functions, 391
    of identity function, 22
    intercept of, 4
    of inverse hyperbolic functions, 395
    of inverse trigonometric functions, 374
    orthogonal, 147
    of parametric equations, 711
    polar, 733
        points of intersection, 743
        special polar graphs, 737
    of rational function, 22
    of sine function, 22
    of square root function, 22
    of squaring function, 22
    symmetry of, 5
Gravitational field, 1059
Greatest integer function, 72
Green, George (1793–1841), 1094

Green's Theorem, 1093
  alternative forms of, 1098, 1099
Gregory, James (1638–1675), 666
Gudermannian function, 404
Guidelines
  for analyzing the graph of a function, 209
  for evaluating integrals involving secant and tangent, 539
  for evaluating integrals involving sine and cosine, 536
  for finding extrema on a closed interval, 167
  for finding intervals on which a function is increasing or decreasing, 180
  for finding an inverse function, 346
  for finding limits at infinity of rational functions, 201
  for finding a Taylor series, 682
  for implicit differentiation, 142
  for integration, 337
  for integration by parts, 527
  for making a change of variables, 301
  for solving applied minimum and maximum problems, 219
  for solving the basic equation, 560
  for solving related-rate problems, 150
  for testing a series for convergence or divergence, 645
  for using the Fundamental Theorem of Calculus, 283
Gyration, radius of, 1017

H

Half-life, 362, 417
Hamilton, William Rowan (1805–1865), 766
Harmonic equation, 1141
Harmonic series, 621
  alternating, 634, 636, 638
Heaviside, Oliver (1850–1925), 39
Heaviside function, 39
Helix, 835
Heron's Formula, 981
Herschel, Caroline (1750–1848), 705
Higher-order derivative, 125
  of a vector-valued function, 843
  partial, 912
Homogeneous of degree n, 425, 931
Homogeneous differential equation, 425
  change of variables for, 426
Homogeneous function, 425, 931
Hooke's Law, 491
Horizontal asymptote, 199
Horizontal component of a vector, 769
Horizontal line, 14
Horizontal Line Test, 345
Horizontal shift of a graph of a function, 23
Horizontally simple region of integration, 986
Huygens, Christian (1629–1795), 478
Hypatia (370–415 A.D.), 696
Hyperbola, 696, 703
  asymptotes of, 703
  center of, 703
  conjugate axis of, 703
  eccentricity of, 704
  foci of, 703
  rotated, 146
  standard equation of, 703
  transverse axis of, 703
  vertices of, 703
Hyperbolic functions, 390
  derivatives of, 392
  graphs of, 391
  identities, 391, 392
  integrals of, 392
  inverse, 394
    differentiation involving, 396
    graphs of, 395
    integration involving, 396
Hyperbolic identities, 391, 392
Hyperbolic paraboloid, 813, 815
Hyperboloid
  of one sheet, 813, 814
  of two sheets, 813, 814
Hypocycloid, 720

I

Identities, hyperbolic, 391, 392
Identity function, 22
If and only if, 14
Image of x under f, 19
Implicit derivative, 142
Implicit differentiation, 141, 930
  Chain Rule, 930
  guidelines for, 142
Implicit form of a function, 19
Implicitly defined function, 141
Implied domain, 21
Improper integral, 580
  comparison test for, 588
  with infinite discontinuities, 583
    convergence of, 583
    divergence of, 583
  with infinite integration limits, 580
    convergence of, 580
    divergence of, 580
  special type, 586
Incidence, angle of, 698
Inclination of a plane, angle of, 949
Incompressible, 1066, 1129
Increasing function, 179
  test for, 179
Increment of z, 918
Increments of x and y, 918
Indefinite integral, 249
  pattern recognition, 297
  of a vector-valued function, 846
Indefinite integration, 249
Independence of path and conservative vector fields, 1086
Independent of path, 1086
Independent variable, 19
  of a function of two variables, 886
Indeterminate form, 63, 85, 200, 214, 569, 572
Index of summation, 259
Inductive reasoning, 601
Inequality
  Cauchy-Schwarz, 791
  Napier's, 342
  preservation of, 278, A11
  triangle, 769
Inertia, moment of, 1016, 1032
  polar, 1016
Infinite discontinuities, 580
  improper integrals with, 583
    convergence of, 583
    divergence of, 583
Infinite integration limits, 580
  improper integrals with, 580
    convergence of, 580
    divergence of, 580
Infinite interval, 198
Infinite limit(s), 83
  at infinity, 204
  from the left and from the right, 83
  properties of, 87
Infinite series (or series), 608
  absolutely convergent, 636
  alternating, 633
    geometric, 633
    harmonic, 634, 636
    remainder, 635
  conditionally convergent, 636
  convergence of, 608
  convergent, limit of nth term, 612
  divergence of, 608
    nth term test for, 612
  geometric, 610
  guidelines for testing for convergence or divergence of, 645
  harmonic, 621
    alternating, 634, 636, 638
  nth partial sum, 608
  properties of, 612
  p-series, 621
  rearrangement of, 637
  sum of, 608
  telescoping, 609
  terms of, 608
Infinity
  infinite limit at, 204
  limit at, 198, 199, A10
Inflection point, 192, 193

Initial condition(s), 253, 407
Initial point, directed line segment, 764
Initial value, 416
Inner partition, 992, 1027
    polar, 1005
Inner product
    of two functions, 544
    of two vectors, 783
Inner radius of a solid of revolution, 461
Inscribed rectangle, 263
Inside limits of integration, 985
Instantaneous velocity, 114
Integrability and continuity, 273
Integrable function, 273, 994
Integral(s)
    definite, 273
        properties of, 277
        two special, 276
    double, 992, 993, 994
    flux, 1118
    elliptic, 317
    of hyperbolic functions, 392
    improper, 580
    indefinite, 249
    involving inverse trigonometric functions, 382
    involving secant and tangent, guidelines for evaluating, 539
    involving sine and cosine, guidelines for evaluating, 536
    iterated, 985
    line, 1070
    Mean Value Theorem, 285
    of $p(x) = Ax^2 + Bx + C$, 313
    single, 994
    of the six basic trigonometric functions, 339
    surface, 1112
    trigonometric, 536
    triple, 1027
Integral Test, 619
Integrand(s), procedures for fitting to basic rules, 523
Integrating factor, 434
Integration
    as an accumulation process, 453
    Additive Interval Property, 276
    basic rules of, 250, 385, 522
    change of variables, 300
        guidelines for, 301
    constant of, 249
    of even and odd functions, 305
    guidelines for, 337
    indefinite, 249
        pattern recognition, 297
    involving inverse hyperbolic functions, 396
    Log Rule, 334
    lower limit of, 273
    of power series, 666
    preservation of inequality, 278, A11
    region R of, 985
    rules for exponential functions, 356
    upper limit of, 273
    of a vector-valued function, 846
Integration by parts, 527
    guidelines for, 527
    summary of common integrals using, 532
    tabular method, 532
Integration by tables, 563
Integration formulas
    reduction formulas, 565
    special, 549
    summary of, 1136
Integration rules
    basic, 250, 385, 522
    General Power Rule, 302
    Power Rule, 250
Integration techniques
    basic integration rules, 250, 385, 522
    integration by parts, 527
    method of partial fractions, 554
    substitution for rational functions of sine and cosine, 566
    tables, 563
    trigonometric substitution, 545
Intercept(s), 4
    x-intercept, 4
    y-intercept, 4
Interest formulas, summary of, 366
Interior point of a region R, 898, 904
Intermediate Value Theorem, 77
Interpretation of concavity, 190, A9
Interval of convergence, 662, A18
Interval, infinite, 198
Inverse function, 343
    continuity and differentiability of, 347, A13
    derivative of, 347, A14
    existence of, 345
    guidelines for finding, 346
    Horizontal Line Test, 345
    properties of, 363
    reflective property of, 344
Inverse hyperbolic functions, 394
    differentiation involving, 396
    graphs of, 395
    integration involving, 396
Inverse square field, 1059
Inverse trigonometric functions, 373
    derivatives of, 376, A15
    graphs of, 374
    integrals involving, 382
    properties of, 375
Irrotational vector field, 1064
Isobars, 148, 889
Isothermal curves, 428
Isothermal surface, 892
Isotherms, 889
Iterated integral, 985
    evaluation by, 1028
    inside limits of integration, 985
    outside limits of integration, 985
Iteration, 229
ith term of a sum, 259

J

Jacobi, Carl Gustav (1804–1851), 1045
Jacobian, 1045
Jerk function, 162

K

Kappa curve, 145, 147
Kepler, Johannes, (1571–1630), 753
Kepler's Laws, 753
Kinetic energy, 1089
Kirchhoff's Second Law, 438
Kovalevsky, Sonya (1850–1891), 898

L

Lagrange, Joseph-Louis (1736–1813), 174, 970
Lagrange form of the remainder, 656
Lagrange multiplier, 970, 971
Lagrange's Theorem, 971
Lambert, Johann Heinrich (1728–1777), 390
Lamina, planar, 502
Laplace, Pierre Simon de (1749–1827), 1038
Laplace Transform, 590
Laplace's equation, 1141
Laplacian, 1141
Lateral surface area over a curve, 1081
Latus rectum, of a parabola, 697
Law of Conservation of Energy, 1089
Leading coefficient
    of a polynomial function, 24
    test, 24
Least squares
    method of, 964
    regression, 7
        line, 964, 965
Least upper bound, 603
Left-hand limit, 72
Left-handed orientation, 775
Legendre, Adrien-Marie (1752–1833), 965
Leibniz, Gottfried Wilhelm (1646–1716), 238
Leibniz notation, 238
Lemniscate, 40, 144, 147, 737
Length
    of an arc, 478, 479
        parametric form, 724
        polar form, 745
    of a directed line segment, 764
    of the moment arm, 499

  - of a scalar multiple, 768
  - of a vector in the plane, 765
  - of a vector in space, 777
  - on x-axis, 1021
- Level curve, 889
  - gradient is normal to, 940
- Level surface, 891
  - gradient is normal to, 950
- L'Hôpital, Guillaume (1661–1704), 570
- L'Hôpital's Rule, 570, A17
- Limaçon, 737
  - convex, 737
  - dimpled, 737
  - with inner loop, 737
- Limit(s), 45, 48
  - basic, 59
  - of a composite function, 61, A4
  - definition of, 52
  - ε-δ definition of, 52
  - evaluating
    - direct substitution, 59, 60
    - divide out like factors, 63
    - rationalize the numerator, 63
  - existence of, 73
  - of a function involving a radical, 60, A4
  - of a function of two variables, 899
  - indeterminate form, 63
  - infinite, 83
    - from the left and from the right, 83
    - properties of, 87
  - at infinity, 198, 199, A10
    - infinite, 204
    - of a rational function, guidelines for finding, 201
  - of integration
    - inside, 985
    - lower, 273
    - outside, 985
    - upper, 273
  - involving e, 366, A15
  - from the left and from the right, 72
  - of the lower and upper sums, 265
  - nonexistence of, common types of behavior, 51
  - of nth term of a convergent series, 612
  - one-sided, 72
  - of polynomial and rational functions, 60
  - properties of, 59, A2
  - of a sequence, 597
    - properties of, 598
  - strategy for finding, 62
  - of trigonometric functions, 61
  - two special trigonometric, 65
  - of a vector-valued function, 837
- Limit Comparison Test, 628
- Line(s)
  - contour, 889
  - as a degenerate conic, 696
  - equation of
    - general form, 14
    - horizontal, 14
    - point-slope form, 11, 14
    - slope-intercept form, 13, 14
    - summary, 14
    - vertical, 14
  - equipotential, 889
  - least squares regression, 964, 965
  - moment about, 499
  - normal, 945, 946
    - at a point, 147
  - parallel, 14
  - perpendicular, 14
  - radial, 731
  - secant, 45, 97
  - slope of, 10
  - in space
    - direction number of, 800
    - direction vector of, 800
    - parametric equations of, 800
    - symmetric equations of, 800
  - tangent, 45, 97
    - approximation, 235
    - at the pole, 736
    - with slope m, 97
    - vertical, 99
- Line of impact, 945
- Line integral, 1070
  - for area, 1096
  - differential form of, 1077
  - evaluation of as a definite integral, 1071
  - of f along C, 1070
  - independent of path, 1086
  - summary of, 1121
  - of a vector field, 1074
- Line segment, directed, 764
- Linear approximation, 235, 920
- Linear combination of **i** and **j**, 769
- Linear function, 24
- Local maximum, 165
- Local minimum, 165
- Locus, 696
- Log Rule for Integration, 334
- Logarithmic differentiation, 329
- Logarithmic function, 24, 324
  - to base a, 363
    - derivative of, 364
  - common, 363
  - natural, 324
    - derivative of, 328
    - properties of, 325, A12
- Logarithmic properties, 325
- Logarithmic spiral, 749
- Logistic curve, 429, 562
- Logistic differential equation, 245, 429
  - carrying capacity, 429
- Logistic growth function, 367
- Lorenz curves, 456
- Lower bound of a sequence, 603
- Lower bound of summation, 259
- Lower limit of integration, 273
- Lower sum, 263
  - limit of, 265
- Lune, 553

M

- Macintyre, Sheila Scott (1910–1960), 536
- Maclaurin, Colin, (1698–1746), 678
- Maclaurin polynomial, 652
- Maclaurin series, 679
- Magnitude
  - of a directed line segment, 764
  - of a vector in the plane, 765
- Major axis of an ellipse, 699
- Marginal productivity of money, 973
- Mass, 498, 1118
  - center of, 499, 500, 501
    - of a one-dimensional system, 499, 500
    - of a planar lamina, 502
      - of variable density, 1014, 1032
    - of a solid region Q, 1032
    - of a two-dimensional system, 501
  - moments of, 1014
  - of a planar lamina of variable density, 1012
  - pound mass, 498
  - total, 500, 501
- Mathematical model, 7, 964
- Mathematical modeling, 33
- Maximum
  - absolute, 164
  - of f on I, 164
  - of a function of two variables, 954
  - global, 164
  - local, 165
  - relative, 165
- Mean Value Theorem, 174
  - alternative form of, 175
  - Extended, 245, 570, A16
  - for Integrals, 285
- Measurement, error in, 237
- Mechanic's Rule, 233
- Method of
  - Lagrange Multipliers, 970, 971
  - least squares, 964
  - partial fractions, 554
- Midpoint Formula, 776
- Midpoint Rule, 269, 313
- Minimum
  - absolute, 164
  - of f on I, 164
  - of a function of two variables, 954
  - global, 164
  - local, 165
  - relative, 165

- Minor axis of an ellipse, 699
- Mixed partial derivatives, 912
  - equality of, 913
- Möbius Strip, 1111
- Model
  - exponential growth and decay, 416
  - mathematical, 7, 964
- Modeling, mathematical, 33
- Moment(s)
  - about a line, 499
  - about the origin, 499, 500
  - about a point, 499
  - about the x-axis
    - of a planar lamina, 502
    - of a two-dimensional system, 501
  - about the y-axis
    - of a planar lamina, 502
    - of a two-dimensional system, 501
  - arm, length of, 499
  - first, 1032
  - of a force about a point, 796
  - of inertia, 1016, 1032, 1141
    - polar, 1016
    - for a space curve, 1082
  - of mass, 1014
    - of a one-dimensional system, 500
    - of a planar lamina, 502
  - second, 1016, 1032
- Monotonic sequence, 602
  - bounded, 603
- Monotonic, strictly, 180, 345
- Motion of a liquid, 1136
- Mutually orthogonal, 428

N

- n factorial, 599
- Napier, John (1550–1617), 324
- Napier's Inequality, 342
- Natural equation for a curve, 883
- Natural exponential function, 352
  - derivative of, 354
  - integration rules, 356
  - operations with, 353, A15
  - properties of, 353
  - series for, 684
- Natural logarithmic base, 327
- Natural logarithmic function, 324
  - base of, 327
  - derivative of, 328
  - properties of, 325, A12
  - series for, 684
- Negative of a vector, 766
- Net change, 291
- Net Change Theorem, 291
- Newton, Isaac (1642–1727), 96, 229
- Newton's Law of Cooling, 419
- Newton's Law of Gravitation, 1059
- Newton's Law of Universal Gravitation, 491
- Newton's Method for approximating the zeros of a function, 229
  - convergence of, 231, 232
  - iteration, 229
- Newton's Second Law of Motion, 437, 854
- Nodes, 844
- Noether, Emmy (1882–1935), 768
- Nonexistence of a limit, common types of behavior, 51
- Nonremovable discontinuity, 71, 902
- Norm
  - of a partition, 272, 992, 1005, 1027
    - polar, 1005
  - of a vector in the plane, 765
- Normal component
  - of acceleration, 862, 863, 877
  - of a vector field, 1118
- Normal line, 945, 946
  - at a point, 147
- Normal probability density function, 355
- Normal vector(s), 785
  - principal unit, 860, 877
  - to a smooth parametric surface, 1105
- Normalization of $\mathbf{v}$, 768
- Notation
  - antiderivative, 249
  - derivative, 99
  - for first partial derivatives, 909
  - function, 19
  - Leibniz, 238
  - sigma, 259
- nth Maclaurin polynomial for f at c, 652
- nth partial sum, 608
- nth Taylor polynomial for f at c, 652
- nth term
  - of a convergent series, 612
  - of a sequence, 596
- nth-Term Test for Divergence, 612
- Number, critical, 166
- Number e, 327
  - limit involving, 366, A15
- Numerical differentiation, 103

O

- Octants, 775
- Odd function, 26
  - integration of, 305
  - test for, 26
- Ohm's Law, 241
- One-dimensional system
  - center of gravity of, 500
  - center of mass of, 499, 500
  - moment of, 499, 500
  - total mass of, 500
- One-sided limit, 72
- One-to-one function, 21
- Onto function, 21
- Open disk, 898
- Open interval
  - continuous on, 70
  - differentiable on, 99
- Open region R, 898, 904
  - continuous in, 900, 904
- Open sphere, 904
- Operations
  - with exponential functions, 353, A15
  - with power series, 673
- Order of a differential equation, 406
- Orientable surface, 1117
- Orientation
  - of a curve, 1069
  - of a plane curve, 712
  - of a space curve, 834
- Oriented surface, 1117
- Origin
  - moment about, 499, 500
  - of a polar coordinate system, 731
  - reflection about, 23
  - symmetry, 5
- Orthogonal
  - functions, 544
  - graphs, 147
  - trajectory, 147, 428
  - vectors, 785
- Ostrogradsky, Michel (1801–1861), 1124
- Ostrogradsky's Theorem, 1124
- Outer radius of a solid of revolution, 461
- Outside limits of integration, 985

P

- Padé approximation, 333
- Pappus
  - Second Theorem of, 508
  - Theorem of, 505
- Parabola, 2, 146, 696, 697
  - axis of, 697
  - directrix of, 697
  - focal chord of, 697
  - focus of, 697
  - latus rectum of, 697
  - reflective property of, 698
  - standard equation of, 697
  - vertex of, 697
- Parabolic spandrel, 507
- Parallel
  - lines, 14
  - planes, 802
  - vectors, 778
- Parameter, 711
  - arc length, 870, 871
  - eliminating, 713
- Parametric equations, 711
  - graph of, 711
  - finding, 715
  - of a line in space, 800
  - for a surface, 1102

Parametric form
  of arc length, 724
  of the area of a surface of revolution, 726
  of the derivative, 721
Parametric surface, 1102
  area of, 1106
  equations for, 1102
  partial derivatives of, 1105
  smooth, 1105
    normal vector to, 1105
  surface area of, 1106
Partial derivatives, 908
  first, 908
  of a function of three or more variables, 911
  of a function of two variables, 908
  higher-order, 912
  mixed, 912
    equality of, 913
  notation for, 909
  of a parametric surface, 1105
Partial differentiation, 908
Partial fractions, 554
  decomposition of $N(x)/D(x)$ into, 555
  method of, 554
Partial sums, sequence of, 608
Particular solution of a differential equation, 253, 407
Partition
  general, 272
  inner, 992, 1027
    polar, 1005
  norm of, 272, 992, 1027
    polar, 1005
  regular, 272
Pascal, Blaise (1623–1662), 509
Pascal's Principle, 509
Path, 899, 1069
Pear-shaped quartic, 161
Percent error, 237
Perigee, 708
Perihelion, 708, 757
Perpendicular
  lines, 14
  planes, 802
  vectors, 785
Piecewise smooth curve, 716, 1069
Planar lamina, 502
  center of mass of, 502
  moment of, 502
Plane
  angle of inclination of, 949
  distance between a point and, 805
  region
    area of, 265
    simply connected, 1062, 1093
  tangent, 946
    equation of, 946
  vector in, 764
Plane curve, 711, 834
  orientation of, 712
  smooth, 1069
Plane in space
  angle between two, 802
  equation of
    general form, 801
    standard form, 801
  parallel, 802
    to the axis, 804
    to the coordinate plane, 804
  perpendicular, 802
  trace of, 802
Planimeter, 1140
Point
  as a degenerate conic, 696
  of diminishing returns, 227
  fixed, 233
  of inflection, 192, 193
  of intersection, 6
    of polar graphs, 743
  moment about, 499
  in a vector field
    incompressible, 1129
    sink, 1129
    source, 1129
Point-slope equation of a line, 11, 14
Polar axis, 731
Polar coordinate system, 731
  polar axis of, 731
  pole (or origin), 731
Polar coordinates, 731
  area in, 741
  area of a surface of revolution in, 746
  converting to rectangular, 732
  Distance Formula in, 739
Polar curve, arc length of, 745
Polar equations of conics, 751
Polar form of slope, 735
Polar graphs, 733
  cardioid, 736, 737
  circle, 737
  convex limaçon, 737
  dimpled limaçon, 737
  lemniscate, 737
  limaçon with inner loop, 737
  points of intersection, 743
  rose curve, 734, 737
Polar moment of inertia, 1016
Polar sectors, 1004
Pole, 731
  of cylindrical coordinate system, 822
  tangent lines at, 736
Polynomial
  Maclaurin, 652
  Taylor, 161, 652
Polynomial approximation, 650
  centered at c, 650
  expanded about c, 650
Polynomial function, 24, 60
  constant term of, 24
  degree of, 24
  leading coefficient of, 24
  limit of, 60
  of two variables, 887
  zero, 24
Position function, 32, 113, 125
  for a projectile, 855
Potential energy, 1089
Potential function for a vector field, 1061
Pound mass, 498
Power Rule
  for differentiation, 108, 136
  for integration, 250, 302
  for Real Exponents, 365
Power series, 661
  centered at c, 661
  convergence of, 662, A18
  convergent, form of, 678
  differentiation of, 666
  domain of, 662
  for elementary functions, 684
  endpoint convergence, 664
  geometric, 671
  integration of, 666
  interval of convergence, 662, A18
  operations with, 673
  properties of functions defined by, 666
    interval of convergence of, 666
    radius of convergence of, 666
  radius of convergence, 662, A18
  representation of functions by, 671
Preservation of inequality, 278, A11
Pressure, fluid, 509
Primary equation, 218, 219
Principal unit normal vector, 860, 877
Probability density function, 355
Procedures for fitting integrands to basic rules, 523
Product
  of two functions, 25
    inner, 544
  of two vectors in space, 792
Product Rule, 119, 136
  differential form, 238
Projectile, position function for, 855
Projection form of work, 789
Projection of **u** onto **v**, 787
  using the dot product, 788
Prolate cycloid, 723
Propagated error, 237
Properties
  of continuity, 75, A6
  of the cross product
    algebraic, 793

    - geometric, 794
  - of definite integrals, 277
  - of the derivative of a vector-valued function, 844
  - of the dot product, 783
  - of double integrals, 994
  - of functions defined by power series, 666
  - of the gradient, 937
  - of infinite limits, 87
  - of infinite series, 612
  - of inverse functions, 363
  - of inverse trigonometric functions, 375
  - of limits, 59, A2
  - of limits of sequences, 598
  - logarithmic, 325
  - of the natural exponential function, 325, 353
  - of the natural logarithmic function, 325, A12
  - of vector operations, 767
- Proportionality constant, 416
- p-series, 621
  - convergence of, 621
  - divergence of, 621
  - harmonic, 621
- Pulse function, 94
  - unit, 94
- Pursuit curve, 395, 397

Q

- Quadratic function, 24
- Quadric surface, 813
  - ellipsoid, 813, 814
  - elliptic cone, 813, 815
  - elliptic paraboloid, 813, 815
  - general form of the equation of, 813
  - hyperbolic paraboloid, 813, 815
  - hyperboloid of one sheet, 813, 814
  - hyperboloid of two sheets, 813, 814
  - standard form of the equations of, 813, 814, 815
- Quaternions, 766
- Quotient, difference, 20, 97
- Quotient Rule, 121, 136
  - differential form, 238
- Quotient of two functions, 25

R

- Radial lines, 731
- Radical, limit of a function involving a, 60, A4
- Radicals, solution by, 232
- Radioactive isotopes, half-lives of, 417
- Radius
  - of convergence, 662, A18
  - of curvature, 874
  - function, 818
  - of gyration, 1017
  - inner, 461
  - outer, 461
- Ramanujan, Srinivasa (1887–1920), 675
- Range of a function, 19
  - of two variables, 886
- Raphson, Joseph (1648–1715), 229
- Rate of change, 12, 911
  - average, 12
  - instantaneous, 12
- Ratio, 12
  - golden, 606
- Ratio Test, 641
- Rational function, 22, 25
  - guidelines for finding limits at infinity of, 201
  - limit of, 60
  - of two variables, 887
- Rationalize the numerator, 63
- Real Exponents, Power Rule, 365
- Real numbers, completeness of, 77, 603
- Real-valued function f of a real variable x, 19
- Reasoning, inductive, 601
- Recovering a function from its gradient, 1065
- Rectangle
  - area of, 261
  - circumscribed, 263
  - inscribed, 263
  - representative, 448
- Rectangular coordinates
  - converting to cylindrical, 822
  - converting to polar, 732
  - converting to spherical, 825
  - curvature in, 874, 877
- Rectifiable curve, 478
- Recursively defined sequence, 596
- Reduction formulas, 565
- Reflection
  - about the origin, 23
  - about the x-axis, 23
  - about the y-axis, 23
  - angle of, 698
  - in the line $y = x$, 344
- Reflective property
  - of an ellipse, 701
  - of inverse functions, 344
  - of a parabola, 698
- Reflective surface, 698
- Refraction, 226, 977
- Region of integration R, 985
  - horizontally simple, 986
  - r-simple, 1006
  - θ-simple, 1006
  - vertically simple, 986
- Region in the plane
  - area of, 265, 986
    - between two curves, 449
  - centroid of, 503
  - connected, 1086
- Region R
  - boundary point of, 898
  - bounded, 954
  - closed, 898
  - differentiable function in, 919
  - interior point of, 898, 904
  - open, 898, 904
    - continuous in, 900, 904
  - simply connected, 1062, 1093
- Regression, least squares, 7, 964, 965
- Regular partition, 272
- Related-rate equation, 149
- Related-rate problems, guidelines for solving, 150
- Relation, 19
- Relative error, 237
- Relative extrema
  - First Derivative Test for, 181
  - of a function, 165, 954
  - occur only at critical numbers, 166
  - occur only at critical points, 955
  - Second Derivative Test for, 194
  - Second Partials Test for, 957
- Relative maximum
  - at $(c, f(c))$, 165
  - First Derivative Test for, 181
  - of a function, 165, 954, 957
  - Second Derivative Test for, 194
  - Second Partials Test for, 957
- Relative minimum
  - at $(c, f(c))$, 165
  - First Derivative Test for, 181
  - of a function, 165, 954, 957
  - Second Derivative Test for, 194
  - Second Partials Test for, 957
- Remainder
  - alternating series, 635
  - of a Taylor polynomial, 656
- Removable discontinuity, 71
  - of a function of two variables, 902
- Representation of antiderivatives, 248
- Representative element, 453
  - disk, 458
  - rectangle, 448
  - shell, 469
  - washer, 461
- Resultant force, 770
- Resultant vector, 766
- Return wave method, 544
- Review
  - of basic differentiation rules, 378
  - of basic integration rules, 385, 522
- Revolution
  - axis of, 458
  - solid of, 458
  - surface of, 482
    - area of, 483, 726, 746

- volume of solid of
  - disk method, 458
  - shell method, 469, 470
  - washer method, 461
- Riemann, Georg Friedrich Bernhard (1826–1866), 272, 638
- Riemann sum, 272
- Riemann zeta function, 625
- Right cylinder, 812
- Right-hand limit, 72
- Right-handed orientation, 775
- Rolle, Michel (1652–1719), 172
- Rolle's Theorem, 172
- Root Test, 644
- Rose curve, 734, 737
- Rotated ellipse, 146
- Rotated hyperbola, 146
- Rotation of **F** about **N**, 1135
- *r*-simple region of integration, 1006
- Rulings of a cylinder, 812

S

- Saddle point, 957
- Scalar, 764
  - field, 889
  - multiple, 766
  - multiplication, 766, 777
  - product of two vectors, 783
  - quantity, 764
- Secant function
  - derivative of, 123, 136
  - integral of, 339
  - inverse of, 373
    - derivative of, 376
- Secant line, 45, 97
- Second derivative, 125
- Second Derivative Test, 194
- Second Fundamental Theorem of Calculus, 289
- Second moment, 1016, 1032
- Second Partials Test, 957
- Second Theorem of Pappus, 508
- Secondary equation, 219
- Second-degree equation, general, 696
- Separable differential equation, 423
- Separation of variables, 415, 423
- Sequence, 596
  - Absolute Value Theorem, 600
  - bounded, 603
  - bounded above, 603
  - bounded below, 603
  - bounded monotonic, 603
  - convergence of, 597
  - divergence of, 597
  - Fibonacci, 606, 617
  - least upper bound of, 603
  - limit of, 597
    - properties of, 598
  - lower bound of, 603
  - monotonic, 602
  - *n*th term of, 596
  - of partial sums, 608
  - pattern recognition for, 600
  - recursively defined, 596
  - Squeeze Theorem, 599
  - terms of, 596
  - upper bound of, 603
- Series, 608
  - absolutely convergent, 636
  - alternating, 633
    - geometric, 633
    - harmonic, 634, 636, 638
  - Alternating Series Test, 633
  - binomial, 683
  - conditionally convergent, 636
  - convergence of, 608
  - convergent, limit of *n*th term, 612
  - Direct Comparison Test, 626
  - divergence of, 608
    - *n*th term test for, 612
  - finite Fourier, 544
  - Fourier Sine, 535
  - geometric, 610
    - alternating, 633
    - convergence of, 610
    - divergence of, 610
  - guidelines for testing for convergence or divergence, 645
  - harmonic, 621
    - alternating, 634, 636, 638
  - infinite, 608
    - properties of, 612
  - Integral Test, 619
  - Limit Comparison Test, 628
  - Maclaurin, 679
  - *n*th partial sum, 608
  - *n*th term of convergent, 612
  - power, 661
  - *p*-series, 621
  - Ratio Test, 641
  - rearrangement of, 637
  - Root Test, 644
  - sum of, 608
  - summary of tests for, 646
  - Taylor, 678, 679
  - telescoping, 609
  - terms of, 608
- Serpentine, 127
- Shell method, 469, 470
  - and disk method, comparison of, 471
- Shift of a graph
  - horizontal, 23
  - vertical, 23
- Sigma notation, 259
  - index of summation, 259
  - *i*th term, 259
  - lower bound of summation, 259
  - upper bound of summation, 259
- Signum function, 82
- Simple curve, 1093
- Simple Power Rule, 108, 136
- Simple solid region, 1125
- Simply connected plane region, 1093
- Simpson's Rule, 314
  - error in, 315
- Sine function, 22
  - derivative of, 112, 136
  - integral of, 339
  - inverse of, 373
    - derivative of, 376, A15
  - series for, 684
- Sine integral function, 322
- Sine Series, Fourier, 535
- Single integral, 994
- Singular solution, differential equation, 406
- Sink, 1129
- Slant asymptote, 211
- Slope(s)
  - field, 256, 306, 325, 408
  - of the graph of *f* at $x = c$, 97
  - of a line, 10
  - of a surface in *x*- and *y*-directions, 909
  - of a tangent line, 97
    - parametric form, 721
    - polar form, 735
- Slope-intercept equation of a line, 13, 14
- Smooth
  - curve, 478, 716, 844, 859
    - on an open interval, 844
    - piecewise, 716
  - parametric surface, 1105
  - plane curve, 1069
  - space curve, 1069
- Snell's Law of Refraction, 226, 977
- Solenoidal, 1066
- Solid region, simple, 1125
- Solid of revolution, 458
  - volume of
    - disk method, 458
    - shell method, 469, 470
    - washer method, 461
- Solution
  - curves, 407
  - of a differential equation, 406
    - Bernoulli, 439
    - Euler's Method, 410
    - first-order linear, 435
    - general, 249, 406
    - particular, 253, 407
    - singular, 406
  - point of an equation, 2
  - by radicals, 232
- Some basic limits, 59

Somerville, Mary Fairfax (1780–1872), 886
Source, 1129
Space curve, 834
  arc length of, 869
  moments of inertia for, 1082
  smooth, 1069
Spandrel, parabolic, 507
Special integration formulas, 549
Special polar graphs, 737
Special type of improper integral, 586
Speed, 114, 850, 851, 875, 877
  angular, 1017
Sphere, 776
  astroidal, 1111
  open, 904
  standard equation of, 776
Spherical coordinate system, 825
  converting to cylindrical coordinates, 825
  converting to rectangular coordinates, 825
Spiral
  of Archimedes, 725, 733, 749
  cornu, 761, 883
  logarithmic, 749
Spring constant, 34
Square root function, 22
Squared errors, sum of, 964
Squaring function, 22
Squeeze Theorem, 65, A5
  for Sequences, 599
Standard equation of
  an ellipse, 699
  a hyperbola, 703
  a parabola, 697
  a sphere, 776
Standard form of the equation of
  an ellipse, 699
  a hyperbola, 703
  a parabola, 697
  a plane in space, 801
  a quadric surface, 813, 814, 815
Standard form of a first-order linear differential equation, 434
Standard normal probability density function, 355
Standard position of a vector, 765
Standard unit vector, 769
  notation, 777
Step function, 72
Stirling's approximation, 529
Stirling's Formula, 360
Stokes, George Gabriel (1819–1903), 1132
Stokes's Theorem, 1098, 1132
Strategy for finding limits, 62
Strictly monotonic function, 180, 345
Strophoid, 761
Substitution for rational functions of sine and cosine, 566
Sufficient condition for differentiability, 919, A19
Sum(s)
  ith term of, 259
  lower, 263
    limit of, 265
  nth partial, 608
  Riemann, 272
  Rule, 111, 136
    differential form, 238
  of a series, 608
  sequence of partial, 608
  of the squared errors, 964
  of two functions, 25
  of two vectors, 766
  upper, 263
    limit of, 265
Summary
  of common integrals using integration by parts, 532
  of compound interest formulas, 366
  of curve sketching, 209
  of differentiation rules, 136
  of equations of lines, 14
  of first-order differential equations, 440
  of integration formulas, 1136
  of line and surface integrals, 1121
  of tests for series, 646
  of velocity, acceleration, and curvature, 877
Summation
  formulas, 260, A10
  index of, 259
  lower bound of, 259
  upper bound of, 259
Surface
  closed, 1124
  cylindrical, 812
  isothermal, 892
  level, 891
  orientable, 1117
  oriented, 1117
  parametric, 1102
  parametric equations for, 1102
  quadric, 813
  reflective, 698
  trace of, 813
Surface area
  of a parametric surface, 1106
  of a solid, 1020, 1021
Surface integral, 1112
  evaluating, 1112
  summary of, 1121
Surface of revolution, 482, 818
  area of, 483
    parametric form, 726
    polar form, 746
Symmetric equations, line in space, 800
Symmetry
  tests for, 5
  with respect to the origin, 5
  with respect to the point (a, b), 403
  with respect to the x-axis, 5
  with respect to the y-axis, 5

T

Table of values, 2
Tables, integration by, 563
Tabular method for integration by parts, 532
Tangent function
  derivative of, 123, 136
  integral of, 339
  inverse of, 373
    derivative of, 376
Tangent line(s), 45, 97
  approximation of f at c, 235
  to a curve, 860
  at the pole, 736
  problem, 45
  slope of, 97
    parametric form, 721
    polar form, 735
  with slope m, 97
  vertical, 99
Tangent plane, 946
  equation of, 946
Tangent vector, 850
Tangential component of acceleration, 862, 863, 877
Tautochrone problem, 717
Taylor, Brook (1685–1731), 652
Taylor polynomial, 161, 652
  error in approximating, 656
  remainder, Lagrange form of, 656
Taylor series, 678, 679
  convergence of, 680
  guidelines for finding, 682
Taylor's Theorem, 656, A17
Telescoping series, 609
Terminal point, directed line segment, 764
Terms
  of a sequence, 596
  of a series, 608
Test(s)
  comparison, for improper integrals, 588
  for concavity, 191, A9
  conservative vector field in the plane, 1062
  conservative vector field in space, 1065
  for convergence
    Alternating Series, 633
    Direct Comparison, 626
    geometric series, 610
    guidelines, 645
    Integral, 619

    - Limit Comparison, 628
    - p-series, 621
    - Ratio, 641
    - Root, 644
    - summary, 646
  - for even and odd functions, 26
  - First Derivative, 181
  - Horizontal Line, 345
  - for increasing and decreasing functions, 179
  - Leading Coefficient, 24
  - Second Derivative, 194
  - for symmetry, 5
  - Vertical Line, 22
- Theorem
  - Absolute Value, 600
  - of Calculus, Fundamental, 282
    - guidelines for using, 283
  - of Calculus, Second Fundamental, 289
  - Cavalieri's, 468
  - Darboux's, 245
  - existence, 77, 164
  - Extended Mean Value, 245, 570, A16
  - Extreme Value, 164, 954
  - Intermediate Value, 77
  - Mean Value, 174
    - alternative form, 175
    - Extended, 245, 570, A16
    - for Integrals, 285
  - Net Change, 291
  - of Pappus, 505
    - Second, 508
  - Rolle's, 172
  - Squeeze, 65, A5
    - for sequences, 599
  - Taylor's, 656, A17
- Theta, θ
  - simple region of integration, 1006
- Third derivative, 125
- Three-dimensional coordinate system, 775
  - left-handed orientation, 775
  - right-handed orientation, 775
- Top half of circle, 138
- Topographic map, 889
- Torque, 500, 796
- Torricelli's Law, 445
- Torsion, 884
- Total differential, 918
- Total distance traveled on $[a, b]$, 292
- Total mass, 500, 501
  - of a one-dimensional system, 500
  - of a two-dimensional system, 501
- Trace
  - of a plane in space, 802
  - of a surface, 813
- Tractrix, 333, 395, 396
- Trajectories, orthogonal, 147, 428
- Transcendental function, 25, 378
- Transformation, 23, 1046
- Transformation of a graph of a function, 23
  - basic types, 23
  - horizontal shift, 23
  - reflection about origin, 23
  - reflection about x-axis, 23
  - reflection about y-axis, 23
  - reflection in the line $y = x$, 344
  - vertical shift, 23
- Transverse axis of a hyperbola, 703
- Trapezoidal Rule, 312
  - error in, 315
- Triangle inequality, 769
- Trigonometric function(s), 24
  - and the Chain Rule, 135
  - cosine, 22
  - derivative of, 123, 136
  - integrals of the six basic, 339
  - inverse, 373
    - derivatives of, 376, A15
    - graphs of, 374
    - integrals involving, 382
    - properties of, 375
  - limit of, 61
  - sine, 22
- Trigonometric integrals, 536
- Trigonometric substitution, 545
- Triple integral, 1027
  - in cylindrical coordinates, 1038
  - in spherical coordinates, 1041
- Triple scalar product, 796
  - geometric property of, 797
- Two-dimensional system
  - center of gravity of, 501
  - center of mass of, 501
  - moment of, 501
  - total mass of, 501
- Two-Point Gaussian Quadrature Approximation, 321
- Two special definite integrals, 276
- Two special trigonometric limits, 65

U

- Unit pulse function, 94
- Unit tangent vector, 859, 877
- Unit vector, 765
  - in the direction of **v**, 768, 777
  - standard, 769
- Universal Gravitation, Newton's Law, 491
- Upper bound
  - least, 603
  - of a sequence, 603
  - of summation, 259
- Upper limit of integration, 273
- Upper sum, 263
  - limit of, 265
- u-substitution, 297

V

- Value of f at x, 19
- Variable
  - dependent, 19
  - dummy, 275
  - force, 490
  - independent, 19
- Vector(s)
  - acceleration, 862, 877
  - addition, 766, 767
    - associative property of, 767
    - commutative property of, 767
  - Additive Identity Property, 767
  - Additive Inverse Property, 767
  - angle between two, 784
  - component
    - of **u** along **v**, 787
    - of **u** orthogonal to **v**, 787
  - component form of, 765
  - components, 765, 787
  - cross product of, 792
  - difference of two, 766
  - direction, 800
  - direction angles of, 786
  - direction cosines of, 786
  - Distributive Property, 767
  - dot product of, 783
  - equal, 765, 777
  - horizontal component of, 769
  - initial point, 764
  - inner product of, 783
  - length of, 765, 777
  - linear combination of, 769
  - magnitude of, 765
  - negative of, 766
  - norm of, 765
  - normal, 785
  - normalization of, 768
  - operations, properties of, 767
  - orthogonal, 785
  - parallel, 778
  - perpendicular, 785
  - in the plane, 764
  - principal unit normal, 860, 877
  - product of two vectors in space, 792
  - projection of, 787
  - resultant, 766
  - scalar multiplication, 766, 777
  - scalar product of, 783
  - in space, 777
  - standard position, 765
  - standard unit notation, 777
  - sum, 766
  - tangent, 850
  - terminal point, 764
  - triple scalar product, 796
  - unit, 765

    in the direction of $\mathbf{v}$, 768, 777
    standard, 769
  unit tangent, 859, 877
  velocity, 850, 877
  vertical component of, 769
  zero, 765, 777
Vector field, 1058
  circulation of, 1135
  conservative, 1061, 1083
    test for, 1062, 1065
  continuous, 1058
  curl of, 1064
  divergence of, 1066
  divergence-free, 1066
  incompressible, 1129
  irrotational, 1064
  line integral of, 1074
  normal component of, 1118
  over a plane region R, 1058
  over a solid region Q, 1058
  potential function for, 1061
  rotation of, 1135
  sink, 1129
  solenoidal, 1066
  source, 1129
Vector space, 768
  axioms, 768
Vector-valued function(s), 834
  antiderivative of, 846
  continuity of, 838
  continuous on an interval, 838
  continuous at a point, 838
  definite integral of, 846
  derivative of, 842
    higher-order, 843
    properties of, 843
  differentiation of, 843
  domain of, 835
  indefinite integral of, 846
  integration of, 846
  limit of, 837
Velocity, 114, 851
  average, 113
  escape, 94
  function, 125
  instantaneous, 114
  potential curves, 428
Velocity field, 1058, 1059
  incompressible, 1066
Velocity vector, 850, 877
Vertéré, 201
Vertex
  of an ellipse, 699
  of a hyperbola, 703
  of a parabola, 697
Vertical asymptote, 84, 85, A7
Vertical component of a vector, 769
Vertical line, 14
Vertical Line Test, 22
Vertical shift of a graph of a function, 23
Vertical tangent line, 99
Vertically simple region of integration, 986
Volume of a solid
  disk method, 459
  with known cross sections, 463
  shell method, 469, 470
  washer method, 461
Volume of a solid region, 994, 1027

W

Wallis, John (1616–1703), 538
Wallis's Formulas, 538, 544
Washer, 461
Washer method, 461
Weierstrass, Karl (1815–1897), 955
Weight-densities of fluids, 509
Wheeler, Anna Johnson Pell (1883–1966), 435
Witch of Agnesi, 127, 146, 201, 841
Work, 489, 789
  done by a constant force, 489
  done by a variable force, 490
  dot product form, 789
  force field, 1074
  projection form, 789

X

x-axis
  moment about, of a planar lamina, 502
  moment about, of a two-dimensional system, 501
  reflection about, 23
  symmetry, 5
x-intercept, 4
xy-plane, 775
xz-plane, 775

Y

y-axis
  moment about, of a planar lamina, 502
  moment about, of a two-dimensional system, 501
  reflection about, 23
  symmetry, 5
y-intercept, 4
Young, Grace Chisholm (1868–1944), 45
yz-plane, 775

Z

Zero factorial, 599
Zero of a function, 26
  approximating
    bisection method, 78
    Intermediate Value Theorem, 77
    with Newton's Method, 229
Zero polynomial, 24
Zero vector, 765, 777

ALGEBRA

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, and $0 \le b^2 - 4ac$, then the real zeros of p are $x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$.

Special Factors

$$x^2 - a^2 = (x - a)(x + a)$$

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

$$x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$$

Binomial Theorem

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$$

$$(x - y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$$acx^3 + adx^2 + bcx + bd = ax^2(cx + d) + b(cx + d) = (ax^2 + b)(cx + d)$$

Arithmetic Operations

$$ab + ac = a(b + c)$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc}$$

$$\frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc}$$

$$\frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c}$$

$$\frac{a - b}{c - d} = \frac{b - a}{d - c}$$

$$\frac{ab + ac}{a} = b + c$$

Exponents and Radicals

$$a^0 = 1, \quad a \ne 0$$

$$(ab)^x = a^x b^x$$

$$a^x a^y = a^{x+y}$$

$$\sqrt{a} = a^{1/2}$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$\sqrt[n]{a} = a^{1/n}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$\sqrt[n]{a^m} = a^{m/n}$$

$$a^{-x} = \frac{1}{a^x}$$

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$$

$$(a^x)^y = a^{xy}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

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FORMULAS FROM GEOMETRY

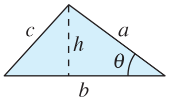
Triangle

$h = a \sin\theta$

$\text{Area} = \frac{1}{2}bh$

(Law of Cosines)

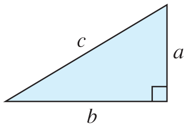
$c^2 = a^2 + b^2 - 2ab\cos\theta$



Right Triangle

(Pythagorean Theorem)

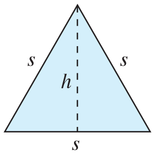
$c^2 = a^2 + b^2$



Equilateral Triangle

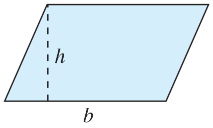
$h = \frac{\sqrt{3}s}{2}$

$\text{Area} = \frac{\sqrt{3}s^2}{4}$



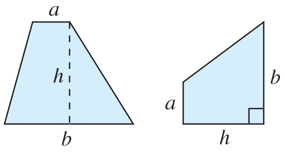
Parallelogram

$\text{Area} = bh$



Trapezoid

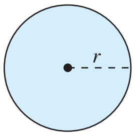
$\text{Area} = \frac{h}{2}(a + b)$



Circle

$\text{Area} = \pi r^2$

$\text{Circumference} = 2\pi r$

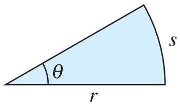


Sector of Circle

(θ in radians)

$\text{Area} = \frac{\theta r^2}{2}$

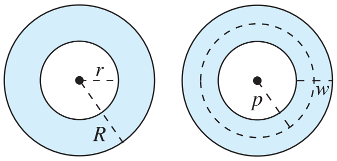
$s = r\theta$



Circular Ring

(p = average radius,
w = width of ring)

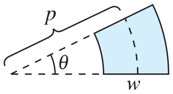
$\text{Area} = \pi(R^2 - r^2)$
$= 2\pi pw$



Sector of Circular Ring

(p = average radius,
w = width of ring,
θ in radians)

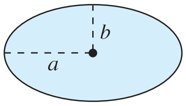
$\text{Area} = \theta pw$



Ellipse

$\text{Area} = \pi ab$

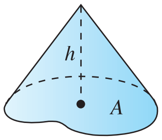
$\text{Circumference} \approx 2\pi\sqrt{\frac{a^2 + b^2}{2}}$



Cone

(A = area of base)

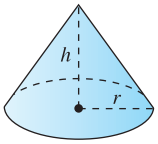
$\text{Volume} = \frac{Ah}{3}$



Right Circular Cone

$\text{Volume} = \frac{\pi r^2 h}{3}$

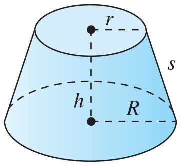
$\text{Lateral Surface Area} = \pi r\sqrt{r^2 + h^2}$



Frustum of Right Circular Cone

$\text{Volume} = \frac{\pi(r^2 + rR + R^2)h}{3}$

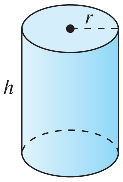
$\text{Lateral Surface Area} = \pi s(R + r)$



Right Circular Cylinder

$\text{Volume} = \pi r^2 h$

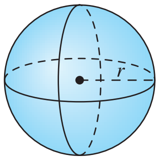
$\text{Lateral Surface Area} = 2\pi rh$



Sphere

$\text{Volume} = \frac{4}{3}\pi r^3$

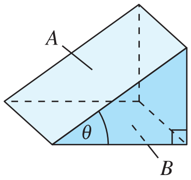
$\text{Surface Area} = 4\pi r^2$



Wedge

(A = area of upper face,
B = area of base)

$A = B\sec\theta$



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Tear out Formula Cards for Homework Success.

Index of Applications (continued from front inside cover)

Sound intensity, 40, 333, 422
Speed, 29, 177, 880, 969
  of sound, 287
Statics problems, 506
Stopping distance, 118, 129, 159, 241
Surface area, 154, 160
  of a dome, 1111
  of an oil spill, 455
  of a pond, 515
  of a satellite-signal receiving dish, 708
Suspension bridge, 488
Temperature, 18, 177, 207, 350, 413, 445
  of a house, 309, 310
  at which water boils, 333
Temperature distribution, 891, 896, 916, 938, 943, 944, 977, 981
Topography, 889, 944
Torque, 796, 798, 799, 829
Tossing bales, 857
Velocity, 118, 177, 294, 318
  of a diver, 114
  of a piston, 153
  of a rocket, 594
Velocity and acceleration, 318, 322
  on the moon, 162
Velocity in a resisting medium, 578
Vertical motion, 117, 158, 176, 177, 254, 257, 389, 400, 444
Vibrating spring, 158
Vibrating string, 535
Volume, 82, 118, 127, 154
  of a box, 30, 921
  of fluid in a storage tank, 552
  of a goblet, 879
  of the Great Salt Lake, 1056
  of ice, 1011
  of a pond, 476
  of a pontoon, 473
  of a pyramid, 464
  of a shampoo bottle, 225
  of a spherical ring, 517
  of a trough, 924
  of water in a conical tank, 149
  of water in a fire truck tank, 709
Water depth in a tank, 467
Water flow, 296
Water running into a vase, 196
Wave equation, 915, 982
Wind chill, 924
Work, 317, 516
  done by aircraft engines, 1139
  done in closing a door, 789
  done by an expanding gas, 494
  done by a force field, 1080, 1082, 1091, 1139, 1142
  done by a hydraulic cylinder, 568
  done in lifting a chain, 494, 496, 516
  done in moving a particle, 1100, 1142
  done by a person walking up a staircase, 1082
  done in pulling a sled, 791
  done in pulling a toy wagon, 791
  done in splitting a piece of wood, 497
  done in towing a car, 791
Wrinkled and bumpy spheres, 1044

Business and Economics

Annuities, 617
Average production, 1002
Average profit, 1052
Average sales, 294
Break-even analysis, 37
Break-even point, 9
Capitalized cost, 589
Cash flow, 308
Cobb-Douglas production function, 891, 896, 973, 981
Compound interest, 367, 370, 402, 421, 578, 605, 690, 691
Consumer price index, 9
Consumer and producer surpluses, 518
Cost, 140, 295, 350
Declining sales, 418
Demand, 968
Demand function, 244
Depreciation, 308, 359, 370, 402, 616, 690
Eliminating budget deficits, 456
Federal debt, 606
Home mortgage, 333, 404
Inflation, 369, 606
Inventory cost, 197, 243
Inventory management, 82, 118
Inventory replenishment, 127
Investment, 896, 916
Investment growth, 440, 441
Manufacturing, 310, 463, 468
Marginal costs, 916
Marginal productivity, 916
Marginal productivity of money, 973
Marginal revenue, 916
Marginal utility, 916
Marketing, 616
Maximum profit, 227, 963, 967, 980
Maximum revenue, 967
Minimum cost, 966, 967, 977, 980
Personal income, 606
Present value, 535, 616
Profit, 38, 457
Revenue, 456, 790
Salary, 617, 691
Sales, 29, 177, 309, 342, 443, 445
  Wal-Mart, 897
Sales growth, 197, 243
Value of a mid-sized sedan, 360

Social and Behavioral Sciences

Cellular phone subscribers, 9
Crime, 234
Health maintenance organizations, 35
Learning curve, 421, 422, 441
Memory model, 535
Outlays for national defense, 243
Population, 421, 1011
  of Colorado, 12
  of United States, 16, 422
Population growth, 440, 443
Psychology, intelligence test, 916
Women in the work force, 968
World population, 969
World record times for running one mile, 207

Life Sciences

Bacterial culture growth, 367, 421, 433
Blood flow, 294
Carbon dioxide concentration, 7
Circulatory system, 139
Concentration of a tracer drug in a fluid, 446
DNA molecule, 835
Endangered species, 433
Epidemic model, 562
Forestry, 422, 896
Intravenous feeding, 441
Models for tumors, 1044
Organ transplants, 371
Population, 568
Population growth, 694
  of bacteria, 127, 256, 342
  of brook trout, 444
  of coyotes, 427
  of fish, 371
  of fruit flies, 418
Respiratory cycle, 294, 320
Trachea contraction, 188

General

Applicants to a university, 916
Average typing speed, 197, 207
Dental inlays, 832
Folding paper, 246
Möbius Strip, 1111
Probability, 309, 361, 589, 616, 677, 688, 1003, 1011, 1052
Queuing model, 896
School commute, 27, 28
Sphereflake, 617
Spiral staircase, 881
Throwing a dart, 270